Computation of Equilibrium of the Stochastic Growth Model with Leuisure

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Abstract

We compute a stochastic growth model with leisure. We first define the problem. Then we show four different ways of solving this baby-step problem: (i) Value Function Iteration (VFI); (ii) Linear-Quadratic Model (LQ Method); (iii) Using Ricatti's Equation; (iv) Vaughan's Method. The code can be found here.

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1 The Model

$$\max_{c_t, k_{t+1}, h_t, x_t} \mathbb{E} \sum_{t=0}^{\infty} \beta^t \left[log(c_t) + \psi log(1-h_t) \right] N_t$$

$$s.t.\ 0 = k_t^{\alpha}[(1+\gamma_z)^t z_t h_t]^{1-\alpha} - c_t - x_t (\text{Feasibility Condition})$$

$$0 = [(1-\delta)k_t + x_t]N_t - N_{t+1}k_{t+1} \ \ (Aggregate\ Law\ of\ Motion)$$

$$log(z_t) = \rho log(z_{t-1}) + \epsilon_t \ \ (Shocks)$$

$$N_t = (1+\gamma_n)^t \ \ (Population\ Growth)$$

2 The Steady State

The trick to getting rid of γ_z: $log(c_t) = log(\frac{c_t(1+\gamma_z)^t}{(1+\gamma_z)^t}) = log(\hat{c_t}) + tlog(1+\gamma_z)$ Do this trick for all the variables, then put resource and law of motion constraints into the objective function and you get:

(Note: One beauty of this trick is that since the $tlog(1+\gamma_z)$) the part does not change the policy function, we can get rid of it in the objective function, which we will do in VFI)

$$\max_{\hat{k_{t+1}}, h_t} \mathbb{E} \sum_{t=0}^{\infty} \hat{\beta}^t [log(\hat{k_t}^{\alpha}(z_t h_t)^{1-\alpha} + (1-\delta)\hat{k_t} - \hat{\gamma}\hat{k_{t+1}}) + \psi log(1-h_t) + tlog(1+\gamma_z)]$$

$$s.t. \hat{\beta} = \beta(1+\gamma_n), \ \ \hat{\gamma} = (1+\gamma_n)(1+\gamma_z)$$

FOC:

$$(h_t): \quad \frac{(1-\alpha)\hat{k}_t^{\alpha}(z_t)^{1-\alpha}h_t^{-\alpha}}{\hat{k}_t^{\alpha}(z_t h_t)^{1-\alpha} + (1-\delta)\hat{k}_t - \hat{\gamma}\hat{k}_{t+1}} = \frac{\psi}{1-h_t}$$

$$(\hat{k}_{t+1}): \quad \frac{\hat{\gamma}}{\hat{k}_t^{\alpha}(z_t h_t)^{1-\alpha} + (1-\delta)\hat{k}_t - \hat{\gamma}\hat{k}_{t+1}} = \hat{\beta}\mathbb{E}[\frac{\alpha \hat{k_{t+1}}^{\alpha-1}(z_{t+1}h_{t+1})^{1-\alpha} + (1-\delta)}{\hat{k}_{t+1}^{\alpha}(z_{t+1}h_{t+1})^{1-\alpha} + (1-\delta)\hat{k}_{t+1} - \hat{\gamma}\hat{k}_{t+2}}]$$

Let's find the steady state (it will be useful for the LQ model).

Now, first of all, let's find z: $log(z_t) = \rho log(z_{t-1}) + \epsilon_t$ and when we are computing for steady state, we put means of the random variables. Let's assume $\epsilon \sim N(0, \sigma)$, then $log(z) = \rho log(z)$. Assuming $\rho < 1$, then z needs to be 1.

$$z_{ss} = 1$$

$$log(z_{ss}) = 0$$

Now, let's find a steady state for k and h: To do this use FOC equations and get rid of subscript t:

$$(h): \quad \frac{(1-\alpha)\hat{k}^{\alpha}(z)^{1-\alpha}h^{-\alpha}}{\hat{k}^{\alpha}(zh)^{1-\alpha} + (1-\delta-\hat{\gamma})\hat{k}} = \frac{\psi}{1-h}$$

$$(\hat{k}): \quad \hat{\gamma} = \hat{\beta}[\alpha\hat{k}^{\alpha-1}h^{1-\alpha} + (1-\delta)]$$

Plug z = 1 and it is two unknown two equation problem. After some calculations we have:

$$\Lambda = (\frac{\frac{\hat{\gamma}}{\hat{\beta}} - (1-\delta)}{\alpha})^{\frac{1}{1-\alpha}}$$

$$\Theta = \frac{(1-\alpha)\Lambda^{-1}}{\Lambda^{1-\alpha} + (1-\delta-\hat{\gamma})}$$

$$h_{ss} = [1 + \frac{\psi}{\Lambda\Theta}]^{-1}$$

$$k_{ss} = \frac{1 - h_{ss}}{\psi}\Theta$$

So, our steady state is:

$$(z_{ss}, h_{ss}, k_{ss}) = (1, [1 + \frac{\psi}{\Lambda\Theta}]^{-1}, \frac{1 - h_{ss}}{\psi}\Theta)$$

3 VFI

$$V(z_t, \hat{k}_t) = max_{\{\hat{k}_{t+1}, h_t\}}\{[log(\hat{k}_t^{\alpha}(z_t h_t)^{1-\alpha} + (1-\delta)\hat{k}_t - \hat{\gamma}\hat{k}_{t+1}) + \psi log(1-h_t)] + \hat{\beta}\mathbb{E}[V(z_{t+1}, \hat{k}_{t+1})]\}$$

Now, how to solve the maximization problem with two variables? We can write h_t as a function of $z_t, \hat{k}_t, \hat{k}_{t+1}$, i.e. solve h_t first. Then the rest is the same as what we did in chapter 1, just plug in h_t for each k_{t+1} on K-grid and calculate the maximum.

But solving h_t for each possible $z_t, \hat{k}_t, \hat{k}_{t+1}$ took forever. Instead of 'solving' the h, we create a grid for h values and for each possible $\hat{k}, \hat{k}', z$ it will take the maximum from that grid.

We also know that since h is labor, it should be bounded between 0 and 1. Therefore, we will create a 100x1 grid between 0 and 1 for h.

After solving for h, then all the rest will be the same as what we did in chapter 1.

4 LQ Method

Now our state variables: $x_t = \begin{bmatrix} log(z_t) \\ \hat{k}_t \end{bmatrix}_{2x1}$ and control variables: $u_t = \begin{bmatrix} \hat{k}_{t+1} \\ h_t \end{bmatrix}_{2x1}$.

Define: **steady state values** for state and control variables as: $\bar{x}_t = \begin{bmatrix} log(z_{ss}) \\ \hat{k}_{ss} \end{bmatrix}_{2x1}$; $\bar{u}_t = \begin{bmatrix} \hat{k}_{ss} \\ h_{ss} \end{bmatrix}_{2x1}$.

First of all, **the one-period return function** is the following:

$$r(x_t, u_t) = r(z_t, \hat{k}_t, \hat{k}_{t+1}, h_t) = log(\hat{k}_t^{\alpha}(exp(log(z_t))h_t)^{1-\alpha} + (1-\delta)\hat{k}_t - \hat{\gamma}k_{t+1}) + \psi log(1-h_t)$$

Now, define **normalized around the steady state (including constant 1 to state variables to incorporate $r(\bar{x}, \bar{u})$)**:

state variables as: $\tilde{x}_t = \begin{bmatrix} 1 \\ log(z_t) - log(z_{ss}) \\ \hat{k}_t - k_{ss} \end{bmatrix}_{3x1} = x_t - \bar{x}$; and the control variables as: $\tilde{u}_t = \begin{bmatrix} \hat{k}_{t+1} - k_{ss} \\ h_t - h_{ss} \end{bmatrix}_{2x1} = u_t - \bar{u}$.

Then second order Taylor approximation of the one-period return function around the steady state, $\bar{x}, \bar{u}$ is:

$$r(x_t, u_t) \simeq \frac{1}{2} \begin{bmatrix} \tilde{x}_t & \tilde{u}_t \end{bmatrix}_{1x5} \begin{bmatrix} 2r(\bar{x}, \bar{u}) & J_r^T \\ J_r & H_r \end{bmatrix}_{5x5} \begin{bmatrix} \tilde{x}_t \\ \tilde{u}_t \end{bmatrix}_{5x1}$$

Where J_r is the Jacobian of the return function at the steady state, H_r is the Hessian of the return function at the steady state, $r(\bar{x}, \bar{u})$ is the return function evaluated at the steady state.

Then we can also write down the same equation:

$$r(x_t,u_t) \simeq \begin{bmatrix} \tilde{x}_t & \tilde{u}_t \end{bmatrix}_{1x5} \begin{bmatrix} Q_{3x3} & W_{3x2} \\ W^T_{2x3} & R_{2x2} \end{bmatrix}_{5x5} \begin{bmatrix} \tilde{x}_t \\ \tilde{u}_t \end{bmatrix}_{5x1}$$

(don't forget to multiply the middle matrix with $\frac{1}{2}$)

Thus, we have found Q, W, and R.

Now, let's look at the law of motion to find A, B, and C:

$$\tilde{x}_{t+1} = A\tilde{x}_t + B\tilde{u}_t + C\epsilon_{t+1}$$

$$\begin{bmatrix} 1 \\ log(\tilde{z}_{t+1}) \\ \tilde{k}_{t+1} \end{bmatrix}_{3x1} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & \rho & 0 \\ 0 & 0 & 0 \end{bmatrix}_{3x3}}_{A} \begin{bmatrix} 1 \\ log(\tilde{z}_t) \\ \tilde{k}_t \end{bmatrix}_{3x1} + \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}_{3x2}}_{B} \begin{bmatrix} \tilde{k}_{t+1} \\ \tilde{h}_t \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}_{3x1}}_{C} \epsilon_{t+1}$$

After finding A, B, C, R, Q, W the original problem can be written as the following LQ problem:

$$\max_{\{\tilde{u}_t,\tilde{x}_{t+1}\}} \mathbb{E}\sum_{t=0}^{\infty} \hat{\beta}^t[\tilde{x}_t^T Q\tilde{x}_t + \tilde{u}_t^T R\tilde{u}_t + 2\tilde{x}_t^T W\tilde{u}_t]$$
$$s.t. \quad \tilde{x}_{t+1} = A\tilde{x}_t + B\tilde{u}_t + C\epsilon_{t+1}$$

In order to get rid of the discount factor $\hat{\beta}^t$ and the cross product $2\tilde{x}_t^T W\tilde{u}_t$, we will re-define our variables as:

$$\hat{x}_t = \hat{\beta}^{\frac{t}{2}}\tilde{x}_t$$

$$\hat{u}_t = \hat{\beta}^{\frac{t}{2}}[\tilde{u}_t + R^{-1}W^T\tilde{x}_t]$$

$$\hat{\epsilon}_{t+1} = \hat{\beta}^{\frac{t}{2}}\epsilon_{t+1}$$

$$\hat{A} = \sqrt{\hat{\beta}}[A - BR^{-1}W^T]$$

$$\hat{B} = \sqrt{\hat{\beta}}B$$

$$\hat{Q} = Q - WR^{-1}W^T$$

$$\hat{C} = \sqrt{\hat{\beta}}C$$

Then the problem becomes:

$$\max_{\{\hat{u}_t, \hat{x}_{t+1}\}} \mathbb{E}\sum_{t=0}^{\infty}[\hat{x}_t^T\hat{Q}\hat{x}_t + \hat{u}_t^T R\hat{u}_t]$$

$$s.t. \quad \hat{x}_{t+1} = \hat{A}\hat{x}_t + \hat{B}\hat{u}_t + \hat{C}\hat{\epsilon}_{t+1}$$

5 Solving by Using Ricatti Equation

The problem above can be written in a functional form:

$$V(\hat{x}) = \max_{\{\hat{u}, \hat{x}_{+1}\}} \hat{x}'\hat{Q}\hat{x} + \hat{u}'R\hat{u} + \mathbb{E}[V(\hat{x}_{+1})]$$

$$s.t. \quad \hat{x}_{+1} = \hat{A}\hat{x} + \hat{B}\hat{u} + \hat{C}\hat{\epsilon}_{+1}$$

We will solve this dynamic problem by guessing and verifying. Guess the value function as: $V_t(\hat{x}) = \hat{x}'P\hat{x} + \hat{\beta}^t c$

Then:

$$V_t(\hat{x}) = \max_{\{\hat{u}, \hat{x}_{+1}\}} \hat{x}'\hat{Q}\hat{x} + \hat{u}'R\hat{u} + \mathbb{E}[\hat{x}_{+1}'P\hat{x}_{+1}] + \hat{\beta}^{t+1}c$$

$$s.t. \quad \hat{x}_{+1} = \hat{A}\hat{x} + \hat{B}\hat{u} + \hat{C}\hat{\epsilon}_{+1}$$

Substitute the resource constraint into the objective function and we get:

$$V_t(\hat{x}) = \max_{\{\hat{u}\}} \hat{x}'\hat{Q}\hat{x} + \hat{u}'R\hat{u} + \mathbb{E}[(\hat{A}\hat{x} + \hat{B}\hat{u} + \hat{C}\hat{\epsilon}_{+1})'P(\hat{A}\hat{x} + \hat{B}\hat{u} + \hat{C}\hat{\epsilon}_{+1})] + \hat{\beta}^{t+1}c$$

$$= \max_{\{\hat{u}\}} \hat{x}'\hat{Q}\hat{x} + \hat{u}'R\hat{u}\hat{x}\hat{A}'P\hat{A}\hat{x} + 2\hat{x}\hat{A}'P\hat{B}\hat{u} + \hat{u}'\hat{B}'P\hat{B}\hat{u} + \mathbb{E}[\hat{\epsilon}^2_{+1}]\hat{C}'P\hat{C} + \hat{\beta}^{t+1}c$$

$$= \max_{\{\hat{u}\}} \hat{x}'(\hat{Q} + \hat{A}'P\hat{A})\hat{x} + \hat{u}'(R + \hat{B}'P\hat{B})\hat{u} + 2\hat{x}\hat{A}'P\hat{B}\hat{u} + \mathbb{E}[\hat{\epsilon}^2_{+1}]\hat{C}'P\hat{C} + \hat{\beta}^{t+1}c$$

FOC wrt $\hat{u}$ is:

$$2[R + \hat{B}'P\hat{B}]\hat{u} + 2\hat{B}'P\hat{A}\hat{x} = 0$$

$$\hat{u} = -\underbrace{[(R + \hat{B}'P\hat{B})^{-1}\hat{B}'P\hat{A}]}_{F}\hat{x}$$

Notice that F depends on P. So we need to find P. How to find it? Replace your LHS with your guess as well and substitute $\hat{u} = -F\hat{x}$ in the maximization problem on the RHS:

$$\hat{x}'P\hat{x}+\hat{\beta}^t c = \hat{x}'(\hat{Q}+\hat{A}'P\hat{A})\hat{x}+\hat{x}'F'(R+\hat{B}'P\hat{B})F\hat{x}-2\hat{x}'\hat{A}'P\hat{B}F\hat{x}+\mathbb{E}[\hat{\epsilon}^2_{+1}]\hat{C}'P\hat{C}+\hat{\beta}^{t+1}c$$

$$\hat{x}'P\hat{x}+\hat{\beta}^t c = \hat{x}'[(\hat{Q}+\hat{A}'P\hat{A})+F'(R+\hat{B}'P\hat{B})F-2\hat{A}'P\hat{B}F]\hat{x}+\mathbb{E}[\hat{\epsilon}^2_{+1}]\hat{C}'P\hat{C}+\hat{\beta}^{t+1}c$$

(Note that we can write down this equation with the $\sim$ notation of x as the following:)

$$\tilde{x}'P\tilde{x}+c = \tilde{x}'[(\hat{Q}+\hat{A}'P\hat{A})+F'(R+\hat{B}'P\hat{B})F-2\hat{A}'P\hat{B}F]\tilde{x}+\mathbb{E}[\epsilon_{+1}^2]\hat{C}'P\hat{C}+\hat{\beta}c \quad (*)$$

From the last equation, we can find P as:

$$P = (\hat{Q} + \hat{A}'P\hat{A}) + F'(R + \hat{B}'P\hat{B})F - 2\hat{A}'P\hat{B}F$$

Substitute F:

$$P = (\hat{Q} + \hat{A}'P\hat{A}) - \hat{A}'P\hat{B}F$$

$$P = (\hat{Q} + \hat{A}'P\hat{A}) - \hat{A}'P\hat{B}[(R + \hat{B}'P\hat{B})^{-1}\hat{B}'P\hat{A}]$$

This is the Ricatti equation. We can iterate the last equation to find P.

5.1 Summary

We can iterate the following iteration to get P:

$$P_{t+1} = (\hat{Q} + \hat{A}'P_t\hat{A}) - \hat{A}'P_t\hat{B}[(R + \hat{B}'P_t\hat{B})^{-1}\hat{B}'P_t\hat{A}]$$

Then our F is:

$$F = (R + \hat{B}'P\hat{B})^{-1}\hat{B}'P\hat{A}$$

Then our policy function is:

$$\hat{u}_t = -F\hat{x}_t$$

and our law of motion is:

$$\hat{x}_{t+1} = (\hat{A} - \hat{B}F)\hat{x}_t + \hat{C}\epsilon_{t+1}$$

After finding $\hat{}$ values, we can find $\sim$ values of $\tilde{x}_t, \tilde{u}_t$ by the reverse transformation of the transformation we did to get rid of the discount factor and cross product term.

Now, to find the value function, we need to find c. To find it equate the RHS of constant and LHS of constant in the equation (*):

$$c = \mathbb{E}[\epsilon_{+1}^2]\hat{C}'P\hat{C} + \hat{\beta}c = \frac{\hat{\beta}}{1-\hat{\beta}}\sigma^2\hat{C}'P\hat{C}$$

After finding c as well, we can write down our value function:

$$V(\tilde{x}) = \tilde{x}'P\tilde{x} + c$$

5.2 Solution Steps

- Find steady states.
- Find Q, W, R by writing down the one-period return function as a quadratic function (by taking second-order Taylor approximation). To do this, find Jacobian and Hessian at the steady state.
- Find A, B, and C from the law of motion.
- Normalize (transform) your $\sim$ variables to get $\hat{}$ variables to get rid of the discount factor and cross product.
- Iterate the Ricatti equation to get P. Then get F as well.
- Reverse transform to get $\sim$ variables.

6 Vaughan's Method

This method can be used after getting rid of the discount factor and cross-product of the LQ model (i.e. after transformation). Recall that after transformation, our problem became (disregarding the stochastic part, since by certainty equivalence P does not change and here our goal is to find P):

$$\max_{\{\hat{u}_t, \hat{x}_{t+1}\}} \mathbb{E} \sum_{t=0}^{\infty} [\hat{x}_t^T \hat{Q} \hat{x}_t + \hat{u}_t^T R \hat{u}_t]$$

$$s.t. \quad \hat{x}_{t+1} = \hat{A}\hat{x}_t + \hat{B}\hat{u}_t$$

Let $(2\lambda_{t+1})$ be the lagrange multiplier of the constraint and take FOC's. Then we get:

$$(\hat{u}_t) \quad 2R\hat{u}_t = -2\hat{B}'\lambda_{t+1}$$

$$(\hat{x}_{t+1}) \quad 2\hat{Q}\hat{x}_{t+1} = 2\lambda_{t+1} - 2\hat{A}'\lambda_{t+2}$$

$$(2\lambda_{t+1}) \quad \hat{x}_{t+1} = \hat{A}\hat{x}_t + \hat{B}\hat{u}_t$$

This system of equations can be reduced by lagging the second equation by one period and substituting others:

$$\lambda_t = \hat{Q}\hat{x}_t + \hat{A}'\lambda_{t+1}$$

$$\hat{x}_{t+1} + \hat{B}R^{-1}\hat{B}\lambda_{t+1} = \hat{A}\hat{x}_t$$

If $\hat{A}$ is invertible, we can express this system as:

$$\lambda_t = \hat{Q}\hat{x}_t + \hat{A}'\lambda_{t+1}$$

$$\hat{A}^{-1}\hat{x}_{t+1} + \hat{A}^{-1}\hat{B}R^{-1}\hat{B}\lambda_{t+1} = \hat{x}_t$$

Then it becomes:

$$\lambda_t = \hat{Q}(\hat{A}^{-1}\hat{x}_{t+1} + \hat{A}^{-1}\hat{B}R^{-1}\hat{B}\lambda_{t+1}) + \hat{A}'\lambda_{t+1}$$

$$\implies \lambda_t = \hat{Q}\hat{A}^{-1}\hat{x}_{t+1} + (\hat{Q}\hat{A}^{-1}\hat{B}R^{-1}\hat{B} + \hat{A}')\lambda_{t+1}$$

$$\hat{A}^{-1}\hat{x}_{t+1} + \hat{A}^{-1}\hat{B}R^{-1}\hat{B}\lambda_{t+1} = \hat{x}_t$$

We can now form Hamiltonian Matrix as:

$$\begin{bmatrix} \hat{x}_t \\ \lambda_t \end{bmatrix} = \underbrace{\begin{bmatrix} \hat{A}^{-1} & \hat{A}^{-1}\hat{B}R^{-1}\hat{B}' \\ \hat{Q}\hat{A}^{-1} & \hat{Q}\hat{A}{-1}\hat{B}R^{-1}\hat{B}' + \hat{A}' \end{bmatrix}}_{\mathbb{H}} \begin{bmatrix} \hat{x}_{t+1} \\ \lambda_{t+1} \end{bmatrix}$$

$$\begin{bmatrix} \hat{x}_t \\ \lambda_t \end{bmatrix} = \mathbb{H} \begin{bmatrix} \hat{x}_{t+1} \\ \lambda_{t+1} \end{bmatrix}$$

Now decompose H in such a way that the eigenvalues which exceed the unit value will be on the top half of the eigenvalue matrix and the bottom half is the reciprocal of the top:

$$\mathbb{H} = V\Lambda V^{-1}$$

$$\mathbb{H} = \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix} \begin{bmatrix} \Lambda & 0 \\ 0 & \Lambda^{-1} \end{bmatrix} \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix}^{-1}$$

Then, we have:

$$P = V_{21}V_{11}^{-1}$$

After finding P, we can find F as how we have found in the Ricatti case:

$$F = (R + \hat{B}'P\hat{B})^{-1}\hat{B}'P\hat{A}$$

Then our policy function is:

$$\hat{u}_t = -F\hat{x}_t$$

6.1 Solution Steps

- Find steady states.
- Find Q, W, R by writing down the one-period return function as a quadratic function (by taking second-order Taylor approximation). To do this, find Jacobian and Hessian at the steady state.
- Find A, B, and C from the law of motion.
- Normalize (transform) your $\sim$ variables to get $\hat{}$ variables to get rid of discount factor and cross product.
- Form Hamiltonian.
- Decompose Hamiltonian in such a way that the eigenvalues which exceed unit value will be on the top half of the eigenvalue matrix and the bottom half is the reciprocal of the top.
- Compute P as $P = V_{21}V_{11}^{-1}$. Then compute F.
- Reverse transform to get $\sim$ variables.

7 Side Quest 1: Solving by Using Consumption and Leisure

(I'm going to use LQ Method to solve this problem)

Now let's do the same model using consumption and leisure being control variables.

Now the state variable is: $x_t = \begin{bmatrix} 1 \\ log(z_t) \\ \hat{k}_t \end{bmatrix}$, control variable is: $u_t = \begin{bmatrix} \hat{c}_t \\ l_t \end{bmatrix}$

and the problem is:

$$\max_{\hat{c}_t, l_t} \mathbb{E} \sum_{t=0}^{\infty} \hat{\beta}^t [log(\hat{c}_t) + \psi log(l_t) + tlog(1 + \gamma_z)]$$

$$s.t.\hat{\beta} = \beta(1+\gamma_n), \ \ \hat{\gamma} = (1+\gamma_n)(1+\gamma_z)$$

$$\hat{c_t} = \hat{k_t}^{\alpha}(exp(log(z_t))(1-l_t))^{1-\alpha} + (1-\delta)\hat{k_t} - \hat{\gamma}\hat{k_{t+1}}$$

We can get rid of the $tlog(1+\gamma_z)$ part because it won't affect the decision. and simpler version of the model is the following:

$$\max_{\hat{c}_t, l_t} \mathbb{E}\sum_{t=0}^{\infty} \hat{\beta}^t[log(\hat{c_t}) + \psi log(l_t)]$$

$$s.t.\hat{\beta} = \beta(1+\gamma_n), \ \ \hat{\gamma} = (1+\gamma_n)(1+\gamma_z)$$

$$\hat{c_t} = \hat{k_t}^{\alpha}(exp(log(z_t))(1-l_t))^{1-\alpha} + (1-\delta)\hat{k_t} - \hat{\gamma}\hat{k_{t+1}}$$

Now, the return function is the following:

$$r(u_t, x_t) = log(\hat{c_t}) + \psi log(l_t)$$

First of all, we have found a steady state for capital and hours above. So we can easily calculate the steady state values of consumption and labor: (remember $z_{ss} = 1$ again):

$$(log(z_{ss}), c_{ss}, l_{ss}) = (0, (k_{ss}^{\alpha}(z_{ss}h_{ss})^{1-\alpha} + (1-\delta)k_{ss} - \hat{\gamma}k_{ss}), 1 - h_{ss})$$

$$\text{where, } (h_{ss}, k_{ss}) = ([1 + \frac{\psi}{\Lambda\Theta}]^{-1}, \frac{1-h_{ss}}{\psi}\Theta)$$

$$\text{where, } \Lambda = (\frac{\frac{\hat{\gamma}}{\hat{\beta}} - (1-\delta)}{\alpha})^{\frac{1}{1-\alpha}}, \Theta = \frac{(1-\alpha)\Lambda^{-1}}{\Lambda^{1-\alpha} + (1-\delta-\hat{\gamma})}$$

We can find R, Q, and W by taking a second-order Taylor approximation of the return function:

Define: **difference from the steady state variables**:

state variables as: $\tilde{x}_t = \begin{bmatrix} 1 \\ log(z_t) - log(z_{ss}) \\ \hat{k}_t - k_{ss} \end{bmatrix}_{3x1} = x_t - \bar{x}$; and the control variables as: $\tilde{u}_t = \begin{bmatrix} \hat{c}_t - c_{ss} \\ l_t - l_{ss} \end{bmatrix}_{2x1} = u_t - \bar{u}$.

Then second order Taylor approximation of the one-period return function around the steady state, $\bar{x}, \bar{u}$ is:

$$r(x_t, u_t) \simeq \frac{1}{2} \begin{bmatrix} \tilde{x}_t & \tilde{u}_t \end{bmatrix}_{1x5} \begin{bmatrix} 2r(\bar{x}, \bar{u}) & J_r^T \\ J_r & H_r \end{bmatrix}_{5x5} \begin{bmatrix} \tilde{x}_t \\ \tilde{u}_t \end{bmatrix}_{5x1}$$

Where J_r is the Jacobian of the return function at the steady state, H_r is the Hessian of the return function at the steady state, $r(\bar{x}, \bar{u})$ is the return function evaluated at the steady state.

Then we can also write down the same equation:

$$r(x_t, u_t) \simeq \begin{bmatrix} \tilde{x}_t & \tilde{u}_t \end{bmatrix}_{1x5} \begin{bmatrix} Q_{3x3} & W_{3x2} \\ W_{2x3}^T & R_{2x2} \end{bmatrix}_{5x5} \begin{bmatrix} \tilde{x}_t \\ \tilde{u}_t \end{bmatrix}_{5x1}$$

So far so good.

Now here is the tricky part: How can we find A, B, and C's? Remember that in the LQ model, the constraint should be linear. But our constraint is:

$$x_{t+1}(\hat{k}_{t+1}) = \frac{1}{\hat{\gamma}}[\hat{k_t}^{\alpha}(exp(log(z_t))(1 - l_t))^{1-\alpha} + (1 - \delta)\hat{k}_t - \hat{c}_t]$$

i.e. x_{t+1}'s $\hat{k}_{t+1}$ variable is not linearly related to x_t's $\hat{k}_t, z_t$ and u_t's l_t. What should we do now?

Answer: Take a first-order approximation of this law of motion of capital in a steady state! So that we can linearize the production function and that allows us to have a linear constraint. Then we can easily define the A, B, and C matrices.

Remember, the variables of this law of motion are c_t, l_t, z_t, k_t. You need to take jacobian wrt these variables.

Solution Steps

- First, calculate steady-state values (use the steady-state values that you found above)
- Take the second-order Taylor approximation of the return function to get R, Q, W
- Take first-order Taylor approximation of the production function to make the constraint linear. Then find A, B, and C matrices.
- Do the Ricatti iteration to find P and correspondingly F

8 Side Quest 2: Solving by Using Only Capital Today and Tomorrow

We would have the same steady-state values for z and k.

Now we will use FOCs of the original problem and solve for h and substitute the solution of h into the return function.

In this case the state variables are the same: $x_t = \begin{bmatrix} log(z_t) \\ \hat{k}_t \end{bmatrix}_{2x1}$ and control variables: $u_t = \begin{bmatrix} \hat{k}_{t+1} \end{bmatrix}_{1x1}$.

the one-period return function is the following:

$$r(x_t, u_t) = r(z_t, \hat{k}_t, \hat{k}_{t+1}) = log(\hat{k}_t^{\alpha}(exp(log(z_t))h_t)^{1-\alpha} + (1-\delta)\hat{k}_t - \hat{\gamma}k_{t+1}) + \psi log(1-h_t), \text{(where h is solved}$$

Now, define: **normalized around the steady state (including con-**

stant 1 to state variables to incorporate $r(\bar{x}, \bar{u})$):

state variables as: $\tilde{x}_t = \begin{bmatrix} 1 \\ log(z_t) - log(z_{ss}) \\ \hat{k}_t - k_{ss} \end{bmatrix}_{3x1} = x_t - \bar{x}$; and the control variables as: $\tilde{u}_t = \begin{bmatrix} \hat{k}_{t+1} - k_{ss} \end{bmatrix}_{1x1} = u_t - \bar{u}$.

Then second order Taylor approximation of the one-period return function around the steady state, $\bar{x}, \bar{u}$ is:

$$r(x_t, u_t) \simeq \frac{1}{2} \begin{bmatrix} \tilde{x}_t & \tilde{u}_t \end{bmatrix}_{1x4} \begin{bmatrix} 2r(\bar{x}, \bar{u}) & J_r^T \\ J_r & H_r \end{bmatrix}_{4x4} \begin{bmatrix} \tilde{x}_t \\ \tilde{u}_t \end{bmatrix}_{4x1}$$

Where J_r is the Jacobian of the return function at the steady state, H_r is the Hessian of the return function at the steady state, $r(\bar{x}, \bar{u})$ is the return function evaluated at the steady state.

Then we can also write down the same equation:

$$r(x_t, u_t) \simeq \begin{bmatrix} \tilde{x}_t & \tilde{u}_t \end{bmatrix}_{1x4} \begin{bmatrix} Q_{3x3} & W_{3x1} \\ W_{1x3}^T & R_{1x1} \end{bmatrix}_{4x4} \begin{bmatrix} \tilde{x}_t \\ \tilde{u}_t \end{bmatrix}_{4x1}$$

(don't forget to multiply the middle matrix with $\frac{1}{2}$**)**

Thus, we have found Q, W, and R.

Now, let's look at the law of motion to find A, B, and C:

$$\tilde{x}_{t+1} = A\tilde{x}_t + B\tilde{u}_t + C\epsilon_{t+1}$$

$$\begin{bmatrix} 1 \\ log(\tilde{z}_{t+1}) \\ \tilde{k}_{t+1} \end{bmatrix}_{3x1} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & \rho & 0 \\ 0 & 0 & 0 \end{bmatrix}_{3x3}}_{A} \begin{bmatrix} 1 \\ log(\tilde{z}_t) \\ \tilde{k}_t \end{bmatrix}_{3x1} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}_{3x1}}_{B} \begin{bmatrix} \tilde{k}_{t+1} \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}_{3x1}}_{C} \epsilon_{t+1}$$

As you can see, it is the same as capital today, tomorrow, and leisure case. The only difference is that, here, we solve for h and substitute it into the return function to get rid of that variable. But how should we solve for h? h should be such that it should solve the FOC wrt h in the original problem:

$$(h): \quad \frac{(1-\alpha)\hat{k}^{\alpha}(z)^{1-\alpha}h^{-\alpha}}{\hat{k}^{\alpha}(zh)^{1-\alpha}+(1-\delta-\hat{\gamma})\hat{k}} = \frac{\psi}{1-h}$$

Let h_t solve the above condition given variables k_t, k_{t+1}, z_t, and parameters γ and ψ.

Then, actually, this remaining part of the question is the same as what we did in chapter 1.

9 Same Model with Various Parameters

I will use the LQ method to solve each case. For each case, since $\psi = 0$, leisure is not valued by the consumer, so the optimal leisure is 0 and optimal labor is 1. This turns the problem that we have faced in chapter 1 with Population and TFP growth. So we can slightly modify the model that we have done in chapter 1. Use the same tricks that we have done for question 1 and we get:

$$\max_{\hat{k}_{t+1}} \mathbb{E}\sum_{t=0}^{\infty}\hat{\beta}^t[log(\hat{k_t}^{\alpha}z_t^{1-\alpha}+(1-\delta)\hat{k}_t-\hat{\gamma}\hat{k_{t+1}})+tlog(1+\gamma_z)]$$

$$s.t.\hat{\beta}=\beta(1+\gamma_n), \;\; \hat{\gamma}=(1+\gamma_n)(1+\gamma_z)$$

FOC:

$$\begin{aligned}(\hat{k}_{t+1}): \quad & \frac{\hat{\gamma}}{\hat{k}_t^{\alpha}(z_t h_t)^{1-\alpha}+(1-\delta)\hat{k}_t-\hat{\gamma}\hat{k}_{t+1}} \\ &= \hat{\beta}\mathbb{E}\left[\frac{\alpha \hat{k_{t+1}}^{\alpha-1}z_{t+1}^{1-\alpha}+(1-\delta)}{\hat{k}_{t+1}^{\alpha}z_{t+1}^{1-\alpha}+(1-\delta)\hat{k}_{t+1}-\hat{\gamma}\hat{k}_{t+2}}\right]\end{aligned}$$

In steady state:

$$(\hat{k}): \quad \hat{\gamma} = \hat{\beta}[\alpha \hat{k}^{\alpha-1} + (1-\delta)]$$

$$K_{ss} = \left[\frac{\frac{\hat{\gamma}}{\hat{\beta}} - (1-\delta)}{\alpha}\right]^{\frac{1}{\alpha-1}}$$

and as before,

$$Z_{ss} = 1$$

So the steady state is:

$$(Z_{ss}, K_{ss}) = (1, \left[\frac{\frac{\hat{\gamma}}{\hat{\beta}} - (1-\delta)}{\alpha}\right]^{\frac{1}{\alpha-1}})$$

Remember that in this question, the state variables are $x_t = \begin{bmatrix} z_t \\ \hat{k}_t \end{bmatrix}$ and control variables are $u_t = \hat{k}_{t+1}$, so R is 1x1 scalar, Q is 3x3 matrix, W is 3x1 vector.

10 The Same Model with Power Utility

Let's find the steady state and then do the same analysis.

The problem is:

$$\max_{\hat{k_{t+1}}, h_t} \mathbb{E} \sum_{t=0}^{\infty} \hat{\beta}^t \frac{[(\hat{k_t}^{\alpha}(z_t h_t)^{1-\alpha} + (1-\delta)\hat{k_t} - \hat{\gamma}\hat{k}_{t+1})(1-h_t)^{\psi}(1+\gamma_z)^t]^{1-\sigma}}{1-\sigma}$$

$$s.t.\hat{\beta} = \beta(1+\gamma_n), \;\; \hat{\gamma} = (1+\gamma_n)(1+\gamma_z)$$

Now, first of all, let's find z: $log(z_t) = \rho log(z_{t-1}) + \epsilon_t$ and when we are computing for steady state, we put means of the random variables. Let's assume $\epsilon \sim N(0, \sigma)$, then $log(z) = \rho log(z)$. Assuming $\rho < 1$, then z needs to be 1.

$$z_{ss} = 1$$

FOC:

$$(h_t): \quad \hat{\beta}^t((1+\gamma_z)^t)^{1-\sigma}\Big[-\psi(1-h_t)^{\psi-1}[\hat{k}_t^{\alpha}(z_t h_t)^{1-\alpha} + (1-\delta)\hat{k}_t - \hat{\gamma}k_{t+1}] + (1-\alpha)\hat{k}_t^{\alpha} z_t^{1-\alpha} h_t^{-\alpha}(1-h_t)^{\psi}\Big]\Big[(\hat{k}_t^{\alpha}(z_t h_t)^{1-\alpha} + (1-\delta)\hat{k}_t - \hat{\gamma}\hat{k}_{t+1})(1 - h_t)^{\psi}\Big]^{-\sigma} = 0$$

If we continue to compute FOC we would have the following equations:

$$(h_t): \quad \frac{(1-\alpha)\hat{k}_t^{\alpha}(z_t)^{1-\alpha}h_t^{-\alpha}}{\hat{k}_t^{\alpha}(z_t h_t)^{1-\alpha} + (1-\delta)\hat{k}_t - \hat{\gamma}\hat{k}_{t+1}} = \frac{\psi}{1-h_t}$$

$$(\hat{k}_{t+1}): \quad \frac{\hat{\gamma}}{\hat{k}_t^{\alpha}(z_t h_t)^{1-\alpha} + (1-\delta)\hat{k}_t - \hat{\gamma}\hat{k}_{t+1}} = \hat{\beta}\mathbb{E}[\frac{\alpha \hat{k_{t+1}}^{\alpha-1}(z_{t+1}h_{t+1})^{1-\alpha} + (1-\delta)}{\hat{k}_{t+1}^{\alpha}(z_{t+1}h_{t+1})^{1-\alpha} + (1-\delta)\hat{k}_{t+1} - \hat{\gamma}\hat{k}_{t+2}}]$$

And you can see that these are the same FOC equations of the first question. (Note: It makes sense to have the same steady values since when $\sigma \to 1$ the function turns into the function in the first question). Thus, they need to have the same steady-state values.

So our steady state is:

$$(z_{ss}, h_{ss}, k_{ss}) = (1, [1 + \frac{\psi}{\Lambda\Theta}]^{-1}, \frac{1-h_{ss}}{\psi}\Theta)$$

where,

$$\Lambda = \left(\frac{\frac{\hat{\gamma}}{\hat{\beta}} - (1-\delta)}{\alpha}\right)^{\frac{1}{1-\alpha}}$$

$$\Theta = \frac{(1-\alpha)\Lambda^{-1}}{\Lambda^{1-\alpha} + (1-\delta-\hat{\gamma})}$$

References

[1] McGrattan, E. R. (1990). Solving the Stochastic Growth Model by Linear-Quadratic Approximation. Journal of Business & Economic Statistics, 8(1), 41–44. https://doi.org/10.2307/1391750

[2] McGrattan, E. R. (1994). A note on computing competitive equilibria in linear models. Journal of Economic Dynamics and Control, 18(1), 149-160. https://doi.org/10.1016/0165-1889(94)90073-6