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18.03 Hour Exam I

February 24, 2010

Your Name
Your Recitation Leader's Name
Your Recitation Time

Problem	Points
1	
2	
3	
4	
5	
Total	

Do not open this booklet till told to do so. There are five problems. Use your test-taking skills—be sure you get to all the problems. Do all your work on these pages. No calculators or notes may be used. The point value (out of 100) of each problem is marked in the margin. Solutions will be available on the web after 4:00 today, and at recitation.

[8] **1. (a)** In a perfect environment, the population of Norway rat that breeds on the MIT campus increases by a factor of $e \simeq 2.718281828459045\ldots$ each year. Model this natural growth by a differential equation.

What is the growth rate k?

[4] **(b)** MIT is a limited environment, with a maximal sustainable Norway rat population of $R = 1000$ rats. Write down the logistic equation modeling this. (You may use "k" for the natural growth rate here if you failed to find it in **(a)**.)

[8] **(c)** The MIT pest control service intends to control these rats by killing them at a constant rate of a rats per year. If it wants to limit the rat population to 75% of the maximal sustainable population, what rate a it should aim for (in rats per year)?

2. For the autonomous equation $\dot{x} = x(x-1)(x+2)$, please sketch:

[4] **(a)** the phase line, identifying the critical points and whether they are stable, unstable, or neither.

[4] **(b)** at least one solution of each basic type (so that every solution is a time-translate of one you have drawn)

Below is a diagram of a direction field of the differential equation $y' = (1/4)(x - y^2)$. On it please plot and label:

[3] **(c)** the nullcline

[3] **(d)** at least two quite different solutions

[3] **(e)** the separatrix (if there is one)

[3] **(f)** True or false: If $y(x)$ is a solution with a minimum, then for all large enough x, $y(x) < \sqrt{x}$. (No explanation needed: just circle one.)

[10] **3. (a)** Use Euler's method with stepsize $h = 1/2$ to estimate the value at $x = 3/2$ of the solution to $y' = x + y$ such $y(0) = 1$.

[10] **(b)** Find the solution of $t\dot{x} + x = \cos t$ such that $x(\pi) = 1$.

[3] **4.** **(a)** Find real a, b such that $\dfrac{1}{3+2i} = a + bi$.

[3] **(b)** Find real r, θ such that $1 - i = re^{i\theta}$.

[3] **(c)** Find real a, b such that $(1-i)^8 = a + bi$.

[3] **(d)** Find real a, b such that $b > 0$ and $a + bi$ is a cube root of -1.

[3] **(e)** Find real a, b such that $e^{\ln 2 + i\pi} = a + bi$.

[5] **(f)** Write $f(t) = 2\cos(4t) - 2\sin(4t)$ in the form $A\cos(\omega t - \phi)$.

[5] **5.** **(a)** Find a particular solution to the equation $\dot{x} + 3x = e^{2t}$.

[5] **(b)** Find the solution to the same equation such that $x(0) = 1$.

[5] **(c)** Write down a linear equation with exponential right hand side of which $\dot{x} + 3x = \cos(2t)$ is the real part.

[5] **(d)** Find a particular solution to the equation $\dot{x} + 3x = \cos(2t)$.

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18.03SC Final Exam

1. This problem concerns the differential equation

$$\frac{dy}{dx} = x^2 - y^2 \qquad (*)$$

Let $y = f(x)$ be the solution with $f(-2) = 0$.

(a) Sketch the isoclines for slopes -2, 0, and 2, and sketch the direction field along them.

(c) On the same diagram, sketch the graph of the solution $f(x)$. What is its slope at $x = -2$?

(d) Estimate $f(100)$.

(e) Suppose that the function $f(x)$ reaches a maximum at $x = a$. What is $f(a)$?

(f) Use two steps of Euler's method to estimate $f(-1)$.

2. In **(a)–(c)** we consider the autonomous equation $\dot{x} = 2x - 3x^2 + x^3$.

(a) Sketch the phase line of this equation.

(b) Sketch the graphs of some solutions. Be sure to include at least one solution with values in each interval above, below, and between the critical points.

(c) Some solutions have points of inflection. What are the possible values of $x(a)$ if a non-constant solution $x(t)$ has a point of inflection at $t = a$?

(d) A radioactive isotope of the element Cantabrigium, Ct, decays with half life of two years. The MIT reactor runs on Cantabrigium.
At $t = 0$ there is no Ct in it, but starting at $t = 0$, Ct is added in such a way that the cumulative total amount inserted by time t years is t kg.

Write down a differential equation for the number of moles of Ct in the reactor as a function of time. What is the initial condition?

(e) Solve the initial value problem $\quad x\frac{dy}{dx} + 3y = x^2, y(1) = 1.$

3. (a) Find non-negative real numbers A, ω, and ϕ such that $\text{Re}\left(\frac{ie^{2it}}{1+i}\right) = A\cos(\omega t - \phi)$.

(b) Sketch the trajectory of $e^{(1-\pi i)t}$.

(c) Express the cube roots of $8i$ in the form $a + bi$ (with a and b real).

4. (a)–(c) Find one solution to $\ddot{x} + 2\dot{x} + 2x = q(t)$ for

(a) $q(t) = t^2 + 1$.

(b) $q(t) = e^{-2t} + 1$.

(c) $q(t) = \sin t$. What is the amplitude of the sinusoidal solution?

In **(d)** and **(e)**, suppose that that t^3 is a solution to $\ddot{x} + 2\dot{x} + 2x = q(t)$.

(d) What is $q(t)$?

(e) What is the general solution to $\ddot{x} + 2\dot{x} + 2x = q(t)$?

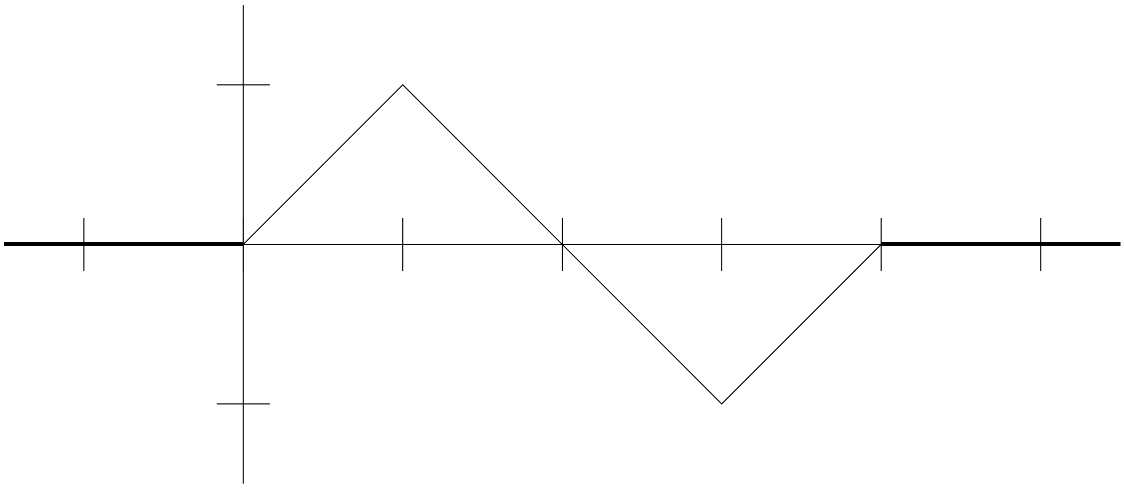
5. (a)–(b) concern the function $f(t) = \text{sq}(t + \frac{\pi}{2})$.

(a) Graph $f(t)$.

(b) What is its Fourier series? (Simplify the trig functions.)

(c) Find a solution to $\ddot{x} + x = \mathrm{sq}(t)$.

6. (a)–(d) In a recent game of Capture the Flag, a certain student was observed to move according to the following graph, in which the hashmarks are at unit spacing.



(a) Graph the generalized derivative $v(t)$.

(b) Write a formula for $v(t)$ in terms of the unit step and (if necessary) the delta function.

(c) Still with the same function as in **(a)**: Graph the generalized derivative $\dot{v}(t)$.

(d) Write a formula for the acceleration $\dot{v}(t)$ in terms of the unit step and (if necessary) the delta function.

(e) Suppose that the unit impulse response of a certain operator $p(D)$ is $w(t)$. Let $q(t) = 0$ for $t < 0$ and $t > 1$, and $q(t) = 1$ for $0 < t < 1$. Please find functions $a(t)$, $b(t)$ so that the solution $x(t)$ to $p(D)x = q(t)$, with rest initial conditions, is given by

$$x(t) = \int_{a(t)}^{b(t)} w(\tau)\, d\tau$$

7. This problem concerns the operator $p(D) = 2D^2 + 8D + 16I$.

(a) What is the transfer function of the operator $p(D)$?

(b) What is the unit impulse response of this operator?

(c) What is the Laplace transform of the solution to $p(D)x = \sin(t)$ with rest initial conditions?

8. In **(a)** and **(b)**, $A = \begin{bmatrix} 2 & 12 \\ 3 & 2 \end{bmatrix}$.

(a) What are the eigenvalues of A?

(b) For each eigenvalue, find a nonzero eigenvector.

(c) Suppose that the matrix B has eigenvalues 1 and 2, with eigenvectors $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ respectively. Calculate e^{Bt}.

(d) What is the solution to $\dot{\mathbf{u}} = B\mathbf{u}$ with $\mathbf{u}(0) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$?

9. (a) Suppose again that the matrix B has eigenvalues 1 and 2, with eigenvectors $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ respectively. Sketch the phase portrait on the graph below.

(b) Let $A = \begin{bmatrix} a & -2 \\ 2 & 1 \end{bmatrix}$, and consider the homogeneous linear system $\dot{\mathbf{u}} = A\mathbf{u}$. For each of the following conditions, determine all values of a (if any) which are such that the system satisfies the condition.

(i) Saddle
(ii) Star
(iii) Stable node
(iv) Stable spiral. What is the direction of rotation?
(v) Unstable spiral.
(vi) Unstable defective node

10. Parts **(a)–(c)** deal with the nonlinear autonomous system $\begin{cases} \dot{x} = x^2 - y^2 \\ \dot{y} = x^2 + y^2 - 8 \end{cases}$.

(a) Find the equilibria of this system.

(b) There is one equilibrium in the south-west quadrant. Find the Jacobian at this equilibrium.

(c) The equilibrium you found in **(b)** is a stable spiral. For large t, the solutions which converge to this equilibrium have x-coordinate which are well-approximated by the function $Ae^{at}\cos(\omega t - \phi)$ for some constants A, ϕ, a, and ω. Some of these constants depend upon the particular solution, and some are common to all solutions of this type. Find the values of the ones which are common to all such solutions.

(d) Finally, return to the autonomous equation $\dot{x} = 2x - 3x^2 + x^3$ that you studied in problem **2**. Write down a formula approximating the solutions converging to the stable equilibrium when t is large.

Operator Formulas

- Exponential Response Formula: $x_p = Ae^{rt}/p(r)$ solves $p(D)x = Ae^{rt}$ provided $p(r) \neq 0$.
- Resonant Response Formula: If $p(r) = 0$ then $x_p = Ate^{rt}/p'(r)$ solves $p(D)x = Ae^{rt}$ provided $p'(r) \neq 0$.

Defective matrix formula

If A is a defective 2×2 matrix with eigenvalue λ_1 and nonzero eigenvector $\mathbf{v}_1$, then you can solve for $\mathbf{w}$ in $(A - \lambda_1 I)\mathbf{w} = \mathbf{v}_1$ and $\mathbf{u} = e^{\lambda_1 t}(t\mathbf{v}_1 + \mathbf{w})$ is a solution to $\dot{\mathbf{u}} = A\mathbf{u}$.

Properties of the Laplace transform

0. Definition: $\mathcal{L}[f(t)] = F(s) = \int_{0-}^{\infty} f(t)e^{-st}\,dt$ for $\operatorname{Re} s >> 0$.

1. Linearity: $\mathcal{L}[af(t) + bg(t)] = aF(s) + bG(s)$.

2. Inverse transform: $F(s)$ essentially determines $f(t)$.

3. s-shift rule: $\mathcal{L}[e^{at}f(t)] = F(s-a)$.

4. t-shift rule: $\mathcal{L}[f_a(t)] = e^{-as}F(s)$, $\quad f_a(t) = \begin{cases} f(t-a) & \text{if } t > a \\ 0 & \text{if } t < a \end{cases}$.

5. s-derivative rule: $\mathcal{L}[tf(t)] = -F'(s)$.

6. t-derivative rule: $\mathcal{L}[f'(t)] = sF(s)$ [generalized derivative]
$\mathcal{L}[f_r'(t)] = sF(s) - f(0+)$ [$f(t)$ continuous for $t > 0$]

7. Convolution rule: $\mathcal{L}[f(t) * g(t)] = F(s)G(s)$, $f(t) * g(t) = \int_0^t f(\tau)g(t-\tau)\,d\tau$.

8. Weight function: $\mathcal{L}[w(t)] = W(s) = \dfrac{1}{p(s)}$, $\quad w(t)$ the unit impulse response.

Formulas for the Laplace transform

$$\mathcal{L}[1] = \frac{1}{s} \qquad \mathcal{L}[e^{at}] = \frac{1}{s-a} \qquad \mathcal{L}[t^n] = \frac{n!}{s^{n+1}}$$

$$\mathcal{L}[\cos(\omega t)] = \frac{s}{s^2+\omega^2} \qquad \mathcal{L}[\sin(\omega t)] = \frac{\omega}{s^2+\omega^2}$$

$$\mathcal{L}[u_a(t)] = \frac{e^{-as}}{s} \qquad \mathcal{L}[\delta_a(t)] = e^{-as}$$

where $u(t)$ is the unit step function $u(t) = 1$ for $t > 0$, $u(t) = 0$ for $t < 0$.

Fourier series

$$f(t) = \frac{a_0}{2} + a_1\cos(t) + a_2\cos(2t) + \cdots + b_1\sin(t) + b_2\sin(2t) + \cdots$$

$$a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(mt)\,dt, \qquad b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(mt)\,dt$$

$$\int_{-\pi}^{\pi}\cos(mt)\cos(nt)\,dt = \int_{-\pi}^{\pi}\sin(mt)\sin(nt)\,dt = 0 \qquad \text{for} \quad m \neq n$$

$$\int_{-\pi}^{\pi}\cos^2(mt)\,dt = \int_{-\pi}^{\pi}\sin^2(mt)\,dt = \pi \qquad \text{for} \quad m > 0$$

If $\operatorname{sq}(t)$ is the odd function of period 2π which has value 1 between 0 and π, then

$$\operatorname{sq}(t) = \frac{4}{\pi}\left(\sin(t) + \frac{\sin(3t)}{3} + \frac{\sin(5t)}{5} + \cdots\right)$$

18.03SC Differential Equations
Fall 2011

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exam_11.pdf

Practice Final Exam

1. For the DE $\dfrac{dy}{dx} = -\dfrac{y}{x} + 3x$:
a) Sketch the direction field for this DE, using (light or dotted) isoclines for the slopes -1 and 0.

b) For the solution curve passing through the point (1,2): if Euler's method with step-size $h = 0.1$ was used to approximate $y(1.1)$, would the approximation come out too high or too low? Explain.

c) Compute the Euler approximation to $y(1.1)$ using step-size $h = 0.1$.

d) The functions $y_1 = x^2$ and $y_2 = x^2 + \frac{1}{x}$ are solutions to this DE. If $y = y(x)$ is the solution satisfying the IC $y(1) = 1.5$, show that $100 \leq y(10) \leq 100.1$. Do we need to include the equal signs in this inequality? Why or why not?

e) Find the general solution the DE and verify the prediction of part (b).

2. Suppose that a population of variable size (in some suitable units) $p(t)$ follows the growth law $\dfrac{dp}{dt} = p^3 - 4p^2 + 4p$. Without solving the DE explicitly:

a) Find all critical points and classify each according to its stability type using a phase-line diagram.

b)) Draw a rough sketch (on p-vs.-t axes) of the family of solutions. What happens to the population in the long-run if it starts out at size 1 unit; at size 3 units ?

c) Explain why the rate information given by the DE was all we needed to get the answer to part (b).

3. Let $p(D) = D^2 + b\,D + 5$ where $D = \frac{d}{dt}$. a) For what range of the values of b will the solutions to $p(D)y = 0$ exhibit oscillatory behavior?

b) For $b = 4$, solve the DE's (i) $p(D)y = 4\,e^{2t}\sin t$
(ii) $p(D)y = 4\,e^{2t}\cos t$
using the Exponential Response formula. Write your answers in both amplitude-phase and rectangular form.

c) Given $b = 2$, for what ω does $p(D)y = \cos\omega t$ have the biggest response?

4. Find the general solution to the DE $(D^3 - 1)\,y = e^x$
Express the answer using *real* - valued functions only.

5. For $f(t) = t$ on $-1 < t < 1$, periodic with period $P = 2$:
a) Sketch f over three or more full periods P

Choose endpoint values that show where the Fourier series expansion will converge (*without* computing the Fourier series).

b) Compute the Fourier series of f

c) Compute the steady-periodic solution to the DE $x''(t) + 10\,x(t) = f(t)$. Does near-resonance occur in this situation? If so, which frequency in the 'driving force' $\tilde{f}_{\text{odd}}$ produces it?

6. a) Suppose that starting at $t = 0$ a radioactive material is continuously flowing into a container at a rate $f(t)$ in curies per unit time, and that one uses the standard exponential model for continuous radioactive decay, with rate constant k (in $\frac{1}{\text{time}}$). Let $R = R(t)$ denote the total amount of radioactive material in the container at time t. Give the DE for $R(t)$ and solve it (in terms of k, $f(t)$ and $R_0 = R(0)$).

b) Show that the solution satisfying the IC $R(0) = 0$ can be written as a convolution integral. What is the weight function w in this case? What DE does w satisfy?

7. Let $\mathcal{L}$ denote the Laplace transform (as usual). Derive the formula for $\mathcal{L}(\cos(t))$, by expressing $\cos(t)$ in terms of the complex exponential.

8. Compute $f(t) = \mathcal{L}^{-1}\left(\frac{1}{s} - \frac{e^{-s}}{s(s+1)}\right)$ in both "u -form" and in "cases" form, and sketch the graph of f.

9. Use the Laplace transform method to solve the IVP's

a) $y'' - y' - 2y = 0, \;\; y(0) = 0, \;\; y'(0) = 2$

b) $y'' + 4y = \cos t, \;\; y(0) = y'(0) = 0$

10. Consider an undamped spring-mass system $Lx = x'' + x = f(t)$, where $f(t)$ is an external applied force, and suppose that the system starts out at time $t = 0$ at its equilibrium position $x = 0$ with a velocity $x'(0) = 1$ (in some suitable units). Using the Laplace transform method, solve for the position function $x = x(t)$ for the following forcing function $f(t)$:

$f(t)$ is an *impulsive* force of magnitude F_0 at time $t = \pi$ and $f(t) = 0$ otherwise.
Graph the general solution. What happens in the special case $F_0 = 1$, and why?
Express your answer in u-form, and in 'cases' form as well.

11. Find the general real solution to the DE system $x' = x - 2y \qquad y' = 4x + 3y$ using the eigenvalue/eigenvector method.

12. For the DE system $\mathbf{x}' = A_a \mathbf{x}$ with $A_a = \begin{bmatrix} a & 1 \\ 1 & a \end{bmatrix}$:

a) Find the range of the values of a for which the critical point at (0,0) will be:
(i) a source node (ii) a sink node (iii) a saddle.

b) Choose a convenient value for a for each of the types above, solve, and sketch the trajectories in the vicinity of the critical point, showing the direction of increasing t.

13. For the DE system $x' = x - 2y \qquad y' = 4x - x^3$:

a) Compute the critical points of this system.

b) Find the type of each of the critical points using the linearized system which approximates this non-linear system and classify them according to their stability type and also their structural stability type. (Use the Jacobian.)

c) Using the results of part(b), compute the eigenvectors as needed. Now put it all together into a reasonable sketch of the phase-plane portrait of this system. Is there more than one possiblity for the general shape and stability type of the trajectories around each of the critical points in this case? Why/why not?

14. Same instructions as **13** for the DE system $x' = y \qquad y' = 2x - x^2$.

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exam_12.pdf

ES.1803 Linear Algebra Practice, Spring 2024

Problem 1.

(a) Find the eigenvalues and eigenvectors of $A = \begin{bmatrix} 1 & -10 \\ 3 & -1 \end{bmatrix}$.

(b) Find the eigenvalues and eigenvectors of $A = \begin{bmatrix} -8 & 7 \\ 1 & -2 \end{bmatrix}$.

Problem 2.

Suppose that the matrix B has eigenvalues 2, 7 and 7, with eigenvectors

$$\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix}$$

respectively.

(a) Calculate e^{Bt}.

(b) What are the eigenvalues and eigenvectors of e^{Bt}?

(c) Give an argument based on transformations why $B = \begin{bmatrix} 1 & 1 & 5 \\ -1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 7 \end{bmatrix} \begin{bmatrix} 1 & 1 & 5 \\ -1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}^{-1}$ has the eigenvalues and eigenvectors given in Part (a).

(d) What is the solution to $\mathbf{x}' = B\mathbf{x}$ with $\mathbf{x}(0) = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$?

(e) Decouple the system $\mathbf{x}' = B\mathbf{x}$. That is, make a change of variables and write the DE in the new variables.

Problem 3.

Let $R = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ and suppose R is the reduced row echelon form for A.

(a) What is the rank of A?

(b) Find a basis for the null space of A.

(c) Suppose the column space of A has basis $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$. Find a possible matrix for A. That is, give a matrix with RREF R and the given column space.

(d) Find a matrix with the same reduced echelon form but such that $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ are in its column space.

Problem 4.

Suppose $A = \begin{bmatrix} a & b & c \\ 0 & 2 & e \\ 0 & 0 & 3 \end{bmatrix}$.

(a) What are the eigenvalues of A?

(b) For what value (or values) of a, b, c, e is A singular (non-invertible)?

(c) What is the minimum rank of A (as a, b, c, e vary)? What's the maximum?

(d) Suppose $a = -5$. In the system $\mathbf{x}' = A\mathbf{x}$, is the equilibrium at the origin stable or unstable.

Problem 5.

Suppose that $A = S \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} S^{-1}$.

(a) What are the eigenvalues of A?

(b) Express A^2 and A^{-1} in terms of S.

(c) What would I need to know about S in order to write down the most rapidly growing exponential solution to $\mathbf{x}' = A\mathbf{x}$?

Problem 6.

(a) An orthogonal matrix is one where the columns are orthonormal (mutually orthogonal and unit length). Equivalently, S is orthogonal if $S^{-1} = S^T$.

Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$. Find an orthogonal matrix S and a diagonal matrix Λ such that $A = S\Lambda S^{-1}$

(b) Decouple the equation $\mathbf{x}' = A\mathbf{x}$, with $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$.

Problem 7.

Suppose A has eigenvalues -2 and -3 with corresponding eigenvectors $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

(a) Compute A^{-1} explicitly.

(b) Consider the system $\begin{bmatrix} x' \\ y' \end{bmatrix} = A \begin{bmatrix} x \\ y \end{bmatrix}$. Find a change of coordinates

$$u = ax + by, \qquad v = cx + dy$$

so that in these new coordinates the system becomes $u' = r_1 u$ and $v' = r_2 v$. Also give the values of r_1 and r_2.

Problem 8.

Let $A = \begin{bmatrix} 1 & 4 & 2 & 2 \\ 2 & 8 & 1 & 9 \\ 1 & 4 & 1 & 7 \end{bmatrix}$

(a) Put A in reduced row echelon form.

(b) Give a basis for the column space of A.

Problem 9.

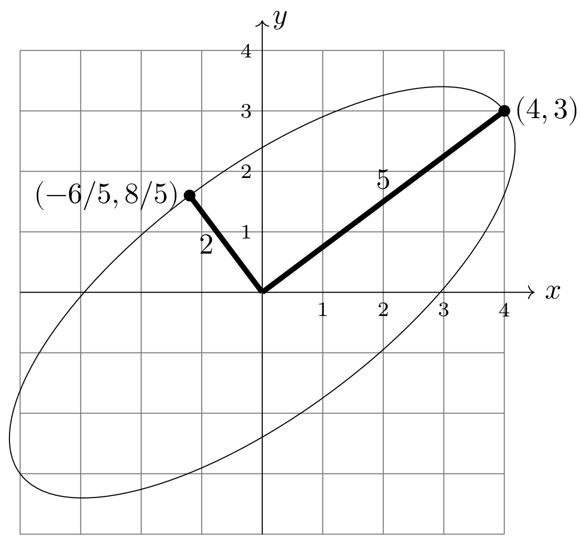
The matrix A has reduced row echelon form $R = \begin{bmatrix} 1 & 5 & 0 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

(a) What is the rank of A?

(b) Find a basis for the null space of A.

(c) Find a matrix A with reduced row echelon form R and such that the equations $A\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ and $A\mathbf{x} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ can both be solved.

Problem 10. (a) Consider the ellipse shown. The axes are drawn in with their lengths and endpoints.



Find a matrix A such that multiplication by A transforms this ellipse into the unit circle.

(b) Suppose A is a matrix with eigenvalue λ and corresponding eigenvector $\mathbf{v}$. Show that the block matrix $\begin{bmatrix} 0 & A \\ A & 0 \end{bmatrix}$ has eigenvalues $\pm\lambda$ and find an eigenvector for each one.

End linear algebra practice for final

Next file:

exam_13.pdf

ES.1803 Practice Questions – Final Quiz, Spring 2024

Important: Not every topic is covered here. When preparing for the final be sure to look over other review materials as well as old psets and exams.

On the final quiz, you will be given the formula posted alongside these problems.

Problem 1. For the DE $\dfrac{dy}{dx} = -\dfrac{y}{x} + 3\,x$:

(a) Sketch the direction field for this DE, using (light or dotted) isoclines for the slopes -1 and 0.

(b) For the solution curve passing through the point (1,2): If Euler's method with step-size $h = 0.1$ were used to approximate $y(1.1)$, would the approximation come out too high or too low? Explain.

(c) For the solution with $y(1) = 2$, compute the Euler approximation to $y(1.1)$ using step-size $h = 0.1$.

(d) The functions $y_1 = x^2$ and $y_2 = x^2 + \frac{1}{x}$ are solutions to this DE. If $y = y(x)$ is the solution satisfying the IC $y(1) = 1.5$, show that $100 \le y(10) \le 100.1$. Do we need to include the equal signs in this inequality? Why or why not?

(e) Find the general solution the DE and verify the prediction of Part (b).

Problem 2. Let $P(D) = D^2 + bD + 5I$ where $D = \frac{d}{dt}$.

(a) For what range of the values of $b \ge 0$ will the solutions to $P(D)y = 0$ exhibit oscillatory behavior?

(b) For $b = 4$, solve the DEs (i) $P(D)y = 4\,e^{2t}\sin(t)$ (ii) $P(D)y = 4\,e^{2t}\cos(t)$

Write your answers in both amplitude-phase and rectangular form.

(c) Given $b = 2$, for what ω does $P(D)y = \cos(\omega t)$ have the biggest response?

Problem 3. Find the general solution to the DE $(D^3 - I)\,y = e^x$

Express the answer using *real*-valued functions only.

Problem 4. Let L denote the differential operator $Ly = D^2y - \frac{1}{x}\,Dy + 4x^2y$, where $D = \frac{d}{dx}$.

(a) Show that the DE $Ly = 0$ has solutions $y_1(x) = \cos\left(x^2\right)$ and $y_2(x) = \sin\left(x^2\right)$.

(b) Show that the initial value problem

$$Ly = 0, \quad y(0) = 0,\ y'(0) = 0$$

has *more than one* solution. Why doesn't this contradict the Existence and Uniqueness Theorem? On what intervals does existence and uniqueness hold?

Problem 5. Suppose that a population of variable size (in some suitable units) $p(t)$ follows the growth law $\dfrac{dp}{dt} = p^3 - 4p^2 + 4p$. Without solving the DE explicitly:

(a) Find all critical points and classify each according to its stability type using a phase line diagram.

(b) Draw a rough sketch (on p-vs.-t axes) of the family of solutions. What happens to the population in the long-run if it starts out at size 1 unit; at size 3 units ?

(c) Explain why the rate equation given by the DE was all we needed to get the answer to Part (b).

(d) Now we'll add a harvesting parameter to the system: $p' = p^3 - 4p^2 + 4p - r$.

(i) Draw the bifurcation diagram for this system.

(ii) Give the bifurcation points.

(iii) For what values of r is the population sustainable?

Problem 6. (a) Solve $2y'' - 2y' - 4y = \delta(t)$, with rest IC.

(b) Solve $2y'' + 2y = \delta(t-3)$, with rest IC.

(c) Solve $x' + tx = \delta(t-5)$

Problem 7. Let $f(t)$ be 0 for $t < 0$ and $3e^{-2t}$ for $t > 0$. Compute the generalized derivative of $f(t)$.

Problem 8. For $f(t) = t$ on $0 < t < 1$:

(a) Sketch the following periodic extensions of f over three or more full periods in the cases.

(i) Even period 2 extension (ii) Odd period 2 extension (iii) Period 1 extension.

In all three cases chose endpoint values that show where the Fourier series expansion will converge. (Do this *without* computing the Fourier series).

(b) Compute the Fourier sine series of f.

(c) Find the periodic solution to the DE $x'' + 10x = \tilde{f}_{\text{odd}}(t)$. Does near-resonance occur in this situation? If so, which frequency in the 'driving force' $\tilde{f}_{\text{odd}}(t)$ produces it?

(d) Solve the DE $x' + 10x = \tilde{f}_{\text{odd}}(t)$.

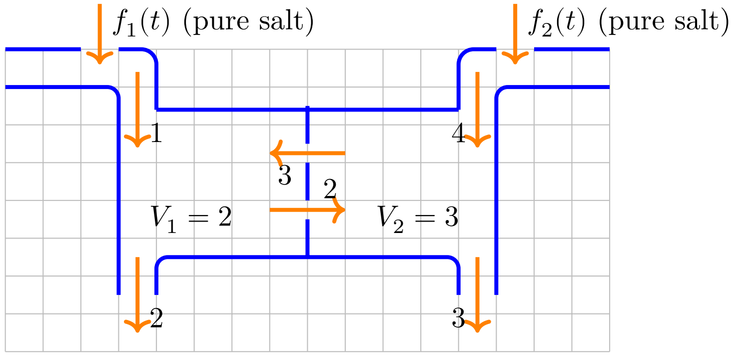
Problem 9. (a) Write down the wave equation with IC's and BC's for the string of length 1, with clamped ends, wave speed 2, initially at equilibrium, struck at time 0. Then derive the Fourier series solution using separation of variables.

(b) Give the explicit solution to the equation of Part (a) when the initial velocity is given by $f(x) = x$ on $0 < x < 1$ (as if that were possible!).

Problem 10. Find the general real-valued solution to the system of DEs:

$$x' = x - 2y, \quad y' = 4x + 3y.$$

Problem 11. Given the following two-tank mixing system with flow rates, inputs and volumes as shown. (All unit are compatible; $f_1(t)$ and $f_2(t)$ denote salt rates in $\frac{\text{mass}}{\text{time}}$.)



(a) Let x and y be the amount of salt in tanks 1 and 2 respectively. Set up a system of DEs modeling x, y.

(b) Suppose the input salt rates $f_1(t)$ and $f_2(t)$ are constant. Show that the system approaches a state in which the final concentrations are constant.

For more problems on linear and nonlinear systems see Practice Quiz 7, Problems 8-11.

End practice final

Next file:

exam_14.pdf

Review of entire semester, Spring 2024

This is a large set of problems covering all the topics. Most are taken from the problem section worksheets.

Topic 1. Modeling; separable DEs

Problem 1. (Here's the second geometry example in the notes for Topic 1.)

$y = y(x)$ is a curve in the first quadrant. The part of the tangent line in the first quadrant is bisected by the point of tangency. Find and solve the DE for this curve.

Problem 2. Consider the family of all lines whose y-intercept is twice the slope.

(a) Find a DE which has this family as its solutions.

(b) Find the orthogonal trajectories to the curves in Part (a). That is, find a family of functions whose graphs intersect all the lines in Part (a) orthogonally.

(c) Sketch both families.

Problem 3. You deposit money in a bank at the rate of \$1000/year. The money earns (continuous) 8% interest. Construct a DE to model the amount of money in the bank as a function of time; then solve the DE. Assume that at time 0 there is no money in the bank.

Topic 2. Linear DEs

Problem 4. (Linear homogeneous)
(a) Solve $y' + ky = 0$.

(b) Solve $y' + ty = 0$.

Problem 5. Solve $y' + ty = t^3$. (Hint: use Part (b) of the previous problem.

Problem 6. (a) Solve $y' + 2y = 2$.

(b) Solve $y' + 2y = 2t$.

(c) Solve $y' + 2y = 5 + 2t$.

Problem 7. (IVP using definite integrals)
Solve $xy' - e^x y = 0, \quad y(1) = 2$ using definite integrals.

Problem 8. Solve $y' + 2y = 2$; $y(1) = 1$.

Problem 9. Show that $y' + y^2 = q$ does not satisfy the superposition principle.

Topic 3. Input response models

Problem 10. Solve the DE $x' + 2x = f(t)$, $x(0) = 0$, where $f(t) = \begin{cases} 6 & \text{for } 0 \le t < 1 \\ 0 & \text{for } 1 \le t < 2 \\ 6 & \text{for } 2 \le t. \end{cases}$

Topic 4. Complex arithmetic and exponentials

Problem 11. Polar coordinates: Write $z = -2 + 3i$ in polar form.

Problem 12. Write $3e^{i\pi/6}$ in rectangular coordinates.

Problem 13. (Trig triangle)
Draw and label the triangle relating rectangular with polar coordinates.

Problem 14. Compute $\dfrac{1}{-2+3i}$ in polar form. Convert the denominator to polar form first. Be sure to describe the polar angle precisely.

Problem 15. Find a formula for $\cos(3\theta)$ in terms of $\cos(\theta)$ and $\sin(\theta)$.

Problem 16. (Roots)
Find all fifth roots of -2. Give them in polar form. Draw a figure showing the roots in the complex plane.

Problem 17. Compute $I = \displaystyle\int e^{2x}\cos(3x)\,dx$ using complex techniques.

Problem 18. (a) Show $\cos(t) = (e^{it} + e^{-it})/2$ and $\sin(t) = (e^{it} - e^{-it})/2i$.

(b) Find all the real-valued functions of the form $\tilde{c}_1 e^{it} + \tilde{c}_2 e^{-it}$.

Problem 19. Find all the real-valued functions of the form $x = \tilde{c}e^{(2+3i)t}$.

Problem 20. Find the 3 cube roots of 1 by locating them on the unit circle and using basic trigonometry.

Problem 21. Express in the form $a + bi$ the 6 sixth roots of 1.

Problem 22. Use Euler's formula to derive the trig addition formulas for sin and cos.

Topic 5. Constant coefficient linear homogeneous DEs; Damping

Problem 23. (a) Solve $x'' - 8x' + 7x = 0$ using the characteristic equation method.

(b) Solve $x'' + 2x' + 5x = 0$ using the characteristic equation method.

(c) Assume the polynomial $r^5 + a_4 r^4 + a_3 r^3 + a_2 r^2 + a_1 r + a_0 = 0$ has roots

$$0.5, \quad 1, \quad 1, \quad 2 \pm 3i.$$

Give the general real-valued solution to the homogeneous constant coefficient DE

$$x^{(5)} + a_4 x^{(4)} + a_3 x^{(3)} + a_2 x'' + a_1 x' + a_0 x = 0.$$

Problem 24. (Unforced second-order physical systems)
The DE $x'' + bx' + 4x = 0$ models a damped harmonic oscillator. For each of the values $b = 0, 1, 4, 5$ say whether the system is undamped, underdamped, critically damped or overdamped.

Sketch a graph of the response of each system with initial condition $x(0) = 1$ and $x'(0) = 0$. (It is not necessary to find exact solutions to do the sketch.)

Say whether each system is oscillatory or non-oscillatory.

Problem 25. State and verify the superposition principle for $mx'' + bx' + kx = 0$, (m, b, k constants).

Problem 26. A constant coefficient, linear, homogeneous DE has characteristic roots

$$-1 \pm 2i,\ -2,\ -2, -3 \pm 4i.$$

(a) What is the order of the DE? (Notice the $\pm$ in the list of roots.)

(b) What is the general, real-valued solution.

(c) Draw the pole diagram for this system. Explain why it shows that all solutions decay exponentially to 0. What is the exponential decay rate of the general solution?

Topic 6. Exponential Response Formula

Problem 27. Let $P(D) = D^2 + 8D + 7$. Find the general real-valued solution to the following.

For oscillatory answers your particular solutions should be in amplitude-phase form.

(a) $P(D)x = e^{2t}$.

(b) $P(D)x = \cos(3t)$.

(c) $P(D)x = e^{2t}\cos(3t)$.

(d) $P(D)x = e^{-t}$.

Topic 7. Undetermined coefficients; Theory

Problem 28. Find the general solution to $x' + 3x = t^2 + 3$

Problem 29. Find one solution to $x''' + 3x'' + 2x' + 5x = 4$.

Problem 30. Find the general solution to $x'' + 3x' = t + 1$.

Topic 8. Stability

Stability is about the system not the input.

Problem 31. Is the system $x'' + x' + 4x = 0$ stable?

Problem 32. Is a 4th order system with roots $\pm 1, -2 \pm 3i$ stable. Which solutions to the homogeneous DE go to 0 as $t \to \infty$?

Problem 33. For what k is the system $x' + kx = 0$ stable?

Topic 9. Amplitude response, resonance and practical resonance

Problem 34. Consider the system $x'' + 8x = F_0 \cos(\omega t)$.

(a) Why is this called a driven undamped system?

(b) Solve this using the sinusoidal response formula (SRF). Then do it again using complex replacement and the exponential response formula (ERF).

(c) Consider the right hand side of the DE to be the input. Graph the amplitude response function.

(d) What is the resonant frequency of the system?

(e) Why is this called the natural frequency?

Problem 35. Consider the forced damped system: $x'' + 2x' + 9x = \cos(\omega t)$.

(a) What is the natural frequency of the system?

(b) Find the response of the system in amplitude-phase form.

(c) Consider the right hand side of the DE to be the input. What is the amplitude response of the system? Draw its graph –be sure to label your axes correctly

(d) What is the practical resonant frequency?

(e) When $\omega = \sqrt{7}$ by how many radians does the output peak lag behind the input peak?

(f) For the forced undamped system $x'' + 9x = \cos(\omega t)$ give a detailed description of the phase lag for different input frequencies?

Problem 36. Consider the driven first-order system: $x' + kx = kF_0 \cos(\omega t)$. We'll take the input to be $F_0 \cos(\omega t)$. Solve the DE. Find the amplitude response. Show there is never practical resonance.

Topic 10. Direction fields, integral curves
Topic 11. Numerical methods: Euler's method

Problem 37. Consider $y' = x^2 - y^2$

(a) Sketch the nullcline. Use it to label the regions of the plane where the slope field has positive slope as + and negative slope as −. Use this to give a very rough sketch of some solution curves.

Note: the nullcline consists of two lines.

(b) Start a new graph. Add the nullcline, some isoclines with direction field elements, and sketch some solution curves.

(c) Add some integral curves to the plot in Part (b). Include the one with $y(2) = 0$.

(d) Use squeezing to estimate $y(100)$ for the solution with IC $y(2) = 0$.

(e) Use Euler's method with $h = 0.5$ to estimate $y(3)$ for the solution with $y(2) = 0$.

(f) Is the estimate in Part (e) too high or too low?

Topic 12. Autonomous first-order DEs

Problem 38. Let $x' = x(x-a)(x-3)$

(a) Let $a = 1$ draw phase line. Identify type of each critical point, sketch solution graphs.

(b) Considering a to be a parameter: draw the bifurcation diagram: identify the stable and unstable branches.

(c) If this models a population, for what a is the population sustainable?

Topic 13. Linear algebra: linearity, vector spaces, connection to DEs
Topic 14. Linear algebra: row reduction, column space, pivots

Problem 39. Solve this system of linear equations. How many methods can you think of to solve this system?

$$\begin{aligned} x + y &= 5 \\ 3x + 2y &= 7 \end{aligned}$$

Problem 40. Let $R = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Suppose R is the row reduced echelon form for A.

(a) What is the rank of A?

(b) Find a basis for the null space of A.

(c) Suppose the column space of A has basis $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$. Find a possible matrix for A. That is, give a matrix A with RREF R and the given column space.

(d) Find a matrix with the same row reduced echelon form, but such that $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ are in its column space.

Problem 41. Consider the following system of equations:

$$\begin{aligned} x + y + z &= 5 \\ x + 2y + 3z &= 7 \\ x + 3y + 6z &= 11 \end{aligned}$$

(a) Write this system of equations as a matrix equation.

(b) Use row reduction to get to row echelon form. What is the solution set?

Problem 42. Solve the following equation using row reduction:

$$\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

(a) At the end of the row-reduction process, was the last column pivotal or free? Is this related to the absence of solutions?

(b) Find a new vector $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ such that $\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ has a solution.

Problem 43. Consider the matrix $A = \begin{bmatrix} 1 & 2 & 3 & 1 & 2 \\ 2 & 4 & 6 & 2 & 4 \\ 0 & 0 & 10 & 3 & 6 \end{bmatrix}$. Put A in row reduced echelon form. Find the column space, null space, rank, a basis for the column space, a basis for the null space, the dimension of each of the spaces.

Problem 44. **(a)** Suppose we have a matrix equation

$$\begin{bmatrix} \bullet \\ \bullet \end{bmatrix} \begin{bmatrix} \bullet & \bullet \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & x \end{bmatrix}$$

Can you specify x? For any value of x you think is allowable, find such an equation. Can any of the $\bullet$'s be 0?

(b) Suppose we have a matrix equation

$$\begin{bmatrix} \bullet & 3 \\ \bullet & 4 \\ \bullet & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Can you specify the $\bullet$'s?

(c) Suppose we have a matrix equation

$$\begin{bmatrix} x & 3 \\ y & 4 \\ z & 5 \end{bmatrix} \mathbf{c} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

and all we know about the vector $\mathbf{c}$ is that $\mathbf{c} \neq \mathbf{0}$. What can we say about $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$?

Problem 45. Suppose we have a matrix equation

$$\begin{bmatrix} 1 & x & 2 \\ 3 & y & 4 \\ 5 & z & 6 \end{bmatrix} \mathbf{c} = \mathbf{0}$$

and all we know about the vector $\mathbf{c}$ is that $\mathbf{c} \neq \mathbf{0}$. What can we say about $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$?

Problem 46. For what values of y is it the case that the columns of $\begin{bmatrix} 1 & 1 & 2 \\ 3 & y & 4 \\ 5 & 1 & 6 \end{bmatrix}$ form a linearly independent set?

Problem 47. For the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix}$:

(a) Find the row reduced echelon form of A; call it R.

(b) The last column of R should be a linear combination of the first columns in an obvious way. This is a linear relation among the columns of R. Find a vector $\mathbf{x}$, such that $R\mathbf{x} = \mathbf{0}$, which expresses this linear relationship.

(c) Verify that the same relationship holds among the columns of A.

(d) Explain why the linear relations among the columns af R are the same as the linear relations among the columns of A. In fact, explain why, if A and B are related by row transformations, the linear relations among the columns of A are the same as the linear relations among the columns of B.

Problem 48. This continues the previous problem. Now, suppose we want to solve $A\mathbf{x} = \mathbf{b}$, where, again, $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix}$.

(a) When is this possible? Answer this in the form: "$\mathbf{b}$ must be a linear combination of the two vectors …"

(b) $A\mathbf{x} = \mathbf{b}$ is certainly solvable for $\mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. (What is the obvious particular solution?) Describe the general solution to this equation, as $\mathbf{x} = \mathbf{x_p} + \mathbf{x_h}$.

Problem 49. Suppose that the row reduced echelon form of the 4×6 matrix B is

$$R = \begin{bmatrix} 0 & 1 & 2 & 3 & 0 & 5 \\ 0 & 0 & 0 & 1 & 0 & 7 \\ 0 & 0 & 0 & 0 & 1 & 9 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) Find a linearly independent set of vectors of which every vector in the null space of B is a linear combination.

(b) Write the columns of B as $\mathbf{b_1}, \mathbf{b_2}, \ldots, \mathbf{b_6}$. What is $\mathbf{b_1}$? What can we say about $\mathbf{b_2}$? Which of these vectors are linearly independent of the preceding ones? Express the ones which are not independent as explicit linear combinations of the previous ones. Describe a linearly independent set of vectors of which every vector in the column space of B is a linear combination.

Topic 15. Linear algebra: transpose, inverse, determinant

Problem 50. Compute the transpose of the following matrices.

(a) $A = \begin{bmatrix} 6 & 5 \\ 1 & 2 \end{bmatrix}$, $\quad B = \begin{bmatrix} 1 & 2 & 4 \\ 8 & 16 & 32 \end{bmatrix}$, $\quad C = \begin{bmatrix} 1 & 2 \\ 4 & 8 \\ 16 & 32 \end{bmatrix}$

(b) Verify that $(AB)^T = B^T A^T$ where A and B are from Part (a).

Summary of properties of the determinant

(0) $\det A$ is a number determined by a square matrix A.
(1) $\det I = 1$.
(2) Adding a multiple of one row to another does not change the determinant.
(3) Multiplying a row by a number a multiplies the determinant by a.
(4) Swapping two rows reverses the sign of the determinant.
(5) $\det(AB) = \det(A)\det(B)$.
(6) A is invertible exactly when $\det A \neq 0$.

Problem 51. Compute the determinants of the following matrices, and if the determinant is nonzero find the inverse.

(i) $\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}$ (iii) $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ (iv) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$.

Problem 52. (Rotation matrices)
Let $R(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$
Compute $\det R(\theta)$ and $R(\theta)^{-1}$.

Topic 16. Linear algebra: eigenvalues, diagonal matrices, decoupling

Problem 53. (a) Find the eigenvalues and basic eigenvectors of $A = \begin{bmatrix} -3 & 13 \\ -2 & -1 \end{bmatrix}$.

(b) Find the eigenvalues and eigenvectors of $B = \begin{bmatrix} -3 & 4 \\ 2 & -5 \end{bmatrix}$.

Problem 54. Suppose that the matrix B has eigenvalues 1 and 7, with eigenvectors

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix} \text{ and } \begin{bmatrix} 5 \\ 1 \end{bmatrix}$$

respectively.

(a) What is the solution to $\mathbf{x}' = Bx$ with $x(0) = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$?

(b) Decouple the system $\mathbf{x}' = B\mathbf{x}$. That is, make a change of variables so that system is decoupled. Write the DE in the new variables.

(c) Give an argument based on transformations why $B = \begin{bmatrix} 1 & 5 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 7 \end{bmatrix} \begin{bmatrix} 1 & 5 \\ -1 & 1 \end{bmatrix}^{-1}$ has the eigenvalues and eigenvectors given above.

Problem 55. Suppose $A = \begin{bmatrix} a & b & c \\ 0 & 2 & e \\ 0 & 0 & 3 \end{bmatrix}$.

(a) What are the eigenvalues of A?

(b) For what value (or values) of a, b, c, e is A singular (non-invertible)?

(c) What is the minimum rank of A (as a, b, c, e vary)? What's the maximum?

(d) Suppose $a = -5$. In the system $\mathbf{x}' = A\mathbf{x}$, is the equilibrium at the origin stable or unstable.

Problem 56. Suppose that $A = S \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} S^{-1}$.

(a) What are the eigenvalues of A?

(b) Express A^2 and A^{-1} in terms of S.

(c) What would I need to know about S in order to write down the most rapidly growing exponential solution to $\mathbf{x}' = A\mathbf{x}$?

Problem 57. We didn't cover orthogonal matrices. They won't be on the final.

(a) An orthogonal matrix is one where the columns are orthonormal (mutually orthogonal and unit length). Equivalently, S is orthogonal if $S^{-1} = S^T$.

Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$. Find an orthogonal matrix S and a diagonal matrix Λ such that $A = S\Lambda S^{-1}$

(b) Decouple the equation $\mathbf{x}' = A\mathbf{x}$, with $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$.

Topic 17. Matrix methods for solving systems of DEs. The companion matrix

Problem 58. **(a)** Let $A = \begin{bmatrix} 4 & -3 \\ 6 & -7 \end{bmatrix}$. Solve $\mathbf{x}' = A\mathbf{x}$.

Problem 59. Solve $x' = -3x + y, \quad y' = 2x - 2y$.

Problem 60. (**Complex roots**) Solve $\mathbf{x}' = \begin{bmatrix} 7 & -5 \\ 4 & 3 \end{bmatrix} \mathbf{x}$ for the general real-valued solution.

Problem 61. *Don't dwell on the computations for this problem. Just look at the final result.*

(**Repeated roots**) Solve $\mathbf{x}' = \begin{bmatrix} 3 & 1 \\ -1 & 1 \end{bmatrix} \mathbf{x}$.

Problem 62. Solve the system $x' = x + 2y; \quad y' = -2x + y$.

Topic 20. Step and delta functions

Problem 63. Compute the following integrals.

(a) $\displaystyle\int_{-\infty}^{\infty} \delta(t) + 3\delta(t-2)\, dt$

(b) $\displaystyle\int_1^5 \delta(t) + 3\delta(t-2) + 4\delta(t-6)\,dt.$

Problem 64. Compute the following integrals.

(a) $\displaystyle\int_{0^-}^{\infty} \cos(t)\delta(t) + \sin(t)\delta(t-\pi) + \cos(t)\delta(t-2\pi)\,dt.$

(b) $\displaystyle\int \delta(t)\,dt.$ (Indefinite integral)

(c) $\displaystyle\int \delta(t) - \delta(t-3)\,dt.$ Graph the solution

(d) Make up others.

Problem 65. Solve $x' + 2x = \delta(t) + \delta(t-3)$ with rest IC

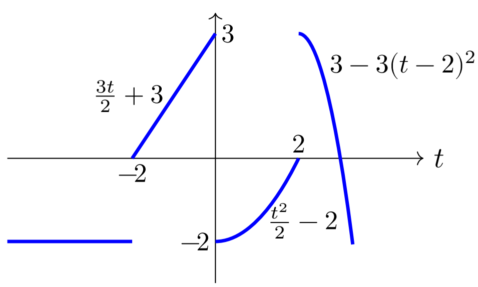
Problem 66. (Second-order systems) Solve $4x'' + x = 5\delta(t)$ with rest IC.

Problem 67. Solve $x' + 3x = \delta(t) + e^{2t}u(t) + 2\delta(t-4)$ with rest IC.
(The $u(t)$ is there to make sure the input is 0 for $t < 0$.)

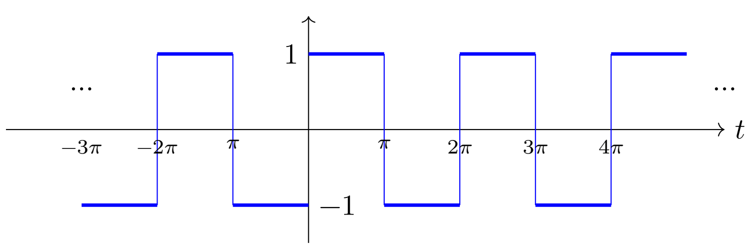
Problem 68. **(a)** Solve $2x'' + 8x' + 6x = \delta(t)$ with rest IC.

(b) Plug your solution into the DE and verify that it is correct

Problem 69. The graph of the function $f(t)$ is shown below. Compute the generalized derivative $f'(t)$. Identify the regular and singular parts of the derivative.



Problem 70. Derivative of a square wave
The graph below is of a function $\mathrm{sq}(t)$ (called a square wave). Compute and graph its generalized derivative.



Graph of $\mathrm{sq}(t)$ = square wave

Topic 21. Fourier series
Topic 22. Fourier series continuation

Problem 71. For each of the following:

(i) Find the Fourier series (no integrals needed)

(ii) Identify the fundamental frequency and corresponding base frequency.

(iii) Identify the Fourier coefficients a_n and b_n

(a) $\cos(2t)$

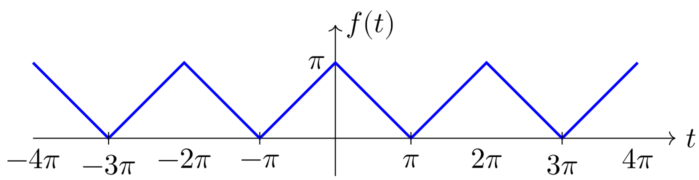
(b) $3\cos(2t - \pi/6)$

(c) $\cos(t) + 2\cos(5t)$

(d) $\cos(3t) + \cos(4t)$

Problem 72. Compute the Fourier series for the odd, period 2, amplitude 1 square wave. (Do this by computing integrals –not starting with the period 2π square wave.)

Problem 73. Compute the Fourier series for the period 2π triangle wave shown.



Topic 23. Fourier: calculation; sine and cosine series

Problem 74. Find the Fourier cosine series for the function $f(x) = x^2$ on $[0, 1]$. Graph the function and its even period 2 extension.

Problem 75. Find the Fourier series for the standard square wave shifted to the left so it's an even function, i.e., $sq(t + \pi/2)$.

Problem 76. Find the Fourier sine series for $f(t) = 30$ on $[0, \pi]$.

Topic 24. Fourier: ODEs

Problem 77. Solve $x' + kx = f(t)$, where $f(t)$ is the period 2π triangle wave with $f(t) = |t|$ on $[-\pi, \pi]$. (You can use the known series for $f(t)$.)

Problem 78. Solve $x'' + x' + 8x = g(t)$, where $g(t)$ is the period 2 triangle wave with $g(t) = |t|$ on $[-1, 1]$.

Problem 79. Solve $x'' + 16x = \displaystyle\sum_{n=1}^{\infty} \frac{\cos(nt)}{n^2(n^2-2)^2}$. Look out for resonance.

Topic 25. PDEs: separation of variables
Topic 26. Continuation

Problem 80. Let L=1. Solve the wave equation with boundary and initial conditions.

PDE: $y_{tt} = y_{xx}$

BC: $y(0,t) = 0$, $y(1,t) = 0$

IC: $y(x,0) = 30$, $y_t(x,0) = 0$.

Problem 81. Solve the heat equation with insulated ends.

(This problem uses a cosine series, so the $\lambda = 0$ case is important.)

(PDE) $u_t = 5u_{xx}$ for $0 \leq x \leq \pi$, $t > 0$.

(BC) $u_x(0,t) = 0$, $u_x(\pi,t) = 0$

(IC) $u(x,0) = x$.

Topic 27. Qualitative behavior of linear systems, phase portraits

Problem 82. Draw a phase protrait of $\mathbf{x}' = A\mathbf{x}$, where $A = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$. What type of critical point is at the origin? Is it dynamically stable?

Problem 83. Draw a phase protrait of $\mathbf{x}' = A\mathbf{x}$, where $A = \begin{bmatrix} -1 & 2 \\ -2 & -1 \end{bmatrix}$. What type of critical point is at the origin? Is it dynamically stable?

Problem 84. Draw a phase protrait of $\mathbf{x}' = A\mathbf{x}$, where $A = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$. What type of critical point is at the origin? Is it dynamically stable?

Problem 85. Draw a phase protrait of $\mathbf{x}' = A\mathbf{x}$, where $A = \begin{bmatrix} 1 & 1 \\ 3 & 1 \end{bmatrix}$. What type of critical point is at the origin? Is it dynamically stable?

Problem 86. Draw the trace-determinant diagram. Label all the parts with the type and dynamic stability of the critical point at the origin. Which types represent structurally stable systems?

(b) Give the equation for the parabola in the diagram. Explain where is comes from.

Problem 87. Consider the linear system $\mathbf{x}' = A\mathbf{x}$.

(a) Suppose A has $\text{tr}(A) = -2.5$ and $\det(A) = 1$. Locate this system on the trace-determinant diagram. For this system, what is the type of the critical point at the origin?

(b) Compute the eigenvalues of this system and verify your answer in Part (a).

Problem 88. For each of the following linear systems, sketch phase portraits. Give the dynamic stability of the critical point at the origin. Give the structural stability of the system.

(a) $\mathbf{x}' = \begin{bmatrix} 5 & 1 \\ -4 & 10 \end{bmatrix}$

(b) $\mathbf{x}' = \begin{bmatrix} -7 & -3 \\ 3 & -17 \end{bmatrix}$

(c) $\mathbf{x}' = \begin{bmatrix} 5 & 3 \\ 0 & -2 \end{bmatrix}$

(d) $\mathbf{x}' = \begin{bmatrix} 5 & 5 \\ -5 & -1 \end{bmatrix}$

(e) $\mathbf{x}' = \begin{bmatrix} 3 & -4 \\ 4 & -3 \end{bmatrix}$

(f) $\mathbf{x}' = \begin{bmatrix} -4 & 4 \\ -1 & 0 \end{bmatrix}\mathbf{x}$

Topic 28. Qualitative behavior of non-linear systems
Topic 29. Structural stability

Problem 89. **(a)** Sketch the phase portrait for $x' = -x + xy$, $y' = -2y + xy$.

(b) Consider x and y to be the sizes of two interacting populations. Tell a story about the populations.

Problem 90. Sketch the phase portrait for $x' = x^2 - y$, $y' = x(1-y)$.

Draw one phase portrait for each possibility for the non-structurally stable critical point.

Problem 91. Consider the system: $x' = x - 2y + 3$, $y' = x - y + 2$.

(a) Find the one critical point and linearize at it. For the linearized system, what is the type of the critical point?

(b) In Part (a) you should have found that the linearized system is a center. Since this is not structurally stable, it is not necessarily true that the nonlinear system has a center at the critical point. Nonetheless, in this case, it does turn out to be a nonlinear center. Prove this.

Problem 92. For the following system, draw the phase portrait by linearizing at the critical points.

$$x' = 1 - y^2, \quad y' = x + 2y.$$

Problem 93. For the following system, draw the phase portrait by linearizing at the critical points.

$$x' = x - y - x^2 + xy, \quad y' = -y - x^2.$$

Topic 30. Population models

Problem 94. Let $x(t)$ be the population of sharks off the coast of Massachusetts and $y(t)$ the population of fish. Assume that the populations satisfy the Volterra predator-prey equations

$$x' = ax - pxy; \qquad y' = -by + qxy, \qquad \text{where } a,\ b,\ p,\ q, \text{ are positive.}$$

Assume time is in years and a and b have units 1/years.

Suppose that, in a few years, warming waters start killing 10% of both the fish and the sharks each year. Show that the shark population will actually increase.

Problem 95. Consider the system of equations

$$x'(t) = 39x - 3x^2 - 3xy; \qquad y'(t) = 28y - y^2 - 4xy.$$

The four critical points of this system are (0,0), (13,0), (0,28), (5,8).

(a) Show that the linearized system at (0,0) has eigenvalues 39 and 28. What type of critical point is (0,0)?

(b) Linearize the system at (13,0); find the eigenvalues; give the type of the critical point.

(c) Repeat Part (b) for the critical point (0,28).

(d) Repeat Part (b) for the critical point (5,8).

(e) Sketch a phase portrait of the system. If this models two species, what is the relationship between the species? What happens in the long-run?

Problem 96. The system for this equation is

$$\begin{aligned} x' &= 4x - x^2 - xy \\ y' &= -y + xy \end{aligned}$$

(a) This models two populations with a predator-prey relationship. Which variable is the predator population?

(b) What would happen to the predator population in the absence of prey? What about the prey population in the absence of predators?

(c) There are three critical points. Find and classify them

(d) Sketch a phase portrait of this system. What is the relationship between the species? What happens in the long-run?

Problem 97. The equations for this system are

$$\begin{aligned} x' &= x^2 - 2x - xy \\ y' &= y^2 - 4y + xy \end{aligned}$$

(a) If this models two populations, what would happen to each of the populations in the absence of the other?

(b) There are four critical points. Find and classify them

(c) Sketch a phase portrait of the system. What is the relationship between the species? What happens in the long-run?

Topic 31. Physical models: the pendulum

We didn't cover physical models in class. This is still good practice for creating and interpreting phase portraits.

Problem 98. Nonlinear Spring
The following DE models a nonlinear spring:

$$m\ddot{x} = -kx + cx^3 \qquad \begin{cases} \textbf{hard if } c < 0 & \text{(cubic term adds to linear force)} \\ \textbf{soft if } c > 0 & \text{(cubic term opposes linear force).} \end{cases}$$

(a) Convert this to a companion system of first-order equations.

(b) Sketch a phase portrait of the system for both the hard and soft springs. You can use the fact that the linearized centers are also nonlinear centers. (This follows from energy considerations.)

(c) (Challenge! For anyone who is interested. This is not part of the ES.1803 syllabus.) Find equations for the trajectories of the system.

Problem 99. Nonlinear Spring
The following DE models a nonlinear spring:

$$m\ddot{x} = -kx + cx^3 \qquad \begin{cases} \textbf{hard if } c < 0 & \text{(cubic term adds to linear force)} \\ \textbf{soft if } c > 0 & \text{(cubic term opposes linear force).} \end{cases}$$

(a) Convert this to a companion system of first-order equations.

(b) Sketch a phase portrait of the system for both the hard and soft springs. You can use the fact that the linearized centers are also nonlinear centers. (This follows from energy considerations.)

(c) (Challenge! For anyone who is interested. This is not part of the ES.1803 syllabus.) Find equations for the trajectories of the system.

Next file:

exam_2.pdf

18.03 Hour Exam II
March 17, 2010

Your Name
Your Recitation Leader's Name
Your Recitation Time

Problem	Points
1	
2	
3	
4	
5	
Total	

Do not open this booklet till told to do so. There are five problems. Use your test-taking skills—be sure you get to all the problems. Do all your work on these pages. No calculators or notes may be used. The point value (out of 100) of each problem is marked in the margin. Solutions will be available on the web after 4:00 today, and at recitation.

[8] **1. (a)** For what value of k is the system represented by $\ddot{x}+\dot{x}+kx=0$ critically damped?

[4] **(b)** For k greater than that value, is the system overdamped or underdamped?

[8] **(c)** Suppose a solution of $\ddot{x}+\dot{x}+kx=0$ vanishes at $t=1$, and then again for $t=2$ (but not in between). What is k?

[10] **2. (a)** Find a solution of $\ddot{x} + x = 5te^{2t}$.

[10] **(b)** Suppose that $y(t)$ is a solution of the same equation, $\ddot{x}+x = 5te^{2t}$, such that $y(0) = 1$ and $\dot{y}(0) = 2$. (This is probably *not* the solution you found in **(a)**.) Use $y(t)$ and other functions to write down a solution $x(t)$ such that $x(0) = 3$ and $\dot{x}(0) = 5$.

[10] **3. (a)** Consider the equation $\ddot{x} + b\dot{x} + kx = \cos(\omega t)$. We will vary the spring constant but keep b fixed. For what value of k is the amplitude of the sinusoidal solution of $\ddot{x} + b\dot{x} + kx = \cos(\omega t)$ maximal? (Your answer will be a function of ω and may depend upon b as well.)

[10] **(b)** (Unrelated to the above.) Find the general solution of $\dfrac{d^3x}{dt^3} - \dfrac{dx}{dt} = 0$.

4. A certain system has input signal y and system response x related by the differential equation $\ddot{x} + \dot{x} + 6x = 6y$. It is subjected to a sinusoidal input signal.

[10] **(a)** Calculate the complex gain $H(\omega)$.

[5] **(b)** Compute the gain at $\omega = 2$.

[5] **(c)** Compute the phase lag at $\omega = 2$.

5. Suppose that $\frac{1}{2}t\sin(2t)$ is a solution to a certain equation $m\ddot{x} + b\dot{x} + kx = 4\cos(2t)$.

[4] **(a)** Write down a solution to $m\ddot{x} + b\dot{x} + kx = 4\cos(2t-1)$.

[4] **(b)** Write down a solution to $m\ddot{x} + b\dot{x} + kx = 8\cos(2t)$.

[12] **(c)** Determine m, b, and k.

Next file:

exam_3.pdf

18.03 Hour Exam III

April 23, 2010

Your Name
Your Recitation Leader's Name
Your Recitation Time

Problem	Points
1	
2	
3	
4	
5	
Total	

Do not open this booklet till told to do so. There are five problems. Use your test-taking skills—be sure you get to all the problems. Do all your work on these pages. No calculators or notes may be used. The point value (out of 100) of each problem is marked in the margin. Solutions will be available on the web after 4:00 today, and at recitation.

There is a page of formulas at the back of the exam.

1. A certain periodic function has Fourier series

$$f(t) = 1 + \frac{\cos(\pi t)}{2} + \frac{\cos(2\pi t)}{4} + \frac{\cos(3\pi t)}{8} + \frac{\cos(4\pi t)}{16} + \cdots$$

[4] **(a)** What is the minimal period of $f(t)$?

[4] **(b)** Is $f(t)$ even, odd, neither, or both?

[8] **(c)** Please give the Fourier series of a periodic solution (if one exists) of

$$\ddot{x} + \omega_n^2 x = f(t)$$

[4] **(d)** For what values of ω_n is there no periodic solution?

2. Let $f(t) = (u(t+1) - u(t-1))t$.

[6] **(a)** Sketch a graph of $f(t)$.

[6] **(b)** Sketch a graph of the generalized derivative $f'(t)$.

[8] **(c)** Write a formula for the generalized derivative $f'(t)$, and identify in your formula the regular part $f'_r(t)$ and the singular part $f'_s(t)$.

3. Let $p(D)$ be the operator whose unit impulse response is given by $w(t) = e^{-t} - e^{-3t}$.

[10] **(a)** Using convolution, find the unit step response of this operator: the solution to $p(D)v = u(t)$ with rest initial conditions.

[5] **(b)** What is the transfer function $W(s)$ of the operator $p(D)$?

[5] **(c)** What is the characteristic polynomial $p(s)$?

[10] **4 (a)** Find a generalized function $f(t)$ with Laplace transform $F(s) = \dfrac{e^{-s}(s-1)}{s}$.

[10] **(b)** Find a function $f(t)$ with Laplace transform $F(s) = \dfrac{s+10}{s^3+2s^2+10s}$.

5. Let $W(s) = \dfrac{s+10}{s^3+2s^2+10s}$.

[10] **(a)** Sketch the pole diagram of $W(s)$.

[10] **(b)** If $p(D)$ is the operator with transfer function $W(s)$, what is the Laplace transform of the solution, with rest initial conditions, of $p(D)x = \sin(2t)$?

Properties of the Laplace transform

0. Definition: $\mathcal{L}[f(t)] = F(s) = \int_{0-}^{\infty} f(t)e^{-st}\,dt$ for $\operatorname{Re} s \gg 0$.

1. Linearity: $\mathcal{L}[af(t) + bg(t)] = aF(s) + bG(s)$.

2. Inverse transform: $F(s)$ essentially determines $f(t)$.

3. s-shift rule: $\mathcal{L}[e^{at}f(t)] = F(s-a)$.

4. t-shift rule: $\mathcal{L}[f_a(t)] = e^{-as}F(s)$, $\quad f_a(t) = \begin{cases} f(t-a) & \text{if } t > a \\ 0 & \text{if } t < a \end{cases}$.

5. s-derivative rule: $\mathcal{L}[tf(t)] = -F'(s)$.

6. t-derivative rule: $\mathcal{L}[f'(t)] = sF(s)$, where $f'(t)$ denotes the generalized derivative.

$\mathcal{L}[f_r'(t)] = sF(s) - f(0+)$ if $f(t)$ is continuous for $t > 0$.

7. Convolution rule: $\mathcal{L}[f(t) * g(t)] = F(s)G(s)$, $f(t) * g(t) = \int_0^t f(t-\tau)g(\tau)d\tau$.

8. Weight function: $\mathcal{L}[w(t)] = W(s) = 1/p(s)$, $w(t)$ the unit impulse response.

Formulas for the Laplace transform

$$\mathcal{L}[1] = \frac{1}{s} \qquad \mathcal{L}[e^{at}] = \frac{1}{s-a} \qquad \mathcal{L}[t^n] = \frac{n!}{s^{n+1}}$$

$$\mathcal{L}[\cos(\omega t)] = \frac{s}{s^2+\omega^2} \qquad \mathcal{L}[\sin(\omega t)] = \frac{\omega}{s^2+\omega^2}$$

$$\mathcal{L}[t\cos(\omega t)] = \frac{2\omega s}{(s^2+\omega^2)^2} \qquad \mathcal{L}[t\sin(\omega t)] = \frac{s^2-\omega^2}{(s^2+\omega^2)^2}$$

Fourier coefficients for periodic functions of period 2π:

$$f(t) = \frac{a_0}{2} + a_1\cos(t) + a_2\cos(2t) + \cdots + b_1\sin(t) + b_2\sin(2t) + \cdots$$

$$a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(mt)\,dt, \qquad b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(mt)\,dt$$

If $\operatorname{sq}(t)$ is the odd function of period 2π which has value 1 between 0 and π, then

$$\operatorname{sq}(t) = \frac{4}{\pi}\left(\sin(t) + \frac{\sin(3t)}{3} + \frac{\sin(5t)}{5} + \cdots\right)$$

Next file:

exam_4.pdf

18.03 Final Examination

9:00–12:00, May 18, 2010

Your Name	
Recitation Leader	
Recitation Time	

Do not turn the page until you are instructed to do so.

Write your name, your recitation leader's name, and the time of your recitation. Show all your work on this exam booklet. When a particular method is requested you must use it. No calculators or notes may be used, but there is a table of Laplace transforms and other information at the end of this exam booklet. Point values (out of a total of 360) are marked on the left margin. The problems are numbered 1 through 10.

1	
2	
3	
4	
5	
6	
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8	
9	
10	

1. **(a)** and **(b)** concern the tritium that is leaking from the Oyster Creek Nuclear Generating Station into the aquifer in New Jersey at a certain rate, which we will assume is one kilogram per year. The half life of tritium is 12 years.

[6] **(a)** Ignoring other effects (other sources or sinks of the tritium), set up a differential equation for the amount of tritium in the aquifer as a function of time. (For full credit, determine any constants in the equation.)

[6] **(b)** If this leak goes on for a long time, what will the tritium load in the aquifer be? (How many kilograms?)

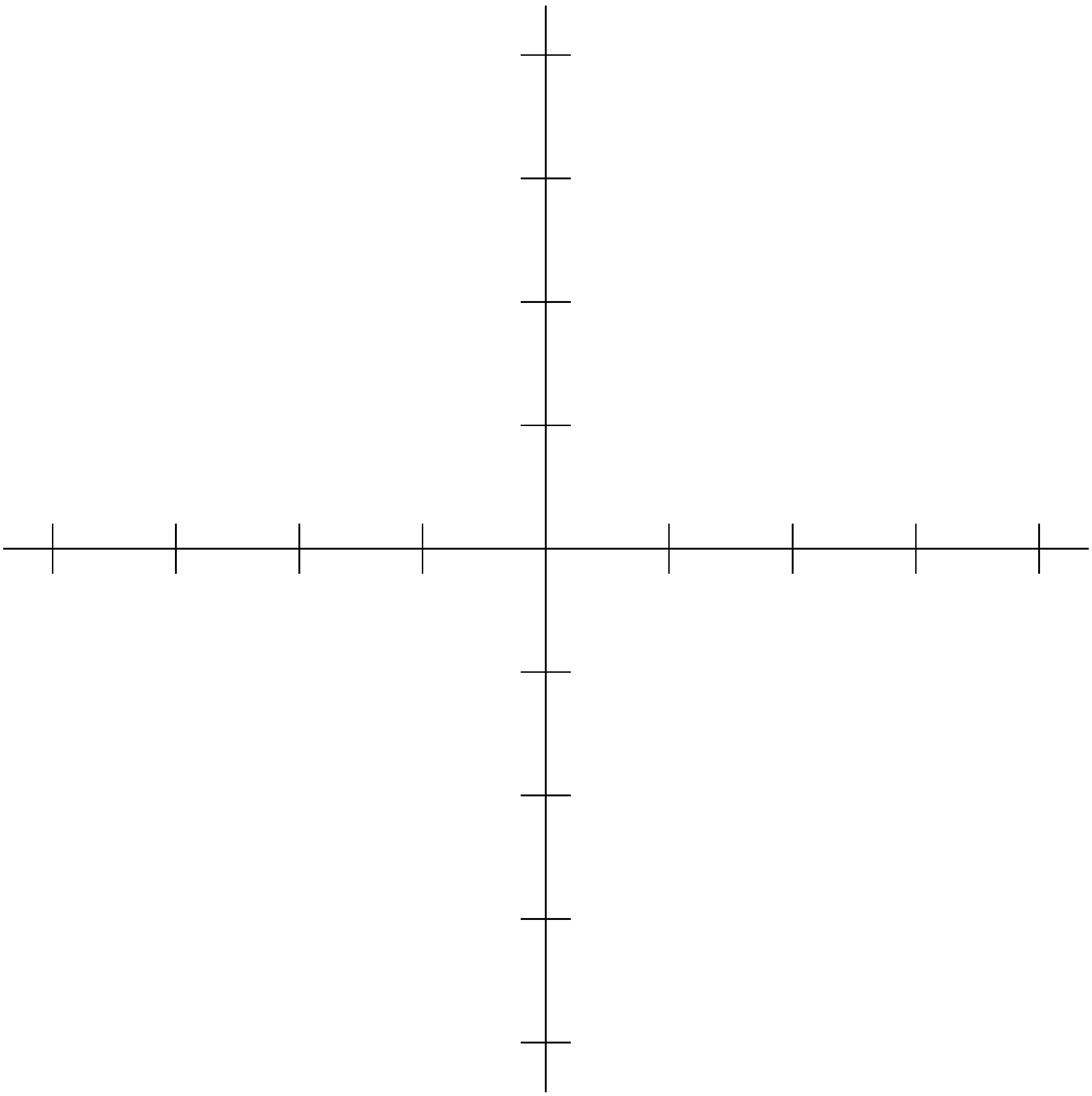
Parts **(c)**–**(g)** of this problem will concern the differential equation $\dfrac{dy}{dx} = x - \dfrac{y^2}{4}$.

[6] **(c)** Let $y(x)$ denote the solution to this equation such that $y(1) = 0$. Use Euler's method with step size $\frac{1}{2}$ to estimate $y(2)$.

1. Continuing with $\dfrac{dy}{dx} = x - \dfrac{y^2}{4}$

[3] **(d)** Sketch the isoclines for slopes $m = -1$, $m = 0$, and $m = 1$, on the plane below.

[3] **(e)** On the same plane, sketch the graph of the solution $y(x)$ with $y(1) = 0$.



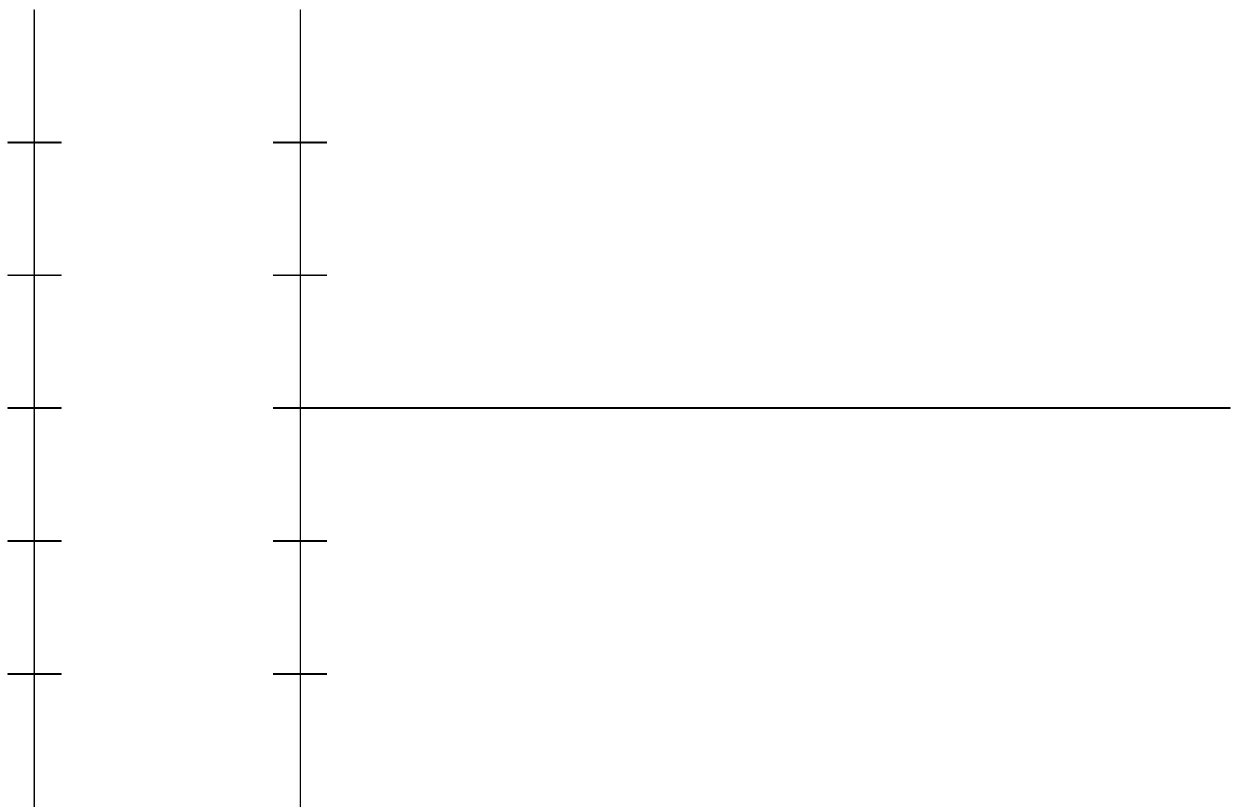
[6] **(f)** For this same solution, suppose that $y(x)$ achieves a minimum at $x = a$. What is $y(a)$ (in terms of a)?

[6] **(g)** Estimate the value of $y(100)$.

2. In **(a)**–**(c)** we consider the autonomous equation $\dot{x} = x^3 - x^2 - 2x$.

[10] **(a)** On the vertical line below, sketch the phase line of this equation.

[10] **(b)** Sketch the graphs of some solutions. Be sure to include at least one solution with values in each interval above, below, and between the critical points.



[6] **(c)** Suppose $x(0)$ is quite small, say 0.1. For $t > 0$, $x(t)$ is best approximated by $(0.1)e^{at}$ for what value of a?

[10] **(d)** Solve the initial value problem $x\dfrac{dy}{dx} + 2y = -\dfrac{\sin(x)}{x}$, $y(\pi) = 0$.

[6] **3. (a)** Find a complex number r (expressed as $r = a + bi$ with a, b real) and a positive real number ω such that $\mathrm{Re}\left(\frac{e^{i\omega t}}{r}\right) = 2\cos\left(2t - \frac{\pi}{4}\right)$.

[6] **(b)** Express the cube roots of $-8i$ in the form $a + bi$ (with a and b real).

[8] **(c)** What is the pseudoperiod of a nonzero solution to $\ddot{x} + 2\dot{x} + 10x = 0$?

[8] **3.** (continued) **(d)** At what circular frequency $\omega = \omega_r$ does the sinusoidal solution to $\ddot{x} + 2\dot{x} + 10x = \cos(\omega t)$ have the largest amplitude?

[8] **(e)** At what circular frequency $\omega = \omega_p$ is the phase lag of the solution to $\ddot{x}+2\dot{x}+10x = \cos(\omega t)$ equal to $\frac{\pi}{2}$?

4. Let $p(D) = D^2 + 4D + 8I$.

[10] **(a)** Find one solution to $p(D)x = 8t^2$.

[10] **(b)** Find one solution to $p(D)x = e^{-t}$.

In **(c)** and **(d)**, suppose that $x = t^3$ is a solution to $p(D)x = q(t)$ (where still $p(D) = D^2 + 4D + 8I$).

[8] **(c)** What is $q(t)$?

[8] **(d)** What is the solution to $p(D)x = q(t)$ such that $x(0) = 0$ and $\dot{x}(0) = 2$?

[15] **5.** **(a)** Find a periodic solution to $\ddot{x} + \omega_n^2 x = \mathrm{sq}(2t)$ if one exists.

(b)–(c) concern the function $f(t)$, periodic of period 2π, with $f(t) = |t| - \frac{\pi}{2}$ for $|t| \leq \pi$.

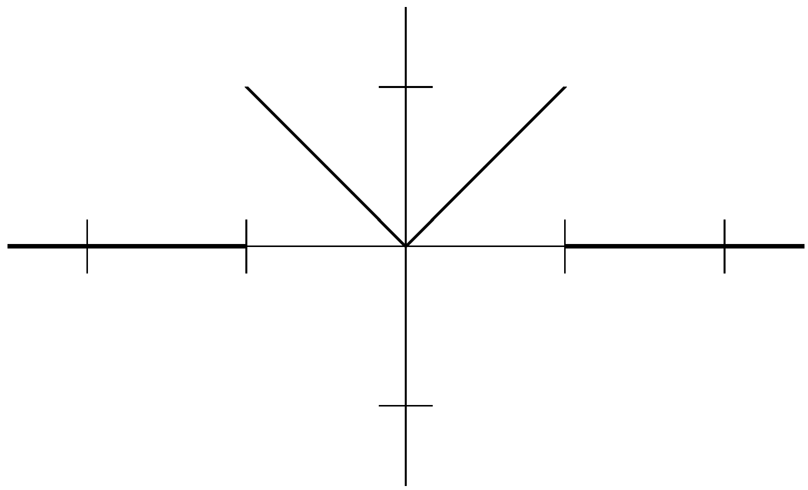
[6] **(b)** Is $f(t)$ even, odd, or neither?

What is its average value?

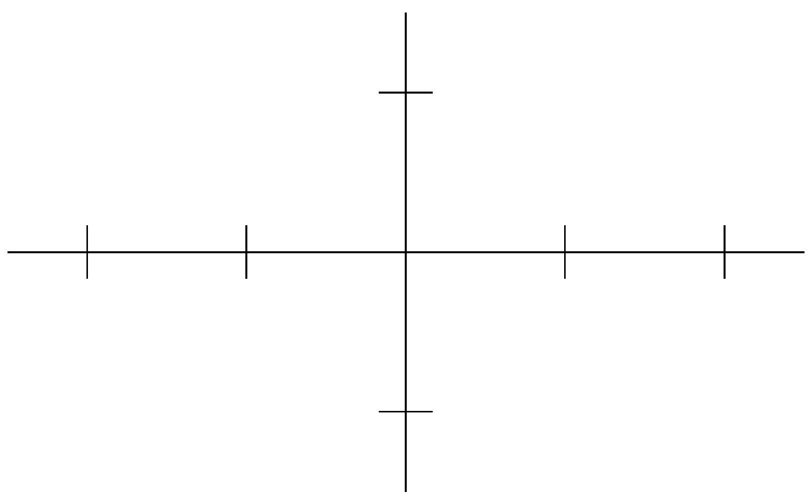
Graph $f(t)$.

[15] **(c)** What is the Fourier series of $f(t)$?

6. (a)–(b) The function $f(t)$ is defined by the following graph, in which the hashmarks are at unit spacing.



[6] **(a)** Graph the generalized derivative $f'(t)$.



[8] **(b)** Write a formula for $f'(t)$, using step and delta functions as necessary. Identify in your formula the regular part $f'_r(t)$ and the singular part $f'_s(t)$.

6. (continued) Suppose that the unit impulse response of a certain operator $p(D)$ is $w(t) = u(t)e^{-t}\sin(2t)$.

[8] **(c)** Please calculate the solution to $p(D)x = u(t)e^{-t}$, with rest initial conditions.

[6] **(d)** What is the solution to $p(D)x = \delta(t-1)$ with rest initial conditions?

[8] **(e)** What is the characteristic polynomial $p(s)$?

7. **(a)–(c)** concern the operator $p(D) = D^3 + D$.

[8] **(a)** What is the transfer function of the operator $p(D)$?

[10] **(b)** What is the unit impulse response of this operator?

[10] **(c)** What is the Laplace transform $X(s)$ of the solution to $p(D)x = \cos(2t)$ with rest initial conditions?

[8] **(d)** Sketch the pole diagram of $\dfrac{(s+1)e^{-s}}{s^3 + 2s^2 + 2s}$

8. In **(a)** and **(b)**, $A = \begin{bmatrix} 5 & -1 \\ -1 & 5 \end{bmatrix}$.

[8] **(a)** What are the eigenvalues of A?

[8] **(b)** For each eigenvalue, find a nonzero eigenvector.

(c)–(e) concern a certain 2×2 matrix B, whose eigenvalues are known to be 2 and -2.

[6] **(c)** What is the trace of B?

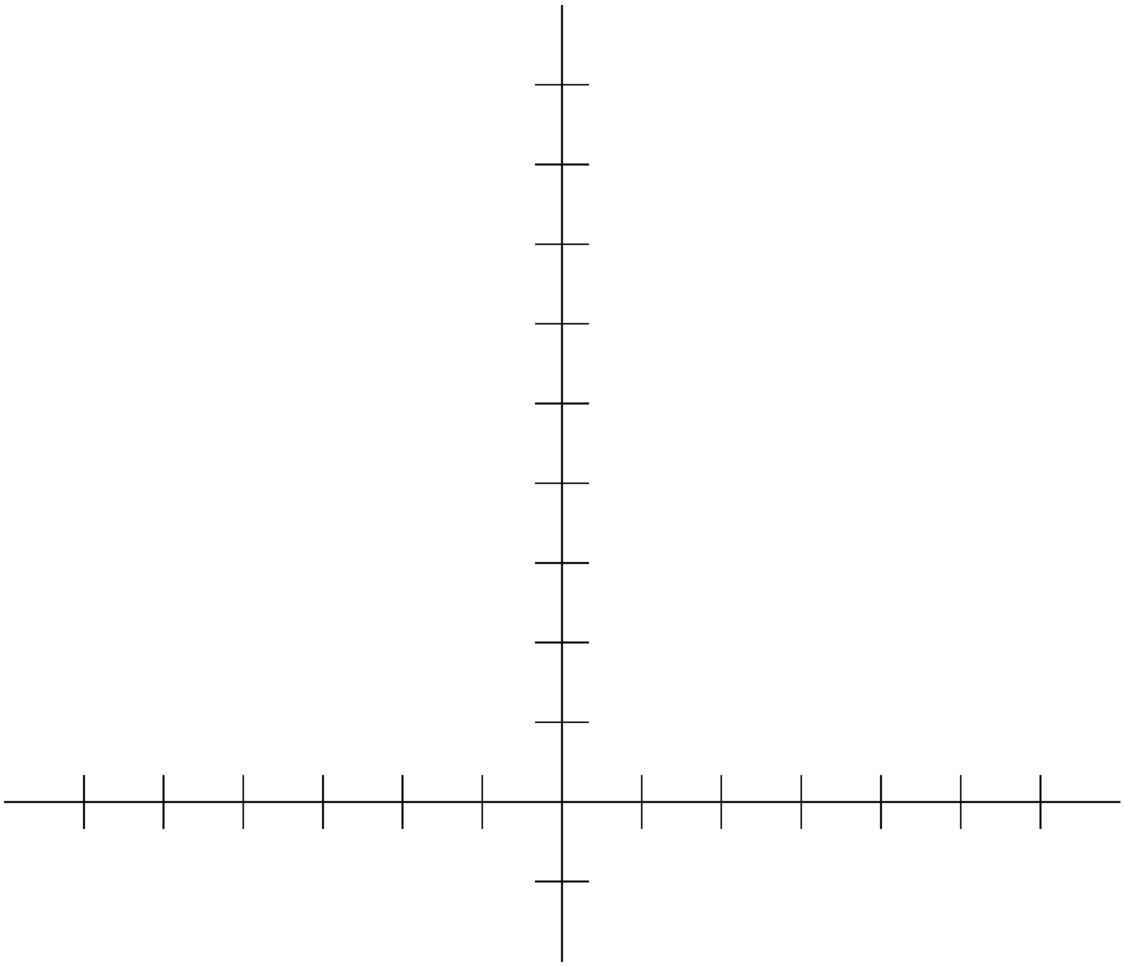
What is the determinant of B?

[8] **(d)** Suppose also that $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is an eigenvector for value 2 and $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$ is an eigenvector for value -2. Calculate e^{Bt}.

[6] **(e)** What is the solution to $\dot{\mathbf{u}} = B\mathbf{u}$ with $\mathbf{u}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$?

9. Let $A = \begin{bmatrix} 1 & 2 \\ -2 & a \end{bmatrix}$, and consider the homogeneous linear system $\dot{\mathbf{u}} = A\mathbf{u}$.

[6] **(a)** On the (Tr,Det) plane below, sketch the critical parabola (where the matrix has repeated eigenvalues) and sketch line corresponding to the matrices A as a varies.



[30] **(b)** For each of the following conditions, indicate the values of a (if any) for which the phase portrait satisfies the condition.

(i) Saddle

(ii) Stable node = nodal sink

(iii) Unstable node = nodal source

(iv) Stable spiral = spiral sink

(v) Unstable spiral = spiral source

(vi) Stable defective node = defective nodal sink

(vii) Unstable defective node = defective nodal source

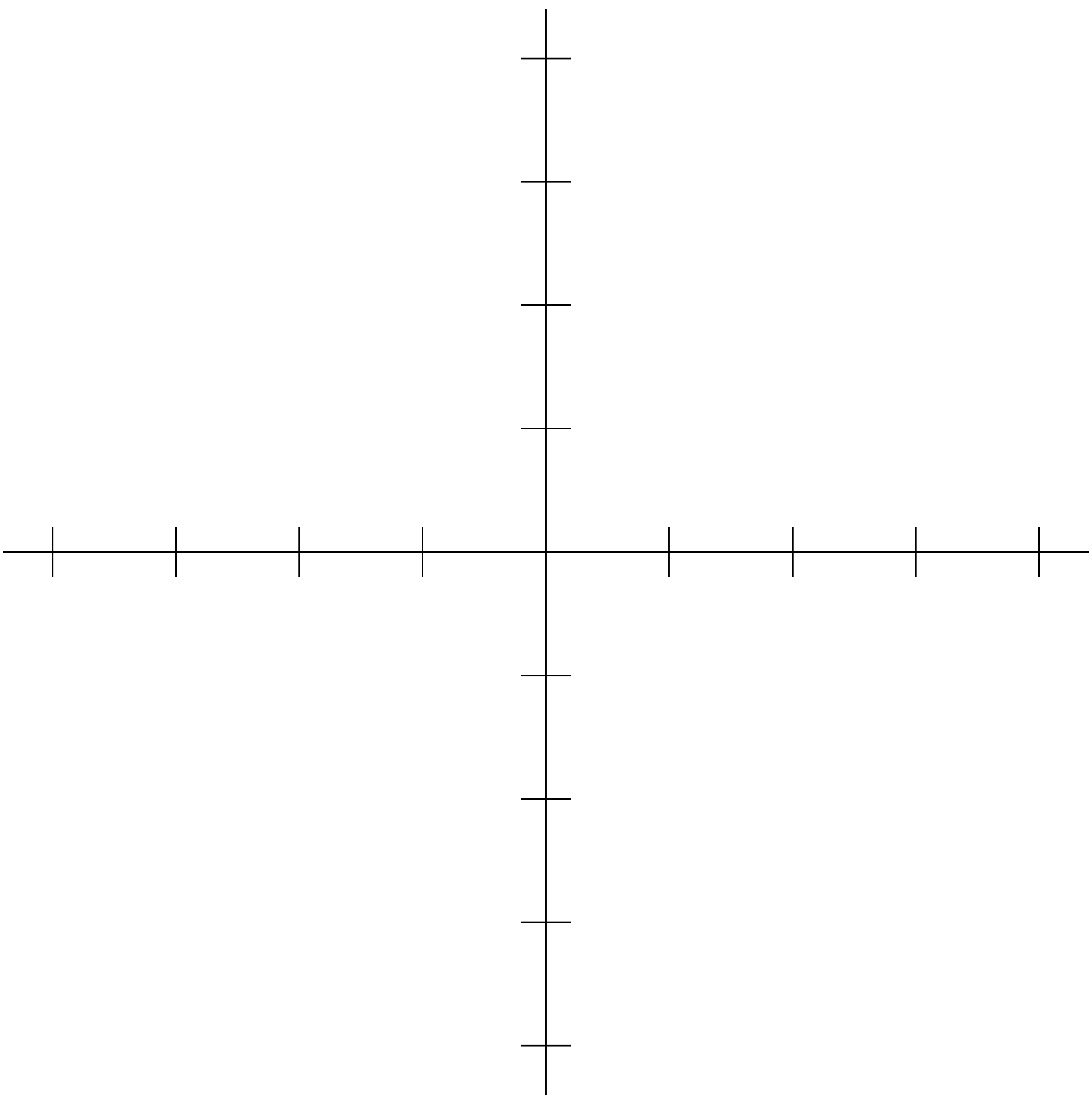
(viii) Star

(ix) Center

(x) Degenerate case

10. This problem concerns the nonlinear autonomous system $\begin{cases} \dot{x} = (y+1)(y-x+1) \\ \dot{y} = (x-3)(x+y-1) \end{cases}$.

[12] **(a)** Find the equilibria of this system and plot them on the (x, y) plane below.



10. (continued) $\begin{cases} \dot{x} = (y+1)(y-x+1) \\ \dot{y} = (x-3)(x+y-1) \end{cases}$.

[12] **(b)** One equilibrium is at $(1,0)$. Find the Jacobian matrix at this equilibrium.

[12] **(c)** The equilibrium at $(1,0)$ is a stable spiral. For large t, the solutions which converge to this equilibrium have y-coordinate well-approximated by the function $Ae^{at}\cos(\omega t - \phi)$ for some constants A, ϕ, a, and ω. Some of these constants depend upon the particular solution, and some are common to all solutions of this type. Find the values of the ones which are common to all such solutions.

Operator Formulas

• Exponential Response Formula: $x_p = Ae^{rt}/p(r)$ solves $p(D)x = Ae^{rt}$ provided $p(r) \neq 0$.

• Resonant Response Formula: If $p(r) = 0$ then $x_p = Ate^{rt}/p'(r)$ solves $p(D)x = Ae^{rt}$ provided $p'(r) \neq 0$.

Defective matrix formula

If A is a defective 2×2 matrix with eigenvalue λ_1 and nonzero eigenvector $\mathbf{v}_1$, then you can solve for $\mathbf{w}$ in $(A - \lambda_1 I)\mathbf{w} = \mathbf{v}_1$ and $\mathbf{u} = e^{\lambda_1 t}(t\mathbf{v}_1 + \mathbf{w})$ is a solution to $\dot{\mathbf{u}} = A\mathbf{u}$.

Properties of the Laplace transform

0. Definition: $\mathcal{L}[f(t)] = F(s) = \int_{0-}^{\infty} f(t)e^{-st}\,dt$ for $\mathrm{Re}(s) >> 0$.

1. Linearity: $\mathcal{L}[af(t) + bg(t)] = aF(s) + bG(s)$.

2. Inverse transform: $F(s)$ essentially determines $f(t)$.

3. s-shift rule: $\mathcal{L}[e^{at}f(t)] = F(s-a)$.

4. t-shift rule: $\mathcal{L}[f_a(t)] = e^{-as}F(s)$, $\quad f_a(t) = \begin{cases} f(t-a) & \text{if } t > a \\ 0 & \text{if } t < a \end{cases}$.

5. s-derivative rule: $\mathcal{L}[tf(t)] = -F'(s)$.

6. t-derivative rule:
$\mathcal{L}[f'(t)] = sF(s)$ [generalized derivative]
$\mathcal{L}[f_r'(t)] = sF(s) - f(0+)$ [$f(t)$ continuous for $t > 0$]

7. Convolution rule: $\mathcal{L}[f(t) * g(t)] = F(s)G(s)$, $f(t) * g(t) = \int_0^t f(\tau)g(t-\tau)\,d\tau$.

8. Weight function: $\mathcal{L}[w(t)] = W(s) = \dfrac{1}{p(s)}$, $\quad w(t)$ the unit impulse response.

Formulas for the Laplace transform

$$\mathcal{L}[1] = \frac{1}{s} \qquad \mathcal{L}[e^{at}] = \frac{1}{s-a} \qquad \mathcal{L}[t^n] = \frac{n!}{s^{n+1}} \qquad \mathcal{L}[\delta_a(t)] = e^{-as}$$

$$\mathcal{L}[\cos(\omega t)] = \frac{s}{s^2+\omega^2} \qquad \mathcal{L}[\sin(\omega t)] = \frac{\omega}{s^2+\omega^2} \qquad \mathcal{L}[u_a(t)] = \frac{e^{-as}}{s}$$

Fourier series $f(t) = \frac{a_0}{2} + a_1\cos(t) + a_2\cos(2t) + \cdots + b_1\sin(t) + b_2\sin(2t) + \cdots$

$$a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(mt)\,dt, \qquad b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(mt)\,dt$$

If $\mathrm{sq}(t)$ is the odd function of period 2π which has value 1 between 0 and π, then

$$\mathrm{sq}(t) = \frac{4}{\pi}\left(\sin(t) + \frac{\sin(3t)}{3} + \frac{\sin(5t)}{5} + \cdots\right)$$

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18.03 Practice Hour Exam I (2010)

1. A certain computer chip sheds heat at a rate proportional to the difference between its temperature and that of its environment.

(a) Write down a differential equation controlling the temperature of the chip, as a function of time measured in minutes, if the temperature in the environment is a constant 20°C. Your equation will have a constant in it which can't be determined from the data given so far.

(b) What is the general solution of this equation? (This will still involve the unknown constant).

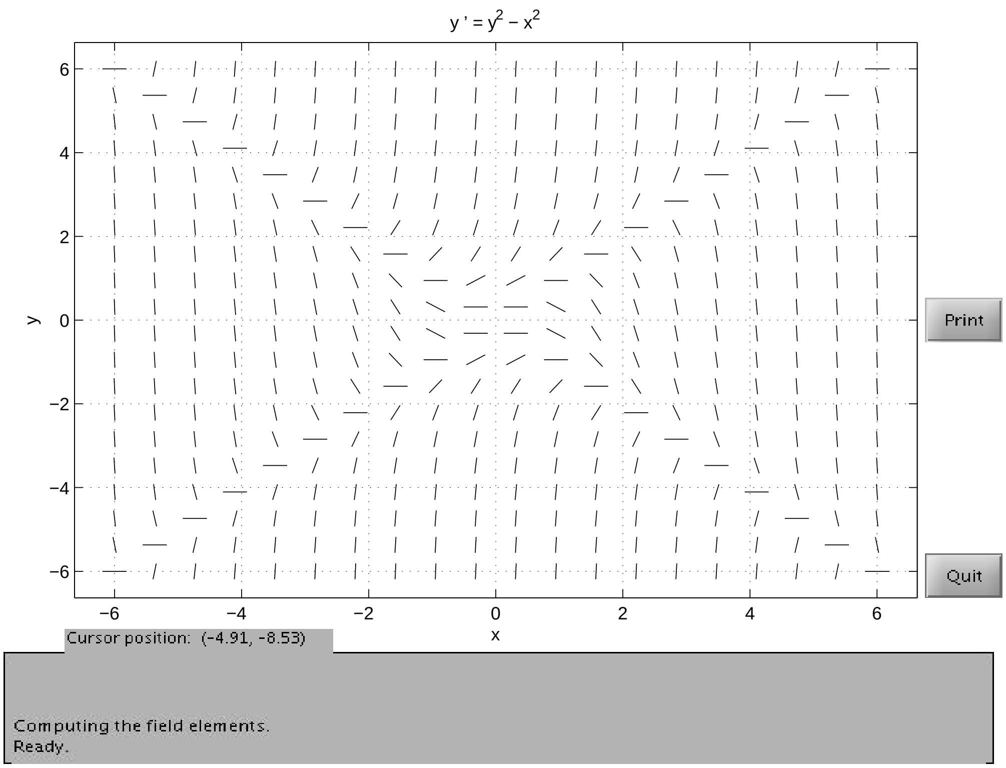
(c) It is observed that if the chip is powered down at $t = 0$ at a temperature of 70°C in a room at 20°C, its temperature at $t = 10$ minutes is 60°C. Use this new information to complete the determination of the differential equation.

2. Estimate $y(2.2)$ where y is the solution of the differential equation $y' = y^2 - x^2$ with $y(2) = 0$, using Euler's method with step size 0.1.

3. This problem concerns the differential equation $y' = y^2 - x^2$. Part of its direction field is shown below.

(a) On the diagram, sketch and label the isoclines for slope $m = -4$, $m = 0$, and $m = 4$.

(b) On the diagram, sketch the graph of the solution of the equation with $y(2) = 0$.



(c) Estimate the value $y(100)$ of the solution with $y(2) = 0$. Is your estimate too large or too small?

(d) A certain solution y has a local extremum at $x = -1$. What can you say about $y(-1)$? Is the extremum a maximum or a minimum? For full credit, make a relevant calculation, rather than merely relying on the picture.

4. (a) Find the general solution of $t\dot{x} + 2x = t^2$.

(b) Find a sinusoidal solution to the differential equation $\dot{x} + 2x = \cos(2t)$. Express your answer as a sum of sines and cosines. You may use any method to find this solution.

5. (a) Express each of the cube roots of $-8i$ first in the form $Ae^{i\theta}$ and then in the form $a + bi$.

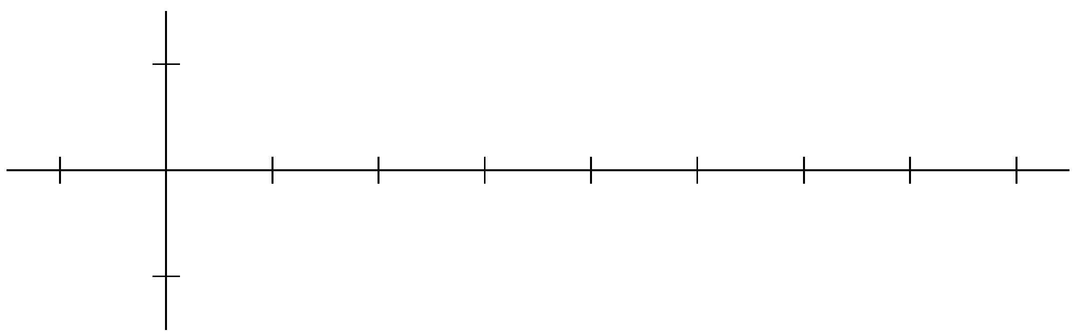
(b)–(e) relate to the sinusoidal function $f(t) = -\cos(\frac{\pi}{2}t) - \sin(\frac{\pi}{2}t)$.

(b) Find positive real numbers A and ϕ such that $f(t) = A\cos(\frac{\pi}{2}t - \phi)$.

(c) What is the period P of this sinusoidal function?

(d) What is the time lag t_0 of this sinusoidal function?

(e) Please sketch a graph of this function below, marking on the diagram A, P, and t_0.



6. This problem concerns the autonomous equation $\dot{y} = y^3 - y$.

(a) Sketch the phase line for this equation. Mark on it all critical points. Label each critical point as stable, unstable, or neither.

(b) Sketch some solutions for this equation, enough so that for any b between -2 and $+2$ you show a solution y such that $y(t) = b$ for some t.

(c) Determine where points of inflection occur in solutions to this equation. (A function $f(t)$ has a point of inflection at (a, b) if $f(a) = b$ and $f''(a) = 0$.)

Solutions

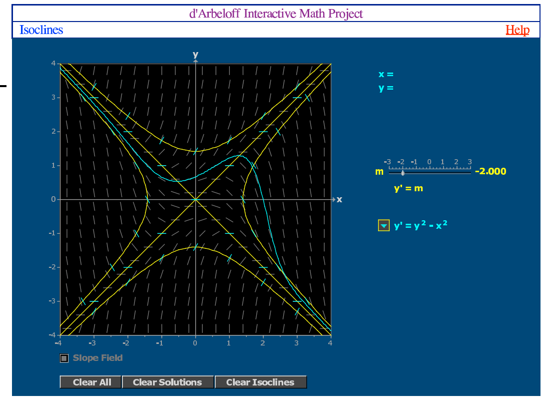
1. (a) Let $x(t)$ be the temperature of the chip in degrees C. $\dot{x} = (20 - x)$, or $\dot{x} + kx = 20$.

(b) $x = 20 + Ce^{-kt}$.

(c) The data gives $70 = x(0) = 20 + C$, so $C = 50$ and $60 = x(10) = 20 + 50e^{-10k}$, so $-10k = \ln(40) - \ln(50)$ or $k = (\ln(50) - \ln(40))/10$.

2.

n	x_n	y_n	$A_n = y_n^2 - x_n^2$	$hA_n = A_n$
0	2	0	-4.0	-0.4
1	2.1	-0.4	$0.16 - 4.41 = -4.25$	-0.425
2	2.2	-0.825		



3. (a) The nullcline is the pair of crossed lines $y = \pm 1$. The $m = 2$ isocline is the upper/lower hyperbola; the $m = -2$ isocline is the left/right hyperbola.

(c) The graphed solution is trapped by the funnel having the nullcline as its lower fence and the $m = -2$ isocline as the upper fence, $y(100)$ is very near to -100, but slightly larger.

(d) Extrema occur when $\dot{y} = 0$; that is, along the nullcline. $\ddot{y} = 2y\dot{y} - 2x$, which is $-2x$ on the nullcline. At $x = -1$ this is positive, so we have a minimum.

4. (a) Multiply through by t: $\dfrac{d}{dt}(t^2 x) = t^3$. Thus $t^2 x = t^4/4 + c$ so $x = t^2/4 + c/t^2$.

(b) First solve the complex-valued equation $\dot{z} + 2z = e^{2it}$. One way to solve this is to try $z_p = Ae^{2it}$ and solving for A: $(2i+2)A = 1$, or $A = \frac{1}{2+2i} = \frac{2-2i}{8} = \frac{1-i}{4}$. The real part of $\frac{1-i}{4}e^{2it}$ is $x_p = \frac{1}{4}(\cos(2t) + \sin(2t))$.

5. (a) $-8i = 8e^{3\pi i/2}$ so its cube roots all have modulus equal to the postive number whose cube is 8, namely 2. The arguments are $\frac{\pi}{2}$, $\frac{\pi}{2} + \frac{2\pi}{3} = \frac{7\pi}{6}$, and $\frac{\pi}{2} + \frac{4\pi}{3} = \frac{11\pi}{6}$, so the roots are $2e^{\pi i/2} = 2i$, $2e^{7\pi i/6} = -\sqrt{3} - i$, and $2e^{11\pi i/6} = \sqrt{3} - i$.

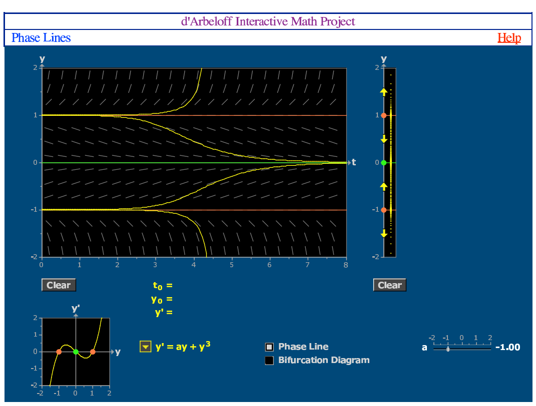
(b) The point $(a, b) = (-1, -1)$ has polar coordinates $A = \sqrt{2}$ and $\phi = 5\pi/4$.

(c) $P = \frac{2\pi}{\omega} = \frac{2\pi}{\pi/2} = 4$.

(d) $t_0 = \phi/\omega = \frac{5\pi}{4}/\frac{\pi}{2} = \frac{5}{2}$.

(e) Amplitude $\sqrt{2}$, period 4, trough at 1/2, peak at 5/2.

6. (a), (b)



(c) $\ddot{y} = 3y^2\dot{y} - \dot{y} = (3y^2 - 1)\dot{y}$ is zero if either $\dot{y} = 0$—that is, $y = 0$ or $y = \pm 1$—or $3y^2 - 1 = 0$—that is, $y = \pm 1/\sqrt{3}$.

18.03 Pactice Final Exam, 2010

1. This problem concerns the differential equation

$$\frac{dy}{dx} = x^2 - y^2 \qquad (*)$$

Let $y = f(x)$ be the solution with $f(-2) = 0$.

(a) Sketch the isoclines for slopes -2, 0, and 2, and sketch the direction field along them.

(c) On the same diagram, sketch the graph of the solution $f(x)$. What is its slope at $x = -2$?

(d) Estimate $f(100)$.

(e) Suppose that the function $f(x)$ reaches a maximum at $x = a$. What is $f(a)$?

(f) Use two steps of Euler's method to estimate $f(-1)$.

2. In **(a)–(b)** we consider the autonomous equation $\dot{x} = 2x - 3x^2 + x^3$.

(a) Sketch the phase line of this equation.

(b) Sketch the graphs of some solutions. Be sure to include at least one solution with values in each interval above, below, and between the critical points.

(c) Some solutions have points of inflection. What are the possible values of $x(a)$ if a non-constant solution $x(t)$ has a point of inflection at $t = a$?

(d) A radioactive isotope of the element Bostonium, Bo, decays with half life of two years to an isotope of Cantabrigium, Ct. The MIT reactor is loaded with this material. At $t = 0$ there is no Ct in it, but starting at $t = 0$ Ct is added in such a way that the cumulative total amount inserted by time t is t moles.

Write down a differential equation for the number of moles of Ct in the reactor as a function of time. What is the initial condition?

(e) Solve the initial value problem $\quad x\dfrac{dy}{dx} + 3y = x^2$, $y(1) = 1$.

3. **(a)** Find non-negative real numbers A, ω, and ϕ such that $\mathrm{Re}\left(\dfrac{ie^{2it}}{1+i}\right) = A\cos(\omega t - \phi)$.

(b) Sketch the trajectory of $e^{(1-\pi i)t}$.

(c) Express the cube roots of $8i$ in the form $a + bi$ (with a and b real).

4. **(a)–(c)** Find one solution to $\ddot{x} + 2\dot{x} + 2x = q(t)$ for

(a) $q(t) = t^2 + 1$.

(b) $q(t) = e^{-2t} + 1$.

(c) $q(t) = \sin t$. What is the amplitude of the sinusoidal solution?

In **(d)** and **(e)**, suppose that that t^3 is a solution to $\ddot{x} + 2\dot{x} + 2x = q(t)$.

(d) What is $q(t)$?

(e) What is the general solution to $\ddot{x} + 2\dot{x} + 2x = q(t)$?

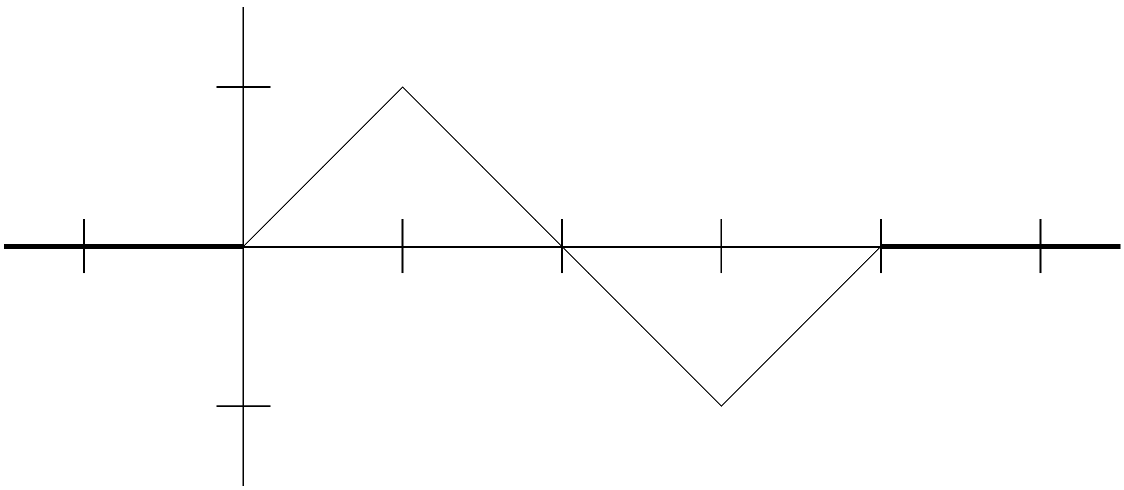
5. **(a)–(b)** concern the function $f(t) = \mathrm{sq}(t + \frac{\pi}{2})$.

(a) Graph $f(t)$.

(b) What is its Fourier series? (Simplify the trig functions.)

(c) Find a solution to $\ddot{x} + x = \mathrm{sq}(t)$.

6. (a)–(d) In a recent game of Capture the Flag, a certain student was observed to move according to the following graph, in which the hashmarks are at unit spacing.



(a) Graph the generalized derivative $v(t)$.

(b) Write a formula for $v(t)$ in terms of the unit step and (if necessary) the delta function.

(c) Still with the same function as in **(a)**: Graph the generalized derivative $\dot{v}(t)$.

(d) Write a formula for the acceleration $\dot{v}(t)$ in terms of the unit step and (if necessary) the delta function.

(e) Suppose that the unit impulse response of a certain operator $p(D)$ is $w(t)$. Let $q(t) = 0$ for $t < 0$ and $t > 1$, and $q(t) = 1$ for $0 < t < 1$. Please find functions $a(t)$, $b(t)$ so that the solution $x(t)$ to $p(D)x = q(t)$, with rest initial conditions, is given by

$$x(t) = \int_{a(t)}^{b(t)} w(\tau)\, d\tau$$

7. This problem concerns the operator $p(D) = 2D^2 + 8D + 16I$.

(a) What is the transfer function of the operator $p(D)$?

(b) What is the unit impulse response of this operator?

(c) What is the Laplace transform of the solution to $p(D)x = \sin(t)$ with rest initial conditions?

8. In **(a)** and **(b)**, $A = \begin{bmatrix} 2 & 12 \\ 3 & 2 \end{bmatrix}$.

(a) What are the eigenvalues of A?

(b) For each eigenvalue, find a nonzero eigenvector.

(c) Suppose that the matrix B has eigenvalues 1 and 2, with eigenvectors $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ respectively. Calculate e^{Bt}.

(d) What is the solution to $\dot{\mathbf{u}} = B\mathbf{u}$ with $\mathbf{u}(0) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$?

9. (a) Suppose again that the matrix B has eigenvalues 1 and 2, with eigenvectors $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ respectively. Sketch the phase portrait on the graph below.

(b) Let $A = \begin{bmatrix} a & -2 \\ 2 & 1 \end{bmatrix}$, and consider the homogeneous linear system $\dot{\mathbf{u}} = A\mathbf{u}$. For each of the following conditions, determine all values of a (if any) which are such that the system satisfies the condition.

(i) Saddle
(ii) Star
(iii) Stable node
(iv) Stable spiral. What is the direction of rotation?
(v) Unstable spiral.
(vi) Unstable defective node

10. Parts **(a)**–**(c)** deal with the nonlinear autonomous system $\begin{cases} \dot{x} = x^2 - y^2 \\ \dot{y} = x^2 + y^2 - 8 \end{cases}$.

(a) Find the equilibria of this system.

(b) There is one equilibrium in the south-west quadrant. Find the Jacobian at this equilibrium.

(c) The equilibrium you found in **(b)** is a stable spiral. For large t, the solutions which converge to this equilibrium have x-coordinate which are well-approximated by the function $Ae^{at}\cos(\omega t - \phi)$ for some constants A, ϕ, a, and ω. Some of these constants depend upon the particular solution, and some are common to all solutions of this type. Find the values of the ones which are common to all such solutions.

(d) Finally, return to the autonomous equation $\dot{x} = 2x - 3x^2 + x^3$ that you studied in problem **2**. Write down a formula approximating the solutions converging to the stable equilibrium when t is large.

Operator Formulas

- Exponential Response Formula: $x_p = Ae^{rt}/p(r)$ solves $p(D)x = Ae^{rt}$ provided $p(r) \neq 0$.
- Resonant Response Formula: If $p(r) = 0$ then $x_p = Ate^{rt}/p'(r)$ solves $p(D)x = Ae^{rt}$ provided $p'(r) \neq 0$.

Defective matrix formula

If A is a defective 2×2 matrix with eigenvalue λ_1 and nonzero eigenvector $\mathbf{v}_1$, then you can solve for $\mathbf{w}$ in $(A - \lambda_1 I)\mathbf{w} = \mathbf{v}_1$ and $\mathbf{u} = e^{\lambda_1 t}(t\mathbf{v}_1 + \mathbf{w})$ is a solution to $\dot{\mathbf{u}} = A\mathbf{u}$.

Properties of the Laplace transform

0. Definition: $\mathcal{L}[f(t)] = F(s) = \int_{0-}^{\infty} f(t)e^{-st}\,dt$ for $\mathrm{Re}\, s >> 0$.

1. Linearity: $\mathcal{L}[af(t) + bg(t)] = aF(s) + bG(s)$.

2. Inverse transform: $F(s)$ essentially determines $f(t)$.

3. s-shift rule: $\mathcal{L}[e^{at}f(t)] = F(s-a)$.

4. t-shift rule: $\mathcal{L}[f_a(t)] = e^{-as}F(s)$, $\quad f_a(t) = \begin{cases} f(t-a) & \text{if } t > a \\ 0 & \text{if } t < a \end{cases}$.

5. s-derivative rule: $\mathcal{L}[tf(t)] = -F'(s)$.

6. t-derivative rule: $\mathcal{L}[f'(t)] = sF(s)$ [generalized derivative]
$\mathcal{L}[f_r'(t)] = sF(s) - f(0+)$ [$f(t)$ continuous for $t > 0$]

7. Convolution rule: $\mathcal{L}[f(t) * g(t)] = F(s)G(s)$, $f(t) * g(t) = \int_0^t f(\tau)g(t-\tau)\,d\tau$.

8. Weight function: $\mathcal{L}[w(t)] = W(s) = \dfrac{1}{p(s)}$, $\quad w(t)$ the unit impulse response.

Formulas for the Laplace transform

$$\mathcal{L}[1] = \frac{1}{s} \qquad \mathcal{L}[e^{at}] = \frac{1}{s-a} \qquad \mathcal{L}[t^n] = \frac{n!}{s^{n+1}}$$

$$\mathcal{L}[\cos(\omega t)] = \frac{s}{s^2+\omega^2} \qquad \mathcal{L}[\sin(\omega t)] = \frac{\omega}{s^2+\omega^2}$$

$$\mathcal{L}[u_a(t)] = \frac{e^{-as}}{s} \qquad \mathcal{L}[\delta_a(t)] = e^{-as}$$

where $u(t)$ is the unit step function $u(t) = 1$ for $t > 0$, $u(t) = 0$ for $t < 0$.

Fourier series

$$f(t) = \frac{a_0}{2} + a_1\cos(t) + a_2\cos(2t) + \cdots + b_1\sin(t) + b_2\sin(2t) + \cdots$$

$$a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(mt)\,dt, \qquad b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(mt)\,dt$$

$$\int_{-\pi}^{\pi}\cos(mt)\cos(nt)\,dt = \int_{-\pi}^{\pi}\sin(mt)\sin(nt)\,dt = 0 \qquad \text{for} \quad m \neq n$$

$$\int_{-\pi}^{\pi}\cos^2(mt)\,dt = \int_{-\pi}^{\pi}\sin^2(mt)\,dt = \pi \qquad \text{for} \quad m > 0$$

If $\mathrm{sq}(t)$ is the odd function of period 2π which has value 1 between 0 and π, then

$$\mathrm{sq}(t) = \frac{4}{\pi}\left(\sin(t) + \frac{\sin(3t)}{3} + \frac{\sin(5t)}{5} + \cdots\right)$$

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18.03SC Unit 1 Exam

1. (a) In a perfect environment, the population of Norway rat that breeds on the MIT campus increases by a factor of $e \simeq 2.718281828459045\ldots$ each year. Model this natural growth by a differential equation. [8]

What is the growth rate k?

(b) MIT is a limited environment, with a maximal sustainable Norway rat population of $R = 1000$ rats. Write down the logistic equation modeling this. (You may use "k" for the natural growth rate here if you failed to find it in **(a)**.) [4]

(c) The MIT pest control service intends to control these rats by killing them at a constant rate of a rats per year. If it wants to limit the rat population to 75% of the maximal sustainable population, what rate a it should aim for (in rats per year)? [8]

2. For the autonomous equation $\dot{x} = x(x-1)(x+2)$, please sketch:

(a) the phase line, identifying the critical points and whether they are stable, unstable, or neither. [4]

(b) at least one solution of each basic type (so that every solution is a time-translate of one you have drawn) [4]

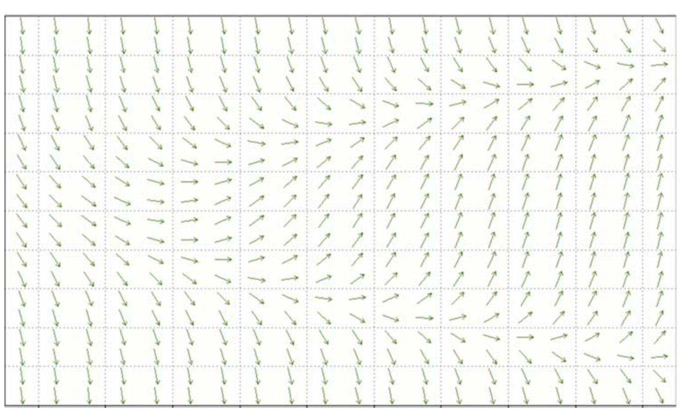
Below is a diagram of a direction field of the differential equation $y' = (1/4)(x - y^2)$. On it please plot and label:

(c) the nullcline [3]

(d) at least two quite different solutions [3]

(e) the separatrix (if there is one) [3]

(f) True or false: If $y(x)$ is a solution with a minimum, then for all large enough x, $y(x) < \sqrt{x}$. (No explanation needed: just circle one.) [3]



3. **(a)** Use Euler's method with stepsize $h = 1/2$ to estimate the value at $x = 3/2$ of the solution to $y' = x + y$ such $y(0) = 1$. [10]

(b) Find the solution of $t\dot{x} + x = \cos t$ such that $x(\pi) = 1$. [10]

4. (a) Find real a, b such that $\dfrac{1}{3+2i} = a + bi$. [3]

(b) Find real r, θ such that $1 - i = re^{i\theta}$. [3]

(c) Find real a, b such that $(1 - i)^8 = a + bi$. [3]

(d) Find real a, b such that $b > 0$ and $a + bi$ is a cube root of -1. [3]

(e) Find real a, b such that $e^{\ln 2 + i\pi} = a + bi$. [3]

(f) Write $f(t) = 2\cos(4t) - 2\sin(4t)$ in the form $A\cos(\omega t - \phi)$. [5]

5. (a) Find a particular solution to the equation $\dot{x} + 3x = e^{2t}$. [5]

(b) Find the solution to the same equation such that $x(0) = 1$. [5]

(c) Write down a linear equation with exponential right hand side of which $\dot{x} + 3x = \cos(2t)$ is the real part. [5]

(d) Find a particular solution to the equation $\dot{x} + 3x = \cos(2t)$. [5]

18.03SC Differential Equations
Fall 2011

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18.03SC Unit 2 Exam

1. (a) For what value of k is the system represented by $\ddot{x} + \dot{x} + kx = 0$ critically damped? [8]

(b) For k greater than that value, is the system overdamped or underdamped? [4]

(c) Suppose a solution of $\ddot{x} + \dot{x} + kx = 0$ vanishes at $t = 1$, and then again for $t = 2$ (but not in between). What is k? [8]

2. (a) Find a solution of $\ddot{x} + x = 5te^{2t}$. [10]

(b) Suppose that $y(t)$ is a solution of the same equation, $\ddot{x} + x = 5te^{2t}$, such that $y(0) = 1$ and $\dot{y}(0) = 2$. (This is probably *not* the solution you found in **(a)**.) Use $y(t)$ and other functions to write down a solution $x(t)$ such that $x(0) = 3$ and $\dot{x}(0) = 5$. [10]

3. (a) Consider the equation $\ddot{x} + b\dot{x} + kx = \cos(\omega t)$. We will vary the spring constant but keep b fixed. For what value of k is the amplitude of the sinusoidal solution of $\ddot{x} + b\dot{x} + kx = \cos(\omega t)$ maximal? (Your answer will be a function of ω and may depend upon b as well.) [10]

(b) (Unrelated to the above.) Find the general solution of $\dfrac{d^3x}{dt^3} - \dfrac{dx}{dt} = 0$. [10]

4. A certain system has input signal y and system response x related by the differential equation $\ddot{x} + \dot{x} + 6x = 6y$. It is subjected to a sinusoidal input signal.

(a) Calculate the complex gain $H(\omega)$. [10]

(b) Compute the gain at $\omega = 2$. [5]

(c) Compute the phase lag at $\omega = 2$. [5]

5. Suppose that $\frac{1}{2}t\sin(2t)$ is a solution to a certain equation $m\ddot{x} + b\dot{x} + kx = 4\cos(2t)$.

(a) Write down a solution to $m\ddot{x} + b\dot{x} + kx = 4\cos(2t - 1)$. [4]

(b) Write down a solution to $m\ddot{x} + b\dot{x} + kx = 8\cos(2t)$. [4]

(c) Determine m, b, and k. [12]

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18.03SC Unit 3 Exam

1. A certain periodic function has Fourier series

$$f(t) = 1 + \frac{\cos(\pi t)}{2} + \frac{\cos(2\pi t)}{4} + \frac{\cos(3\pi t)}{8} + \frac{\cos(4\pi t)}{16} + \cdots$$

(a) What is the minimal period of $f(t)$? [4]

(b) Is $f(t)$ even, odd, neither, or both? [4]

(c) Please give the Fourier series of a periodic solution (if one exists) of [8]

$$\ddot{x} + \omega_n^2 x = f(t)$$

(d) For what values of ω_n is there no periodic solution? [4]

2. Let $f(t) = (u(t+1) - u(t-1))t$.

(a) Sketch a graph of $f(t)$. [6]

(b) Sketch a graph of the generalized derivative $f'(t)$. [6]

(c) Write a formula for the generalized derivative $f'(t)$. [8]

3. Let $p(D)$ be the operator whose unit impulse response is given by $w(t) = e^{-t} - e^{-3t}$.

(a) Using convolution, find the unit step response of this operator: the solution to $p(D)v = u(t)$ with rest initial conditions. [10]

(b) What is the transfer function $W(s)$ of the operator $p(D)$? [5]

(c) What is the characteristic polynomial $p(s)$? [5]

4 (a) Find a generalized function $f(t)$ with Laplace transform $F(s) = \dfrac{e^{-s}(s-1)}{s}$. [10]

(b) Find a function $f(t)$ with Laplace transform $F(s) = \dfrac{s+10}{s^3+2s^2+10s}$. [10]

5. Let $W(s) = \dfrac{s+10}{s^3 + 2s^2 + 10s}$.

(a) Sketch the pole diagram of $W(s)$. [10]

(b) If $W(s)$ is the transfer function of an LTI system , what is the Laplace transform of the response from rest initial conditions to the input $\sin(2t)$? [10]

Properties of the Laplace transform

0. Definition: $\mathcal{L}[f(t)] = F(s) = \int_{0^-}^{\infty} f(t)e^{-st}\,dt$ for $\operatorname{Re} s \gg 0$.

1. Linearity: $\mathcal{L}[af(t) + bg(t)] = aF(s) + bG(s)$.

2. Inverse transform: $F(s)$ essentially determines $f(t)$.

3. s-shift rule: $\mathcal{L}[e^{at}f(t)] = F(s-a)$.

4. t-shift rule: $\mathcal{L}[f_a(t)] = e^{-as}F(s)$, $\quad f_a(t) = u(t-a)f(t-a) = \begin{cases} f(t-a) & \text{if } t > a \\ 0 & \text{if } t < a \end{cases}$.

5. s-derivative rule: $\mathcal{L}[tf(t)] = -F'(s)$.

6. t-derivative rule: $\mathcal{L}[f'(t)] = sF(s) - f(0^-)$, where $f'(t)$ denotes the generalized derivative.

7. Convolution rule: $\mathcal{L}[f(t) * g(t)] = F(s)G(s)$, $f(t) * g(t) = \int_{0^-}^{t^+} f(t-\tau)g(\tau)d\tau$.

8. Weight function: $\mathcal{L}[w(t)] = W(s) = 1/p(s)$, $w(t)$ the unit impulse response.

Formulas for the Laplace transform

$$\mathcal{L}[1] = \frac{1}{s} \qquad \mathcal{L}[e^{at}] = \frac{1}{s-a} \qquad \mathcal{L}[t^n] = \frac{n!}{s^{n+1}}$$

$$\mathcal{L}[\cos(\omega t)] = \frac{s}{s^2+\omega^2} \qquad \mathcal{L}[\sin(\omega t)] = \frac{\omega}{s^2+\omega^2}$$

$$\mathcal{L}[t\cos(\omega t)] = \frac{2\omega s}{(s^2+\omega^2)^2} \qquad \mathcal{L}[t\sin(\omega t)] = \frac{s^2-\omega^2}{(s^2+\omega^2)^2}$$

Fourier coefficients for periodic functions of period 2π:

$$f(t) = \frac{a_0}{2} + a_1\cos(t) + a_2\cos(2t) + \cdots + b_1\sin(t) + b_2\sin(2t) + \cdots$$

$$a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(mt)\,dt, \qquad b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(mt)\,dt$$

If $\operatorname{sq}(t)$ is the odd function of period 2π which has value 1 between 0 and π, then

$$\operatorname{sq}(t) = \frac{4}{\pi}\left(\sin(t) + \frac{\sin(3t)}{3} + \frac{\sin(5t)}{5} + \cdots\right)$$