9 Aiyagari (1994)

Individual's Problem:

$$
\begin{aligned}
&E_0\left\{\sum_{t=0}^{\infty}\beta^t U\left(c_t\right)\right\}\\
&\text{s.t}\\
&c_t + a_{t+1} = wl_t + (1+r)a_t\\
&c_t \geq 0, a_t \geq -b
\end{aligned}
$$

where b is the natural limit $(wl_{\min}/r)$ or an ad hoc one and l_t is a stochastic with bounded support.

There is a competitive in this firm with constant returns to scale st:

$$r = \alpha(K/L)^{(\alpha-1)} \text{ and } w = (1-\alpha)(K/L)^{-\alpha}$$

We can re-state the problem as:

$$
\begin{aligned}
\widehat{a}_t &\equiv a_t + \phi\\
z_t &= wl_t + (1+r)\widehat{a}_t - r\phi
\end{aligned}
$$

Graphically:

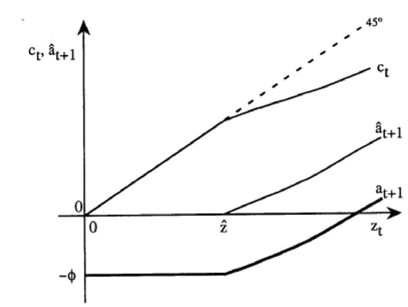

FIGURE Ia
Consumption and Assets as Functions of Total Resources

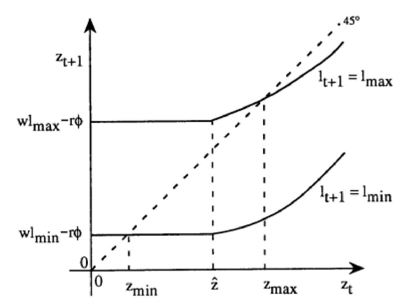

FIGURE Ib
Evolution of Total Resources

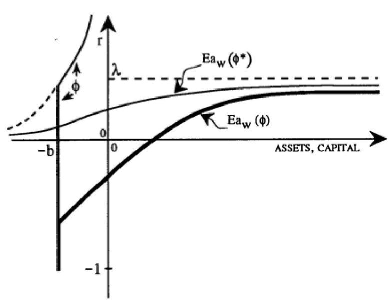

FIGURE IIa
Interest Rate versus Per Capita Assets

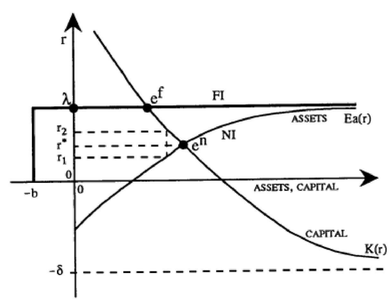

FIGURE IIb
Steady-State Determination

9.1 Solving the model by simulation:

Algorithm 1. Steps:

1. Solve the consumption-savings problem $\underline{\text{for a given } K_r^D}$ level (r and w implied) and obtain $a_{t+1}\left(a_t, z_t; r, w\right)$.

   Assuming that the borrowing limit is zero $\phi = 0$, the normalization has no effect since $\hat{a} = a$, Let;

$$V(z,a;w,r)=\max_{a'}\left\{U(z-a')+\beta\Pi(z'|z)V(z',a;w,r)\right\}$$

st:

$$z'=wl'+(1+r)a'$$

From this step, we can construct the policy $a'(z,a,w,r)$. Why can't we just keep track of z? Since the process for income is not i.i.d then we always need a way to back up income in the past.

2. Starting from an initial distribution $\Gamma^0(z,a)$, simulate a single long time series of asset values: $\{a_t\}_{t=1}^T$.

   Take any (z,a) that has positive probability according to Γ^0. Note that is basically taking values for (l,a). Then we can compute $a'(z,a;w,r)$. After this, we can draw a realization for l' given l and compute (z^{prime},a'). We can iterate this process until we obtain a sequence for $\{a_t\}_{t=1}^T$.

3. Discard the first T^0 periods $\left(0\ll T^0\ll T\right)$ and use $\{a_t\}_{t=T^0+1}^T$, to calculate the implied aggregate asset supply $K_r^S=\left(\frac{1}{T-T^0}\right)\sum_{t=T^0+1}^T a_t$.
4. Check if $K_r^D=K_r^S$. If so, we have a steady state equilibrium and $K^*=K_r^D$
5. If $K_r^D\lesseqgtr K_r^S$, reduce/increase K_r^D and go back to step 1. Iterate until convergence.

Remarks: Notice that we are not simulating a panel of individuals, but a single time series. And in step 3, we are taking the time series average (instead of cross-sectional) to find steady state K^s. Why are we allowed to do that? Because the model's solution is ergodic for the mean of capital. If, for example, the income process had a fixed effect we couldn't do that. So, steps 2 and 3 would have to be replaced by simulating an $N\times T$ panel and taking an average over N once the model reaches a steady state. Moreover, we are not checking if the distribution has converged, just checking if the mean has converged!

Note that as is it, there is no gain in writing the problem in terms of (z,a). We could just rewrite the problem in terms of (l,a). Every step should follow directly.

Algorithm 2. Steps:

1. Solve the consumption-savings problem for a given K_r^D level (r and w implied) and obtain $a_{t+1}\left(a_t,z_t;r,w\right)$- We already know how to do this.
2. Starting from an initial distribution Γ^0, simulate an $N\times T(N,T\gg 0)$ panel of asset values: $\{a_{i,t}\}_{i=1,t=1}^{N,T}$.

   If we make the series large enough we can even start from the same initial condition for every simulation.

3. Use the last cross section, $\{a_t\}_{i=1,t=T}^{N,T}$, to calculate the implied aggregate asset supply $K_r^S=\left(\frac{1}{N}\right)\sum_{i=1}^N a_{i,T}$.
4. Check if $K_r^D=K_r^S$. If so, we have a steady state equilibrium and $K^*=K_r^D$. Here we can also check if $\Gamma^{T_1}=\Gamma^T$ for some $0<T_1<T$ to see if the entire distribution has converged to a steady state.
5. If $K_r^D\lesseqgtr K_r^S$, reduce/increase K_r^D and go back to step 1 . Iterate until convergence.

Remark: T here can be much smaller than the T when we simulate a single time series.

It is more comparable to the T^0 in that context- we require the model to have converged to a steady state, nothing more. Think about 1000 periods or so minimum though. It is a good idea to calculate K_r^s for more than one cross-section and compare to ensure they are not moving over time. If the solution has truly converged to a steady state, the mean should remain (virtually) constant. In the last step, to check if the distribution remains constant, you can compute a histogram with 10 or 20 equal size bins (or more finely at the top if it matters more for your problem) and calculate the fraction of individuals in each bin in periods T_1 and T and make sure the fractions are (virtually) the same.