

Massachusetts Institute of Technology

Department of Electrical Engineering and Computer Science

6.011: Introduction to Communication, Control and Signal Processing

FINAL EXAM Question Booklet, May 18, 2010

Your Full Name:	
Recitation Instructor & Time :	at o'clock

- This exam is **closed book**, but 4 sheets of notes are allowed. Calculators and other electronic aids will not be necessary and are not allowed.
- Check that this question booklet has pages numbered up to 10. The accompanying answer booklet contains spaces for all relevant reasoning and answers; **DO NOT use this question booklet for answers!**
- Neat work and clear explanations count; show all relevant work and reasoning!
- There are **5 problems, weighted as shown, for a total of 100 points**. (The indicated weightings on subparts of problems are nominal, and may be altered slightly prior to the start of grading.)

**Problem 1 (17 points)**

In the block diagram in Figure 1,

$$x[n] = x_c(nT_1) ; \quad y[n] = h[n] * x[n] ;$$

$$r(t) = \sum_{-\infty}^{\infty} y[n]p(t - nT_2) ; \quad q[n] = r(nT_2) .$$

Also, assume the continuous-time input  $x_c(t)$  is bandlimited to frequencies  $|\omega| < 2\pi \times 10^3$ , so

$$X_c(j\omega) = 0 \quad \text{for } |\omega| \geq 2\pi \times 10^3 .$$

Take  $h[n]$  to be the unit sample response of an ideal lowpass filter, with frequency response  $H(e^{j\Omega})$  that is 1 in the passband  $|\Omega| < (\pi/2)$ , and 0 for  $(\pi/2) \leq |\Omega| \leq \pi$ , as shown in Figure 2.

**Note that part (d) below does not require the answers to parts (a)–(c), and can therefore be done independently of them.**

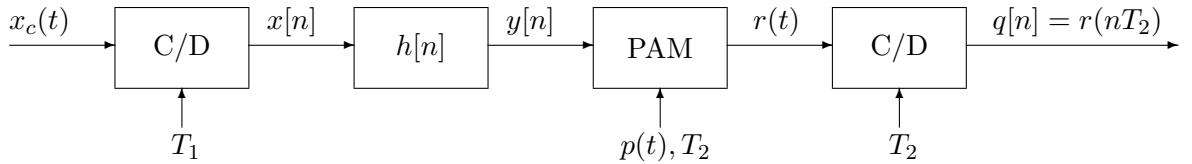


Figure 1: Block diagram.

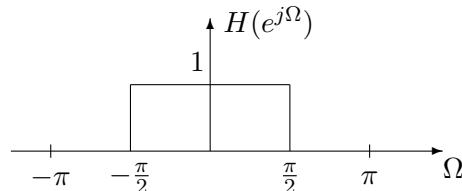


Figure 2: Frequency response  $H(e^{j\Omega})$  of the LTI filter.

- (a) (4 points) Determine the *largest* value of  $T_1$  to ensure that

$$y[n] = x_c(nT_1) .$$

- (b) (6 points) With  $T_1$  picked as in (a), determine a choice for  $T_2$  and  $p(t)$  to ensure that

$$r(t) = x_c(t) .$$

(It is fine — especially if you are not sure of your answer in (a)! — to leave your expressions for  $T_2$  and  $p(t)$  in terms of  $T_1$ , instead of substituting in the numerical value you obtained in (a) for  $T_1$ .)

Also determine if there is another choice of  $T_2$  and  $p(t)$  that could ensure the equality  $r(t) = x_c(t)$ . Explain your answer carefully.

- (c) (3 points) With  $T_1$  picked as in (a), how would you modify your choice of  $T_2$  and  $p(t)$  from (b) to ensure that

$$r(t) = x_c(2.7t) .$$

- (d) (4 points) Assume that  $p(t)$  is now chosen so that its CTFT,  $P(j\omega)$ , is as shown in Figure 3. Determine a value of  $T_2$  to ensure that  $q[n] = y[n]$ .

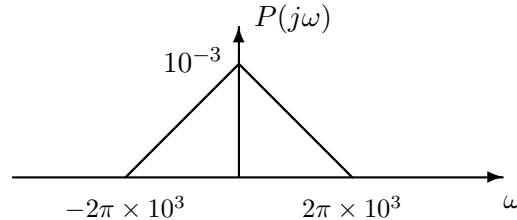


Figure 3: Transform  $P(j\omega)$  of  $p(t)$ , for part (d).

## Problem 2 (18 points)

For each of the following parts, write down whether the statement is **True** or **False**, giving a clear explanation or counterexample.

**Caution:** The fact that these are True/False questions does **not** mean that a quick and intuitive answer will suffice! — you will need to think carefully and present your answer clearly. Your explanation/counterexample is an essential part of your answer, and may take more than a sentence. Sloppy work here will mean you lose 18% of your points on the exam — consider yourself adequately warned!

- (a) (4 points) Suppose  $x[n]$  is a zero-mean discrete-time (DT) wide-sense stationary (WSS) random process. If its autocorrelation function  $R_{xx}[m]$  is 0 for  $|m| \geq 2$  but nonzero for  $m = -1, 0, 1$ , then the linear minimum mean-square-error (LMMSE) estimator of  $x[n+1]$  from measurements of  $x[n]$  and  $x[n-1]$ , namely

$$\hat{x}[n+1] = a_0x[n] + a_1x[n-1],$$

will necessarily have  $a_1 = 0$ .

- (b) (4 points) If the power spectral density  $S_{yy}(j\omega)$  of a continuous-time (CT) WSS random process  $y(t)$  is given by

$$S_{yy}(j\omega) = \frac{17 + \omega^2}{23 + \omega^2}$$

then the mean value of the process is zero, i.e.,  $\mu_y = E[y(t)] = 0$ .

- (c) (4 points) If the autocovariance function  $C_{vv}[m]$  of a DT WSS random process  $v[n]$  is given by

$$C_{vv}[m] = \left(\frac{1}{3}\right)^{|m|},$$

then the LMMSE estimator of  $v[n+1]$  from all past measurements, which we write as

$$\hat{v}[n+1] = \left( \sum_{k=0}^{\infty} h_k v[n-k] \right) + d,$$

will have  $h_k = 0$  for all  $k \geq 1$ , i.e., only the coefficients  $h_0$  and  $d$  can be nonzero.

- (d) (3 points) The process  $v[n]$  in (c) is ergodic in mean value.

- (e) (3 points) If  $z[n] = v[n] + W$ , where  $v[n]$  is the process in (c), and where  $W$  is a random variable with mean 0 and variance  $\sigma_W^2 > 0$ , then the process  $z[n]$  is ergodic in mean value.

### Problem 3 (25 points)

A causal discrete-time system is governed by the following state evolution equation and associated output equation:

$$\begin{aligned}\mathbf{q}[n+1] &= \mathbf{A}\mathbf{q}[n] + \mathbf{b}x[n] + \mathbf{h}w[n], \\ y[n] &= \mathbf{c}^T\mathbf{q}[n] + v[n].\end{aligned}$$

where

$$\mathbf{q}[n] = \begin{bmatrix} q_1[n] \\ q_2[n] \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} \frac{1}{2} & \frac{3}{4} \\ 0 & 2 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}, \quad \mathbf{h} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \mathbf{c}^T = [0 \ 1],$$

Here  $x[n]$  denotes a known (i.e., available or accessible) input signal, while  $w[n]$  is an unknown (i.e., unavailable or inaccessible) *disturbance* input signal that is modeled as a (zero-mean) wide-sense stationary (WSS) white-noise process, with variance  $\sigma_w^2$  at each instant. The unknown (unavailable or inaccessible) *measurement* noise  $v[n]$  is also modeled as (zero-mean) WSS white noise, with variance  $\sigma_v^2$  at each instant. We also assume the noise processes  $v[\cdot]$  and  $w[\cdot]$  are uncorrelated with each other.

- (a) (8 points) Determine the two natural frequencies of the system (i.e., the eigenvalues  $\lambda_1$  and  $\lambda_2$  of  $\mathbf{A}$ ), and for each of them specify whether the associated mode satisfies the following properties:
  - (i) decays asymptotically to 0 in the zero-input response;
  - (ii) is reachable from the input  $x[n]$  (with  $w[n]$  kept at zero);
  - (iii) is reachable from the input  $w[n]$  (with  $x[n]$  kept at zero);
  - (iv) is observable from the output  $y[n]$ .
  
- (b) (2 points) Suppose you wish to implement an observer to estimate the state  $\mathbf{q}[n]$ , and you ask a friend for advice. Your friend, who denotes the state estimate by  $\hat{\mathbf{q}}_f[n]$ , suggests the following (causal) specification for the observer:

$$\hat{\mathbf{q}}_f[n+1] = \mathbf{A}\hat{\mathbf{q}}_f[n] + \mathbf{b}x[n] + \mathbf{h}w[n] - \ell(y[n] - \mathbf{c}^T\hat{\mathbf{q}}_f[n] - v[n]),$$

where  $\ell = \begin{bmatrix} \ell_1 \\ \ell_2 \end{bmatrix}$  denotes the observer gain vector. Write down what you think is the appropriate specification of an observer that you can implement, explaining your reasoning. (Agreeing with your friend is one option, of course.)

**Here and for the rest of this problem,** use the notation  $\hat{\mathbf{q}}[n]$  to denote the state estimate in *your* observer (whether or not you agree with your friend's specification).

- (c) (2 points) Denote the state estimation error by  $\tilde{\mathbf{q}}[n] = \mathbf{q}[n] - \hat{\mathbf{q}}[n]$ . Now explain carefully why the components  $\tilde{q}_1[n]$  and  $\tilde{q}_2[n]$  of  $\tilde{\mathbf{q}}[n]$  at time  $n$  are uncorrelated with the noise terms  $w[n]$  and  $v[n]$  at time  $n$  (or — equivalently, of course! — explain why the components of  $\tilde{\mathbf{q}}[n+1]$  are uncorrelated with  $w[n+1]$  and  $v[n+1]$ ).
- (d) (4 points) The state estimation error in (c) is governed by a state-space model of the form

$$\tilde{\mathbf{q}}[n+1] = \mathbf{B}\tilde{\mathbf{q}}[n] + \mathbf{f}w[n] + \mathbf{g}v[n].$$

Determine  $\mathbf{B}$ ,  $\mathbf{f}$  and  $\mathbf{g}$  in terms of previously specified quantities.

- (e) (5 points) Is it possible to *arbitrarily* vary the natural frequencies of the state estimation error evolution equation in (d) by controlling the observer gains  $\ell_1$  and  $\ell_2$ ? Explicitly note how your answer here is consistent with your answer to (a)(iv). Also specify what constraints, if any, on  $\ell_1$  and  $\ell_2$  must be satisfied to make the error evolution equation asymptotically stable. In particular, would the choice  $\ell_2 = 0$  allow you to obtain a good state estimate? — explain.

If you have done things correctly, you should find that choosing  $\ell_1 = -\frac{3}{4}$  makes the matrix  $\mathbf{B}$  in part (d) a diagonal matrix. **Keep  $\ell_1$  fixed at  $-\frac{3}{4}$  for the rest of this problem**, and also assume  $\ell_2$  is chosen so that the error evolution equation is asymptotically stable (you should find that this is possible by proper choice of  $\ell_2$ , even with  $\ell_1 = -\frac{3}{4}$ ).

- (f) (4 points) Under the preceding assumption of asymptotic stability, and assuming that the system and the observer have been running since  $n = -\infty$ , it turns out that the mean-squared estimation errors  $E(\tilde{q}_1^2[n])$  and  $E(\tilde{q}_2^2[n])$  attain constant steady-state values at any finite  $n$ ; denote these constant values by  $\sigma_{q1}^2$  and  $\sigma_{q2}^2$  respectively. Find explicit expressions for  $\sigma_{q1}^2$  and  $\sigma_{q2}^2$ , expressing them as functions of  $\ell_2$ . [Hint: At steady state,  $E(\tilde{q}_1^2[n+1]) = E(\tilde{q}_1^2[n])$  and  $E(\tilde{q}_2^2[n+1]) = E(\tilde{q}_2^2[n])$ .]

### Problem 4 (20 points)

Assume we have to decide between hypotheses  $H_0$  and  $H_1$  based on a measured random variable  $X$ . The conditional densities for  $X$  given  $H_0$  and  $H_1$  respectively are shown Figure 4.

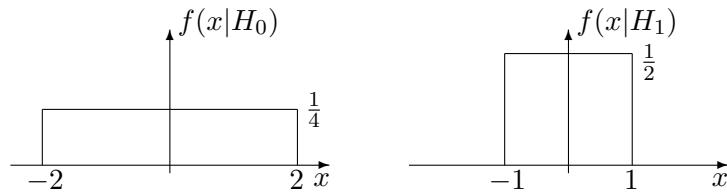


Figure 4: The conditional densities for  $X$  given  $H_0$  and  $H_1$ .

We would like to design a decision rule that will maximize the conditional probability of “detection”  $P_D = P(\cdot|H_1|H_1)$ , subject to the conditional probability of “false alarm”  $P_{FA} = P(\cdot|H_1|H_0)$  not exceeding some limit  $\beta$ . The Neyman-Pearson result tells us that the optimal test will, on obtaining the measurement  $X = x$ , compute the likelihood ratio

$$\Lambda(x) = \frac{f_{X|H}(x|H_1)}{f_{X|H}(x|H_0)}$$

and announce ‘ $H_1$ ’ if  $\Lambda(x) > \eta$ , ‘ $H_0$ ’ if  $\Lambda(x) < \eta$  for some suitably chosen threshold  $\eta$ . However, as this problem shows, a closer look may be needed in the case where  $\Lambda(x)$  — or, more correctly,  $\Lambda(X)$  — can be *exactly at* the threshold  $\eta$  with a nonzero probability.

- (a) (4 points) Sketch  $\Lambda(x)$  as a function of  $x$  for  $-2 < x < 2$ . (You needn’t spend time wondering what  $\Lambda(x)$  is at the edges of the pdf’s or for  $|x| > 2$ , since the probability that  $X$  will take any of these specific values is 0.)
- (b) (6 points) For  $\eta$  fixed at some value in each of the following ranges, specify the corresponding  $P_D$  and  $P_{FA}$ :
  - (i)  $\eta$  at some value strictly above 2;
  - (ii)  $\eta$  at some value strictly between 0 and 2;
  - (iii)  $\eta$  at some value strictly below 0.
- (c) (2 points) If the specified limit on  $P_{FA}$  is  $\beta = 0.3$ , which of the choices in (b) can we pick, and what is the associated  $P_D$ ?

The reason we don’t seem to be able to do too well in (c) is that with  $\eta$  restricted to the ranges in (b), we will only get three possible values of  $P_{FA}$ , with the three values of  $P_D$  that go along with these. In other words, the “receiver operating characteristic” (ROC) that plots  $P_D$  as a function of  $P_{FA}$  will only have three points on it. Here’s how to do better:

- (d) (8 points) Suppose we choose  $\eta = 0$ . What is the probability that we get  $\Lambda(X) = 0$  if  $H_0$  holds? And what is the probability we get  $\Lambda(X) = 0$  if  $H_1$  holds?

With  $\eta = 0$ , we will of course never get  $\Lambda(x) < \eta$  (as the likelihood ratio is always nonnegative), but we might well get  $\Lambda(x) = \eta$  or  $\Lambda(x) > \eta$ . Suppose we announce ' $H_0$ ' when  $\Lambda(x) = 0$ ; however, when  $\Lambda(x) > 0$  we shall announce ' $H_1$ ' with probability  $\alpha$ , and otherwise announce ' $H_0$ '. What are  $P_D$  and  $P_{FA}$  with this *randomized* decision rule?

Note how the choice of  $\alpha$  can give you new points on the ROC, between the points you found in (b)(i) and (b)(ii). What choice of  $\alpha$  will allow you to maximize  $P_D$  while keeping  $P_{FA} \leq 0.3$  with this decision rule?

### Problem 5 (20 points)

The diagram in Figure 5 represents a received signal  $r[n]$  as the sum of two components. Parts (a) and (b) of this problem have distinct specifications for what these two components are, so read the problem statements carefully! In both cases, assume the parameter  $\mu$  that defines the transfer function  $K(z)$  is known to you, and satisfies the condition  $|\mu| < 1$ .

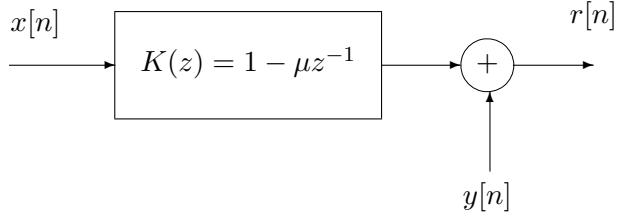


Figure 5: Generation of the received signal  $r[n]$ .

- (a) (10 points) Suppose  $x[n]$  is a signal that we are interested in, while  $y[n]$  is a zero-mean, i.i.d., Gaussian noise process, with variance  $\sigma^2$  at each instant of time. The block in the figure denotes a communication channel — modeled as LTI with transfer function  $K(z) = 1 - \mu z^{-1}$  — through which  $x[n]$  has to pass. We have the following two hypotheses regarding the signal  $x[n]$ :

$$\begin{aligned} H_0 : x[n] &= 0, & P(H_0) &= p_0, \\ H_1 : x[n] &= \delta[n], & P(H_1) &= p_1 = 1 - p_0. \end{aligned}$$

You are to design a receiver that takes  $r[n]$  as input and decides between  $H_0$  and  $H_1$  with minimum probability of error (MPE). We have shown that optimum processing at the receiver involves the sequence of steps shown in Figure 6: LTI (but possibly noncausal) filtering of  $r[n]$  using a filter with unit sample response  $f[n]$ ; sampling the output  $g[n]$  of the filter at some appropriate time  $n_0$ ; and deciding in favor of  $H_0$  or  $H_1$ , based on where the sample value is relative to a threshold  $\gamma$ .

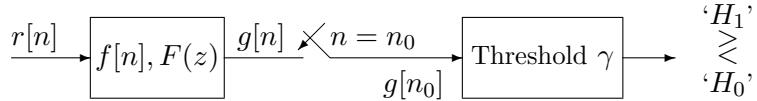


Figure 6: Structure of optimal receiver for this problem.

- (i) Fully specify the MPE receiver in Figure 6 when  $n_0 = 0$ , i.e., specify  $f[n]$  or  $F(z)$  and the value of  $\gamma$ .

- (ii) Write down an expression for  $P('H_1'|H_0)$  and for the minimum probability of error in the case where the two hypotheses are equally likely,  $p_0 = p_1$ . You can write these in terms of the standard function

$$Q(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{\alpha}^{\infty} e^{-\frac{t^2}{2}} dt$$

- (iii) If the value of  $\mu$  is changed to a new value  $\bar{\mu} = \mu/2$ , we can get the same probability of error as prior to the change if the noise variance changes to some new value  $\bar{\sigma}^2$ . Express  $\bar{\sigma}$  in terms of  $\sigma$ .
- (b) (10 points) Suppose now that  $x[n]$  in Figure 5 is a zero-mean, i.i.d., Gaussian noise process, with variance  $\sigma^2$  at each instant of time, and that  $y[n]$  is the signal we are interested in. We have the following two hypotheses regarding  $y[n]$ :

$$\begin{aligned} H_0 : y[n] &= 0 , & P(H_0) &= p_0 , \\ H_1 : y[n] &= \delta[n] , & P(H_1) &= p_1 = 1 - p_0 . \end{aligned}$$

Fully specify the MPE receiver in Figure 6 when  $n_0 = 0$ , i.e., specify  $f[n]$  or  $F(z)$  and the value of  $\gamma$  for this case. Also write down (in terms of  $\mu$  and  $\sigma$ ) the relevant “signal energy to noise power” ratio that governs the performance of this system. Be sure to explain your reasoning throughout!

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