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Single-Period Multiproduct Inventory Models with Substitution

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SINGLE-PERIOD MULTIPRODUCT INVENTORY MODELS WITH SUBSTITUTION

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We study a single period multiproduct inventory problem with substitution and proportional costs and revenues. We consider N products and N demand classes with full downward substitution, i.e., excess demand for class i can be satisfied using product j for $i \geqslant j$. We first discuss a two-stage profit maximization formulation for the multiproduct substitution problem. We show that a greedy allocation policy is optimal. We use this to write the expected profits and its first partials explicitly. This in turn enables us to prove additional properties of the profit function and several interesting properties of the optimal solution. In a limited computational study using two products, we illustrate the benefits of solving for the optimal quantities when substitution is considered at the ordering stage over similar computations without considering substitution while ordering. Specifically, we show that the benefits are higher with high demand variability, low substitution cost, low profit margins (or low price to cost ratio), high salvage values, and similarity of products in terms of prices and costs.

1. INTRODUCTION

On this paper we study a single period periodic review multiproduct inventory model with stochastic demands, proportional revenues and costs, *substitution*, and arbitrary starting inventory. We assume that there are N products and N demand classes. We say that a *substitution* occurs whenever demand from class i is met using stocks of product j, $i \neq j$. Though various structures of substitution arise in real life, in this study we will focus on full downward substitution. That is, demands from class i can be satisfied only using stocks of product j for all $i \geqslant j$. A downward substitution structure arises, for example, in the semiconductor industry producing similar integrated circuits with varying performance characteristics; circuits with higher performance characteristics (e.g., speed) could substitute for demand for circuits with lower performance characteristics but not vice versa. Another example (Leachman 1987) in the same industry relates to memory chips where a higher capacity (4 megabytes) chip can be used to satisfy demands for lower capacity memory (say, 2 megabytes). An example from the steel industry is stated in Wagner and Whitin (1958) where steel beams of a greater strength can substitute for beams of lesser strength. Potential advantages of recognizing substitution structures to effectively manage inventories and reduce costs are also discussed by Fuller et al. (1983).

We assume that the demands for each class are stochastic. The order, holding, penalty, and salvage costs are proportional. The revenue earned is also linear in the quantity sold. We assume no delivery lags and no capacity constraints. There is a substitution cost proportional to the quantity substituted. The allocation of products to demands is carried out after observing the demand, however, orders have to be placed before demands are realized. Thus we have a two-stage decision problem. We call the first stage ordering decision the *ordering* problem and the second stage allocation decision the *allocation* problem. The main contributions of this paper are as follows:

1. We characterize the structure of the optimal policy for a general N-product downward substitution problem with arbitrary starting inventory.
2. We develop an expression for the first differentials of the profit function for the general N-product problem, which in turn allows us to prove several interesting properties of the optimal solution. These expressions are also useful to develop any gradient based algorithm to solve the problem.
3. For a two-product problem, we demonstrate that significant gains (depending upon cost and demand parameters) could accrue from solving for optimal quantities when substitution is considered at the ordering stage

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over similar computations without considering substitution while ordering.

Veinott (1965) was the earliest work on optimal policies for multiproduct inventory models. This study was generalized by Ignall and Veinott (1969) and extended to perishable inventories by Deuermeyer (1980). Ignall and Veinott (1969) give conditions under which the optimal order quantity will be monotone in the initial inventory. This condition requires that the cost function be convex and its Hessian a *substitute matrix* (see Definition 1 in Appendix A). For a two-product problem it is very easy to verify if the Hessian is a substitute matrix; however, verification of these conditions for the specific substitution structure of the N-product problem being discussed here is extremely difficult. In contrast, we take a different approach. First we show that the profit function is concave and submodular. We then develop explicit expressions for the first partial of the profit function for the general N-product problem and show that the optimal order quantity will be monotone in the initial inventory.

Analysis of single period two product substitution problems appear in McGillivray and Silver (1978), Parlar and Goyal (1984), Pasternack and Drezner (1991), and Gerchak and Wang (1996). Bitran and Dasu (1992) consider end-product substitution problems with single input component and random yields. Recently, Hsu and Bassok (1994) use techniques similar to the ones discussed here to solve a production/inventory problem similar to the one studied by Bitran and Dasu (1992). Our basic approach follows after Veinott (1965) and Ignall and Veinott (1969).

The rest of the paper is organized as follows. In § 2, we develop a general profit maximization model for single period multiproduct substitution problem. We show that the profit maximization problem is concave and submodular. We present a greedy algorithm for allocation of products to demand classes. For this greedy allocation policy, we give a new and compact notation of writing the first differentials of the profit function with respect to stock levels and prove several interesting properties of the optimal policy. In § 3 we develop some bounds on the optimal order points, present an iterative algorithm to compute the order points for a two-product problem, and present a computational study for the two-product problem to illustrate benefits of substitution. We conclude in § 4 with a summary, extensions, and directions for future research.

2. A MODEL

In this section, we give a profit maximization formulation for the multiproduct product, single period substitution problem. We suggest a greedy algorithm that solves the allocation problem optimally. We prove that the profit function is concave and submodular (see Definition 2 in Appendix A). We derive expressions for the first partials of the profit function and prove the optimality of a base stock policy.

2.1. Notation and Assumptions

There are N products and N demand classes. The demand for class i is a random variable with marginal density of $f_i(\cdot)$ and marginal distribution of $F_i(\cdot)$. The products and demand classes are numbered such that demand for class i can be satisfied using stock of product j as long as $i \geqslant j$; if product $j, j < i$ is used to satisfy demand from class i, then we allow for a cost of substitution denoted by b. We assume that the revenue earned, p_i, for each unit of satisfied demand in class i depends only on i and not on the type of product j used to satisfy this demand. Furthermore, we assume the following sequence of events:

1. At the beginning of the period, we first observe the inventory on-hand, x_i of product i and place order(s) for the various products.
2. The order(s) are delivered immediately. The inventory of product i after order delivery is y_i.
3. Demands for all classes are realized.
4. Demands are satisfied (using an allocation algorithm discussed later) earning revenue, p_i for a unit of satisfied demand for class i, backorder costs of π_i per unit for shortages in class i, and storage costs of $\tilde{h}_i$ per unit for leftover units of product i.
5. Excess stock, if any, of product i is salvaged at the end of the period for $\tilde{s}_i$ per unit.

Let c_i be the per unit purchase cost of product i and s_i the effective per unit salvage value of product i ($=\tilde{s}_i - \tilde{h}_i$). The net revenue from using product i to satisfy a unit of demand class j is denoted by $a_{i,j}$ and equals $p_j - b$ for $j > i$ and p_i for $i = j$. We make the following assumptions about various costs and revenues.

ASSUMPTION 1. $\pi_i + p_i \geqslant \pi_j + p_j$, *for* $i < j$.

ASSUMPTION 2. $s_i \geqslant s_j$, *for* $i < j$.

ASSUMPTION 3. $a_{ij} + \pi_j - s_i \geqslant 0$, *for* $i \leqslant j$.

Assumption 1 states that it is more profitable to satisfy unmet demand of class i than of class j, $i < j$. Assumption 2 states that the effective salvage value (salvage value less holding cost) of product i is not less than that of product j, for $i < j$. This ensures that it is not optimal to substitute product i for product j whenever there is excess inventory of product j. Assumption 3 states that substitution of product i for demand class j is profitable.

Throughout our discussion we will use the following convention: Random variables will be denoted by an uppercase letter and their realizations by the corresponding lowercase letter; vectors will be denoted by boldface type. All vectors have N-components unless specified otherwise.

2.2. The Profit Function

Let $I(\vec{x})$ be the maximum single period profits and $P(\vec{x}, \vec{y})$ be the expected single period profits when starting inventory before ordering is $\vec{x}$ and after ordering is raised to $\vec{y}$. Then

Figure 1. Allocation Algorithm (A).

```
For i = 1, 2, ..., N
{   u_i = d_i;
    j ← i;
    v_j = y_j;
    While "u_i > 0" AND "j > 0" do
    {     w_ji = min(u_i, v_j);
          u_i ← u_i − w_ji; v_j ← v_j − w_ji;
          j ← j − 1;
    }
}
```

$$I(\vec{x}) \equiv \max_{\vec{y} \geqslant \vec{x}} P(\vec{x}, \vec{y}). \tag{1}$$

Let $\vec{d} = (d_1, \ldots, d_N)$ be a vector of realized demands. Define $F(\vec{d}) = F_{1,2,\ldots,N}(d_1, d_2, \ldots, d_N)$ as the joint distribution of demands from class 1 to N. Let $G(\vec{y}, \vec{d})$ be the profits for a given stock level, $\vec{y}$, and realized demand, $\vec{d}$. Let w_{ij} be the quantity of product i allocated to demand class j. Then

$$P(\vec{x}, \vec{y}) = -\sum_{k=1}^{N} c_k(y_k - x_k) + \int_{\bar{R}_+^N} G(\vec{y}, \vec{d})\,dF(\vec{d}), \tag{2}$$

where

$$G(\vec{y}, \vec{d}) = \max_{u_i, v_i, w_{ji}} \sum_{i=1}^{N} \sum_{j=1}^{i} a_{ji} w_{ji} + \sum_{i=1}^{N} s_i v_i - \sum_{i=1}^{N} \pi_i u_i, \tag{3a}$$

subject to

$$u_i + \sum_{j=1}^{i} w_{ji} = d_i \quad \text{for } i = 1, \ldots, N, \tag{3b}$$

$$v_i + \sum_{i=1}^{N} w_{ij} = y_i \quad \text{for } i = 1, \ldots, N, \tag{3c}$$

$$w_{ij} \geqslant 0 \quad \text{for all } i, j = 1, \ldots, N, \tag{3d}$$

$$u_i, v_i \geqslant 0 \quad \text{for all } i = 1, \ldots, N. \tag{3e}$$

In (3a) the first term is the net revenue for satisfying demands, the second term is the storage cost incurred minus the salvage value earned on excess stock (v_i), and the last term gives the penalty costs incurred when there are shortages (v_i). Equation (3b) is the inventory balance equation for each product, and (3c) ensures that allocation quantities are equal to the available inventories for each product.

We present a greedy allocation algorithm (A) in Figure 1. The algorithm works sequentially starting from product 1 going down to N. It basically satisfies demand for class i using stock of product i first and then if necessary, uses leftover stocks (if available) to satisfy remaining unmet demand of i.

Proposition 1. *Under Assumptions 1–3 and a given starting inventory level, the allocation algorithm (A) will maximize profits in* $G(\vec{y}, \vec{d})$.

Proof. Observe that the allocation problem in (3) can be formulated as a restricted transportation problem in which the products represent source nodes, demand classes represent destination nodes, arcs with appropriate costs represent feasible allocations, and arcs with negative infinity costs represent forbidden allocations. It has been shown that a greedy solution (similar to the one presented in Figure 1) to a general transportation problem (Hoffman 1963) or to a restricted transportation problem (Shamir and Dietrich 1990) that rank-orders the arcs is optimal if and only if this rank-ordered sequence of arcs is a *Monge sequence* (see Definition 3 in Appendix A). Observe that our algorithm (A) also rank-orders arcs connecting the products and demand classes, and it can be shown that this rank-ordered sequence follows a Monge sequence, given Assumptions 1–3. □

2.3. Optimal Policy and Properties

In this subsection, we show that the profit function for the single period substitution problem is concave and submodular. Concavity and submodularity of the profit function enables us to prove several properties of the optimal policy. Suppose $\vec{x}$ denotes the initial inventory vector and $\vec{y}(\vec{x})$ be the optimal inventory levels after ordering. We then show that

1. There exists a $\vec{y}^*$ such that if $\vec{x} < \vec{y}^*$, then the policy is order up to $\vec{y}^*$.
2. No order is placed for product $i \in I$ where $I = \{i | x_i \geqslant y_i^*\}$.
3. The optimal inventory level after ordering, $\vec{y}(\vec{x})$, is a nonincreasing function of the starting inventory vector, $\vec{x}$.

The following proposition establishes the concavity and submodularity and of the profit function.

Proposition 2. *The profit function* $P(\vec{x}, \vec{y})$ *is concave and submodular.*

Proof. See Appendix B. □

Concavity implies that the constraint qualification condition holds and the Kuhn-Tucker conditions can be used to derive the policy structure and its properties. To do so, we need to write the expression for the first partials of the profit function.

Before we present a new technique to write the first partials of the profit function with respect to the stock position, we need some notation to define shortages and excesses of various subproblems. For any given stock level $\vec{y}$, apply the allocation algorithm (A). Define *a subproblem with only products* $i, \ldots, m$ as a snapshot of the substitution problem just after allocations w_{kj} to demand class $j \in \{i, \ldots, m\}$ from stocks of product $k \in \{i, \ldots, m\}$; that is, before using more expensive products $k \in \{1, \ldots, i-1\}$. Let S_j^i (a scalar) be the shortage of product j in a subproblem with only products $i, \ldots, j$. Let the vector, $\vec{S}_{j,n}^i = (S_j^i, S_{j+1}^i, \ldots, S_N^i)$, and $T_k = p_k + \pi_k - b$ for $k = 1, \ldots, N$.

In the sequel, we use the following convention. Suppose $\vec{x} \in \bar{R}^N$. Whenever we say that $\vec{x} > 0$, we mean $x_i \geqslant 0$ for all $i = 1, \ldots, N$ and $x_i > 0$ for at least one i; $\vec{x} \geqslant 0$ implies that $x_i \geqslant 0$ for all i and $x_i = 0$ for some i.

We then write:

$$\frac{\partial P^{(N)}}{\partial y_i} = -c_i + s_i \Pr\{\vec{S}^{\,i}_{i,N} = 0\} \tag{4a}$$

$$+ s_{i-1} \Pr\{\vec{S}^{\,i-1}_{i,N} = 0,\ \vec{S}^{\,i}_{i,N} > 0\}$$

$$+ \ldots + s_1 \Pr\{\vec{S}^{\,1}_{i,N} = 0,\ \vec{S}^{\,2}_{i,N} > 0\} \tag{4b}$$

$$+ b \Pr\{S^1_i = 0,\ S^i_i > 0\} \tag{4c}$$

$$+ (T_i + b) \Pr\{S^1_i > 0\}$$

$$+ \ldots + T_N \Pr\{\vec{S}^{\,1}_{i,N-1} = 0,\ S^1_N > 0\}. \tag{4d}$$

The above expression gives the increase in profits for a unit increase in the inventory of product i. The terms have the following interpretation. The first term on the right-hand side of (4a) is the order cost. The second term on the right-hand side of (4a) states that if there is no shortage of product i in a subproblem with only products $i, \ldots, N$, we will recover its salvage value. However, if there is a shortage of product i in a subproblem with only products $i, \ldots, N$, i.e., $\vec{S}^{\,i}_{i,N} > 0$, but no shortage of it in a subproblem with only products $i - 1, \ldots, N$, i.e., $\vec{S}^{\,i-1}_{i,N} = 0$, then we will recover the salvage value of product $i - 1$; this is given by the first term in (4b). Other terms in (4b) have a similar interpretation. Equation (4c) states that if there is a shortage of product i in a single product subproblem with product i but no shortage of it in a subproblem with products 1, $\ldots$, i, then a unit increase in inventory of product i will save substitution cost in addition to recovering salvage value (captured by the salvage terms for the corresponding product). On the other hand, if there is a shortage of product i in a subproblem with only products $1, \ldots\ i$, then this additional unit will bring additional revenue and save penalty cost for product i; this is given by the first term in Equation (4d). The last term of (4d) states that if there is no shortage for only products $1, \ldots, N - 1$ and there is a shortage of product N in a subproblem of products $1, \ldots,$ N, then this additional unit of i will be used to substitute for N incurring a substitution cost, earning revenue and saving penalty for product N.

The properties of the optimal policy are stated in theorems that follow. Suppose $\vec{x}$ denotes the initial inventory vector and $\vec{y}(\vec{x})$ be the optimal inventory levels after ordering.

THEOREM 1. *There exists a $\vec{y}^*$ such that for $\vec{x} < \vec{y}^*$, $\vec{y}(\vec{x}) = \vec{y}^*$.*

PROOF. Notice that

$$\left.\frac{\partial P^{(N)}}{\partial y_i}\right|_{\vec{y}\to\vec{0}} > 0 \quad \text{and} \quad \left.\frac{\partial P^{(N)}}{\partial y_i}\right|_{\vec{y}\to\infty} < 0$$

for all $i = 1, \ldots, N$. Observe that $P^{(N)}$ is concave (Proposition 2) and differentiable and hence has continuous derivatives. This implies there exists a $\vec{y}^* \geqslant \vec{0}$, independent of $\vec{x}$, such that $\partial P^{(N)}/\partial y_i|_{\vec{y}^*} = 0$.

An immediate implication is that the policy is order up to $\vec{y}^*$ whenever $\vec{x} < \vec{y}^*$.

The proof of the properties of the policy structure when $\vec{x} \nleq \vec{y}^*$ is a little involved. The basic idea is to do pairwise comparison of the partial derivatives of the profit function. For example, suppose $x_i \geqslant y^*_i$ and $x_j < y^*_j$. We then prove that $\partial P^{(N)}/\partial y_i \leqslant \partial P^{(N)}/\partial y_j$. That is, it is optimal to increase the stock of product j, if at all, but not of product i. For ease of exposition we first state some results as lemmas. The proofs of these lemmas follow easily from the Allocation Algorithm (A).

LEMMA 1. *For $k < i$,*

$$\Pr\{S^i_i = 0\} + \Pr\{S^i_i > 0,\ \vec{S}^{\,i-1}_{i-1,i} = 0\}$$

$$+ \ldots + \Pr\{S^{k+1}_i > 0,\ \vec{S}^{\,k}_{k,i} = 0\} = 1 - \Pr\{S^k_i > 0\}.$$

Lemma 1 establishes an identity for the probability of no shortage of product i in a subproblem with only products $k, \ldots, i$. The following lemma states that as the stock of product i increases, the probability of no stock-out of product i in a subproblem with only products $k, \ldots, i$ cannot decrease.

LEMMA 2.

$$\frac{\partial}{\partial y_i}\{\Pr\{S^i_i = 0\} + \Pr\{S^i_i > 0,\ \vec{S}^{\,i-1}_{i-1,i} = 0\}$$

$$+ \ldots + \Pr\{S^{k+1}_i > 0,\ \vec{S}^{\,k}_{k,i} = 0\}\} \geqslant 0.$$

The next two lemmas again establish identities for expressions of the shortage probabilities.

LEMMA 3.

$$\Pr\{\vec{S}^{\,j+1}_{i,N} > 0,\ \vec{S}^{\,j}_{i,N} = 0\} - \Pr\{\vec{S}^{\,j+1}_{i+1,N} > 0,\ \vec{S}^{\,j}_{i+1,N} = 0\}$$

$$= \Pr\{S^{j+1}_i > 0,\ \vec{S}^{\,j}_{j,i} = 0\} \Pr\{\vec{S}^{\,i+1}_{i+1,N} = 0\}.$$

LEMMA 4.

$$\Pr\{S^i_i > 0,\ S^1_i = 0\} - \Pr\{S^{i+1}_{i+1} > 0,\ S^1_{i+1} = 0\}$$

$$- \Pr\{S^1_{i+1} > 0\} = \Pr\{S^{i+1}_{i+1} = 0\}[1 - \Pr\{S^1_i > 0\}]$$

$$- \Pr\{S^i_i = 0,\ S^1_i = 0\}.$$

The following lemma establishes an upper bound on the savings in shortage costs for an additional unit of product i in a subproblem with only $1, \ldots, i$ products.

LEMMA 5.

$$\frac{\partial}{\partial y_i}\{s_i \Pr\{S^i_i = 0\} + s_{i-1} \Pr\{S^i_i > 0,\ \vec{S}^{\,i-1}_{i-1,i} = 0\}$$

$$+ \ldots + s_1 \Pr\{S^2_i > 0,\ \vec{S}^{\,1}_{1,i} = 0\}\}$$

$$\leqslant s_1 \frac{\partial}{\partial y_i}\{\Pr\{S^i_i = 0\} + \ldots + \Pr\{S^2_i > 0,\ \vec{S}^{\,1}_{1,i} = 0\}\}.$$

Now we are ready to prove an important property of the optimal policy. It states that if the on-hand inventory of any product is greater than its optimal base-stock level, then no order is placed for this product.

Theorem 2. *If* $x_i \geqslant y_i^*$ *then* $\bar{y}_i = x_i$.

To prove this theorem, we need the expression for the difference between the first partials of the profit function with respect to y_i and y_{i+1}. This is given by the following lemma.

Lemma 6.

$$\frac{\partial P^{(N)}}{\partial y_i} - \frac{\partial P^{(N)}}{\partial y_{i+1}} = -c_i + c_{i+1} + s_{i+1}\Pr\{\vec{S}_{i+1,N}^{i+1} = 0\}$$

$$+ b\Pr\{S_{i+1}^{i+1} = 0\} \tag{5a}$$

$$- b\Pr\{S_i^i = 0,\ S_i^1 = 0\} \tag{5b}$$

$$+ \Pr\{S_i^1 > 0\}[T_i + b - T_{i+1}\Pr\{S_{i+1}^{i+1} > 0\}$$

$$- \cdots - T_N\Pr\{\vec{S}_{i+1,N-1}^{i+1} = 0, S_N^{i+1} > 0\} - b\Pr\{S_{i+1}^{i+1} = 0\}] \tag{5c}$$

$$+ \Pr\{\vec{S}_{i+1,N}^{i+1} = 0\}[s_i\Pr\{S_i^i = 0\} + s_{i-1}\Pr\{S_i^i > 0,\ \vec{S}_{i-1,i}^{i-1}$$

$$= 0\} + \cdots + s_1\Pr\{S_i^2 > 0,\ \vec{S}_{1,i}^1 = 0\}]. \tag{5d}$$

Proof. See Appendix B.

Proof of Theorem 2. If $x_i \geqslant y_i^*$ for every $i = 1, \ldots, N$, then from the concavity of $P^{(N)}$ we have that $\partial P^{(N)}/\partial y_i \leqslant 0$ giving $\bar{y}_i = x_i$. Therefore, we assume that $x_i \geqslant y_i^*$ and $x_j < y_j^*$. We then prove that it is optimal to Increase the stock of product j, if at all, but not of product i; that is,

$$\left.\frac{\partial P^{(N)}}{\partial y_i}\right|_{y_i = x_i \geqslant y_i^*} \leqslant \left.\frac{\partial P^{(N)}}{y_j}\right|_{y_j = x_j < y_j^*}. \tag{6}$$

For ease of exposition, we will show (6) for $j = i + 1$. The proof of result for any arbitrary j follows along similar lines.

Assume that $x_i = y_i^*$ and $x_{i+1} < y_{i+1}^*$. Using Theorem 1 it is clear that the difference in marginal benefits given by (5) is negative. Thus, it is sufficient to show that (5) is a decreasing function of x_i for $x_i > y_i^*$.

Now the right-hand side of (5a) is independent of x_i. It is also clear that

$$\frac{\partial}{\partial y_i}\{\Pr\{S_i^1 > 0\}\} < 0 \quad \text{and}$$

$$\frac{\partial}{\partial y_i}\{\Pr\{S_i^i = 0,\ S_i^1 = 0\}\} \geqslant 0. \tag{7}$$

However, Assumption 1 implies that

$$T_1 \geqslant T_2 \geqslant \cdots \geqslant T_N. \tag{8}$$

Then, we can bound the bracketed expression in term (5c) as follows:

$$T_i + b - T_{i+1}\Pr\{S_{i+1}^{i+1} > 0\}$$

$$- \cdots - T_N\Pr\{\vec{S}_{i+1,N-1}^{i+1} = 0,\ S_N^{i+1} > 0\}$$

$$- b\Pr\{S_{i+1}^{i+1} = 0\}$$

$$\leqslant T_i - T_N\{\Pr\{S_{i+1}^{i+1} > 0\}$$

$$+ \cdots + \Pr\{\vec{S}_{i+1,N-1}^{i+1} = 0,\ S_N^{i+1} > 0\}\}$$

$$+ b\Pr\{S_{i+1}^{i+1} > 0\} \leqslant T_i - T_N\{1 - \Pr\{\vec{S}_{i+1,N}^{i+1} = 0\}\}$$

$$+ b\Pr\{S_{i+1}^{i+1} > 0\}. \tag{9}$$

Using (5d) and Lemmas 1, 2, and 5, we can bound the partial derivative of the bracketed expression in term (5d) with respect to y_i as follows:

$$\frac{\partial}{\partial y_i}\{s_i\Pr\{S_i^i = 0\} + s_{i-1}\Pr\{S_i^i > 0,\ \vec{S}_{i-1,i}^{i-1} = 0\}$$

$$+ \cdots + s_1\Pr\{S_i^2 > 0,\ \vec{S}_{1,i}^1 = 0\}\}$$

$$\leqslant s_1\frac{\partial}{\partial y_i}\{1 - \Pr\{S_i^1 > 0\}\} = s_1\left\{-\frac{\partial}{\partial y_i}\Pr\{S_i^1 > 0\}\right\}. \tag{10}$$

Using (7), (9), and (10) in Equation (5) we get

$$\frac{\partial}{\partial y_i}\left\{\frac{\partial P^{(N)}}{\partial y_i} - \frac{\partial P^{(N)}}{y_{i+1}}\right\} \leqslant \frac{\partial}{\partial y_i}\Pr\{S_i^1 > 0\}$$

$$\cdot[T_i - T_N\{1 - \Pr\{\vec{S}_{i+1,N}^{i+1} = 0\}\} - s_1\Pr\{\vec{S}_{i+1,N}^{i+1} = 0\}$$

$$+ b\Pr\{S_{i+1}^{i+1} > 0\}] = \frac{\partial}{\partial y_i}\Pr\{S_i^1 > 0\}[(T_i - T_N) + b$$

$$+ (T_N - s_1)\Pr\{\vec{S}_{i+1,N}^{i+1} = 0\} + b\Pr\{S_{i+1}^{i+1} > 0\}] < 0.$$

The last step follows from (7) and the fact that $T_i \geqslant T_N$ by (8); $T_N \geqslant s_1$ by Assumption 3.

Thus, for $x_i > y_i^*$ and $x_{i+1} < y_{i+1}^*$, $\partial P^{(N)}/\partial y_i < \partial P^{(N)/\partial y_{i+1}}$, which proves the theorem.

The following theorem establishes the monotone property of the optimal inventory levels after ordering.

Theorem 3. $\vec{y}(\vec{x})$ *is a nonincreasing function of* $\vec{x}$.

Proof. The theorem is trivially true for $\vec{x} \leqslant \vec{y}^*$. Now suppose that $x_i < y_i^*$ and $x_j > y_j^*, j \neq i$ and $j = 1, \ldots, N$. From Theorem 2.2 we have that $\bar{y}_j = x_j$. Because the profit function is submodular (Proposition 2), $\partial P^{(N)}/\partial y_i$ is decreasing in x_j. Suppose $\bar{y}_i$ solves $\partial P^{(N)}/\partial y_i = 0$ for $x_j < y_j^*$. Then $\bar{y}_i(\vec{x})$ is nonincreasing in x_j. □

3. BOUNDS, ALGORITHM, AND COMPUTATIONS

In this section we give some results on the bounds for the optimal order points $\vec{y}^*$. We then use these bounds to develop an iterative algorithm. Using this algorithm we report computational results that demonstrate the gains of computing the optimal solution incorporating substitution effects.

3.1. Bounds and Algorithm

The bounds are based on a decomposition idea. To solve the general N-product problem, we decompose the problem into a one-product problem consisting of product 1 only and an $(N - 1)$-product problem consisting of products $2, \ldots, N$. The next property we prove states that (i) the solution of the (single) product 1 problem is a lower bound on the optimal order point for product 1 in the general N-product problem; and (ii) the optimal order points for the $N - 1$ products in the $(N - 1)$-product problem is an upper bound for the order points of the $N -$

1 products in the N-product problem. This generalizes a similar result for a two product problem proved by Pasternack and Drezner (1991). Formally, we have Proposition 3.

PROPOSITION 3. *Let $y_1^*, \ldots, y_N^*$ be the optimal solution of the N-product problem and $\tilde{y}_2, \ldots, \tilde{y}_N$ the optimal solution of the $(N - 1)$-product problem. Let the $\tilde{y}_1$ be the optimal solution of the single product problem consisting only of product 1. Then $\tilde{y}_1 \leqslant y_1^*$ and $\tilde{y}_i \geqslant y_i^*$, for $i = 2, \ldots, N$.*

PROOF. Observe that the optimal solution of the single product problem consisting of product 1 can be obtained by setting the inventory level, y_i for $i = 2, \ldots, N$ to infinity in the N-product problem and solving for the optimal stock level for product 1. Similarly, the solution of the $(N - 1)$-product problem can be obtained by solving for the stock levels of products $2, \ldots, N$ in an N-product problem with $y_1 = 0$. The proposition then follows from Theorem 3. □

We now describe an iterative algorithm to compute the optimal order points $\vec{y}^*$ for a two-product problem. Our interest here is to give an algorithm to compute the order points used to illustrate the benefits of incorporating substitution effects in solving for the order points. Other, perhaps better, gradient based algorithms may be devised. Any such algorithm, however, would still require computation of the gradient of the profit function for which we have given an explicit expression in (4). Now, observe that a simple newsvendor solution (y_i) to a single product (say i) problem is:

$$F(y_i) = \frac{p_i + \pi_i - c_i}{p_i + \pi_i - s_i}. \tag{11}$$

Applying Proposition 3 to a two-product problem, we know that y_i obtained from (11) gives us a lower bound for y_1^* and an upper bound for y_2^*. We then iteratively get the solution for the two-product problem as follows:

1. Update the stock level for product 2 by solving the first-order condition for, product 2 and assuming the stock level of product 1 derived in the previous iteration, and
2. Update the stock level for product 1 by solving the first-order condition for product 1 and assuming the just updated stock level of product 2,

until the first order conditions for both products are satisfied. Concavity of the profit function guarantees convergence. The basic idea of our algorithm can be extended to solve an N-product problem. Such an algorithm will first compute the newsboy solution for product 1 only, then solve the $(N - 1)$-product problem and finally, iterate until convergence. Observe that such an implementation of the algorithm will perform search along the axes.

3.2. Computational Study

The main purposes of this limited computational study are to (i) demonstrate the gains from computing the optimal solution, and (ii) explore conditions of the parameter ranges for which this gain is significant. To that extent we limit this study to the two-product problem. Specifically, we compare the gains in profits between the optimal solution with the profits obtained when the order points are derived using the standard newsvendor model. In both cases, however, we assume that substitutions occur once demands are realized. Let $\vec{y}^{nb}$ and $\vec{y}^*$ be the order quantities obtained using the newsboy and optimal procedures, respectively. The iterative algorithm outlined in the previous section was used to compute $\vec{y}^*$. Implementation of this algorithm requires computation of the gradient of the profits function given by (4). The initial bounds are simply the independent newsvendor solutions, given by (11), for the two products, individually.

Let $P(\vec{y})$ be the profit from a stock of $\vec{y}$. Then the percentage gain is

$$\text{Percentage Gain } = \frac{P(\vec{y}^*) - P(\vec{y}^{nb})}{P(\vec{y}^*)} \times 100.$$

The parameters used for this study are as follows. The order cost c_1 of product 1 was fixed at 4.0. The order cost c_2 of product 2 took values from {2.0, 2.5, 3.0, 3.5, 4.0}. The penalty cost π_i for both products was set to 10.0. The per unit price of product i was chosen such that the price to cost ratio of p_i/c_i took values from {2.0, 3.0, 4.0}. The holding cost $\tilde{h}_i$ of product i was set to 0.2. The substitution cost, b, was varied from 0.0 to 3.0 in steps of 1.0 and the salvage value $\tilde{s}_i$ of product i was varied from 0.0 to c_i. Finally, the average demand for each product was fixed at 100.0 and the coefficient of variation was chosen from {0.125, 0.25, 0.4, 0.5}.

The basic idea behind the selection of the range of parameter values was to test effects of the coefficient of variation of demand, substitution costs, and similarity of products in terms of the price to cost ratio. Because we have a profit maximization formulation, a shortage results in lost revenue in single period, and thus the penalty cost needs only to capture the loss of goodwill, assumed here to be same for both products. We use Monte Carlo methods for computing the first partials of the profit function. Observe that to evaluate the gradient in (4), we need to compute nested integrals. The number of nested integrals increases exponentially with the number of products. Hence numerical integration will not be efficient, and we use Monte Carlo methods. Our iterative algorithm combined with Monte Carlo methods can be used to solve large scale problems. We solve here only a two-product problem to illustrate the benefits of incorporating substitution effects in solving for the order points.

The demand points were generated using the polar method of normal deviates due to G.E.P Box, M.E. Muller, and G. Marsaglia (see Knuth 1981, p. 117). The number of demand points generated for each product with the underlying normal distribution was 4096. The computational time required to solve the two-product problem varied from approximately 5 seconds to 116 seconds, depending on the parameters. In general the computation time will also be greatly affected by the number of demand points chosen. The program was written in UNIX/C and run on SUN SPARC 2.[1] We make the following observations:

1. *Salvage value*: The gains *increase* with increasing salvage value from 14.71% for $\tilde{s}_i = 0.0$ *to* 33.0% *for* $\tilde{s}_i = c_i$, for $i = 1, 2$. The rest of the parameters were fixed at $b = 0.0$, $c_1 = 4.0$, $c_2 = 2.0$, $p_i/c_i = 2.0$, and $v = 0.5$.

2. *Coefficient of variation of demand*: The gains *increase* with increasing coefficient of variation from 6.11% for $v = 0.125$ to 27.95% for $v = 0.5$. The other parameters were fixed at $b = 0.0$, $c_1 = 4.0$, $c_2 = 2.0$, $p_i/c_i = 2.0$, $\tilde{s}_i = 0.9\ c_i$, and $v = 0.5$.

3. *Substitution cost*: The gains *decrease* with increasing substitution cost from 27.95% for $b = 0.0$ to 25.44% for $b = 3.0$. The other parameters were fixed at $c_1 = 4.0$, $c_2 = 2.0$, $p_i/c_i = 2.0$, $\tilde{s}_i = 0.9\ c_i$, and $v = 0.5$.

4. *Price to cost ratio*: The percentage gains *decrease* with increasing price to cost ratio from 27.95% for a ratio of 2.0 to 8.35% for a ratio of 4.0. The other parameters were fixed at $c_i = 4.0$, $c_2 = 2.0$, $b = 0.0$, $\tilde{s}_i = 0.9\ c_i$, and $v = 0.5$.

5. *Similarity of products in terms of prices and costs*: The effect of product similarity on percentage gains is a function of the salvage value of the products. The change in percentage gains is significant when salvage values are low. For example, for salvage value = 0.0 for both products, keeping the cost and price of product 1 fixed at $c_1 = 4.0$ and a price to cost ratio of 2.0 for both products, the percentage gains *increase* as the cost of product 2 increases from 14.51% at $c_2 = 2.0$ to 23.0% at $c_2 = 4.0$. Whereas for the infinite horizon problem, the percentage gains *decrease* 27.95% at $c_2 = 2.0$ to 26.03% at $c_2 = 4.0$. The other parameters of the problem were $b = 0.0$, $\tilde{s}_i = 0.9\ c_i$, and $v = 0.5$.

Thus we conclude that most gains accrue in a problem with high salvage value of products, high demand variability, low substitution cost, low profit margins (or low price to cost ratio), and similarity of products in terms of prices and costs. Thus computation of optimal policies is important when the problem domain has some or all of the above characteristics.

4. CONCLUSION, EXTENSIONS, AND FUTURE RESEARCH

In this paper we studied a single period multiproduct inventory problem with downward substitution, zero leadtime of supply, and proportional costs and revenues. We developed several interesting properties of the optimal solution. While we assumed a constant marginal cost of substitution, the specific techniques used here can be extended to more general substitution problems that satisfy the following condition on the substitution cost b_{ij}: for all $i, j, i < j$, $b_{ij} = \alpha_i + \beta_j$. That is, the cost of substituting product i for demand class j, b_{ij} can be written as a sum of two independent costs each of which depends only on i and j, respectively. The techniques used here depend critically on our ability to write the profit function and its first partials explicitly. This in turn is possible because of optimality of a greedy allocation algorithm. If b_{ij} is additive in i and j as mentioned, then we can rank-order the arcs connecting products and demand classes and show that they form a Monge sequence implying optimality of a greedy allocation algorithm.

An obvious extension of the current work is to study multiperiod and infinite horizon versions of the problem (Bassok et al. 1997). Other interesting extensions are as follows.

1. While there appear to be gains in solving for the optimal policy incorporating substitution effects, obtaining such solutions for the general multiproduct problem is time consuming. Thus, it might be worthwhile to consider other approximate structures of downward substitution which may be relatively easier to solve and which could give us most of the gains of substitution. For example, one could consider a model with only one level substitution, i.e., product i only substitutes for $i + 1$. Analysis of such approximate structures and comparisons with optimal solutions discussed here merit a separate study.

2. We assume zero leadtime in the supply process here. Problems with nonzero leadtimes, both deterministic and stochastic, and uncertainties in supply is an area of future research. Clearly, with nonzero leadtimes the optimal allocation problem will not be myopic. However, one could assume a myopic allocation algorithm and derive the corresponding "optimal" order policy. Similar assumptions of a myopic allocation algorithm in nonzero leadtime situations, e.g., in the single warehouse multiretailer settings (see Federgruen and Zipkin 1984), have been shown to perform well.

APPENDIX A

Definitions

DEFINITION 1 *(Ignall and Veinott 1969). Let H* be a $N \times N$ matrix and let H^{ij} be formed from H by interchanging the ith and jth columns of H. We say H is a *substitute matrix* if it is symmetric and positive definite, and if each principal minor of H^{ij} that contains elements of exactly one of the columns i and j is nonnegative for all $i < j$.

DEFINITION 2 *(Heyman and Sobel 1984). A twice differentiable function is submodular (supermodular) if all its cross partials are nonpositive (nonnegative), respectively.*

DEFINITION 3 *(Hoffman 1963). Consider a network with N* sources and M destinations. An ordering of the indices *(i,*

[1] Lembeke and Bassok (1994) present computation times for eight-product substitution problems similar, but not identical, to the one presented here. They show that using the algorithm presented in this paper (with minor modifications) it is possible to solve eight-product problems in 126 CPU seconds on average. Hsu and Bassok (1994) show that solving similar medium size substitution problems using stochastic LP is impractical. On Sun Sparcstation running Minos 5.3 as the LP solver, because of memory limitations it was impossible to solve the LP formulation for 7–10 products with 1,000 random scenarios.

j) is a Monge sequence if for every $1 \leqslant i, r \leqslant N$ and $1 \leqslant j, s \leqslant M$ if (i, j) proceeds both (i, s) and (r, j) then $c_{i,j} + c_{r,s} \leqslant c_{i,s} + c_{r,j}$, where $c_{i,j}$ is the cost of shipping a unit from source i to destination j.

APPENDIX B

Proofs

To prove concavity and submodularity of the profit function in (2), we first prove concavity and submodularity of the expected profits of the allocation subproblem (the second term in (2)) given by (3). We achieve this in two steps. We first define

$$\tilde{G}(\vec{y}, d_1, \ldots, d_{N-1}) \equiv \int_{R_+} G(\vec{y}, d_1, \ldots, d_N) dF_N(d_N)$$

as the expected profits for a given initial stock, $\vec{y}$, and realized demands, $(d_1, \ldots, d_{N-1})$, of the $N-1$ products; that is, the expectation of the profit is taken over the product N demand. We prove (in Lemma B.2) that $\tilde{G}(\vec{y}, d_1, \ldots, d_{N-1})$ is concave and submodular in $\vec{y}$. We then invoke known results in the literature (discussed later) to show that the profit function $P(\vec{x}, \vec{y})$ is concave and submodular in $\vec{y}$.

We first develop an explicit expression for $\tilde{G}(\vec{y}, d_1, \ldots, d_{N-1})$ using the allocation algorithm (A). To do so, we need some further notations. Let

H_i = inventory on-hand of product $i = 1, \ldots, N-1$ after realization of the demands and allocation of product i to demand class i but before substitution from excess stock of product i to satisfy excess demands in class j, $i < j$.

B_i = shortage of product $i = 1, \ldots, N-1$ after realization of demands and allocations of product i to demand class j, $i \leqslant j$.

R_i = actual quantity of product $i = 1, \ldots, N-1$ sold before substitution from excess stock of product i to satisfy excess demands in class j, $i < j$.

Observe that H_i and B_i are nonnegative. Further $R_i = \min(y_i, D_i)$. Also define $H_{i,j} = \Sigma_{k=i}^{j} H_k$. Finally w_{ij} the allocation of product i for demand j is always nonnegative. Then, using the profit maximization framework discussed in § 2, we have

$$\tilde{G}(\vec{y}, d_1, \ldots, d_{N-1})$$

$$= \sum_{i=1}^{N-1} p_i R_i + p_N \int_0^{y_N} D_N dF_N(D_N) + p_N y_N (1 - F_N(y_N)) \tag{B-1a}$$

$$+ s_N \int_0^{y_N} (y_N - D_N) dF_N(D_N)$$

$$+ \sum_{i=1}^{N-1} s_i H_{i,i} F_N(H_{i+1,N-1} + y_N) \tag{B-1b}$$

$$+ \sum_{i=1}^{N-1} s_i \int_{H_{i+1,N-1}+y_N}^{H_{i,N-1}+y_N} (H_{i,N-1} + y_N - D_N) dF_N\,(D_N) \tag{B-1c}$$

$$+ \sum_{i=1}^{N-1} \sum_{j=1, i \neq j}^{N-1} a_{i,j} w_{ij} + \sum_{i=1}^{N-1} a_{i,n} \int_{H_{i+1,N-1}+y_N}^{H_{i,N-1}+y_N} \tag{B-1d}$$

$$\cdot (D_N - H_{i+1,N-1} - y_N) dF_N(D_N) \tag{B-1d}$$

$$+ \sum_{i=1}^{N-1} a_{i,n} H_{i,i} (1 - F_N(H_{i,N-1} + y_N)) \tag{B-1e}$$

$$- \sum_{i=1}^{N-1} \pi_i B_i - \pi_N \int_{H_{1,N-1}}^{\infty} (D_N - H_{1,N-1} - y_N) dF_N(D_N). \tag{B-1f}$$

We assume if there is a shortage of product i we use product $j < i$ to substitute for i, such that product j is the first product with excess inventory and no product k, $j < k < i$ has excess inventory. The first part of (B-1a) gives the revenue from products $i = 1, \ldots, N-1$; the second part of (B-1a) gives the expected (expectation over product N demand) revenue of product N when demand for product N is less than stock-on-hand; the last part of (B-1a) gives the expected revenue from product N when its demand exceeds stock-on-hand. Notice that these revenue terms exclude substitution. The expected salvage value (expectation taken over product N demand) is given by (B-1b) and (B-1c). The first part of (B-1d) is the benefit of substitutions amongst products $1, \ldots, N-1$ and the second part of (B-1d) is the expected benefit of substitution *from* products $1, \ldots, N-1$ to N when there is no stock-out for product N *after* substitutions. The expected benefit of substitution from products $1, \ldots, N-1$ to N when product N stocks out even after substitutions is given by (B-1e). Finally, (B-1f) gives the penalty cost.

We need to establish the first and second derivatives of H_i, B_i, R_i and w_{ij}.

LEMMA B.1. *(a) The first derivatives of H_i, B_i and R_i with respect to y_j are:*

$$\frac{\partial H_i}{\partial y_j} = \begin{cases} 0 & \text{if } j < i, \\ 0 \text{ or } 1 & \text{if } j \geqslant i; \end{cases} \tag{B-2}$$

$$\frac{\partial B_i}{\partial y_i} = \begin{cases} 0 & \text{if } j > i, \\ 0 \text{ or } -1 & \text{if } j \leqslant i; \end{cases} \tag{B-3}$$

$$\frac{\partial R_i}{\partial y_i} = \begin{cases} 0 & \text{if } j > i, \\ 0 \text{ or } 1 & \text{if } j \leqslant i; \end{cases} \tag{B-4}$$

(b) $\partial H_{i,j}/\partial y_j \leqslant 1$ *for* $i \leqslant j$,

(c) If $\partial H_{i,j}/\partial y_j = 1$ *then* $\partial H_{i-1,j}/\partial y_j = 1$ *for* $i \leqslant j$,

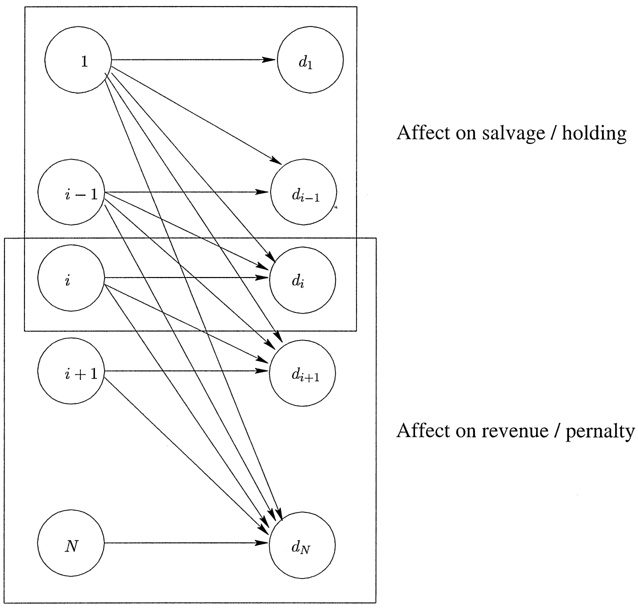
(d) If $\partial H_i/\partial y_i = 1$ *then* $\partial H_{i,j}/\partial y_j = 1$ *for* $i \leqslant j$,

(e) For all i, j,

$$\frac{\partial^2 H_i}{\partial y_i, \partial y_j} = \frac{\partial^2 B_i}{\partial y_i \partial y_j} = \frac{\partial^2 R_i}{\partial y_i \partial y_j} = \frac{\partial^2 w_{ij}}{\partial y_i \partial y_j} = 0.$$

The proofs of the results are straightforward and we will not discuss them. They follow from a fundamental structural property exhibited by this problem. A unit increase in inventory of, say product i, can affect only the stocks and hence the holding/salvage of products "higher" in the substitution hierarchy, i.e., products $1, \ldots, i$; and, it can affect

Figure 2. Effect of Unit Increase in Inventory of Product i



only the revenues/penalty of products "lower" in the substitution hierarchy, i.e., products $i, \ldots, N$. This is shown in Figure 2.

The results have the following intuitive interpretation. Recall that the H_i's represent excess inventory of product i after satisfying demands $j \geqslant i$. In part (a) of Lemma B.1, Equation (B-2) states that for $i > j$ an additional unit of product j is not going to affect the inventory of product i. That is, products "higher" in the substitution hierarchy cannot affect the inventories of products below it. An additional unit of product "lower" in the hierarchy may affect the inventory of products higher than itself. For $i \leqslant j$ if the inventory of product i were negative, then an additional unit of product j would not make the inventory of product i positive; on the other hand, if product i inventory were nonnegative, then a unit increase in the inventory of product j would increase inventory of product i by one unit. Similar results hold for shortages given by (B-3) and for revenues given by (B-4). Part (b) of the lemma states that a unit increase in the inventory of product j can make the inventory of *at most one* product $i < j$ increase. If inventories of all products $i, \ldots, j$ are negative,No inventory becomes positive. Part (c) of the lemma states that if there exists a product $i < j$ whose inventory *does* increase, then inventories of products k, $k < i < j$, remain unaffected; i.e., the "propagation" stops. Part (d) of the lemma states that that the propagation will stop at the first i whose inventory is positive. Finally, the last part of the lemma states that all the second derivatives are zero. This follows from the fact that the corresponding first derivative are all constant.

We need the following results, well known in the literature.

RESULT 1 (see Avriel 1976, Theorem 4.15, p. 76). *The function $P(\vec{x})$, $\vec{x} \in \vec{R}^N$ is concave if and only if the function*

$$\Phi(\lambda) \equiv P(\lambda\vec{x} + (1-\lambda)\vec{y})$$

is concave for every $\vec{x}, \vec{y} \in \vec{R}^N$ and for every $0 \leqslant \lambda \leqslant 1$.

RESULT 2 (see Van Slyke 1966, Proposition 7). *If $\int P(\vec{x}, \tilde{D}, D_N) dF_N(D_N)$ is concave for every realization of $\tilde{D} \in \vec{R}^{N-1}$, then $\int_{\tilde{D}\in\vec{R}^N} P(\vec{x}, \tilde{D}) dF(\tilde{D})$ is also concave.*

The following lemma establishes the concavity of $G(\cdot)$.

LEMMA B.2. *$\tilde{G}(\vec{y}, d_1, \ldots, d_{N-1})$ is concave and submodular in $\vec{y}$.*

PROOF. Let $\Phi(\lambda, \vec{u}, \vec{v}) = \tilde{G}[\lambda\vec{u} + (1-\lambda)\vec{v}] = \tilde{G}[\lambda(\vec{u} - \vec{v}) + \vec{v}]$ for $\vec{u}, \vec{v} \in \vec{R}^N$. We will show that $\partial^2\Phi/\partial\lambda^2 \leqslant 0$. Now

$$\frac{\partial\Phi}{\partial\lambda} = \frac{\partial\tilde{G}}{\partial\lambda}[\lambda(\vec{u} - \vec{v}) + \vec{v}] = \sum_{i=1}^{N} \frac{\partial\tilde{G}}{\partial z_i}\frac{\partial z_i}{\partial\lambda},$$

where $z_i = \lambda(u_i - v_i) - v_i$. Then

$$\sum_{i=1}^{N} \frac{\partial\tilde{G}}{\partial z_i}\frac{\partial z_i}{\partial\lambda} = \sum_{i=1}^{N} \frac{\partial\tilde{G}}{\partial z_i}(u_i - v_i),$$

and

$$\frac{\partial^2\tilde{G}}{\partial\lambda^2} = \sum_{i=1}^{N} \frac{\partial^2\tilde{G}}{\partial z_i^2}(u_i - v_i)^2 + 2\sum_{i=1}^{N}\sum_{j=i+1}^{N} \frac{\partial^2\tilde{G}}{\partial z_i \partial z_j}(u_i - v_i)(u_j - v_j). \qquad \text{(B-5)}$$

Using the results of Lemma B.1 for simplification of terms, the second derivatives of $\tilde{G}(\cdot)$ can be written as follows:

$$\frac{\partial^2\tilde{G}}{\partial z_i^2} = -(a_{1,N} + \pi_N - s_N)\left(\frac{\partial H_{1,i}}{\partial z_i}\right)^2 f_N(H_{1,N-1} + y_N) + \sum_{k=2}^{i}(a_{k-1,N} - s_{k-1} - a_{k,N} + s_k) \cdot \left(\frac{\partial H_{k,i}}{\partial z_i}\right)^2 f_N(H_{k,N-1} + y_N),$$

$$\frac{\partial^2\tilde{G}}{\partial z_1 \partial z_j} = -(a_{1,N} + \pi_N - s_N)\left(\frac{\partial H_{1,1}}{\partial z_1}\right) \cdot \left(\frac{\partial H_{1,j}}{\partial z_j}\right) f_N(H_{1,N-1} + y_N).$$

For $i > 1$,

$$\frac{\partial^2\tilde{G}}{\partial z_i \partial z_j} = -(a_{1,N} + \pi_N - s_N)\left(\frac{\partial H_{1,i}}{\partial z_i}\right)\left(\frac{\partial H_{1,j}}{\partial z_j}\right) \cdot f_N(H_{1,N-1} + y_N) + \sum_{k=2}^{i}(a_{k-1,N} - s_{k-1} - a_{k,N} + s_k)\left(\frac{\partial H_{k,i}}{\partial z_i}\right)\left(\frac{\partial H_{k,j}}{\partial z_j}\right) f_N(H_{k,N-1} + y_N).$$

Substituting into (B-5) and simplifying, we get

$$\frac{\partial^2 \tilde{G}}{\partial \lambda^2} = \sum_{k=1}^{N} B_k f_N(H_{k,N-1} + y_N) \cdot \left[\sum_{i=k}^{N} \left(\frac{\partial H_{k,i}}{\partial z_i}\right)^2 (u_i - v_i)^2 + 2\sum_{i=k}^{N} \sum_{j=i+1}^{N} \left(\frac{\partial H_{k,i}}{\partial z_i}\right) \cdot \left(\frac{\partial H_{k,j}}{\partial z_j}\right) (u_i - v_i)(u_j - v_j) \right], \quad \text{(B-6)}$$

where B_k is a constant and is defined by

$$B_k = \begin{cases} -(a_{1,N} + \pi_N - s_1), & k = 1, \\ (a_{k-1,N} - s_{k-1} - a_{k,N} + s_k), & k = 2, \ldots, N. \end{cases}$$

We will now show that any kth term in (B-6) in nonpositive. First of all, $f_N(H_{k,N-1} + y_N) \geqslant 0$. Clearly $B_k \leqslant 0$ for all k. So from (B-6) it is sufficient to show that for any $k = 1, \ldots, N$,

$$\sum_{i=k}^{N} \left(\frac{\partial H_{k,i}}{\partial z_i}\right)^2 (u_i - v_i)^2 + 2\sum_{i=k}^{N} \sum_{j=i+1}^{N} \left(\frac{\partial H_{k,i}}{\partial z_i}\right) \left(\frac{\partial H_{k,j}}{\partial z_j}\right) \cdot (u_i - v_i)(u_j - v_j) \geqslant 0. \quad \text{(B-7)}$$

There are two cases to consider.

Case 1: There exist minimal $l_1 \geqslant k$ such that $\partial H_{k,l_1}/\partial z_{l_1} = 1$ and $\partial H_{k,q}/\partial z_q = 0$ for $q < l_1$. It is also true that $\partial H_{l_1,q}/\partial z_q = 1$ for $l_1 \leqslant q \leqslant N$. Then (B-7) reduces to

$$\sum_{i=l_1}^{N} (u_i - v_i)^2 + 2 \sum_{i=l_1}^{N} \sum_{j=i+1}^{N} (u_i - v_i)(u_j - v_j) = \left[\sum_{i=l_1}^{N} (u_i - v_i) \right]^2 \geqslant 0,$$

which implies nonpositivity of (B-6).

Case 2: If no such minimal l_1 exists, then all terms are zero and (B-6) is nonpositive.

Concavity of $\tilde{G}(\cdot)$ follows from Result 1. To prove that $\tilde{G}(\cdot)$ is submodular,Notice that the cross-partials are all nonpositive. This follows from the fact that $B_k \leqslant 0$, and from Lemma B.1 the partials of $H_{i,j}$ are all nonnegative.

PROOF OF PROPOSITION 2. From Lemma B.2 $\tilde{G}(\cdot)$ is concave and submodular. The right-hand side of (2) is a sum of linear and concave functions. Using Result 2 we have that $P(\vec{x}, \vec{y})$ is concave. It is easily seen that a property similar to Result 2 holds for submodularity. This gives us the submodularity of the profit function $P(\vec{x}, \vec{y})$. $\square$

PROOF OF LEMMA 6. Similar to Equation (4), we can write the marginal benefit from product $i + 1$ as

$$\frac{\partial P^{(N)}}{\partial y_{i+1}} = -c_{i+1} + s_{i+1} \Pr\{\vec{S}_{i+1,N}^{i+1} = 0\} + \ldots + s_1 \Pr\{\vec{S}_{i+1,N}^{1} = 0, \vec{S}_{i+1,N}^{2} > 0\} + b\Pr\{S_{i+1}^{i+1} > 0, S_{i+1}^{1} = 0\} + (T_{i+1} + b)\Pr\{S_{i+1}^{1} > 0\} + \ldots + T_N \Pr\{\vec{S}_{i+1,N-1}^{1} = 0, S_N^1 > 0\}.$$

We need to evaluate the term $\partial P^{(N)/\partial y_i} - \partial P^{(N)}/\partial y_{i+1}$. Using Lemma 3, we can write the difference in salvage terms for product j as

$$s_j \Pr\{S_i^{j+1} > 0, \vec{S}_{j,i}^{j} = 0\} \Pr\{\vec{S}_{i+1,N}^{i+1} = 0\}.$$

The revenue and penalty term for product $i + 1$ in the difference of marginal benefits is given by

$$\begin{aligned} &T_{i+1}\Pr\{S_i^1 = 0, S_{i+1}^1 > 0\} - T_{i+1} \Pr\{S_{i+1}^1 > 0\} \\ &= -T_{i+1} \Pr\{S_i^1 > 0, S_{i+1}^1 > 0\} \\ &= -T_{i+1}\Pr\{S_i^1 > 0, S_{i+1}^{i+1} > 0\}. \end{aligned}$$

Then,

$$\begin{aligned} \frac{\partial P^{(N)}}{\partial y_i} - \frac{\partial P^{(N)}}{\partial y_{i+1}} &= -c_i + c_{i+1} + s_{i+1}\Pr\{\vec{S}_{i+1,N}^{i+1} = 0\} \\ &\quad + s_i\Pr\{S_i^i = 0, \vec{S}_{i+1,N}^{i+1} = 0\} && \text{(B-8a)} \\ &+ s_{i-1}\Pr\{S_i^i > 0, \vec{S}_{i-1,N}^{i-1} = 0\}\Pr\{\vec{S}_{i+1,N}^{i+1} = 0\} \\ &\quad + \ldots + s_1\Pr\{S_i^2 > 0, \vec{S}_{1,i}^{1} = 0\}\Pr\{S_{i+1}^{i+1} = 0\} && \text{(B-8b)} \\ &+ b\Pr\{S_i^i > 0, S_i^1 = 0\} - b\Pr\{S_{i+1}^{i+1} > 0, S_{i+1}^1 = 0\} && \text{(B-8c)} \\ &+ (T_i + b)\Pr\{S_i^1 > 0\} - b\Pr\{S_{i+1}^1 > 0\} && \text{(B-8d)} \\ &- T_{i+1}\Pr\{S_i^1 > 0, S_{i+1}^{i+1} > 0\} \\ &\quad - \ldots - T_N\Pr\{S_i^1 > 0, \vec{S}_{i+1,N-1}^{1} = 0, S_N^1 > 0\}. && \text{(B-8e)} \end{aligned}$$

Using Lemma 4, term (B-8c) and the second part of term (B-8d) can be reduced to

$$\Pr\{S_{i+1}^{i+1} = 0\}[1 - \Pr\{S_i^i > 0\}] - \Pr\{S_i^i = 0, S_i^1 = 0\}.$$

The last term in Equation (B-8) can be written as

$$\begin{aligned} &T_N\Pr\{S_i^1 > 0, \vec{S}_{i+1,N-1}^{1} = 0, S_N^1 > 0\} \\ &= T_N\Pr\{S_i^1 > 0, \vec{S}_{i+1,N-1}^{i+1} = 0, S_N^{i+1} > 0\}. \end{aligned}$$

Using these simplifications (B-8) can be written as

$$\begin{aligned} \frac{\partial P^{(N)}}{\partial y_i} - \frac{\partial P^{(N)}}{\partial y_{i+1}} &= -c_i + c_{i+1} + s_{i+1}\Pr\{\vec{S}_{i+1,N}^{i+1} = 0\} + b\Pr\{S_{i+1}^{i+1} = 0\} \\ &\quad - b\Pr\{S_i^i = 0, S_i^1 = 0\} \\ &\quad + \Pr\{S_i^1 > 0\}[T_i + b - T_{i+1}\Pr\{S_{i+1}^{i+1} > 0\} \\ &\quad - \cdots + T_N\Pr\{\vec{S}_{i+1,N-1}^{i+1} = 0, S_N^{i+1} > 0\} \\ &\quad - b\Pr\{S_{i+1}^{i+1} = 0\}] + \Pr\{\vec{S}_{i+1,N}^{i+1} = 0\} \\ &\quad \cdot [s_i\Pr\{S_i^i = 0\} + s_{i-1}\Pr\{S_i^i > 0, \vec{S}_{i-1,i}^{i-1} = 0\} \\ &\quad + \cdots + s_1\Pr\{S_i^2 > 0, \vec{S}_{1,i}^{1} = 0\}]. \quad \square \end{aligned}$$

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