Optimal harvest of a forest in the presence of uncertainty

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A method is described for finding logging levels to maximize harvest in a finite horizon type II model. Uncertainty is considered in the form of the risk of forest fires and other environmental hazards, which may destroy a random fraction of the existing forest. Numerical results include upper and lower bound approximations to the original problem.

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Une méthode est décrite pour trouver le niveau d'exploitation forestière pour maximiser la coupe dans un modèle à horizon fini de type II. L'élément d'incertitude est pris en considération sous forme de risque d'incendie forestier ou d'autres dangers reliés à l'environnement qui peuvent détruire une portion aléatoire de la forêt existante. Les résultats numériques tiennent compte des approximations des limites supérieures et inférieures du problème original.

[Traduit par la revue]

Introduction

This paper describes the application of a computational algorithm of stochastic programming to a problem that arises in forest management. The objective is to determine a harvesting sequence that maximizes the expected total volume of wood harvested from the forest over some time periods to come. Uncertainty enters the model in the form of forest fires and other environmental hazards that may destroy a portion of the standing timber. Following the work of Reed and Errico (1986*a*, 1986*b*, 1986*c*) and Gunn (1986), the entire forest is differentiated into smaller units by the introduction of age-classes comprising all the area on which trees of ages between given bounds grow, regardless of location in the forest. As trees mature, areas move from one age-class to another, with new trees of age zero replacing trees that have been harvested. This model is referred to as the Type II model in the forestry literature (Johnson and Scheurman 1977). The advantage in this model lies in the fact that after logging, stands that previously belonged to different classes may be considered as one class from that point on, which keeps the problem size small. On the other hand, the Type II model contains implicit homogeneity assumptions about growth rates and logging costs that will rarely be satisfied in practice.

Other approaches are possible: Johnson and Scheurman (1977) also presents a Type I model in which all possible harvesting sequences must be specified beforehand by the decision maker. This may lead to rather large problems, especially if the planning horizon is long, but heterogenous data can be incorporated rather easily, because the separation of stands present in period 1 is maintained through the entire time horizon.

Lembersky and Johnson (1975) present a probabilistic model that uses Markov decision processes and Van Wagner (1978,1983) describes a simulation study based on 1000 stands that are treated as indivisible units in his model. A stochastic dynamic programming model is given by Martell (1980), who finds a (deterministic) rotation age in the presence of uncertainty. In contrast to this, the Type II model employed in the present actually provides a full contingency plan that prescribes harvest levels in each period based on the fire rates observed up to that point.

The data used in the present paper describe one particular forest of timber near Fort Nelson, British Columbia, comprising a total area of nearly 100 000 ha. The model is idealized in the sense that it assumes a single species forest of pure white spruce (*Picea glauca* (Moench) Voss), uniform quality of timber and land, uniform accessibility and logging cost throughout the forest, and constant technology. The forest is split into a number of different age-classes, and the biological yield is assumed to be a known function of the age of the stand. The transition pattern between the various age-classes is simple: part or all of a stand moves on to the next age-class at the end of each period, the rest is harvested or destroyed, in which case a new stand grows in its place in age-class 1. The width of an age-class is assumed equal to the length of one time period, taken to be 20 years so as to keep the size of the problem manageable. This leads to eight age-classes, including one absorbing state, into which all stands that are of age greater than 140 years are lumped together.

Determining the transition matrix

Suppose the state of the forest at the beginning of a time period can be described by the vector $s = (s_1, \ldots, s_K)^T$, where s_k denotes the area of the forest covered with timber in age-class k ($k = 1, \ldots K$). Then an area $x = (x_1, \ldots, x_K)^T$ is harvested, leaving the remainder susceptible to forest fires, which destroy a random proportion $(p_1, \ldots, p_K)^T$ of the timber left standing. Assuming instant reforestation of the cleared area, either naturally or by human intervention, the state s' of the forest at the beginning of the next period is given by

$$s' = \mathbf{P}(s - x) + \mathbf{Q}x$$

with

$$\mathbf{P} = \begin{bmatrix} p_1 & p_2 & \cdots & & p_{K-1} & p_K \\ 1 - p_1 & 0 & \cdot & & \cdot & 0 \\ 0 & 1 - p_2 & 0 & \cdot & \cdot & 0 \\ \cdot & & \ddots & & & \cdot \\ \cdot & & & \ddots & & \cdot \\ \cdot & & & & \ddots & \cdot \\ 0 & \cdot & \cdot & \cdot & 1 - p_{K-1} & 1 - p_K \end{bmatrix}$$

$$\mathbf{Q} = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ 0 & \cdot & \cdots & 0 \end{bmatrix}$$

TABLE 1. Annual fire rates and their cumulative probability distribution based on historic data

Percent destroyed	Cumulative distribution	Percent destroyed	Cumulative distribution	Percent destroyed	Cumulative distribution
0.0	0.47	0.001 8	0.68	0.007 77	0.85
0.000 1	0.50	0.002 0	0.71	0.010 4	0.88
0.000 2	0.56	0.002 37	0.74	0.011 8	0.91
0.000 37	0.59	0.002 66	0.76	0.014 8	0.94
0.001 2	0.62	0.003 15	0.79	0.026 6	0.97
0.001 7	0.65	0.005 46	0.82	0.031 5	1.00

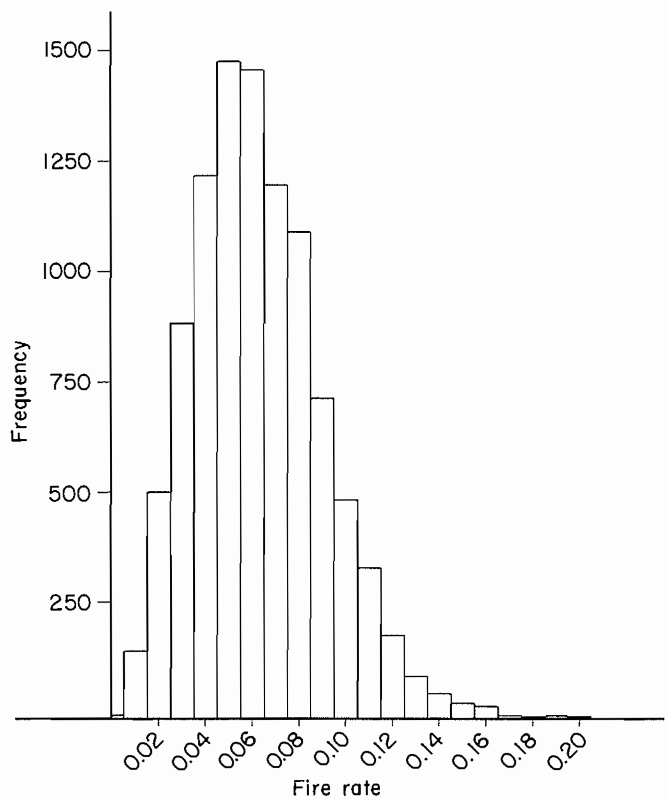

FIG. 1. Histogram of simulated fire rates.

The loss rates p_K are random variables, and historic data are available for their distribution. Ordinarily one would expect the p_k to have different distributions: a tree's ability to survive a forest fire in all likelihood depends on the age of the tree. Unfortunately, the information available from the British Columbia Forest Service as given in Reed and Errico (1986*c*) (see Table 1) aggregates all these figures into one single variable p describing the annual proportion of forest land destroyed by fire. In other words, one assumes $p_1 = p_2 = \ldots = p_K = p$. This is a restrictive assumption, and more work should be done to disaggregate the loss rates.

A further technical assumption made in the present study is that p is stationary over time and independent from period to period. This is not an unreasonable assumption, but it neglects the possibility of factors such as improved fire-fighting equipment and early warning systems, introduction or eradication of noxious pests and tree diseases, climatic shifts, and the like. Nonstationary processes could be accommodated by the same techniques, provided that reasonable distributions for the random elements are available.

Information about the incidence of forest fires is available in the form of yearly data, such that the distribution of loss rates for a 20-year period has to be computed using the yearly information. To that end, let $p_1, \ldots, p_{20}$ be the loss rates for years 1 through 20, so that the loss rate p over the entire period can be calculated from

$$p = 1 - \prod_{j=1}^{20} (1 - p_j)$$

One could try to find a closed-form approximation to the distribution of p based on the fact that the survival rate $q = 1 - p$ satisfies

$$q = \exp(\sum_{j=1}^{20} \ln q_j)$$

Because the q_j are independently identically distributed and possess a mean and variance, the central limit theorem implies that their sum is approximately normal so long as enough terms are used in the sum; that is, each time period is long enough. Here we have 20-year periods, and the distribution of q_j is most irregular and has a discontinuity at 0. A sample size of 20 might not be large enough to smooth out this effect. To settle the question, 10 000 sets of yearly loss rates were randomly generated on a computer and the resulting ln q plotted. As evidenced in Fig. 1, the distribution of the sample is decidedly skewed, and a normal distribution does not fit the data all that well.

Since the linear programming model, described later, requires discrete approximations regardless, I did not pursue this line any further. Instead, the true population distribution was assumed to be equal to the sample distribution from the simulation study. This seems defensible in light of the large sample size and the fact that the 1-year distributions are also based on a sample of historical data. The sample distribution was then discretized in various ways, as shown in Table 2. The upper bound discretizations correspond to the well-known Jensen's inequality (see, e.g., Madansky (1959) or Gassmann and Ziemba (1986)), and the lower bound discretizations yield an approximation due to Edmundson (1957) and Madansky (1959). Several deterministic versions were also tried: fixing the loss rate at one particular value, assumed constant over the time, and calculating the optimal solution for this value.

Present value of standing timber

Finite time horizons quite frequently lead to erratic behavior in the last period. This is not hard to understand: if the world ends tomorrow, one might as well cut down all the forest while one can still derive some profit from it.

TABLE 2. Discretizations and probabilities for fire rates

	Upper bound discretizations				
1 point					
Fire rate	0.062 58				
Probability	1.000				
2 point					
Fire rate	0.086 12	0.042 40			
Probability	0.461 6	0.538 4			
3 point					
Fire rate	0.104 99	0.073 54	0.042 40		
Probability	0.184 7	0.276 9	0.538 4		
4 point					
Fire rate	0.104 99	0.073 54	0.052 78	0.030 17	
Probability	0.184 7	0.276 9	0.291 2	0.247 2	
5 point					
Fire rate	0.103 49	0.075 36	0.059 97	0.046 33	0.027 77
Probability	0.2000	0.2000	0.2000	0.2000	0.2000
	Lower bound discretizations				
2 point					
Fire rate	0.00 0	0.202 68			
Probability	0.691 2	0.308 8			
3 point					
Fire rate	0.00 0	0.062 58	0.202 68		
Probability	0.173 6	0.748 8	0.077 6		
4 point					
Fire rate	0.00 0	0.062 58	0.086 12	0.202 68	
Probability	0.173 6	0.512 8	0.283 7	0.029 9	
5 point					
Fire rate	0.00 0	0.042 40	0.062 58	0.086 12	0.202 68
Probability	0.071 3	0.317 3	0.297 8	0.283 7	0.029 9

There are several ways to circumvent this undesirable last period effect.

Reed and Errico (1986*c*) solved various deterministic scenarios with 35 time periods and simply discarded the final five periods from their model. This is justifiable on theoretical grounds because of "turnpike theorems," which ensure the existence of an optimal infinite horizon solution, the turnpike, with the property that finite horizon "trips" (provided they are far enough apart in time) will optimally proceed to the turnpike as rapidly as possible and follow the turnpike until the last moment before they have to turn. This approach permits the specification of arbitrary starting and end points.

Since it is inconceivable to solve a stochastic problem over such a long time horizon, and also because I saw no way to justify that the assumptions would hold statically over a period of 7 centuries, I decided to proceed differently. The stochastic problems considered all have seven time periods, and I attempted to impose a steady-state solution at the end, i.e., force an entry into the turnpike after seven periods. For some of the test problems, it turned out that this was not enough time to reach the turnpike, and infeasibility resulted.

The advantage of this method is that it does not require discounting, that is, future harvests are just as valuable as harvests today, and future constraint violations are penalized as heavily as infeasibilities in the first period. It has long been recognized (Clark 1976) that discounting in renewable resources leads to nonconservative harvesting policies and lower long-run sustained yield, because the biomass is depleted early and never permitted to recover. This is particularly true of slow-growing classes, such as trees.

The alternative is to assign a value to timber left standing at the end of the problem horizon, which does require a nonzero discount rate, for otherwise the present value of standing timber is infinite. Ordinary accounting rules work with interest rates of 10 and even 15% per year, but in problems with 20-year periods, such high interest rates simply cannot be used. Reed and Errico (1986*a*, 1986*b*, 1986*c*) experimented with values of 2, 3, and 4% annually; I decided to use initially a rate of 0.5% in order to include maximum information about the future into the stochastic solution. Some consistency checks were also done for yearly interest rates of 1, 2, 3, and 4%. The rate of 0.5% per year leads to a one-period discount factor of about 0.905, which is reasonably close to the accounting rules for yearly rates. (The discount factor δ is defined by $\delta = (1 - i)^{20}$, where i is the discount or interest rate.)

For the computation of present values of standing timber, one proceeds as follows. Let $v_1, \ldots, v_8$ be the values of standing timber in each of the eight age-classes and $y_1, \ldots, y_8$ be the yields if one unit of area is cut down. These yields are assumed to be stationary over time and known with certainty, so that they can be scaled to be 1. In other words, 1 m^3 of wood is assumed to be worth one unit of currency.

Then for each age-class, there are two possible actions to be considered. One can either cut down the trees, realizing an immediate yield of y_k plus anticipated yield δv_1 once the logged-off area has been reforested. The alternative is to postpone harvesting, in which case a proportion q of the area will still be standing one time period later, representing a value $\delta q v_{k+1}$, the rest will have been destroyed and have value $\delta p v_1$. Thus

$$\begin{aligned} v_k &= \max[y_k + \delta v_1, \mathrm{E}(\delta q v_{k+1} + \delta p v_1)] \\ &= \max[y_k + \delta v_1, \delta(1 - \bar{p}) v_{k+1} + \delta \bar{p} v_1] \end{aligned}$$

In particular, since $v_9 = v_8$, the optimal strategy in age-

TABLE 3. Age-specific data for the problem

Age-class	Yield (m^3/ha)	Initial area (ha)	Value of standing timber				
			i = 0.005	i = 0.01	i = 0.02	i = 0.03	i = 0.04
1	0	2 004	320.3417	132.5536	43.1161	17.6345	9.1245
2	0	1 404	356.1874	163.6890	65.4628	32.7989	20.7185
3	16	125	398.4370	204.2164	100.8833	62.0158	47.8189
4	107	241	448.2349	256.9687	157.0264	118.3075	111.1642
5	217	9 768	506.9294	325.6336	246.0159	226.7638	221.1642
6	275	16 385	564.9294	383.6336	304.0159	284.7638	279.1642
7	298	2 815	587.9294	406.6336	327.0159	307.7638	302.1642
8	306	61 995	595.9294	414.6336	335.0159	315.7638	310.1642

class 8 will always be to harvest, unless the constraints of the problem specifically forbid it. Because of the structure of the yields y_k, the optimal solution is then

$$v_k = \begin{cases} y_k + \delta v_1, & \text{if } k \geq k_0 \\ \delta(1 - \bar{p})v_{k+1} + \delta\bar{p}v_1, & \text{if } k < k_0 \end{cases}$$

for some k_0, which describes the optimal harvesting age. Once k_0 is known, the defining equations for $k = 1,\ldots,$ $k_0 - 1$, together with $v_{k_0} = y_{k_0} + \delta v_1$, form a nonsingular system of equations that can be solved for the v_k. In the present case, one finds $k_0 = 5$ for interest rates of 0.5, 1, 2, and 3%, and $k_0 = 4$ if the interest rate equals 4% per annum. The corresponding values v_k can now be read from Table 3, along with the yields y_k and the initial sizes of the various age-classes.

Additional constraints are placed into the model to ensure a controlled flow of timber. It might, for instance, be desirable on economic terms to avoid large fluctuations in the harvest from one period to the next. Without further constraints, the optimal solution might well be to cut down the entire forest and wait for 100 years or so before the entire area is harvested again. For small stands of timber, this may constitute a defensible strategy, but for an entire forest of some 1000 km^2, it probably does not.

Possible harvest flow constraints could be of three types: (*i*) absolute limits of the form

$$\alpha \leq y'x_t \leq \beta, \qquad t = 1,\ldots,T$$

(*ii*) absolute limits on the change of the harvest

$$\alpha \leq y'(x_t - x_{t-1}) \leq \beta, \qquad t = 1,\ldots,T$$

(*iii*) relative limits on the change of the harvest

$$\alpha\, y'x_{t-1} \leq y'x_t \leq \beta\, y'x_{t-1}, \qquad t = 1,\ldots,T$$

Following Reed and Errico (1986*b*), the constraints employed are exclusively of type *iii*), although in principle any type and even a mixture of them could be accommodated. One could further envisage, as in Gunn (1986), that flow constraints are separated into various categories of wood, corresponding to different uses in the wood industry. Immature trees that are too small for a sawmill might, for example, still be usable in a pulpmill. The entire complex of a wood-using industry with its capacity constraints and input requirements has been ignored in the present formulation.

Formulation of the problem and numerical results

Putting all the previous information together, one has the following mathematical formulation for the problem

$$[1.1] \quad \max \sum_{t=1}^{T} \delta^t y'x_t + \delta^{T+1}v's_{T+1}$$

subject to availability constraints

$$[1.2] \quad x_t - s_t \leq 0, \qquad \text{for } t = 1,\ldots,T$$

inventory constraints

$$[1.3] \quad (\mathbf{P} - \mathbf{Q})x_t - \mathbf{Q}s_t + s_{t+1} = 0, \qquad \text{for } t = 1,\ldots,T$$

and flow constraints

$$[1.4] \quad \begin{aligned} \alpha y'x_{t-1} - y'x_t &\leq 0 \\ \beta y'x_{t-1} - y'x_t &\geq 0 \end{aligned}$$

where the x_t are the decision variables, s_t are the state variables describing the age distribution of the forest at time t, s_1 is the original state of the forest, assumed to be known, y is the vector of biological yields, also assumed to be known and constant over time, and v is the vector of present values of timber left standing, discounted back to time period 1. The primes are used to indicate row vectors.

The solution method employed differs greatly from Reed and Errico's approach, who performed a "scenario analysis" of eq. 1 for some 35 time periods. Each scenario uses a typical realization p to define fixed matrices **P** and **Q**. The resulting deterministic linear programming (LP) problem was then solved by an LP solver. With eight age-classes per time period, each problem has $(34 \times 18) + 8 = 620$ constraints and $35 \times 16 = 560$ decision variables, but any connection between the scenarios is lost. Since their results showed period 1 harvest decreasing from over 400 000 m^3 to less than half that amount if the risk of forest fire is increased from 0 to 0.02 annually, this connection is an important point of consideration. Moreover, the planning horizon of 700 years appears excessively long if all one is interested in is a harvest policy to be implemented during the first period. The analysis presented below will attempt to address this latter point specifically.

To this end, the long planning horizon is replaced by a much shorter one of seven time periods. Now the problem can be solved using a special computer program described in Gassmann (1987), which is based on the Dantzig–Wolfe decomposition principle (Dantzig and Wolfe 1960) and is designed to exploit the special structure of problem [1].

This gave the results reported in Tables 4, 5, and 6. These tables show the optimal objective function values for the problem formulated in the previous section, and the area harvested in age-class 8 during the first period. (Owing to the overabundance of fully mature trees, no other age-class will be harvested until at least period 3.)

The central processor (CPU) times were measured on a

TABLE 4. Optimal first-period harvest for various deterministic fire rates

	Fire rate, p						
	0.00	0.05	0.062 58	0.10	0.15	0.20	0.202 68
Objective value	46 487.1	41 800.5	41 133.1	39 166.4	36 589.0	34 049.9	33 914.7
Harvest (ha) (age-class 8)	23 072.8	21 006.3	20 488.4	19 111.5	17 052.1	14 888.7	14 773.9
CPU time(s)	24.32	26.09	25.93	24.35	22.78	22.16	21.61
No. of rows	124	124	124	124	124	124	124

TABLE 5. Optimal first-period harvest and computation results for various upper bound discretizations

	Node structure				
	1.111.111	1.222.111	1.333.111	1.444.111	1.555.111
Objective value	41 133.1	41 008.7	40 908.7	40 897.2	40 870.3
Harvest (ha) (age-class 8)	20 488.4	20 313.1	20 202.5	20 110.5	20 007.9
CPU time(s)	25.93	82.15	208.64	256.18	634.72
No. of rows	124	700	2176	4984	9556

TABLE 6. Optimal first-period harvest and computation results for some lower bound discretizations

	Node structure			
	1.222.222	1.322.222	1.332.222	1.432.222
Objective value	38 825.6	39 683.3	39 784.8	39 528.4
Harvest (ha) (age-class 8)	15 564.9	15 564.9	15 564.9	15 564.9
CPU time(s)	350.91	404.48	547.17	744.46
No. of rows	2284	3436	5092	6784

VAX 11/780 computer at the Technical University of Nova Scotia, using the VMS FORTRAN compiler with full optimization; times are in seconds and refer to the solution phase only, not counting time for input–output. The final number supplied for each problem refers to the number of rows of the resulting linear program; this depends on the number of realizations used.

An attempt was made to condense the relevant probabilistic information to the absolute minimum. This information is conveyed via the "node structure," which is a string of seven integers giving the number of realizations used at each of the seven stages. The first stage is deterministic, so each string starts with a 1. The actual realizations and probabilities for each stage can then be read off from Table 2, for upper bound and lower bound problems.

Several deterministic versions were tried first, fixing the loss rate at one particular value, assumed constant over time, which resulted in the optimal solutions given in Table 4. This scenario analysis demonstrably does not work on this problem: the optimum fluctuates wildly, and without specific probabilistic information on how to assemble the different scenarios into one unifying picture, one is likely to arrive at a harvesting schedule that is quite far from the true optimum.

The problems shown in Table 5 were solved as four-stage problems by using one-point discretizations for periods 5–7, which permits aggregating period 4–7 into one stage. The first four stages use the realizations and probabilities given in Table 2.

These discretizations all give upper bounds for the optimal value, and they prove reasonably consistent. Not only the optimal value, but also the optimal first-stage decision variables stabilize, and an implementable first period solution suggests itself. Almost all the information about the probability distribution of the loss rates is contained in the mean, and consequently the expected value of perfect information appears insignificant.

Table 6 shows results for some lower bound discretizations. Here the smallest discretization available has two points for each period, so these problems have to be solved with the full seven-stage formulation. These solutions suggest a somewhat different picture. The lower bound on the optimal value of the problem is reasonably close to its upper bound, but the optimal decision variables are quite far off. The optimal harvest in period 1 is identical for all four problems tested, but this value is about 25% below the optimal harvest for the upper bound problems.

This is caused by feasibility problems if all seven periods exhibit bad years and p_t is at its lowest observed value, 0.797 32, for all periods. The flow constraints are so tight in this case that there is only one feasible solution. Indeed at the end of period 7 there will be no escapement whatsoever, all the harvestable forest has been destroyed despite the imposition of a value for trees left standing. This solution is forced by the problem constraints no matter how small the probability of observing 7 lean years in succession and illustrates what Madansky (1960) called the "fat solution" to a stochastic programming problem, namely a solution that is feasible for all possible future events.

The difficulty should be recognized as the modelling problem it represents: it is unavoidable to use a finite time horizon, but this finite horizon forces the use of discount rates, which in turn necessitate some balancing restrictions. It is interesting to note that this phenomenon is of course also present in the scenario analysis described in Reed and Errico (1986*c*), but there the effect is marked and may easily

TABLE 7. Optimal first-period harvest and computation results for upper bound problems with soft constraints

	Node structure					
	1.111.111	1.222.222	1.322.222	1.332.222	1.333.222	1.333.322
Objective value	41 132.0	40 914.3	40 897.0	40 864.2	40 835.8	40 703.1
Harvest (ha) (age-class 8)	20 495.8	20 047.9	20 076.9	19 952.8	19 947.4	19 726.6
CPU time(s)	19.86	266.45	347.81	515.79	818.40	1140.57
No. of rows	124	2284	3436	5092	7522	10 924

TABLE 8. Optimal first-period harvest and computation results for lower bound problems with soft constraints

	Node structure						
	1.222.222	1.322.222	1.422.222	1.332.222	1.442.222	1.333.222	1.333.322
Objective value	38 944.8	39 142.6	39 161.1	39 356.9	39 556.4	39 664.4	39 847.8
Harvest (ha) (age-class 8)	16 859.1	16 909.7	16 909.7	17 585.8	17 975.7	18 143.2	18 248.4
CPU time(s)	257.52	409.29	519.37	473.74	857.40	1166.02	1326.49
No. of rows	2284	3436	4552	5092	9016	7522	10 924

(and wrongly) be lumped together with the true effects of stochasticity.

In an attempt to rectify the situation, a linear penalty term was included that allows violations of the flow constraints, but imposes a penalty on the objective function for doing so. This device is known as "complete recourse" in stochastic programming. After some experimentation, I set the penalty parameter at 50, so that each unit of violation will decrease the objective function by 50 units. This number was chosen completely arbitrarily, but it is well known that a sufficiently large penalty parameter will force exact solutions. Some more fine tuning of the method revealed that the exact penalty parameter lies somewhere between 100 and 150.

Tables 7 and 8 show both upper and lower bound solutions with the soft constraints. Now it would appear that both upper and lower bounds are converging towards a joint limit, with the lower bounds showing much the larger fluctuations.

It is worth pointing out that the solution times for these problems appear to increase more or less linearly with the size of the equivalent LP problem. This result indicates that the algorithm employed in the computations is quite useful. (SIMPLEX-based LP algorithms (Dantzig 1963) exhibit solution times that typically increase more or less quadratically with the problem size.)

Some consistency checks are reported in Table 9. The motivation was to study whether and in what form the assumed interest rate would influence the results. For annual interest rates of 0.5, 1, 2, 3, and 4%, five model formulations were fed into the computer program. First, the mean value problem was solved using the one-point discretization from Table 2. The mean value solution seems quite a plausible harvesting policy to implement, and it will give an upper bound for the problem, but how good is it? Tighter upper and lower bound solutions were computed with three realizations in periods 2 through 6 and two realizations in period 7. (This is also referred to as node structure 1.333.332 for short.) This represents the largest decision tree the computer program can handle in its present configuration. It consists of 850 nodes in the decision tree, and the equivalent LP formulation has 15 000 rows. Finally, the same node structure was used to calculate new upper and lower bounds on the value of the problem once the mean value solution has been implemented for one period.

The two figures reported in each instance represent the optimal value of the decision problem and the harvest in age-class 8 during period 1. Owing to an abundance of fully mature trees (see Table 3), no other age-classes will be harvested before period 3.

For each event tree, the recommended harvest increases as the interest rate increases, which is just what one would expect to see. It is somewhat counterintuitive to find that uncertainty results in a more conservative harvest policy than the mean value problem. The reason for this is that hedging takes place to guard against rapid drops in timber yield if a period is encountered with a larger fire rate. Since lower bound problems work with more extreme distributions, they lead to even more conservative policies than upper bound problems. The optimal value lies somewhere in the middle.

Because the mean value problem leads to suboptimal policies, the possible range for the optimal value after implementation expectedly decreases for all interest rates, but the size of the drop is surprising nevertheless. The last column in Table 9, corresponding to an interest rate of 4% annually, is the most drastic and merits a closer look.

If only deterministic analysis is being used, a decision maker might implement the mean value solution, cutting down 21.44 units of area in the 1st year and expecting a long-run value of 11 113. This figure represents an upper bound on the expected yield, but it is reasonably close to the true figure, which lies somewhere between 9626 and 10 522, the better lower and upper bounds, respectively. Both these models indicate more conservative harvest levels, and the effect of overharvesting with the mean value solution is underscored in the last two rows of Table 9. Once the mean value solution is implemented, the best the decision maker can hope for is a long-run yield of 5301, which is about half

TABLE 9. Optimal value and first-period harvest dependence on annual discount rates before and after (underscored values) implementing the mean value solution for one period

	i = 0.005		i = 0.01		i = 0.02		i = 0.03		i = 0.04	
	Total yield	Period 1 harvest (ha)	Total yield	Period 1 harvest (ha)	Total yield	Period 1 harvest (ha)	Total yield	Period 1 harvest (ha)	Total yield	Period 1 harvest (ha)
Mean value problem	41 132	20.50	25 080	20.88	16 376	21.36	12 996	21.44	11 113	21.44
Upper bound problem	40 826	20.05	24 747	20.09	15 827	20.17	12 399	20.22	10 522	20.22
Lower bound problem	39 978	18.52	23 695	18.61	14 815	18.71	11 438	18.71	9 626	18.71
Upper bound implemented	40 406	20.50	23 570	20.88	11 379	21.36	7 187	21.44	5 301	21.36
Lower bound implemented	36 618	20.50	19 076	20.88	6 750	21.36	2 382	21.44	288	21.44

of what he might have expected. At worst, future losses become so high that they essentially wipe out any immediate gains.

Conclusions

This paper has shown that stochastic methods can be a computationally viable alternative to deterministic methods in the management of a forest. It has explored the connection between the discount rate and the value of information and demonstrates that the value of information increases as the discount rate increases. Stochastic models lead to more conservative harvesting policies than deterministic ones, and the mean value solution becomes more and more unrealistic with increasing discount rates.

Much work remains to be done before the model itself could possibly be implemented in practice. Better stochastic information, such as a breakdown of the loss rate distributions for individual age-classes, is clearly desirable. It might also be interesting to incorporate the possibility of expanding or receding forests into the model. This could be done by allowing for a separate random variable r describing the probability that a cleared area is reforested (in the present formulation $r \equiv 1$).

A more realistic model would further include a rudimentary representation of the woodworking industry, as demonstrated by Gunn (1986) in the deterministic case. Some of the homogeneity assumptions could be removed by a refined age-class differentiation, which might distinguish between qualitatively different stands or even different species of trees growing in the same forest. The transition matrices $\mathbf{P}$ and $\mathbf{Q}$ could be set up to account for that rather easily, at the expense of a larger problem size.

The possibility of stochastic yields should be analyzed more closely. This leads to stochastic cost coefficients in the LP problem. In principle, the decomposition algorithm remains applicable, but the uncertainty in the prices will complicate the calculation of the value of timber left standing.

A final point of interest concerns the dependence of the solutions on the length of the planning horizon. It is true that the decision tree for the problem exhibits geometric growth if more than one realization is used per period. This precludes the use of Reed and Errico's (1986*a*, 1986*b*, 1986*c*) 35-period model. On the other hand, there is nothing magic about the number seven, and it would be interesting to see how the solution changes if the planning horizon is extended to eight, nine, or perhaps ten periods. This problem has been set aside for a future study.

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CLARK, C.W. 1976. Mathematical bioeconomics. John Wiley & Sons, New York.

DANTZIG, G.B. 1955. Linear programming under uncertainty. Manage. Sci. **1**: 197–206.

______ 1963. Linear programming and extensions. Princeton University Press, Princeton, NJ.

DANTZIG, G.B., and WOLFE, P. 1960. Decomposition principle for linear programs. Oper. Res. **8**: 101–111.

EDMUNDSON, H.P. 1957. Bounds on the expectation of a convex function of a random variable. The Rand Corporation, Santa Monica, CA.

GASSMANN, H.I. 1987. Multi-period stochastic programming. Ph.D. thesis, Faculty of Commerce and Business Administration, University of British Columbia, Vancouver.

GASSMANN, H., and ZIEMBA, W.T. 1986. A tight upper bound for the expectation of a convex function of a multivariate random variable. Math. Program. Stud. **27**: 39–53.

GUNN, E.A. 1986. A model of long-term forest management and a decomposition approach to its solution. Working paper, Department of Industrial Engineering, Technical University of Nova Scotia, Halifax.

JOHNSON, K.N., and SCHEURMAN, H.L. 1977. Techniques for prescribing optimal timber harvest and investment under different objectives: discussion and synthesis. For. Sci. Monogr. No. 18.

LEMBERSKY, M.R., and JOHNSON, K.N. 1975. Optimal policies for managed stands: an infinite horizon Markov decision approach. For. Sci. **21**: 109–122.

MADANSKY, A. 1959. Bounds on the expectation of a convex function of a multivariate random variable. Ann. Math. Stat. **30**: 743–746.

______ 1960. Inequalities for stochastic linear programming problems. Manage. Sci. **6**: 197–204.

MARTELL, D.L. 1980. The optimal rotation of a flammable forest stand. Can. J. For. Res. **10**: 30–34.

REED, W.J., and ERRICO, D. 1986*a*. Optimal harvest scheduling at the forest level in the presence of the risk of fire. Can. J. For. Res. **16**: 266–278.

______ 1986*b*. Harvest scheduling in the risk of fire—further results. Working paper, Department of Mathematics, University of Victoria, Victoria, B.C.

______ 1986*c*. A new look at the model II harvest scheduling problem. Working paper, Department of Mathematics, University of Victoria, Victoria, B.C.

VAN WAGNER, C.E. 1978. Age-class distribution and the forest fire cycle. Can. J. For. Res. **8**: 220–227.

______ 1983. Simulating the effect of forest fire on long-term annual timber supply. Can. J. For. Res. **13**: 451–457.