Mitigating Uncertainty via Compromise Decisions in Two-stage Stochastic Linear Programming

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Stochastic Programming (SP) has long been considered as a well-justified yet computationally challenging paradigm for practical applications. Computational studies in the literature often involve approximating a large number of scenarios by using a small number of scenarios to be processed via deterministic solvers, or running Sample Average Approximation on some genre of high performance machines so that statistically acceptable bounds can be obtained. In this paper we show that for a class of stochastic linear programming problems, an alternative approach known as Stochastic Decomposition (SD) can provide solutions of similar quality, in far less computational time using ordinary desktop or laptop machines of today. In addition to these compelling computational results, we also provide a stronger convergence result for SD, and introduce a new solution concept which we refer to as the compromise decision. This new concept is attractive for algorithms which call for multiple replications in sampling-based convex optimization algorithms. For such replicated optimization, we show that the difference between an average solution and a compromise decision provides a natural stopping rule. Finally our computational results cover a variety of instances from the literature, including a detailed study of SSN, a network planning instance which is known to be more challenging than other test instances in the literature.

Key words: Stochastic Linear Programming, Stochastic Decomposition, Computational Experiments

1. Introduction

For certain stochastic linear programming (SLP) models, the associated probability space can be so large that identifying a deterministically verifiable optimum is impossible (in reasonable time) using any foreseeable computer. Nevertheless, such models arise in a variety of applications, and new notions of approximate (or near)- optimality, supported by statistically verifiable bounds, are important for decision support. While statistical optimality bounds have been studied in the literature for a while (e.g. Higle and Sen (1991b), Higle and Sen (1996a), Mak et al. (1999), Kleywegt et al. (2002), Bayraksan and Morton (2011), Glynn and Infanger (2013)), their use in identifying near-optimal *decisions* for realistic instances has been limited. It is important to note the emphasis on "decisions" as opposed to identifying bounds within which the optimal value might belong. Prior attempts to use statistical optimality bounds have either required some genre of high performance computing or they have been limited to relatively small instances (few decision variables/constraints and a small number of random variables or scenarios). The goal of this paper is to demonstrate that for certain classes of two-stage SLP models, we are able to provide statistically verifiable, near-optimal decisions even if uncertainty is modeled using continuous random variables. Building on previous work connected to stochastic decomposition, we introduce the notion of a "compromise" decision, which allows us to not only confirm statistical bounds, but also recommend decisions with significant confidence. We report computational results for a suite of SLP test problems from the literature, and show that statistically verifiable decisions can be obtained within a few minutes of computing on desktops, laptops, and similar "run-of-the-mill" computing devices.

The class of SLP models discussed below fall under a category known as fixed-and-relatively-complete recourse models, and may be stated as follows.

$$\text{Min} \quad f(x) = c^\top x \; + \; E[h(x,\tilde{\omega})] \tag{1a}$$

$$\text{s.t. } x \in X = \{Ax \;\leq\; b\} \subseteq \Re^{n_1} \tag{1b}$$

where, $\tilde{\omega}$ denotes an appropriately defined (vector) random variable, and the function h is defined as follows.

$$h(x,\omega) = \text{Min} \quad d^\top y \tag{2a}$$

$$\text{s.t.} \quad Dy = \xi(\omega) - C(\omega)x \tag{2b}$$

$$y \geq 0, y \in \Re^{n_2}. \tag{2c}$$

The notation ω denotes any outcome of the random variable $\tilde{\omega}$. The fixed-and-relatively-complete recourse assumption of the above model implies that the matrix D is deterministic, and the function h has finite value for all solutions x satisfying $Ax \leq b$. As is common in decision-making under uncertainty, it is necessary to make decisions (x) in the first stage before an outcome ω is observed, and subsequently, the second stage decisions (y) are made. The quantity $\xi(\omega) - C(\omega)x$ often denotes the "position of resources" after the decision x has been made, and the outcome ω has been observed. In the terminology of Approximate Dynamic Programming (ADP, Powell (2007)), this vector may be looked upon as the "post-decision state" of a two-stage model.

In order to motivate this paper, we will begin by presenting some computational results for a well known instance called SSN (Sonet Switched Network, Sen et al. (1994)). This model has been used for a variety of purposes in the past; it was one of the early SLP models validated using discrete-event simulation; others have used it to illustrate whether certain algorithms scale well (Linderoth et al. (2006) and Nemirovski et al. (2009)), and still others have used it to illustrate what makes certain SLP models difficult for sampling-based algorithms (Shapiro et al. (2002)). More recently the Defense Science Board Report (Grasso and LaPlante (2011)) recommends SSN and models of this genre for DoD trade-off studies in which a very large number of contingency scenarios are necessary as part of the analysis to accompany recommendations for investment in new technologies. For these reasons, and because of its roots as an industrial-sized instance (SSN grew out of a 1990's Metropolitan network in Atlanta, GA), we use the performance on SSN to illustrate that decision-makers need not shy away from some classes of SLP models; certain current algorithms are up to the task of providing statistical optimality bounds within reasonable time using ordinary computing machinery. This evidence is provided in section 2.

Following a discussion of SSN, section 3 presents the methodology, which is based on Stochastic Decomposition (SD, Higle and Sen (1991a), Higle and Sen (1994), Higle and Sen (1996b)). While SD has close ties to some Simulation Optimization approaches (e.g. Shapiro and de Mello (1998), and approaches described in Kim et al. (2011)), these methods are more general in scope than SD because the latter focuses strictly on two-stage SLP models. This focus facilitates computations with large scale problems arising in practical applications. In addition to the ability to scale up using linear programming, focusing on SLP models also allows us to show that the algorithm produces a sequence which converges to a unique limit (with probability one). Moreover, as outputs of the algorithm, our procedure provides decision-makers two alternative choices which either reinforce each other, or provide indicators of "indecision". One of the alternative decisions will be the "compromise" decision, and the other will be an "average" solution. We will show that when these decisions are similar (i.e. they reinforce), they are both very close to optimum. On the other hand, if these decisions are not similar, then we suggest greater precision in calculating solutions for each replication, and these should be undertaken by using the "warm starting" capability that is naturally available via SD. Following our presentation of the methodology, we present further computational results in section 4. The test instances, which are available in the literature cover a variety of applications ranging from inventory control and supply chain planning to power generation planning, freight transportation, and others. All of the test instances reported here require an algorithmic treatment of multi-dimensional random vectors, and hence instances with the simple recourse property (e.g. news-vendor models) are not included in this study.

This paper addresses several questions of relevance to the SP community.

1. Given that SP problems can be demanding, greater accuracy may call for re-runs that should re-use previously discovered structures of an instance. How can such structures be re-used for the purposes of warm-starting?
2. Given that SLP models have a very special (convex) structure, should sampling-based methods be designed to take advantage of such structure?
3. Sampling-based SP methods borrow variance reduction techniques from the simulation literature. Are there other variance reduction techniques that are appropriate for SP, but are not considered in the simulation literature?
4. Parallel architectures in SP have traditionally been used to process bunches of scenarios. Are there other ways to use parallel architectures which permit the solution of industrial-strength models?
5. Should SP algorithms report lower bound estimates for the "true" problem so that the quality of a recommended decision can be ascertained?

We will present the conclusions of this paper by placing our contributions in the context of these questions. For now, we begin by applying sampling-based algorithms to SSN.

2. Motivation: Computations with a Practical SLP

Formally speaking, SSN is a two-stage stochastic linear programming model. The basic "operations"-issue in the SSN model is to recommend link sizes of a given network so that the network will experience the least number of "lost calls" (in expectation), while operating under a given budget constraint. We refer the reader to Appendix A for its mathematical formulation. In the SP literature, such models are often classified as "here-and-now" models because the link capacities must be decided before actual demands are known. Models of this type, which are based on introducing randomness to linear programming models, must contend with multi-dimensional random vectors, which, in the SSN model represent point-to-point demand uncertainty. In this particular example, there are 86 point-to-point pairs, which, by standards of LP models, is modest. As is common today, these demands are available through forecasting systems, and *errors* in forecasts may be treated as independent random variables. For the model presented in Sen et al. (1994), each marginal error random variable was deemed to be sufficiently approximated by a discretization using about 5-9 outcomes per demand pair. It is not difficult to see that the total number of scenarios involves an astronomical number of parametric LPs (approximately $O(10^{71})$). Even if one had access to an exascale (10^{18} flops) computing platform, it would be pointless trying to seek a solution whose optimality could be verified in a deterministic sense. It is therefore pragmatic to seek approximate solutions which are near-optimum in a statistical sense. Other approximations to SLP have been suggested via linear decision rules (Kuhn et al. (2011), Chen et al. (2008)). However, these approaches are motivated by scalability, rather than statistical bounds of optimality.

The remainder of this section uses the SSN model to illustrate the level of computing resources that may be necessary to provide decision support using sampling approaches for two-stage SLP models. The two classes of algorithms presented below are Sample Average Approximation (SAA) and Stochastic Decomposition (SD).

2.1. Sample Average Approximation

Conceptually, the SAA approach consists of replacing the “Expectation”, the objective function (1a), by a collection of instances which optimize a sample average approximation defined by a relatively small number of outcomes, say N. Then the function $F_N(x)$ below is known as the sample average approximation, and the following optimization problem is an SAA *instance*.

$$\operatorname*{Min}_{x\in X} F_N(x) \equiv c^\top x + \frac{1}{N}\sum_{i=1}^{N} h(x,\omega^i) \tag{3}$$

For the SSN model, one has $c = 0$, and in this sense the statement of (3) is agnostic to this fact. The SAA process may be summarized as follows.

1. Choose a sample size N, and choose a number M denoting the number of replications (batches).
2. (Optimization Step). For $m = 1, \ldots, M$ create an approximation $F_N^m(x)$, and solve an SAA instance (3). Let $\hat{F}_N^m$ denote the optimal value of replication m.
3. (Statistical Validation Step). Using $\{\hat{F}_N^m\}_{m=1}^M$ estimate a Lower Bound confidence interval. Using M solutions (i.e. potential decisions) from the Optimization Step, estimate the best Upper Bound confidence interval to a specified level of accuracy.
4. If the lower end of the estimated Lower Bound confidence interval is acceptably close to the upper end estimated Upper Bound confidence interval, then stop. Else, increase the sample size N and repeat from step 2.

Because the sampling step is independent of the optimization step, SAA is sometimes referred to as an "external" sampling algorithm. Some presentations in the literature refer to the "Optimization Step" as the "Training Step", and the "Statistical Validation Step" as simply the "Validation Step" (see Boyd (2013)). While the notion of replications is often not emphasized in some segments of the literature, decision-makers who have experience with sample-based algorithms (e.g. simulation) seek variance estimates of any metric reported by an algorithm, which in this case consists of lower bound and upper bound estimates.

The calculation in step 4 reflects the worst case optimality gap, which we refer to as the Pessimistic-Gap. We should also note that variance reduction techniques, such as Latin Hypercube Sampling (LHS), have been found to reduce variance of SAA estimates (Linderoth et al. (2006)). When M is in the neighborhood of 30, one typically invokes the central limit theorem for estimating a Lower Bound confidence interval; however, when M is much smaller (say 10), then, confidence intervals should be derived using the χ^2.

In order to complete the numerical illustration of SAA for SSN, we now draw upon results from Linderoth et al. (2006) where the Lower and Upper confidence bounds (Table 1) were reported using $M \in [7, 10]$ and Latin Hypercube Sampling. The Optimization Step for their study was performed using the Asynchronous Trust Region algorithm of Linderoth and Wright (2003), and the Statistical Validation is in line with the Mak et al. (1999). It is well known that the optimality gap estimates reduce with increases in sample size as shown in Table 1, and with a significant increase in sample size (from 1000 to 5000), we see a significant improvement in the "pessimistic gap" in Table 1. The chance that the actual gap exceeds the pessimistic gap is very small. The last row corresponding to a sample size of 5000 yields a pessimistic gap of about 2%, suggesting near optimality with very high probability.

Sample Size (N)	Lower Bound	Upper Bound	Pessimistic-Gap
50	10.10(+/-0.81)	11.380(+/-0.023)	2.113
100	8.90(+/-0.36)	10.542(+/-0.021)	2.023
500	9.87(+/-0.22)	10.069(+/-0.026)	0.445
1000	9.83(+/-0.29)	9.996(+/-0.025)	0.445
5000	9.84(+/-0.10)	9.913(+/-0.022)	0.195

Table 1 SAA with Latin Hypercube Sampling

The results of Linderoth et al. (2006) were obtained using a computational grid with hundreds of desktop PCs, although no more than one hundred machines were in operation at any one time. Even so, each SAA instance of the final row (with $N = 5000$) required about 30-45 minutes of wall clock time for solving one SAA instance of SSN. As it turned out, the solutions provided by the replications (about 6) were quite disparate even though these experiments were done using Latin-Hypercube Sampling. The use of a computational grid to establish the above results was a remarkable feat, solving millions of LPs on geographically dispersed and architecturally diverse machines. However, it also underscores the challenge of using SAA for real-world applications; if one resorts to sampling/simulation without exploiting the structure of the optimization problem, then the computing resources required to solve these instances can easily out-strip the available resources, thus restricting the potential of the SLP modeling paradigm.

2.2. Stochastic Decomposition (SD)

In keeping with the "high-level" description of SAA above, we provide a "high-level" description of SD. As with SAA, one may choose the number of replications M, but instead of choosing a sample size, we allow the SD algorithm to determine what is a sufficiently large sample size during the Optimization Step. Unlike SAA, SD does not optimize one sample average function F_N^m; instead it optimizes a sequence of Value Functions (VF) approximations $f_k^m(x)$, where k denotes an iteration counter during replication m. A VF approximation in iteration k will consist of two terms: the linear first stage cost $c^\top x$, and the second term will be the pointwise maximum of a finite number of linear (formally affine) functions representing the second stage expected recourse function. We refer to each linear piece as a *sample average price function* which represents a subgradient of some sample average approximation observed by the algorithm. These sample average price functions will resemble Benders' cuts, but there are several differences as summarized in the Remark provided in the following section. We will discuss further algorithmic details there; for the moment however, we note that each VF approximation will satisfy the following minorizing property

$$f_k^m(x) \leq F_k^m(x), \tag{4}$$

where F_k^m denotes a sample average function with a sample size of k (as in (3)). For iteration $k+1$, the next sample (of size $k+1$) will use all k previously generated outcomes and add one more (generated independently of previous outcomes) to arrive at a sample size of $k+1$. The earliest version of SD (Higle and Sen (1991a)) optimized the VF approximation of iteration k to obtain the next candidate solution $x^{m,k}$. More recent versions, including this paper, are based on Higle and Sen (1994) where optimization of a regularized version produces the next candidate solution. A subset of the sequence of candidate solutions, denoted $\hat{x}^{m,k}$, will be refereed to as "incumbents" (or incumbent solutions). In these earlier papers, it has been shown that if $k \to \infty$ then, we have asymptotic consistency of the values i.e., if $\hat{x}^{m,k} \to \mathbf{x}^m$, then $\lim_{k\to\infty} f_k^m(\hat{x}^{m,k}) = \lim_{k\to\infty} F_k^m(\hat{x}^{m,k}) = E[h(\mathbf{x}^m, \tilde{\omega})]$ (wp1). A few more details regarding the algorithm are provided in the following section (see also Birge and Louveaux (1997)).

While results like consistency are based on long-run behavior of an algorithm, one stops each replication after a finite number of steps, which will be based on detecting that the approximations obtained for the current replication have stabilized sufficiently, based on a given tolerance level. This test is known as an In-Sample stopping rule (Higle and Sen (1999)), and signals whether a particular replication has enough information to propose a solution which we denote by $\mathbf{x}^m$. If $m < M$ (the desired number of replications), then, we proceed to the next replication; otherwise SD recommends a *"compromise decision"* $\mathbf{x}^c$ which presents a compromise between all replicated decisions $\mathbf{x}^m$. Using $\mathbf{x}^c$ as the proposed decision, we calculate a 95% confidence interval for the upper bound $f(\mathbf{x}^c)$. In addition, a 95% confidence interval for a lower bound estimate on the optimal value is also reported. As with SAA computations in the previous subsection, we will report the pessimistic gap. A high-level description of SD is provided next.

1. (Initialize). Let m denote a counter of completed replications, and set $m=0$.
2. (Out-of-Sample loop). If the current number of completed replications is less than M, then increment m, and initialize (or continue) the next replication (go to step 3). Else, go to step 6.
3. (In-Sample loop). Update the available sample by adding one sampled outcome (independent of previously generated outcomes), and update empirical frequencies.
4. (Update Value Function (VF) Approximation). Using previously generated approximations and the new outcome, update the VF approximation $f_k^m(x) = c^\top x + \max\{h_{t,k}^m(x), t \in J_k\}$, where $h_{t,k}$ denote sample average price functions (see (8)), and J_k is an index set with $|J_k| \leq n_1 + 3$.
5. (Optimization Step). Optimize a regularization of the VF approximation (see (5)), and update an incumbent decision for the first stage.
6. (In-Sample Stopping Rule). If an In-Sample stopping rule is satisfied, then output a lower bound estimate $\hat{\ell}^m$, an incumbent $\mathbf{x}^m$ for replication m, and continue to step 7. Else repeat from step 3.
7. (Out-of-Sample Stopping Rule). If the number of replications is less than M, then go to step 2. Else, using the replicated solutions $\{\mathbf{x}^m\}$ calculate a compromise decision (denoted $\mathbf{x}^c$) and with this solution estimate a 95% Upper Bound confidence interval of specified accuracy. Using $\{\hat{\ell}^m\}_{m=1}^M$, the lower bound estimates calculate a 95% Lower Bound confidence interval. If the Pessimistic-Gap is acceptably small then stop. Else, decrease the tolerance of the In-Sample Stopping Rule, reset $m=0$, and resume all M replications from step 2.

This algorithm was executed for the SSN instance using three increasingly tighter relative tolerances: loose (0.01), nominal (0.001), and tight (0.0001), and the results for each run appear in Table 2.

Stopping Tolerance	Avg. Sample Size (Std Dev)	Lower Bound	Upper Bound	Pessimistic-Gap	CPU Time (s) (Std Dev)
Loose	1030.83(182.31)	9.345(+/-0.240)	9.951(+/-0.05)	0.896	30.11(6.63)
Nominal	2286.90(341.71)	9.736(+/-0.118)	9.927(+/-0.05)	0.359	90.50(20.56)
Tight	3305.47(617.17)	9.852(+/-0.107)	9.923(+/-0.05)	0.228	162.01(55.43)

Table 2 SD with Common Random Numbers (on a desktop PC with CPLEX12.4)

Upon examining Table 2, we first observe that the average sample size (per replication) increases with increased precision, as expected. The solution quality (as seen in the upper bound) does not improve dramatically from the first row to the third. However, the improvements in lower bounds are significant, ultimately, mitigating the uncertainty about the quality of the solution. In this sense, it reinforces a common observation in difficult optimization models: proving optimality is what requires extensive computations for difficult instances. Nevertheless, the average CPU secs. for even the row with "tight" tolerance is under 3 minutes on a desktop PC with the following specifications: Intel Core i7-3770S CPU@3.10GHz (Quad-Core), and 8 GB Memory @1600MHz. Since these processors are faster than the ones used in the computational grid study (Linderoth et al. (2006)), we also ran the same SD code on a Mac Book Air running Intel Core i5 CPU@1.8GHz (Dual-Core) with 4 GB Memory @1600 MHz. Such (laptop) processors of today could be considered on par with (or slightly slower) than the standard Pentium IV processors of 2004/2005 vintage. The average solution times and solution quality for such a processor is reported in Table 3. Notice that the solution quality is very similar to that reported in Tables 1 and 2. Indeed, the average lower bound as well as the pessimistic lower bound (in Table 3) are the best reported to date for SSN. Finally, the increase in CPU time (secs.) is marginal with "tight" tolerance, requiring just over 3 mins. of CPU time on average. In contrast, the grid study (Linderoth et al. (2006)) reported wall-clock time of the order of 30-45 minutes per replication using about 100 processors at any given time.

We recognize that it is impossible to provide precise characterization of the level of speed-up for several reasons: a) in addition to processor speeds, LP software has also made significant progress since 2004/2005, and b) there is communications overhead involved with using the grid. Nevertheless, it is worth noting, as in a recent PCAST report Holdren et al. (2010), that software advances are just as meaningful for challenging numerical problems as improvements in processing power. As a rough estimate of such advances, let η represent the fraction of time that processors are either idle or communicating during the shortest run (30 mins. of wall-clock time) on the grid. Then the speed-up factor for software (LP-solvers and SD) using a laptop could be approximated as $\frac{30}{3} \times 100(1-\eta)$. To see how this might compare with the rate of Moore's Law, first note that the speed of processors used for Table 3 is about the same or slower than standard processors of 2004/2005 vintage. Hence the speed-up can be attributed entirely to software progress, which implies that as long as $\eta \leq 0.872$ (i.e. idling/communications are less than 87.2%) speed-ups in LP/SLP software outpace Moore's Law which calls for a factor of at least 128 in 10.5 years.

Stopping Tolerance	Avg. Sample Size (Std Dev)	Lower Bound	Upper Bound	Pessimistic- Gap	CPU Time (s) (Std Dev)
Loose	1023.33(167.62)	9.366(+/-0.244)	9.953(+/-0.050)	0.881	32.73(6.97)
Nominal	2353.43(343.33)	9.764(+/-0.120)	9.928(+/-0.050)	0.334	109.96(26.31)
Tight	3137.50(605.17)	9.876(+/-0.107)	9.925(+/-0.050)	0.206	189.79(74.57)

Table 3 SD with Common Random Numbers (on a MacBook Air with CPLEX12.4)

Let us now return to a closer examination of Table 2. The stepped curve in Figure 1 shows the spread of objective function estimates obtained for each tolerance level reported in Table 2. Despite the fact that SD is an asymptotically convergent algorithm, the objective function estimates (for each terminal incumbent) show variability due to the fact that each run is terminated upon achieving some level of accuracy in finitely many iterations. For each tolerance setting (Loose, Nominal and Tight), the objective function estimates are in the range [10.100, 10.460], [9.994, 10.330] and [9.950, 10.310] respectively, and note that both upper and lower limits of these ranges move in the appropriate direction (lower). Moreover, from Table 2 we notice that lower bounds increase steadily, starting with 9.345 (for loose tolerance) rising to 9.736 (for nominal tolerance), and finally 9.852 (for tight tolerance). Thus increasing precision in SD leads to less biased estimates of lower bounds. Moreover, comparing these lower bounds to the last two rows ($N = 1000, 5000$) in Table 1, we observe that lower end of confidence intervals for lower bounds in Table 1 are in fact weaker than those reported in Tables 2 and 3.

Figure 2 also shows three lines on the far left of the figure. These correspond to the upper bound (average value) reported in Table 2. They correspond to objective estimates for compromise decisions at each tolerance setting. As shown in Figure 1 the compromise decisions for each setting yields a lower objective function estimate than the incumbent solutions for the corresponding run. The compromise decisions are not only superior, but they also possess another important property: when the compromise decision and the average solution are reasonably close, we can also conclude that both decisions are reasonably good. We will establish this result in the following section. For now, examining the specific instance at hand (i.e. SSN), we present the maximum relative error between the compromise and average solutions in Figure 2. For variables that are almost zero we report the absolute error instead of the relative error. In these figures, the horizontal axis displays

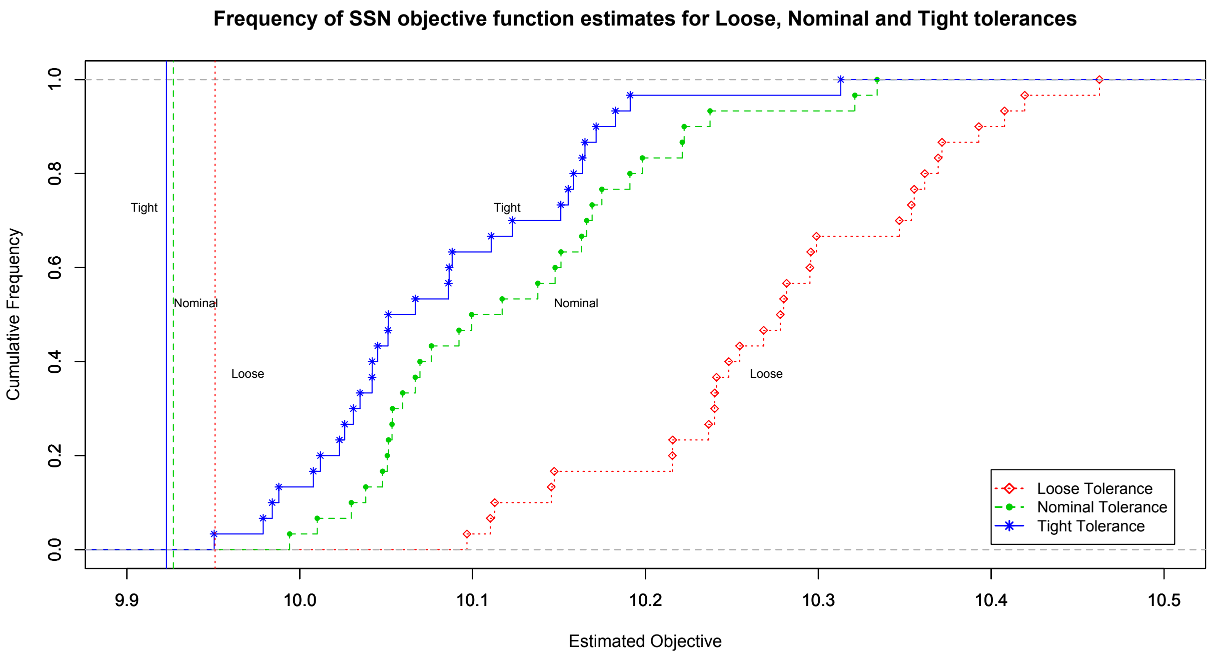

Figure 1 **Cumulative frequency of SSN objective function estimates for Loose, Nominal and Tight tolerances**

the index of first stage decision variables, and the vertical axis represents the difference between compromise and average solutions for each first stage variable. Observe that for each tolerance setting, the maximum relative differences are on the order of 10^{-4} to 10^{-5}. Since the relative error shown for all tolerance levels is pretty minimal, we can infer high quality decisions from compromise solutions for each tolerance level, even though the Lower Bound confidence intervals are weaker for the loose and nominal tolerances. Thus even for SSN, an instance considered to be ill-conditioned, the nominal setting provides reasonable accuracy. Such decision support is intended to mitigate the effects of uncertainty, without requiring extensive computational resources.

3. Algorithmic Concepts in Stochastic Decomposition

This section provides the algorithmic background for the computational results presented for SSN, as well as the more extensive computations presented in the next section. The algorithmic content will be presented in two subsections: one dealing with algorithm design and convergence, and another on stopping rules. The latter will be divided into two further subsections dealing with In-Sample and Out-of-Sample aspects of stopping within the SD framework. Before getting into the details, we mention some of the critical *assumptions for SD.*

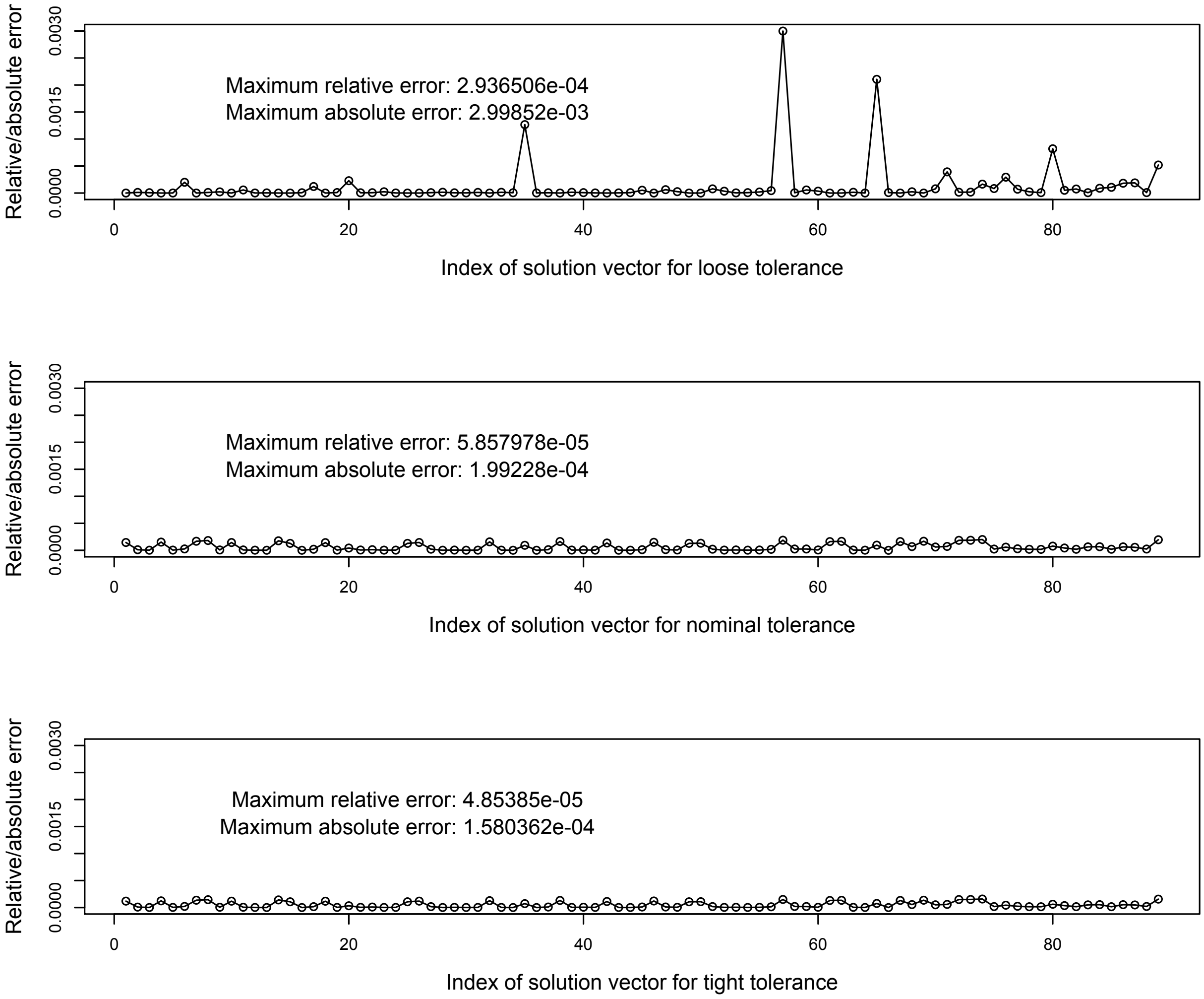

Figure 2 **Relative/absolute error between compromise and average solutions calculated for Loose, Nominal and Tight tolerances**

Assumptions. In addition to the fixed, and relatively complete recourse assumption, the set of first stage solutions, X, and the set of outcomes Ω are assumed to be compact and moreover, the recourse function h is assumed to be non-negative. The last assumption can be easily dropped by recognizing that a lower bound can always be added to the recourse function so as to ensure non-negativity. ■

3.1. Algorithmic Details of SD and Convergence

For this, and the following section, we will suppress the index m because our focus will be on calculations during any replication. At iteration k, a VF approximation will be given by the pointwise maximum of some linear (formally affine) functions, that is, $f_{k-1}(x) =$

$c^\top x + \max\{h_{j,k-1}(x), j \in J_{k-1}\}$. With each index j, we will record $t(j)$ as the iteration at which the linear function was created. This quantity $t(j)$ will also remind us of the sample size used to create the j^{th} function. It was shown in Higle and Sen (1994) that one only needs $n_1 + 3$ indices (at most) in J_{k-1}, where n_1 denotes the number of decision variables in the first stage, and functions that are deleted will be "forgotten" for all future iterations.

Any iteration of the SD algorithm works like an election. At iteration k, we start with a previously chosen incumbent decision and a VF approximation $f_{k-1}(x)$. The algorithmic strategy is to present a challenge to the incumbent by finding a solution to the following optimization problem.

$$x^k \in \operatorname{argmin}\{f_{k-1}(x) + \frac{\sigma}{2}\|x - \hat{x}^{k-1}\|^2 \mid x \in X\} \tag{5}$$

The decision x^k is referred to as the candidate. The quadratic term in the above problem is variously referred to as a proximal term or Tikhonov regularization. The quantity $\sigma \geq 1$ and is chosen adaptively depending upon the progress observed during the algorithm. Formally, it should also be indexed by k, but since we will not discuss the procedure to update σ, we prefer to maintain an un-indexed parameter.

We will pit the two competing decisions $\hat{x}^{k-1}$ (incumbent) and x^k (candidate) against each other using an *updated* value function $f_k(x)$. If the candidate happens to be significantly better (lower value) than the incumbent (see (9)), then we accept it as the new incumbent. Otherwise, there is no incumbent update. The first question at this point is: how does one calculate and update the VF approximations $f_k(x)$? We accomplish the update in the following steps:

1. Generate a new outcome ω^k, independent of all previous outcomes.

2. Let π_k denote the optimal conditional shadow price for the second stage LP, given inputs (x^k, ω^k). Assuming V_{k-1} (possibly empty) is available from previous iterations, $V_k \leftarrow V_{k-1} \cup \pi_k$.

3. Derive two sample average price functions denoted $h_{\nu,k}, h_{0,k}$, with the former representing the candidate, and the latter representing the incumbent. We simply present calculations for $h_{\nu,k}$, and recognize that $h_{0,k}$ is calculated similarly (by replacing x^k by $\hat{x}^{k-1}$ in (7)).

$$h_{\nu,k}(x) = \frac{1}{k}\sum_{i=1}^{k} \pi_{i,k}^\top [\xi(\omega^i) - C(\omega^i)x] \tag{6}$$

where, $i = 1, \ldots, k$ and $\pi_{i,k}$ is a conditional shadow price for outcome i, and is calculated as follows.

$$\pi_{i,k} \in \ \operatorname{argmax}\ \{\pi^\top[\xi(\omega^i) - C(\omega^i)x^k] \mid \pi \in V_k\} \tag{7}$$

4. Delete one of those linear functions $h_{j,k-1}$ for which the dual multiplier obtained by solving (5) is zero. Create an updated index set J_k. (One can ensure that at most n_1+3 indices are present in J_k.)

5. Let $t(j)$ denote the iteration at which inequality $j \in J_k$ was created. Under the assumption that $h \geq 0$ for all (x,ω), the functions $j \notin \{0,\nu\}$ are updated as follows.

$$h_{j,k}(x) \leftarrow \left(\frac{t(j)}{k}\right) h_{j,t(j)}(x), \quad j \in J_k \backslash \{0,\nu\}. \tag{8}$$

Remark. The approximations used in SD depends critically on sample average price functions (6) derived for both the candidate and the incumbent solutions. These price functions are similar in spirit to Benders' cuts; however, they are different in several important ways: a) they use the empirical distribution induced by the sample, b) as seen in (7) they can be derived by using approximately optimum shadow prices (in V_k), and c) as iterations proceed, the sample average price functions that were generated in past iterations are given less weight because they were created using a smaller sample size (than k). Thus unlike cuts in Benders' decomposition, the sample average price functions ultimately fade away, thus avoiding the persistence of poor (sampled) approximations. And, because sample average price functions evolve with every iteration, we do not refer to them as "cuts" (as in traditional Benders' decomposition) because they do not cut away any part of the epigraph permanently. ■

Let $\Delta_k = f_{k-1}(x^k) - f_{k-1}(\hat{x}^{k-1})$ denote the predicted change under the VF approximation f_{k-1}. We allow $\sigma \geq 1$ to be chosen in such a way that $\Delta_k \leq 0$. Then the winner of the election using the approximation f_k will be the next incumbent $\hat{x}^k$ as obtained below, with a fixed parameter $\rho \in (0,1)$ (set at 0.2 in our experiments).

$$\hat{x}^k = \begin{cases} x^k & \text{if} \quad f_k(x^k) < f_k(\hat{x}^{k-1}) + \rho\Delta_k \\ \hat{x}^{k-1} & \text{otherwise} \end{cases} \tag{9}$$

In the event that the incumbent changes, the positioning of the incumbent and candidate inequalities in J_k must also be swapped. We refer to Higle and Sen (1994) for a basic proof of asymptotic convergence (wp1). We now present a uniqueness result to give the reader a sense of the type of convergence that is possible.

Theorem 1. Assuming X is a compact set, and the fixed-and-complete recourse assumption holds. In addition, assume that $\sigma \geq 1$, and the recourse function $h(x,\omega)$ is non-negative for all $x \in X$ almost surely. Then the sequence converges to a unique solution with probability one.

PROOF. The only case of interest is one in which the incumbent sequence is infinite. Since X is compact, the sequence of incumbent solutions must have a convergent subsequence. In addition, noting that subgradient inequalities are generated at incumbents in every iteration, and the sample size increases with k, it follows that for any convergent incumbent subsequence $\{\hat{x}^{k-1}\}_{k\in K_1} \to \hat{x}$ we have $\{f(\hat{x}^{k-1})\}_{k\in K_1} \to f(\hat{x}) = \hat{f}$ with probability one (i.e., consistency of SD approximations). Moreover, for any other convergent subsequence, indexed by K_2 say, (9), and the continuity of the expectation functional imply that $\{f(\hat{x}^{k-1})\}_{k\in K_2} \to \hat{f}$ (wp1).

Now let K denote the sequence of solutions where the incumbent changes, and for $\tau > 1$, let $k_0, k_1, k_2, \ldots, k_\tau \in K$. Consider the quantity

$$\gamma_\tau = \frac{1}{\tau}\sum_{\ell=1}^{\tau}\big(f_{k_\ell-1}(\hat{x}^{k_\ell}) - f_{k_\ell-1}(\hat{x}^{k_{\ell-1}})\big) \tag{10a}$$

$$= \frac{1}{\tau}[f_{k_\tau-1}(\hat{x}^{k_\tau}) - f_{k_1-1}(\hat{x}^{k_0})] + \frac{1}{\tau}\sum_{\ell=1}^{\tau-1}\big(f_{k_\ell-1}(\hat{x}^{k_\ell}) - f_{k_{\ell+1}-1}(\hat{x}^{k_\ell})\big) \tag{10b}$$

Because of the consistency of objective estimates for incumbent solutions shown above, the summation term in (10b) must approach zero (wp1) as $\tau \to \infty$. Moreover the compactness of X and the relatively complete recourse assumption implies that the difference $f_{k_\tau-1}(\hat{x}^{k_\tau}) - f_{k_1-1}(\hat{x}^{k_0})$ is bounded. Hence if $\tau \to \infty$, then $\gamma_\tau \to 0$ (wp1).

Now with $\sigma \geq 1$, the optimality conditions for (5) at a candidate solution x^k imply that (see equation (2.6) on page 115 of Higle and Sen (1996b)).

$$f_{k-1}(x^k) - f_{k-1}(\hat{x}^{k-1}) \leq -\|x^k - \hat{x}^{k-1}\|^2 \leq 0. \tag{11}$$

Focusing on those iterations in which incumbents change (as in the index set K above), and using (10) and (11), and noting that $\hat{x}^{k_{\ell-1}} = \hat{x}^{k_\ell - 1}$ for $k_\ell \in K$, we conclude that

$$\gamma_\tau \leq -\frac{1}{\tau}\sum_{\ell=1}^{\tau}\|\hat{x}^{k_\ell} - \hat{x}^{k_{\ell-1}}\| \leq 0, \quad k \in K.$$

Hence as $\tau \to \infty$, we have $\gamma_\tau \to 0$ (wp1), and therefore we conclude that the average change between all incumbent solutions vanishes (wp1). This proves the result. ■

A few comments on analytical predictions (as opposed to computational experiments) are also in order. In this regard, recall that any SAA implementation separates the computational work along two dimensions: numerical optimization and statistical validation. Royset and Szechtman (2013) explores a variety of combinations based on asymptotic rates for using numerical optimization within SAA. A related method, known as Retrospective Approximations (RA), has been studied in Pasupathy (2010) where the sample average function is allowed to use larger sample sizes as iterations proceed. It is easy to see that RA has similarities with both SAA as well as SD. Like SAA, it seeks a near-optimal solution to an approximate problem of the same form as SAA (3); although like SD, it uses a growing set of outcomes. Indeed, RA is perhaps closest in spirit to a precursor of SD known as a Stochastic Cutting Plane method (SCP, Higle and Sen (1996b)), and for smooth problems, one might expect SCP to have similar convergence rates as RA, asymptotically.

3.2. Stopping Rules

As with any decision-making algorithm, SD must be terminated in finitely many iterations. Because the expectation operator requires multi-dimensional integration, providing deterministic certificates of optimality for practical instances (with several random variables) is intractable in general. As a result, a tandem of stopping rules, one based on In-Sample estimates, and the other on Out-of-Sample tests have been studied previously in a series of papers (Higle and Sen (1991b, 1996a, 1999)). We will comment on the performance of these tests in the appropriate subsections below. At this point, it suffices to say that previously known hurdles (e.g. the inability to reconcile multiple solutions from replications, relatively large gap estimates, and in some cases time-consuming LP-based bootstrap processes) have been overcome through a fresh view of the Out-of-Sample tests. These stopping rules are presented next.

3.2.1. In-Sample Stopping Rule. As proposed in Higle and Sen (1999), the In-Sample rule is intended to address two issues:

1. (Shadow Price Stability.) To assess whether the approximation due to (7) exhibits any sensitivity to additional information in the form of new shadow prices.
2. (Primal-Dual Gap Stability.) To recognize whether the estimated primal and dual solutions associated with the (5) are sensitive to variability due to sampling.

Shadow Price Stability. At iteration k, we assess the impact of new information (new outcomes, new first-stage candidate solutions and most importantly new shadow prices) on VF approximations. Note however that the first term of any VF approximation is linear ($c^\top x$) and c is known. Hence the predictive capacity of a VF approximation depends on how well the shadow prices in V_k predict the recourse function realizations $h(x,\omega)$ for any pair (x,ω) encountered by the algorithm. For any set of runs, suppose that we fix a tolerance level denoted ϵ. For such a run, let $q \in [2, k-1]$ define a window of iterations which will be used to ascertain whether further iterations are meaningful for the purpose of improving the approximation of the recourse function. Towards this end we observe the difference in approximation-quality when the recourse function is estimated using V_q versus the larger set V_k in (7). We will choose q to be large enough so that some recourse function observations have positive value, and calculate the following ratio.

$$S_k(x) = \sum_{i=1}^{k} \max\{\pi^\top[\xi(\omega^i) - C(\omega^i)x] | \pi \in V_k\} \tag{12a}$$

$$S_k^-(x) = \sum_{i=1}^{k} \max\{\pi^\top[\xi(\omega^i) - C(\omega^i)x] | \pi \in V_q\} \tag{12b}$$

$$R_k(x) = \frac{S_k^-(x)}{S_k(x).} \tag{12c}$$

These ratios are calculated whenever we calculate a subgradient at either a candidate (x^k) or an incumbent ($\hat{x}^{k-1}$). By assumption, $S_k(x) > 0$, and since we have $S_k^- \leq S_k$, we have $0 \leq R_k(x) \leq 1$. Then, we assess the stability of $R(k)$ by examining its sample mean and variance over the most recent $w(\epsilon)$ iterations. When these measures are sufficiently close to target thresholds (0.95 for the sample mean, and 0.00001 for the sample variance), then we declare the set of shadow prices to be stable. Appendix B provides figures showing the evolution of R_k associated with candidate and incumbent solutions for the industrial-strength instances in our study. In our implementation, we use $w(\epsilon) \in \{64, 256, 512\}$ for $\epsilon \in \{\text{Loose (0.01)}, \text{Nominal(0.001)}, \text{Tight(0.0001)}\}$ respectively.

Primal-Dual Gap Stability. Formally we wish to estimate the probability $P(f(\hat{\mathbf{x}}) - f^* \leq \epsilon)$ where f^* denotes the optimal value of the "true" problem. There are several computational hurdles with this calculation, all of which can be overcome using our non-parametric statistical approach based on bootstrapping. Recall that (5) is defined using several sample average price functions. As a result, this problem and its dual are random objects due to variability of each sample average price function. We will consider a primal-dual pair of solutions $(\hat{x}^{k-1}, \hat{\theta}^k, \hat{\lambda}^k)$ which are primal and dual optimum for (5) in iteration k. Since this instance is subject to variability (due to sampling), we wish to ascertain whether the variability of the gap estimate is small enough that this pair of primal and dual solutions is close enough to optimality. To do so, we use the general concept of bootstrapping as set forth in Efron (1979) (see also Singh (1981)). In the context of SD, bootstrapping involves re-sampling each sample average price function in J_k to create a re-sampled instance of (5). This application of bootstrapping was first used in Higle and Sen (1991b) where primal and dual pairs were both linear programs. Because re-sampled versions of linear programming approximations may render the dual multipliers $(\hat{\theta}^k, \hat{\lambda}^k)$ infeasible for the re-sampled approximation, the above implementation of the bootstrapping procedure required solving each re-sampled dual problem, thus making it somewhat computationally burdensome. In a subsequent paper, Higle and Sen (1999) proposed the primal-dual pair of QPs below which overcome the computational demands posed by the earlier LP counterpart.

Let B_k denote the matrix of sample average price functions (subgradients) $\{\beta^j\}_{j\in J_k}$ and H_k denote the vector of scalars $h_{j,k}(\hat{x}^{k-1})$. As shown in Higle and Sen (1994), the set J_k has cardinality of at most $n_1 + 3$, thus maintaining a finite bound on the size of the primal master problem. Let A and b define the polyhedron $X = \{Ax \leq b\}$ and let b_k denote the quantity $b - A\hat{x}^{k-1}$. Then using primal decisions as $d = x - \hat{x}^{k-1}$ the primal and dual problems may be stated as

$$u = \text{Min} \quad h + c^\top d + \frac{\sigma}{2}\|d\|^2 \tag{13a}$$

$$\text{s.t. } h - (\beta^j)^\top d \geq h_{j,k}(\hat{x}^{k-1}) \quad \forall j \in J_k \tag{13b}$$

$$\hat{x}^{k-1} + d \in X \tag{13c}$$

$$\ell = \text{Max} \quad H_k^\top \theta + b_k^\top \lambda - \frac{1}{2\sigma}\|c + B_k^\top \theta + A^\top \lambda\|^2 \tag{14a}$$

$$\text{s.t.} \quad \mathbf{1}^\top \theta = 1, \theta \geq 0, \lambda \geq 0 \tag{14b}$$

With the above formulations, the point $\hat{x}^{k-1}$ gets mapped to $d = 0$ in the primal, and the multipliers $(\hat{\theta}^k, \hat{\lambda}^k)$ are to be used for the dual. The gap estimate for this pair of solutions will be estimated by constructing a primal-dual pair in which each sample average price function is represented by a re-sampled estimate. Very briefly, the idea is as follows. Let $\{\omega^1, \ldots, \omega^k\}$ be a random i.i.d. sample of size k with distribution $\mathcal{F}$ and let $\mathcal{F}_k$ denote the empirical distribution of $\{\omega^1, \ldots, \omega^k\}$. Define a random object $T(\omega^1, \ldots, \omega^k; \mathcal{F})$, which depends upon distribution $\mathcal{F}$. The bootstrap method is to approximate the distribution $T(\omega^1, \ldots, \omega^k; \mathcal{F})$ by $T(\psi^1, \ldots, \psi^k; \mathcal{F}_k)$ under $\mathcal{F}_k$, where $\{\psi^1, \ldots, \psi^k\}$ denotes a random sample of size k under distribution $\mathcal{F}_k$. Next, we summarize an important theorem by Singh (1981).

Lemma 1. Let $\mu = E_{\mathcal{F}}[\omega]$, $\bar{\omega}^k = 1/k \sum_{i=1}^k \omega^i$, $\bar{\psi}^k = 1/k \sum_{i=1}^k \psi^i$ and assume $E[\omega^2] < \infty$. Let P and P_k denote the probabilities under $\mathcal{F}$ and $\mathcal{F}_k$ respectively. Then

$$\lim_{k \to \infty} |P(k^{1/2}(\bar{\omega}^k - \mu) \leq s) - P_k(k^{1/2}(\bar{\psi}^k - \bar{\omega}^k) \leq s)| = 0 \quad a.s. \tag{15}$$

Proof. See Singh (1981). ■

Basically, Lemma 1 studies the convergence (to zero) of the discrepancy between distribution $k^{1/2}(\bar{\omega}^k - \mu)$ and the bootstrap approximation. Essentially, pivotal statistics like the sample average are appropriate for bootstrapping because they are based on linear operators. In order to apply this idea to our setting, we re-sample every sample average price function defining the primal (13) to obtain a re-sampled primal. The dual corresponds to this re-sampled primal, and therefore has the form (14) using the re-sampled data. Thus in our re-sampling process we setup primal and dual problems from which we compare the gap associated with the given primal-dual pair. Note that this procedure re-samples pivotal statistics (i.e. sample averages) and not the duality gap directly because the latter is not necessarily pivotal. Moreover, since the primal solution $\hat{x}^{k-1}$ and the dual solution $(\hat{\theta}^k, \hat{\lambda}^k)$ are feasible to the respective re-sampled problems, we are able to calculate the re-sampled gap estimates by simply computing the primal and dual objective functions for each re-sampled instance. This eliminates the need to solve LPs as in the original bootstrapping method of Higle and Sen (1991b). Let $u_i - \ell_i$ denote the gap obtained for the i^{th} re-sampled instance, and from these we compute the empirical frequency distribution $\mathcal{F}_k(\epsilon)$ to estimate $P(u - \ell \geq \epsilon)$. Thus when $\mathcal{F}_k(\epsilon) \leq 1 - \delta$, then the replication is terminated, and the next SD replication can begin. In fact, the actual implementation uses a relative tolerance, so that a replication terminates when duality gap is small relative to the current incumbent value. For our computational experiments, we use $\delta = 0.95$, so that the In-Sample test requires 95% of the bootstrapped gap estimates to pass the test.

3.2.2. Out-of-Sample Stopping Rule. The idea of replication in stochastic programming has been adopted in many papers dealing with sample-based algorithms (Mak et al. (1999)). Because sampling introduces randomness into the algorithm, it is important to characterize errors in a manner that provides statistical performance guarantees. As the reader may have observed from Figure 1, the variability of solutions as well as objective values can be significant. Indeed, Linderoth et al. (2006) also report wide disparity of solutions of sampled instances with a sample size of 5000. A common suggestion is to obtain a preliminary objective estimate for solutions associated obtained from each replication, and then to successively prune (solutions) and refine objective estimates, until the subset of solutions is small enough to recommend a decision (Linderoth et al. (2006)). In general, this strategy can be extremely computationally intensive, requiring on the order of $\frac{1}{\epsilon^r}$ computations in the worst case. (Here r denotes the number of random variables, and ϵ is the desired accuracy.) In order to overcome issues related to the complexity of multiple replications, Bayraksan and Morton (2011) proposed a sequential sampling scheme, where the increase in sample size was controlled. To the best of our knowledge, this idea has been tested on some of the smaller instances (e.g. GBD, PGP2, 4TERM) and the computational results suggest that for instances with higher variability (e.g. PGP2), multiple replications provide more reliable estimates. In contrast, Nesterov and Vial (2008) suggest multiple replications with small increments (e.g. 1) to the sample size, and then to use an average solution, instead of trying to find the best among the replications. This idea has some similarity with the concept of compromise decisions which we report below.

Our Out-of-Sample test will leverage not only the primal solutions $\{\mathbf{x}^m\}_{m=1}^M$, but also the value function approximations $\{f^m(x)\}_{m=1}^M$. Towards this end, consider the following problem

$$\underset{x\in X}{\text{Min}} \quad \frac{1}{M}\sum_{m=1}^{M}[f^m(x)+\frac{\bar{\sigma}}{2}\|x-\mathbf{x}^m\|^2] \tag{16}$$

where $\bar{\sigma}$ is the sample average of $\{\sigma^m\}$. We refer to (16) as Compromise Problem and its solution as Compromise Decision. The Compromise Problem represents two objectives: the average value function approximation and the sum of squared errors. Let $\mathbf{x}^c$ denote the compromise decision, and $\bar{\mathbf{x}} = \frac{1}{M}\sum_m \mathbf{x}^m$. Intuitively, $\bar{\mathbf{x}}$ is an optimal solution to the squared errors part of the objective and if $\mathbf{x}^c$ and $\bar{\mathbf{x}}$ agree, then clearly, both are optimal to (16). We formalize this intuition below.

Let $K_m(\epsilon)$ denote the sample size used to construct the sample average price function at $\mathbf{x}^m$ prior to termination for a given trial m and $\epsilon > 0$. Define $N = \min\{K_m(\epsilon), m = 1, \ldots, M\}$, the smallest sample size of the sample average price functions used by any of the M approximations at their terminal solutions x^m. Of course, N depends on ϵ, but we suppress this dependence for notational convenience.

Lemma 2. Suppose that the fixed and relatively complete recourse assumptions hold, and assume that both the set of first stage solutions, X, and the set of outcomes Ω are compact. Furthermore assume that the the recourse function is non-negative.
a) If $\mathbf{x}^c = \bar{\mathbf{x}}$, then $\mathbf{x}^c$ solves the following

$$\operatorname*{Min}_{x \in X} \quad \bar{F}_M(x) = \frac{1}{M} \sum_{m=1}^{M} f^m(x). \tag{17}$$

b) Under the same hypothesis as in a), if $[f(\mathbf{x}^m) - f^m(\mathbf{x}^m)] = O_p(N^{-\frac{1}{2}})$ for all m, then,

$$f(\mathbf{x}^c) \le \frac{1}{M} \sum_{m=1}^{M} f^m(\mathbf{x}^m) + O_p((NM)^{-\frac{1}{2}}) \le \bar{F}_M(\mathbf{x}^c) + O_p((NM)^{-\frac{1}{2}}). \tag{18}$$

Proof. a) The optimization problem in (16) is a convex program and the optimality conditions imply that

$$0 \in \frac{1}{M} \sum_{m=1}^{M} \partial f^m(\mathbf{x}^c) + \mathcal{N}_X(\mathbf{x}^c) + \bar{\sigma}(\mathbf{x}^c - \frac{1}{M} \sum_{m=1}^{M} \mathbf{x}^m) \tag{19}$$

where $\mathcal{N}_X(\mathbf{x}^c)$ denotes the normal cone at $\mathbf{x}^c$. If $\mathbf{x}^c = \bar{\mathbf{x}}$, then the above equation reduces to the optimality conditions of (17).
b) We have

$$\begin{aligned}
f(\mathbf{x}^c) = f(\bar{\mathbf{x}}) &\le \frac{1}{M} \sum_{m=1}^{M} f(\mathbf{x}^m) && \text{(20a)}\\
&= \frac{1}{M} \sum_{m=1}^{M} f^m(\mathbf{x}^m) + \frac{1}{M} \sum_{m=1}^{M} [f(\mathbf{x}^m) - f^m(\mathbf{x}^m)] && \text{(20b)}\\
&\le \frac{1}{M} \sum_{m=1}^{M} f^m(\mathbf{x}^c) + O_p((NM)^{-\frac{1}{2}}) && \text{(20c)}\\
&= \bar{F}_M(\mathbf{x}^c) + O_p((NM)^{-\frac{1}{2}}). && \text{(20d)}
\end{aligned}$$

Here (20a) is due to convexity of f and (20c) follows from the fact that $\mathbf{x}^m$ is a minimizer of f^m. ■

Lemma 2 simply states under the hypothesis of b), the errors may be attributed to sampling, not optimization. While the above result motivates our Out-of-Sample stopping rule (i.e. measuring $\|\mathbf{x}^c - \bar{\mathbf{x}}\|$), we caution that testing the inequalities in (18) could, in fact, take a large number of samples because they depend on the sampling error, which can be very slow to reduce. Nevertheless, when $\epsilon \to 0$, we have $K_m(\epsilon) \to \infty$, and $N \to \infty$. As the reader might recall, N depends on ϵ, and in fact, so does $\bar{F}_M$. If we make this dependence explicit by using $\bar{F}_{\epsilon,M}$ instead, then the fact that sample average price functions are asymptotic minorants implies the following theorem.

Theorem 2. Let f^* denote the optimal value of the problem. Let $\mathbf{x}^c(\epsilon)$ and $\mathbf{x}^m(\epsilon)$ denote the quantities analogous to those in Lemma 2 for a given ϵ. Suppose that $\lim_{\epsilon\to 0} \mathbf{x}^c(\epsilon) = \mathbf{x}^c$ and $\lim_{\epsilon\to 0} \mathbf{x}^m(\epsilon) = \mathbf{x}^m$ and $\bar{\mathbf{x}} = \frac{1}{M}\sum_{m=1}^{M} \mathbf{x}^m$. If $\mathbf{x}^c = \bar{\mathbf{x}}$, then, $\lim_{\epsilon\to 0} P(\bar{F}_{\epsilon,M}(\mathbf{x}^c) = f^*) = 1$.

Proof. Because $\epsilon \to 0$, we have $K_m(\epsilon) \to \infty$, and $N \to \infty$, the sample average price functions provide asymptotic lower bounds, and therefore

$$\lim_{\epsilon\to 0} P(\bar{F}_{\epsilon,M}(\mathbf{x}^c) \leq f^*) = 1. \tag{21}$$

Moreover using arguments similar to Lemma 2, we have $\lim_{\epsilon\to 0} P(\bar{F}_{\epsilon,M}(\mathbf{x}^c) \geq f(\mathbf{x}^c)) = 1$ because $N \to \infty$. Since $f(\mathbf{x}^c) \geq f^*$, it follows that

$$\lim_{\epsilon\to 0} P(\bar{F}_{\epsilon,M}(\mathbf{x}^c) \geq f^*) = 1. \tag{22}$$

Hence combining (21) and (22) the result follows. ■

This theorem justifies our use of the compromise decision. Accordingly, we report confidence intervals for upper and lower bounds objective values, as well as the pessimistic gap. This provides the statistical analog of stopping deterministic algorithms when the estimated gap falls below a pre-specified error tolerance.

The increased reliability offered by compromise decisions reported for SSN (see Figure 2) might have come as a surprise for a problem that has been characterized as "ill-conditioned". The reader might recall that for the loose tolerance run, Figure 2 already showed good agreement between the average and compromise solutions, and a decision-maker might have been satisfied with an objective function upper bound of 9.951 (see Table 2). However, if the decision-maker chooses to obtain greater accuracy, then, it is easy to re-start the SD process for the nominal tolerance using all the information obtained in the course of the previous run (with loose tolerance). Thus, while greater reliability may require more sampling (for instances like SSN), the marginal effort is simply the additional iterations and not an entire run from scratch. Such warm-starting is critical for those decision-makers who seek greater accuracy in decision support.

On a related note, we observe that while it might seem that (16) is even more onerous than the individual replications, that is not the case. Because the In-Sample rule already provides $\mathbf{x}^m$, the sample average solution $\bar{\mathbf{x}}$ is already available, and so are the VF approximations $\{f^m\}$. Hence all that is required is to warm-start the quadratic program (16) using $\bar{\mathbf{x}}$, and if it declares optimality at the start, one can directly use $\bar{\mathbf{x}}$ as the compromise solution. However, since the marginal amount of computations associated with solving the compromise problem (independently of the sample average decision) is not very resource intensive, we calculate both $\mathbf{x}^c$ and $\bar{\mathbf{x}}$ independently so that the user can verify similarities between these solutions. To give the reader a sense of the computational time involved for each of tolerance level of SSN, we note that the CPU seconds for the compromise problems were 5.78 (Loose Tol.), 5.86 (Nominal Tol.) and 6.21 (Tight Tol.) respectively. When compared with the effort required to estimate the objective function at one or more of the points $\{\mathbf{x}^m\}$, the solution time of the Compromise Problem is minimal. To close this section, we note that there are several instances in the literature (e.g. CEP1 and STORM) for which longer runs (and even replications) are an overkill. By monitoring the coefficient of variation of each decision variable across successive replications, several instances can be accurately solved without excessively long runs. We shall report these computational results in the following section.

4. Further Computational Results

Table 4 summarizes all test instances solved by SD under nominal tolerance. BAA99 is a single period multi-product inventory model (see Bassok et al. (1999)) and BAA99-20 is a larger version (20 products) of BAA99. CEP is a capacity expansion model appears in Higle and Sen (1996b). LandS, LandS2 and LandS3 are three versions of a power generation planning problem (Louveaux and Smeers (1988)). The original LandS instance has one random variable with 3 outcomes. LandS2 has 3 random variables with a total of 64 scenarios. LandS3 has 3 random variables each descritized to a finer normal distribution with a total of 10^6 scenarios. PGP2 is the same genre of instances, but has greater objective function variability in value as well as solution variability (Higle and Sen (1996b)). The multi-location transshipment model called RETAIL can be found in Herer et al. (2006). SSN is the network expansion model presented at the beginning of this paper. STORM is an air freight scheduling model (Mulvey and Ruszczyński (1995)). 4NODE and 20TERM are both freight fleet scheduling problems, with the former data set appearing on a web site due to A. Felt (http://www4.uwsp.edu/math/afelt/slptestset/download.html), and the latter came to us from Infanger (1999). Along with SSN, STORM and 20TERM are based on industrial applications, and are indicated by an asterisk in Table 4. One of the larger data sets in this set is STORM. While it is large both in size and number of scenarios, the variability of its objective function is relatively small, as confirmed by the ratios $R_k(x)$ reported for STORM in Appendix B.

Because the early computational tests in the Stochastic Programming literature were performed using deterministic algorithms, it became customary to report performance by creating alternative instances using different sample sizes. For example, Zverovich et al. (2012) reported 16 different sample sizes for 4NODE, corresponding to scenarios ranging from $2^0, 2^1, \ldots, 2^{15}$. In our study, we do not consider these as 16 different instances, but one instance with 2^{15} scenarios, and our goal is to provide statistical guarantees for objective value upper and lower bounds, together with recommendations of the compromise and average solutions with the former being generated without warm-starting (using the average solution). We made this choice to avoid prompting the compromise problem in any way.

From Table 5 we see that for all instances, the confidence levels obtained using the nominal tolerance is very reasonable, and moreover, the average and compromise solutions also provide similar values. For most instances, these solutions obtained with nominal tolerance should be considered acceptable, although the user has the option to be more demanding, and seek solutions with tighter tolerance without having to re-start the process "from scratch". This is because SD records all shadow prices, and sample average price functions for each run, and they can be used immediately to continue further iterations if the user desires. For large scale models like SSN, this is a powerful setup. Another aspect of our study that is unique to SD is its ability to report the sample sizes that were necessary to obtain the quality of solution. This is particularly important in large scale applications where optimization and simulation are tightly coupled (see the wind energy case study for the state of Illinois in Gangammanavar et al. (2013)).

Instance Name	1^{st} stage variables/constraints	2^{nd} stage variables/constraints	# of random variables	Universe of scenarios
BAA99	2/0	7/4	2	615
BAA99-20	20/0	250/40	20	O(10^{34})
CEP	8/5	15/7	3	216
LandS	4/2	12/7	1	3
LandS2	4/2	12/7	3	64
LandS3	4/2	12/7	3	O(10^{6})
PGP2	4/2	16/7	3	576
RETAIL	7/0	70/22	7	O(10^{11})
SSN*	89/1	706/175	86	O(10^{70})
STORM*	121/185	1259/528	117	O(10^{81})
4NODE	52/14	186/74	12	32768
20TERM*	63/3	764/12	40	O(10^{12})

Table 4 Test instances with SD under Nominal Tolerance

In Table 6 we report the sample sizes used for each instance (together with its variability over 30 replications). From this table one can observe that although STORM is very large in terms of the variables, constraints and scenarios, it requires a sample size of only 300 to get reasonable solutions. This is about the same number of samples required for much smaller instances such as PGP2 and BAA99. From our study, we also observe that 20TERM and RETAIL take more iterations, and in fact, the latter takes longer for objective function estimation even though the instance (RETAIL) is smaller than 20TERM in terms of decision variables and constraints. As expected it is the variability that makes instances take longer. Finally, it is clear that SSN is by far the most demanding of these instances, but each replication can be completed in about 90 secs for nominal tolerance. It was shown in Figure 2 that this solution has very low error from the average solution. From this point of view, the user may decide to accept the decision based on the fact that the difference between the compromise and average solution is small enough. However, as shown in section 2 tighter statistical bounds can also be obtained by demanding greater precision on the bounds. Such solution reports are likely to provide decision makers a great deal of confidence in adopting decisions recommended by the SLP solver.

Instance Name	Upper(UB) and Lower Bounds(LB)	Average Solution		Compromise Solution	
		Average Values	95% CI's	Average Value	95% CI's
BAA99	OBJ_UB	-236.204	±5.451	-236.203	±5.451
	OBJ_LB	-240.864	±5.988	-240.864	±5.988
BAA99-20	OBJ_UB	-318393.648	±3994.053	-318451.800	±3994.502
	OBJ_LB	-322870.496	±1337.854	-322870.496	±1337.854
CEP	OBJ_UB	354175.320	±1687.537	354175.320	±1687.537
	OBJ_LB	353080.809	±10530.916	353080.809	±10530.916
LandS	OBJ_UB	381.761	±1.309	381.761	±1.309
	OBJ_LB	381.120	±1.174	381.120	±1.174
LandS2	OBJ_UB	227.393	±0.668	227.395	±0.668
	OBJ_LB	227.789	±1.628	227.789	±1.628
LandS3	OBJ_UB	225.541	±0.640	225.541	±0.640
	OBJ_LB	225.712	±1.319	225.712	±1.319
PGP2	OBJ_UB	447.955	±1.406	447.928	±1.405
	OBJ_LB	447.339	±2.157	447.339	±2.157
RETAIL	OBJ_UB	154.411	±0.772	154.406	±0.772
	OBJ_LB	153.995	±2.573	153.995	±2.573
SSN	OBJ_UB	9.927	±0.050	9.927	±0.050
	OBJ_LB	9.736	±0.118	9.736	±0.118
STORM	OBJ_UB	15481852.286	±48193.646	15481760.131	±48208.190
	OBJ_LB	15493958.503	±8826.814	15493958.503	±8826.814
4NODE	OBJ_UB	447.058	±0.392	446.951	±0.394
	OBJ_LB	446.979	±0.068	446.979	±0.068
20TERM	OBJ_UB	254515.479	±1005.008	254514.672	±1004.934
	OBJ_LB	253649.385	±168.573	253649.385	±168.573

Table 5 Objective upper and lower bounds and corresponding 95% confidence intervals of all instances solved by SD

Instance Name	Average Sample Size (Std Dev) for SD	Average CPU secs (Std Dev) for SD	CPU secs to Solve Compromise Problem	CPU secs to Estimate Obj of Compromise Solution	CPU secs to Estimate Obj of Average Solution
BAA99	298.03(58.82)	0.89(0.23)	0.01	0.25	0.21
BAA99-20	330.50(13.68)	1.49(0.13)	0.04	0.13	0.14
CEP	263.03(6.73)	0.79(0.08)	0.01	5.61	5.43
LandS	260.27(2.77)	0.76(0.07)	0.01	0.21	0.22
LandS2	264.27(4.86)	0.83(0.08)	0.01	1.29	1.26
LandS3	263.57(5.99)	0.78(0.07)	0.00	0.78	0.74
PGP2	284.63(27.60)	0.74(0.10)	0.01	0.31	0.31
RETAIL	460.47(150.55)	1.53(0.67)	0.00	4.01	3.98
SSN	2286.90(341.71)	90.50(20.56)	5.86	112.31	111.52
STORM	300.50(5.33)	19.56(1.46)	8.83	0.06	0.05
4NODE	406.07(49.44)	5.66(0.85)	3.04	0.01	0.01
20TERM	453.07(5.98)	7.28(0.39)	1.13	0.13	0.12

Table 6 Sample size (std dev), solution time for SD(std dev), solution time for Compromise Problem and evaluation time (of compromise solution and average solution) for all instances solved by SD

5. Conclusions

Our contributions to the SP literature may be viewed through several lenses. These points of view are captured by the questions that were posed in section 1, and in this context, we offer the following comments.

1. Given that SP problems can be demanding, greater accuracy may call for re-runs that should re-use previously discovered structures of an instance. How can such structures be re-used for the purposes of warm-starting?

• Because SD uses "sampling-on-the-fly" (i.e. adaptive sampling), it creates a sequence of sample average price functions based on re-using shadow prices. This approach records information gathered during the course of the algorithm (e.g. shadow prices, and sample average price functions), so that future iterations can be resumed without having to re-discover them from scratch. This capability is also important in the context of Stochastic MIPs, just as fast LP solvers, and warm-starting are important for deterministic MIPs.

2. Given that SLP models have a very special (convex) structure, should sampling-based methods be designed to take advantage of such structure?

• Convex non-smooth optimization forms the backbone of SD, which derives its stability from using a proximal term. The latter also suggests a quadratic programming dual problem which is used for the In-Sample (Primal-Dual Gap Stability) as well as the Out-of-Sample tests (with the Compromise Problem). Such integration of numerical optimization and statistical computing promotes algorithmic synergy for more efficient solution.

3. Sampling-based SP methods borrow variance reduction techniques from the simulation literature. Are there other variance reduction techniques that are appropriate for SP, but are not considered in the simulation literature?

• Variance reduction in simulation is primarily aimed at performance (objective function) estimation. While this is important for SP models, it needs to go beyond performance estimation because of the emphasis on decision-making. In order to appreciate this point, note that past attempts at solving SSN have not only required high performance computing, but disagreement between alternative runs can leave users in a quandary. Instead, concurrence between compromise and sample average decisions, provides a principled (see Theorem 2), and computationally effective approach. Figure 2 highlights the empirical success of this approach.

4. Parallel architectures in SP have traditionally been used to process bunches of scenarios. Are there other ways to use parallel architectures which permit the solution of industrial-strength models?

• The speed of replications can be improved significantly by using parallelization. This approach to SP (and of course SD) needs greater attention.

5. Should SP algorithms report lower bound estimates for the "true" problem so that the quality of a recommended decision can be ascertained?

• It is difficult to establish the quality of a solution without both upper bound and lower bound estimates. The former is usually a statistical estimation problem, if a decision is available. The latter (lower bounds) is an optimization issue since it is the optimization step that provides the lower bounding estimate. Clearly this is an essential ingredient of any optimization method.

In summary we have demonstrated that two-stage SLP models are now solvable to very reasonable levels of accuracy, even for industrial-strength models, in reasonable time, using “run-of-the-mill” computing devices. This bodes well for users of the SLP modeling paradigm.

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Appendix A: The SSN Instance - Back to the Future

The subtitle of this Appendix (“Back to the Future”) is intended to reflect the fact that there are certain instances from the past which need to be fully explored before we can claim to have a thorough understanding of the two-stage SLP paradigm. SSN is one such instance. The subtitle conveys the fact that we went back two decades to revive our interest in the SSN model, and the results of this paper have shown us how to solve such problems fairly accurately, and without requiring computing platforms that are beyond the ordinary.

SSN is a network planning model which seeks to size links (i.e. choose link capacities) before demand is known with certainty. This model is limited to adding capacity to links that have been identified for expansion, and in this sense, it does not recommend what additional links may be useful. The model does not consider the possibility of adding new links, which would require stochastic integer programming. Thus given a collection of links, we wish to choose additional capacities under a budget constraint, while ensuring that the expected unserved demand is minimized. The model uses the following notation.
First stage decision variables:

$x_j \equiv$ the amount of capacity to be added to the j^{th} link.

First stage data/parameters:

$n \equiv$ the number of links that are to be considered for capacity expansion.

$b \equiv$ the total capacity budget for the entire network.

$\tilde{\omega} \equiv$ the m dimensional random variable that represents demands associated with the m point to point pairs served by the network.

The first stage problem can be summarized as follows:

$$\text{Min} \quad E[h(x,\tilde{\omega})] \tag{23a}$$

$$\text{s.t.} \quad \sum_j x_j \leq b \tag{23b}$$

$$x \geq 0 \tag{23c}$$

The function $h(x,\tilde{\omega})$ is a random variable representing unserved demand. It is a function of the capacity expansion plan x and a random vector $\tilde{\omega}$. For each outcome ω following model is used to estimate unserved demand, and the generic name referring to this piece of the model is the second stage problem. The second stage decision variables are

$f_{ir} \equiv$ the number of calls associated with point to point pair i that are served via route $r \in R(i)$.

$s_i \equiv$ the number of unserved requests associated with point to point pair i.

Second stage data/parameters:

$m \equiv$ the number of point to point pairs served by the network.

$R(i) \equiv$ the set of routes that can be used to connect point to point pair i.

$A_{ir} \equiv$ is an incidence vector in $\mathbb{R}^n$ whose j^{th} element is 1 if link j belongs to route $r \in R(i)$, and is 0 otherwise.

$e \equiv$ is a vector in $\mathbb{R}^n$ of current link capacities.

And the second stage problem is the following:

$$h(x,\omega) \quad = \quad \text{Min} \quad \sum_{i=1}^{m} s_i \tag{24a}$$

$$\text{s.t.} \quad \sum_i \sum_{r \in R(i)} A_{ir} f_{ir} \leq x + e \tag{24b}$$

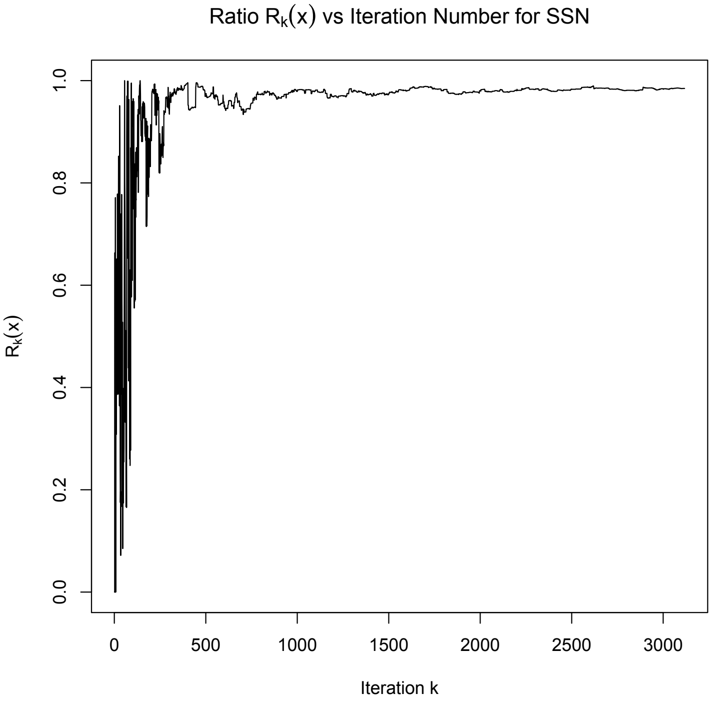
$$\sum_{r \in R(i)} f_{ir} + s_{ir} = \omega_i \quad i = 1, ..., m \tag{24c}$$

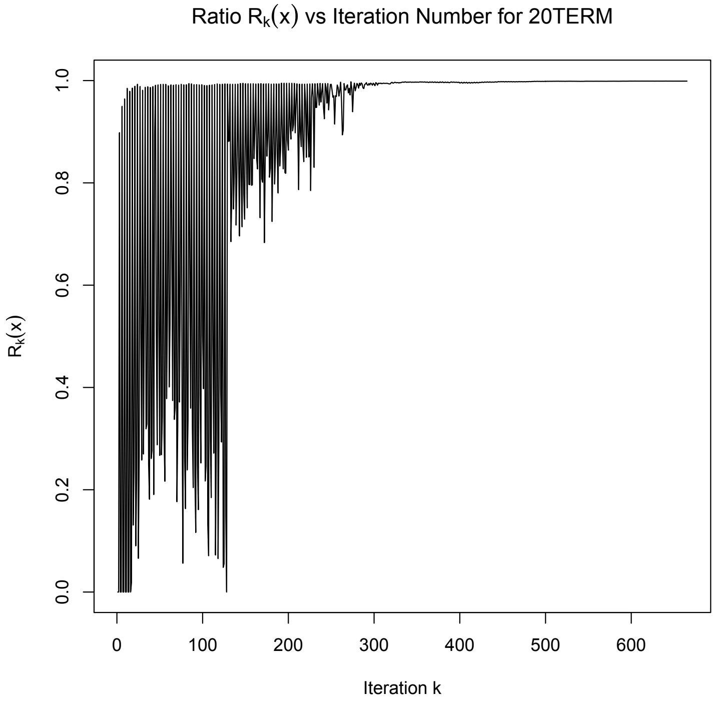
$$f, s \geq 0 \tag{24d}$$

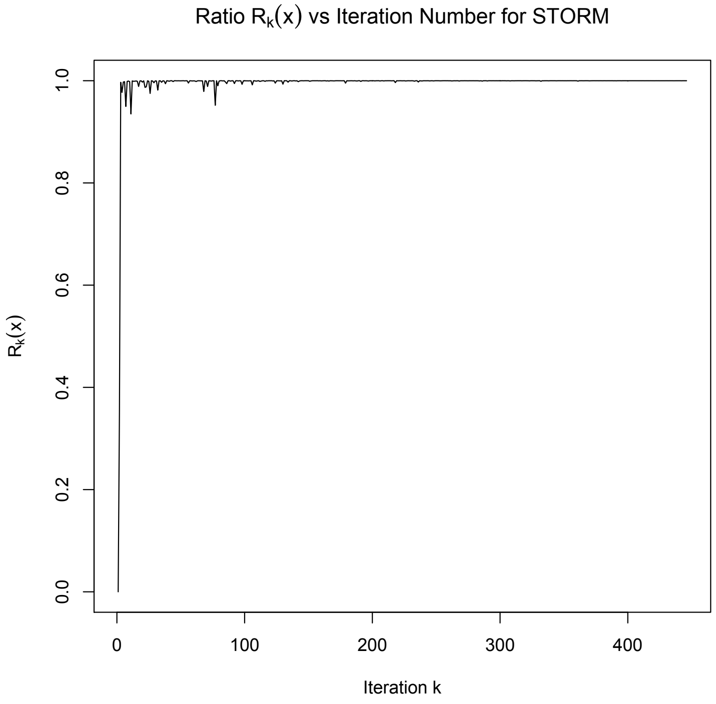
Constraints (23b) limits the expansion plan under a total capacity level b while constraints (24b) ensure that routing is achieved without violating any link capacity. Constraints (24c) are flow balance equations of the network.

Appendix B: Figures of Shadow Price Stability

The following figures show that for the three "industrial-strength" instances included in this paper, the VF approximations are poor representations of the expected recourse function in early iterations. As the SD algorithm proceeds, the ratios R_k approach 1 (see Shadow Price Stability in subsection 3.2.1), indicating vastly improved approximations in the vicinity of candidate and incumbent solutions, and they are all steady by the time that the In-Sample rule accepts an incumbent. The number of data points in these graphs exceed the number of iterations because SD updates incumbent objective estimates periodically, and the ratios are calculated for those updates too.







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