Plasma Physics with Applications

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What is a plasma

A plasma is defined via the **ionization degree** α defined as the fraction of ionized particles. At $\alpha > 0.01$ we say it is **fully ionized**, while for smaller α:s its **weakly / partially ionized**. One important property of a plasma is the ability to quickly screen out changes in the electric potential, so called **Debye Screening**. Debye screening is described by the **Debye Length** and **Debye potential**

$$\lambda_{\mathrm{D}} = \sqrt{\frac{\varepsilon_0 k_{\mathrm{B}} T}{n_0 e^2}}$$
$$\phi_{\mathrm{D}} = \frac{q}{4\pi\varepsilon_0 r}\exp\left(-\frac{r}{\lambda_{\mathrm{D}}}\right),$$

(p. 4 – 5). With the plasma frequency $\omega_{\mathrm{p}}^2 = n_0 e^2/m_e\varepsilon_0$, the collision frequency ν and the macroscopic dimension of the plasma L, the definition of a plasma requires

$$\lambda_{\mathrm{D}} \ll L$$
$$\frac{4\pi}{3}\lambda_{\mathrm{D}}^3 n_0 \gg 1$$
$$\nu \ll \omega_{\mathrm{p}}.$$

Single particles in EM-field

The most fundamental equation of motion for a charged particle in an EM-field is the **Lorentz Force Equation**:

$$m\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = q(\boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{B})$$

A constant, nonzero E-field with no B-field gives constant acceleration along $\boldsymbol{E}$. A constant, nonzero B-field with no E-field gives rise to helical motion with **cyclotron frequency** $\omega_{\mathrm{c}} = |q|B/m$ and **Larmor radius** $r_{\mathrm{L}} = v_\perp/\omega_{\mathrm{c}}$ (p. 8 – 9, trick is to divide velocity into one component parallel and one perpendicular to $\boldsymbol{B}$). In this circular orbit we have the current and magnetic moment

$$I = \frac{|q|\omega_{\mathrm{c}}}{2\pi} = \frac{1}{2\pi}\frac{q^2 B}{m}$$
$$\mu = I\pi r_{\mathrm{L}}^2 = \frac{m v_\perp^2}{2B} = \left|\frac{q r_{\mathrm{L}} v_\perp}{2}\right|,\ \boldsymbol{\mu} = \frac{q\boldsymbol{r}_{\mathrm{L}}\times\boldsymbol{v}_\perp}{2}.$$

A constant, nonzero B-field with a constant force $\boldsymbol{F}$ gives a constant acceleration from the component of $\boldsymbol{F}$ along the B-field, while the component perpendicular to $\boldsymbol{B}$ gives rise to a constant **drift velocity**

$$\boldsymbol{v}_D = \frac{1}{q}\frac{\boldsymbol{F}_\perp\times\boldsymbol{B}}{B^2}$$

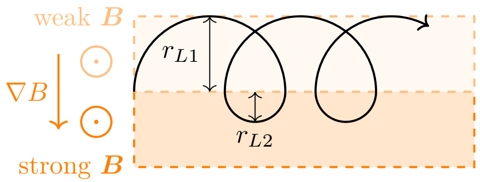
(p. 11, trick is to divide $\boldsymbol{v}_\perp$ into a constant drift velocity $\boldsymbol{v}_{\mathrm{D}}$ and a varying component u which solves the case without a force). An *inhomogeneous* B-field gives rise to the so called **grad B drift**. The motion parallel to the B-field lines is governed by

$$m\frac{\mathrm{d}\boldsymbol{v}_\parallel}{\mathrm{d}t} = -\mu\nabla_\parallel B,$$

and for the motion perpendicular we get a drift velocity

$$\boldsymbol{v}_{\nabla B} = \frac{\mu}{q}\frac{\boldsymbol{B}\times\nabla B}{B^2} = \frac{m v_\perp^2}{2qB^2}\frac{\boldsymbol{B}\times\nabla B}{B}$$

in addition to the Larmor rotation, just like before (p. 13 – 16).



When the magnetic field lines are curved, $|B|$ cannot be constant according to Maxwell's equations. Thus this such a magnetic field must be inhomogenous and this gives rise both to a **grad B drift** and a **curvature drift**

$$\boldsymbol{v}_{\mathrm{c}} = \frac{m v_\parallel^2}{qB^2}\frac{\boldsymbol{B}\times\nabla B}{B}$$
$$\implies \boldsymbol{v}_{\nabla B} + \boldsymbol{v}_{\mathrm{c}} = \frac{m}{qB^2}\left(\frac{1}{2}v_\perp^2 + v_\parallel^2\right)\frac{\boldsymbol{B}\times\nabla B}{B},$$

(p. 19). The final relevant drift velocity is the **polarization drift** occuring in a slowly varying electric field

$$\boldsymbol{v}_{\mathrm{p}} = \frac{m}{qB^2}\frac{\mathrm{d}\boldsymbol{E}}{\mathrm{d}t},$$

(p. 42). For a inhomogeneous magnetic field constant in time the magnetic moment is invariant (p. 23). For a slowly varying magnetic field , this is also true which is called the first adiabatic invariant (p. 44 – 45). The second adiabatic invariant is the integral

$$J = \oint v_\parallel ds$$

(p. 46 – 47).

The Vlaslov and Boltzmann equations

Firstly we define a **distribution function**, $f(\boldsymbol{r},\boldsymbol{v},t)$. This gives the expected number of particles occupying the volume in phase space $\mathrm{d}\boldsymbol{r}\mathrm{d}\boldsymbol{v}$ around the point $(\boldsymbol{r},\boldsymbol{v})$ at time t. To understand how this f evolves argue that the rate of change of the number of particles inside some volume in phase space Ω must be the flux of particles through the walls.

$$\frac{\partial}{\partial t}\int_\Omega f\mathrm{d}\boldsymbol{r}\mathrm{d}\boldsymbol{v} = -\int_{\partial_r\Omega} f\boldsymbol{v}\cdot\hat{n}_r\mathrm{d}S_r - \int_{\partial_v\Omega} f\boldsymbol{a}\cdot\hat{n}_v\mathrm{d}S_v.$$

$\partial_r\Omega$ means the boundry of Ω in r-space. Plopping the $\frac{\partial}{\partial t}$ into the integral and using Gauss' theorem to make the surface integral a volume integral we obtain

$$\mathrm{d}\boldsymbol{r}\mathrm{d}\boldsymbol{v}\left[\frac{\partial f}{\partial t} + \frac{\partial}{\partial\boldsymbol{r}}\cdot\boldsymbol{v}f + \frac{\partial}{\partial\boldsymbol{v}}\cdot\boldsymbol{a}f\right] = 0,$$

which has to be true for all volumes, meaning that the integrand has to be 0 everywhere. Further noting that $\frac{\partial}{\partial r}\cdot\boldsymbol{v} = 0$ and plugging in the Lorentz force for the acceleration we get the **Vlaslov equation**

$$\frac{\partial f}{\partial t} + \boldsymbol{v}\cdot\frac{\partial f}{\partial\boldsymbol{r}} + \frac{q}{m}(\boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{B})\cdot\frac{\partial f}{\partial\boldsymbol{v}} = 0.$$

The LHS is really the change of f along a particle trajectory (if you let $\boldsymbol{r}(t)$ and $\boldsymbol{v}(t)$ be the particle position and velocity at time t), so you can let f be the sum of a bunch of delta functions and reduce this to the particle description of the plasma. But this would not be very useful, instead we let f be some smooth distribution. This is okay if the collective effect of far away particles are more important than the effects of collisions. If we want to include the effects of collisions, we add a mysterious term called the collision operator to obtain the **Boltzmann equation**

$$\frac{\partial f}{\partial t} + \boldsymbol{v}\cdot\frac{\partial f}{\partial\boldsymbol{r}} + \frac{q}{m}(\boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{B})\cdot\frac{\partial f}{\partial\boldsymbol{v}} = \left(\frac{\partial f}{\partial t}\right)_c.$$

The $\left(\frac{\partial f}{\partial t}\right)_c$ is interpreted as the rate of change of f that is due to collisions, and in general we don't have closed form for it. For plasmas in equilibrium, $\frac{\partial f}{\partial t} = 0$ since they are by definition stationary in time, and $\left(\frac{\partial f}{\partial t}\right)_c = 0$ as well since the rate of change due to collisions must also be 0. The most probablye distribution function describing plasmas in equilibrium is the Maxwell distribution function

$$f_{\mathrm{M}} = n\left(\frac{m}{2\pi k_{\mathrm{B}}T}\right)^{3/2}\exp\left(-\frac{mv^2}{2k_{\mathrm{B}}T}\right),$$

(p. 61 – 62, trick is to maximise entropy under the constraints that mean velocity is zero, particle density is constant and mean particle energy is constant using Lagrange multipliers).

The two fluid model

In this model, we try to simplify the equations above by not caring about the true f, but rather we treat the plasma as a fluid and only care about some of its moments. A moment $\langle\psi\rangle$ is defined as the velocity average of the function ψ:

$$\langle\psi\rangle = \frac{1}{n}\int \psi f \mathrm{d}\boldsymbol{v},$$

where n is the density of particles $n(\boldsymbol{r}) = \int f\mathrm{d}\boldsymbol{v}$. Using the Boltzmann equation we can derive the **general moment equation**

$$\frac{\partial}{\partial t}(n\langle\psi\rangle) + \nabla\cdot(n\langle\boldsymbol{v}\psi\rangle) - \frac{nq}{m}\left\langle(\boldsymbol{E}+\boldsymbol{v}\times\boldsymbol{B})\cdot\frac{\partial\psi}{\partial\boldsymbol{v}}\right\rangle = \frac{\partial}{\partial t}(n\langle\psi\rangle)_c,$$

(p. 64 – 65). This gives a way of solving the time evolution of these microscopic properties instead of dealing with f directly. However, the problem as you can see in the second term is that the equation for the order k moment contains the order $k+1$ moment.
As an easy example, consider the equation for the zero order moment ($\psi = 1$). It becomes

$$\frac{\mathrm{d}n_\alpha}{\mathrm{d}t} + \nabla\cdot(n_\alpha\boldsymbol{u}_\alpha) = 0. \tag{1}$$

$\boldsymbol{u}_\alpha$ is here the average velocity of the particles of species α at $\boldsymbol{r}$. The is basically the continuity equation. As you can see, it does depend on the average velocity. The equation for the velocity (first order moment, $\psi = \boldsymbol{v}$) becomes

$$n_\alpha m_\alpha\left[\frac{\partial\boldsymbol{u}_\alpha}{\partial t} + (\boldsymbol{u}_\alpha\cdot\nabla)\boldsymbol{u}_\alpha\right] = n_\alpha q_\alpha(\boldsymbol{E}+\boldsymbol{u}_\alpha\times\boldsymbol{B}) - \underbrace{\nabla\cdot\boldsymbol{\Pi}_\alpha}_{=\nabla p \text{ if isotropic}} + \sum_{\beta\neq\alpha} P_{\alpha\beta}, \tag{2}$$

after a bit of massaging (p. 65 – 66). Here $\boldsymbol{P}_{\alpha\beta}$ is the effect of collisions between particles of different species on the average velocity and $\boldsymbol{\Pi}_\alpha \equiv n_\alpha m_\alpha\langle(\boldsymbol{v}_\alpha - \boldsymbol{u}_\alpha)(\boldsymbol{v}_\alpha - \boldsymbol{u}_\alpha)\rangle$ is the pressure tensor (a central second order moment). If f is isotropic (I think even if just the velocity distribution is isotropic, but maybe that is what they meant), then $\nabla\cdot\boldsymbol{\Pi}$ becomes the gradient of the scalar pressure $p = k_BTn$.

An aside: wtf is a *pressure tensor*?

If the pressure tensor of a fluid is $\boldsymbol{\Pi}$, then $\boldsymbol{\Pi}\hat{n}$ is the force per unit area on a surface oriented normal to $\hat{n}$. It is the same as the negative of the expectation value of the stress tensor: $-\langle\boldsymbol{\sigma}\rangle$. Why is $\boldsymbol{\Pi} = nm\langle\boldsymbol{ww}\rangle$, where $\boldsymbol{w} = \boldsymbol{v} - \langle\boldsymbol{v}\rangle$? Idk. It's kinda the Irving-kirkwood formula, but I don't know where that comes from either.

In deriving the second order moment $\langle mv^2/2\rangle$ (p. 67 – 69) we get a Big Boring Expression™. If we ignore the heat exchange between different fluids, and the **heat conduction vector** $\boldsymbol{Q} \equiv nm\langle\boldsymbol{w}w^2\rangle/2$ where $\boldsymbol{w}$ is the deviation from the mean velocity, then some shuffling around gives

$$\frac{p_\alpha}{n_\alpha^{5/3}} = \text{const}.$$

In general

$$\frac{p_\alpha}{n_\alpha^\gamma} = \text{const}., \tag{3}$$

where $\gamma = (d+2)/d$ and d represents degrees of freedom. For isotropic pressure, as assumed above, $d = 3$. The full equation for the second order moments contain a third order moment and, for the third a forth and so on. Somewhere we have to cut it, and here we chose to ignore the third, thus stopping the chain.
The reason for calling it the *two* fluid model is that we the proceed to model the plasma using two species: electrons and ions.

MHD / Single fluid model

In **magnetohydrodynamics** (MHD) we treat the plasma as a single conducting fluid characterized by quantities like the mass density ρ_m, the charge density ρ, the center of mass velocity $\boldsymbol{V}$ and the current density $\boldsymbol{j}$. These are gotten from the two fluid model in a straight forward manner like

$$\rho_m \equiv \sum_\alpha m_\alpha n_\alpha$$
$$\rho \equiv \sum_\alpha q_\alpha n_\alpha$$
$$\boldsymbol{V} \equiv \frac{1}{\rho_m}\sum_\alpha m_\alpha n_\alpha\boldsymbol{u}_\alpha$$
$$\boldsymbol{j} \equiv \sum_\alpha q_\alpha n_\alpha\boldsymbol{u}_\alpha$$

The one-fluid equations are then gotten from the two-fluid equations by clever manipulations, e.g. multiplying the continuity equations by the masses / charges to get

$$\frac{\partial\rho_m}{\partial t} + \nabla\cdot(\rho_m\boldsymbol{V}) = 0$$
$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\boldsymbol{j}) = 0$$

or conservation of mass and charge.
Similar manipulations of the two-fluid momentum equations yield

$$\rho_m\left(\frac{\partial}{\partial t} + \boldsymbol{V}\cdot\nabla\right)\boldsymbol{V} = \boldsymbol{j}\times\boldsymbol{B} - \nabla\cdot\boldsymbol{\Pi}^*$$

and the **Generalized Ohms law** is yet another Big Boring Expression™, but it can be approximated as

$$\boldsymbol{E} + \boldsymbol{V}\times\boldsymbol{B} = \eta\boldsymbol{j}.$$

In very hot plasmas, the rhs can be neglected (p. 70 – 73).

Waves

What is a wave? Simply put, when we derive the waves we assume them to be small perturbations in the underlying equilibrium state. The basic derivation technique is to first linearise the equations governing the evolution of the plasma, e.g. equations (1), (2) and (3), around the equilibrium point, say that the wave is a small perturbation (hence waranting the linearisation). We then usually look for monochromatic waves: waves where all of the variables (density, electric field, mean velocity) vary with the same frequency. Thus we plug in the ansatz $X = X_1\exp[i(\boldsymbol{k}\cdot\boldsymbol{r} - wt)]$ for all of the quantities and take the equations to first order. This yields a solution and a **dispersion relation**.
For **Electrostatic** (meaning no oscillating B-field $\iff$ the waves are longitudinal: $\boldsymbol{k}\parallel\boldsymbol{E}$) **Electron Waves** (meaning

high oscillation frequencies so that the ions can be treated as static) we obtain (p.79 – 81) the dispersion relation

$$\omega^2 = \omega_p^2\left(1 + 3\frac{3v_{\rm th}^2}{2\omega_{\rm p}}k^2\right) = \omega_p^2(1 + 3\lambda_D^2 k^2)$$

This is only valid for long wavelengths $\lambda_D^2 k^2 \ll 1$. If $\lambda_D^2 k^2 \approx 1$ we get **Landau Dampening** from the interaction of the wave with the electrons with velocities close to the phase velocity of the wave. Using the kinetic approach to derive the **dispersion relation** (p. 82 – 84), an imaginary component is added and we get $\omega = \omega_{\rm re} + i\omega_{\rm im}$ where

$$\omega_{\rm re}^2 = \omega_{\rm p}^2 + \frac{3}{4}v_{\rm th}^2 k^2$$
$$\omega_{\rm im} = \frac{\pi}{2}\frac{\omega_{\rm p}^3}{k^2}\frac{{\rm d}\hat{f}}{{\rm d}v}\bigg|_{v=\omega_{\rm p}/k},$$

where $\hat{f} = f/n_0$. For **Electrostatic Ion Acoustic Waves** we instead assume that the electrons instantly adjust to the potential field and obtain (p.86-87)

$$\omega^2 = \frac{k^2 c_s^2}{1 + k^2\lambda_D^2}$$

where $c_s^2 \equiv k_B T/m_i$ (p. 86 – 87).
For **Transverse Electromagnetic Waves** (i.e. $\boldsymbol{k} \perp \boldsymbol{E}$) with no background magnetic field and high-frequency waves (static ions) we obtain

$$\omega^2 = \omega_p^2 + k^2 c^2,$$

(p. 88 – 89). For frequencies below the plasma frequency, waves cannot propagate and are reflected. The relation then says that $k = i/\delta$, where δ is the penetration depth, because the spatial variation is $\exp[ikx] = \exp[-x/\delta]$.
For transverse, high frequency electromagnetic waves in a magnitized plasma, the velocity generated from the electric field $\boldsymbol{E}_1$ will together with the static magnetic field give rise to a velocity due to the Lorentz force. This velocity will be perpendicular to the electric field induced velocity and the dynamics in these two perpendicular directions will be coupled. The electric field can thus not be linearly polarized. The dispersion relation for this case can be written in terms of the refractive index

$$N^2 \equiv \frac{k^2c^2}{\omega^2} = 1 - \frac{\omega_{\rm p}^2}{\omega(\omega \pm \omega_{\rm c})},$$

and the waves are cut off when $N = 0$ or

$$\omega = \mp\frac{\omega_{\rm c}}{2} + \sqrt{\frac{\omega_{\rm c}^2}{4} + \omega_{\rm p}^2} \tag{4}$$

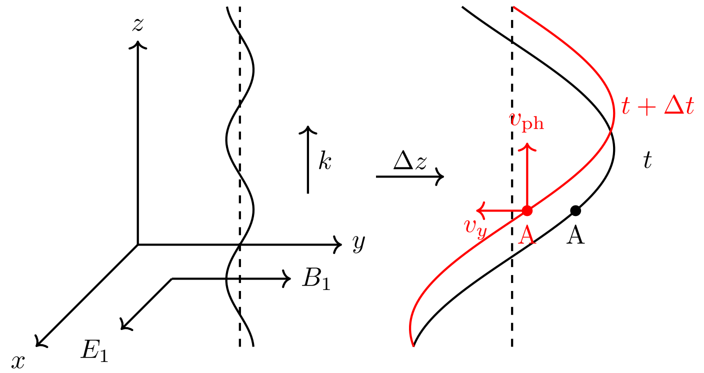
where you get the frequency for the L-wave $\omega_{\rm L}$ using $-$ in the equation above, and using $+$ you get the frequency for the R-wave (p. 97 – 98).
Alfvén waves are low frequency, transverse electromagnetic waves in a magnitized plasma, and due to the frequency being much lower than the cyclotron frequency the particle velocity perturbations can be approximated to only consist of the $\boldsymbol{E} \times \boldsymbol{B}$ and polarization drifts. With the Alfvén velocity $v_{\rm A} \equiv \varepsilon_0 B_0 c^2/n_0 m_i$ the dispersion relation for Alfvén waves is

$$\omega 2 = \frac{k^2 v_{\rm A}^2}{1 + v_{\rm A}^2/c^2},$$

(p. 105 – 106). If $\boldsymbol{B}_0 \parallel \boldsymbol{k}$, the total magnetic field lines become rippled like waves on a string as the electromagnetic wave propagates. With $\boldsymbol{B}_0 \parallel \hat{z}$, $\boldsymbol{E}_1 \parallel \hat{x}$ and $\boldsymbol{B}_1 \parallel \hat{y}$, we have

$$\frac{v_y}{v_{\rm ph}} = -\frac{B_1}{B_0},$$

and as the wave propagate past a point A through which a field line passes, the field lines moves in the same direction as the particle – the field line and particle oscillate together.



If $\boldsymbol{B}_0 \perp \boldsymbol{k}$, the two magnetic fields are parallel and thus the direction of the magnetic field does not change but the magnitude does.

Diffusion

Adding collisions to the description will induce diffusion. We assume that the plasma looses it's entire momentum in a collision. This is probably because after the collision, the momentum points in an essentially random direction, and thus the expectation value of the momentum is zero. Therefore the plasma species α looses momentum at a rate of $nm\nu\boldsymbol{u}_\alpha$, where ν is the collision frequency. Adding this effect to equation (2) gives

$$n_\alpha m_\alpha\left[\frac{\partial\boldsymbol{u}_\alpha}{\partial t} + (\boldsymbol{u}_\alpha\cdot\nabla)\boldsymbol{u}_\alpha\right] = n_\alpha q_\alpha\boldsymbol{E} - \nabla p - mn\nu\boldsymbol{u}_\alpha \tag{5}$$

assuming a non-magnetized isotropic plasma. In a stationary situation, this gives a drift velocity

$$\boldsymbol{v}_D = \frac{q}{m\nu}\boldsymbol{E} - \frac{k_B T}{m\nu}\frac{\nabla n}{n} \tag{6}$$

which naturally leads us to define the **electron mobility**, μ; the **diffusion coefficient**, D, and the **particle flux**, $\boldsymbol{\Gamma}$ as

$$\mu \equiv \frac{q}{m\nu} \qquad D \equiv \frac{k_B T}{m\nu} \qquad \boldsymbol{\Gamma} \equiv n\boldsymbol{v}_D$$

Since electrons are much lighter than ions, their velocities are higher and thus their collisions more frequent. Therefore they diffuse much faster which means that an electric field forms opposing the diffusion of the electrons and boosting the diffusion of the ions. In equilibrium, when the electron and ion fluxes are the same, the electric field is

$$\boldsymbol{E} = \frac{D_i - D_e}{\mu_i - \mu_e}\frac{\nabla n}{n} \tag{7}$$

which gives the common flux

$$\boldsymbol{\Gamma}_a = -\frac{\mu_i D_e - \mu_e D_i}{\mu_i - \mu_e}\nabla n \equiv -D_a\nabla n \tag{8}$$

Using $\mu_i \ll |\mu_e|$ and the definitions of μ and D we find that $D_a \approx 2D_i \ll D_e$, given that the ion and electron temperatures are the same. Plugging this into the continuity equation gives **the diffusion equation**

$$\frac{\partial n}{\partial t} = D\nabla^2 n \tag{9}$$

Maxwells Equations

Just for reference, I keep forgetting them all of the time...

$$\nabla\cdot\boldsymbol{E} = \frac{\rho}{\varepsilon_0}$$
$$\nabla\times\boldsymbol{E} = -\frac{\partial\boldsymbol{B}}{\partial t}$$
$$\nabla\cdot\boldsymbol{B} = 0$$
$$\nabla\times\boldsymbol{B} = \mu_0\left(J + \varepsilon_0\frac{\partial\boldsymbol{E}}{\partial t}\right)$$