The first problem, DB1, is a stochastic vehicle-allocation model in a single-commodity network from Donohue and Birge [7]. They use this problem, with fixed first-stage variables, to evaluate a deterministic bound. We allow first-stage variables x to position a fleet of vehicles subject to a "resource constraint" $x \in X = \{x | \sum_i x_i = h, x_i \geq 0, \forall i\}$. (For ease of solution, we treat first-stage decisions as continuous in all four problems.) The

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Dist. Number, i	L_i	H_i	$\Pr(X_i = L_i)$	$\Pr(X_i = L_i + 1)$	$\Pr(X_i = L_i + 2)$	$\Pr(X_i = L_i + 3)$
1	1	4	0.35	0.30	0.25	0.10
2	2	5	0.40	0.20	0.20	0.20
3	3	6	0.30	0.25	0.25	0.20
4	4	6	0.50	0.30	0.20	-
5	5	8	0.40	0.30	0.15	0.15
6	6	8	0.45	0.35	0.20	-
7	7	9	0.50	0.25	0.25	-
8	8	11	0.35	0.30	0.20	0.15
9	9	11	0.45	0.40	0.15	-
10	10	12	0.40	0.35	0.25	-

Table 1: Possible Distributions for Links with Stochastic Upper Capacity for Transportation Problem.

Similarly, by letting t_i denote the initial node of link i, the same inequality results from applying Proposition 4.1 over each set of links with a common initial node. Refinements of the bound in Proposition 4.1 are found in [8] which can be applied here to improve the effectiveness of this bound.

5 Experimental Result

To test the effectiveness of this upper bound, two test problems were created and solved. The first problem is a transportation problem with 15 source nodes, 15 sink nodes and 105 links connecting the sources and sinks. Three different versions of this problem were considered. The second problem is a vehicle allocation problem with uncertain demands. This type of problem determines the expected value of a specific initial vehicle distribution, given uncertainty about the demand for vehicle routings along several or all links. This problem has 20 nodes and 92 links.

Three versions of the transportation problem were considered. First, all 105 links were assumed to be stochastic, making finding an upper bound using Edmundson-Madansky computationally impossible. Next, 30 of the links were randomly selected to have stochastic upper capacities, and each of the other links were given capacity near the mean of its respective distribution. These 30 links had 9 common terminal nodes, allowing an upper bound to be found with $2^9 = 512$ function evaluations. Finally, all of the links out of node 8 were considered stochastic, giving a total of 9 stochastic links. Here, an upper bound is established with just two function evaluations. Further, a refinement of the upper bound by partitioning was calculated for comparison. Table 2 lists the cost per unit flow between all sources and sinks, where flow between the corresponding nodes is possible. Table 3 lists the supply and demand requirements at each of the source and sink nodes. Each link has a lower capacity bound of zero. The distribution of the random variable corresponding to the upper capacity of each link was one of ten possibilities. Let X_i denote the random variable with distribution i, and let L_i (H_i) denote the lowest (highest) possible value of distribution i. The ten possible distributions are listed in Table 1. Table 4 gives the distribution that was assigned to each link.

The second problem is a vehicle allocation problem, one stage of a dynamic vehicle allocation problem with uncertain demands (see [13]). Figure 5.1 depicts the network over which the fleet of vehicles must be routed, given the initial distribution of vehicles at the source nodes 1 through 5. The problem only

	t_1	t_2	t_3	t_4	t_5	t_6	t_7	t_8	t_9	t_{10}	t_{11}	t_{12}	t_{13}	t_{14}	t_{15}
s_1	109	323	-	96	107	230	380	-	254	352	-	-	-	-	385
s_2	-	105	-	99	-	405	-	-	-	-	-	-	428	249	158
s_3	-	-	-	315	-	-	217	66	-	139	73	157	-	-	-
s_4	202	-	175	277	-	-	402	290	-	-	-	233	-	-	-
s_5	-	123	-	-	-	-	-	-	167	94	169	296	154	447	-
s_6	-	-	-	143	-	-	216	382	-	112	229	149	-	-	-
s_7	291	-	127	-	-	-	279	-	226	436	-	-	-	-	-
s_8	-	-	388	137	280	239	-	-	-	81	318	201	245	-	97
s_9	384	-	-	-	-	334	339	-	137	317	-	-	-	65	-
s_{10}	197	-	-	-	-	357	-	415	362	272	246	223	-	393	-
s_{11}	-	442	-	257	242	307	420	-	119	-	384	-	149	-	209
s_{12}	172	-	136	-	-	186	-	111	-	322	-	106	-	-	312
s_{13}	159	267	-	-	67	76	-	362	427	401	-	-	174	-	-
s_{14}	-	-	244	-	110	291	-	211	-	-	292	228	-	205	-
s_{15}	352	-	-	-	-	-	426	350	-	-	-	-	97	94	355

Table 2: Cost per unit flow from source s_i to sink t_j in Transportation Problem.

$i =$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
s_i	55	46	35	26	44	38	24	58	38	44	32	46	50	34	30
t_i	47	32	23	43	13	59	50	41	44	72	38	47	30	30	31

Table 3: Source and sink node supply and demand requirements in Transportation Problem.

	t_1	t_2	t_3	t_4	t_5	t_6	t_7	t_8	t_9	t_{10}	t_{11}	t_{12}	t_{13}	t_{14}	t_{15}
s_1	8	5	-	9	1	9	5	-	9	10	-	-	-	-	3
s_2	-	10	-	7	-	10	-	-	-	-	-	-	1	10	10
s_3	-	-	-	2	-	-	7	8	-	5	6	9	-	-	-
s_4	8	-	3	6	-	-	3	8	-	-	-	1	-	-	-
s_5	-	7	-	-	-	-	-	-	7	6	9	7	8	2	-
s_6	-	-	-	9	-	-	4	3	-	6	10	8	-	-	-
s_7	6	-	6	-	-	-	6	-	7	2	-	-	-	-	-
s_8	-	-	7	6	2	8	-	-	-	10	7	8	10	-	3
s_9	1	-	-	-	-	4	8	-	7	10	-	-	-	9	-
s_{10}	10	-	-	-	-	2	-	1	10	9	7	8	-	1	-
s_{11}	-	2	-	9	1	4	9	-	3	-	2	-	2	-	4
s_{12}	6	-	1	-	-	9	-	7	-	8	-	7	-	-	10
s_{13}	10	9	-	-	5	9	-	6	2	9	-	-	3	-	-
s_{14}	-	-	7	-	8	7	-	3	-	-	2	2	-	8	-
s_{15}	2	-	-	-	-	-	10	7	-	-	-	-	8	6	3

Table 4: Distribution for upper capacity of link from source s_i to sink t_j in Transportation Problem.

D_i	L_i	H_i	$\Pr(X_i = L_i)$	$\Pr(X_i = L_i + 1)$	$\Pr(X_i = L_i + 2)$	$\Pr(X_i = L_i + 3)$	$\Pr(X_i = L_i + 4)$
1	0	2	0.40	0.30	0.30	-	-
2	0	3	0.40	0.25	0.20	0.15	-
3	0	3	0.35	0.25	0.20	0.20	-
4	0	4	0.30	0.25	0.20	0.15	0.10
5	0	4	0.35	0.20	0.15	0.15	0.15
6	1	3	0.45	0.35	0.20	-	-
7	1	3	0.40	0.35	0.25	-	-
8	1	4	0.45	0.20	0.20	0.15	-
9	1	4	0.40	0.25	0.20	0.15	-
10	2	4	0.40	0.30	0.30	-	-

Table 5: Possible Distributions for Stochastic Links in Vehicle Allocation Problem.

requires that flow is conserved throughout the network, given the initial distribution of vehicles. Thus, Node 20 is actually an artificial sink node and the links into Node 20 are artificial as well. These links are uncapacitated and add no value to the problem. All other links shown in the diagram handle the flow of loaded vehicles between the adjacent nodes.

Since the need to move loads is governed by demand, the upper capacities of these links are stochastic. Flow along these links generate positive revenue. Running parallel to each of these links, but not in the diagram, is another link which handles the flow of empty vehicles. These links are uncapacitated. Flow along these links generate positive cost. The goal is to minimize negative profits (cost - revenue). The given initial distribution of vehicles is (10,12,12,10,7). The links in this problem were numbered by considering each node in numerical order (as seen in the diagram) and numbering consecutively from top to bottom. Table 6 gives the cost per unit of loaded flow along each link (r_i), the cost of per unit of empty flow along each link (c_i), and the distribution given to the random variable corresponding to that link's upper capacity(D_i). As before, the distributions were chosen from among 10 different possibilities, which are shown in Table 5.

Again, three versions of this problem were considered. In the first version, all of the links handling the flow of loaded vehicles were assumed to have stochastic upper capacities. In the second version, 15 links were chosen at random to have stochastic capacities. These 15 links have 8 distinct initial nodes and 11 distinct terminal nodes, so that the upper bound from links with a common terminal node differs only slightly from the Edmundson-Madansky bound, both in value and computation. The upper bound from links with common initial nodes, though, is small enough that that probability space can be partitioned and the initial upper bound can be refined. Finally, all links into and out of Node 8 are assumed to have stochastic upper capacities. Thus, a total of eight links have stochastic capacities. Since the network is acyclic, none of the links into Node 8 are affected when the bound for links with a common initial node is used over the links out of Node 8. Hence, the bound for links with a common terminal node can be used on all the links into Node 8. Thus, only four function evaluations are necessary.

A program was written in C which recursively updated the right-hand side of each network problem as needed, then passed the new program to be solved by IBM's Optimization Subroutine Library (OSL). The program was run on an IBM RS\6000 workstation.

The results are shown in the table below. The table includes information about the number of stochastic links in each version, and the number of distinct problems that had to be solved. If two values are

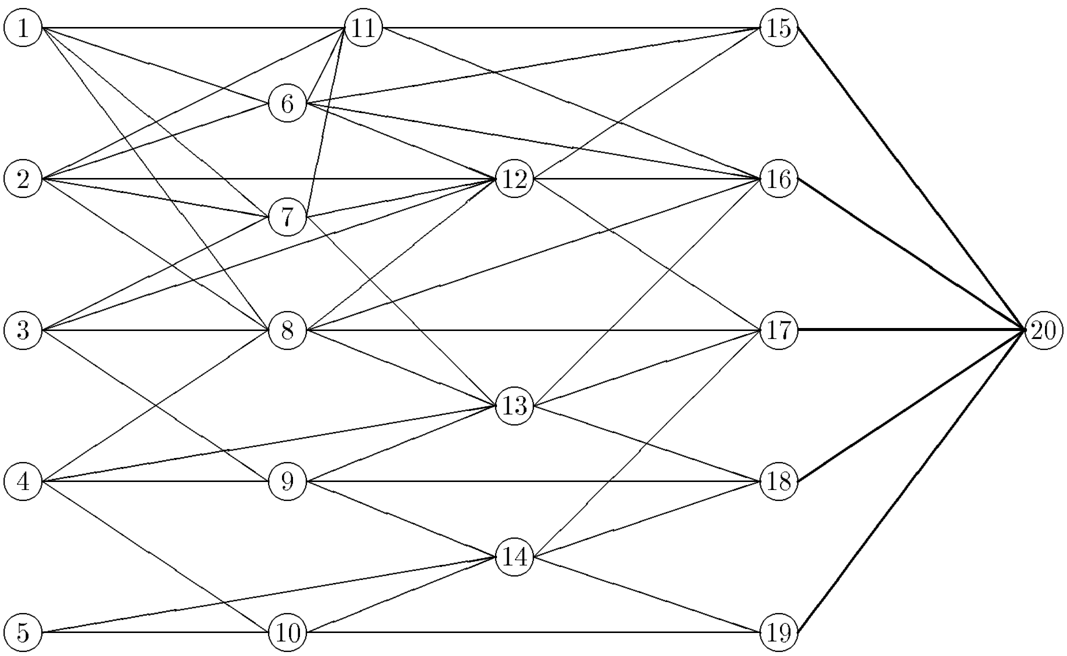

Figure 3: Vehilcle Allocation Problem with Uncertain Demands

i	r_i	c_i	D_i	i	r_i	c_i	D_i	i	r_i	c_i	D_i	i	r_i	c_i	D_i
1	327	81	4	13	255	73	7	25	177	63	9	36	263	93	4
2	211	67	10	14	265	76	3	26	348	91	10	37	285	98	7
3	243	70	1	15	360	105	9	27	148	51	10	38	281	99	7
4	268	70	7	16	161	60	10	28	312	76	8	39	242	87	1
5	301	75	10	17	264	78	8	29	332	83	9	40	301	108	6
6	204	73	3	18	348	86	10	30	284	75	3	41	277	93	4
7	356	95	8	19	267	78	8	31	317	76	6	42	168	77	5
8	190	63	5	20	146	54	7	32	367	103	6	43	175	78	7
9	225	69	1	21	330	86	4	33	329	81	2	44	273	94	8
10	198	66	1	22	352	93	2	34	214	65	9	45	252	87	3
11	343	85	2	23	188	61	6	35	326	78	1	46	252	88	2
12	261	80	3	24	225	74	3								

Table 6: Revenue for loaded flow, cost of empty flow, and distribution of capacity for links in Vehicle Allocation Problem.

shown for the number of problems solved, the first is for the upper bound for links with a common terminal node, the second for links with a common initial node. If only one value is shown, the values are the same. Also shown is the optimal objective value when all stochastic components are at their lowest value (Greatest Objective Value), and the optimal objective value when all stochastic components are at their highest value (Least Objective Value). This was done to show the range of possible values. Finally, the table shows the lower bound found using Jensen's Inequality, the upper bound using links with a common terminal node, and the upper bound using links with a common initial node. The problem Trans.3.a shows the effect of the refining the bound in problem Trans.3. Problem Veh.Allo.3 shows the bound obtained by clustering both the links into Node 8 and links out of Node 8.

Problem	Number of Stochastic Links	Number of Problems Solved	Greatest Objective Value	Least Objective Value	Lower Bound (Jensen)	Upper Bound (Terminal Node)	Upper Bound (Initial Node)
Trans.1.	105	2^{15}	132,095	114,190	124,154.90	126,838.00	126,671.59
Trans.2.	30	2^{9}	132,095	126,007	129,519.45	130,064.32	130,098.17
Trans.3.	9	2	130,303	127,165	128,766.40	-	129,126.25
Trans.3.a	9	3	130,303	127,165	128,766.40	-	129,054.28
Veh.Allo.1	46	2^{14}	-2058	-33270	-17,835.50	-16,082.53	-14,831.20
Veh.Allo.2	15	2^{11},2^{8}	-8038	-20,258	-13,179.10	-12,912.32	-12,558.45
Veh.Allo.3	8	4	-6861	-16952	-11086.65	-10787.42	

The bounds for the Transportation problem are remarkably tight, as the upper bound is within 2% of the lower bound in all cases, and as close as .224% in problem Trans.3.a. This would suggest that the