Notes on Computation of a Stochastic Growth Model with Wedges

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Abstract

We compute a stochastic growth model with two factors of production with multiple wedges. We first define the problem. Then we show four different ways of solving this baby-step problem: (i) Using Ricatti's Equation; (ii) Vaughan's Method. Then we show why VFI is not a good method to compute this model. Finally, we simulate the model. The code can be found here.

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1 The Problem

Assume that the utility function is one that we have used in chapter 2: $u(c_t, h_t) = ln(c_t) + \psi ln(1 - h_t)$. Let's write down the maximization problem of the economy with distortions:

$$\max_{c_t, h_t} \mathbb{E}_0 \sum_{t=0}^{\infty} \hat{\beta}^t [ln(c_t) + \psi ln(1 - h_t)]$$

$$\text{s.t. } c_t = \frac{1}{1+\tau_{ct}}[r_t k_t + w_t h_t + T_t - \tau_{ht} w_t h_t - \tau_{pt}(r_t k_t - \delta k_t) - x_t - \tau_{dt}(r_t k_t - x_t - \tau_{pt}(r_t k_t - \delta k_t))]$$

$$x_t = (1 + \gamma_n) k_{t+1} - (1 - \delta) k_t$$

$$0 \leq h_t \leq 1$$

$$\hat{\beta}^t = ((1 + \gamma_n)\beta)^t$$

$$S_t = P S_{t-1} + Q \epsilon_t$$

$$c_t, x_t \geq 0$$

where: $S_t = \begin{bmatrix} ln(z_t) \\ \tau_{ct} \\ \tau_{ht} \\ \tau_{dt} \\ \tau_{pt} \\ ln(g_t) \end{bmatrix}$ is the vector of exogenous state variables.

Note that in this model, the state and control vectors will be the following:

- State vector: $X_t := \begin{bmatrix} 1 \\ k_t \\ ln(z_t) \\ \tau_{ct} \\ \tau_{ht} \\ \tau_{dt} \\ \tau_{pt} \\ ln(g_t) \\ K_t \\ H_t \\ T_t \end{bmatrix}_{11x1}$

- Control vector: $u_t := \begin{bmatrix} k_{t+1} \\ h_t \end{bmatrix}_{2x1}$ where:
  - k_t: Per capita individual capital stock
  - h_t: Per capita individual hours worked
  - z_t: Efficiency shock/wedge
  - τ_{ct}: Consumption tax/wedge
  - τ_{ht}: Labor income tax/wedge
  - τ_{pt}: Profit tax/wedge
  - τ_{dt}: Dividend tax/wedge
  - g_t: Per capita government consumption/wedge
  - K_t: Aggregate capital stock
  - H_t: Aggregate hours worked
  - T_t: Transfer payments (Lump-sum)

2 Strategy

We'll do more or less the same analysis that we did in chapter 2. (And in chapter 2 we did more or less the same analysis that we did in chapter 1). Here are the steps:

- Step 1: Calculate the non-stochastic steady state
- Step 2: Take second-order Taylor approximation around the steady state to calculate R, Q, W
- Step 3: If the resource constraint is not linear, make it linear by taking first-order Taylor approximation and eventually get A, B, C
- Step 4: Calculate P and F using Ricatti Equation

3 The Non-Stochastic Steady State

Assume in steady state the exogenous state variables have the following steady-state values:

$$S_{ss} = \begin{bmatrix} ln(z_{ss}) = 0 \\ \tau_{c_{ss}} = 0 \\ \tau_{h_{ss}} = 0 \\ \tau_{d_{ss}} = 0 \\ \tau_{p_{ss}} = 0 \\ ln(g_{ss}) = ln(\bar{g}) \end{bmatrix}$$

Note that if $\bar{g} = 0$ then we have the same steady state variables for consumption, capital, and labor as what we calculated in chapter 2.

Then let's write down the planner's problem to solve for steady state values:

$$\max_{c_t,h_t} \mathbb{E}_0 \sum_{t=0}^{\infty} \hat{\beta}^t [ln(c_t) + \psi ln(1 - h_t)]$$

$$\text{s.t. } [\lambda_t] \;\; c_t = [r_t k_t + w_t h_t + T_t - [(1+\gamma_n)k_{t+1} - (1-\delta)k_t]] \quad \forall t$$

$$\hat{\beta}^t = \beta^t(1+\gamma_n)^t$$

First, note that this problem is written in per capita terms (everything is divided by N_t) for simplicity. Secondly, note that since we assumed that $\bar{g} \neq 0$, we put T_t in the budget constraint to have a government budget balance.

Household First order conditions:

$$[c_t]: \quad \hat{\beta}^t \frac{1}{c_t} = \lambda_t$$

$$[h_t]: \quad \hat{\beta}^t \frac{\psi}{1-h_t} = \lambda_t w_t h_t$$

$$[k_{t+1}]: \quad \lambda_t(1+\gamma_n) = \lambda_{t+1}[(1-\delta) + r_{t+1}]$$

Now, the firm's problem is:

$$\max_{K_t^f, H_t^f} {K_t^f}^{\theta} {H_t^f}^{1-\theta} - r_t K_t^f - w_t H_t^f$$

To make the firm problem in per capita terms, divide everything by N_t as well:

$$\max_{k_t^f, h_t^f} {k_t^f}^{\theta} {h_t^f}^{1-\theta} - r_t k_t^f - w_t h_t^f$$

Then firm's First order condition:

$$[k_t^f]: \quad r_t = \theta \frac{{k_t^f}^{\theta} {h_t^f}^{1-\theta}}{k_t^f} = \theta \frac{y_t}{k_t}$$

$$[h_t^f]: \quad w_t = (1-\theta) \frac{{k_t^f}^{\theta} {h_t^f}^{1-\theta}}{h_t^f} = (1-\theta) \frac{y_t}{h_t}$$

where y_t is the per capita production of the firm.

Then, from market clearing, we have:

$$k_t = k_t^f$$

$$h_t = h_t^f$$

And finally, the government budget balance holds:

$$T_t = -g_t$$

Now let's make some algebraic $\mathbb{ABRACADABRA}$'s:

$$\lambda_t(1+\gamma_n) = \lambda_{t+1}[(1-\delta) + r_{t+1}]$$

$$\hat{\beta}^t \frac{1}{c_t}(1+\gamma_n) = \hat{\beta}^{t+1}\frac{1}{c_{t+1}}[(1-\delta) + \theta \frac{{k_t}^\theta h_t^{1-\theta}}{k_t}]$$

Then in steady state: (erase all subscript t's and $c_t = c_{t+1}$)

$$\frac{1}{\beta} = (1-\delta) + \theta \frac{k^\theta h^{1-\theta}}{k}$$

$$\frac{h}{k} = \underbrace{\left[\frac{1}{\theta}\left[\frac{1}{\beta} - (1-\delta)\right]\right]^{\frac{1}{1-\theta}}}_{\Lambda_1}$$

Also, combining BC, HH FOC, and MC, we have:

$$c = (1-\delta-(1+\gamma_n))k + k^\theta h^{1-\theta} - g_{ss}$$

From the intertemporal condition:

$$\psi \frac{c}{1-h} = \underbrace{(1-\theta)\frac{k^\theta h^{1-\theta}}{h}}_{\equiv w_{ss}}$$

$$\psi \frac{c}{1-h} = (1-\theta)\Lambda_1^{-\theta}$$

Now, for simplicity, assume that the steady state of government spending is a constant fraction of the production:

$$g_{ss} = \phi y$$

Then:

$$c = (1-\phi)k^{\theta}h^{1-\theta} - (\delta+\gamma_n)k$$

$$c = \underbrace{\left[(1-\phi)\Lambda_1^{1-\theta} - (\delta+\gamma_n)\right]}_{\Lambda_2} k$$

$$c = \Lambda_2 k$$

Then, we have:

$$\psi\frac{c}{1-h} = (1-\theta)\Lambda_1^{-\theta}$$

$$\psi\Lambda_2 k = (1-\theta)\Lambda_1^{-\theta}(1-h)$$

$$\psi\Lambda_2 k = (1-\theta)\Lambda_1^{-\theta} - (1-\theta)\Lambda_1^{-\theta}\underbrace{h}_{\Lambda_1 k}$$

$$k\left[\psi\Lambda_2 + (1-\theta)\Lambda_1^{1-\theta}\right] = (1-\theta)\Lambda_1^{-\theta}$$

$$k = \frac{(1-\theta)\Lambda_1^{-\theta}}{\left[\psi\Lambda_2 + (1-\theta)\Lambda_1^{1-\theta}\right]}$$

Therefore the steady states are:

- $\Lambda_1 = \left[\frac{1}{\theta}\left[\frac{1}{\beta} - (1-\delta)\right]\right]^{\frac{1}{1-\theta}}$
- $\Lambda_2 = \left[(1-\phi)\Lambda_1^{1-\theta} - (\delta+\gamma_n)\right]$
- $k_{ss} = \dfrac{(1-\theta)\Lambda_1^{-\theta}}{\left[\psi\Lambda_2 + (1-\theta)\Lambda_1^{1-\theta}\right]}$
- $h_{ss} = \Lambda_1 k_{ss}$
- $c_{ss} = \Lambda_2 k_{ss}$
- $l_{ss} = 1 - h_{ss}$

- $g_{ss} = \phi k_{ss}^{\theta} h_{ss}^{1-\theta}$
- $w_{ss} = (1-\theta)\dfrac{k_{ss}^{\theta} h_{ss}^{1-\theta}}{h_{ss}}$
- $r_{ss} = \theta\dfrac{k_{ss}^{\theta} h_{ss}^{1-\theta}}{k_{ss}}$
- $T_{ss} = -g_{ss}$

4 Getting R, Q, W

Now that we have found the steady state, we can take a second-order Taylor approximation of the return function around the steady state. Note that, from the budget constraint, we have:

$$\begin{aligned}
c_t = \frac{1}{1+\tau_{ct}} & [r_t k_t + w_t h_t + T_t - \tau_{ht} w_t h_t - \tau_{pt}(r_t k_t - \delta k_t) \\
& - [(1+\gamma_n)k_{t+1} - (1-\delta)k_t] \\
& - \tau_{dt}(r_t k_t - [(1+\gamma_n)k_{t+1} - (1-\delta)k_t] - \tau_{pt}(r_t k_t - \delta k_t))] \\
= \frac{1}{1+\tau_{ct}} & [r_t(k_t - \tau_{pt}k_t - \tau_{dt}(k_t - \tau_{pt}k_t)) + w_t(h_t - \tau_{ht}h_t) + T_t + \tau_{ptt} \\
& + \tau_{dt}[(1+\gamma_n)k_{t+1} - (1-\delta)k_t - \tau_{pt}\delta k_t]]
\end{aligned}$$

where: $w_t = (1-\alpha)\dfrac{Y_t}{H_t}, \quad r_t = \alpha\dfrac{Y_t}{K_t}, \quad y_t = K_t^{\alpha}(z_t H_t)^{1-\alpha}$

This means:

$$\begin{aligned}
c_t = \frac{1}{1+\tau_{ct}} & \left[\alpha\frac{Y_t k_t}{K_t}[(1-\tau_{dt})(1-\tau_{pt})] + (1-\alpha)\frac{Y_t h_t}{H_t}[1-\tau_{ht}] + T_t + \tau_{pt}[\delta k_t]\right. \\
& \left. + \tau_{dt}[(1+\gamma_n)k_{t+1} - (1-\delta)k_t - \tau_{pt}(\delta k_t)]\right]
\end{aligned}$$

Thus, we can get rid of r_t, w_t and can write consumption c as a function of state and control variables. This enables us to take a Taylor approximation of the objective return function around the steady state.

Note that our return function is the following:

$$\begin{aligned}
r(X_t, u_t) &= r(k_t, ln(z_t), \tau_{ct}, \tau_{ht}, \tau_{dt}, \tau_{pt}, ln(g_t), K_t, H_t, T_t, k_{t+1}, h_t) \\
&= log(\frac{1}{1+\tau_{ct}}\left[\alpha\frac{Y_t k_t}{K_t}[(1-\tau_{dt})(1-\tau_{pt})] + (1-\alpha)\frac{Y_t h_t}{H_t}[1-\tau_{ht}] + T_t\right. \\
&\quad \left. + \tau_{pt}[\delta k_t] + \tau_{dt}[(1+\gamma_n)k_{t+1} - (1-\delta)k_t - \tau_{pt}(\delta k_t)]\right]) + \psi log(1-h_t)
\end{aligned}$$

Now, define: **normalized around steady state**:

state variables as: $\tilde{x}_t = \begin{bmatrix} X_t - X_{ss} \end{bmatrix}_{11x1} = x_t - \bar{x}$; and the control variables as: $\tilde{u}_t = \begin{bmatrix} k_{t+1} - k_{ss} \\ h_t - h_{ss} \end{bmatrix}_{2x1} = u_t - \bar{u}$.

Then second order Taylor approximation of the one-period return function around the steady state, $\bar{x}, \bar{u}$ is:

$$r(x_t, u_t) \simeq \frac{1}{2} \begin{bmatrix} \tilde{x}_t & \tilde{u}_t \end{bmatrix}_{1x13} \begin{bmatrix} 2r(\bar{x}, \bar{u}) & J_r^T \\ J_r & H_r \end{bmatrix}_{13x13} \begin{bmatrix} \tilde{x}_t \\ \tilde{u}_t \end{bmatrix}_{13x1}$$

Where J_r is the Jacobian of the return function at the steady state, H_r is the Hessian of the return function at the steady state, $r(\bar{x}, \bar{u})$ is the return function evaluated at the steady state.

Then we can also write down the same equation:

$$r(x_t, u_t) \simeq \begin{bmatrix} \tilde{x}_t & \tilde{u}_t \end{bmatrix}_{1x13} \begin{bmatrix} Q_{11x11} & W_{11x2} \\ W_{2x11}^T & R_{2x2} \end{bmatrix}_{13x13} \begin{bmatrix} \tilde{x}_t \\ \tilde{u}_t \end{bmatrix}_{13x1}$$

(don't forget to multiply the middle matrix with $\frac{1}{2}$)

5 Getting A, B, C

Finally, we need to get A, B, C to form a linear law of motion of the state vector X_t. If we can find these matrices as well, then we are done.

Remember, the state vector is $X_t := \begin{bmatrix} 1 \\ k_t \\ ln(z_t) \\ \tau_{ct} \\ \tau_{ht} \\ \tau_{dt} \\ \tau_{pt} \\ ln(g_t) \\ K_t \\ H_t \\ T_t \end{bmatrix}_{11x1}$ (i.e. I will write down the A, B, C according to this order of the state variables)

$$X_{t+1} = AX_t + Bu_t + C_{t+1}$$

$$
\begin{bmatrix} 1 \\ k_{t+1} \\ ln(z_{t+1}) \\ \tau_{ct+1} \\ \tau_{ht+1} \\ \tau_{dt+1} \\ \tau_{pt+1} \\ ln(g_{t+1}) \\ K_{t+1} \\ H_{t+1} \\ T_{t+1} \end{bmatrix}_{11x1}
=
\underbrace{\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & {}_c & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & {}_h & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & {}_d & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & {}_p & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & {}_g & 0 & 0 & 0 \\
0 & 0 & ? & ? & ? & ? & ? & ? & ? & ? & ? \\
0 & 0 & ? & ? & ? & ? & ? & ? & ? & ? & ? \\
0 & 0 & ? & ? & ? & ? & ? & ? & ? & ? & ?
\end{bmatrix}}_{A}
\begin{bmatrix} 1 \\ k_t \\ ln(z_t) \\ \tau_{ct} \\ \tau_{ht} \\ \tau_{dt} \\ \tau_{pt} \\ ln(g_t) \\ K_t \\ H_t \\ T_t \end{bmatrix}_{11x1}
+
\underbrace{\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}}_{B}
\begin{bmatrix} k_{t+1} \\ h_t \end{bmatrix} + C_{11x1}\epsilon_{t+1}
$$

Let

- $X_{1t} = \begin{bmatrix} 1 \\ k_t \end{bmatrix}$: individual states,

- $X_{2t} = \begin{bmatrix} ln(z_t) \\ \tau_{ct} \\ \tau_{ht} \\ \tau_{dt} \\ \tau_{pt} \\ ln(g_t) \end{bmatrix}$: aggregate or exogenous states with **known** law of motion,
- $X_{3t} = \begin{bmatrix} K_t \\ H_t \\ T_t \end{bmatrix}$: aggregate or exogenous states with **unknown** law of motion.

Define: $y_t = \begin{bmatrix} X_{1t} \\ X_{2t} \end{bmatrix}$.

Then, define:

$$\tilde{A}_y = \sqrt{\hat{\beta}}(A_y - B_y R^{-1} W'_y)$$

$$\tilde{A}_z = \sqrt{\hat{\beta}}(A_z - B_y R^{-1} W'_z)$$

$$\tilde{B}_y = \sqrt{\hat{\beta}} B_y$$

$$\tilde{Q} = Q - W R^{-1} W'$$

where, $A_y = A[0:8, 0:8]_{8x8}$, $B_y = B[0:8, 0:2]_{8x2}$, $A_z = A[0:8, 8:]_{8x3}$, $W_y = W[0:8, :]_{8x2}$, $W_z = W[8:, :]_{3x2}$

Then we can write down the "known" law of motions as:

$$\tilde{y}_{t+1} = \tilde{A}_y \tilde{y}_t + \tilde{B}_y \tilde{u}_t + \tilde{A}_z \tilde{X}_{3t}$$

Now, how do we proceed? We need to impose new information to be able to solve the unknown law of motion.

We impose the market clearing condition and we have:

$$
\underbrace{\begin{bmatrix} K_t \\ H_t \\ T_t \end{bmatrix}}_{[X_{3t}]} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ ? & ? & ? & ? & ? & ? & ? & ? \end{bmatrix}}_{\Theta} \underbrace{\begin{bmatrix} 1 \\ k_t \\ ln(z_t) \\ \tau_{ct} \\ \tau_{ht} \\ \tau_{dt} \\ \tau_{pt} \\ ln(g_t) \end{bmatrix}}_{[X_{1t}X_{2t}]} + \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 1 \\ ? & ? \end{bmatrix}_{3x2}}_{\Psi} \underbrace{\begin{bmatrix} k_{t+1} \\ h_t \end{bmatrix}}_{u_t}
$$

Note that we can find? parameters by solving:

$$
T_t = \tau_{ct}c_t + \tau_{ht}w_t h_t + \tau_{pt}(r_t k_t - \delta k_t) + \tau_{dt}(r_t k_t - \tau_{pt}(r_t k_t - \delta k_t)) - g_t
$$

Substituting c_t:

$$
\begin{aligned}
T_t = \tau_{ct} \Bigg[\frac{1}{1+\tau_{ct}} \Big[& \alpha K_t^{\alpha}(z_t H_t)^{1-\alpha}[(1-\tau_{dt})(1-\tau_{pt})] \\
& + (1-\alpha)K_t^{\alpha}(z_t H_t)^{1-\alpha}[1-\tau_{ht}] + T_t + \tau_{pt}[\delta k_t] \\
& + \tau_{dt}[(1+\gamma_n)k_{t+1} - (1-\delta)k_t - \tau_{pt}(_t)]\Big]\Bigg] \\
& + \tau_{ht}w_t h_t + \tau_{pt}(r_t k_t - \delta k_t) + \tau_{dt}(r_t k_t - \tau_{pt}(r_t k_t - \delta k_t)) - g_t
\end{aligned}
$$

Then by market clearing condition:

$$
\begin{aligned}
T_t = \tau_{ct} \Bigg[\frac{1}{1+\tau_{ct}} \Big[& \alpha k_t^{\alpha}(z_t h_t)^{1-\alpha}[(1-\tau_{dt})(1-\tau_{pt})] + (1-\alpha)k_t^{\alpha}(z_t h_t)^{1-\alpha}[1-\tau_{ht}] \\
& + T_t + \tau_{pt}[\delta k_t] + \tau_{dt}[(1+\delta_n)k_{t+1} - (1-\delta)k_t - \tau_{pt}(\delta k_t)]\Big]\Bigg] \\
& + \tau_{ht}w_t h_t + \tau_{pt}(r_t k_t - \delta k_t) + \tau_{dt}(r_t k_t - \tau_{pt}(r_t k_t - \delta k_t)) - g_t
\end{aligned}
$$

Putting all T_t's on LHS, we have:

$$
\begin{aligned}
T_t = \tau_{ct} \Big[& \alpha k_t^{\alpha}(z_t h_t)^{1-\alpha}[(1-\tau_{dt})(1-\tau_{pt})] + (1-\alpha)k_t^{\alpha}(z_t h_t)^{1-\alpha}[1-\tau_{ht}] \\
& + \tau_{pt}[\delta k_t] + \tau_{dt}[(1+\gamma_n)k_{t+1} - (1-\delta)k_t - \tau_{pt}(\delta k_t)]\Big] \\
& + (1+\tau_{ct})\left[\tau_{ht}w_t h_t + \tau_{pt}(r_t k_t - \delta k_t) + \tau_{dt}(r_t k_t - \tau_{pt}(r_t k_t - \delta k_t)) - g_t\right]
\end{aligned}
$$

Since this is non-linear, we will take a first-order Taylor approximation around the steady state to make it linear.

After finding the parameters of T_t as well, we finish constructing Θ and Ψ. These were for the level variables. Do the following normalization:

- $\tilde{\Theta} = (I + \Psi R^{-1} W_z')^{-1}(\Theta - \Psi R^{-1} W_y')$
- $\tilde{\Psi} = (I + \Psi R^{-1} W_z')^{-1}\Psi$

One last $\mathbb{MAGIC}$ trick:

- $\hat{A} = \tilde{A}_y + \tilde{A}_z \tilde{\Theta}$
- $\hat{Q} = \tilde{Q}_y + \tilde{Q}_z \tilde{\Theta}$
- $\hat{B} = \tilde{B}_y + \tilde{A}_z \tilde{\Psi}$
- $\bar{A} = \tilde{A}_y - \tilde{B}_y R^{-1} \tilde{\Psi}' \tilde{Q}_z'$

Then we can find P and F as follows:

6 Computing P, F

6.1 Solving via Ricatti Equation

- Ricatti Iteration: $P_n = \hat{Q} + \bar{A}' P_{n+1} \hat{B}(R + \tilde{B}_y' P_{n+1} \hat{B})^{-1} \tilde{B}_y' P_{n+1} \hat{A}$ Iterate it until convergence to find P.
- $\tilde{u}_t = -\left[(R + \tilde{B}_y' P \hat{B})^{-1} \tilde{B}_y' P \hat{A}\right] y_t$. The expression in square brackets is F.

6.2 Solving via Vaughan's Method

- After doing the $\mathbb{MAGIC}$ trick, and putting $\tilde{X}_{3t}$ in the first order conditions, we have the following system:

$$\begin{bmatrix} y \\ \lambda \end{bmatrix}_t = \underbrace{\begin{bmatrix} \hat{A}^{-1} & \hat{A}^{-1}\hat{B}R^{-1}\tilde{B}_y' \\ \hat{Q}\hat{A}^{-1} & \hat{Q}\hat{A}^{-1}\hat{B}R^{-1}\tilde{B}_y' + \bar{A}' \end{bmatrix}}_{H} \begin{bmatrix} y \\ \lambda \end{bmatrix}_{t+1}$$

- Then, as before, $P = V_{21}V_{11}^{-1}$ and $F = (R + \tilde{B}_y' P\hat{B})^{-1}\tilde{B}_y' P\hat{A}$

7 Why VFI Sucks?

Let's write the Bellmann equation:

$$
\begin{aligned}
V(S,k) = \max_{k',h} \Bigg\{ \frac{1}{1+\tau_c} \Bigg[& \theta \frac{K^\theta H^{1-\theta}}{K} k[(1-\tau_p)(1-\tau_d)] \\
& + (1-\theta)\frac{K^\theta H^{1-\theta}}{H} h[1-\tau_h] + T - [(1+\gamma_n)k' - (1-\delta)k](1-\tau_d) \\
& + [\tau_p - \tau_p\tau_d] \Bigg] + \psi log(1-h) + \beta \mathbb{E}\{V(S',k')|S\} \Bigg\}
\end{aligned}
$$

$$
\text{s.t. } S = [log(z), \tau_c, tau_h, \tau_d, \tau_p, log(g)]
$$

Here, we have only two decision variables. So it is not a problem. The main problem of brute force VFI is that **we have so many exogenous stochastic variables**. $z, \tau_c, \tau_h, \tau_d, \tau_p, g$

Let's assume we use the Rouwenhorst method and discretized all these shocks into 20 grids. Then, in every iteration, the VFI process needs to calculate 20^6 calculations, which is an absurdly big number. This is the main issue of VFI in this problem.

8 Simulation

To simulate we will assume the shocks for the exogenous stochastic variables are normally distributed. And we will use $u_t = -Fy_t$ to simulate the control variables. Lastly, to simulate the aggregate variables, we will use Θ, Ψ, and the following relation: $X_{3t} = \Theta y_t + \Psi u_t$

And we will assume that in the initial period, the variables start from their steady states.

We will use the following version of u to compute the variables:

$$u_t = -\left(R + W_z'\Psi\right)^{-1}\left(RF + W_y' + W_z'\Theta\right)\left[1, \hat{k}_t, s_t\right]'$$

8.1 Dividends, Accounting Profits, and Stock Valuation Simulation

Dividends: $d_t = K_t^{\theta}(z_t H_t)^{1-\theta} - w_t H_t - (1+\tau_{dt})X_t$, where $X_t = (1+\gamma_n)K_{t+1} - (1-\delta)K_t$

Accounting Profits: $Pr_t = d_t + k_{t+1} - k_t$

Stock Valuation: $v_t = (1+\tau_{dt})K_t$

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