Counting
Sheep
Sets
Goal: Avoid counting!

Set Size

The number of elements in a set S is called its size, or cardinality, and denoted |S| or # S

Set of size n: n-set

ation

Bits	$\lvert\{0,1\}\rvert = 2$		
Coin	$\lvert\{\text{heads, tails}\}\rvert = 2$	Die	$\lvert\{1,2,3,4,5,6\}\rvert = 6$
Digits	$\lvert\{0,1,\ldots,9\}\rvert = 10$	Letters	$\lvert\{a,\ldots,z\}\rvert = 26$
Empty set	$\lvert\varnothing\rvert = 0$		
Integers	$\lvert\mathbb{Z}\rvert = \lvert\mathbb{N}\rvert = \lvert\mathbb{P}\rvert = \infty$	Countably infinite $\aleph_0$	
Reals	$\lvert\mathbb{R}\rvert = \infty$	Uncountably infinite $\aleph$	

Integer Intervals

$m \le n$ $\{m,\ldots,n\}$ = {integers from m to n, inclusive} $\{3,\ldots, 5\} = \{3,4,5\}$

$$\boxed{|\{m,\ldots,n\}| = n-m+1}$$

+1? First Try small #'s

$|\{5,\ldots,5\}| = |\{5\}| = 1 = 5 - 5 + 1$

$|\{1,\ldots,3\}| = |\{1,2,3\}| = 3 = 3 - 1 + 1$

$\{3,4,5\}$

points = #intervals+1= (5-3)+1= 3

5-3 = 2 = length = #intervals

Integer Multiples

$_d(n] = \{ 1 \le i \le n : d \mid i \}$

$(n] = [n] = \{ 1, \ldots, n \}$

$_3(8] = \{ 3, 6 \} = \{1 \cdot 3, 2 \cdot 3\}$

$_3(9] = \{ 3, 6, 9 \} = \{1 \cdot 3, 2 \cdot 3, 3 \cdot 3\}$

$| \, _d(n] \, | = \lfloor n/d \rfloor$

$| \, _3(8] \, | = \lfloor 8/3 \rfloor = 2$

$| \, _3(9] \, | = \lfloor 9/3 \rfloor = 3$

Set Size

Size **len**

```
print(len({-1, 1}))
2
```

Sum **sum**

```
print(sum({-1, 1}))
0
```

Minimum **min**

```
print(min({-1, 1}))
-1
```

Maximum **max**

```
print(max({-1, 1}))
1
```

Loops

for <var> in <set>

```
A = {1, 2, 3}
print(len(A))
3
num=0
for i in A:
    num += 1
print(num)
3
```

Disjoint Unions

Addition Rule

Subtraction Rule

Pets

A Hairy Problem

Disjoint Unions

$|A| = 2$ $|B| = 3$

$|A \cup B| = 2 + 3 = 5$

A union of disjoint sets is called a disjoint union

Use ●● to emphasize a disjoint union

$\{0\} \overset{\bullet\bullet}{\cup} \{1\} = \{0,1\}$

For disjoint sets,
the size of the union
is the sum of the sizes

$$|A \overset{\bullet\bullet}{\cup} B| = |A| + |B|$$

Addition Rule ⊕

Numerous applications & implications

Reason $\cup \approx +$

Kids Play

Class has 2 boys and 3 girls

students = ?

∪ ⊕ # students = 2 + 3 = 5

Jar with Marbles

1 blue, 2 green, 3 red

marbles = ?

∪ of 3 sets

⊕ twice

=1+2+3 = 6

Complements

Quintessential disjoint sets

A and A^c

$A \cup A^c = \Omega$

$|\Omega| = |A \cup A^c| = |A| + |A^c|$

$\oplus$

Subtraction (or complement) Rule

$\ominus$

$|A^c| = |\Omega| - |A|$

$D = \{ i \in [6] : 3 \mid i \} = \{ 3, 6 \}$

$|D| = 2$

Application of $\oplus$ rule

$D^c = \{ i \in [6] : 3 \nmid i \} = \{1, 2, 4, 5\}$

$|D^c| = 4$

$\Omega = [6] = \{1,..,6\}$

$|D^c| = 4 = 6 - 2 = |\Omega| - |D|$

✅

Reason set difference ≈ -

Think Outside the Circle

$|A^c| = |\Omega| - |A|$

$|A| = |\Omega| - |A^c|$

Handy for large or complex sets!

$A = \{ i \in [100] : 3 \nmid i \} = \{1,2,4,5,7,\ldots,100\}$

integers between 1 and 100 not divisible by 3

$\Omega = \{1,\ldots,100\}$

$A^c = \{i \in [100] : 3 \mid i\} = \{ 3, 6, 9,\ldots, 99 \}$

$|A^c| = 33$

$|A| = |\Omega| - |A^c| = 100 - 33 = 67$

General Subtraction Rule

$A \subseteq B$ → $B = A \cup (B-A)$

$|B| = |A| + |B-A|$

$|B-A| = |B| - |A|$

Subtraction Rule

Disjoint Unions

Addition Rule

Subtraction Rule

Pets

A Hairy Problem

Another Hairy Problem

A friend claims she can determine the size of any set instantly and exactly

She can determine the exact # of hairs on your head

Can you ask her some questions to be fairly certain if she tells the truth?

Hint

It doesn't just work for humans

See you next video

General Unions (Piece of PIE)

Size of a general union

Principle of Inclusion Exclusion

Multiple sets

Start: Hairy Problem

Hairy Problem

A friend claims she can determine the size of any set instantly and exactly

She can determine the exact # of hairs on your head

Can you ask her some questions to be fairly certain if she tells the truth?

Hint

It doesn't just work for humans

Simple Solution

Subtraction Rule

Ask her how many hairs you have

Remove a small number of hairs, say 8

Ask her how many hairs you have now

− Difference between her answers should be # hairs removed (8)

Can you ask a single question?

Ask just how many hairs you removed

8 hairs

Zero-knowledge proofs

Prove identity without revealing password

General Unions

Disjoint A and B: $|A \cup B| = |A| + |B|$

Size of union = sum of sizes

In general: $|A \cup B| \neq |A| + |B|$

$|\{a\} \cup \{a\}| = |\{a\}| = 1 \neq 2 = |\{a\}| + |\{a\}|$

Can we determine $|A \cup B|$ in general?

$|A \cup B| = |A| + |B| - |A \cap B|$

Principle of Inclusion-Exclusion (PIE)

Pokémons

= 4

$|W \cup F| = |W| + |F| - |W \cap F| = 3+2-1 = 4$ ✅

Divisibility by 2 Numbers

$D_{2 \vee 3} = \{ i \in [100] : 2|i \vee 3|i \} = \{2,3,4,6,8,\ldots,100\}$

$| D_{2 \vee 3} | = ?$

$D_2 = \{ i \in [100] : 2|i \} = \{2, 4, 6,\ldots, 100\}$

$D_3 = \{ i \in [100] : 3|i \} = \{3, 6, 9,\ldots, 99\}$

$D_{2 \vee 3} = D_2 \cup D_2$

$|D_2 \cup D_3| = |D_2| + |D_3| - |D_2 \cap D_3|$

$D_2 \cap D_3 = \{ i \in [100] : 2|i \wedge 3|i \} = \{ i \in [100] : 6|i \}$

$| D_{2 \vee 3} | = |D_2| + |D_3| - |D_2 \cap D_3| = 50 + 33 - 16 = 67$

Multiple Sets

Two sets

$$|A \cup B| = |A| + |B| - |A \cap B|$$

Three sets

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

n sets

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{1 \le i \le n} |A_i| - \sum_{1 \le i < j \le n} |A_i \cap A_j| + \cdots + (-1)^{n-1} \left|\bigcap_{i=1}^{n} A_i\right|$$

Polyglots

8 students in class, each "speaks" **C**, **R**, or **Π**thon

$|C \cup R \cup \Pi| = 8$

Each language spoken by 5 students

$|C| = |R| = |\Pi| = 5$

Every language *pair* is spoken by 3 students

$|C \cap R| = |C \cap \Pi| = |R \cap \Pi| = 3$

How many students speak all 3 languages?

$|C \cap R \cap \Pi| = ?$

$$|C \cup R \cup \Pi| = |C| + |R| + |F| - |C \cap R| - |C \cap \Pi| - |R \cap \Pi| + |C \cap R \cap \Pi|$$

$$8 = 5 + 5 + 5 - 3 - 3 - 3 + ?$$

$$? = 8 - 5 - 5 - 5 + 3 + 3 + 3$$

$$= 2$$

With that, can say more

Sanity Checks

Compare to some expected outcomes

A, B disjoint $|A \cup B| = |A| + |B| - |A \cap B| = |A| + |B|$ ✅

0

Equal sets $|A \cup A| = |A| + |A| - |A \cap A| = |A|$ ✅

$|A|$

$\max\{|A|, |B|\} \leq |A \cup B| \leq |A| + |B|$ ✅

= iff nested

= iff disjoint

Counting
Cartesian
Products

Cartesian Products

$|\{a,b\}| = 2$ $\quad$ $|\{1,2,3\}| = 3$

$$\{a,b\} \times \{1,2,3\} = \begin{Bmatrix} (a,1) & (a,2) & (a,3) \\ (b,1) & (b,2) & (b,3) \end{Bmatrix} \begin{matrix} \leftarrow 3 \\ \leftarrow 3 \end{matrix}$$

$\oplus$

$|\{a,b\} \times \{1,2,3\}| = 3+3 = 2x3 = 6$

$$\begin{array}{c} \overbrace{\hphantom{(a,1)\quad(a,2)\quad(a,3)}}^{3} \\ 2\left\{\begin{array}{|c|c|c|}\hline (a,1) & (a,2) & (a,3) \\ \hline (b,1) & (b,2) & (b,3) \\ \hline \end{array}\right. \end{array}$$

area 2x3 = 6

The size of a Cartesian Product is the product of the set sizes

$|A \times B| = |A| \times |B|$

Product Rule $\otimes$

Another application of $\oplus$ rule

Tables

Cartesian product

3 attributes

5 records

Adam	M	3.5
Eve	F	3.7
John	M	3.4
Lisa	F	3.2
Mary	F	3.9

X

5x3=15 cells

Three Sets

$A \times B$ $\{(a,b): a \in A, b \in B\}$ rectangle $|A \times B| = |A| \times |B|$

$A \times B \times C$ $\{(a,b,c): a \in A, b \in B, c \in C\}$ "cuboid" $|A \times B \times C| = |A| \times |B| \times |C|$

Dandy Dresser

3 shirts

2 pants

5 pairs of shoes

How many outfits can he have?

Outfit = (shirt, pants, shoes)

( )

3-tuple

{Outfits} = {(shirt, pants, shoes)} = {shirts} x {pants} x {shoes}

Cartesian Product

| {Outfits} | = | {shirts} | x | {pants} | x | {shoes} | = 3 x 2 x 5 = 30

$\otimes$

Useful?

X 3 x 3 x 4 36

n Sets

For n sets, by ⊗ and induction

$$|A_1 \times \ldots \times A_n| = |A_1| \times \ldots \times |A_n|$$

Subway

How many sandwiches can Subway make?

Bread = {Wheat, Italian}

Meat = {Turkey, Ham, Chicken, Bacon, Beef}

Cheese = {American, Monterey, Cheddar}

Veggie = {Cucumbers, Lettuce, Spinach, Onions}

Sauce = {Ranch, Mustard, Mayonnaise}

Sandwiches = Bread x Meat x Cheese x Veggie x Sauce

|Sand's| = |Bread| x |Meat| x |Cheese| x |Veggie| x |Sauce|

$\otimes$ = 2 x 5 x 3 x 4 x 3 = 360

Cartesian Products

Product rule $|A \times B| = |A| \times |B|$

Multiple sets $|A_1 \times \ldots \times A_n| = |A_1| \times \ldots \times |A_n|$

Cartesian Powers

sequences

subsets

functions

exponential growth

+, -, X, ...

Analogies between number and set operations

Numbers	Sets
Addition	Disjoint union
Subtraction	Complement
Multiplication	Cartesian product
Exponents	?

Cartesian Powers of a Set

Cartesian product of a set with itself is a Cartesian power

$A^2 = A \times A$ Cartesian square

$A^n \stackrel{\text{def}}{=} \underbrace{A \times A \times \ldots \times A}_{n}$ n'th Cartesian power

$$|A^n| = |\, A \times A \times \ldots \times A \,| = |A| \times |A| \times \ldots \times |A| = |A|^n$$

Practical and theoretical applications

California License Plates

Till 1904 — no registration

1905-1912 — various registration formats — one-time $2 fee

1913

≤6 digits — 10^6 = 1 million — If all OK

1956

$26^3 \times 10^3 \approx 17.6$ m

Sam?

1969

$26^3 \times 10^4 \approx 176$ m

Binary Strings

$\{0,1\}^n$ = { length-n binary strings }

n-bit strings

n	Set	Strings	Size
0	$\{0,1\}^0$	Λ	1
1	$\{0,1\}^1$	0, 1	2
2	$\{0,1\}^2$	00, 01, 10, 11	4
3	$\{0,1\}^3$	000, 001, 011, 010, 100, 110, 101, 111	8
...	...	...	...
n	$\{0,1\}^n$	0…0, …, 1…1	2^n

$\{0,1\}^1$

$\{0,1\}^2$

$\{0,1\}^3$

$|\{0,1\}^n| = |\{0,1\}|^n = 2^n$

Subsets

The power set of S, denoted $\mathbb{P}(S)$, is the collection of all subsets of S

$\mathbb{P}(\{a,b\}) = \{ \{\}, \{a\}, \{b\}, \{a,b\} \}$

$\mathbb{P}(\{a,b\})$ and $\{0,1\}^2$

$\mathbb{P}(\{a,b\})$	a	b	$\{0,1\}^2$
{}	❌	❌	00
{b}	❌	✅	01
{a}	✅	❌	10
{a,b}	✅	✅	11

$|\mathbb{P}(S)| = ?$

Subsets of S

Binary strings of length |S|

1-1 correspondence between $\mathbb{P}(S)$ and $\{0,1\}^{|S|}$

$$|\mathbb{P}(S)| = |\{0,1\}^{|S|}| = 2^{|S|}$$

The size of the power set is the power of the set size

Functions

A function from A to B maps every element $a \in A$ to an element $f(a) \in B$

Define a function f: specify f(a) for every $a \in A$

f from {1,2,3} to {p, u} specify f(1), f(2), f(3) f(1)=p, f(2)=u, f(3)=p

f: 3-tuple (f(1), f(2), f(3)) (p, u, p)

{ functions from {1,2,3} to {p,u} **}** {p,u} x {p,u} x {p,u}

functions from {1,2,3} to {p,u} $= 2 \times 2 \times 2 = 2^3 = |\{p,u\}|^{|\{1,2,3\}|}$

{ functions from A to B **}** $\underbrace{B \times B \times \ldots \times B}_{|A|} = B^{|A|}$

functions from A to B $= |B^{|A|}| = |B|^{|A|}$ X

Exponential Growth

2^n

n^2

n

Cartesian Powers

Cartesian Powers & Exponentials

Cartesian power

Again use **product** function in **itertools** library

```
import itertools
print(set(itertools.product({2, 5, 9}, repeat = 2)))
{(5,9),(5,5),(2,9),(9,2),(9,9),(2,2),(9,5),(2,5),(5,2)}
```

Exponent

**

```
print(3**2)
9
```

Notebook

Compute exponentials

Compare to other growth rates

Chess-Rice Legend

(Indian & Persian versions)

Chess

- Invented by poor yet clever peasant
- Became very popular
- King liked it
- Offered peasant any reward he wished

Peasant

- Poor and humble farmer
- Just need a little rice
- Kindly place a single rice grain on first square
- Double on each subsequent square

King

- Such a modest request
- Granted!

Chess-Rice Legend (ctd.)

King

Placed one (2^0) grain on first square

Two (2^1) on second

Four (2^2) on third..

All Americans' worth $\$90 \cdot 10^{12}$

All humans' worth $\$600 \cdot 10^{12}$

World richest person worth $\$90 \cdot 10^{9}$

64th square: $2^{63} \approx 1{,}000{,}000{,}000{,}000{,}000{,}000{,}000{,}000{,}000$

Two endings

Peasant became king

Peasant beheaded

Moral

Be peasant or be King: beware of exponentials!

Jeopardy

Counting questions → Answer

$$\# \left\{ \begin{array}{l} \text{n-bit sequences} \\ \text{Subsets of } \{1,\ldots, n\} \\ \text{Functions: } \{1,\ldots, n\} \text{ to } \{0,1\} \end{array} \right\} = 2^n$$

? ← 2^{2^n}

Find a natural counting question whose answer is a double exponential

Find a natural counting question whose answer is a double exponential

2 solutions | Subsets | Functions | $\exists$ more

Power set of S - set of subsets of S $\mathbb{P}(S)$

$\mathbb{P}(\{a, b\}) = \{ \{\}, \{a\}, \{b\}, \{a,b\} \}$

$| \mathbb{P}(S) | = 2^{|S|}$

$| \mathbb{P}(\{a,b\}) | = 4 = 2^2 = 2^{|\{a,b\}|}$ ✅

$\mathbb{P}(S)$ is a set

What about power set of $\mathbb{P}(S)$?

Find a natural counting question whose answer is a double exponential

$\mathbb{P}(\mathbb{P}(S))$ - set of subsets of $\mathbb{P}(S)$

$|\mathbb{P}(S)| = 2^{|S|}$

$|\mathbb{P}(\mathbb{P}(S))| = 2^{|\mathbb{P}(S)|} = 2^{2^{|S|}}$

$\mathbb{P}(\{a, b\}) = \{ \{\}, \{a\}, \{b\}, \{a,b\} \}$

$\mathbb{P}(\mathbb{P}(\{a, b\})) = \mathbb{P}(\{ \{\}, \{a\}, \{b\}, \{a,b\} \})$

$= \{ \{\}, \{ \{\} \}, \{ \{a\} \}, \ldots, \{ \{\}, \{a\} \}, \ldots, \{\{\},\{a\},\{b\},\{a,b\}\} \}$

$|\mathbb{P}(\mathbb{P}(\{a,b\}))| = 2^{|\mathbb{P}(\{a,b\})|} = 2^{2^{|\{a,b\}|}}$

$|\mathbb{P}(\mathbb{P}([n]))| = 2^{2^n}$

Double exponential

Solution 2: Binary Functions

Functions from A to B B^A $\# = |B|^{|A|}$

Binary functions of n binary variables

Functions from $\{0,1\}^n$ to $\{0,1\}$ $\{0,1\}^{\{0,1\}^n}$

$$\# = |\{0,1\}|^{|\{0,1\}^n|} = 2^{2^n}$$

Double exponential

x	f(x)	
000	0	2
001	0	2
⋮		⋮
111	1	2

2^n (rows), n (bits), 2^{2^n}

Circuit with n binary inputs, one binary output

Can implement 2^{2^n} functions $2^{63} = ½ \cdot 2^{2^6}$

Which Came First

Numbers

Addition

Subtraction

Multiplication

Exponent

Sets

Disjoint union

Complement

Cartesian product

Cartesian power

#1

Like all innovations

Necessity is the mother of math!

Cartesian Powers

sequences

subsets

functions

exponential growth

Cartesian Powers

Cartesian powers $A^n \stackrel{\text{def}}{=} A \times A \times \ldots \times A$ $|A^n| = |A|^n$

Applications binary strings subsets functions

Python A^k `itertools.product(A, repeat = k)`

n^k `n**k`

Variations

Counting Variations

Find a natural counting question whose answer is a double exponential

2 solutions | Subsets | Functions | $\exists$ more

Power set of S - set of subsets of S $\mathbb{P}(S)$

$\mathbb{P}(\{a, b\}) = \{ \{\}, \{a\}, \{b\}, \{a,b\} \}$

$| \mathbb{P}(S) | = 2^{|S|}$

$| \mathbb{P}(\{a,b\}) | = 4 = 2^2 = 2^{|\{a,b\}|}$ ✅

$\mathbb{P}(S)$ is a set

What about power set of $\mathbb{P}(S)$?

Find a natural counting question whose answer is a double exponential

$\mathbb{P}(\mathbb{P}(S))$ - set of subsets of $\mathbb{P}(S)$

$|\mathbb{P}(S)| = 2^{|S|}$

$|\mathbb{P}(\mathbb{P}(S))| = 2^{|\mathbb{P}(S)|} = 2^{2^{|S|}}$

$\mathbb{P}(\{a, b\}) = \{ \{\}, \{a\}, \{b\}, \{a,b\} \}$

$\mathbb{P}(\mathbb{P}(\{a, b\})) = \mathbb{P}(\{ \{\}, \{a\}, \{b\}, \{a,b\} \})$

$= \{ \{\}, \{ \{\} \}, \{ \{a\} \}, \ldots, \{ \{\}, \{a\} \}, \ldots, \{\{\},\{a\},\{b\},\{a,b\}\} \}$

$|\mathbb{P}(\mathbb{P}(\{a,b\}))| = 2^{|\mathbb{P}(\{a,b\})|} = 2^{2^{|\{a,b\}|}}$

$|\mathbb{P}(\mathbb{P}([n]))| = 2^{2^n}$

Double exponential

Solution 2: Binary Functions

Functions from A to B B^A $\# = |B|^{|A|}$

Binary functions of n binary variables

Functions from $\{0,1\}^n$ to $\{0,1\}$ $\{0,1\}^{\{0,1\}^n}$

$$\# = |\{0,1\}|^{|\{0,1\}^n|} = 2^{2^n}$$

Double exponential

x	f(x)	
000	0	2
001	0	2
⋮		⋮
111	1	2

2^n (rows), n (bits), 2^{2^n}

Circuit with n binary inputs, one binary output

Can implement 2^{2^n} functions $2^{63} = ½ \cdot 2^{2^6}$

Combine

addition

subtraction

multiplication

Determine size of more complex sets

PINs with various constraints

Counting Variations

PIN

Personal Identification Number 2174

4-digit PIN's = ?

$D = \{0,\ldots,9\}$ Set of digits

{4-digit PIN's} $= D^4$ Cartesian Power

$|D^4| = |D|^4 = 10^4 = 10{,}000$

Variable Length

3-5 digit PINs

314 2246 79380

$D = \{0,\ldots,9\}$

$\{PINs\} = D^3 \cup D^4 \cup D^5$

Disjoint Union

$\oplus$

$$\# \text{ PINs} = |D^3 \cup D^4 \cup D^5| = |D^3| + |D^4| + |D^5|$$

$$= 10^3 + 10^4 + 10^5$$

$$= 1{,}000 + 10{,}000 + 100{,}000$$

$$= 111{,}000$$

Forbidden Patterns

4-digit PIN's

Forbidden	All same	3333	0000, 1111, …, 9999	# = 10
	Consecutive	3456	0123, 1234, …, 6789	# = 7

Forbidden All same ∪ Consecutive ⊕ 10 + 7 = 17

Allowed D^4 - Forbidden ⊖ 10,000 - 17 = 9,983

4-digit PINs

PINs Containing Zero

PIN's containing 0

8093

~~2534~~

Use latex for formulas
→ slides less colorful

D = {0,1, …, 9}

$Z = \{0\}$ $\quad \overline{Z} = \{1, 2, ..., 9\}$ ← $\overline{Z} = Z^c$ set of non-zero digits

$x^n \triangleq x_1, \ldots, x_n$ ← n-digit sequence

$\exists Z = \{x^n \in D^n : \exists i \; x_i \in Z\}$ ← {n-digit PINs containing 0}

$|\exists Z| = ?$

Start with 2 digits

2 ways

Inclusion exclusion

Subtraction rule

2-Digits: Inclusion-Exclusion

$\exists Z = \{x_1 x_2 \colon \exists i \; x_i = 0\}$ 00 03 50 ~~73~~

$Z_1 = \{x_1 x_2 \colon x_1 = 0\}$ 00 03 ~~50~~ ~~73~~ $|Z_1| = 10$

$Z_2 = \{x_1 x_2 \colon x_2 = 0\}$ 00 ~~03~~ 50 ~~73~~ $|Z_2| = 10$

$$\exists Z = Z_1 \cup Z_2$$

$$|\exists Z| = |Z_1| + |Z_2| - |Z_1 \cap Z_2|$$

$$= 10 + 10 - 1 = 19$$

$$Z_1 \cap Z_2 = \{00\}$$

$$|Z_1 \cap Z_2| = 1$$

2-Digits: Complement Rule

$\exists Z = \{x_1 x_2 \colon \exists i \; x_i = 0\}$ 00 03 50 ~~73~~

$\Omega = D^2$ All 2-digit PINs

$\overline{\exists Z} = \overline{\{x_1 x_2 : \exists i \; x_i = 0\}} = \{x_1 x_2 : \forall i \; x_i \neq 0\} = \overline{Z} \times \overline{Z}$ 73 44 19 ~~50~~

both digits nonzero

{1,..,9} x {1,..,9}

$$|\overline{\exists Z}| = |\overline{Z} \times \overline{Z}| = |\overline{Z}|^2 = 9^2 = 81$$

$$\exists Z = D^2 - \overline{\exists Z}$$

$$|\exists Z| = |D^2| - |\overline{\exists Z}| = 100 - 81 = 19$$

or ⊖ Better?

$\exists Z$

00	10	20	30	…	…	…	…	80	90
01	11								
02									
03									
…									
…									
…									
…									
08									
09									99

Z_2

D^2

$\overline{\exists Z}$

Z_1

Inclusion-Exclusion

$|Z_1| + |Z_2| - |Z_1 \cap Z_2|$

$= 10 + 10 - 1 = 19$

Complement $|\overline{\exists Z}| = 9^2 = 81$

100 - 81 = 19

n Digit: Inclusion Exclusion

$$\exists Z = \{x^n \colon \exists i \; x_i = 0\} \qquad x^n \triangleq x_1, \ldots, x_n$$

$$Z_i = \{x^n \colon x_i = 0\}$$

n=4 $\quad Z_2 = \{x0yz\} \quad Z_4 = \{xyz0\}$

$$\exists Z = Z_1 \cup \ldots \cup Z_n$$

$$\begin{aligned}
|\exists Z| \;=\; & |Z_1| + |Z_2| + \ldots + |Z_n| \\
& - |Z_1 \cap Z_2| - |Z_1 \cap Z_3| - \ldots - |Z_{n-1} \cap Z_n| \\
& + |Z_1 \cap Z_2 \cap Z_3| + \ldots + |Z_{n-2} \cap Z_{n-1} \cap Z_n| \\
& \ldots \\
& + (-1)^{n-1} |Z_1 \cap Z_2 \cap \ldots \cap Z_n|
\end{aligned}$$

n Digits: Complement

$$\overline{\exists Z} = \overline{\{x^n | \exists i \; x_i \in Z\}} = \{x^n | \forall i \; x_i \notin Z\} = (\overline{Z})^n \triangleq \forall \overline{Z}$$

all digits nonzero

$$|\forall \overline{Z}| = |\overline{Z}|^n = 9^n$$

$$\exists Z = D^n - \forall \overline{Z}$$

$$|\exists Z| = |D^n| - |\forall \overline{Z}| = 10^n - 9^n$$

Visualize

Inclusion exclusion

Complement

Counting Variations

Combined

addition

subtraction

multiplication

Determine size of more complex sets

PINs with various constraints

Cartesian products as trees

Trees generalize products

Systematic tree counts

Paths in graphs

Useful for modeling randomness

Counting Trees

Cartesian Products as Trees

Tree ← Sequences ← Cartesian product

a1
a2
a3
b1
b2
b3

$\{a,b\} \times \{1,2,3\}$

2 x 3 = 6 6

$|\{a,b\} \times \{1,2,3\}| = 2 \times 3 = 6$

Used only

At any level, all nodes have same degree

Trees are More General

San-Diego University of Data Science (SUDS)

3 departments: CS, EE, Math

Each offers two courses

courses = ?

Departments offer *different* courses

Not Cartesian Product

Still 2 courses / department

Each level, all degrees equal

courses = 3 x 2 = 6

“X”

Why Trees

A tree can represent any set of sequences, not just Cartesian Products

Enable systematic counting technique

Useful in modeling random phenomena

Best of n

Many sports

Two teams or players compete to determine stronger

Single competition too random

Play odd # games

n

NBA Playoffs

n = 7 games

Tennis matches

n = 3 or 5 sets

Goal

Win majority of n games

Once someone wins > n/2

Stop

Paths to Victory

Roger and Serena

3-set match

Stop when one wins two sets

win sequences = ?

More later

Paths from Source to Destination

Generalize to directed acyclic graph

paths from source to destination

Recursively determine # paths from a node to destination

+ recursively

Look for Structure

Don’t lose sight of the forest

Cartesian products as trees

Trees generalize products

Systematic tree counts

Paths in graphs

Useful for modeling randomness

Counting Trees

Counting Sets

Disjoint unions ⊕

Complements ⊖

General unions

Cartesian products ⊗

Cartesian powers

Variations

Sequences

Trees

Graphs

Combinatorics

“Advanced counting”

Useful for determining probabilities

Combinatorics Permutations

Musical Chairs, Bryant Park, NYC, Summer 2013

Permutations

A permutation is an ordering of a set of objects

permutations of n objects = ?

Objects can be anything

For most excitement

letters!

# letters	permu-tations	# permu-tations
1	a	1
2	a b b a	2
3	a b c a c b b a c b c a c a b c b a	6

General n?

Counting Permutations

2 objects

2 choices

1 choice

a b

b a

2 x 1 = 2

3 objects

2 choices

3 choices

1 choice

a b c

a c b

b a c

b c a

c a b

c b a

"⊗"

3 x 2 x 1 = 6

permutations of n objects = n × (n-1) × … × 2 × 1 ≜ n!

n factorial

0 Factorial

For $n \geq 1$ $\quad$ $n! = \#$ permutations of n objects $= n \times (n-1) \times \ldots \times 2 \times 1$

What about $0!$?

How many ways can you permute 0 objects?

a,b: ab, ba	a: a	: ?
a,b: (ab), (ba)	a: (a)	: ( )

$0! = 1$

Exact same reason as $2^0=1$

Alternative Factorial View

Write | Left to right → Smallest to largest

One position for the 1st object

Two for 2nd

Three for 3rd

Four for 4th

1 x 2 x 3 x… x n = n!

Recursive Definition

n! can be defined recursively

$$n! = n \cdot (n-1) \cdot \ldots \cdot 2 \cdot 1$$

$$= n \cdot [(n-1) \cdot \ldots \cdot 2 \cdot 1]$$

$$= n \cdot (n-1)! \qquad \forall\, n \geq 1$$

1!=1x0!
Extends to
negatives

n	Product	n!	
0	1	1	x 1
1	1	1	x 2
2	2 × 1	2	x 3
3	3 × 2 × 1	6	x 4
4	4 × 3 × 2 × 1	24	x 5
5	5 × 4 × 3 × 2 × 1	120	x 6
6	6 × 5 × 4 × 3 × 2 × 1	720	

Examples and applications

Basic Permutations

orders to visit 3 cities

LA, SD, SF

$$\left.\begin{array}{l} \text{LA SD SF} \\ \cdots\cdots \\ \text{SF SD LA} \end{array}\right\} \quad 3! = 3 \times 2 \times 1 = 6$$

anagrams of 5 distinct letters

PEARS

$$\left.\begin{array}{l} \text{SPEAR} \\ \cdots\cdots \\ \text{EAPRS} \end{array}\right\} \quad 5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$$

Constrained Anagrams of PEARS

A,R stay adjacent in order

PARSE
.....
SEPAR

Permutations of P E AR S $4! = 4 \times 3 \times 2 \times 1 = 24$

A,R are adjacent in any order

SPARE
.....
RAESP

2 orders 24 anagrams each "⊗" $2 \times 24 = 48$

A,R are not adjacent

AESPR
.....
SRPAE

⊖ $5! - 48 = 120 - 48 = 72$

More Constrained Permutations

ways 3 distinct boys and 2 distinct girls can stand in a row

Unconstrained

$(3+2)! = 5! = 120$

Alternating boys and girls

$3! \times 2! = 6 \times 2 = 12$

Boys together and girls together

$2 \times 3! \times 2! = 24$

Unconstrained, but orientation (left to right) doesn't matter

$5! / 2 = 60$

Circular Arrangements

ways 5 people can sit at a round table = ?

Rotations matter

$\neq$

5! = 120

Rotations don't matter

$=$

5! / 5 = 4! = 24

Partial Permutations

Partial Permutations

orders of n objects = n!

orders of some of the n objects = ?

2-digit PINs

any digits

11 45

10 x 10

distinct digits

05 32 ~~33~~

10 x 9

1st digit 2nd digit

3-letter words

any letters

mom xyz

26 x 26 x 26

distinct letters

abc ~~dad~~

26 x 25 x 24

1st 2nd 3rd

Sequences without Repetition

k-permutation of [n]

Length-k sequences over [n]

with repetition n^k

without repetition $n^{\underline{k}}$

k-Permutations

Set of size n: n-set

ation

An ordering of k elements in a set S is a k-permutation of S

2-permutations of {a,b,c}

ab, ac, ba, bc, ca, cb

n-permutation of an n-set is just a permutation of the set

k-permutations of an n-set

$$n \cdot (n-1) \cdot \ldots \cdot (n-k+1) \stackrel{\text{def}}{=} n^{\underline{k}}$$

k	$n^{\underline{k}}$
1	n
2	n(n-1)
3	n(n-1)(n-2)
…	…
k	n(n-1)…(n-k+1)

k^{th} falling power of n

Also denoted P(n,k) or $(n)_k$

Falling Powers and Factorials

Falling powers are simply related to factorials

$$n^{\underline{k}} \qquad n \cdot (n-1) \cdot \ldots \cdot (n-k+1) \quad = \frac{n!}{(n-k)!}$$

Factorials and Permutations

4 programming, 5 probability, 6 machine-learning books

ordered lists with 2 books from each subject where same subject books are listed consecutively = ?

Prob 3, Prob1, ML 5, ML 2, Prog 1, Prog 4

ML 2, ML 6, Prog 1, Prog 2, Prob 5, Prob 1

$3! \cdot 4^{\underline{2}} \cdot 5^{\underline{2}} \cdot 6^{\underline{2}}$ = 6 x (4x3) x (5x4) x (6x5) = 43,200

Subject Order

Progra-mming

Proba-bility

Machine Learning

Stirling's Approximation

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$

N	N!	Stirling approximation	Error %
1	1	1.00	0.227447%
2	2	2.00	0.038602%
3	6	6.00	0.009986%
4	24	24.00	0.004266%
5	120	120.00	0.002198%
6	720	720.01	0.001276%
7	5040	5040.04	0.000805%
8	40320	40320.22	0.000540%
9	362880	362881.38	0.000380%
10	3628800	3628810.05	0.000277%
11	39916800	39916883.11	0.000208%
12	479001600	479002368.48	0.000160%
13	6227020800	6227028659.89	0.000126%
14	87178291200	87178379323.32	0.000101%
15	1307674368000	1307675442913.47	0.000082%
16	20922789888000	20922804061389.80	0.000068%
17	355687428096000	355687629001078.00	0.000056%
18	6402373705728000	6402376752492220.00	0.000048%
19	121645100408832000	121645149634119000.00	0.000040%
20	2432902008176640000	2432902852332160000.00	0.000035%

Combinations

k-subsets

A k-element set is called a k-set, a k-element subset is a k-subset

$\binom{[n]}{k}$ collection of k-subsets of [n] = {1,2,…,n}

$\binom{[3]}{1}$ { {1}, {2}, {3} }

$\binom{[3]}{2}$ { {1,2}, {1,3}, {2,3} }

$\binom{[4]}{2}$ { {1,2}, {1,3}, {1,4}, {2,3}, {2,4}, {3,4} }

Sequences with k 1's

$\binom{[n]}{k}$ collection of k-subsets of [n] = {1,2,…,n}

1-1 correspondence to n-bit sequences with k 1's

	Subsets	Binary Sequences
$\binom{[3]}{1}$	{1}, {2}, {3}	100, 010, 001
$\binom{[3]}{2}$	{1, 2}, {1, 3}, {2, 3}	110, 101, 011
$\binom{[4]}{2}$	{1, 2}, {1, 3}, …, {3, 4}	1100, 1010, …, 0011

Two Interpretations

$\binom{[n]}{k}$ k-subsets of an n-set n-bit sequences with k 1's

$\binom{[3]}{2}$ {{1,2}, {1,3}, {2,3}} 110, 101, 011

1 - 1 correspondence Same number of elements

Mostly count sequences Same applies to subsets

Number of n-Bit Sequences with k 1's

$$\binom{n}{k} \triangleq \left|\binom{[n]}{k}\right|$$ = # n-bit sequences with k 1's

binomial coefficient

$$\binom{3}{2} = \left|\binom{[3]}{2}\right|$$ = $|\{110, 101, 011\}| = 3$

Locations of 1's: Ordered Pairs from {1,2,3}

$$\# = 3^{\underline{2}} = 6$$

12	110
13	101
21	110
23	011
31	101
32	011

$$\binom{3}{2} = \frac{3^{\underline{2}}}{2} = \frac{6}{2} = 3$$

Calculating the Binomial Coefficients

$$\binom{n}{k} \triangleq \left| \binom{[n]}{k} \right| = ?$$

number set

Each location ∈ [n]

Specify locations of the k 1's in order

123, 531, 213, …

ordered locations $n^{\underline{k}}$

Every binary sequence with k 1's corresponds to k! ordered locations

10101 ↔ 1,3,5 1,5,3 3,1,5 3,5,1 5,1,3 5,3,1

$$k!\binom{n}{k} = n^{\underline{k}} \qquad \binom{n}{k} = \frac{n^{\underline{k}}}{k!} = \frac{n!}{k!(n-k)!}$$

$$\left|\binom{[n]}{k}\right| = \binom{n}{k} = \frac{n!}{k!(n-k)!}$$ **for small n,k**

$$\binom{[3]}{1} = \left\{\begin{matrix} 001 \\ 010 \\ 100 \end{matrix}\right\} \qquad \binom{3}{1} = \frac{3!}{1!\,2!} = 3$$

Choose location of 1

$$\binom{[3]}{2} = \left\{\begin{matrix} 011 \\ 101 \\ 110 \end{matrix}\right\} \qquad \binom{3}{2} = \frac{3!}{2!\,1!} = 3$$

Choose location of 0

$$\binom{[4]}{2} = \left\{\begin{matrix} 0011 \\ 0101 \\ 0110 \\ 1001 \\ 1010 \\ 1100 \end{matrix}\right\} \qquad \binom{4}{2} = \frac{4!}{2!\,2!} = 6$$

Choose location of 1's
1st location: 4 choices
2nd location: 3 choices
Each chosen twice

Simple $\binom{n}{k}$

$$\binom{n}{0} = \frac{n!}{0!\,n!} = 1$$

All-zero sequence

$$\binom{n}{n} = \frac{n!}{n!\,0!} = 1$$

All-one sequence

$$\binom{n}{1} = \frac{n!}{1!\,(n-1)!} = n$$

Choose location of single 1

$$\binom{n}{2} = \frac{n!}{2!\,(n-2)!} = \frac{n(n-1)}{2}$$

1st location: n ways
2nd location: n-1 ways
Each sequence chosen twice

Alternative Explanation of $\binom{n}{2} = \frac{n(n-1)}{2}$

$\binom{[n]}{2} = \{\text{n-bit strings with two 1's}\}$

$A_i = \{x^n\text{: first 1 at location i}\} \quad (1 \leq i \leq n-1)$

n-i locations w/ exactly one 1

$$\overset{1}{\underset{\uparrow}{0}} \ \ldots \ 0 \ \underset{\uparrow\, i}{1} \ \overbrace{x \ y \ \ldots \ \underset{\uparrow\, n}{z}}$$

$|A_i| = n - i \qquad A_i\text{'s disjoint}$

$\binom{[n]}{2} = A_1 \uplus A_2 \uplus \ldots \uplus A_{n-1}$

$\binom{n}{2} = |\binom{[n]}{2}| = |A_1| + \ldots + |A_{n-1}| = (n-1) + \ldots + 1 = \frac{n(n-1)}{2}$

Binomial Coefficients by Hand

Simply calculate binomial coefficients

$$\binom{n}{k} = \frac{n^{\underline{k}}}{k!}$$

$$\binom{7}{2} = \frac{7 \cdot \cancel{6}^{3}}{\cancel{2} \cdot 1} = 7 \cdot 3 = 21$$

$$\binom{n}{k} = \binom{n}{n-k}, \text{ if k} > \frac{n}{2} \text{ calculate } \binom{n}{n-k}$$

$$\binom{7}{5} = \binom{7}{2} = 21$$

Cancel terms

$$\binom{12}{9} = \binom{12}{3} = \frac{\cancel{12}^{2} \cdot 11 \cdot 10}{\cancel{3} \cdot \cancel{2} \cdot 1} = 220$$

Can calculate fairly large binomial coefficients by hand

Permutations vs. Combinations

Permutations

Order matters

Combinations

Order doesn’t matter

Type of lock you used to secure your bike?

Combinations

Applications of Binomial Coefficients

Subsets

k-element subsets of an n-set

1-1 correspondence between such sets and n-bit sequence with k 1's

$$\# = \binom{n}{k}$$

Committees with Constraints

4 boys 3 girls

choose 2 boys and 2 girls

$$\binom{4}{2}\binom{3}{2} = 6 \cdot 3 = 18$$

Conflict Resolution

4 boys and 3 girls

John Mary cannot serve together

How many committees of 4

with John $\binom{5}{3} = 10$

with Mary $\binom{5}{3} = 10$

neither $\binom{5}{4} = 5$

$$\overline{25}$$

Triangles

n points in general position in plane

How many triangles can be formed?

$$\binom{n}{3}$$

How Many Paths?

Every path from Start to End is a length-10 sequence of U,R with 6 R's

$$\#\text{ of paths } = \binom{10}{6} = 210$$

Next: Properties

Properties
of Binomial Coefficients

$$\binom{n}{k} = \binom{n}{n-k}$$

$$\binom{5}{3} = \frac{5!}{3!\,2!} = \frac{5!}{2!\,3!} = \binom{5}{2}$$

Two proofs

Algebraic

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!} = \frac{n!}{(n-k)!\,k!} = \binom{n}{n-k}$$

$$\binom{n}{k} = \binom{n}{n-k}$$

Combinatorial

$$\binom{[n]}{k} \longleftrightarrow \binom{[n]}{n-k}$$

Complement bits to create 1-1 correspondence

$$\binom{[4]}{3} \longleftrightarrow \binom{[4]}{4-3}$$

$$\left\{\begin{matrix}1110\\1101\\1011\\0111\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}0001\\0010\\0100\\1000\end{matrix}\right\}$$

$$\binom{n}{k} = \left|\binom{[n]}{k}\right| = \left|\binom{[n]}{n-k}\right| = \binom{n}{n-k}$$

$$\binom{n}{k} = \frac{n}{k} \cdot \binom{n-1}{k-1}$$

$$\binom{5}{3} = \frac{5!}{3!\,2!} = 10 = \frac{5}{3} \cdot 6 = \frac{5}{3}\binom{4}{2}$$

Recursive definition

choose one of these locations to be 2

choose location of 2

choose k locations of k non-zeros (1's & 2)

$$\binom{n}{k} \cdot k = n \cdot \binom{n-1}{k-1}$$

from remaining n-1 locations choose locations of the k-1 1's

Number of length-n ternary strings with k-1 1's and one 2

n=4 k=3 { 0112, 1012, 2110,… }

$$\sum_{i=0}^{n} \binom{n}{i} = 2^n$$

$$\binom{3}{0} + \binom{3}{1} + \binom{3}{2} + \binom{3}{3} = 1 + 3 + 3 + 1 = 8 = 2^3$$

%2^{[n]} = |2^{[n]}| =
\sum^n_{i=0} \left | \binom{[n]}i
\right | = \sum^n_{i=0} \binom ni
%2^{[n]} = \uplus^n_{i=0} \binom

Combinatorial proof

$$2^{[n]} = \biguplus_{i=0}^{n} \binom{[n]}{i}$$

$$2^n = |2^{[n]}| = \sum_{i=0}^{n} \left| \binom{[n]}{i} \right| = \sum_{i=0}^{n} \binom{n}{i}$$

Algebraic proof: next video

Outside the circle

subsets of [n] of size ≤ n-1

n-bit sequences with ≤ n-1 1's

n = 3

$$\underbrace{000}_{1},\ \underbrace{001,\ 010,\ 100}_{3},\ \underbrace{011,\ 101,\ 110}_{3}$$

1 + 3 + 3 = 7

Two ways

$$\sum_{i=0}^{n-1} \binom{n}{i}$$

$$\sum_{i=0}^{n} \binom{n}{i} = 2^n \qquad \sum_{i=0}^{n-1} \binom{n}{i} = 2^n - \binom{n}{n} = 2^n - 1$$

$2^3 - 1 = 7$

Hockey Stick Identity

$$\sum_{i=0}^{n} \binom{i+k-1}{k-1} = \binom{n+k}{k}$$

$$\binom{k-1}{k-1} + \binom{k}{k-1} + \ldots + \binom{n+k-1}{k-1} = \binom{n+k}{k}$$

k=3, n=2

$$\binom{2}{2} + \binom{3}{2} + \binom{4}{2} = 1 + 3 + 6 = 10 = \binom{5}{3}$$

Combinatorial Proof

$$\binom{k-1}{k-1} + \binom{k}{k-1} + \ldots + \binom{n+k-1}{k-1} = \binom{n+k}{k}$$

A = { binary sequences of length n+k with k 1's }

$$|A| = \binom{n+k}{k}$$

A_i = { (n+k)-bit sequences with k 1's where the last 1 is at location i}

$$|A_i| = \binom{i-1}{k-1}$$

Example

$$A = A_k \uplus A_{k+1} \uplus \ldots \uplus A_{n+k}$$

Next: Pascal Triangle and Binomial Theorem

Pascal Triangle and Binomial Theorem

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1
1 9 36 84 126 126 84 36 9 1
1 10 45 120 210 252 210 120 45 10 1

Pascal's Identity

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$$

$$\binom{4}{3} = \binom{3}{3} + \binom{3}{2}$$

0 1 1 1		0 1 1 $\boxed{1}$
1 0 1 1		1 0 1 $\boxed{1}$
1 1 0 1		1 1 0 $\boxed{1}$
$\underbrace{1\ 1\ 1\ 0}_{\text{three 1's}}$	$\underbrace{1\ 1\ 1}_{\text{three 1's}}\boxed{0}$	$\underbrace{}_{\text{two 1's}}$

Pascal's Triangle

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$$

$$\binom{2}{1} = \binom{1}{1} + \binom{1}{0}$$

$$\binom{3}{1} = \binom{2}{1} + \binom{2}{0}$$

$$\binom{3}{2} = \binom{2}{2} + \binom{2}{1}$$

k

n	0	1	2	3	4	5
0	1					
1	1	1				
2	1	2	1			
3	1	3	3	1		
4	1	4	6	4	1	
5	1	5	10	10	5	1

$$\binom{n}{0} = 1 \qquad \binom{n}{n} = 1$$

Binomial Theorem

$(a + b)^1 = a + b$

$(a + b)^2 = a^2 + 2ab + b^2$

$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

$(a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$

$$1 = \binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4} = 1$$

n \ k	0	1	2	3	4	5
0	1					
1	1	1				
2	1	2	1			
3	1	3	3	1		
4	1	4	6	4	1	
5	1	5	10	10	5	1

$$(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^{n-i} b^i$$

Binomial Theorem

$$(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^{n-i} b^i \quad \forall\, a, b \quad \forall\, n \geq 0$$

So important it gives the binomial coefficients their name

Explanation

$$(a+b)^3 = (\underline{a}+b)(\underline{a}+b)(a+\underline{b})$$

Three (a+b) factors

$$= \underbrace{aaa + aab + aba + abb + baa + bab + bba + bbb}$$

Each term: product of a's and b's
one selected from each factor

Sum of terms

a's + # b's = 3

$$= a^3 + 3a^2b + 3ab^2 + b^3$$

# of terms with i b's = # ways to select i factors out of the 3 = $\binom{3}{i}$

$$= \binom{3}{0}a^3 + \binom{3}{1}a^2b + \binom{3}{2}ab^2 + \binom{3}{3}b^3$$

Generally

$$(a+b)^3 = \binom{3}{0}a^3 + \binom{3}{1}a^2b + \binom{3}{2}ab^2 + \binom{3}{3}b^3 = \sum_{i=0}^{3}\binom{3}{i}a^{3-i}b^i$$

Similarly

$$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \ldots + \binom{n}{n}b^n = \sum_{i=0}^{n}\binom{n}{i}a^{n-i}b^i$$

$$\sum_{i=0}^{n} \binom{n}{i} = 2^n$$

Gave combinatorial proof

Algebraic proof

$$2^n = (1+1)^n = \sum_{i=0}^{n} \binom{n}{i} 1^{n-i} 1^i = \sum_{i=0}^{n} \binom{n}{i}$$

Polynomial Coefficients

$$(1+x)^n = \sum_{i=0}^{n} \binom{n}{i} x^i$$

Coefficient of x^2 in $(1+x)^7$

$$(1+x)^7 = \sum_{i=0}^{7} \binom{7}{i} x^i \qquad \binom{7}{2} = 21$$

Coefficient of x^3 in $(3+2x)^5$

$$(3+2x)^5 = \sum_{i=0}^{5} \binom{5}{i} 3^{5-i} (2x)^i \qquad \binom{5}{3} 3^2 \cdot 2^3 = 720$$

Binomial → Taylor

Taylor expansion

$$e^x = \sum_{i=0}^{\infty} \frac{x^i}{i!}$$

$$\left(1 + \frac{x}{n}\right)^n = \sum_{i=0}^{n} \binom{n}{i} \left(\frac{x}{n}\right)^i$$

$$= \sum_{i=0}^{n} \frac{n^{\underline{i}}}{i!} \left(\frac{x}{n}\right)^i$$

$$= \sum_{i=0}^{n} \frac{x^i}{i!} \cdot \frac{n^{\underline{i}}}{n^i}$$

$$n \to \infty$$

$$e^x = \sum_{i=0}^{\infty} \frac{x^i}{i!}$$

Binomial Distribution

$$\sum_{i=0}^{n} \binom{n}{i} p^{n-i}(1-p)^i = (p + (1-p))^n = 1^n = 1$$

Example

$$\left(\frac{1}{3}\right)^2 + 2\left(\frac{1}{3}\right)\left(\frac{2}{3}\right) + \left(\frac{2}{3}\right)^2 = 1$$

More in binomial distribution

So far: Binary Sequences

Next: Larger Alphabets

Multinomials

Multinomial Coefficients

Beyond binary - ternary alphabets

$$k_1 + k_2 + k_3 = n$$

$$\#\ \{1,\ 2,\ 3\} \text{ sequences with } \begin{cases} k_1 & 1\text{'s} \\ k_2 & 2\text{'s} \\ k_3 & 3\text{'s} \end{cases}$$

$$\underbrace{\binom{n}{k, n-k}}_{\text{sum to n}} = \binom{n}{k}$$

$$\binom{n}{k_1}\binom{n-k_1}{k_2} = \frac{n!}{k_1!\cdot\cancel{(n-k_1)!}} \cdot \frac{\cancel{(n-k_1)!}}{k_2!\cdot\underbrace{(n-k_1-k_2)!}_{k_3}} = \frac{n!}{k_1!\cdot k_2!\cdot k_3!} \triangleq \binom{n}{k_1, k_2, k_3}$$

k_1 location of 1's

k_2 location of 2's out of n-k_1 locations left

(location of k_3 3's is determined)

symmetric in k_1, k_2 ,k_3

Simple Example

sequences over {1,2,3,4}

digit	1	2	3	4
# times	1	4	4	2

length 11

31222334243

$$\binom{11}{1,4,4,2} = \frac{11!}{1!\cdot 4!\cdot 4!\cdot 2!}$$

$$= \frac{11\cdot 10\cdot 9\cdot \cancel{8}\cdot 7\cdot \cancel{6}\cdot 5\cdot \cancel{4!}}{\cancel{4!}(\cancel{4}\cdot \cancel{3}\cdot \cancel{2}\cdot 1)(\cancel{2}\cdot 1)}$$

$$= 11\cdot 10\cdot 9\cdot 7\cdot 5 = 34,650$$

MISSISSIPPI

anagrams = ?

sequences over {M,I,S,P}

letter	M	I	S	P
# times	1	4	4	2

length 11 (SISSISIPPIM)

Same as sequences over {1,2,3,4} in previous slide

$$\binom{11}{1,4,4,2} = 34,650$$

$\uparrow \ \uparrow \ \uparrow \ \uparrow$

M I S P

Students in Class

10 students

3 classes: morning, afternoon, evening

Any number of students in each class

$$3^{10}$$

6 morning

3 afternoon

1 evening

$$\binom{10}{6,3,1} = \frac{10!}{6!\cdot 3!\cdot 1!} = \frac{10\cdot \overset{3}{\cancel{9}}\cdot \overset{4}{\cancel{8}}\cdot 7}{\cancel{3}\cdot \cancel{2}} = 840$$

morning students

afternoon students

evening students

Multinomial Theorem

$$(a_1 + a_2 + ... + a_m)^n = \sum_{\substack{k_1 + k_2 + ... + k_m = n \\ k_1, k_2, ..., k_m \geq 0}} \binom{n}{k_1, k_2, ..., k_m} \prod_{t=1}^{m} a_t^{k_t}$$

$(a+b+c)^2$

$$(a+b+c)^2 = \sum_{\substack{i+j+k=2 \\ i,j,k \geq 0}} \binom{2}{i,j,k} a^i b^j c^k$$

$$= \tbinom{2}{2,0,0} a^2 + \tbinom{2}{0,2,0} b^2 + \tbinom{2}{0,0,2} c^2 + \tbinom{2}{1,1,0} ab + \tbinom{2}{1,0,1} ac + \tbinom{2}{0,1,1} bc$$

$$= a^2 + b^2 + c^2 + 2ab + 2ac + 2bc$$

Sum of Multinomials

Recall binomial identity $$2^n = \sum_{i=0}^{n} \binom{n}{i}$$

Similarly for multinomials

$$m^n = (1 + 1 + ... + 1)^n = \sum_{\substack{k_1 + ... + k_m = n \\ k_1, ..., k_m \geq 0}} \binom{n}{k_1, k_2, ..., k_m}$$

$$3^2 = 9 = \underbrace{\binom{2}{2,0,0}}_{1} + \underbrace{\binom{2}{0,2,0}}_{1} + \underbrace{\binom{2}{0,0,2}}_{1} + \underbrace{\binom{2}{1,1,0}}_{2} + \underbrace{\binom{2}{1,0,1}}_{2} + \underbrace{\binom{2}{0,1,1}}_{2}$$

Students in Class

2 students

3 classes: morning, afternoon, evening $\quad 3^2 = 9$

Broken by class

$$\underbrace{\binom{2}{2,0,0}}_{1} + \underbrace{\binom{2}{0,2,0}}_{1} + \underbrace{\binom{2}{0,0,2}}_{1} + \underbrace{\binom{2}{1,1,0}}_{2} + \underbrace{\binom{2}{1,0,1}}_{2} + \underbrace{\binom{2}{0,1,1}}_{2} = 9$$

same as last slide

Next: Application

Combinatorics Stars and Bars

Final application

Beyond combinatorics - algebra

Simple derivation of unintuitive result

Counting Sums

ways to

write 5 as a sum of 3 positive integers, when order matters

partition 5 items into 3 groups, when order matters

3 + 1 + 1 ☆ ☆ ☆ | ☆ | ☆

2 + 2 + 1 ☆ ☆ | ☆ ☆ | ☆

2 + 1 + 2 ☆ ☆ | ☆ | ☆ ☆

1 + 3 + 1 ☆ | ☆ ☆ ☆ | ☆

1 + 2 + 2 ☆ | ☆ ☆ | ☆ ☆

1 + 1 + 3 ☆ | ☆ | ☆ ☆ ☆

= 6

Addition	Partition
sum to 5	5 stars (items)
3 positive terms	3 consecutive star intervals
2 +'s separating the numbers	2 bars separating the intervals
	4 inter-star gaps
	Choose 2 of 4 gaps

$$\binom{4}{2} = 6$$

k Terms Adding to n

ways to write n as a sum of k positive integers, when order matters

sums = $\binom{n-1}{k-1}$

Simple Examples

ways to write n as sum of k positive integers, when order matters $= \binom{n-1}{k-1}$

k	sums	$\binom{n-1}{k-1}$
1	n	$\binom{n-1}{1-1} = 1$
n	1+1+…+1	$\binom{n-1}{n-1} = 1$
2	1+(n-1), 2+(n-2), …, (n-1)+1	$\binom{n-1}{2-1} = n-1$
n-1	2+1+..+1, …, 1+..+1+2	$\binom{n-1}{n-2} = n-1$

8 as sum of 4 positive integers $\binom{8-1}{4-1} = \binom{7}{3} = \frac{7 \cdot 6 \cdot 5}{3 \cdot 2 \cdot 1} = 35$

Any Sum to n

ways to write n as a sum of (any # of) positive integers

n=3

3	☆ ☆ ☆
1+2	☆ | ☆ ☆
2+1	☆ ☆ | ☆
1+1+1	☆ | ☆ | ☆

= 4 = 2^2

general n

bar or no bar

= 2^{n-1}

n as sum of k∈[n]: $\binom{n-1}{k-1}$

$$\sum_{k=1}^{n} \binom{n-1}{k-1} = \sum_{i=0}^{n-1} \binom{n-1}{i} = 2^{n-1}$$

Nonnegative Terms

ways to write n as sum of k nonnegative integers

2 as sum of 3 nonnegatives

2+0+0	☆ ☆ | |
0+2+0	| ☆ ☆ |
0+0+2	| | ☆ ☆
1+1+0	☆ | ☆ |
1+0+1	☆ | | ☆
0+1+1	| ☆ | ☆

= 6

As before n ☆ k-1 |

Now ☆ and | can appearing any order

sum ↔ ☆ and |

sums = # ☆ and |

ways to order 2 ☆ and 2 |

$$\binom{4}{2}$$ = 6

k Non-negatives Adding to n

ways to write n as a sum of k non-negative integers, when order matters

sequences with n stars and k-1 bars

length n+(k-1) sequences with k-1 bars

$$\boxed{\binom{n+k-1}{k-1}}$$

4-Letter Words

4-letter words when order matters

$26^4 = 456{,}976$

4-letter words when order does not matter

evil = vile = live = veil = eilv = liev = …

doom = mood = odom = …

Determined by composition:

#a, #b, #c, …, #z

#a + #b + #c + … + #z = 4

26 nonnegative terms

k=26

sum to 4

n=4

$$\binom{4+26-1}{26-1} = \binom{29}{25} = \binom{29}{4}$$

= 23,751

a little ≥ $26^4 / 4!$

More Applications

Can derive # positive adding to n

k positive adding to n = # k non-negative adding to n-k

1+2+1+3 = 7 0+1+0+2 = 7-4 = 3

$$\binom{(n-k)+k-1}{k-1} = \binom{n-1}{k-1}$$

Can derive # non-negative adding to ≤ n

k non-negative adding to ≤ n = # k+1 non-negative adding to n

2+0+3 ≤ 7 2+0+3+2 =7

$$\binom{n+(k+1)-1}{(k+1)-1} = \binom{n+k}{k}$$