Probability introduction

Randomness

Probability motivation

Outcomes

Sample spaces

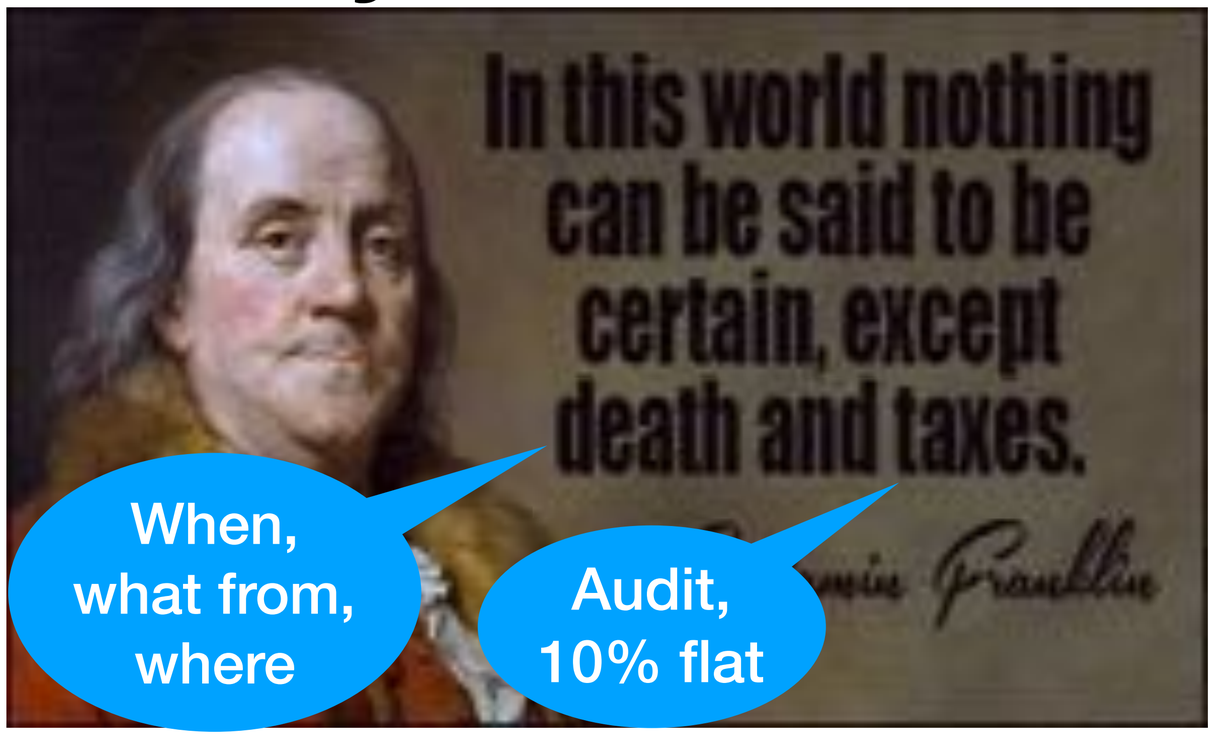
Distributions

Why Probability?

Some things in life are certain

Most are a little less predictable

- Physicians — illness, medication
- Farmers — rain, diet trends
- Investors — stock price, economy
- Advertisers — views, competition
- Consumers — availability, sale
- Students — food line, grade, parents, job, date, game



Random Phenomena

Give up?

Reason intelligently?

Learn	Range	Average	Variability
Infer	Structure	Change	Relations
Predict	Future	Likelihood	Guarantees
Benefit	Compete	Plan	Build

Coming to Terms

As with sets

Need terminology

Discuss

Concisely

Precisely

Process of generating and observing data

Individual and collection of observations

Meaning of probability

Several approaches

Intuitive

Axioms

Data

Experiments

Probability developed in part to aid science

Process

Generating random data

Observing outcome

Experiment

Biology

Engineering

Business

Unified approach

Applies generally

…

Sociology

Understand

Analyze

generalize

Our experiments

Start simple

Get very complex

Outcomes and Sample Space

Possible experiment results are called outcomes, or observations

Set of possible outcomes is the sample space, denoted Ω

Notetation

Some use S or U

Experiment	Ω
Coin	{ h, t }
Die	{ 1, 2, …, 6 }
Gender	{ m, f }
Age	$\mathbb{N}$
Temperature	$\mathbb{R}$

typically lower-case h, t, x

Elements, sets

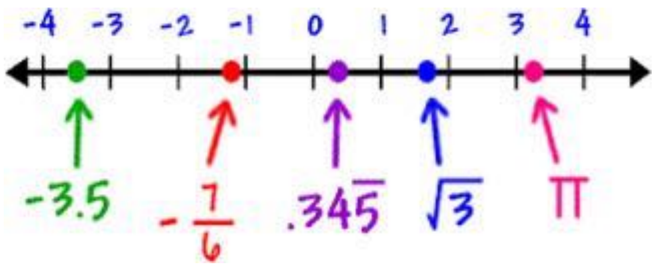
Next lecture: subsets

Two Sample-Space Types

Finite or countably infinite sample space is discrete

$\{ h, t \}$ $\{ 1, 2, \ldots, 6 \}$ $\mathbb{N}$ $\mathbb{Z}$ { words } { cites } { people }

Uncountably infinite sample space - continuous



$\mathbb{R}$ { temperatures } $\underbrace{\text{\{ salaries \} \{ prices \}}}_{\text{upgraded}}$

Discrete spaces Easier to understand, visualize, analyze

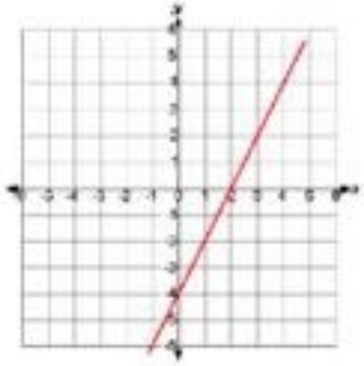
Important First

Next few topics: Discrete

Continuous Important Conceptually harder Later

Random Outcomes

Algebra Unknown value denoted by x $2x - 4 = 0$



Solve Before After

$x \in \mathbb{R}$ $x = 2$

Notetation

If needed use y, z, …

Probability Random value of outcome denoted by X X - coin flip outcome

Experiment Before After

$X \in \Omega$ get h $X = h$

get t $X = t$

Notetation

If needed use Y, Z, …

Probability of Outcome

The probability, or likelihood, of an outcome $x \in \Omega$, denoted $P(x)$, or $P(X=x)$, is the fraction of times x will occur when experiment is repeated many times

Fair coin

As # experiments $\rightarrow \infty$, fraction of heads (or tails) $\rightarrow$ ½

heads has probability ½ | $P(h) = ½$ | $P(X=h) = ½$

tails has probability ½ | $P(t) = ½$ | $P(X=t) = ½$

Fair die

As # experiments $\rightarrow \infty$, fraction of 1's (or 2,…,6) $\rightarrow$ ⅙

1 has probability ⅙ | $P(1) = ⅙$ | $P(X=1) = ⅙$

$P(x)$ | $P(X=x)$ | probability of x | fraction of times x will occur

Probability Distribution Function

P(x) is the fraction of times outcome x occurs

P(h) = ½ P(1) = ⅙

Viewed over the whole sample space, a pattern emerges

Coin P(h) = ½ P(t) = ½

Die P(1) = ⅙ … P(6) = ⅙

Rain P(rain) = 10% P(no rain) = 90%

Sample space Ω and distribution P define the whole probability space

P maps outcomes in Ω to nonnegative values that sum to 1

$P: \Omega \to \mathbb{R}$ $P(x) \geq 0$ $\sum_{x \in \Omega} P(x) = 1$

Probability distribution function (PDF) distribution

Probability introduction

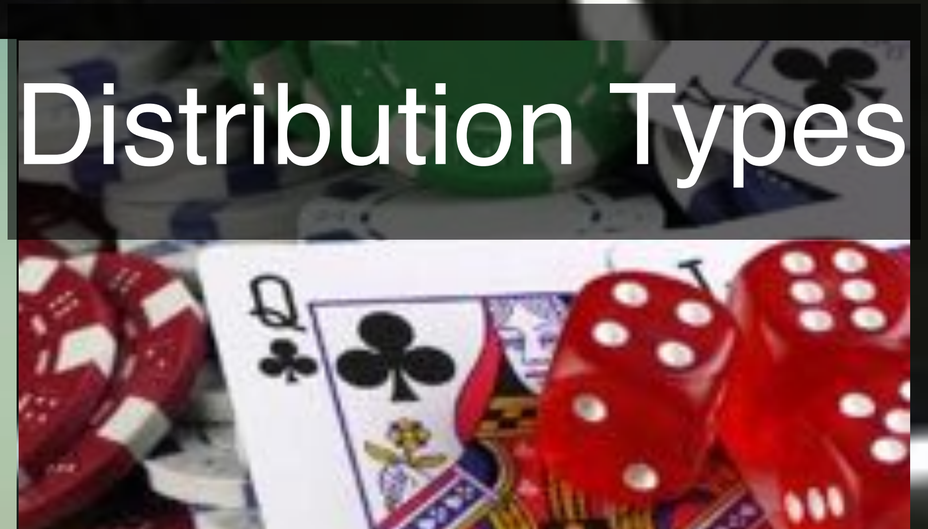
Randomness

Probability motivation

Outcomes

Sample spaces

Distributions



Distribution Types

Uniform sample spaces

Coin, die, cards

Non-uniform spaces

Tetrahedral die

Uniform Probability Spaces

Generally, outcomes may have different probabilities

Rain P(rain) = 10% P(no rain) = 90%

Uniform (equiprobable) spaces Uniform distribution

All outcomes are equally likely

Coin P(h) = P(t) = ½

Drastically simplifies probability specification

Uniform Probability Spaces

All outcomes are equally likely

$\forall\, x \in \Omega \quad P(x) = p$

$1 = \sum_{x \in \Omega} P(x) = \sum_{x \in \Omega} p = |\Omega| \cdot p$

$p = 1/|\Omega|$

Notetation

Draw Uniformly, Randomly

Coin $\quad P(h) = P(t) = p \quad 1 = P(h) + P(t) = 2 \cdot p \quad p = ½$

Uniform spaces $\quad$ Every outcome has probability $1/|\Omega|$

All you need to know is $|\Omega|$!

Ⓤ

Fair Coin

$\Omega = \{ \text{heads, tails} \} = \{ h, t \}$

$|\Omega| = 2$

Flip, or toss

Equally likely $\textcircled{U}$ $P(h) = P(t)$

$$P(h) = P(t) = \frac{1}{|\Omega|} = \frac{1}{2}$$

Fair Die

$\Omega = \{ 1, 2, 3, 4, 5, 6 \}$

$|\Omega| = 6$

Roll

Equally likely $\textcircled{U}$ $P(1) = \ldots = P(6)$

$$P(1) = \ldots = P(6) = \frac{1}{|\Omega|} = \frac{1}{6}$$

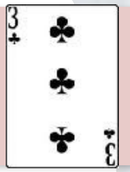
Deck of Cards

$\Omega = \{ \text{cards} \}$

$|\Omega| = 52$

Draw a card

Equally likely $\textcircled{U}$

$P($  $) = \ldots = P($ $) = \frac{1}{|\Omega|} = \frac{1}{52}$

Uniform → Non

Uniform, equiprobable, spaces

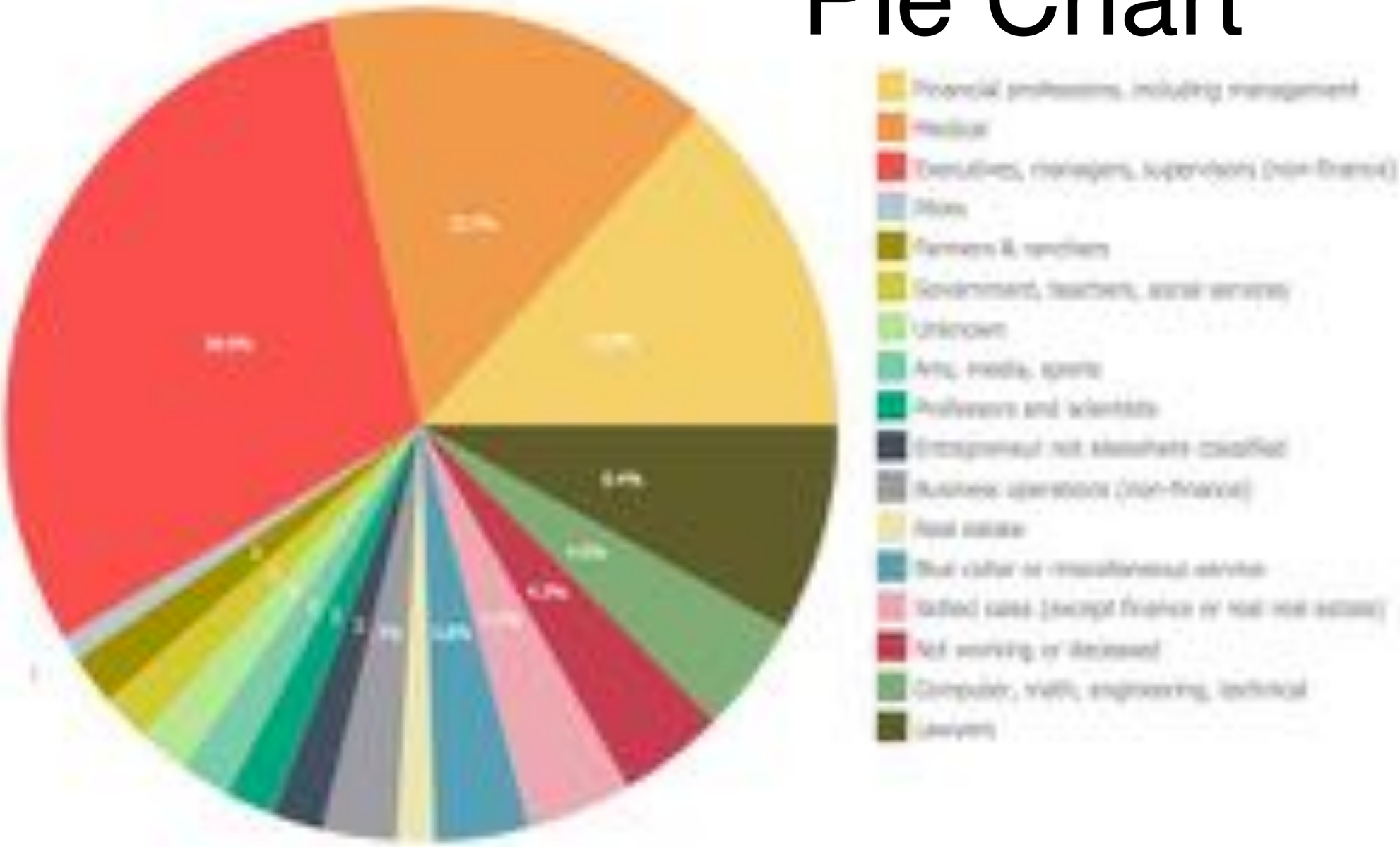
Coin Die Cards

In nature, nonuniform spaces abound

rain grades words illnesses

web pages people ….

Pie Chart



Challenge Non-uniform distribution we can remember

Tetrahedral Die

4-sided, pyramid die

Used in games, D&D

In games die is equiprobable

We will assume different probabilities

Easy to remember

Face	1	2	3	4
Probability	.1	.2	.3	.4

Conveniently, add to 1

Probability distribution

Do’s and Don’ts

Random notation may be confusing at first

Which expressions are valid?

$P(X = 3)$ fair die: ⅙

$P(3)$ $\stackrel{\text{def}}{=} P(X = 3)$

$P(x)$ specify x, e.g., for $\forall x$, $P(x)=¼$

$P(1 = 3)$ 0

$P(X)$ random value

Possible, but less common. Make sure this is what you mean.

$P(x = 3)$

Even less likely, probably wrong

Distribution Types

Uniform sample spaces

- Coin, die, cards

Non-uniform spaces

- Tetrahedral die

Events

Probability Events

Outcomes to subsets

Events

Occurrence

Probability

Events

Social event

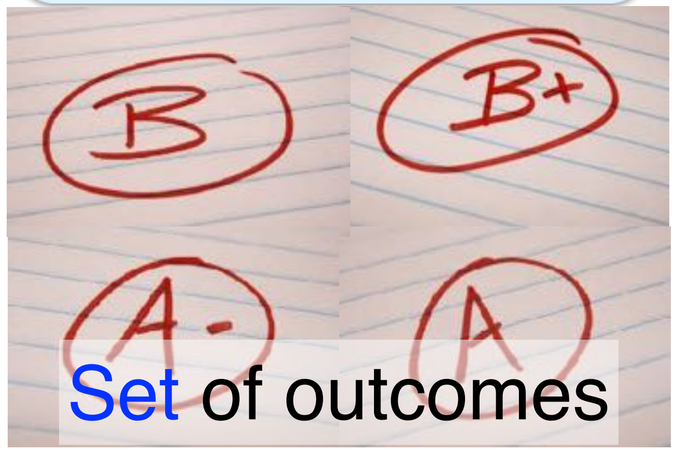
Sometimes Care about one particular outcome

Our horse will win race

Get B+ in class

Usually Interested in a set of possible outcomes

Probabilistic event



Temperature > 10 degrees

Stock will close higher

Pass a course

Set of outcomes

Subset of the sample space Ω

Event

Die Events

Name	Set
Ω (certain)	{1, …, 6}
Even	{2, 4, 6}
Square	{1, 4}
> 4	{5, 6}

Complements

Name	Set
∅ (null)	{ }
Odd	{1, 3, 5}
Non square	{2, 3, 5, 6}
≤ 4	{1, 2, 3}

Tetrahedral Die

Face	1	2	3	4
Probability	.1	.2	.3	.4

Name	Set
Ω (certain)	{1,2,3,4}
Even	{2, 4}
Prime	{2, 3}
∅ (null)	{ }

Event Occurrence

An event occurs if it contains the observed outcome

E occurs if $X \in E$

Face	1	2	3	4
Probability	.1	.2	.3	.4

Name	Set	1	2	3	4
Ω (certain)	{1,2,3,4}	✅	✅	✅	✅
Even	{2, 4}	❌	✅	❌	✅
Prime	{2, 3}	❌	✅	✅	❌
∅ (null)	{ }	❌	❌	❌	❌

Event Probability

Probability of event E

Fraction of experiments where E occurs, as # experiments grows

Even	④	③	②	①	④	③	③	④	②	④	③
	✅		✅		✅			✅	✅	✅	

times E occurs = Sum of # times 2 and 4 occur

Fraction of times E occurs = Sum of fraction of times 2 and 4 occur

$$P(E) = P(X \in E) = \sum_{x \in E} P(x)$$

Tetrahedral Die

Probability	.1	.2	.3	.4

Name	Set	1	2	3	4	P(E)
Ω (certain)	{1,2,3,4}	✅	✅	✅	✅	.1+.2+.3+.4 = 1
Even	{2, 4}	❌	✅	❌	✅	.2 + .4 = .6
Prime	{2, 3}	❌	✅	✅	❌	.2 + .3 = .5
∅ (null)	{ }	❌	❌	❌	❌	0

Uniform Spaces

$$P(x) = \frac{1}{|\Omega|}$$

$$P(E) = \sum_{x \in E} P(x) = \sum_{x \in E} \frac{1}{|\Omega|} = \frac{|E|}{|\Omega|}$$

Die $\Omega = \{1,2,3,4,5,6\}$ $|\Omega| = 6$

Event	Set	$\lvert Event \rvert$	$P(Event) = \frac{\lvert Event \rvert}{6}$
Even	{2,4,6}	3	$\frac{3}{6} = \frac{1}{2}$
Square	{1,4}	2	$\frac{2}{6} = \frac{1}{3}$

Probability that X=2, 4 ,or 6

Terminology

A, B Events

Intersect

Disjoint

Mutually exclusive

Do’s and Don’ts

$P(X \in \text{Even})$

die: $P(X \in \{2,4,6\}) = 3/6 = 1/2$

$P(\text{Even}) \stackrel{\text{def}}{=} P(X \in \text{Even})$

$P(1 \in \text{Even})$ 0

$P(2 \in \text{Even})$ 1

Possible, but less likely
Ensure you meant it

$P(x \in \text{Even})$

Even less likely, probably wrong

Probability Events

Outcomes to subsets

Events

Occurrence

Probability

Repeated Experiments

Repeated Experiments

Compound experiments

Repetition

With and without replacement

Order matters, or not

Composite Experiments

Experiments often consist of several parts

Student | major | year | GPA

Ad | product | audience | cost

Still can be viewed as a single experiment

Outcomes more complex

(CS, senior, 3.8)

(book, teenage, $9.99)

Outcome | 3-tuple

Sample space | Cartesian product

Combine smaller to analyze larger

Start simple

Independent Repetitions

Repetition All experiments of same type

Daily stock prices

Daily temperatures

Coin flips

Die rolls

Card draws

Independecnt Different components unrelated

First coin heads

Second coin 50% heads / tails ✅

Second coin more likely heads ❌

Two Coins

Independent experiments

Outcomes

Probabilities

		coin 2	½	½
			h	t
coin 1	½	h	¼	¼
	½	t	¼	¼

$\Omega = \{ hh, ht, th, tt \} = \{h,t\}^2$ $\quad |\Omega| = 4 = 2^2$

1 coin $\cup$ → 2 coins $\cup$

$P(hh) = P(ht) = P(th) = P(tt) = 1/|\Omega| = 1/4$

Two Dice

Independent experiments

die 1 \ die 2	1	2	…	6
1	1/36	1/36	…	1/36
2	1/36	1/36	…	1/36
⋮	⋮	⋮		⋮
6	1/36	1/36	…	1/36

$\Omega = \{ 11, 12, \ldots, 66 \} = \{1, \ldots, 6\}^2$ $\quad |\Omega| = 6^2 = 36$

1 die Ⓤ → 2 dice Ⓤ

$P(11) = P(12) = P(21) = \ldots = P(66) = 1/|\Omega| = 1/36$

Events

$$P(E) = P(X \in E) = \sum_{x \in E} P(x)$$

Ⓤ → P(E) = |E| / |Ω|

2 coins Ⓤ |Ω| = 2^2 = 4

P(Different outcomes) = P({ht,th}) = 2 / |Ω| = 1/2

P(At least one h) = P({ ht,th,hh }) = 3 / |Ω| = 3/4

3 coins Ⓤ |Ω| = 2^3 = 8

P(Alternating) = P({ hth, tht }) = 2 / 8 = 1/4

P(odd # h's) = P({ htt, tht, tth, hhh}) = 4 / 8 = 1/2

Sampling

Many sources of randomness

Coin

Die

Often sample (select) physical objects

Patients in a study

Customers at a restaurant

Products for quality control

Visitors to web pages

Cards from a deck

Balls from an urn

Two sampling types

With

Without

Replacement

Replacement

Sequentially select physical objects

With replacement

Replace (reuse) selected element

Experiments independent

Outcomes can repeat

Like coins dice

Without replacement

Do not replace selected element

Experiments dependent

Outcomes cannot repeat

Difference largest for small Ω

Replacement and Repetition

Sampling (selection) with without replacement

repeat as if from scratch repeat with smaller set

Same element can cannot be selected again

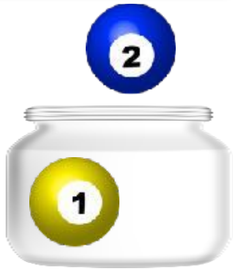
✅ coin cards ❌

✅ die people ❌

Balls in a Jar

1 and 2 ball in a jar

Pick one ball



Pick second ball

→ 2 1

Second selection - from a subset

Selection without replacement

Probabilities

1st ball \ 2nd ball	1	2
½ 1	0	½
½ 2	½	0

$\Omega = \{(1,2), (2,1)\}$

$|\Omega| = 2$

$P(1,2) = P(2,1) = ½$

Drawing Cards

Six cards

2-permutations of {1,..,6}

Draw one

Without replacement, draw a second

Outcomes

$\Omega = \{12,\ldots, 16, 21,\ldots, 26,\ldots, 65\} = [6]^{\underline{2}}$

$|\Omega| = 6^{\underline{2}} = 6\cdot 5 = 30$

Probabilities

$i \neq j$ $\quad P(i,j) = \frac{1}{6} \cdot \frac{1}{5} = 1/30$

Ⓤ $\quad P(i,j) = 1/|\Omega| = 1/30$

card 2

card 1	1	2	…	6
1	0	1/30	…	1/30
2	1/30	0		1/30
⋮	⋮			
6	1/30	1/30	…	0

Summary

2 experiments

	Original	With replacement	Without replacement
		Outcomes can repeat	Outcomes cannot repeat
Sample space	Ω_1	$\{ (x,y) : x,y \in \Omega_1 \} = \Omega_1^2$	$\{ (x,y): x,y \in \Omega_1, x \neq y \} = \Omega_1^{\underline{2}}$
| Sample space |	$\lvert\Omega_1\rvert$	$\lvert\Omega_1\rvert^2$	$\lvert\Omega_1\rvert^{\underline{2}} = \lvert\Omega_1\rvert \cdot (\lvert\Omega_1\rvert - 1)$

If original sample space is uniform

Uniformity		Uniform	Uniform
P (element)	$1 / \lvert\Omega_1\rvert$	$1 / \lvert\Omega_1\rvert^2$	$1 / [\lvert\Omega_1\rvert \cdot (\lvert\Omega_1\rvert - 1)]$

Order Does Not Matter

Sometimes order matters Stock 10 → 50

Sometimes does not matter Elections Dem Rep

When order does not matter

Tuple of outcomes → Set of outcomes

(2,5) (5,2) → {2,5} Event { (2,5), (5,2) }

(4,4) → {4,4}

Probability {1,…,6} twice With and w/o replacement

With Replacement

2 cards $\in$ {1,..,6} with replacement

P({1,2}) = P({ (1,2), (2,1) } = P((1,2)) + P((2,1)) = 2/36

P({1,1}) = P({ (1,1) } = 1/36

	1	2	...	6
1	1/36	1/36	…	1/36
2	1/36	1/36	…	1/36
⋮	⋮	⋮		⋮
6	1/36	1/36	…	1/36

Check

$$\binom{6}{2} \cdot \frac{2}{36} + \binom{6}{1} \cdot \frac{1}{36} = \frac{5}{6} + \frac{1}{6} = 1$$

Without Replacement

2 cards $\in$ {1,..,6} without replacement

	1	2	…	6
1	0	1/30	…	1/30
2	1/30	0		1/30
⋮	⋮			
6	1/30	1/30	…	0

$i \neq j$ $\quad P(\, i,j \,) = \frac{1}{6} \cdot \frac{1}{5} = 1/30$

$P(\{1,2\}) = P(\, \{\, (1,2), (2,1) \,\} = P((1,2))+P((2,1)) = 2/30$

$P(\{1,1\}) = 0$

Check

$$\binom{6}{2} \cdot \frac{2}{30} = 1$$

Alternative View

2 cards $\in$ {1,..,6} sequentially without replacement

P({1,2}) = P(1,2) + P(2,1) = 2/30

	1	2	…	6
1	0	1/30	…	1/30
2	1/30	0		1/30
⋮	⋮			
6	1/30	1/30	…	0

Alternatively Select both cards simultaneously

$\Omega = \{ \{1,2\}, \{1,3\}, \ldots \{5,6\} \} = \binom{[6]}{2}$

$|\Omega| = \binom{6}{2}$ U

P({1,2}) = 1/15

Poker Hands

Deck 52 cards

Hand 5 cards from deck

Card order does not matter

Ω = { , , , …}

$$|\Omega| = \binom{52}{5} = \frac{52!}{5!\cdot 47!} = \frac{52\cdot 51\cdot \overset{10}{\cancel{50}}\cdot 49\cdot \overset{2}{\cancel{48}}}{\cancel{5}\cdot\cancel{4}\cdot\cancel{3}\cdot\cancel{2}\cdot 1} = 2,598,960$$

$\approx$ 2.6 million

P( ) ≈ 1/ 2.6 million

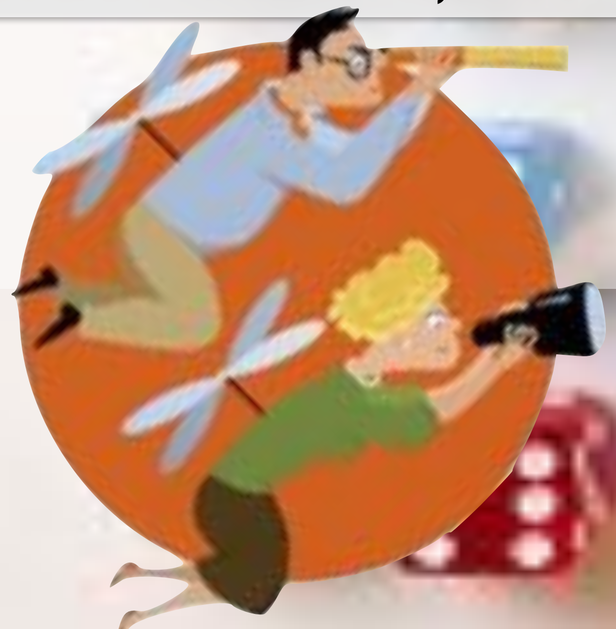
More soon

Compound experiments

Repetition

With and without replacement

Order matters, or not



Games of Chance

Probability

Games of Chance

Uniform probability spaces

Probability of outcomes and events

Several games of chance

Dominoes

Tile | two parts | each with 0 to 6 dots

Only the two numbers matter - not their order

Ω { tiles } $\left\{ \begin{array}{l} \{0,0\}, \{1,1\}, \ldots, \{6,6\} \\ \\ \{0,1\}, \{0,2\}, \ldots, \{5,4\}, \{5,6\} \end{array} \right\}$

Equiprobable

$|\Omega| = \binom{7}{1} + \binom{7}{2} = 7 + \frac{7 \cdot 6}{2} = 7 + 21 = 28$

Max value	0	1	2	...	6
# tiles	1	2	3		7

{0,0} {0,1} {0,2}
{1,1} {1,2}
{2,2}

$$1 + 2 + \ldots + 6 + 7 = \frac{7 \cdot 8}{2} = 28$$

Probabilities

Experiment Select one domino uniformly at random from the set

Outcomes $P(\{0,0\}) = P(\{1,2\}) = 1 / |\Omega| = 1/28$

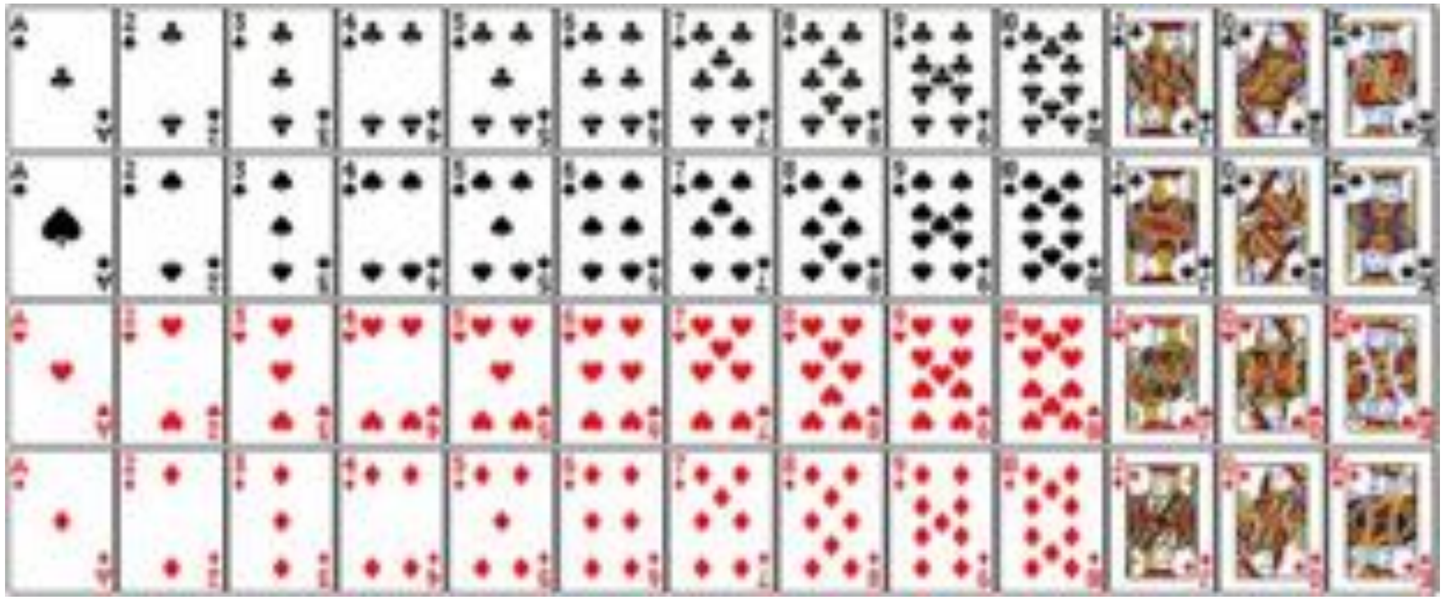
Events $P(6 \text{ is one of the numbers}) = 7 / |\Omega| = 7/28 = 1/4$

$P(\text{Double}) = 7/28 = 1/4$

$P(\text{two different values}) = 21/28 = 3/4$

Cards

Standard deck: 52 cards



4 suits: clubs (♣), diamonds (♢), hearts (♡), and spades (♠)

13 ranks: 2, 3, …, 10, Jack (J), Queen (Q), King(K), and Ace (A)

Single Card

Ω =

|Ω| = 52

Equiprobable

$$P(\heartsuit) = \frac{|\heartsuit|}{52} = \frac{13}{52} = \frac{1}{4}$$

$$P(K) = \frac{|K|}{52} = \frac{4}{52} = \frac{1}{13}$$

$$P(Facecard) = P(J, Q, K) = \frac{3 \cdot 4}{52} = \frac{3}{13}$$

Poker Hands

Deck | 52 cards

Hand | 5 cards from deck

Card order does not matter

$\Omega = \{$, , , $\ldots\}$

Equiprobable

$|\Omega| = \binom{52}{5} = 2,598,960$

Probability of several hands

Start with rarest

smaller sets are easier to count

Straight Flush

Straight — Consecutive rank sequence

Ace can precede 2 (A,2,3,4,5), or follow the K (10,J,Q,K,A)

Flush — Same suit

Straight flush — Five consecutive same-suit cards

$$|SF| = \binom{10}{1} \cdot \binom{4}{1} = 40$$

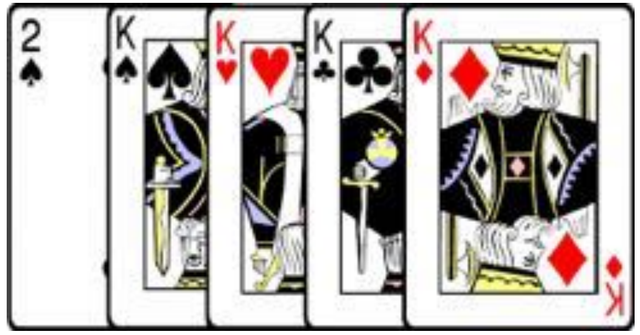
start location
A,2,…,10

suit

$$P(SF) = \frac{40}{\binom{52}{5}} = 0.0015\%$$

Four of a Kind

Four cards of the same rank, and another card



$$|FK| = \binom{13}{1} \cdot \binom{48}{1} = 624$$

rank

another card

$$P(FK) = \frac{624}{\binom{52}{5}} = 0.0240\%$$

Full House

Three cards of one rank and two cards of another rank

$$|FH| = 13^{\underline{2}} \cdot \binom{4}{3} \cdot \binom{4}{2} = 3744$$

rank of triple, rank of pair

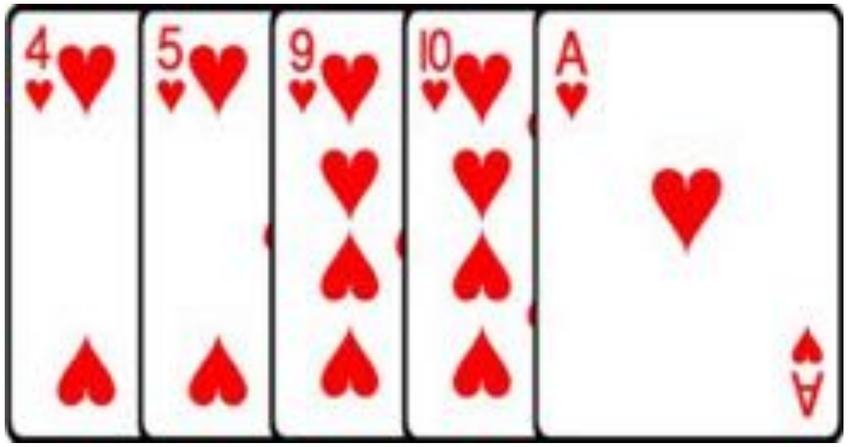
3 cards from rank of triple

2 cards from rank of pair

Falling power 13·12

$$P(FH) = \frac{3744}{\binom{52}{5}} = 0.1441\%$$

Flush



Five nonconsecutive same-suit cards

Nonconsecutive distinguishes from a straight flush

$$|F| = \binom{4}{1} \cdot \left(\binom{13}{5} - 10\right) = 5108$$

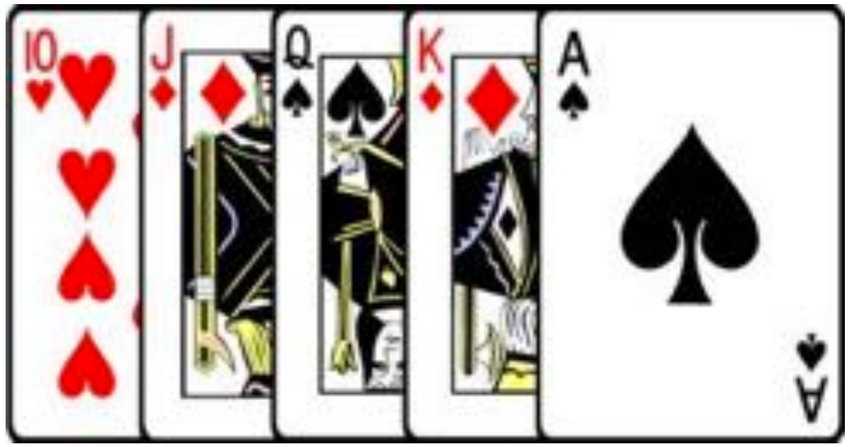
suit

5 cards from suit

5 consecutive same suit cards

$$P(F) = \frac{5108}{\binom{52}{5}} = 0.1965\%$$

Straight



Five consecutive cards, not all of the same suit

Ace can either precede 2 or succeed K

$$|S| = \binom{10}{1} \cdot (4^5 - \binom{4}{1}) = 10,200$$

Start location from ace to 10

Suit for each card

All same suit

$$P(S) = \frac{10200}{\binom{52}{5}} = 0.3925\%$$

Poker Probabilities

Notebook

Simulations for probabilities of several hands

Modify for other hands

Probability

Games of Chance

Uniform probability spaces

Probability of outcomes & events

Several games of chance

General

probability spaces

Probability

Non-Uniform Spaces

Tetrahedral Die

Side	1	2	3	4
Probability	.1	.2	.3	.4

Sum to 1

Even $V = \{2,4\}$ $P(V) = .2 + .4 = .6$

Prime $R = \{2,3\}$ $P(R) = .2 + .3 = .5$

General Subtraction

$P(R - V) = P(\{2,3\} - \{2,4\}) = P(\{3\}) = .3$

$P(R) - P(R \cap V) = P(\{2,3\}) - P(\{2,3\} \cap \{2,4\})$

$= P(\{2,3\}) - P(\{2\})$

$= .2 + .3 - .2 = .3$ ✅

Backgammon

Roll 2 dice

Moves determined by the two faces

order irrelevant

Outcomes

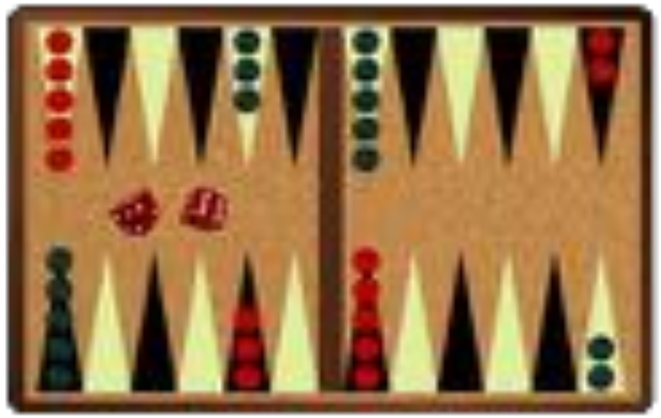
{1,1}, {1,2}, {1,3}, …, {5,6}, {6,6}

possible outcomes = ?

As in Dominos

$$\binom{6}{1} + \binom{6}{2} = 6 + \frac{6 \cdot 5}{2} = 21$$

Outcome Probabilities



$$P(1\,1) = P(1\,2) = \ldots = P(6\,6) = \frac{1}{36}$$

$$\left.\begin{aligned} P(\{1,1\}) &= p(1\,1) = \frac{1}{36} \\ P(\{1,2\}) &= p(1\,2, 2\,1) = \frac{1}{18} \end{aligned}\right\}$$

Note: not 1/21, not equiprobable

Check $\quad \binom{6}{1} \cdot \frac{1}{36} + \binom{6}{2} \cdot \frac{1}{18} = 1$

Event Probabilities

Both reasonable

$$P(\text{double}) = 6 \cdot P(\{1,1\}) = \frac{1}{6}$$

= P(2nd equals 1st)

$$P(\text{not double}) = 1 - \frac{1}{6} = \frac{5}{6}$$

= P(2nd different from 1st)

Event Probabilities

$$P(\text{sum is } 2) = P(\{1,1\}) = \frac{1}{36}$$

$$P(\text{sum is } 3) = P(\{1,2\}) = P(1\,2, 2\,1) = \frac{2}{36}$$

$$P(\text{sum is } 4) = P(1\,3, 2\,2, 2\,1) = \frac{3}{36}$$

Hence sum 7 is the most likely case (6 ways to get it)

$$P(\max = m) = \frac{2m-1}{36} \qquad m = 1, ..., 6$$

$$P(\max \leq m) = \frac{m^2}{36}$$

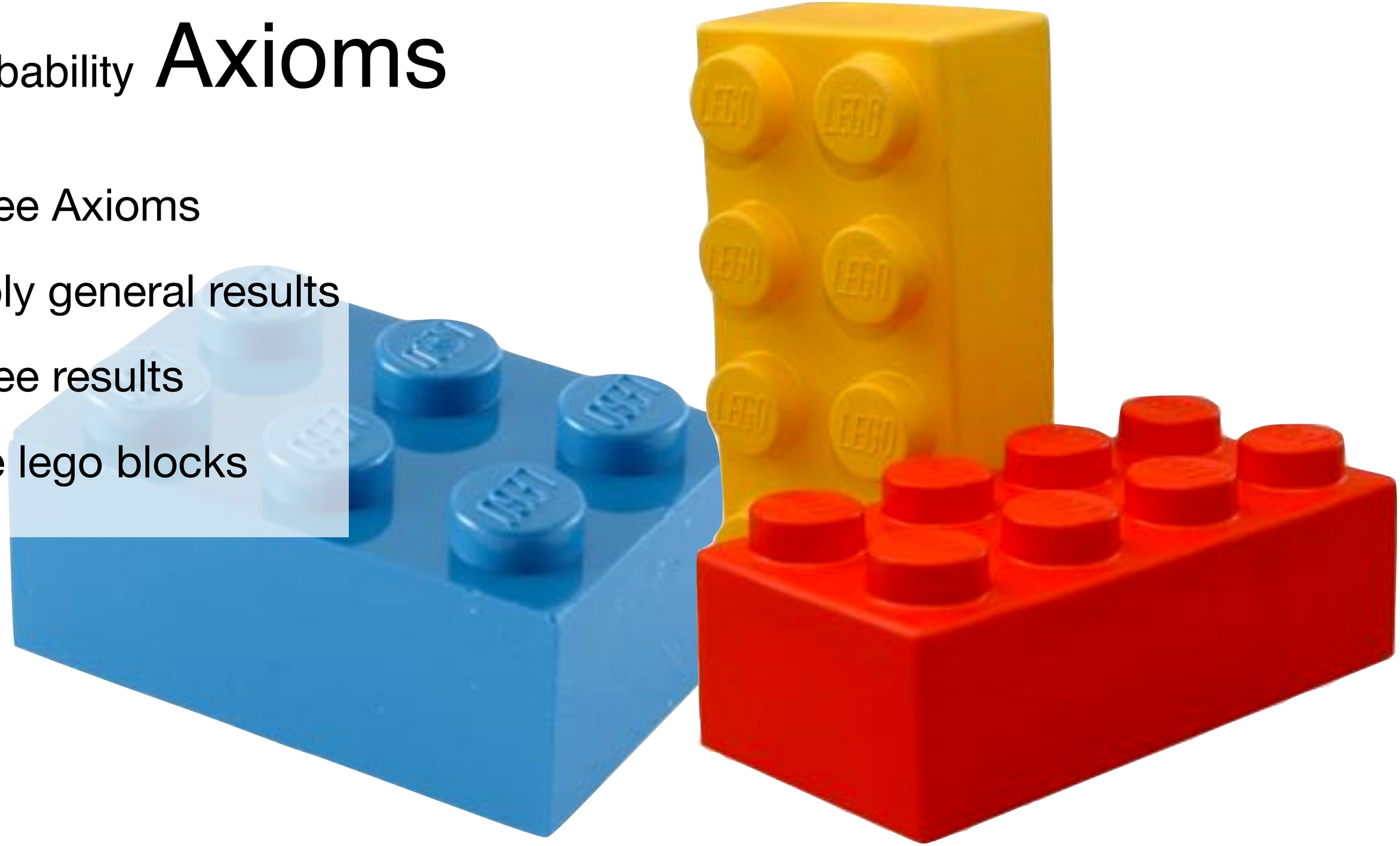
Probability Axioms

Three Axioms

Imply general results

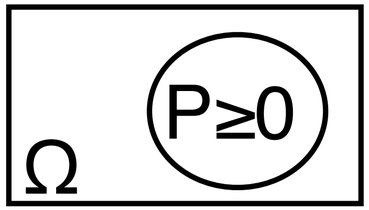
Three results

Use lego blocks



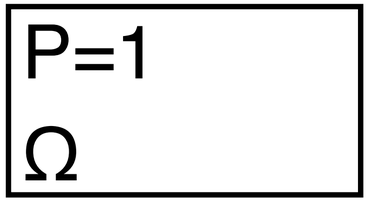
Three Axioms

Non-negativity $P(A) \geq 0$

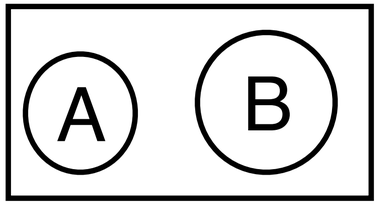


Short for $\forall A \;\; P(A) \geq 0$

Unitarity $P(\Omega) = 1$



Addition rule A, B disjoint → $P(A \cup B) = P(A) + P(B)$



$A_1, A_2, \ldots$ disjoint → $P(A_1 \cup A_2 \cup \ldots) = P(A_1) + P(A_2) + \ldots$

Countable unions only

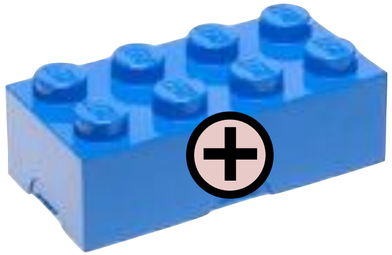
Uncountable later

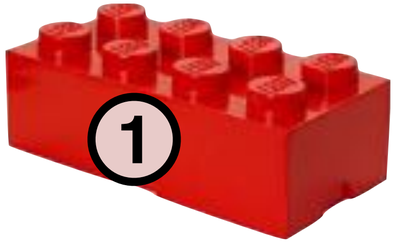
“Building” Results

Use lego blocks

construct three simple results

Complement rule

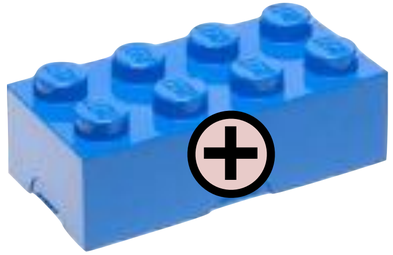




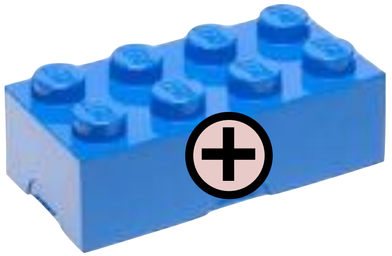
Subtraction

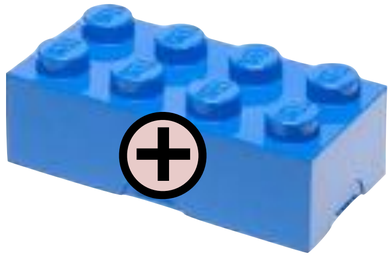
nested sets

general sets



Inclusion-exclusion





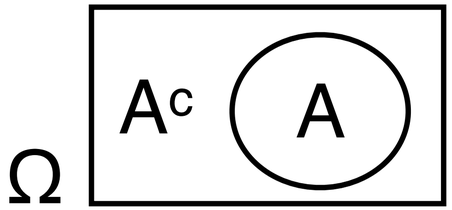
Complement Rule

Complement rule for counting

$|A^c| = |\Omega| - |A|$

Same for

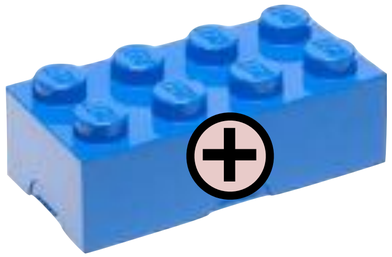
$A \cup A^c = \Omega$

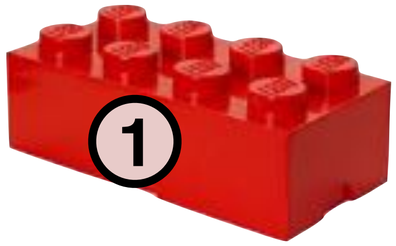


$P(A) + P(A^c) = P(A \cup A^c) = P(\Omega) = 1$

$P(A^c) = 1 - P(A)$

Complement rule for probability



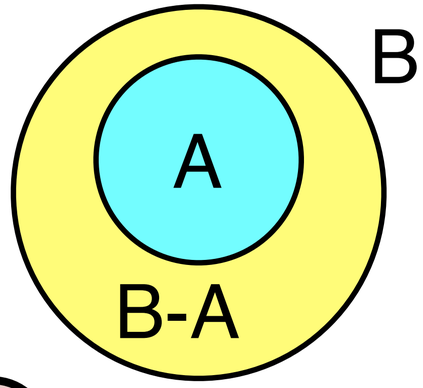


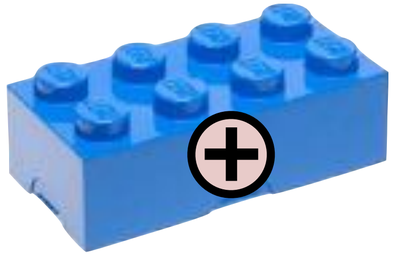
Subtraction Rule - Nested Sets

Complement rule $A \subseteq \Omega$ $P(A^c) = 1 - P(A)$ $P(\Omega - A) = P(\Omega) - P(A)$

Generalize $A \subseteq B$ $\rightarrow$ $P(B-A) = P(B) - P(A)$

$A \subseteq B$ $\rightarrow$ $B = A \cup (B-A)$





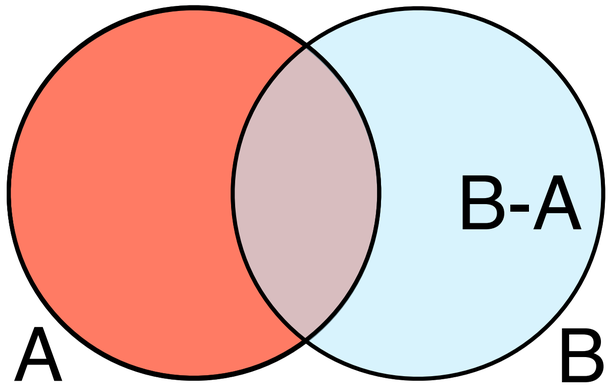
$$P(B) = P(\, A \cup (B-A)\,) = P(A) + P(B-A)$$

$P(B-A) = P(B) - P(A)$ **Subtraction rule for nested sets**

Subtraction Rule - General Sets

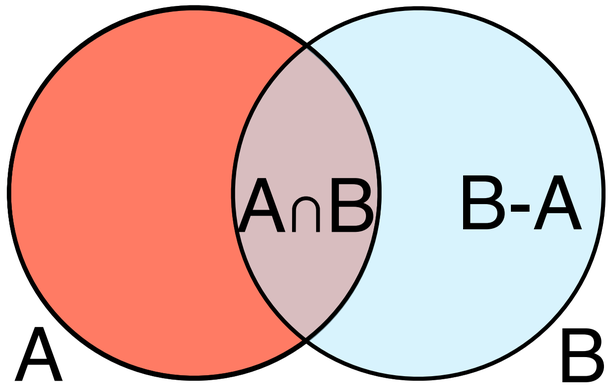
Nested $A \subseteq B$ $P(B-A) = P(B) - P(A)$

General $\forall A, B$ $P(B-A) = P(B) - P(A \cap B)$ ⊖

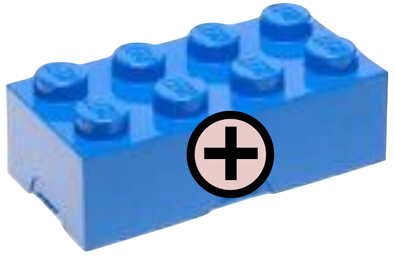


$B - A = B - (A \cap B)$

$A \cap B \subseteq B$

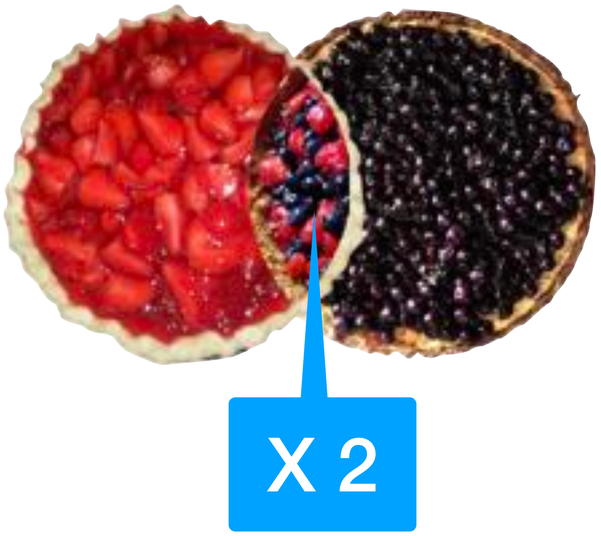


$P(B - A) = P(B - (A \cap B)) = P(B) - P(A \cap B)$

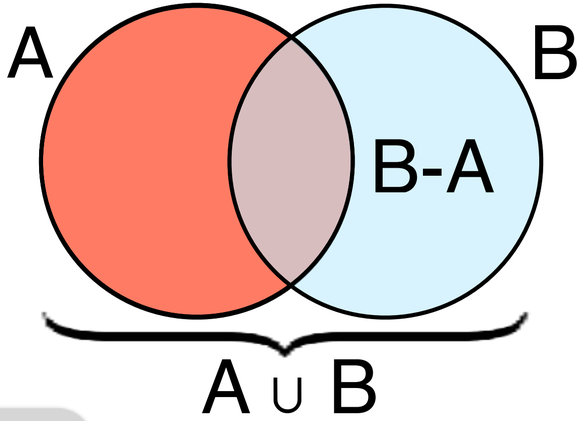


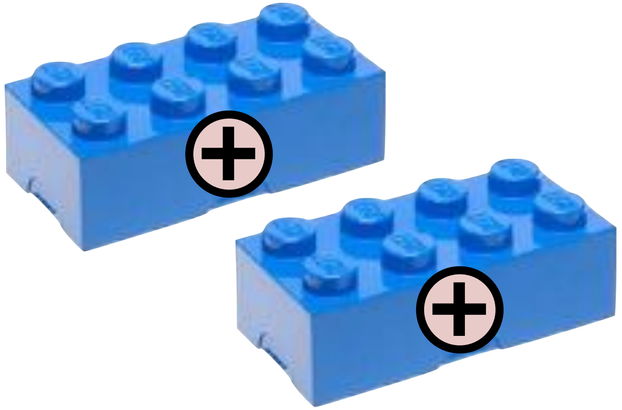
Inclusion Exclusion

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$



$A \cup B = A \cup B\text{-}A$





$P(A \cup B) = P(A \cup B\text{-}A)$

$\oplus$

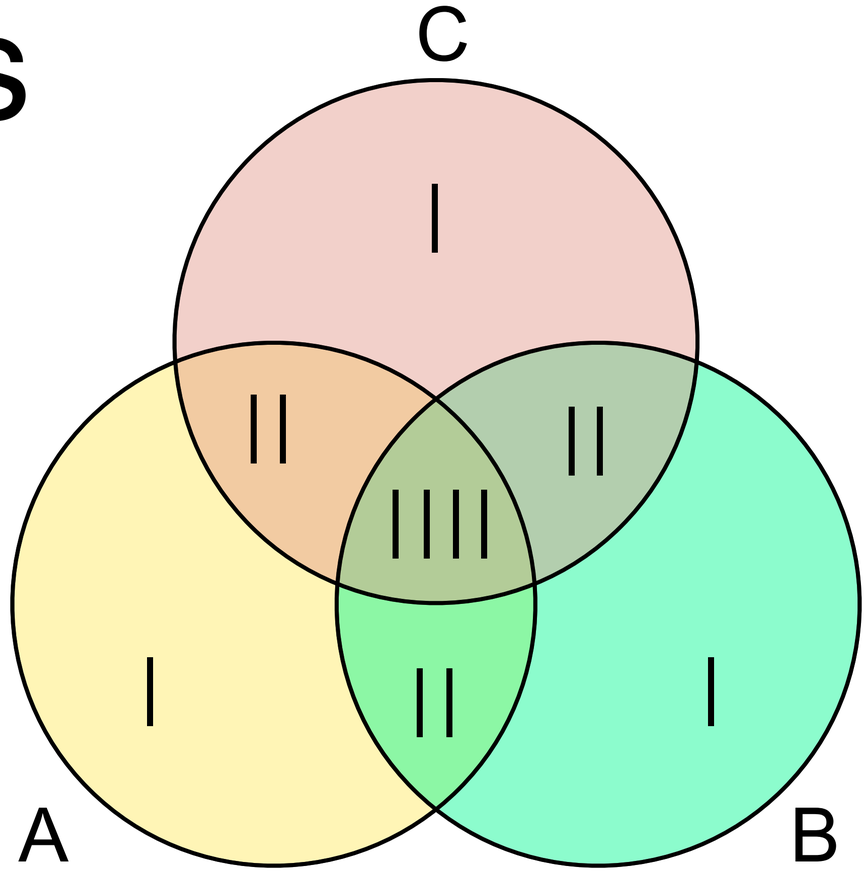
$= P(A) + P(B\text{-}A)$

General subtraction

$P(B\text{-}A) = P(B) - P(A \cap B)$

$= P(A) + P(B) - P(A \cap B)$

More Sets



Two sets

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Three sets

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$$

n sets

$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{1 \le i \le n} P(A_i) - \sum_{1 \le i < j \le n} P(A_i \cap A_j) + \ldots + (-1)^{n-1} P\left(\bigcap_{i=1}^{n} A_i\right)$$

Probability Inequalities

Three Axioms

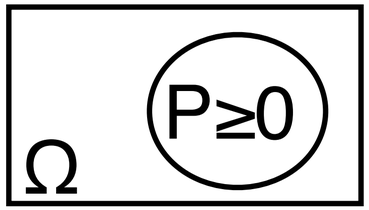
Inequalities

Union bound

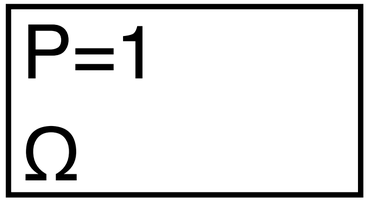
$1.4M question

Three Axioms

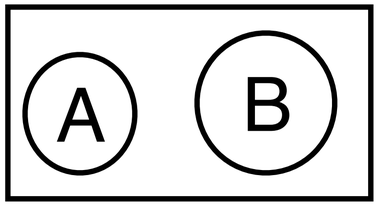
Non-negativity $P(A) \geq 0$



Unitarity $P(\Omega) = 1$



Addition rule A, B disjoint → $P(A \cup B) = P(A) + P(B)$



$A_1, A_2, \ldots$ disjoint → $P(A_1 \cup A_2 \cup \ldots) = P(A_1) + P(A_2) + \ldots$

Equalities → Inequalities

Probability of Null Event

① $P(\Omega) = 1$

What about $P(\varnothing)$?

Show $P(\varnothing) = 0$

$\Omega = \varnothing \cup \Omega$

$$\cancel{P(\Omega)} = P(\varnothing \cup \Omega) = P(\varnothing) + \cancel{P(\Omega)}$$

$P(\varnothing) = 0$

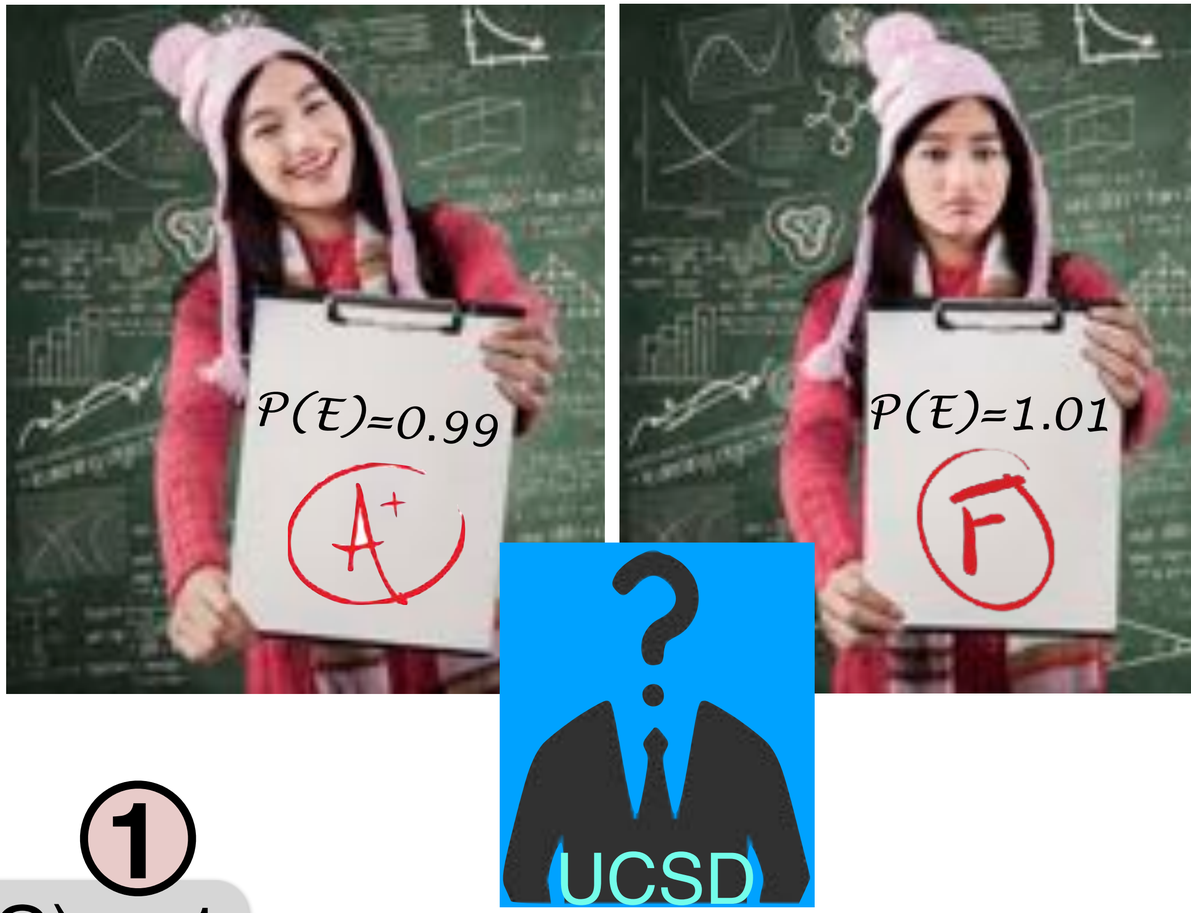
The null event has nil probability

0

$$0 \le P(A) \le 1$$

$\geqslant$ $P(A) \ge 0$

Show $P(A) \le 1$



$A \cup A^c = \Omega$

$\geqslant$ $\oplus$ $\bullet\bullet$ 1

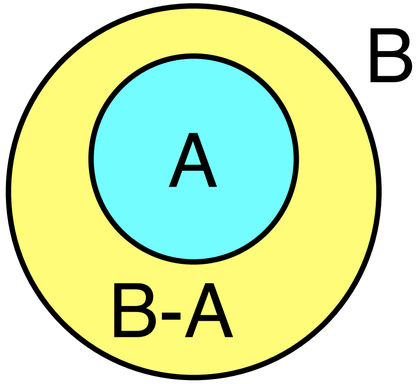
$$P(A) \le P(A) + P(A^c) = P(A \cup A^c) = P(\Omega) = 1$$

$P(A) \le 1$

Probability always between 0 and 1

Subset

$A \subseteq B$ $\quad$ $P(A) \leq P(B)$



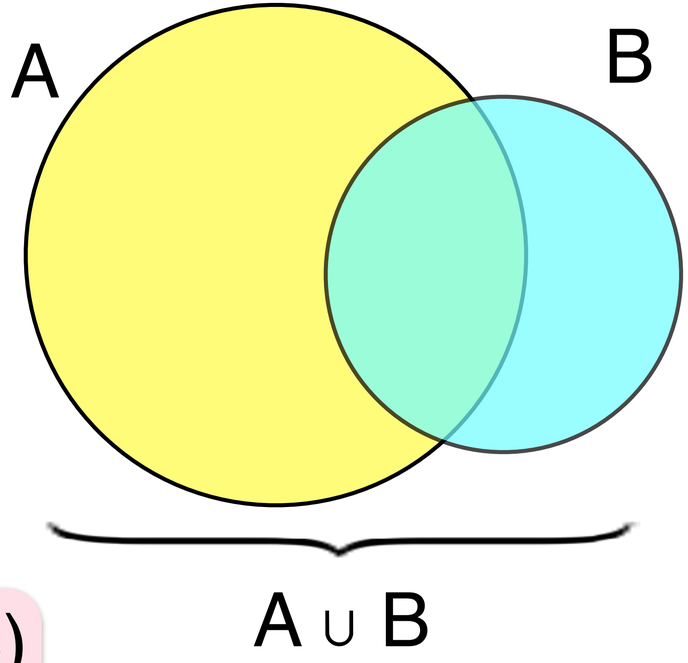
$A \subseteq B$ $\quad$ $B = A \cup (B-A)$

$$P(B) = P(A \cup (B-A)) \overset{+}{=} P(A) + P(B-A) \overset{\geq}{\geq} P(A)$$

Obvious?

Union

$\max(P(A),P(B)) \leq P(A \cup B) \leq P(A) + P(B)$



Left ≤ $A, B \subseteq A \cup B$ $P(A), P(B) \leq P(A \cup B)$

Right ≤ $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$P(A \cup B) \leq P(A) + P(B)$ Union bound Very useful

$1,400,000 Question

Linda is 31 years old, single, outspoken, and very bright. She majored in philosophy. As a student, she was deeply concerned with issues of discrimination and social justice, and also participated in anti-nuclear demonstrations.

Which is more probable?

1. Linda is a bank teller
2. Linda is a bank teller and is active in the feminist movement

Please answer in the poll below

The Linda Problem

Psychological Review

VOLUME 90 NUMBER 4 OCTOBER 1983

Extensional Versus Intuitive Reasoning: The Conjunction Fallacy in Probability Judgment

Amos Tversky
Stanford University

Daniel Kahneman
University of British Columbia, Vancouver, British Columbia, Canada

88 UBC Students

85% Bank teller **&** activist more likely

B - Bank teller

A - Active in feminist movement

$P(B)$ vs. $P(B \cap A)$

$B \supseteq B \cap A$

$P(B) \geq P(B \cap A)$

Irrational concept of probability

Related Questions

Several conjunction fallacy problems

Bjorn Borg — Preeminent tennis player of late ’70’s

6 French opens — Wimbledon ’76-‘80

1980 — Suppose Borg reaches the ’81 Wimbledon Finals.

More likely?

A — Borg will lose first set

B — Borg will lose first set but win match

72% chose B — Again $B \subseteq A$ — $P(B) \leq P(A)$

1981 — Reached final — Won first set — Lost match — to

1983 — Retired — Age 26

The Ultimate Reward

T & K — Many probability-perception studies

1996 — Tversky passed away

2002 — Kahneman

best known

"for integrating insights from psychological research into economic science, especially regarding human judgment and decision-making under uncertainty"

Economics Nobel Prize

Criticism

Multiple choice and ranking questions often disjoint

Coffee or Tea

Tea hot or cold

Rank Mon, Tue, or Wed

Often, when not explicitly disjoint, we still interpret them as such

With milk or with milk and sugar → Milk only or Milk and sugar

Meet Monday or Monday and Tuesday → Mon only or Mon and Tue

Students may have similarly interpreted the Linda question

Teller or Teller and activist → Teller only or Teller and activist

Simply thought Linda more likely active in movement than not

Bottom Line

Humans only moderately good at estimating probability

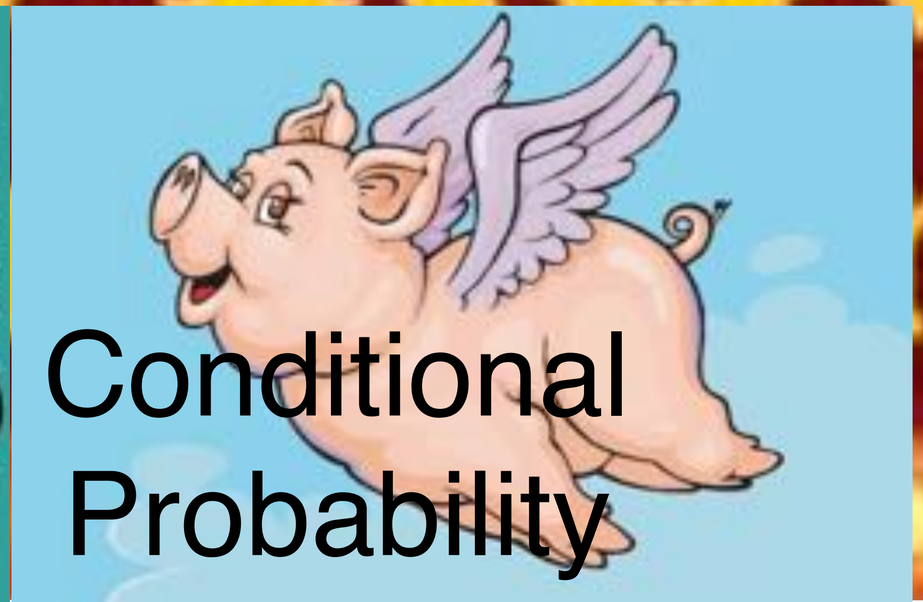
Probability Inequalities

Three Axioms

Inequalities

Union bound

$1.4M question

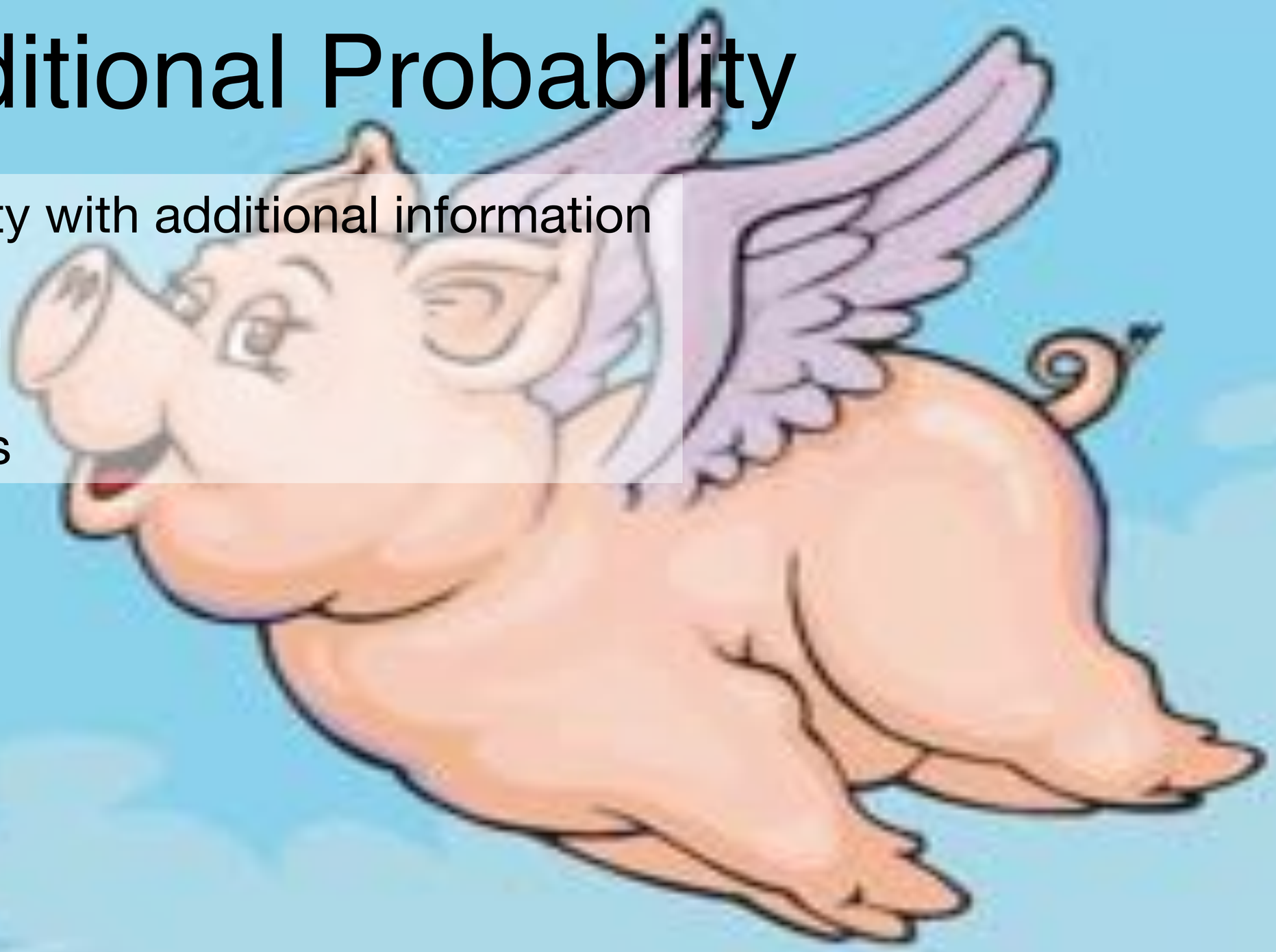


Conditional Probability

Probability with additional information

Definition

Examples



Why Condition

Often have partial information about the world

Modifies event probabilities

- Unemployment numbers - stock prices
- LeBron James injured - Cavaliers game result
- Sunny weekend - beach traffic

Can help

- Improve estimates
- Determine original unconditional probabilities

Back to Basics

Empirical frequency interpretation of probability

The probability P(E) of an event E is the fraction of experiments where E occurs when the number of experiments grows to infinity

To estimate P(E) repeat the experiment many times, calculate the fraction of experiments where E occurs

Fair Die

$$P(2) = \frac{2}{12} = \frac{1}{6}$$

Estimate

$$\underbrace{\boxed{2}\ 1\ 3\ 6\ 4\ \boxed{2}\ 5\ 4\ 3\ 6\ 5\ 1}_{12}$$

Conditional Probability

Let E and F be events. The conditional probability P(F | E) of F given E is the fraction of times F occurs in experiments where E occurs

To estimate P(F | E) take many samples, consider only experiments where E occurs, and calculate the fraction therein where F occurs too

$$P(2 \mid \text{Even}) = \frac{2}{6} = \frac{1}{3}$$

Even = {2, 4, 6}

2 1 3 6 4 2 5 4 3 6 5 1

Die

$P(\{2\}) = P(2) = 1/6$

$P(2) = ?$

$P(2 \mid \text{Odd}) = P(2 \mid \{1,3,5\}) = \frac{0}{6} = 0$

2 1 3 6 4 2 5 4 3 6 5 1

$P(\leq 2) = P(\{1,2\}) = 1/3$

$P(\leq 2) = ?$

$P(\leq 2 \mid \geq 2) = P(\{1,2\} \mid \{2,3,4,5,6\}) = \frac{2}{10} = \frac{1}{5}$

2 1 3 6 4 2 5 4 3 6 5 1

General Events - Uniform Spaces

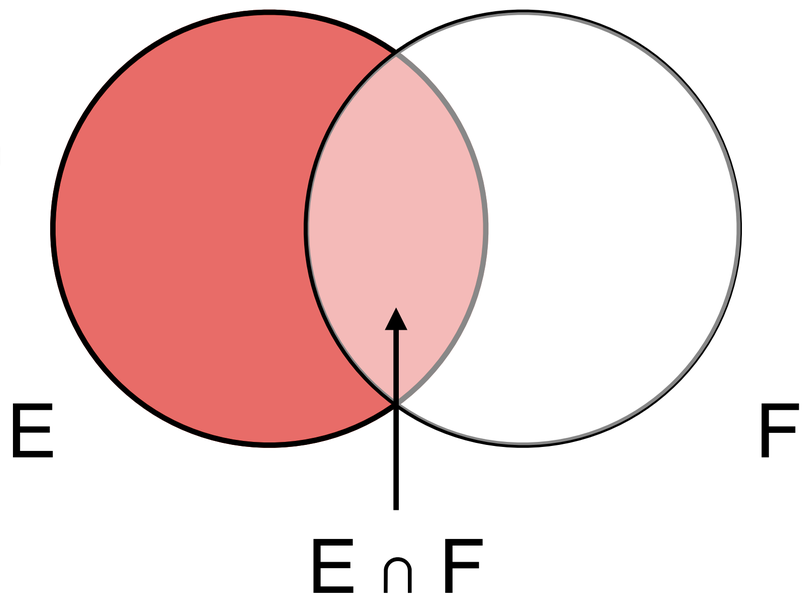
$$P(F \mid E) = P(X \in F \mid X \in E)$$

$$= P(X \in E \text{ and } X \in F \mid X \in E)$$

$$= P(X \in E \cap F \mid X \in E)$$

$$= P(E \cap F \mid E)$$

$$= \frac{|E \cap F|}{|E|}$$



Fair Die Again

$$P(\text{Prime} \mid \text{Odd}) = P(\{2,3,5\} \mid \{1,3,5\})$$

$$= \frac{|\{2,3,5\} \cap \{1,3,5\}|}{|\{1,3,5\}|} = \frac{|\{3,5\}|}{|\{1,3,5\}|} = \frac{2}{3}$$

$$P(\{4\} \mid \text{Prime}) = P(\{4\} \mid \{2,3,5\})$$

$$= \frac{|\{4\} \cap \{2,3,5\}|}{|\{2,3,5\}|} = \frac{|\emptyset|}{|\{2,3,5\}|} = 0$$

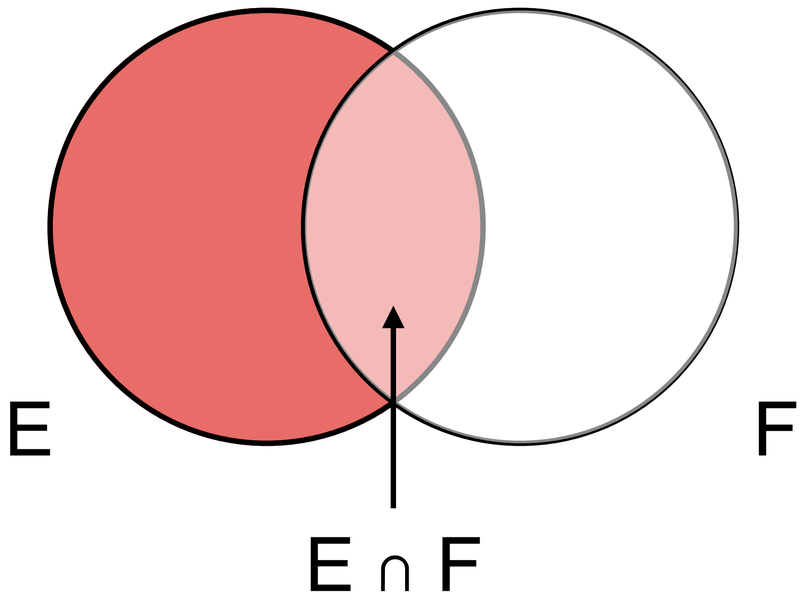
General Spaces

$$P(F \mid E) = P(X \in F \mid X \in E)$$

$$= P[X \in E \cap X \in F \mid X \in E]$$

$$= P[X \in E \cap F \mid X \in E]$$

$$= \frac{P(E \cap F)}{P(E)}$$



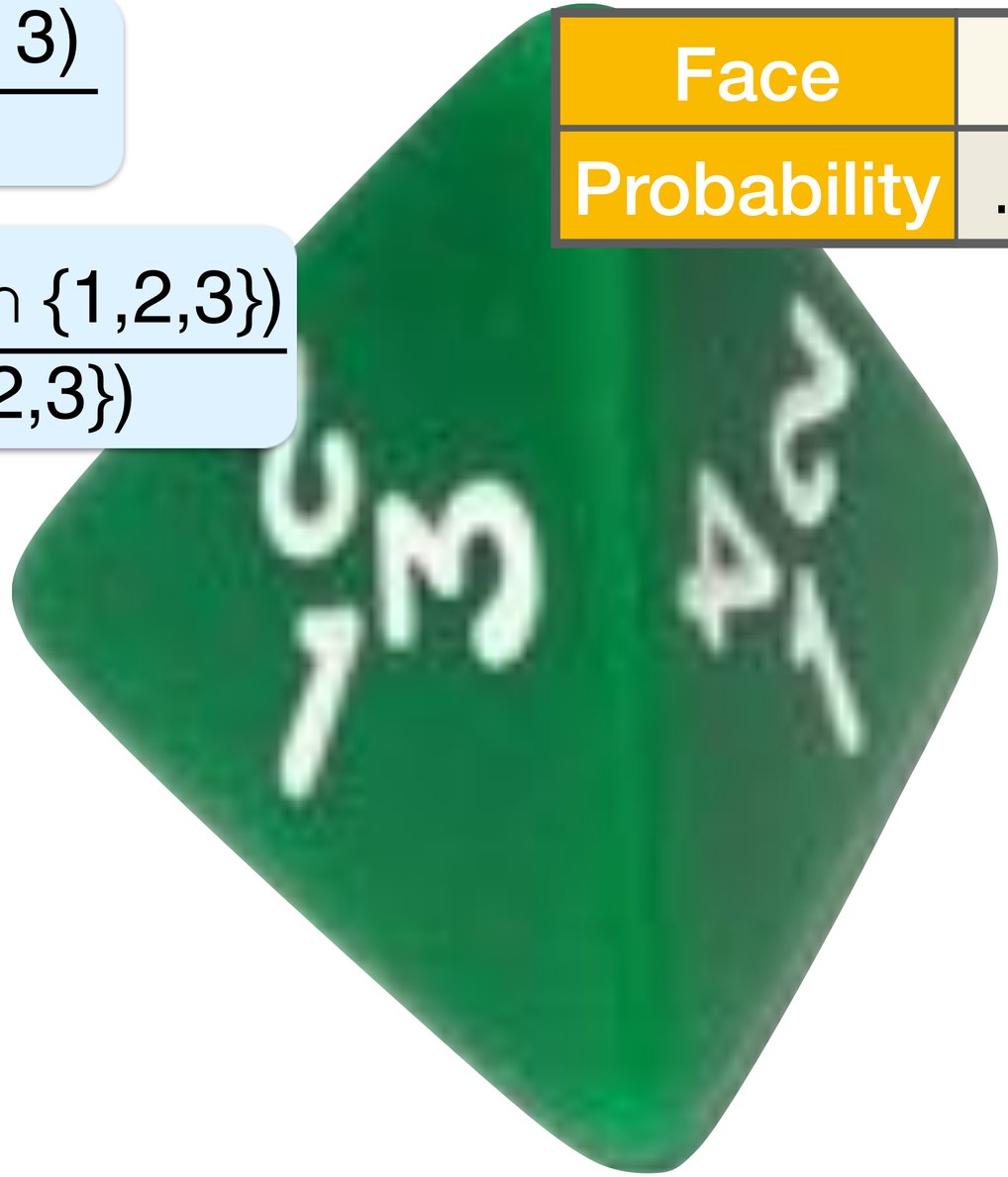
4-Sided Die

$$P(\geq 2 \mid \leq 3) = \frac{P(\geq 2 \cap \leq 3)}{P(\leq 3)}$$

$$= \frac{P(\{2,3,4\} \cap \{1,2,3\})}{P(\{1,2,3\})}$$

$$= \frac{P(\{2,3\})}{P(\{1,2,3\})}$$

$$= \frac{.5}{.6} = \frac{5}{6}$$



Face	1	2	3	4
Probability	.1	.2	.3	.4

KIDS
ARE PEOPLE TOO

Conditionals are Probabilities Too

$P(x \mid a) \geq 0$

$\sum_{x \in \Omega} P(x \mid a) = 1$

$P(x \mid a) \geq 0$

Conditional Probability

Probability with additional information

Definition

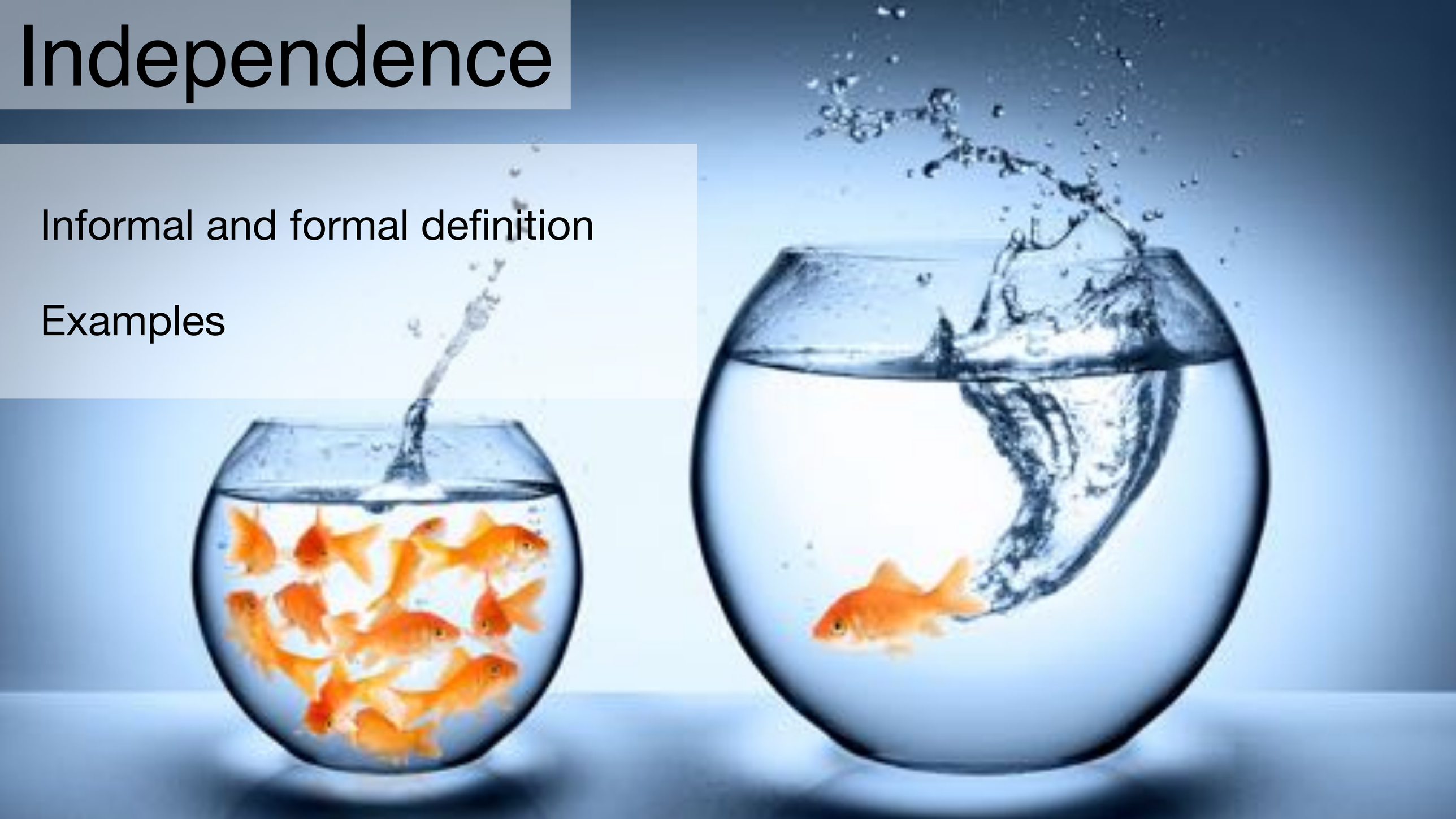
Examples

Independence

Independence

Informal and formal definition

Examples



Motivation

$P(F \mid E) > P(F)$

$P(2 \mid \text{Even}) = \frac{1}{3} > \frac{1}{6} = P(2)$

E ↗ probability of F

$P(F \mid E) < P(F)$

$P(2 \mid \text{Odd}) = 0 < \frac{1}{6} = P(2)$

E ↘ probability of F

$P(F \mid E) = P(F)$

$P(\text{Even} \mid \leq 4) = \frac{1}{2} = P(\text{Even})$

E neither ↗ nor ↘ probability of F

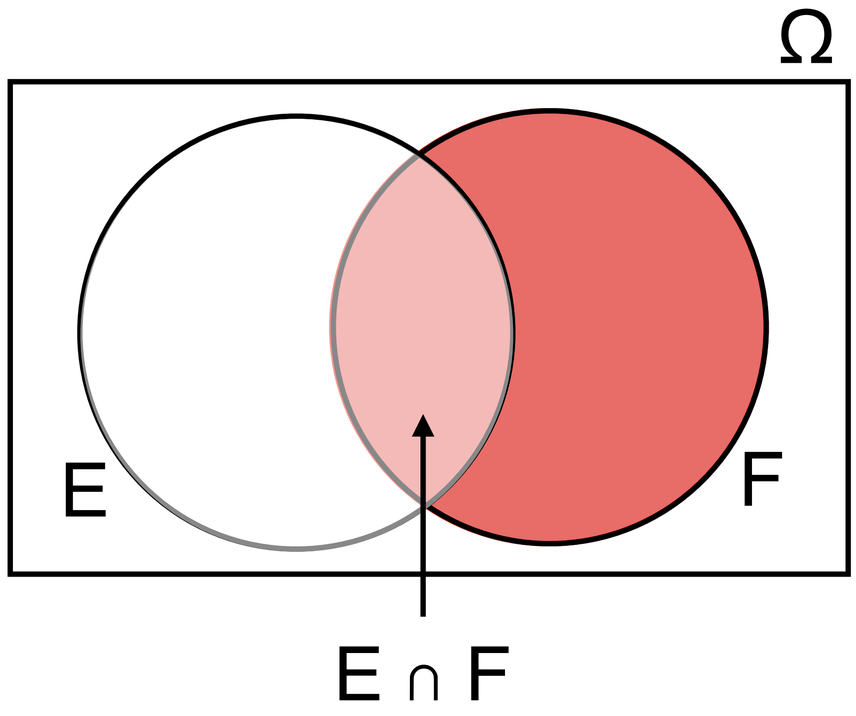
Whether or not E occurs, does not change P(F)

motivation → intuitive definition → formal

Independence - Intuitive

Events E and F are independent, denoted $E \perp\!\!\!\perp F$, if the occurrence of one does not affect the other's probability

$P(F \mid E) = P(F)$



Visually

$$P(F) = \frac{P(F)}{P(\Omega)}$$

F as a fraction of Ω

=

$$P(F \mid E) \triangleq \frac{P(E \cap F)}{P(E)}$$

E ∩ F as a fraction of E

Two issues

Asymmetric

Undefined if P(E)=0

Independence - Formal

Informally

$$P(F) = P(F \mid E) \triangleq \frac{P(E \cap F)}{P(E)}$$

Asymmetric

Undefined if $P(E)=0$

Formally

E and F are independent if $P(E \cap F) = P(E) \cdot P(F)$

Otherwise, dependent

Symmetric ✅

Applies when $P = 0$ ✅

Implies

intuitive def. $P(F|E) = P(F)$

$P(E|F) = P(E)$

$P(F \mid \overline{E}) = P(F)$

$P(E \mid \overline{F}) = P(E)$

Non-Surprising Independence

Two coins

H_1 First coin heads $P(H_1)=½$

H_2 Second coin heads $P(H_2)=½$

$H_1 \cap H_2$ Both coins heads $P(H_1 \cap H_2)=¼$

$P(H_1 \cap H_2) = ¼ = P(H_1) \cdot P(H_2)$ $H_1 \perp\!\!\!\perp H_2$

Not surprising as two separate coins

Can have $\perp\!\!\!\perp$ even for one experiment

Single Die

Three events

Event	Set	Probability
Prime	{ 2, 3, 5 }	½
Odd	{ 1, 3, 5 }	½
Square	{ 1, 4 }	⅓

Which pairs are $\perp\!\!\!\perp$ and $\not\perp\!\!\!\perp$

Intersection	Set	Prob	Product	=?	Independence
Prime ∩ Odd	{ 3, 5 }	⅓	½ · ½ = ¼	≠	dependent
Prime ∩ Square	∅	0	½ · ⅓ = ⅙	≠	dependent
Odd ∩ Square	{ 1 }	⅙	½ · ⅓ = ⅙	=	independent

Three Coins

Three events

Event	Description	Set	Probability
H_1	first coin heads	{h✳✳}	½
H_2	second coin heads	{✳h✳}	½
HH	exactly 2 heads in a row	{hht, thh}	¼

Which pairs are $\perp\!\!\!\perp$ and $\not\perp\!\!\!\perp$

Intersection	Set	Prob	=?	Product	Independence
$H_1 \cap H_2$	{hh✳}	¼	=	½ · ½ = ¼	independent
$H_2 \cap HH$	{hht, thh}	¼	≠	½ · ¼ = ⅛	dependent
$H_1 \cap HH$	{hht}	⅛	=	½ · ¼ = ⅛	independent

Independence of Ω and $\varnothing$

$\forall A$ $\quad P(\Omega \cap A) = P(A) = P(\Omega) \cdot P(A)$ $\quad$ $\Omega \perp\!\!\!\perp$ of any event

A occurring doesn't modify likelihood of Ω

$\forall A$ $\quad P(\varnothing \cap A) = P(\varnothing) = P(\varnothing) \cdot P(A)$ $\quad$ $\varnothing \perp\!\!\!\perp$ of any event

A occurring doesn't modify likelihood of $\varnothing$

Independence

Informal and formal definitions

Examples

Total Probability

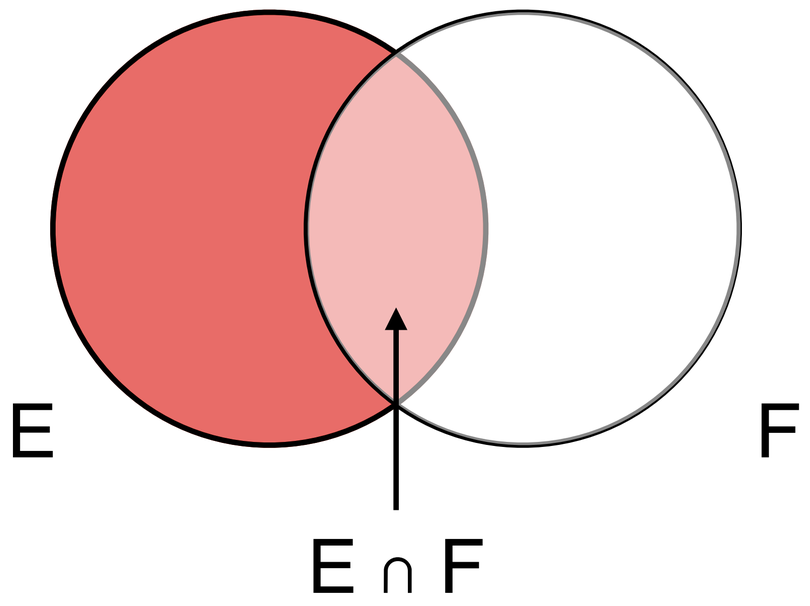
Sequential Probability

Product Rule

Recall conditional probability

$$P(F \mid E) = \frac{P(E \cap F)}{P(E)}$$

$$P(E \cap F) = P(E) \cdot P(F \mid E)$$



Helps calculate regular (not conditional) probabilities

Sequential Selection

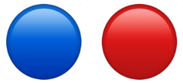
1 blue, 2 red balls

Draw 2 balls without replacement

P(both red) = ?

R_i - i'th ball is red

$P(R_1) = 2/3$



$P(R_2 \mid R_1) = 1/2$

P(both red) $= P(R_1, R_2)$

$= P(R_1) \cdot P(R_2 \mid R_1)$

$= 2/3 \cdot 1/2$ $= 1/3$

General Product Rule

For 3 events

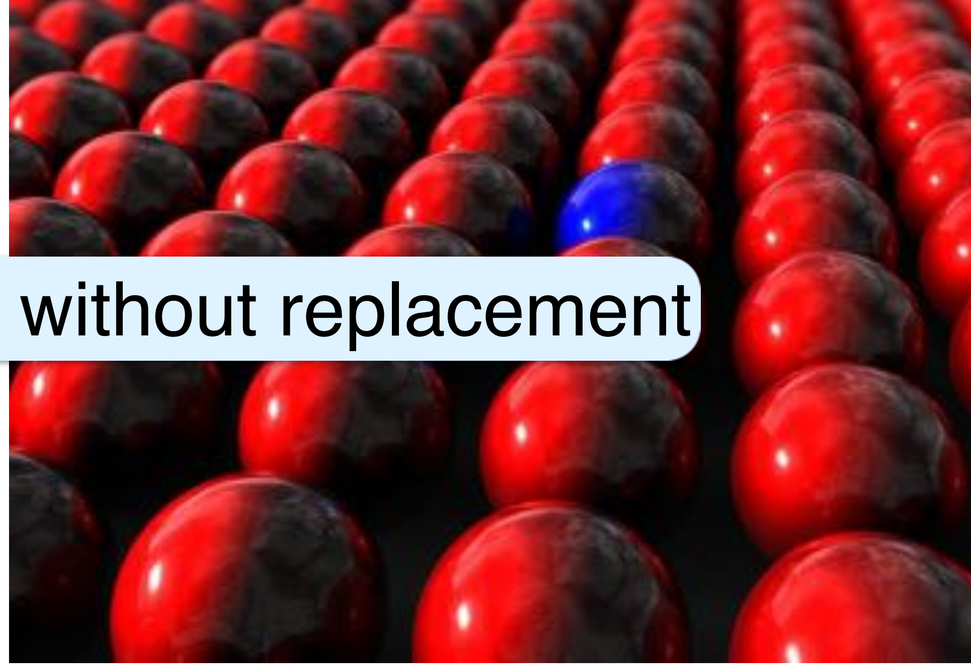
$$P(E \cap F \cap G) = P((E \cap F) \cap G)$$

$$= P(E \cap F) \cdot P(G \mid E \cap F)$$

$$= P(E) \cdot P(F \mid E) \cdot P(G \mid E \cap F)$$

Similarly for more events

Odd Ball



n-1 red balls and one blue ball

Pick n balls without replacement

P(last ball is blue) = ?

R_i - ith ball is red

R^i - $R_1, R_2, \ldots, R_i$

$$P(\text{last ball blue}) = P(R_1)P(R_2|R_1)P(R_3|R^2)\ldots P(R_{n-1}|R^{n-2})$$

$$= \frac{n-1}{n}\,\frac{n-2}{n-1}\,\frac{n-3}{n-2}\cdots\frac{2}{3}\,\frac{1}{2} = \frac{1}{n}$$

Or..

Arrange in row, probability last ball is blue = 1/n

The Birthday Paradox

How many people does it take so that two will likely share a birthday?

Assume that every year has 365 days

Everyone is equally likely to be born on any given day

Probabilistically

Choose n random integers, each $\in \{1, \ldots, 365\}$, with replacement

B(n) - probability that two (or more) are the same

For which n does B(n) exceed, say, 1/2?

Some first think it $n \approx 365$, but in fact much smaller

First Attempt

Consider the n people in order, say alphabetically

List their birthdays 2, 10, 365, 180, 10, …

Selection with replacement

Set of all possible birthdays sequences $\Omega = \{1,2,\ldots, 365\}^n$ $|\Omega| = 365^n$

Individual birthday uniform → Ω uniform

B_n {sequences with repetition}

P(repetition) = $|B_n| / |\Omega|$ Evaluating $|B_n|$ involved

Complement

B_n n people have birthday repetition

B_n^c n people, no two share a birthday

Evaluate sequentially

Person i different b/day from all previous

Calculation

$$1 - x \leq e^{-x}$$

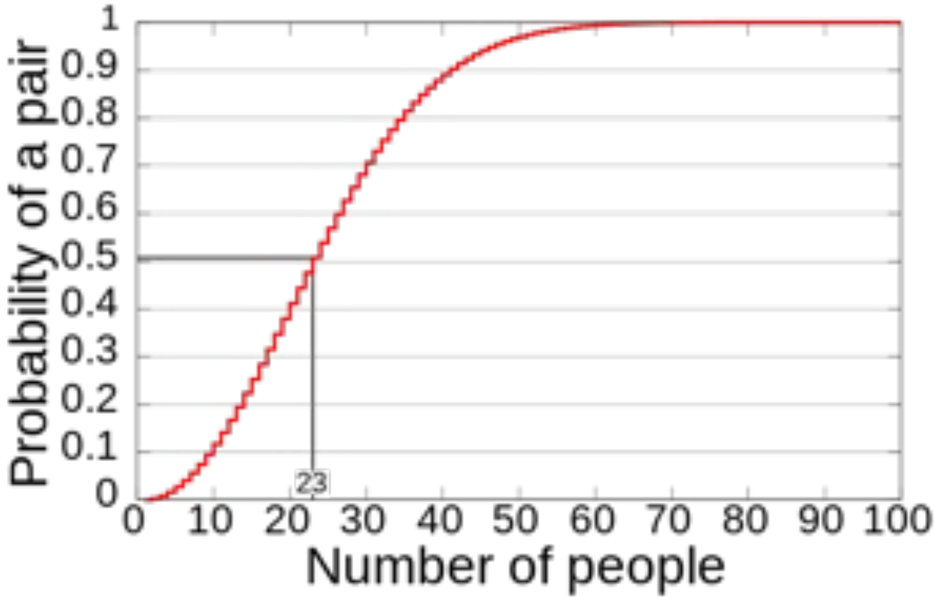
Among n people

P(no two people share a birthday)

When the probability is 0.5

$$-\frac{n^2}{2 \cdot 365} = \ln 0.5 = -\ln 2$$

$$n \approx \sqrt{-2 \cdot 365 \cdot \ln 0.5} = 22.494$$



$$= \frac{364}{365} \cdot \frac{363}{365} \cdot \ldots \cdot \frac{365 - n + 1}{365}$$

$$= \prod_{i=1}^{n-1} \left(1 - \frac{i}{365}\right)$$

$$\leq \prod_{i=1}^{n} e^{-\frac{i}{365}}$$

$$= \exp\left(-\frac{1}{365} \cdot \sum_{i=1}^{n-1} i\right)$$

$$= \exp\left(-\frac{n(n-1)}{2 \cdot 365}\right)$$

$$\approx \exp\left(-\frac{n^2}{2 \cdot 365}\right) = 0.5$$

This lecture: Sequential Probability

Next: Total Probability

Total Probability

Divide and Conquer

When evaluating probability of an event

Sometimes easier to split event into different parts

Calculate probability of each part

Add probabilities

Law of Total Probability

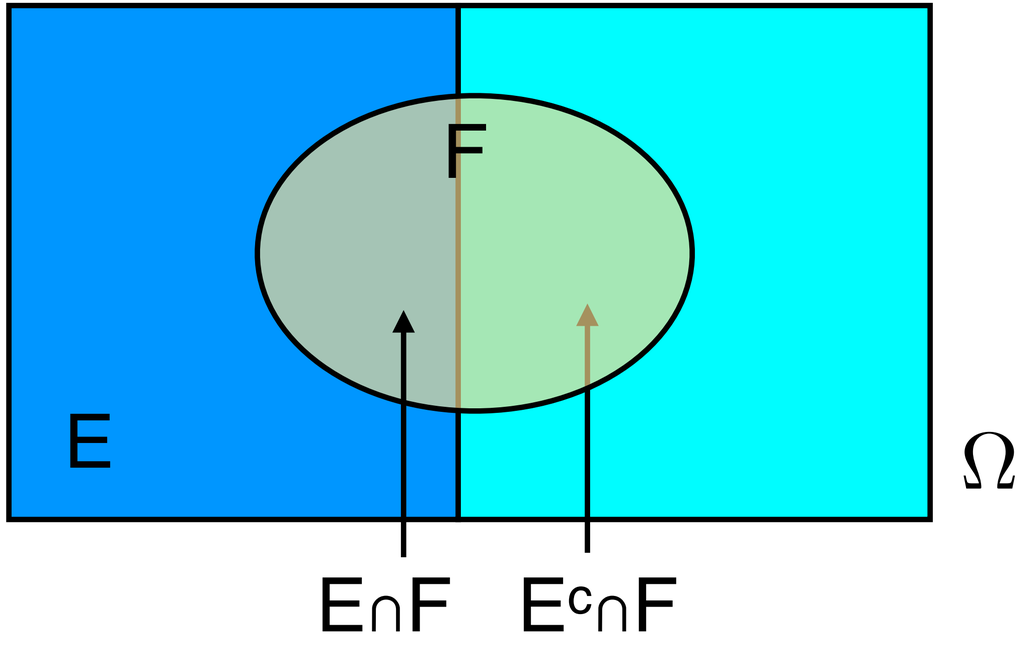
E,F events, P(F)=?

$$F = E \cap F \cup E^c \cap F$$

$$P(F) = P(E \cap F) + P(E^c \cap F)$$

$$= P(E) \cdot P(F \mid E) + P(E^c) \cdot P(F \mid E^c)$$

Product rule



2 Fair Coins

H_i - coin i is h

$\exists H$ - at least one h

$P(\exists H)$?

$$P(\exists H) = \frac{|\exists H|}{|\Omega|} = \frac{3}{4}$$

h h
h t
t h
t t

$$P(\exists H) = P(H_1 \cap \exists H) + P(H_1^C \cap \exists H)$$

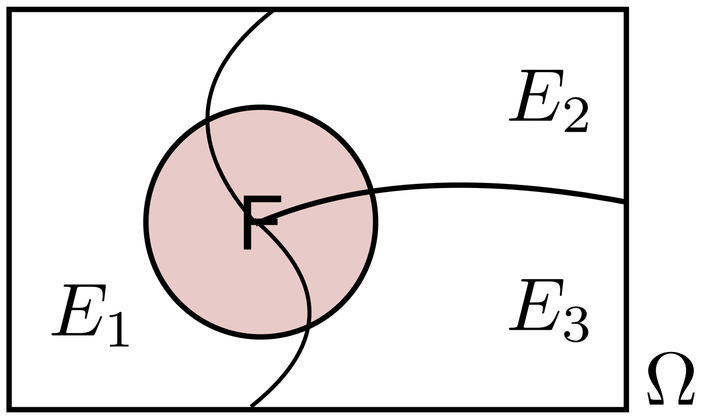
$$= P(H_1) \cdot P(\exists H \mid H_1) + P(H_1^C) \cdot P(\exists H \mid H_1^C)$$

$$= \tfrac{1}{2} \cdot 1 + \tfrac{1}{2} \cdot \tfrac{1}{2} = \tfrac{3}{4}$$

Total Probability - n Conditions

Let $E_1, E_2, \ldots E_n$ partition Ω

$$F = \bigcup_{i=1}^{n} (E_i \bigcap F)$$



$$P(F) = \sum_{i=1}^{n} P(E_i \cap F) \quad = \sum_{i=1}^{n} P(E_i) \cdot P(F|E_i)$$

2 Dice

D_i - outcome of die i

$S = D_1 + D_2$ sum of 2 dice

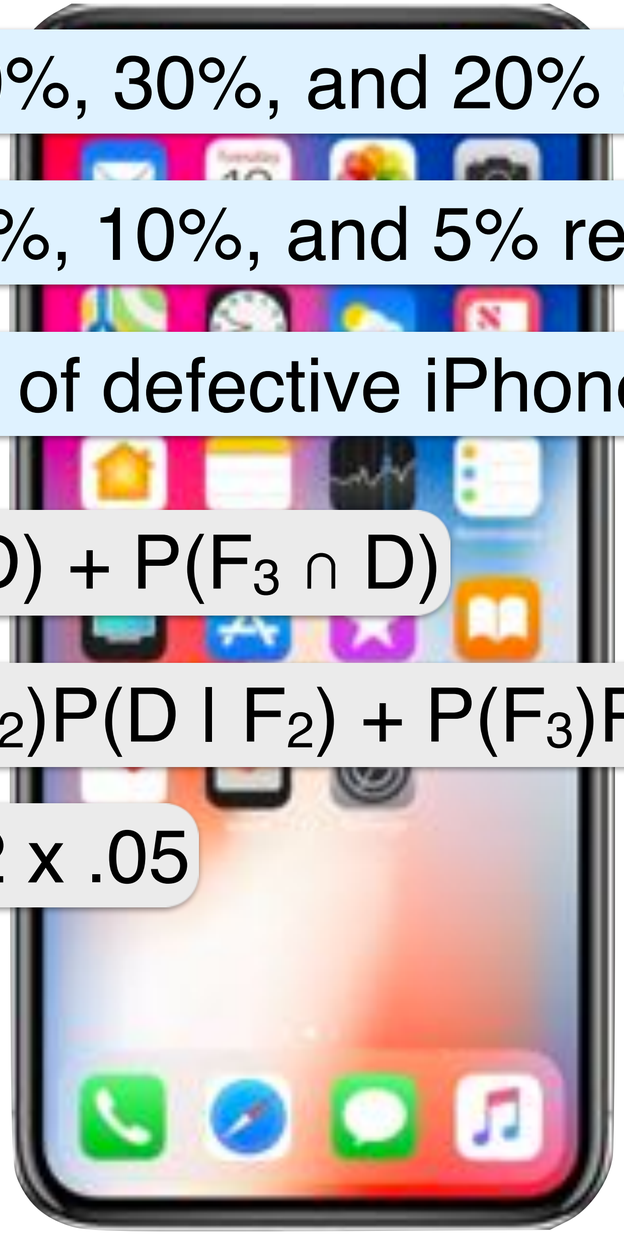
$P(S = 5) = ?$

$$P(S = 5) = \sum_{i=1}^{4} P(D_1 = i) \cdot P(D_2 = 5 - i \mid D_1 = i)$$

$$= \sum_{i=1}^{4} P(D_1 = i) \cdot P(D_2 = 5 - i)$$

$$= 4 \cdot \frac{1}{36} = 1/9$$

iPhone X



Three factories produce 50%, 30%, and 20% of iPhones

Their defective rates are 4%, 10%, and 5% respectively

What is the overall fraction of defective iPhones?

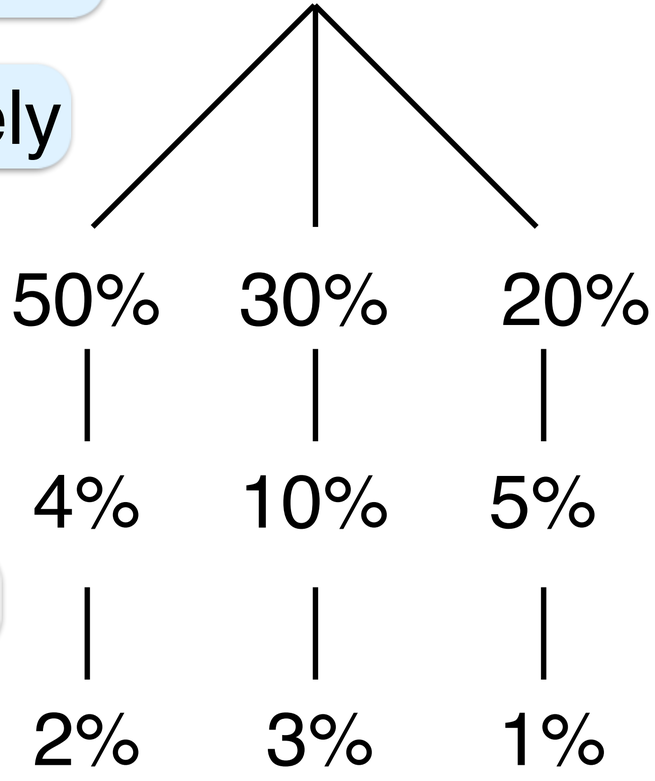
$$P(D) = P(F_1 \cap D) + P(F_2 \cap D) + P(F_3 \cap D)$$

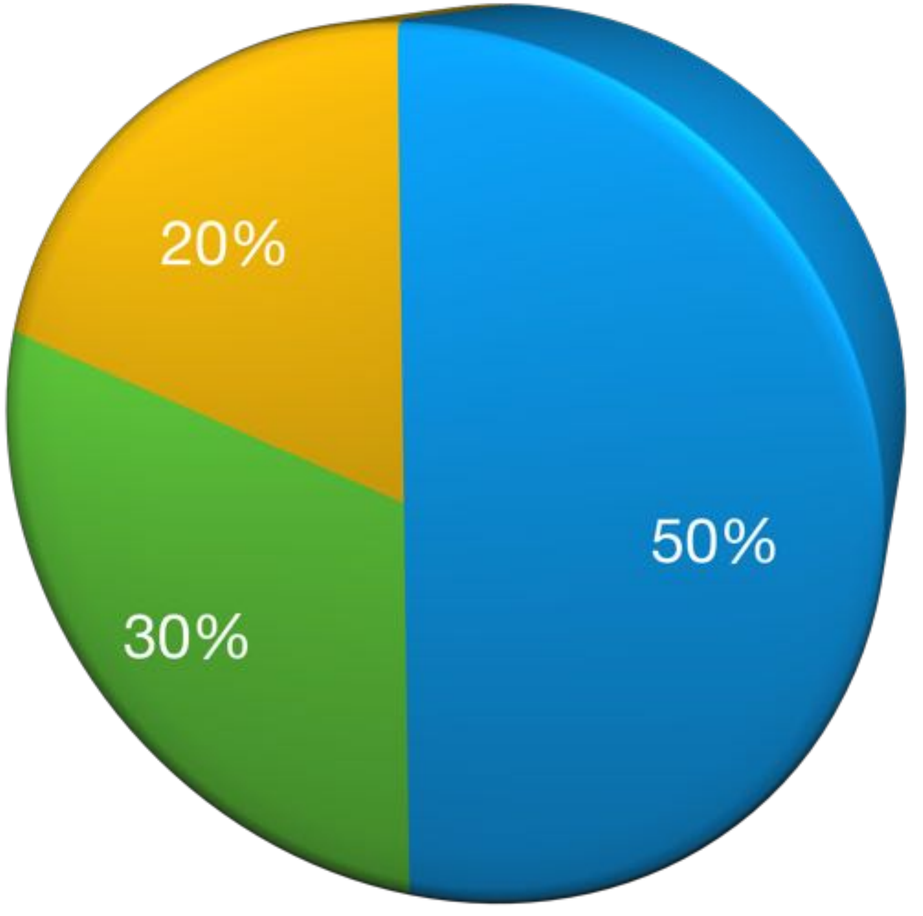
$$= P(F_1)P(D \mid F_1) + P(F_2)P(D \mid F_2) + P(F_3)P(D \mid F_3)$$

$$= .5 \times .04 + .3 \times .1 + .2 \times .05$$

$$= .02 + .03 + .01$$

$$= .06$$




20%
50%
30%

This lecture: Total Probability

Next: Bayes' Rule

Bayes' Rule

Asymmetry

Bought Google at IPO → Wealthy ← ?

Alive today → Born after 1800 ← ?

Forward - Backward

At times

$P(F \mid E)$ - easy

$P(E \mid F)$ - hard

2 coins

H_i - coin i is h

$\exists H$ - at least one h

$P(\exists H \mid H_1) = 1$

$P(H_1 \mid \exists H)$?

2 dice

D_i - face of die i

$S = D_1 + D_2$ sum of 2 faces

$P(S=5 \mid D_1=2) = P(D_2=3) = 1/6$

$P(D_1=2 \mid S=5)$?

Bayes' Rule

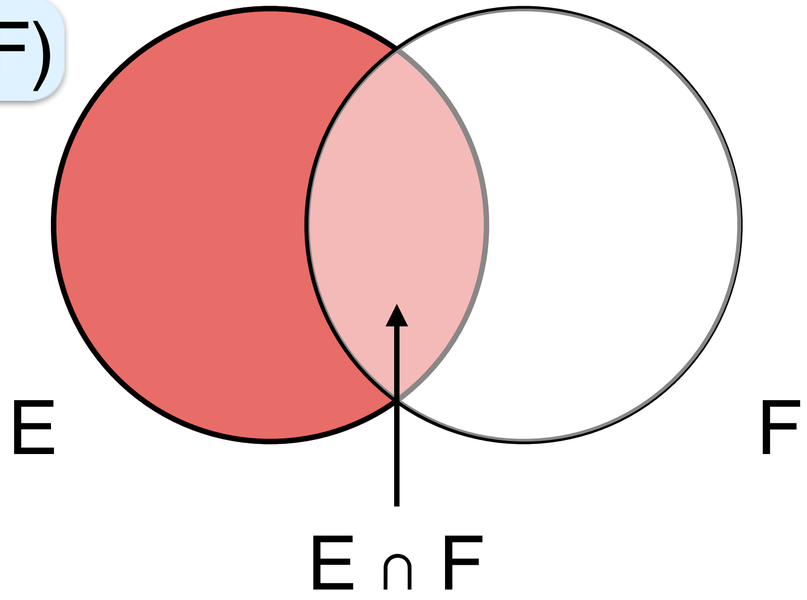
Method for converting $P(F \mid E)$ to $P(E \mid F)$

Bayes' Rule

P(E), P(F)

Given $P(F \mid E)$ (and a bit more) determine $P(E \mid F)$

$$P(E \mid F) = \frac{P(E) \cdot P(F \mid E)}{P(F)}$$



μ-proof

$$P(E \mid F) = \frac{P(E \cap F)}{P(F)} = \frac{P(E) \cdot P(F \mid E)}{P(F)}$$

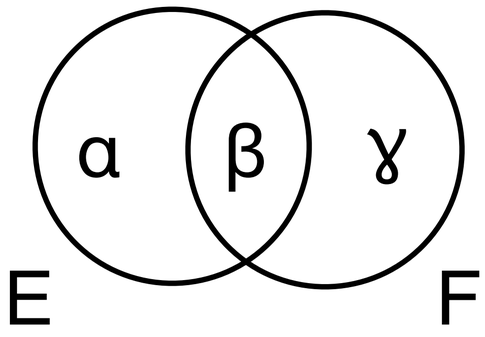
Two More Views

$$\boxed{P(E \mid F) = \frac{P(E) \cdot P(F \mid E)}{P(F)}}$$

$$P(F) \cdot P(E \mid F) = P(E \cap F) = P(E) \cdot P(F \mid E)$$

$$P(F \mid E) = \frac{\beta}{\alpha + \beta}$$

$$P(E \mid F) = \frac{\beta}{\beta + \gamma}$$



$$P(E \mid F) = \frac{\beta}{\beta + \gamma} = \frac{\beta}{\alpha + \beta} \cdot \frac{\alpha + \beta}{\beta + \gamma} = \frac{P(F \mid E) \cdot P(E)}{P(F)}$$

Two Fair Coins

H_i - coin i is h

$\exists H$ - at least one h

$$P(H_1 \mid \exists H) = \frac{|H_1 \cap \exists H|}{|\exists H|}$$

$P(H_1 \mid \exists H) = ?$

$$\exists H \left\{ \begin{matrix} h\ h \\ h\ t \\ t\ h \\ t\ t \end{matrix} \right\} H_1$$

$$P(H_1 \mid \exists H) = P(\exists H \mid H_1) \cdot \frac{P(H_1)}{P(\exists H)} = 1 \cdot \frac{1/2}{3/4} = 2/3$$

$P(\exists H \mid H_1) = 1$

$P(H_1) = 1/2$

$P(\exists H) = 3/4$ Last video

Two Fair Dice

D_i - outcome of die i $\quad S = D_1 + D_2$ Sum of 2 dice

$P(D_1 = 2 \mid S = 5)$?

$$P(D_1 = 2 \mid S = 5) = \frac{P(S = 5 \mid D_1 = 2) \cdot P(D_1 = 2)}{P(S = 5)} = \frac{1/6 \cdot 1/6}{1/9} = 1/4$$

$$P(S = 5 \mid D_1 = 2) = P(D_2 = 3 \mid D_1 = 2) = P(D_2 = 3) = 1/6$$

$$P(D_1 = 2) = 1/6$$

$P(S = 5) = 1/9$ $\quad$ Last video

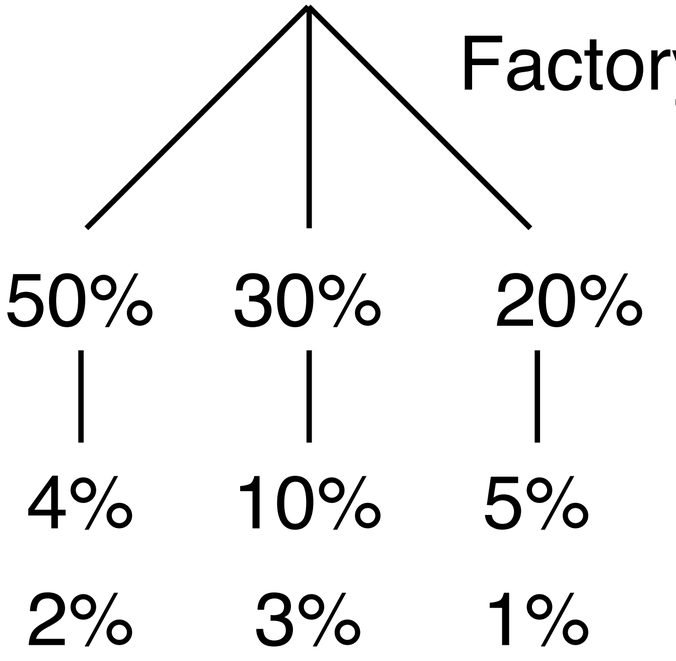
$$P(D_1 = 2 \mid S = 5) = \frac{|D_1 = 2 \cap S = 5|}{|S = 5|}$$

$$S = 5 \begin{cases} 1\ 4 \\ 2\ 3 \\ 3\ 2 \\ 4\ 1 \end{cases} \Big\} D_1 = 2$$

Foxconn

Foxconn has 3 factories producing 50%, 30%, and 20% of its iPhones

Factory defective fractions 4%, 10%, and 5% respectively



Overall fraction of defective phones?

$$
\begin{aligned}
P(D) &= P(D \cap F_1) + P(D \cap F_2) + P(D \cap F_3) \\
&= P(F_1)P(D \mid F_1) + P(F_2)P(D \mid F_2) + P(F_3)P(D \mid F_3) \\
&= .5 \times .04 + .3 \times .1 + .2 \times .05 \\
&= .02 + .03 + .01 \\
&= .06
\end{aligned}
$$

Culprit?

$$P(F_1 \mid D) = \frac{P(D \mid F_1) \cdot P(F_1)}{P(D)} = \frac{.04 \cdot .5}{.06} = \frac{.02}{.06} = ⅓$$

$P(D \mid F_1) = .04$ $\quad$ $P(F_1) = .5$ $\quad$ $P(D) = .06$

$$P(F_2 \mid D) = \frac{.1 \cdot .3}{.06} = \frac{.03}{.06} = ½$$

$$P(F_3 \mid D) = \frac{.05 \cdot .2}{.06} = \frac{.01}{.06} = ⅙$$

Conditional probabilities add to 1

Conditional order determined by both $P(F_i)$ and $P(D \mid F_i)$

Taxi

This Lecture: Bayes' Rule

Next: Random Variables