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A Bank Asset and Liability Management Model

M. I. Kusy, W. T. Ziemba,

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M. I. KUSY

Concordia University, Montreal, Quebec

W. T. ZIEMBA

International Institute for Applied Systems Analysis, Laxenburg, Austria, and University of British Columbia, Vancouver, British Columbia

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In managing its assets and liabilities in light of uncertainties in cash flows, cost of funds and return on investments, a bank must determine its optimal trade-off between risk, return and liquidity. In this paper we develop a multiperiod stochastic linear programming model (ALM) that includes the essential institutional, legal, financial, and bank-related policy considerations, and their uncertainties, yet is computationally tractable for realistically sized problems. A version of the model was developed for the Vancouver City Savings Credit Union for a 5-year planning period. The results indicate that ALM is theoretically and operationally superior to a corresponding deterministic linear programming model, and that the effort required for the implementation of ALM, and its computational requirements, are comparable to those of the deterministic model. Moreover, the qualitative and quantitative characteristics of the solutions are sensitive to the model's stochastic elements, such as the asymmetry of cash flow distributions. We also compare ALM with the stochastic decision tree (SDT) model developed by S. P. Bradley and D. B. Crane. ALM is computationally more tractable on realistically sized problems than SDT, and simulation results indicate that ALM generates superior policies.

The inherent uncertainty of their cash flows, cost of funds and return on investments has prompted banks to seek out greater efficiency in the management of their assets and liabilities. This need has led to studies concerned with how to structure a bank's assets and liabilities to make optimal trade-offs among risk, return and liquidity. These studies focus on determining how funds should be used in various economic scenarios. Important factors in these decisions include: balancing of anticipated sources and uses of funds to meet liquidity and capital adequacy constraints while concurrently maximizing profitability (Chambers and Charnes 1961, Cohen and Hammer 1967, 1972); allocating funds among assets based on risk and liquidity classification, maturity and rate of return (Bradley and Crane 1972, 1973, 1976); and adjusting a bank's financial structure in terms of liquidity, capital adequacy and leverage (Chambers and Charnes 1961, Cohen and Hammer 1967). See also Beazer (1975), Brodt (1979) and Spellman (1982).

Current research has stressed two approaches. The first, based on Markowitz's (1959) theory of portfolio selection, assumes that returns are normally distributed and that bank managers utilize risk-averse utility functions. The value of an asset then depends not only on the expectation and variance of its return but also on the covariance of its return with the returns of all other existing and potential investments. The second approach assumes that a bank seeks to maximize its future stream of profits (or expected profits) subject to portfolio mix constraints.

A general example of the use of the first approach is Pyle (1971), who developed a static model in which the financial intermediary (bank) can select the asset and liability levels it wishes to maintain throughout the period. He considers only the risk of the portfolio and not other possible uncertainties. The model omits trading activity, matching assets and liabilities, transaction costs, and other similar features. It is possible to develop dynamic models using constructs along these lines (see, e.g., Kallberg and Ziemba 1981). However, given the severe computational difficulties due to the complexity of algorithms for these problems, it is not at present possible to develop useful operational models for large organizations such as banks.

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Since our interest is in operational models, we concentrate on the second approach, which has theoretical and empirical support. Myers (1968) attempted to determine which criteria are most suitable for the asset and liability management problem by showing that: a necessary condition for the existence of security market equilibrium is risk independence; security market equilibrium implies risk independence of securities; and risk independence of investment opportunities implies that the maximization of the expected net present value is the appropriate objective criterion.

Thus, if, as is widely believed, the securities held by financial institutions are in a state of equilibrium, and securities purchased do not have synergistic effect (implying the risk independence of securities), then the appropriate objective function for a financial institution is the maximization of the expected net present value (ENPV). Hester and Pierce (1975) used cross-sectional data to analyze the validity of a number of portfolio selection models in bank fund management. They concluded that banks can be well managed using models as a decision aid and that the best objective functions are either ENPV or the maximization of a two-variable function for which ENPV is dominant.

Asset and liability management models using an ENPV criteria can be deterministic or stochastic. Deterministic models use linear programming, assume particular realizations for random events, and are computationally tractable for large problems. The banking industry has accepted these models as useful normative tools (Cohen and Hammer 1967). Stochastic models, however, including the use of chance-constrained programming, dynamic programming, sequential decision theory, and linear programming under uncertainty, have achieved little success. This failure is due to inherent computational difficulties, the oversimplifications needed to achieve computational tractability, and practitioners' unfamiliarity with these models' potential.

Essentially all of the deterministic models, and many of the stochastic models, follow the approach of Chambers and Charnes' linear programming model. They maximize net discounted returns, subject to budget and liquidity constraints, using the Federal Reserve Board's (FRB) capital adequacy formulas (see Section 3). Examples of successful applications of this model are Cohen and Hammer (1967), Komar (1971), and Lifson and Blackman (1973). These models continue to be criticized largely because they omit uncertainty (Bradley and Crane 1976, Cohen and Thore 1970, and Eppen and Fama 1968). Probability distributions can be obtained for different economic scenarios and a linear programming formulation can be applied to each scenario to determine optimal solutions. However, this approach will not generate an optimal solution to the total problem, but rather acts as a deterministic simulation to observe portfolio behavior under various economic conditions. One must use care in defining such models, since it is possible that no scenario leads to an optimal solution (see Birge 1982).

Charnes and Kirby (1965), Charnes and Littlechild (1968), Charnes and Thore (1966), and others developed chance-constrained models that express future deposits and loan repayments as joint, normally distributed random variables, and that replace the capital adequacy formula by chance-constraints on meeting withdrawal claims. These approaches lead to a computationally feasible scheme for realistic situations (see, e.g., Charnes, Gallegos and Yao 1982). However, the chance-constrained procedure cannot handle a differential penalty for either varying magnitudes of constraint violations or different types of constraints. Moreover, multiperiod models raise conceptual difficulties, as yet unresolved in the literature, dealing with the treatment of infeasibility in periods $2, \ldots, n$ (see, e.g., Eisner, Kaplan and Soden 1971).

The second approach to stochastic modeling is dynamic programming. Eppen and Fama (1968, 1969, 1971) modeled two- and three-asset problems. Daellenbach and Archer (1969) extended their work to include one liability. For a survey of this literature, see Ziemba and Vickson (1975). These models are dynamic and account for the inherent uncertainty of the problem. However, given the small number of financial instruments that can be analyzed simultaneously, they are of limited use in practice. See Daellenbach (1974) for estimates of possible gain using these models. For a recent survey of related applications in banking, see Cohen, Maier and Van Der Weide (1981).

The third alternative in considering stochastic models, proposed by Wolf (1969), is a sequential decision theoretic approach that employs sequential decision analysis to find an optimal solution through the use of implicit enumeration. This technique does not find an explicit optimal solution to problems with a time horizon beyond one period, because it is necessary to enumerate all possible portfolio strategies for periods preceding the present decision point in order to guarantee optimality. To explain away this drawback, Wolf makes the dubious assertion that the solution to a one-period model would be equivalent to a solution provided by solving an n-period model. This approach ignores, among other things, the problem of synchronizing the maturities of assets and liabilities. Bradley and Crane (1972, 1973, 1976) have developed a stochastic decision tree model that has many of the desirable features essential to an operational bank portfolio model. Their model is conceptually similar to Wolf's model. To overcome computational difficulties, they reformulated the asset and liability problem and developed a general programming decomposition algorithm that minimizes the computational difficulties. This model is discussed in Section 5.

The fourth approach is stochastic linear programming with simple recourse (SLPSR), which is also called linear programming under uncertainty (LPUU). This technique explicitly characterizes each realization of the random variables by a constraint and leads to large problems in realistic situations. This method greatly handicaps modelers; in fact, Cohen and Thore viewed their model more as a tool for sensitivity analysis (in the aggregate) than a normative decision tool. The computational intractability and the perceptions of the formulation precluded consideration of problems other than those that were limited both in terms of time periods (Cohen and Thore used one, and Crane 1971 used two) and in the number of variables and realizations. Booth (1972) applied this formulation by limiting the number of possible realizations and the number of variables considered, in order to incorporate two time periods. Although relatively efficient solution algorithms are available for solving SLPSR's (Wets 1966), analysts have solved these models using an "extensive representation."

With the possible exception of the Bradley-Crane model, none of the cited models gives an adequate treatment of the essential features necessary for an adequate operational bank asset and liability management model that is computationally tractable. An ideal operational model should contain the following features:

1. multiperiodicity that incorporates: changing yield spreads across time, transaction costs associated with selling assets prior to maturity, and the synchronization of cash flows across time by matching maturity of assets with expected cash outflows;
2. simultaneous considerations of assets and liabilities to satisfy basic accounting principles and match the liquidity of assets and liabilities;
3. transaction costs that incorporate brokerage fees and other expenses incurred in buying and selling securities;
4. uncertainty of cash flows that incorporates the uncertainty inherent in the depositors' withdrawal claims and deposits (to ensure that the structure of the asset portfolio gives the bank the capacity to meet these claims);
5. the incorporation of uncertain interest rates into the decision-making process to avoid lending and borrowing decisions that may ultimately be detrimental to the financial well-being of the bank; and
6. legal and policy constraints appropriate to the bank's operating environment.

In this paper, we develop an SLPSR model that essentially captures these features of asset and liability management while maintaining computational feasibility. Section 2 contains background concerning SLPSR models and the solution algorithm. The asset and liability management (ALM) model is described and formulated in Section 3. In Section 4 we apply ALM to the operations of the Vancouver City Savings Credit Union. Section 5 compares ALM to Bradley and Crane's model. Final remarks and conclusions appear in Section 6.

2. Stochastic Linear Programs with Simple Recourse

The basic (SLPSR) model is

$$\underset{x\geq 0}{\text{minimize}}\ Z(x) \equiv c'x + E_\xi\left[\min_{y^+,y^-\geq 0} (q^{+\prime}y^+ + q^{-\prime}y^-)\right]$$

$$\text{subject to } Ax = b \qquad (1)$$

$$Tx + Iy^+ - Iy^- = \xi$$

where $c, x \in R^n$, $y^+, y^-, q^+, q^- \in R^{m_2}$, A is $m_1 \times n$, T is $m_2 \times n$, I is a m_2 = dimensional identity matrix and ξ is a m_2-dimensional random variable distributed independently of x on the probability space (Ξ, F, F). The SLPSR model is the two-stage process: choose a decision vector x, observe the random vector ξ, then take the corrective action (y^+, y^-). The model is said to have simple recourse because the second stage minimization is fictitious since (y^+, y^-) are effectively unique functions of (x, ξ).

Beale (1955) and Dantzig (1955) independently proposed the SLPSR model as a special case of the general linear recourse model that replaces $Iy^+ - Iy^-$ by Wy for a general matrix W. Detailed presentations of the theory of this model appear in Kall (1976), Parikh (1968), and Ziemba (1974). If we assume that $Ax = b$, $x \geq 0$ has a solution x^0 and $q^+ + q^- \geq 0$, then (1) has an optimal solution and is a separable convex program. If ξ is absolutely continuous, then Z is differentiable and (1) may be solved using modifications of standard feasible direction algorithms (see, e.g., Wets 1966 and Ziemba 1974). If ξ has a finite distribution, then Z is piecewise linear and (1) is equivalent to a large linear program. Wets (1974) noted that the deterministic equivalent linear program can be written in the form

$$\text{minimize } c'x - \sum_{i=1}^{m_2} \sum_{l=1}^{k_i+1} (q_i^- - F_{il} q_i) y_{il} + \sum_{i=1}^{m_2} q_i^+ \bar{\xi}_i$$

$$\text{subject to } Ax = b, \quad x \geq 0 \tag{2}$$

$$\sum_{j=1}^{n} t_{ij} x_j - \sum_{l=1}^{k_i} y_{il} = \alpha_i$$

$$y_{i1} \leq d_{i1}, \quad 0 \leq y_{il} \leq d_{il}, \quad 0 \leq y_{i,k_i+1}$$

where $i = 1, \ldots, m_2$, $l = 2, \ldots, k_i$, $d_{i1} = \xi_{i1}$, $d_{il} = \xi_{il} - \xi_{i,l-1}$, $q_i = q_i^+ + q_i^-$, $\xi_{i1} < \ldots < \xi_{ik_i}$ are the possible values of each ξ_i, the ith component of ξ, with probabilities $f_{i1}, \ldots, f_{ik_i}$ and $F_{is} = \Pr(\xi_i < \xi_{is}) = \sum_{l=1}^{s-1} f_{i1}$.

It is possible to develop an algorithm using generalized upper bounding constructs that will solve (2) in a number of pivots that is of the same order of magnitude as the number of pivots required to solve the mean linear programming approximation problem, i.e., (1) with $\bar{\xi}$ replacing ξ. The linear program (2) has the same number of working basis elements, $(m_1 + m_2)$, as the mean problem. Wets (1974, 1984) has developed an algorithm that has been coded by Collins (1975), Kallberg and Kusy (1976). The code was written to solve problems with up to 70 stochastic constraints, 220 total constraints and 8 realizations per random element. The code can be expanded to solve much larger problems. The International Institute of Applied Systems Analysis, Laxenberg, Austria (IIASA) is currently developing more sophisticated codes to handle larger problems. See Wets (1983) for extension of his algorithm to the convex case.

Formulation (1) is essentially static, while the asset and liability management problem is dynamic. We utilize model (2) and its efficient computational scheme, while at the same time retaining as many of the dynamic aspects of the model as possible. To do so, we utilize the approximation described below. The general n-stage SLPSR problem is

$$\begin{aligned} &\underset{\substack{x^1 \geq 0 \\ Ax = b^1}}{\text{minimize}}\; c^{1\prime} x^1 \\ &+ E_{\xi^1}\left\{ \min_{y^{1+}, y^{1-} \geq 0} \left[q^{1+\prime} y^{1+} + q^{1-\prime} y^{1-} + \ldots + \right.\right. \\ &\qquad + \underset{\substack{x^n \geq 0 \\ A^n x^n = b^n}}{\text{minimize}} \left[c^{n\prime} x^n + E_{\xi^n | \xi^{n-1}, \ldots, \xi^1} \right. \\ &\qquad \left.\left.\left. \left\{ \min_{y^{n+}, y^{n-} \geq 0} [q^{n+\prime} y^{n+} + q^{n-\prime} y^{n-}] \right\} \right] \ldots \right] \right\} \end{aligned} \tag{3}$$

$$\text{subject to } \sum_{j=1}^{i} T_{ij} x^i + I y^{i+} - I y^{i-} = \xi^i, \quad i = 1, \ldots, n.$$

The approximation procedure aggregates $x^2, \ldots, x^n$ with x^1 and $\xi^2, \ldots, \xi^n$ with ξ^1. Thus in (1), one chooses $x \equiv (x^1, \ldots, x^n)'$ in stage one, observes $\xi \equiv (\xi^1, \ldots, \xi^n)'$ at the end of stage one, and these steps together determine $(y^+, y^-) \equiv [(y^{1+}, y^{1-}), \ldots, (y^{n+}, y^{n-})]$ in stage two. This approach yields a feasible procedure for the true dynamic model (3) that is computationally feasible for large problems and incorporates partial dynamic aspects, since penalty costs for periods $2, \ldots, n$ are considered in the choice of $x^1, \ldots, x^n$. Aggregating all future period decision variables into x^1 would make the first period decision function as if all future period decisions were the same regardless of the scenario. The decision maker is primarily interested in the immediate revision of the bank's assets and liabilities. The ALM model incorporates immediate revision by setting apart by an arbitrarily small time period, times 0 and 1. Point 0 is the bank's initial position and point 1 is the bank's position immediately after running the model. In practice, the model is rolled over continuously. To partially overcome the drawbacks of a static solution technique, the decision variables are defined so that a security can be purchased in one time period and sold in one or more subsequent periods.

The feasibility of the SLPSR approximation provides an upper bound on the true optimal solution to (3). A lower bound on (3) is provided by Mandansky's (1960) linear programming mean model, that is, (3) with $\bar{\xi}^i$ replacing ξ^i for all i. Hence one can bound the true optimal solution with two easily computable programs. Specific numerical details of such bounds appear in Kallberg, White and Ziemba (1982) and Kusy (1978).

The recourse aspect of the model gives it a dynamic flavor. The model is two-stage: initially the decision variables are chosen, next the stochastic variables are observed, and these two steps determine the recourse variables (to recover feasibility) and their corresponding penalties. The penalty is a function of both the constraint violated and the magnitude of violation. The recourse cost has the effect of restraining "aggressive" choices of decision variables if the costs involved with regaining feasibility outweigh the benefits. The ALM model's dynamic features are embodied in its continuous rollover, the definition of variables that permits flexibility, and the recourse aspect of SLPSR.

A problem that arises in most deterministic as well as stochastic dynamic planning models is the handling of the end effects. In the formulation of ALM, we have assumed that the model runs for n periods; in the application, n is 5 years. This approach works well for our situation and our goal of obtaining a good first period decision. The most satisfactory theoretical approach to end effects is Grinold's (1977, 1983) dual equilibrium formulation. With this procedure, the dual variables for periods $n + 1, \ldots$ increase in value with the appropriate interest rate. This result yields an equivalent one-period constraint set for the infinite horizon-periods $n + 1, \ldots$. It is possible to extend Grinold's (1977, 1983) deterministic theoretical results and applications to energy planning models (Grinold 1980) to the stochastic case, although this has not appeared in the literature thus far. For ALM, applying Grinold's approach would yield one more set of constraints for a period "6." We did not make this further refinement because its added accuracy did not seem to outweigh the additional computational costs. However, developers of SLPSR models should keep this approach in mind in their choice of model formulation.

3. Formulation of the ALM Model

The ALM model is an intertemporal decision making optimization tool to determine a bank's portfolio of assets and liabilities given deterministic rates of returns and cost (interest rates), and random cash flows (deposits). Although the asset and liability management problem is a continuous decision problem, with portfolios constantly being revised over time, the computations and analysis involved with a continuous time process are infeasible for a practical model. Therefore, the ALM model is developed as a multiperiod decision problem in which portfolios are determined at discrete points in time (e.g., the end of each accounting period).

The ALM model has the following features:

1. Objective function: Maximize the net present value of bank profits minus the expected penalty costs for infeasibility.
2. Constraints:
   a. legal, being a function of the bank's jurisdiction;
   b. budget: initial conditions and the sources and uses of funds;
   c. liquidity and leverage, to satisfy deposit withdrawals on demand;
   d. policy and termination; constraints unique to the bank, and conditions to ensure the bank's continuing existence after the termination of the model; and
   e. deposit flows.

Constraints (a) and (b) are deterministic, (c) consists of both deterministic and stochastic constraints, (d) can consist of either deterministic or stochastic constraints, and (e) contains only stochastic constraints.

Chambers and Charnes, and Cohen and Hammer (1967) have justified the use of linear functions to model a bank's asset and liability management problem. Thus from the point of view of linearity, their analysis establishes the appropriateness of SLPSR. The recourse aspect is justified with the following argument. In the banking business, constraint violations do not imply that the intermediary is put into receivership. Rather, the bank is allowed to restructure its portfolio of assets to regain feasibility at some cost (penalties). With the inherent uncertainties, the asset and liability management problem is well modeled as a stochastic linear program with simple recourse.

3.1. Notation for the ALM Model

x_{ij}^k = amount of asset k purchased in period i sold in period j; $k = 1, \ldots, K$; $i = 0, \ldots, n - 1$; $j = i + 1, \ldots, n$

x_{00}^k = initial holdings of security k

$x_{i\infty}^k$ = amount of security k purchased in period i to be held beyond the horizon of the model

y_i^d = new deposits of type d in period i; $d = 1, \ldots, D$

y_0^d = initial holdings of deposit type d

b_i = funds borrowed in period i

y_{js}^+ = shortage in period j in stochastic constraint s

y_{js}^- = surplus in period j in stochastic constraint s

p_{js}^+ = proportional penalty cost associated with y_{js}^+

p_{js}^- = proportional penalty cost associated with y_{js}^-

β_{ij}^k = parameter for shrinkage, under normal economic conditions, in period j of asset type k purchased in period i

α_{ij}^k = parameter for shrinkage, under severe economic conditions, in period j of asset type k purchased in period i

t_i^k = proportional transaction cost on asset k, which is either purchased or sold in period i

r_i^k = return on asset k purchased in period i

T_j = tax rate on capital gains (losses) in period j

τ_j = marginal tax rate on income in period j

z_{ij}^k = proportional capital gain (loss) on security k purchased in period i and sold in period j

γ_d = the anticipated fraction of deposits of type d withdrawn under adverse economic conditions

c_i^d = rate paid on deposits of type d

p_i = discount rate from period i to period 0

K_0 = set of possible current assets as specified by the British Columbia Credit Union Act

K_1 = set of primary and secondary assets as defined in the capital adequacy formula (caf)

K_2 = set of minimum risk assets as defined in the caf

K_3 = set of intermediate risk assets as defined in the caf

q_i = penalty rate for the potential withdrawal of funds, in period i, that are not covered by assets in $K_1 \cup \ldots \cup K_3$

P_i = liquidity reserves for the potential withdrawal of funds, in period i, not covered by assets in $K_1 \cup \ldots U K_3$

k_{m_i} = m_ith mortgage

ξ_{js} = discrete random variable in period j for stochastic constraint type s, $s \in S$ where S is the set of stochastic constraints.

3.2. The ALM Model

$$\begin{aligned}\underset{x,y,b}{\text{maximize}} \sum_{k=1}^{K} \Bigg[\sum_{j=2}^{n} x_{0j}^k \Bigg\{ \sum_{l=2}^{j} r_0^k(1-\tau_l)p_l + z_{0j}^k(1-T_j)p_j \Bigg\} + x_{01}^k z_{01}(1-T_j) \\ + \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} x_{ij}^k \Bigg\{ \sum_{l=i+1}^{j} r_i^k(1+\tau_l)p_l + z_{ij}^k(1-T_j)p_j \Bigg\} + x_{0\infty}^k \sum_{l=2}^{n} r_0^k(1-T_l)p_j \Bigg] \\ + \sum_{j=\tau_i}^{n} \sum_{l=i+1}^{n} x_{i\infty}^k r_i^k(1-\tau_l)p_l\end{aligned} \quad \text{discounted returns and capital gains (net of taxes) on assets}$$

$$- \sum_{d=1}^{i=1} \Bigg[\sum_{j=1} y_0^d(1-\gamma_d/2)(1-\gamma_d)^{j-1} c_j^d p_j + \sum_{j=1}^{n} 1/2 y_j^d c_j^d p_j + \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} y_i^d(1-\gamma_d/2)(1-\gamma_d)^{j-i} c_j^d p_j \Bigg] \quad \text{net discounted cost of deposits (demand and time)}$$

$$- b_0 c_0^b p_1 - \sum_{j=1}^{n} b_j c_j^b p_{j+1} \quad \text{cost of direct borrowing from other banks and a central bank}$$

$$- E_\xi \min_{y^+, y^-} \sum_{j=1}^{n} \sum_{s \in S} p_{js}^+ y_{js}^+ + p_{js}^- y_{js}^- \quad \text{expected penalty costs for constraint violations}$$

subject to

(a) Legal constraints

$$\sum_{k \in K_0} \sum_{i=0}^{1} \Bigg\{ \sum_{l=2}^{n} x_{il}^k + x_{i\infty}^k \Bigg\} - 0.1 \Bigg[\sum_{d=1}^{D} y_0^d + (1-\gamma_d) y_0^d/2 + y_1^d/2 + b_0 + b_1 \Bigg] \geq 0, \qquad j = 1.$$

$$\sum_{k \in K_0} \sum_{i=0}^{j} \Bigg\{ \sum_{l=j+1}^{n} x_{il}^k + x_{i\infty}^k \Bigg\} - 0.1 \Bigg[\sum_{d=1}^{D} \Bigg\{ \sum_{i=0}^{j-1} y_i^d + (1-\gamma_d) y_0^d/2(1-\gamma_d)^{j-i-1} + y_1^d/2 \Bigg\} + b_j \Bigg] \geq 0, \qquad j = 2, \ldots, n.$$

(b) Budget constraints

i. Initial holdings

$$\sum_{j=1}^{n} x_{0j}^k + x_{0\infty}^k = x_{00}^k, \qquad k = 1, \ldots K.$$

$$y_0^d = y_{00}^d, \qquad d = 1, \ldots, D.$$

ii. Sources and uses

$$\sum_{k=1}^{K} \Bigg[\sum_{l=2}^{n} \{x_{1l}^k + x_{1\infty}^k\}\{1 + t_1^k - x_{01}^k\{1 + z_{01}^k(1-T_1) - t_1^k(1+z_{01}^k)\} \Bigg] + \sum_{d=1}^{D} [\gamma_d y_{0/2}^d - y_1^d/2] - b_1 = 0, \qquad j = 1.$$

$$\begin{aligned}\sum_{k=1}^{K} \Bigg\{ \sum_{l=j+1}^{n} x_{jl}^k + x_{j\infty}^k \Bigg\}\{1 + t_j^k\} - \sum_{i=0}^{j-1} \Bigg[\Bigg\{ \sum_{l=j}^{n} x_{il}^k + x_{i\infty}^k \Bigg\}\{r_i^k(1-\gamma_j)\} + x_{ij}^k\{1 + z_{ij}^k(1-T_j) - t_j^k(1+z_{ij}^k)\} \Bigg] \\ - \sum_{d=1}^{D} \Bigg[\sum_{i=0}^{j-2} y_i^d(1-\gamma_d)^{j-i-2}\{(-\gamma_d)(1-\gamma_d/2)\} + (1-\gamma_d) y_{j-1}^d/2 + y_j^d/2 - (1-\gamma_d/2) c_{j-1}^d - y_{j-1}^d c_{j-1}^d/2 \Bigg] \\ + b_{j-1}(1 + c_{j-1}^b) - b_j = 0, \qquad j = 2, \ldots, n.\end{aligned}$$

(c) Liquidity constraints

i. $$-\sum_{k\in K}\sum_{i=0}^{j}\left[\sum_{l=j+1}^{n} x_{il}^{k}\alpha_{ij}^{k} + x_{i\infty}^{k}\alpha_{ij}^{k}\right] + b_j + \sum_{d=1}^{D}\gamma_d\left[\sum_{i=0}^{j-1} y_i^d(1-\gamma_d)^{j-i-1}(1-\gamma_d/2) + y_j^d/2\right] \leqslant P_{1j}/q_{ij}.$$

ii. $$-\sum_{k\in K_1 U K_2}\sum_{i=0}^{j}\left[\sum_{l=j+1}^{n} x_{il}^{k}\alpha_{ij}^{k} + x_{i\infty}^{k}\alpha_{ij}^{l}\right] + b_j + \sum_{d=1}^{D}\gamma_d\left[\sum_{i=0}^{j-1} y_i^d(1-\gamma_d)^{j-i-1}(1-\gamma_d/2) + y_j^d/2\right] \leqslant P_{2j}/q_{ij}.$$

iii. $$-\sum_{k\in K_1 U K_2 U K_3}\sum_{i=0}^{j}\left[\sum_{l=j+1}^{n} x_{il}^{k}\alpha_{ij}^{k} + x_{i\infty}^{k}\alpha_{ij}^{k}\right] + b_j + \sum_{d=1}^{D}\gamma_d\left[\sum_{i=0}^{j-1} y_i^d(1-\gamma_d)^{j-i-1}(1-\gamma_d/2) + y_j^d/2\right] \leqslant P_{3j}/q_{ij}$$

iv. $$-\sum_{k=1}^{K}\sum_{i=0}^{j}\left[\sum_{l=j+1}^{n}(1-\beta_{ij}^{k})x_{il}^{k} + (1-\beta_{ij}^{k})x_{i\infty}^{k}\right] + y_{js}^{+} - y_{js}^{-} \geqslant P_{1j} + P_{2j} + P_{3j} + b_j$$
$$+\sum_{d=1}^{D}\left[\sum_{i=0}^{j-1} y_i^d(1-\gamma_d)^{j-i-1}(1-\gamma_d/2) + y_j^d/2\right] + \xi_{js}, \qquad j=1,\ldots,n, \quad s\in S.$$

(d) Policy constraints

$$-0.1\sum_{i=0}^{j}\left[\sum_{l=j+1}^{n} x_{il}^{k_{m1}} + x_{i\infty}^{k_{m1}}\right] + \sum_{i=0}^{j}\left[\sum_{l=j+1}^{n} x_{il}^{k_{m2}} + x_{i\infty}^{k_{m2}}\right] + y_{js}^{+} - y_{js}^{-} \leqslant \xi_{js}, \qquad j=1,\ldots,n, \quad s\in S.$$

(e) Deposit flows

$$y_j^d + \sum_{i=0}^{j-1} y_i^d(1-\gamma_d)^{j-i} + y_{js}^{+} + y_{js}^{-} = \gamma_{js}, \qquad j=1,\ldots,n; \quad d=1,\ldots,D, \quad s\in S.$$

(f) Nonnegativity

$$x_{ij}^{k},\ b_i,\ y_i^d,\ y_{js}^{+},\ y_{js}^{-} \geqslant 0 \quad \text{for all} \quad i, j, k, d.$$

There are no discount factors in the constraints since each constraint refers to conditions in one period. The ALM model treats the first two types of constraints, legal and budget, as deterministic. The legal constraint states that the current assets cannot be less than 10% of the total liabilities (as defined by the Credit Union Act, (British Columbia Government, Section 106, 1982a). The legal constraints are peculiar to the locale of the institution being studied. The budget constraints include the initial conditions and the accounting identity—uses and sources of funds are equal. See also Credit Union Regulations (British Columbia Government 1982b and Barr 1984).

The liquidity constraints follow from liquidity calculations in the Federal Reserve Board's capital adequacy formula (caf). The requirement that the market value of a bank's assets is adequate to meet depositor's withdrawal claims during adverse economic conditions is the principal constraint in the caf. The market value of the bank's assets should be not less than the liquidity reserves for disintermediation under severe economic conditions plus liabilities. This constraint is the final liquidity constraint in ALM. Although this constraint is not stochastic, a bank portfolio manager may violate it because the caf is merely a suggested guideline rather than a strict regulation. The penalty for a violation of this constraint is $\sum_{i=1}^{3} q_i$. This *elastic* treatment allows the constraint to be violated when the benefits of violation exceed the costs. In this manner, the criticism, levelled at modellers using FRB's conservative constraints, can be resolved in a systematic manner. See Section 4.1.3 for more discussion concerning these constraints.

The FRB's caf has been used as a suggested disintermediation liquidity guideline for U.S. banks but is not a legal requirement. Because it is quite conservative and tends to reflect depression style financial conditions, its formulas are not currently used by U.S banks. This policy may change in the future, however, given the extreme threat to the world banking system by the many nonperforming loans to countries such as Argentina, Brazil, Chile and Mexico. In British Columbia, however, where the Vancouver City Savings Credit Union operates, the banking establishment is quite conservative and has continued to use the caf's liquidity guidelines. Hence we have used them in ALM. Besides ALM, the caf is also used in British Columbia's Credit Union Reserve Board's credit union solvency model. Ellis (1984), the Board's manager of research services, who operates the solvency model as a regulatory tool, feels that "... the caf reflects the limited experience of the credit union movement in British Columbia when liquidating credit unions." In other environments where models such as ALM may be appropriate, alternative liquidity constraints are easily handled in the SLPWR formulation, assuming that such constraints are linear or can be linearized.

The fourth set of constraints is also elastic. These constraints are introduced to capture the internal policy of the institution modeled. In reality, minor constraint violations of bank policies are usually tolerable, while more severe violations are increasingly less tolerable. The introduction of a piece-wise linear convex penalty function (via additional constraints) can capture the dependency between the penalty costs and the extent of the policy violations. This result is accomplished by the addition of supplementary constraints to reflect the increased seriousness of the magnitude of constraint violations.

The final set of constraints, deposit flows, is stochastic. Since deposit flows are continually turned over and bear various rates of interest, the model must reflect the gross (and not net) flows during an accounting period. This property of the problem was incorporated in the model by establishing a proportional outflow (statistically calculated by the FRB and corroborated for use in British Columbia by White and Dobrzanshi 1976) of old funds during each period.

We now develop the three types of liability expressions in the ALM formulation. The deposit flow constraints represent the total amount of new deposits in the jth period. The total amount of new deposits of type d generated in period j yields the equation

$$y_j^d + \sum_{i=0}^{j-i} y_i^d(1 - \gamma_d)^{j-i} = BS_j^d,$$

where y_j^d is the total amount of new type d deposits in period j, γ_d is the annual rate of withdrawal of type d deposits, and BS_j^d is the discrete random variable representing the balance sheet figure of type d deposits at the end of the jth period.

The second type of liability expression represents the total amount of deposits outstanding during a period. Since the model is discrete, an approximation to the continuous flow is made by assuming that half of a period's net flows arrive at the beginning of the period and the other half arrive at the beginning of the next period. During the first period, the funds available are

$$[y_0^d + (1 - \gamma_d)y_0^d]/2 + y_1^d/2,$$

and for period j, they are

$$\sum_{i=0}^{j-1} [(1 - \gamma)^{j-i-1}y_i^d + (1 - \gamma)^{j-i}y_i^d]/2 + y_j^d/2,$$

or

$$\sum_{i=0}^{j-1} y_i^d(1 - \gamma_d/2)(1 - \gamma_d)^{j-i-1} + y_j^d/2.$$

This expression is used in the objective function and in the legal and liquidity constraints. The third liability expression is the incremental increase (decrease) of deposits from one period to the next. We use this incremental difference in the sources and uses constraint. For period j the incremental difference is

$$\left[\sum_{i=0}^{j-1} y_i^d(1 - \gamma_d/2)(1 - \gamma_d)^{j-i-1} + y_j^d/2\right]$$

$$- \left[\sum_{i=0}^{j-2} y_i^d(1 - \gamma_d/2)(1 - \gamma_d)^{j-i-2} + y_{j-1}^d/2\right]$$

$$= \sum_{i=0}^{j-2} y_i^d(1 - \gamma_d)^{j-i-2}[-\gamma_d(1 - \gamma_d/2)]$$

$$+ (1 - \gamma_d)y_{j-1}^d/2 + y_j^d/2.$$

3.3. Data Required to Implement the ALM Model

To implement the ALM model requires the following data:

1. the identification of assets in which the bank can potentially invest;
2. point estimates of the returns on these assets;
3. point estimates of capital gains (losses) as a function of the time the bank holds the assets;
4. identification of the liabilities that the bank can potentially sell;
5. point estimates of the costs of these liabilities;
6. the rate at which deposits are withdrawn;
7. an estimated weighted cost of funds to determine the discount rate;
8. pertinent legal constraints;
9. parameters used in the development of liquidity constraints;
10. policy constraints used by the bank;
11. estimates of the marginal distributions of the stochastic resources; and
12. unit penalties incurred for shortage or surplus in the stochastic constraints.

Remarks. a. Since the ALM model has a separable objective, only the marginal distributions of the components of the resource vector are needed to find the optimal solution.

b. The shortage (y^+) and surplus (y^-) variables have specific meanings in the ALM formulation. Consider a realization $\xi_{js}^{d\prime}$ of the random deposit ξ_{js}^d. If

$$y_j^d + \sum_{i=0}^{j-i} y_i^d(1 - \gamma_d)^{j-i} < \xi_{js}^{d\prime},$$

then $y^+ > 0$ and $y^- = 0$, assuming $p^+ + p^- > 0$; y^+ is the amount of funds that could have been used for investment purposes in the ALM. Since the cost of deposits is usually lower than the returns on assets, the bank would want to utilize all available funds. A penalty $p^+ > 0$ for the opportunity cost can be determined by assuming that the funds not used can be invested in earning assets. The y^+ dollars would be available at rate c and invested in some asset at a rate r. The penalty, p^+, equals $(r - c)$ discounted to point 0 plus the net discounted returns on $y^+(r - c)$ to the horizon of the model (the profits that could have been generated).

If

$$y_j^d + \sum_{i=0}^{j-i} y_i^d (1 - \gamma_d)^{j-i} > \xi_{js}^{d\prime},$$

then $y^- > 0$ and $y^+ = 0$, and a surplus occurs. The bank would then have to divest itself of some earning assets. The cost, p^-, of this action is $(r - c)$ discounted to point 0 plus the net discounted returns on $y^-(r - c)$ to the horizon of the model (that is, the profits that would have been generated with unavailable funds). Both p^+ and p^- are positive, and profit is lowered if either too little or too much is invested.

4. Application of ALM to the Vancouver City Savings Credit Union

This section considers an application of the ALM model to the asset and liability portfolio problem of Vancouver City Savings Credit Union (VCS). We also discuss procedural aspects of implementing the model for this and related institutions. This study was prompted by the VCS's continual liquidity problem, and focuses on the five year planning period 1970–1974.

During this period, the firm's assets grew at a compound rate of 57%/year from \$26 million to \$160 million, and it developed an aggressive policy of investing in high yielding assets, predominantly mortgages. In 1974, VCS realized that the combination of their aggressive investment policy and changing market conditions was creating serious liquidity problems. Investors were trading low yield term deposits for higher yield deposits. Meanwhile, outstanding mortgage loans were still earning returns on the basis of lower fixed rates. At that point, this study was initiated.

4.1. Model Details

We now describe the input necessary to implement the ALM model at VCS. This discussion focuses on general concepts concerning methods of data collection, choice of decision variables, constraints and objective function. The actual data, a 92 × 257 input matrix, appear in Kusy.

The first-stage variables are assets (x_{ij}^k) and liabilities (y_i^d and b_i). There are eleven asset types:

1. cash;
2. British Columbia Central Credit Union shares;

3–6. federal government bonds maturing in $i = 1, \ldots, 4$ years;

7. federal government bonds maturing in 5–10 years;
8. provincial government bonds maturing in more than 10 years;

9–10. first and second mortgages with a 3-year term; and

11. personal loans.

Six types of liabilities are considered:

1. demand deposits;
2. share capital of VCS;
3. borrowing from banks; and

4–6. term deposits maturing in $i = 1, 3, 5$ years.

These asset and liability types generate 132 and 36 variables, respectively, including initial positions. For example, a 4-year federal government bond purchased at the beginning of the third time period generates decision variables x_{34}^6, x_{35}^6, and $x_{3\infty}^6$, where x_{34}^6 and x_{35}^6 are the amounts of the initial investment to be sold in period four and five, respectively, and $x_{3\infty}^6$ is the amount to be held at the horizon. The choice of assets and liabilities was based on VCS's historical portfolios (1968–1975), so that comparison between actual portfolios and ALM generated portfolios could be easily made. Although cash flows are continuous over time, the model assumes that all transactions occur at the beginning of periods. Cash flows during any period are modeled by assuming that half the flow occurs at the beginning of the present period and the other half at the beginning of the next period. The model has the following constraints.

4.1.1. *Legal Constraints*

The source for legal constraints is the Credit Union Act of British Columbia (British Columbia Government 1982a), which places three operational restrictions on the composition of the portfolio of assets and liabilities. The first constraint requires that credit unions maintain at least 10% of the total of non-equity shares, deposits and borrowings (denoted by the set I) in high liquid assets (denoted by the set I_L):

$$\sum_{i \in I_L} x_{it} \geqslant 0.1 \sum_{i \in I_L} x_{it}.$$

The second requirement states that credit unions must maintain at least 1% of their nonequity shares, deposits and borrowings in cash and term deposits:

$$x_{1t} + x_{2t} \geq 0.01 \sum_{i \in D} y_{it}.$$

The final constraint restricts the credit union's borrowing from opportunities denoted by the set B, to one half of the nonequity shares, deposits and statutory reserve account:

$$\sum_{b \in B} y_{bt} \leq 0.5 \sum_{i \in D} y_{it}.$$

Since the planning horizon has 5 periods, the legal requirements account for 15 constraints.

4.1.2. Budget Constraints

There are 22 budget constraints: 17 establish the initial positions of the 11 asset and 6 liability types, and 5 require the sources and uses of funds to be equal in each period.

4.1.3. Liquidity Constraints

The liquidity constraints ensure that the firm has sufficient capital reserves to meet severe withdrawal claims under adverse economic conditions. The application of the FRB's capital adequacy formula to British Columbia's credit unions is described in the 1973 Credit Union Reserve Board report.

The first three constraints establish capital reserves based upon the structure of the portfolio of assets and liabilities:

$$P_i \geq q_i\left(W - \sum_{k \in K_1 U \,.\, UK_i} \alpha_k\right) \qquad i = 1, 2, 3,$$

where P_i is the required reserve necessary to meet the excess withdrawal claims, q_i measures the reserves required for potential withdrawal claims that exceed the realizable portion of the assets contained in $K_1U \ldots UK_i$, α_k is a parameter that measures the realizable portion of the value of asset k if the asset is to be liquidated quickly under adverse economic conditions, and $W = \sum_{i=1}^{m} \gamma_i y_i$ is the dollar value of the expected withdrawal claims under adverse conditions, where γ_i measures the contraction of liability y_i under adverse economic conditions. The γ's used were 0.47 for demand deposits, 0.36 for term deposits and 1.0 for borrowing; see White and Dobrzanshi for justification.

The assets are classified as follows:

1. "Primary and Secondary Reserves": (K_1), which include cash, treasury bills, and government bonds of less than 5 years' maturity;
2. "Minimum Risk Assets" (K_2), which include government bonds with more than 5 years' maturity, and municipal bonds; and
3. "Intermediate Assets," which include mortgage and personal loans. The principal constraint is

$$\sum_{i=1}^{K} (1 - \beta_i)x_i \geq \sum_{i=1}^{3} P_i + \left.\begin{array}{l}\text{total right-surplus-}\\ \text{equity-hand side}\\ \text{of balance sheet}\end{array}\right\}$$

where β_i is a parameter to measure the shrinkage of asset i, when the asset is to be liquidated quickly. The values used for α_k, q_i, and β_i are those suggested by the FRB (Crosse and Hempel 1973). Since our purpose is not to develop an operational model for VCS, but rather to demonstrate the applicability of the ALM model, the parameter values used provide an adequate proxy. In the development of an operational model, it would be necessary to estimate the parameters. Since these constraints hold for all 5 periods, there are 20 liquidity constraints.

4.1.4. Policy Constraints

Two types of policy constraints are included:

1. personal loans should not exceed 20% of the first mortgage loans in any period t, $x_{tL} \leq 0.2x_{tm}$; and
2. second mortgages should not exceed 12.5% of first mortgages, $x_{tp} \leq 0.125\, x_{tm}$.

The rationale is that, since returns on first mortgages are less risky than second mortgages or personal loans, some first mortgages are desirable (even though they may have lower returns) to respond to management's preference for a less risky portfolio. These constraints may be violated without legal implications and are modeled by treating the constraints as stochastic using $(p^+, p^-) = (0, 1)$. There are 10 such constraints over the 5 periods.

4.1.5. Deposit Flows

The variable y_j^d represents new deposits of type $d = 1, \ldots, 5$ generated in period $j = 1, \ldots, 5$, and ξ_{jd} is a discrete random variable representing the balance sheet of deposit type d at the end of period j. The deposit flow constraints are

$$y_j^d + \sum_{i=0}^{j-i} y_i^d(1 - \gamma_d)^{j-i} + y_{jd}^+ - y_{jd}^- = \xi_{jd},$$

where the γ's (1.0 for demand deposits and 0.36 for term deposits) are included to relfect the gross flow of deposit funds. We estimated the distribution of ξ_{jd}

Table I
Discount Rate Factors

	1970	1971	1972	1973	1974
Average yearly yield	0.0559	0.0356	0.0356	0.0547	0.0782
Multiperiod discount factor $\pi(1/(1 + AYY_t))$	0.9435	0.9110	0.8797	0.8341	0.7736

Table II
Returns on Assets

Type of Asset	Returns on Asset by Year					
	1969	1970	1971	1972	1973	1974
1-year federal government bond (fgb)	0.0725	0.0620	0.0450	0.0510	0.0610	0.0800
2-year fgb	0.0749	0.0657	0.0490	0.0550	0.0654	0.0803
3-year fgb	0.0758	0.0684	0.0525	0.0590	0.0680	0.0807
4-year fgb	0.0767	0.0710	0.0555	0.0626	0.0698	0.0810
5-year fgb	0.0776	0.0758	0.0615	0.0674	0.0717	0.0827
10-year provincial government bond	0.0840	0.0904	0.0803	0.0813	0.0836	0.0991
First mortgage	0.0938	0.1040	0.0943	0.0921	0.0959	0.1124
Second mortgage	0.1050	0.1220	0.1108	0.1083	0.1123	0.1321
Personal loans	0.1040	0.1170	0.1075	0.1050	0.1075	0.1275
BCCU shares	0.0600	0.0600	0.0600	0.0600	0.0700	0.0700

using the balance sheet figures of VCS for 1970–1974; see Kusy for specific estimates.

The penalties for shortages associated with these constraints are:

1. for demand deposits and share capital, p^+ is the total discounted return on a 1-year term deposit minus the discounted cost of the funds calculated to the horizon of the model;
2. for term deposits maturing in 1 or 3 years, p^+ is the total discounted return on a 5-year term deposit minus the discounted cost of the funds over the model's horizon; and
3. for term deposits maturing in 5 years, p^+ is the total discounted return on a 10-year provincial government bond minus the discounted cost of the funds over the horizon.

The penalties p^- for surpluses associated with the deposit flow constraints are the total discounted returns on first mortgages minus the discounted costs of funds over the horizon. The penalty approach attempts to model a conservative management strategy with surplus funds when realized sources exceed uses and when there are shortages.

4.1.6. Objective Function

The objective is to maximize the expected discounted revenues minus expected discounted costs, including penalty costs. The source for data on the returns on the federal and provincial government bonds is the Bank of Canada (1975). The source for the returns on BCCU shares, demand deposits and share capital is Vancouver City and Savings Credit Union (1968–1975).

The discount rate used was the time value of money.

Table III
Returns on 5-Year Bonds

Decision Variable X_{ij}	Return r_{ij}^7
X_{12}^7	$(0.0758)(0.9435) = 0.0720$
X_{13}^7	$(0.0758)(0.9435 + 0.9110) = 0.1410$
X_{14}^7	$(0.0758)(0.9435 + 0.9110 + 0.8797) = 0.2070$
X_{15}^7	$(0.0758)(0.9435 + 0.9110 + 0.8797 + 0.8341) = 0.2700$
$X_{1\infty}^7$	$(0.0758)(0.9435 + 0.9110 + 0.8797 + 0.8341 + 0.7736) = 0.3290$

Table IV
Cost and Liabilities of Credit Union

Type of Liability	Cost of Liability by Year					
	1969	1970	1971	1972	1973	1974
1-year term deposit	0.0712	0.0780	0.0720	0.0680	0.0780	0.0990
3-year term deposit	0.0750	0.0820	0.0760	0.0690	0.0820	0.0980
5-year term deposit	0.0785	0.0850	0.0800	0.0800	0.0850	0.0975
Demand deposit	0.0400	0.0460	0.0410	0.0420	0.0560	0.0770
Share capital	0.0500	0.0500	0.0500	0.0550	0.0575	0.0800

The risk-free rate (the average yield on 3-month treasury bills) (Bank of Canada 1975) is given in Table I.

The returns on the assets (Bank of Canada 1975, Vancouver City Savings Credit Union 1968–1975) are given in Table II.

Purchase of a 5-year federal government bond in 1970 would generate decision variables X_{12}^7, X_{13}^7, X_{14}^7, X_{15}^7, and $X_{1\infty}^7$. The returns are the interest earned each year, discounted to the beginning of the planning horizon, and are given in Table III.

The costs of the liabilities (Vancouver City Savings Credit Union) are given in Table IV.

The cost of a 5-year term deposit (y_1^3) sold during 1970 is shown in Table V.

4.2. Results of the VCS Application

This application demonstrates applicability of the ALM model and tests the sensitivity of the solution generated. To accomplish these goals, we ran the basic model along with several variants that used modified penalty costs and probability distributions, as well as a deterministic model with all random variables replaced by their means.

The basic model has symmetric, 3-point probability distributions (0.2, 0.6, 0.2) for the deposit flow constraints and degenerate probability distributions for the liquidity and policy constraints. The penalties for all stochastic constraints are asymmetric. The optimal value of the basic model is $2,520,316.01 ($8,288,941.53 in expected profits minus $6,282,885 in expected penalties). As shown by Madansky (1960) the mean model provides an upper bound on the optimal value of a stochastic linear program; in this instance, the bound is 8.3% above the optimal value of the basic model, or $2,729,502.24. The solution to the mean model, when used as a feasible point in the basic model, provides the lower bound of $2,278,187, some 9.6% less than the optimal value. The structure of the two portfolios is similar in the initial period; however, the investment patterns differ in later periods. In particular, the basic model invests less heavily in less liquid assets (namely mortgages). See Kusy for specifics.

Table V
Cost of 5-Year Term Deposit

Year	Cost
1970	$(0.5)(0.0850)(0.9435) = 0.0401$
1971	$(0.82)(0.0850)(0.9110) = 0.0635$
1972	$(0.82)(0.64)(0.0850)(0.8797) = 0.0392$
1973	$(0.82)(0.64)^2(0.0850)(0.8341) = 0.0238$
1974	$(0.82)(0.64)^3(0.0850)(0.7736) = 0.0141$
Total Cost	0.1807

The mean model was initially infeasible, since the initial portfolio held by VCS violated the liquidity constraints (a situation known to management). To secure feasibility, we added variables to the liquidity constraints. The objective coefficients of these variables were the same as the penalties associated with violating the stochastic liquidity constraints in the basic model. To develop further insight into the operations of VCS, we could set the penalties arbitrarily high so that the model would vioalte the liquidity constraints only to attain feasibility. The amount by which the constraints are violated will be the amount of liquid reserves that the firm needs to meet the FRB's suggested liquidity requirements.

Variants of the basic model were run in order to ascertain the effects of different probability distributions, penalty costs and parameter values. The initial change was the alteration of the first legal constraint from the requirement that current assets be at least 10% of the liabilities to at least 1% of the liabilities. This change increases the optimal value of $2,906,773.53 ($8,657,619.24 in expected profits minus $5,750,845.71 in expected penalties). For the initial two periods, the investment pattern deviated substantially from that of the basic model in that more of the incremental funds were allocated to longer term assets. After the first two periods, there was no generalizable behavior in the investment patterns of the two models.

The basic model was then further altered to include a change in the probability distributions (0.05, 0.50, 0.45) of the cash flows. The optimal value increases to \$3,256,500.65 (\$8,872,911.53 in expected profits minus \$5,661,410.80 in expected penalties). The expected net profit rises compared with the basic model and the model with the parameter change, while the expected penalty costs decrease in both cases because:

1. all the violations of the stochastic constraints are now feasible only with a probability 0.05 instead of 0.2 (decreased penalties); and
2. constraints that were not violated in the basic model because of excessive penalties are now violated in the modified model by 15%, resulting in more profits.

Although it is not possible to make definitive generalizations from these runs of the ALM model, some conclusions may be inferred. First, the asymmetry of the probability distibutions may have a substantial effect on the optimal solutions and values. Of particular importance is the sensitivity of the estimate of the probability distribution around the value on the left-hand side of the stochastic constraints. Second, the solutions are sensitive to changing penalty costs. Third, the various stochastic models have substantially different solutions than the mean model. This result indicates that reliance on the deterministic models as normative tools can lead to erroneous solutions. Fourth, the implementation of this model is no more difficult than the implementation of a similar deterministic model. Finally, solving the ALM formulation requires the same order of computations necessary for the mean or related deterministic models. (Kallberg, White, and Ziemba found similar conclusions on a SLPSR model of short-term financial planning.) The runs were made on the University of British Columbia's IBM 370/168. The ALM model is 92 variables and 257 constraints, with 40 stochastic constraints. Using the SLPSR code (Kallberg and Kusy), the ALM model's solution took 37 seconds of CPU time. To solve an equivalent sized deterministic problem took 30 seconds using the SLPSR code and 17 seconds on the standard linear programming code UBC LIP. Experience in solving SLPSR models and related deterministic problems indicates that the CPU times are in a ratio of about 1.5–2 to 1. Detailed output appears in Kusy.

5. Comparison of the ALM and SDT Approaches

The asset and liability management problem is a continuous decision problem in which actions (e.g., portfolio revisions) are made continuously on the basis of new information (e.g., differing forecasts of future interest rates, and the like). The ideal way to model the ALM problem would be via a continuous, time-adaptive stochastic dynamic program. At present such a formulation is computationally intractable for the types of problems we are considering. ALM, and Bradley and Crane's stochastic decision tree model (SDT), which is described in this section, constitute two types of operational models that discretize time and probability distributions. In this section, we simulate a large number of economic scenarios to compare these two models.

5.1. The Bradley–Crane Stochastic Decision Tree Model (SDT)

The Bradley–Crane (1972, 1973, 1976) model depends upon the development of economic scenarios that are intended to include the set of all possible outcomes. The scenarios may be viewed as a tree diagram for which each element (economic conditions) in each path has a set of cash flows and interest rates. The problem is formulated as a linear program whose objective is the maximization of expected terminal wealth of the firm. There are four types of constraints:

A. cash flow—the firm cannot purchase more assets than it has funds available;
B. inventory balancing, which ensures that the firm cannot sell and/or hold more of an asset at the end of a period than it held at the beginning;
C. capital loss, which does not allow the net realized capital losses in a period to exceed some prespecified upper bound; and
D. class composition, which limits the holding of a particular asset.

The basic formulation is

$$\text{maximize} \sum_{e_N \in E_N} p(e_N) \sum_{k=1}^{K} \left\{ \sum_{m=0}^{N-1} [y_m^k(e_m) + v_{m,N}^k(e_N)] h_{mN}^k(e_N) + [y_N^k(e_N) + v_{NN}^k(e_N)] b_N^k(e_N) \right\}$$

subject to

$$\text{A)} \quad \sum_{k=1}^{K} b_n^k(e_n) - \sum_{k=1}^{K} \left[\sum_{m=0}^{n-2} y_m^k(e_m) h_{m,n-1}^k(e_{n-1}) + y_{n-1}^k(e_{n-1}) b_{n-1}^k(e_{n-1}) \right] - \sum_{k=1}^{K} \sum_{m=0}^{n-1} [1 + g_{m,n}^k(e_n)] s_{m,n}^k(e_n) = f_n(e_n) \quad \text{(Cash flows)}$$

B) $- h^k_{m,n-1}(e_n - 1) + s^k_{m,n}(e_n) + h^k_{m,n}(e_n) = 0,$

$$m = 0, \ldots, n-2$$

$$- b^k_{n-1}(e_{n-1}) + s^k_{n-1,n}(e_n) + h^k_{n-1,n}(e_n) = 0,$$

$$h^k_{0,0}(e_0) = h^k_0 \qquad \text{(Inventory Balance)}$$

C) $$- \sum_{k=1}^{K} \sum_{m=0}^{n-1} g^k_{m,n}(e_n) s^k_{m,n}(e_n) \le L_n(e_n) \qquad \text{(Capital Losses)}$$

D) $$\sum_{k \in K^i} \left[b^k_n(e_n) + \sum_{m=0}^{n-1} h^k_{m,n}(e_n) \right] (\lesseqgtr) C^i_n(e_n), \qquad i = 1, \ldots I$$

(Category Limits)

$$b^k_{m,n}(e_n) \ge 0, \quad s^k_{m,n}(e_n) \ge 0, \quad h^k_{m,n}(e_n) \le 0,$$

$$m = 1, \ldots, n-1$$

(Nonnegativity)

where

$e_n \in E_n$; $n = 1, \ldots, N$;
$k = 1, \ldots, K$;
e_n is an economic scenario from period 1 to n having probability $p(e_n)$;
E_n is the set of possible economic scenarios from period 1 to n;
K_i is the number of assets of type i, and K is the total number of assets;
N is the number of time periods;
$y^k_m(e_m)$ is the income yield per dollar of purchase price in period m of asset k, conditional on e_m;
$v^k_{m,N}(e_N)$ is the expected terminal value per dollar of purchase price in period m of asset k held at the horizon (period N), conditional on e_N;
$b^k_n(e_n)$ is the dollar amount of asset k purchased in period n, conditional on e_n;
$h^k_{m,n}(e_n)$ is the dollar amount of asset k purchased in period m and held in period n, conditional on e_n;
$s^k_{m,n}(e_n)$ is the dollar amount of asset k purchased in period m and sold in period n, conditional on e_n;
$g^k_{m,n}(e_n)$ is the capital gain (loss) per dollar of purchase price in period m of asset k sold in period n;
$f_n(e_n)$ is the incremental increase (decrease) of funds available for period n;
$L_n(e_n)$ is the dollar amount of maximum allowable net realized capital losses in period n; and
$C^i_n(e_n)$ is the upper (lower) bound in dollars on the amount of funds invested in asset type i in period n.

The SDT formulation is a true dynamic model. The first decision (immediate revision, $h^k_{01}(e_1)$, $b^k(e_1)$, $s^k_{01}(e_1)$) has as its feasible set the intersection of all possible realizations. That is, the current solution must be feasible for the set E_N. This decision is conditional on the realization of economic events in the first period. Similarly, in each succeeding period to the end of the planning horizon, the decisions generated are all conditional on the states of nature that have occurred up to the current decision point.

The model has a number of attractive features, including its dynamic nature and associated clever solution using decomposition. However, the formulation has features that detract from its practicability. The capital loss and category limit constraints have as upper (or lower) bounds amounts (resources) generated arbitrarily by portfolio managers rather than through a systematic procedure. For example, no consideration is given to the portfolio mix in the development of bounds, except in the sense that upper (or lower) bounds are placed on asset categories. At some point, this situation may imply that the bank has invested a disproportionate amount of its available funds in long-term bonds compared to the amount of short-term liabilities held. Also, the formulation does not have any liquidity constraints. Since the capital loss and category limit constraints actually determine the composition of the solution, the arbitrary nature of the choice may bias the solution.

One feature of the SDT model is that first period feasibility is assured for every possible scenario. Such fat formulations (see, e.g., Madansky 1962) shrink the feasible set and give substantial importance to scenarios with low probabilities of occurrence. For example, consider the two-period problem for which an investor:

1. has \$100 in period 1 to invest in asset x_{11} with return $r_{11} = 0.1$ maturing after one period, or x_{12} with return $r_{12} = 0.2$ per period maturing after two periods;
2. receives in period 2 either an additional \$50 to invest with probability 0.9 or loses \$50 with probability 0.1;
3. has, in period 2, the opportunity to invest in a one-period asset x_{21} with return $r_{21} = 0.1$ or can sell off his holdings in x_{12} at a 20% discount; and
4. stipulates that his realized capital losses cannot exceed 10% of the outstanding funds in any period.

The linear programming optimal solution is $b^1_1(l_1) = 11.11$, $b^2_1(l_1) = 88.89$, $b^1_2(l_{21}) = 80.00$, $h^1_2(l_{22}) = 0$, $h^2_{12}(l_{21}) = 88.89$, $h^2_1(l_{12}) = 63.89$, $S^1_{12}(l_{21}) = 11.11$, $S^1_{12}(l_{22}) = 11.11$, $S^2_{12}(l_{21}) = 0$, $S^2_{12}(l_{22}) = 25.00$, with optimal value \$42.87, where "b" means buy, "h" hold, and "s" sell and the l's denote the possible scenario of events. The bound on realized capital losses is binding. If a maximal loss of 15% were allowed, it would be optimal to purchase \$100 of asset x_{12} and sell \$37.50

Table VI
Size of Bradley-Crane Model

	Number of:					
Model	Assets	Asset Classes	Time Periods	Possible Realizations Per Period	Variables	Constraints
(1)	8	1	3	3	656	319
(2)	30	5	3	3	2,460	1,141
(3)	30	5	3	5	6,120	2,827
(4)	30	5	5	5	246,120	116,827

of x_{12} at the end of period 1, if the \$50 is lost. This modification yields an optimal value of \$44.11. Thus consideration of this low probability event significantly alters the optimal solution. By contrast, in the simple recourse ALM formulation, the right-hand sides are not binding; recourse at a penalty cost is allowed to compensate for decision infeasibility. The recourse formulation has more first-period decision flexibility than the decision tree formulation.

To gain computational tractability, the SDT model considers only bonds. If D is the number of possible realizations per period, n is the number of time periods, I is the number of asset claims, and K is the number of assets, then the number of variables is $(3 + 5D + 7D^2 + \ldots + (2n + 1)D^{n-1})$. The number of constraints equals the sum of the cash flow constraints: $(1 + D + D^2 + \ldots + D^{n-1})$, the capital loss constraints $(1 + D + D^2 + \ldots + D^{n-1})$, the category limit constraints $(I)(1 + D + D^2 + \ldots + D^{n-1})$, the inventory balance constraints $(K)(I + 2D + 3D^2 + \ldots + nD^{n-1})$, and the initial conditions K. Table VI shows the effect of differing numbers of assets, possible realizations per period, and number of periods on problem size.

Bradley and Crane (1976) solved model (1) in 68 seconds on an IBM 360/65. The subsequent models add more realism and are much larger. In the case of (4), the basis of the master problem, if one uses decomposition, has 5467 rows and the problem has about 850,000 nonzero elements. The computational and data-handling difficulties of (2) and (3) are less striking but remain formidable. Bradley and Crane (1976, p. 112) are aware of these computational difficulties:

> Unfortunately, taking uncertainty explicitly into account will make an asset and liability management model for the entire bank computationally intractable, unless it is an extremely aggregated model. The complexities of the general dynamic balance sheet management problem are such that the number of constraints and variables needed to accurately model the environment would be very large.

Our aim has been to develop a computationally tractable model that still has some dynamic and other desirable features and represents a practical approach to bank asset and liability management. We now compare the ALM and SDT models via simulation.

5.2. The Economic Scenarios

To maintain computational feasibility for the SDT model, we considered only three assets and one liability over three periods. The assets are a one-period treasury bill, a term deposit maturing beyond the horizon of the model and a long-term mortgage. The liability is a demand deposit. The returns and costs of these financial instruments were generated from 26 consecutive observations using data from the Bank of Canada. To obtain a reasonable correlation of interest rates, the returns and costs were made a function of the prime rate using the following distribution of the prime rate (R):

$\Pr(R = r)$	6/26	3/26	1/26	2/26	1/26	2/26	4/26
r	0.06	0.065	0.0675	0.075	0.0775	0.08	0.085

$\Pr(R = r)$	2/26	2/26	2/26	1/26
r	0.09	0.095	0.11	0.115

The distributions in Table VII were derived for the difference between the prime rate and the rate of return of each of the four financial instruments, where the random variables, M, D, T, and L are defined to be the difference between the prime rate and the

Table VII
Distributions Derived from Difference between the Prime Rate and Rate of Return of Financial Instruments

m	$P(M \leqslant m)$	d	$P(D \leqslant d)$	t	$P(T \leqslant t)$	l	$P(L \leqslant l)$
0.0037	0.0	−0.0104	0.0	−0.0388	0.0	−0.0275	0.0
0.0088	0.2	−0.0072	0.2	−0.306	0.2	−0.025	0.2
0.0198	0.42	+0.0008	0.44	−0.0253	0.5	−0.0225	0.31
0.0235	0.62	+0.004	0.5	−0.0225	0.77	−0.2	0.92
0.0297	0.81	+0.118	0.78	−0.0174	0.81	−0.175	1.00
0.0338	1.00	+0.0195	1.00	−0.0051	1.00		

mortgage rate, term deposit rate, treasury bill rate and liability rate, respectively. At time zero the investor has $100,000 in demand deposits invested equally in the three assets. The demand deposits are assumed to increase (decrease) uniformly from one period to the next on [−20,000, 20,000]. If the demand deposits decrease so that assets must be liquidated, then the FRB's parameters for quick liquidation are used. The discounts for treasury bills, term deposits, and mortgages are 0.5%, 4%, and 6%, respectively. The constraints on the investor are of the BC type and include 1) cash flows, 2) capital losses, 3) class composition, and 4) terminal conditions. The capital loss constraints assume that the investor does not want to realize net losses of more than 3% of the outstanding demand deposits in periods 1 and 2, and 4% in period 3. The class composition constraints limit the investor from having more than $50,000 in total investments in any asset in periods 1 and 2, and $60,000 in period 3. The terminal constraints include a discount on the assets in the current portfolio so that all funds are not simply invested in the highest yielding assets and held to the horizon of the model. These discounts are one-half of the normal discounts. The objective of the model is to maximize the net expected returns.

5.3. Formulations of the Stochastic Dynamic Programming Model

Formulating the SDP requires the establishment of an economic scenario over the three-period horizon. This scenario includes a representative distribution for cash flows and the rate of return for the various financial instruments. Since for computational tractability SDP requires crude approximations of probability distributions, we limited to two the number of possible realizations of the random variables during each time period. With initial demand deposits of $100,000 and an incremental difference in the interval [−20,000, 20,000], a natural 2-point distribution is $90,000 and $110,000 with equal probabilities. Using this distribution, we maintained the mean of the underlying distribution, although the variance was smaller (1.0×10^5 versus 1.33×10^5). The distribution was constructed similarly at the third decision point. The cash flows have the distribution shown in Figure 1.

Using the same approach gives the first period rate of return for a particular financial instrument (assume mortgage rate) as the median prime rate ($\bar{R}$) plus the median of the difference between the prime rate and the rate of return of the mortgages ($\bar{M}$). The two-point estimate in the second period is $\bar{R}$ plus m, where $P(M \leqslant m) = 0.25$. The four rates of return in the third period are $\bar{R} + m$ where $P(M \leqslant m)$ is 0.875, 0.625, 0.375, and 0.125, respectively.

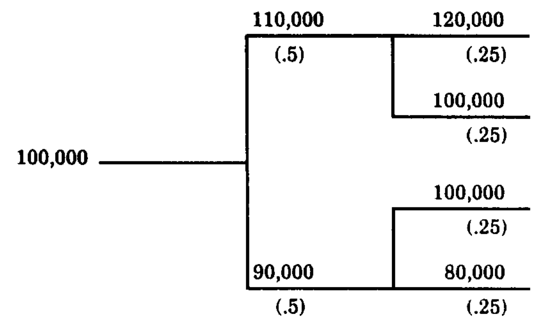

Figure 1. Cash flow distribution.

Figure 2 specifies the distribution of the rates of return used in this example.

For the simulation, we used 70% of the nonchecking rate as the demand deposit rate since the nonchecking rate dominates the treasury bill rate, a situation that would have precluded investment in treasury bills a priori. This ad hoc derivation of the demand deposit rate does not impinge on the usefulness of the simulation because the objective is to demonstrate that one solution technique may be operationally superior.

Treasury bills mature after one period, hence 18 variables define all potential investment opportunities. Since the term deposits and mortgages mature beyond the horizon of the model, 42 variables are required to describe all investment opportunities in each of these categories. The variables necessary to define the demand deposits include the initial position, the demand deposit flows in period 1, two demand deposit flows in period 2, and the 4 demand deposit flows in period 3. In all, 110 variables define the investment opportunities in the problem.

There are 4 types of constraints. Constraints 1 to 7 are the cash flow requirements for each period under each economic scenario; namely, uses of funds equal sources of funds. Constraints 8 to 14 require realized capital losses to be less than 3% of the outstanding demand deposits in period 1 and 2, and 4% in period 3. Constraints 15–35 limit the funds invested in each asset, as prescribed in the problem. Constraints 36–89 (inventory balancing) consist of the initial holdings of each of the four financial instruments and record the transactions in each economic scenario.

The demand deposit flow constraint for period 1 places an upper bound on the funds potentially available for investment. The capital loss constraints and the composition constraints add another 28 slack variables to the formulation. The SDT formulation has 89 constraints and 139 variables.

The objective is to maximize the expected value of the net returns from the portfolio over the horizon of

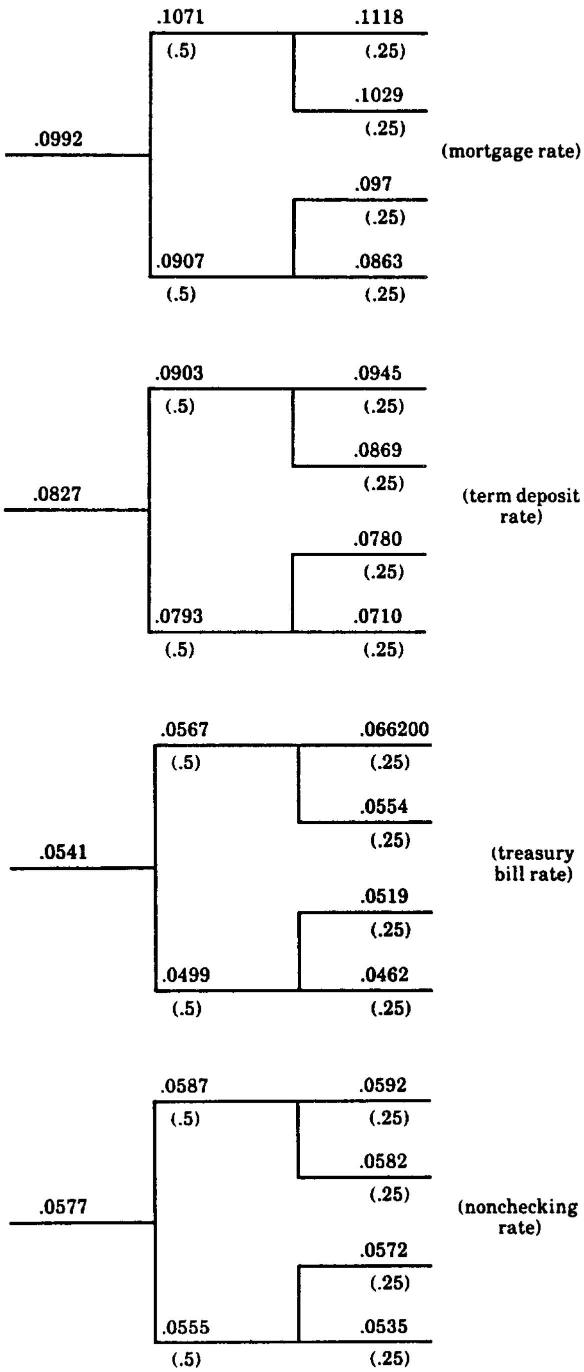

Figure 2. Distribution of rates of return.

the model. The coefficient of each variable is the product of the net return and its probability of occurrence.

5.4. Formulations of the ALM Model

The ALM uses the same information as the SDT model although it has fewer constraints because of its different treatment of uncertainty. The investment opportunities for treasury bills, term deposits, mortgages, and demand deposits are defined by 6, 11, 11 and 4 variables, respectively. There are 25 constraints, of which 5 are stochastic. The key constraints are:

(i) 3 constraints to balance the initial holding of an asset with the future buying and selling of the asset;

(ii) 3 constraints that equate the cash flows for the 3 periods;

(iii) 3 constraints for each of the 3 assets for composition requirements;

(iv) 4 constraints to describe the initial position of the 3 assets and 1 liability;

(v) 3 capital loss constraints of which the first period's is deterministic as constraints (i) to (iv) listed above, and the others are stochastic; and

(vi) 3 stochastic constraints that describe the flow of demand deposits.

When we add 9 slack variables for the class composition constraints and 1 for the deterministic capital loss constraint, the SLPR formulation has 25 constraints and 42 variables, plus the recourse variables.

The right-hand sides of the stochastic demand deposit constraints are representative points from the uniform distribution used in the SDT model. However, because of the ability of the Wets algorithm to handle many realizations without creating computational difficulties, the number of points chosen is larger than in the SDT model. The penalty for violations of any of these constraints is the net return to the horizon of the model, generated by a portfolio consisting of 50% mortgages and 50% term deposits, since their portfolio is considered, a priori, to be potentially the highest yielding portfolio. This penalty is

$$0.5[(1+\bar{r}_m)^{4-n}-1]+0.5[(1+\bar{r}_t)^{4-n}-1]-[(1+\bar{r}_d)^{4-n}-1],$$

where $n = 1, 2, 3$ is the period; $\bar{r}_m$ is the median return on mortgages; $\bar{r}_t$ is the median return on term deposits; and $\bar{r}_d$ is the median cost of demand deposits.

The right-hand sides of the stochastic capital loss constraints are the representative points used in the SDT formulation. A penalty of 4.1% is used for violations of these constraints.

The objective is to maximize the net returns minus the expected penalties for constraint violations. The coefficient of each variable is the net return for the first-stage variables and the penalty for the second-stage variables.

5.5. Results of the Simulation

In normative financial planning models, the objective is generally to determine which portfolio changes should be effected immediately. The multiperiodicity of financial models compensates for the shifting economic scenarios across time. However, the purpose of the model is to determine the changes to be implemented immediately. Hence the simulation is intended to determine which model produces the best first-period solution. In reality, decisions may be made at any point in a period; however, a discrete time model will aggregate information to consider all decisions to be made at the start of each period—facing random rates of return. Again, the incremental cash flows are aggregated so that one-half is available at the beginning of the current period. In both formulations the same initial security holdings are given and the cash flows for the next period are random.

The process starts with an initial portfolio. Both the ALM and SDT models determine an optimal solution for the first period. A random cash flow is then generated. If the amount of funds spent during the first period exeeds the random cash flow, then an amount equal to excess spending is divested from the present portfolio, which consists of 45% of mortgages, 45% term deposits and 10% treasury bills. If random cash flows exceed spending during the first period, then the incremental amount is invested in treasury bills. After this reconciliation, revenues are the sum of the known returns of the assets held since the start of the period and the random returns of the assets bought at the start of the period. The costs are the sum of the random cost of demand deposits and the discount for selling securities prior to maturity. The reconciled portfolio serves as the new initial portfolio, which is then used to generate new solutions for both models. This cycle is repeated 8 times. The process is repeated 50 times, for a total of 400 scenarios. See Kusy for the simulation flowchart, computer program and details of the results.

The simulation results for the ALM and SDT formulations are used to test two hypotheses. The first hypothesis

$$H_1\colon \mu_d = \mu^1_{\text{SDP}} - \mu^1_{\text{ALM}} \geqslant 0$$

is used to test whether or not the initial period profit for ALM is superior to that for SDT. We test this hypothesis by examining the paired differences of the profits for the initial run of the 50 cycles for both models. The specific information used is:

1. the mean of the paired difference ($251.37 in favor of ALM); and
2. the standard deviation of the paired differences ($150.43).

The correlation between the ALM and SDT profits is 0.958. Given the large sample, the significance of the paired differences is tested using the t statistic

$$\frac{-251.37}{1.50.43/\sqrt{50}} = -11.81.$$

The test statistic is significant at the 0.001 level, hence the null hypothesis is rejected. Thus, ALM yields a statistically significant better initial solution than SDT.

The second hypothesis

$$H_2\colon \mu_d = \mu_{\text{SDP}} - \mu_{\text{ALM}} \geqslant 0$$

is used to test whether the mean profit for ALM is superior to that for SDT.

We test this hypothesis by examining the paired differences of the mean profits of the 8 runs of 50 cycles for both models. The specific information used is:

1. the mean of the paired differences ($297.26 in favor of ALM); and
2. the standard deviation of the paired differences ($308.74).

The correlation between the ALM and SDT mean profits is 0.785. The t statistic is

$$\frac{-297.26}{308.74\sqrt{50}} = -6.81.$$

Since this statistic is significant at the 0.001 level, the null hypothesis is rejected. Thus the ALM formulation yields a statistically significant better solution than the SDT formulation.

To test the stability of these summary statistics, we ran a second simulation, using ALM. The results of this simulation are analyzed similarly: 1) a test of the initial solution of the 50 cycles, and 2) a test of the mean profits for the 8 runs of the 50 cycles. The information necessary to test the first hypothesis is 1) the mean profits for the first and second ALM runs ($4645.85 and $4672.23, respectively), and 2) the standard deviations for the 2 runs ($421.11 and $482.15, respectively). We first test the hypothesis that both samples have the same mean

$$H_3\colon \mu_{\text{ALM}_1} = \mu_{\text{ALM}_2}.$$

The standard deviation used for the test statistic is the root of the pooled variance. The test statistic is

$$\frac{4672.23 - 4645.85}{90.53} = 0.291.$$

The test statistic for the final hypothesis is established similarly and is

$$\frac{4783.13 - 4720.15}{86.84} = 0.730.$$

Since the test statistics for H_3 and H_4 are not significant at the 0.10 level, there is no evidence that the mean is not stable.

We used a CDC 6400 at the University of British Columbia to perform the computations. The total CPU time to perform the 400 iterations for ALM was 0.240 hours, and for SDP, 6.385 hours. This difference in computation time explains why only a limited number of financial instruments, time periods and realizations were used in the simulations, and highlights the gap in tractability between the ALM and SDT techniques. Details of the codes used to perform the simulations appear in Kusy.

6. Final Remarks

The existing literature on bank asset and liability management is based on two approaches: the mean variance portfolio selection model, and models that utilize an objective of maximizing expected net returns. Section 1 reviews deterministic and stochastic models that are based on these constraints. In an attempt to determine which of these two approaches is most suitable for asset and liability management problems, Myers (1968) showed that existence of security market equilibrium implies that net present value is the appropriate objective function. The most comprehensive model of this type is the Bradley–Crane stochastic decision tree model (1972, 1973, 1976). They attempted to overcome the crucial obstacle to asset and liability management of incorporating uncertainty while maintining computational tractability. Their model is a useful one and offers many important insights. However, it does not really maintain computational tractability for realistically sized bank asset and liability management problems. It also possesses some undesirable features, notably arbitrary constraints on capital losses, an absence of portfolio mix constraints and an immediate revision that must satisfy all possible forecasted economic constraints.

The ALM model is an attempt to remedy some of these deficiencies and, as shown in Sections 3 and 4, is an implementable model of bank asset and liability management. The results of the application to the Vancouver City Savings Credit Union indicate that the ALM model is superior to related deterministic models, and the simulations in Section 5 indicate that ALM generates better first-period decisions than does the Bradley–Crane SDT model. An additional advantage of ALM over SDT is that the results of small changes in parameter values and previous realizations can easily be determined by calculating penalties, while SDT may become infeasible. The CPU time to solve ALM is 1.5–2 times that of a related deterministic model and much less than that required for the Bradley–Crane model (0.24 hour versus 6.39 hours for simulation, as shown in Section 5). Hence ALM is a feasible option for implementation in large banks. To apply ALM, one must determine essentially the same information as with a deterministic model: 1) deposit flow estimates, 2) estimates fo the term structure of interest rates, 3) estimates of withdrawal rates of deposits under various economic conditions, 4) legal constraints governing the behavior of the financial institution, 5) policy constraints, 6) recommended reserves for maintaining a liquid position, and 7) the initial position of the firm plus discrete probability distributions for random elements and penalty costs. The model is capable of handling useful policy constraints. The major drawback of ALM is that it is not a true dynamic model. Simulations such as those in Section 5 provide confidence in the approach taken in ALM. Birge (1983) delineates bounds on the error associated with aggregation schemes (such as the one used in this paper) in the context of general recourse models. The general problem of the accuracy of various approximations in multiperiod stochastic programs is currently being studied by Ziemba in collaboration with J. Birge, M. A. H. Dempster, H. Gassman, P. Kall and R. Wets. The results of their work will, we hope, provide general guidance regarding solution technique trade-offs in dynamic stochastic modeling. See Birge and Wets (1986) and Kall, Frauendorfer and Ruszczyński (1984) for surveys of the current theory in this area.

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