USAL

جامعة العلوم والآداب اللبنانية

UNIVERSITY OF SCIENCES & ARTS IN LEBANON

Chapter 3:

Conditional Probability and Independence

MATH204: Probability and Statistics I

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Conditional Probability and Independence

Section 1

Conditional probabilities

Section 2

Bayes’s formula

Section 3

Independent events

3.1

Conditional Probabilities

3.1. Conditional Probabilities

Activity 1

Roll a fair die, what is the probability that the outcome is an even number given that it was less than or equal to 3.

Solution

The new sample space is $\{1,2,3\}$ and the event is $\{2\}$. Thus, the probability is $\frac{1}{3}$

3.1. Conditional Probabilities

Activity 2

Suppose that we toss 2 dice. Suppose that we observe that the first die is a 3. Then, given this information, what is the probability that the sum of the 2 dice equals 8?

Solution

The new sample space is $\{\{3,1\},\{3,2\},\{3,3\},\{3,4\},\{3,5\},\{3,6\}\}$ and the event is $\{\{3,5\}\}$. Thus, the probability is $\frac{1}{6}$

3.1. Conditional Probabilities

Definition 1

The probability of an event A occurring when it is known that some event B has occurred is called a **conditional probability** and is denoted by P(A|B). The symbol P(A|B) is usually read “the probability of A, given B.”

$$\boldsymbol{P(A|B) = \frac{P(A \cap B)}{P(B)}}$$

3.1. Conditional Probabilities

Example 1

A coin is flipped twice. Assuming that all four points in the sample space S = {(h, h), (h, t),(t, h),(t, t)} are equally likely, what is the conditional probability that both flips land on heads, given that

i. the first flip lands on heads?

ii. at least one flip lands on heads?

Solution

Let A: “Both flips land on heads”

i. Let B: “The first flip lands on head”. Then: $P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{1}{4}}{\frac{1}{2}} = \frac{1}{2}$

ii. Let C: “At least one flip land on head”. Then, $P(A|C) = \frac{P(A \cap C)}{P(C)} = \frac{\frac{1}{4}}{\frac{3}{4}} = \frac{1}{3}$

3.1. Conditional Probabilities

Example 2

A student is taking a one-hour-time-limit makeup examination. Suppose the probability that the student will finish the exam in less than x hours is $\frac{x}{2}$, for all $0 \leq x \leq 1$.

i. What is the probability that the student will finish the exam in at least x hours?

ii. Given that the student needs at least 0.75 hour to finish, what is the conditional probability that the full hour is used?

Solution

i. Let A_x: "The student will finish the exam in at least x hours".

$P(A_x) = 1 - P(\text{The student will finish the exam in less than } x \text{ hours}) = 1 - \frac{x}{2} = \frac{2-x}{2}$

ii. $P\left(\frac{A_1}{A_{0.75}}\right) = \frac{P(A_1 \cap A_{0.75})}{P(A_{0.75})} = \frac{P(A_1)}{P(A_{0.75})} = \frac{\frac{2-1}{2}}{\frac{2-0.75}{2}} = 0.8$

3.1. Conditional Probabilities

Example 3

Suppose that our sample space S is the population of adults in a small town who have completed the requirements for a college degree. We shall categorize them according to gender and employment status. The data are given in the following table

	Employed	Unemployed	Total
Male	460	40	500
Female	140	260	400
Total	600	300	900

One of these individuals is to be selected at random. Consider the following events:

M: a man is chosen,

F: Female is chosen

E: the one chosen is employed

$\bar{E}$: the one chosen is unemployed

Calculate $P(M)$, $P(E)$, $P(E \cap F)$, $P(E|F)$, and $P(M|\bar{E})$

3.1. Conditional Probabilities

Solution

$$P(M) = \frac{|\mathrm{M}|}{|\Omega|} = \frac{460}{900},\ P(E) = \frac{|E|}{|\Omega|} = \frac{600}{900},\ P(E \cap F) = \frac{|E \cap F|}{|\Omega|} = \frac{140}{900},\ P(E|F) = \frac{P(E \cap F)}{P(F)} = \frac{\frac{140}{900}}{\frac{600}{900}} = \frac{7}{30},\ P\left(M\middle|\overline{E}\right) = \frac{P\left(M \cap \overline{E}\right)}{P(\overline{E})} = \frac{\frac{40}{900}}{\frac{260}{900}} = \frac{2}{13}$$

3.1. Conditional Probabilities

Example 4

In the card game bridge, the 52 cards are dealt out equally to 4 players—called East, West, North, and South. If North and South have a total of 8 spades among them, what is the probability that East has 3 of the remaining 5 spades?

Solution

The new sample space is the set of the ways of distribution of 26 cards, among them there are 5 of spades, on East and West. So $|\Omega| = \binom{26}{13,13} = \frac{26!}{13!13!}$

The cardinal of the event is $|A| = \binom{5}{3}\binom{21}{10}\binom{13}{13}$

Thus, the probability is: $\frac{\binom{5}{3}\binom{21}{10}\binom{13}{13}}{\binom{26}{13,13}}$

3.1. Conditional Probabilities

Example 5

Celine is undecided as to whether to take a French course or a chemistry course. She estimates that her probability of receiving an A grade would be 1/2 in a French course and 2/3 in a chemistry course. If Celine decides to base her decision on the flip of a fair coin, what is the probability that she gets an A in chemistry?

Solution

Consider the following events:
A: “She receive an A”
C: “She choose the chemistry course”
$P(C) = \frac{1}{2}$, $P(A|C) = \frac{2}{3}$
Then, $P(A \cap C) = P(A|C) \times P(C) = \frac{2}{3} \times \frac{1}{2} = \frac{1}{3}$