Plasma Physics with Applications

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What is a plasma

A plasma is defined via the **ionization degree** α defined as the fraction of ionized particles. At $\alpha > 0.01$ we say it is **fully ionized**, while for smaller α:s its **weakly / partially ionized**.

Single particles in EM-field

The most fundamental equation of motion for a charged particle in an EM-field is the **Lorentz Force Equation:**

$$m\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = q(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B})$$

A constant, nonzero E-field with no B-field gives constant acceleration along $\boldsymbol{E}$. A constant, nonzero B-field with no E-field gives rise to helical motion with **cyclotron frequency** $\omega_c = |q|B/m$ and **Larmor radius** $r_L = v_\perp/\omega_c$. A constant, nonzero B-field with a constant force $\boldsymbol{F}$ gives a constant acceleration from the component of $\boldsymbol{F}$ along the B-field, while the component perpendicular to $\boldsymbol{B}$ gives rise to a constant **drift velocity**

$$\boldsymbol{v}_D = \frac{1}{q}\frac{\boldsymbol{F}_\perp \times \boldsymbol{B}}{B^2}$$

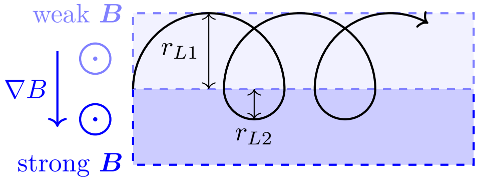
An *inhomogeneous* B-field gives rise to the so called **grad B drift**. The motion parallel to the B-field lines is governed by

$$m\frac{\mathrm{d}\boldsymbol{v}_\parallel}{\mathrm{d}t} = -\mu\nabla_\parallel B,$$

and for the motion perpendicular we get a drift velocity

$$\boldsymbol{v}_D = \frac{\mu}{q}\frac{\boldsymbol{B} \times \nabla B}{B^2}$$

in addition to the Larmor rotation, just like before.



TODO: **Curved fields**

The Vlaslov and Boltzmann equations

Firstly we define a **distribution function**, $f(\boldsymbol{r}, \boldsymbol{v}, t)$. This gives the expected number of particles occupying the volume in phase space $\mathrm{d}\boldsymbol{r}\mathrm{d}\boldsymbol{v}$ around the point $(\boldsymbol{r}, \boldsymbol{v})$ at time t. To understand how this f evolves argue that the rate of change of the number of particles inside some volume in phase space Ω must be the flux of particles through the walls.

$$\frac{\partial}{\partial t}\int_\Omega f \mathrm{d}\boldsymbol{r}\mathrm{d}\boldsymbol{v} = -\int_{\partial_r\Omega} f\boldsymbol{v} \cdot \hat{n}_r \mathrm{d}S_r - \int_{\partial_v\Omega} f\boldsymbol{a} \cdot \hat{n}_v \mathrm{d}S_v.$$

$\partial_r\Omega$ means the boundry of Ω in r-space. Plopping the $\frac{\partial}{\partial t}$ into the integral and using Gauss' theorem to make the surface integral a volume integral we obtain

$$\mathrm{d}\boldsymbol{r}\mathrm{d}\boldsymbol{v}\left[\frac{\partial f}{\partial t} + \frac{\partial}{\partial \boldsymbol{r}} \cdot \boldsymbol{v}f + \frac{\partial}{\partial \boldsymbol{v}} \cdot \boldsymbol{a}f\right] = 0,$$

which has to be true for all volumes, meaning that the integrand has to be 0 everywhere. Further noting that $\frac{\partial}{\partial r} \cdot \boldsymbol{v} = 0$ and plugging in the Lorentz force for the acceleration we get the **Vlaslov equation**

$$\frac{\partial f}{\partial t} + \boldsymbol{v} \cdot \frac{\partial f}{\partial \boldsymbol{r}} + \frac{q}{m}(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}) \cdot \frac{\partial f}{\partial \boldsymbol{v}} = 0.$$

The LHS is really the change of f along a particle trajectory (if you let $\boldsymbol{r}(t)$ and $\boldsymbol{v}(t)$ be the particle position and velocity at time t), so you can let f be the sum of a bunch of delta functions and reduce this to the particle description of the plasma. But this would not be very useful, instead we let f be some smooth distribution. This is okay if the collective effect of far away particles are more important than the effects of collisions. If we want to include the effects of collisions, we add a mysterious term called the collision operator to obtain the **Boltzmann equation**

$$\frac{\partial f}{\partial t} + \boldsymbol{v} \cdot \frac{\partial f}{\partial \boldsymbol{r}} + \frac{q}{m}(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}) \cdot \frac{\partial f}{\partial \boldsymbol{v}} = \left(\frac{\partial f}{\partial t}\right)_c.$$

For plasmas in equilibrium, $\frac{\partial f}{\partial t} = 0$ since they are by definition stationary in time. This also means that the net flux out of any volume in phase space must be 0. TODO why does this mean that the collision operator must be 0?

The two fluid model

In this model, we try to simplify the equations above by not caring about the true f, but rather we treat the plasma as a fluid and only care about some of its moments. A moment $\langle\psi\rangle$ is defined as the velocity average of the function ψ:

$$\langle\psi\rangle = \frac{1}{n}\int \psi f \mathrm{d}\boldsymbol{v},$$

where n is the density of particles $n(\boldsymbol{r}) = \int f \mathrm{d}\boldsymbol{v}$. Using the Boltzmann equation we can derive the **general moment equation**

$$\frac{\partial}{\partial t}(n\langle\psi\rangle) + \nabla \cdot (n\langle\boldsymbol{v}\psi\rangle) - \frac{nq}{m}\left\langle(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}) \cdot \frac{\partial \psi}{\partial \boldsymbol{v}}\right\rangle = \frac{\partial}{\partial t}(n\langle\psi\rangle)_c$$

This gives a way of solving the time evolution of these microscopic properties instead of dealing with f directly. However, the problem as you can see in the second term is that the equation for the order k moment contains the order $k+1$ moment.

As an easy example, consider the equation for the zero order moment ($\psi = 1$). It becomes

$$\frac{\mathrm{d}n_\alpha}{\mathrm{d}t} + \nabla \cdot (n_\alpha \boldsymbol{u}_\alpha) = 0.$$

$\boldsymbol{u}_\alpha$ is here the average velocity of the particles of species α at $\boldsymbol{r}$. The is basically the continuity equation. As you can see, it does depend on the average velocity, and the equation for the average velocity depends on the pressure tensor (a second order moment) and so on. Somewhere we have to cut it.