Chapter 3

Basic Probability

L-ADBD, ISIMS Sfax

February 2024

Objectives

The objectives for this chapter are:

- To understand basic probability concepts.
- To understand conditional probability.
- Use Bayes' theorem to revise probabilities.
- Apply counting rules.

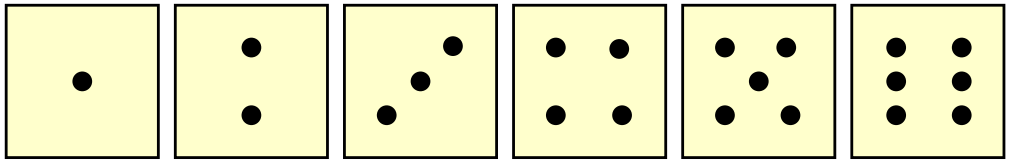
Probability Helps Bridge Descriptive & Inferential Statistics

- Probability principles are the foundation for:
  - Probability distributions.
  - Mathematical expectation.
  - Binomial and Poisson distributions.

- For example, probability can be used on intent-to-purchase survey responses and associated follow-up responses to answer many purchase behavior questions.

The Sample Space Is The Collection Of All Possible Outcomes Of A Variable

e.g. All 6 faces of a die:



e.g. All 52 cards of a bridge deck

Each Possible Outcome Of A Variable Is An Event

- Simple event:
  - An event described by a single characteristic.
  - e.g., A day in January from all days in 2018.
- Joint event:
  - An event described by two or more characteristics.
  - e.g. A day in January that is also a Wednesday from all days in 2018.
- Complement of an event A (denoted A'):
  - All events that are not part of event A.
  - e.g., All days from 2018 that are not in January.

Basic Probability Concepts

- Probability – the numerical value representing the chance, likelihood, or possibility that a certain event will occur (always between 0 and 1).

- Impossible Event – an event that has no chance of occurring (probability = 0).

- Certain Event – an event that is sure to occur (probability = 1).

Mutually Exclusive Events

- Mutually exclusive events:
  - Events that cannot occur simultaneously.

Example: Randomly choosing a day from 2018

A = day in January; B = day in February

  - Events A and B are mutually exclusive.

Collectively Exhaustive Events

- Collectively exhaustive events:
  - One of the events must occur.
  - The set of events covers the entire sample space.

Example: Randomly choose a day from 2018.

A = Weekday; B = Weekend;
C = January; D = Spring;

  - Events A, B, C and D are collectively exhaustive (but not mutually exclusive – a weekday can be in January or in Spring).
  - Events A and B are collectively exhaustive and also mutually exclusive.

Three Approaches To Assessing Probability Of An Event

1. a priori -- *based on prior knowledge of the process*

probability of occurrence $= \frac{X}{T} = \frac{\text{number of ways in which the event occurs}}{\text{total number of possible outcomes}}$

Assuming all outcomes are equally likely

2. empirical probability -- *based on observed data*

probability of occurrence $= \frac{\text{number of ways in which the event occurs}}{\text{total number of possible outcomes}}$

3. subjective probability

based on a combination of an individual's past experience, personal opinion, and analysis of a particular situation.

Example of *a priori* probability

When randomly selecting a day from the year 2018 what is the probability the day is in January?

$$\textbf{Probability of Day In January} = \frac{X}{T} = \frac{\textbf{number of days in January}}{\textbf{total number of days in 2018}}$$

$$\frac{X}{T} = \frac{\textbf{31 days in January}}{\textbf{365 days in 2018}} = \frac{\textbf{31}}{\textbf{365}}$$

Example of empirical probability

Find the probability of selecting a male taking statistics from the population described in the following table:

	Taking Stats	Not Taking Stats	Total
Male	84	145	229
Female	76	134	210
Total	160	279	439

Probability of male taking stats $= \dfrac{\text{number of males taking stats}}{\text{total number of people}} = \dfrac{84}{439} = 0.191$

Subjective Probability Differs From Person To Person

- What is the probability a new ad campaign is successful?
  - A media development team assigns a 60% probability of success to its new ad campaign.
  - The chief media officer of the company is less optimistic and assigns a 40% of success to the same campaign.
- The assignment of a subjective probability is based on a person's experiences, opinions, and analysis of a particular situation.
- Subjective probability is useful in situations when an empirical or a priori probability cannot be computed.

Summarizing Sample Spaces

Contingency Table -- M&R Survey Results.

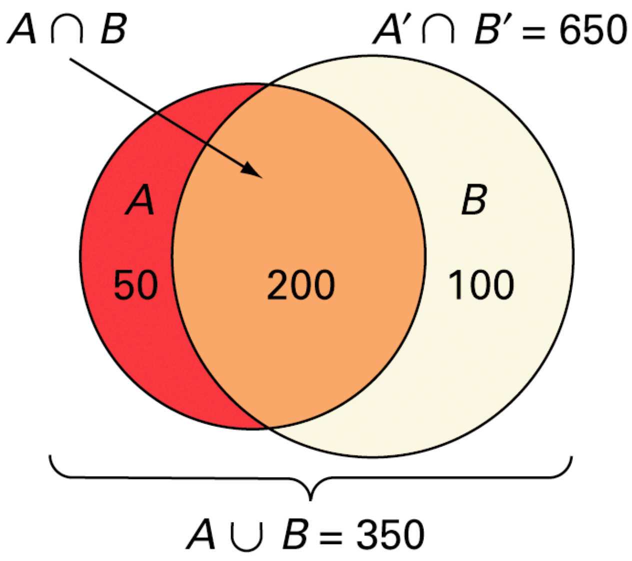
	Actually Purchased TV		
Planned To Purchase TV	Yes	No	Total
Yes	200	50	250
No	100	650	750
Total	300	700	1,000

Total Number
Of Sample
Space Outcomes.

Summarizing Sample Spaces

Venn Diagram -- M&R Survey Results.

A = Planned to Purchase
A' = Did not Plan To Purchase
B = Actually Purchased
B' = Did not Purchase



Planned To Purchase TV		Actually Purchased TV Yes	No	Total
	Yes	200	50	250
	No	100	650	750
	Total	300	700	1,000

Simple Probability: Definition & Computing

- Simple Probability refers to the probability of a simple event.
  - P(Planned to purchase)
  - P(Actually purchased)

$$P(A) = \frac{\textit{number of outcomes satisfying } A}{\textit{total number of outcomes}}$$

P(Planned to Purchase) = 250 / 1000

Planned To Purchase TV	Actually Purchased TV: Yes	No	Total
Yes	200	50	250
No	100	650	750
Total	300	700	1,000

P(Actually Purchased) = 300 / 1000

Joint Probability: Definition & Computing

- Joint Probability refers to the probability of an occurrence of two or more events (joint event).
  - ex. P(Plan to Purchase and Purchase).
  - ex. P(No Plan and Purchase).

$$P(A \text{ and } B) = \frac{\text{number of outcomes satisfying } A \text{ and } B}{\text{total number of outcomes}}$$

P(Plan to Purchase and Purchase) = 200 / 1000

Planned To Purchase TV	Actually Purchased TV: Yes	No	Total
Yes	200	50	250
No	100	650	750
Total	300	700	1,000

P(No Plan and Purchase) = 100 / 1000

Computing A Marginal Probability Via Joint Probabilities

- Computing a marginal (or simple) probability:

$$P(A) = P(A \text{ and } B_1) + P(A \text{ and } B_2) + \cdots + P(A \text{ and } B_k)$$

  - Where B_1, B_2, …, B_k are k mutually exclusive and collectively exhaustive events.

P(Planned) = P(Yes and Yes) + P(Yes and No) =
200 / 1000 + 50 / 1000 = 250 / 1000

Planned To Purchase TV		Actually Purchased TV		
		Yes	No	Total
	Yes	200	50	250
	No	100	650	750
	Total	300	700	1,000

Marginal & Joint Probabilities In A Contingency Table

Event	Event		Total
	B_1	B_2	
A_1	$P(A_1 \text{ and } B_1)$	$P(A_1 \text{ and } B_2)$	$P(A_1)$
A_2	$P(A_2 \text{ and } B_1)$	$P(A_2 \text{ and } B_2)$	$P(A_2)$
Total	$P(B_1)$	$P(B_2)$	1

Joint Probabilities

Marginal (Simple) Probabilities

Probability Summary So Far

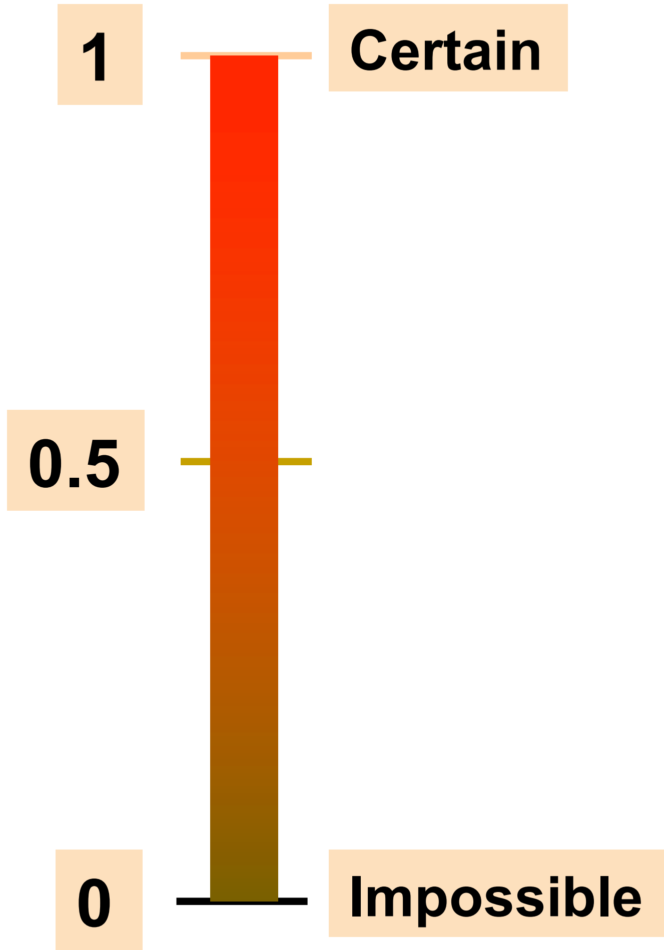
- Probability is the numerical measure of the likelihood that an event will occur.
- The probability of any event must be between 0 and 1, inclusively.

$$0 \leq P(A) \leq 1 \quad \text{For any event A}$$

- The sum of the probabilities of all mutually exclusive and collectively exhaustive events is 1.

$$P(A) + P(B) + P(C) = 1$$

If A, B, and C are mutually exclusive and collectively exhaustive



General Addition Rule

General Addition Rule:

$$\boxed{\mathbf{P(A\ or\ B) = P(A) + P(B) - P(A\ and\ B)}}$$

If A and B are mutually exclusive, then

P(A and B) = 0, so the rule can be simplified:

$$\boxed{\mathbf{P(A\ or\ B) = P(A) + P(B)}}$$

For mutually exclusive events A and B

General Addition Rule Example

P(Planned or Purchased) =

P(Planned) + P(Purchased) – P(Planned and Purchased) =

250 / 1000 + 300 / 1000 – 200 / 1000 = 350 / 1000

Planned To Purchase TV	Actually Purchased TV: Yes	No	Total
Yes	200	50	250
No	100	650	750
Total	300	700	1,000

Computing Conditional Probabilities

- A conditional probability is the probability of one event, given that another event has occurred:

$$P(A \mid B) = \frac{P(A \text{ and } B)}{P(B)}$$

The conditional probability of A given that B has occurred.

$$P(B \mid A) = \frac{P(A \text{ and } B)}{P(A)}$$

The conditional probability of B given that A has occurred.

Where P(A and B) = joint probability of A and B

P(A) = marginal or simple probability of A

P(B) = marginal or simple probability of B

Conditional Probability Example

P(Purchased | Planned) = P(Purchased and Planned) / P(Planned) =

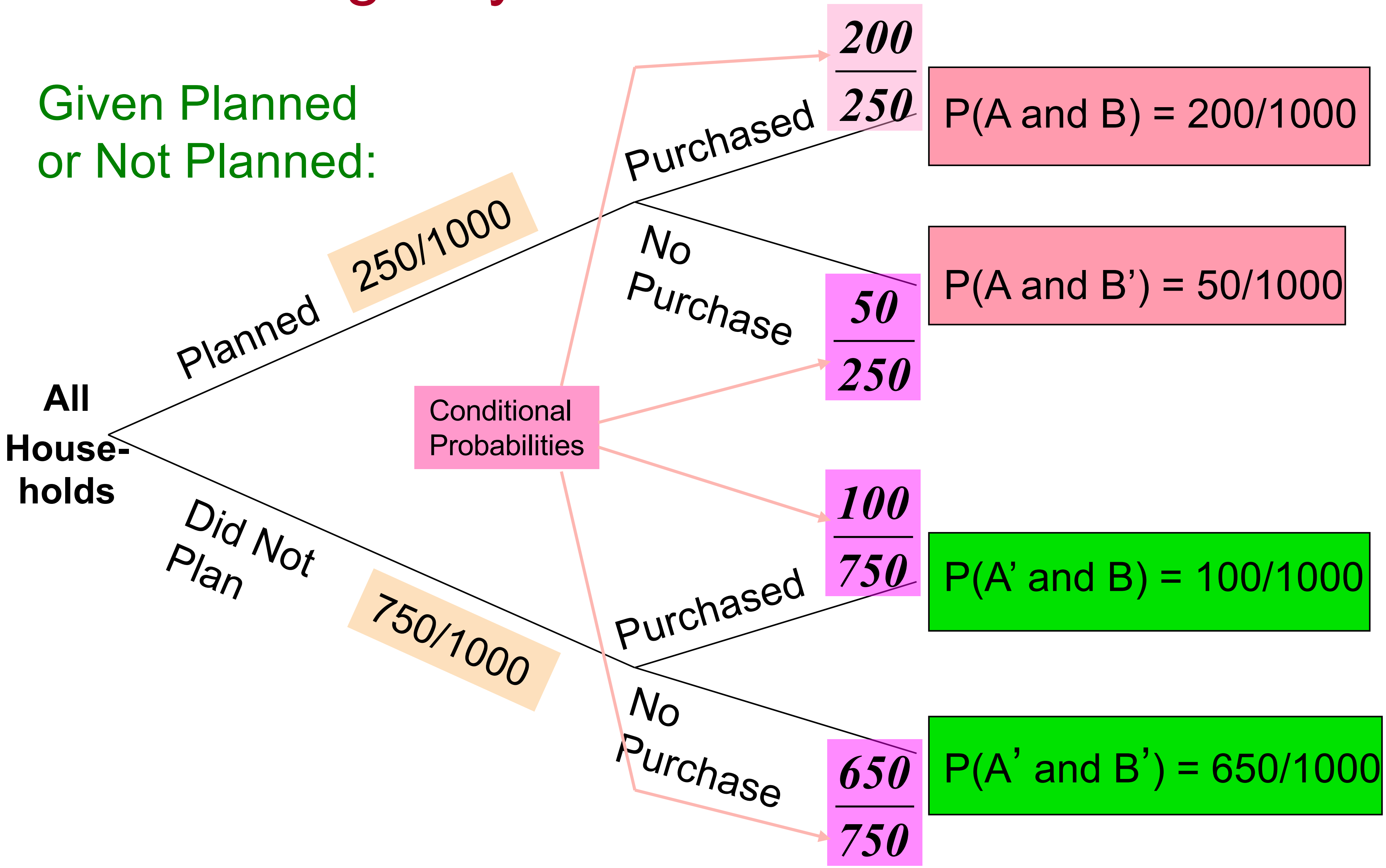
(200 / 1000) / (250 / 1000) = 200 / 250

Planned To Purchase TV	Actually Purchased TV: Yes	No	Total
Yes	200	50	250
No	100	650	750
Total	300	700	1,000

Since Planned is given we only need to consider the top row of the table.

A Decision Tree Is An Alternative To A Contingency Table

Given Planned or Not Planned:



Independent Events

- Two events are independent if and only if:

$$\boxed{P(A \mid B) = P(A)}$$

- Events A and B are independent when the probability of one event is not affected by the fact that the other event has occurred.

Are The Events Planned and Purchased Independent?

Does P(Purchased | Planned) = P(Purchased)?

P(Purchased | Planned) = 200 / 250 = 0.8.

P(Purchased) = 700 / 1000 = 0.7.

Since these two probabilities are not equal, these two events are dependent.

		Actually Purchased TV		
Planned To Purchase TV		Yes	No	Total
	Yes	200	50	250
	No	100	650	750
	Total	300	700	1,000

Multiplication Rules For Two Events

The General Multiplication Rule

$$P(A|B) = P(A \text{ and } B) \,/\, P(B)$$

Solving for P(A and B)

$$P(A \text{ and } B) = P(A|B)\,P(B)$$

Note: If A and B are independent, then $P(A \mid B) = P(A)$ and the multiplication rule simplifies to:

$$P(A \text{ and } B) = P(A)P(B)$$

Marginal Probability Using The General Multiplication Rule

- Marginal probability for event A:

$$P(A) = P(A \mid B_1)P(B_1) + P(A \mid B_2)P(B_2) + \cdots + P(A \mid B_k)P(B_k)$$

  - Where B_1, B_2, …, B_k are k mutually exclusive and collectively exhaustive events.

Let A = Planned, B_1 = Purchase, & B_2 = No Purchase

$P(A) = P(A|B_1)P(B_1) + P(A|B_2)P(B_2)$ =

(200/300)(300/1000) + (50/700)(700/1000) = 0.25

Planned To Purchase TV	Actually Purchased TV: Yes	No	Total
Yes	200	50	250
No	100	650	750
Total	300	700	1,000

Bayes' Theorem

- Bayes' Theorem is used to revise previously calculated probabilities based on new information.
- Developed by Thomas Bayes in the 18th Century.
- It is an extension of conditional probability.

Bayes' Theorem

$$P(B_i \mid A) = \frac{P(A \mid B_i)P(B_i)}{P(A \mid B_1)P(B_1) + P(A \mid B_2)P(B_2) + \cdots + P(A \mid B_k)P(B_k)}$$

- where:

  B_i = i^{th} event of k mutually exclusive and collectively exhaustive events

  A = new event that might impact $P(B_i)$

Bayes’ Theorem Example

- A drilling company has estimated a 40% chance of striking oil for their new well.
- A detailed test has been scheduled for more information. Historically, 60% of successful wells have had detailed tests, and 20% of unsuccessful wells have had detailed tests.
- Given that this well has been scheduled for a detailed test, what is the probability that the well will be successful?

Bayes' Theorem Example

(continued)

- Let S = successful well

  U = unsuccessful well
- $P(S) = 0.4$, $P(U) = 0.6$ (prior probabilities)
- Define the detailed test event as D
- Conditional probabilities:

  $P(D|S) = 0.6$ $P(D|U) = 0.2$
- Goal is to find $P(S|D)$

Bayes' Theorem Example

(continued)

Apply Bayes' Theorem:

$$P(S|D) = \frac{P(D|S)P(S)}{P(D|S)P(S) + P(D|U)P(U)}$$

$$= \frac{(0.6)(0.4)}{(0.6)(0.4) + (0.2)(0.6)}$$

$$= \frac{0.24}{0.24 + 0.12} = 0.667$$

So the revised probability of success, given that this well has been scheduled for a detailed test, is 0.667

Bayes' Theorem Example

(continued)

- Given the detailed test, the revised probability of a successful well has risen to 0.667 from the original estimate of 0.4

Event	Prior Prob.	Conditional Prob.	Joint Prob.	Revised Prob.
S (successful)	0.4	0.6	(0.4)(0.6) = 0.24	0.24/0.36 = 0.667
U (unsuccessful)	0.6	0.2	(0.6)(0.2) = 0.12	0.12/0.36 = 0.333

Sum = 0.36

Counting Rules Are Often Useful In Computing Probabilities

- **In many cases, there are a large number of possible outcomes.**

- **Counting rules can be used in these cases to help compute probabilities.**

Counting Rules

- Rules for counting the number of possible outcomes
- Counting Rule 1:
  - If any one of *k* different mutually exclusive and collectively exhaustive events can occur on each of *n* trials, the number of possible outcomes is equal to

$$\boxed{k^n}$$

  - Example
    - If you roll a fair die 3 times then there are $6^3 = 216$ possible outcomes

Counting Rules

(continued)

- Counting Rule 2:
  - If there are k_1 events on the first trial, k_2 events on the second trial, … and k_n events on the n^{th} trial, the number of possible outcomes is

$$\boxed{(k_1)(k_2)\cdots(k_n)}$$

  - Example:
    - You want to go to a park, eat at a restaurant, and see a movie. There are 3 parks, 4 restaurants, and 6 movie choices. How many different possible combinations are there?
    - Answer: (3)(4)(6) = 72 different possibilities

Counting Rules

(continued)

- Counting Rule 3:
  - The number of ways that *n* items can be arranged in order is

$$\boxed{n! = (n)(n-1)\cdots(1)}$$

  - Example:
    - You have five books to put on a bookshelf. How many different ways can these books be placed on the shelf?
    - Answer: $5! = (5)(4)(3)(2)(1) = 120$ different possibilities.

Counting Rules

(continued)

- Counting Rule 4:
  - Permutations: The number of ways of arranging X objects selected from n objects in order is

$$\boxed{{}_{n}P_{x} = \frac{n!}{(n-X)!}}$$

  - Example:
    - You have five books and are going to put three on a bookshelf. How many different ways can the books be ordered on the bookshelf?
    - Answer: ${}_{n}P_{X} = \frac{n!}{(n-X)!} = \frac{5!}{(5-3)!} = \frac{120}{2} = 60$ different possibilities.

Counting Rules

(continued)

- Counting Rule 5:
  - Combinations: The number of ways of selecting X objects from n objects, irrespective of order, is

$$\boxed{{}_{n}C_{x} = \frac{n!}{X!(n-X)!}}$$

  - Example:
    - You have five books and are going to select three are to read. How many different combinations are there, ignoring the order in which they are selected?
    - Answer: ${}_{n}C_{x} = \frac{n!}{X!(n-X)!} = \frac{5!}{3!(5-3)!} = \frac{120}{(6)(2)} = 10$ different possibilities

Chapter Summary

In this chapter we covered:

- Using basic probability concepts.
- Using conditional probability.
- Using Bayes’ theorem to revise probabilities.
- Using counting rules.