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On dynamic sliding mode control of nonlinear fractional-order systems using sliding observer

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Abstract In this study, a new fractional-order dynamic sliding mode control (FDSMC) for a class of nonlinear systems is presented. In FDSMC, an integrator is placed before the input control signal of the plant, in order to remove the chattering. However, in FDSMC method, the dimension of the resulted system (integrator plus the system) is bigger than the primary system. As a result, the model of the plant is needed to be known completely, in order to stabilize the system. Then, a sliding observer is proposed to extract an appropriate model for the unknown nonlinear system. Then, the chattering free controller can be obtained such that the closed-loop system has the desired properties. Lyapunov theory is used to verify the stability problem of the presented both observer and controller. For practical applications consideration, we have not applied the upper bound of the system dynamic either in controller or in observer. The effectiveness of the proposed method in comparison to the conventional fractional sliding mode control (FSMC) method is addressed. We have utilized a same observer in both control approach, in order to have a valid comparison. The simplicity of the proposed FDSMC method in concept and also in realization can be seen with comparison of the relevant equations. In addition, it is clear that the FDSMC can remove chattering completely, while chattering is the main challenge of the conventional FSMC. Finally, the validity of the proposed method is shown by some simulation examples based on the Arneodo chaotic system.

Keywords Fractional-order dynamic sliding mode control (FDSMC) · Fractional-order sliding mode control (FSMC) · Nonlinear system · Sliding observer · Arneodo chaotic system

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1 Introduction

In sliding mode control (SMC), the dynamics of the system would be altered by application of a discontinuous control signal, which force the state variables of the system from the initial values to the predefined sliding surface and to retain the state variables on it [1]. Due to the invariant characteristic of this manifold or sliding surface, SMC is a powerful controller for real systems [2–4] to cope with uncertainties and disturbances. Notice that invariant is more stronger than robustness [4,5]. Unfortunately, chattering is the main obstacle in implementation of the SMC. In the literature, there are four methods to eliminate or reduce chattering as: boundary Layer (BL) [6,7], adaptive boundary layer (ABL) [8,9], higher order sliding mode control (HOSMC) [10,11], and dynamic sliding mode control (DSMC) [12,13]. Although BL and ABL methods can suppress or reduce the control signal switching or chattering, but the invariance property of the SMC would not preserve with these methods. In HOSMC method, chattering can be removed by transferring the switching to the derivative of the higher-order sliding surface. Up to now, various methods have been proposed for SMC in second order or higher order [14,15]. For example, in [16], a new HOSMC method is designed for a class of unknown nonlinear minimum phase single input single output systems (SISO), whereas, in [17], a HOSMC is presented to control a typical class of multi-input multi-output (MIMO) chaotic systems. For example, in [18], a new HOSMC is utilized to control a DC-DC converter. In [19], a second-order sliding mode differentiator is proposed, which utilizes super-twisting algorithm. A modified super-twisting SMC method is presented in [20], which uses adaptive gain schedule and inner feedback. The important weakness of all these HOSMC methods is the system model derivatives has to be known. For instance, in second-order case, we have to estimate the system model differentiation.

In DSMC, an integral block as the low-pass filter assigns to the before of the input control signal of the system, to remove high frequencies oscillations and chattering due to the sign function [4,14,15,21–23]. However, in DSMC, the augmented system has one dimension more than the actual system, where the augmented system includes system and integrator. Hence, the plant model have to be known completely when one needs to use SMC to control the augmented system. Therefore, in dynamical methods, system model has to be determined but, in higher-order methods, we have to determine the system model differentiation. Therefore, DSMC methods are more interested than HOSMC [21–23].

Fractional-order systems have gained considerable attention in the last few decades due to their intensive applications in precise describing of chaotic systems. Fractional-order systems have important role in various scientific fields as physics, mathematics, engineering and so on [24–26]. Although, in order to control the integer-order systems, fractional calculus along with SMC method has been previously used by researchers [27]. But, controlling the fractional-order systems with SMC method is investigated more recently [28,29]. For example, a fractional adaptation law for SMC method is presented in [27] or in [30], an integral SMC scheme is introduced for tracking the behavior of fractional-order nonlinear chaotic systems. A second-order fractional SMC (FSMC) approach in [31] is designed, which uses proportional integral (PI) controller. A 1st and 2nd order SMC is designed in [32], to address the fractional-order nonlinear systems via diffusive representation. In [33], a new dynamic output feedback SMC method is given to control an unknown fractional-order chaotic system. A FSMC approach is introduced in [34], which is constructed based on the linear-quadratic regulator (LQR) technique. The main goal of this study, is to present a new fractional order based DSMC (FDSMC) method. Then, the plant model should be identified using observer.

System identification is an important subject for determining the dynamical model of an unknown system and for controlling the state variables of the chaotic system [35,36]. In the past few years, many investigations are performed to solve it. The main objective of observer design is to utilize the known system inputs and their corresponding outputs to gain or estimate unmeasurable system states. The output of the observer and the main system is used as a feedback for observer to reform it [37,38]. Knowing the upper bound of uncertainty and system model are the main drawback of these observers, which are not feasible in practical cases. Dynamical state observer has been extensively utilized by the researchers to estimate the unmeasurable states of a system. Easily implementation and robustness in uncertainty conditions are the main advantages of sliding observers [39]. For example, in [40], a sliding observer has been utilized in order to evaluate the uncertainty of the dynamical systems. This observer can estimate the system state variables in special cases, where design of a controller is impossible [41]. For example, in induction motor, the torque on the rotor cannot be measured, while it can be estimated using this observer [42], or in [2] a similar observer is used for actuator fault detection. It is worth to mention that the main drawback of this observer, is their need to know the system model upper bound and/or uncertainty of the system.

Motivated by the above discussion, the goal of this study, is to develop a DSMC method to control the nonlinear fractional-order chaotic systems. To this end, a fractional-order sliding observer has been presented, which can be utilized in estimating the model and dynamics of the system. The Lyapunov stability theory has been utilized to prove the stability of the introduced observer. Hence, chattering removes in the proposed method. Furthermore, there is no need to have any knowledge about the dynamical system upper bound or system model, which is promising for practical applications.

The reminder of this study is presented as follows: Some mathematical modeling are provided in Sect. 2. Then, in Sect. 3, the problem is discussed in details. After that, a new sliding observer is proposed in Sect. 4, in order to estimate the system model. The structure of the introduced controller is presented in Sect. 5. Some simulations related to the proposed FDSMC and the conventional FSMC are provided in Sect. 6. Finally, conclusions of the paper are drawn in Sect. 7.

2 Preliminaries

In the following section, some definitions related to fractional-order derivative is given. Other definitions and mathematical modeling rules are presented as well.

2.1 Fractional calculus

The classical definitions of integer-order differentiation and integration can be extended for real-order ones. The fractional-order differentiation and integration respect to time domain are defined based on a convolution operation, which can be denoted by D^q, as

$$D^q f(t) = \begin{cases} \int_a^t f(\tau)(\mathrm{d}\tau)^{-q}, & q < 0 \\ 1, & q = 0 \\ \frac{\mathrm{d}^q}{\mathrm{d}t^q} f(t), & q > 0 \end{cases} \tag{1}$$

such that $q \in R$ indicate the fractional order of the system.

In the literature, researchers have proposed many methods to compute the fractional-order integration and differentiation of a typical function. Among the proposed methods, Riemann–Liouville method, Grunwald–Letnikov method and Caputo method are more common used methods. Among these methods [25,43], Caputo method is more applicable to real word problems, due to same form of initial conditions for fractional differential equations using Caputo operator with integer-order systems, which have well understood physical meanings. Hence, the Caputo differentiation and integration definitions are adopted in this study.

Definition 1 [24], The Caputo definition for the fractional-order integration of order q for a continuous function $f(t)$ is as

$$I^q f(t) = \frac{1}{\Gamma(q)} \int_{t_0}^t (t-\tau)^{q-1} f(\tau)\mathrm{d}\tau \tag{2}$$

where $t_0 < t$, $q > 0$ and $\Gamma(q)$ represents the Gamma function where $\Gamma(q) = \int_0^\infty t^{q-1} \mathrm{e}^{-t} \mathrm{d}t$.

In addition, the Caputo definition for calculation of the fractional-order differentiation can be defined as

$$D^q f(t) = \frac{1}{\Gamma(n-q)} \int_{t_0}^t \frac{f^{(n)}(\tau)}{(t-\tau)^{1+q-n}} \mathrm{d}\tau \tag{3}$$

where $t > t_0$, $q > 0$ denotes the derivative order for the function $f(t) \in C^n([t_0, \infty), R)$, and n denotes a positive integer number where: $n-1 \leq q < n$. In addition, when $0 < q < 1$, one can get,

$$D^q f(t) = \frac{1}{\Gamma(1-q)} \int_{t_0}^t \frac{f'(\tau)}{(t-\tau)^q} \mathrm{d}\tau. \tag{4}$$

2.2 Mathematical modeling

Lemma 1 [44] *Let $x(t) = [x_1(t), x_2(t), \ldots, x_n(t)]^{\mathrm{T}} \in \boldsymbol{R}^n$, and $q \in (0, 1]$. Then, for any positive time instant t, we have,*

$$\frac{1}{2} D^q \left(x^{\mathrm{T}}(t)x(t)\right) \leq x^{\mathrm{T}}(t) D^q x(t). \tag{5}$$

Based on this lemma, one can conclude the following property of the Caputo fractional derivative as follows.

Theorem 1 *Let $P \in \mathbf{R}^{n\times n}$ is defined as a symmetric positive definite matrix, and $x = [x_1, x_2, \ldots, x_n]^{\mathrm{T}} \in \boldsymbol{R}^n$ is a vector of derivable functions and $q \in (0, 1]$ stands as fractional order. Then, the following inequality can be concluded:*

$$\frac{1}{2} D^q \left(x^{\mathrm{T}} P x\right) \leq x^{\mathrm{T}} P D^q x. \tag{6}$$

Proof When $q = 1$, the proof is straightforward. Now, consider $q \in (0, 1)$. Since $P \in \mathbf{R}^{n\times n}$ is a symmetric positive definite matrix. We can parse it to the $P = Q^{\mathrm{T}} Q$, where $Q \in R^{n\times n}$ would be a positive definite matrix [45]. Then, one can get

$$\frac{1}{2}D^q\left(x^{\mathrm{T}}Px\right)=\frac{1}{2}D^q\left(x^{\mathrm{T}}Q^{\mathrm{T}}Qx\right). \tag{7}$$

Let $y(t)=Qx^{\mathrm{T}}(t)$. Then, Eq. (7) would be simplified by using Lemma. 1 as

$$\begin{aligned}\frac{1}{2}D^q\left(x^{\mathrm{T}}Px\right)&=\frac{1}{2}D^q\left(y^{\mathrm{T}}y\right)\\&\leq y^{\mathrm{T}}D^q y\\&=x^{\mathrm{T}}Q^{\mathrm{T}}D^q\left(Qx\right)\\&=x^{\mathrm{T}}Q^{\mathrm{T}}QD^q x\\&=x^{\mathrm{T}}PD^q x\end{aligned} \tag{8}$$

So, the proof is completed. □

Lemma 2 *Considering the assumptions taken in Theorem 1, the following inequality can be resulted:*

$$D^q\left(x^{\mathrm{T}}Px\right)\leq\left(D^q x\right)^{\mathrm{T}}Px+\boldsymbol{x}^{\mathrm{T}}P\ D^q x \tag{9}$$

Proof Considering the inequality (8), the proof is straightforward. □

Theorem 2 [44,46] *Consider a Lyapunov candidate function $V(t)$ and class-K functions $\gamma_i \ : \ i=1,2,3$ such that:*

$$\gamma_1(\|x\|)\leq V(t)\leq\gamma_2(\|x\|) \tag{10}$$
$$D^q V(t)\leq-\gamma_3(\|x\|) \tag{11}$$

Then the fractional-order system $D^q x=f(x,t)$ is asymptotically stable, where $q\in(0,1)$ and $x=[x_1,x_2,\ldots,x_n]^{\mathrm{T}}$.

Lemma 3 [47] For any positive definite matrix $\mathbf{M}$, we have the following inequalities:

$$0<\lambda_{\min}(\boldsymbol{M})\|x\|^2\leq x^{\mathrm{T}}\boldsymbol{M}x\leq\lambda_{\max}(\boldsymbol{M})\|x\|^2 \tag{12}$$
$$-x^{\mathrm{T}}\boldsymbol{M}x\leq-\lambda_{\min}(\boldsymbol{M})\|x\|^2<0 \tag{13}$$

where $\lambda_{\min}$ and $\lambda_{\max}$ denote the minimum and the maximum eign values, respectively, and the functions $\lambda_{\min}(\boldsymbol{M})\|x\|^2$ and $\lambda_{\max}(\boldsymbol{M})\|x\|^2$ are the class-K functions.

3 Problem formulation

Assume a class of single input nonlinear fractional-order canonical form system as follows:

$$\begin{aligned}D^q x_j&=x_{j+1}\qquad j=1,2,\ldots,n-1\\D^q x_n&=f(x,t)+v\end{aligned} \tag{14}$$

where $x(t)=[x_1(t),x_2(t),...,x_n(t)]^{\mathrm{T}}$ is an accessible system states vector and r denote the input control signal of the system (14). Moreover, $f(x,t)$ represents an uncertain function, which can contain any coupled sentences with any orders. At first, the following state feedback is applied to the system (14)

$$v=\sum_{j=1}^{n}a_j x_j+u \tag{15}$$

such that u denotes the new input. Consequently, the nonlinear system (14) can be represented as follows:

$$D^q x=Ax+Bu+Bf(x,t) \tag{16}$$

and the matrix A and B are as follows:

$$A=\begin{bmatrix}0&1&0&\cdots&0\\0&0&1&\ddots&\vdots\\\vdots&\vdots&\ddots&\ddots&0\\0&0&\cdots&0&1\\a_1&a_2&\cdots&a_{n-1}&a_n\end{bmatrix},\quad B=\begin{bmatrix}0\\0\\\vdots\\0\\1\end{bmatrix} \tag{17}$$

whereas a_i are desired constants. The goal is using of the dynamic sliding mode control to calculate an appropriate chatteringless control u such that the state variables $x(t)=[x_1(t),x_2(t),...,x_n(t)]^{\mathrm{T}}$ and $x_{n+1}(t)=D^q x_n(t)$ converge to zero. To achieve this goal, the integral sliding surface can be defined as

$$\begin{aligned}s&=\lambda x+\lambda_{n+1}x_{n+1}\\\lambda&=[\lambda_1,\lambda_2,\lambda_3,\ldots,\lambda_n]\end{aligned} \tag{18}$$

The sliding surface coefficients have to be selected in order to the zero dynamics of $s=0$, would be stable, meaning that if $s=0$ results in the system states converge to zero. The main challenge is that x_{n+1} is not available to calculate the sliding surface (18), due to uncertainty in function f, as one can see from Eq. (14). To address this problem, a sliding observer has to be derived. Considering the designed sliding observer

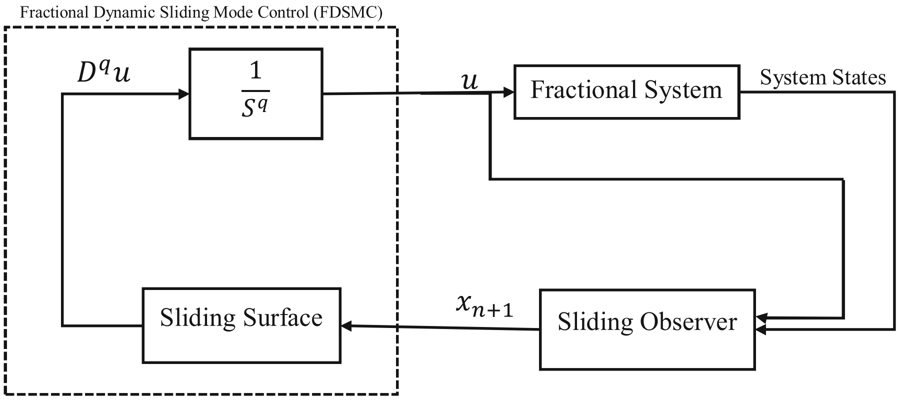

Fig. 1 Diagram of the proposed FDSMC approach

and the sliding surface, the feedback controller can be obtained dynamically, which control law can then be resulted using fractional-order integrator. The diagram of the introduced method is provided in Fig. 1. To this end, the sliding observer and the sliding mode controller are proposed in the following sections.

4 Estimation of system model using sliding observer

Let us assume the function $f(x, t)$ would be bounded as follows

$$|f(x,t)| \leq \rho \tag{19}$$

Now, the dynamical system (15) can be rewritten as

$$\begin{aligned} D^q y_1 &= A_1 y_1 + A_{12} y_2 \\ D^q y_2 &= A_{21} y_1 + A_{22} y_2 + (u + f) \end{aligned} \tag{20}$$

where,

$$A_1 = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}, \quad A_{12} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix},$$

$$A_{21}^{\mathrm{T}} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{n-2} \\ a_{n-1} \end{bmatrix}, \quad A_{22} = [a_n] \tag{21}$$

and $y_1(t) = [x_1(t), x_2(t), ..., x_{n-1}(t)]^{\mathrm{T}}$ and $y_2(t) = x_n(t)$ denote the state variables of the system. For system (20), the following observer is proposed:

$$\begin{aligned} D^q \hat{y}_1 &= A_1 \hat{y}_1 + A_{12} \hat{y}_2 + A_{12} e_2 - P e_1 \\ D^q \hat{y}_2 &= A_{21} \hat{y}_1 + A_{22} \hat{y}_2 + u + v \end{aligned} \tag{22}$$

Such that $e_1 = y_1 - \hat{y}_1$ and $e_2 = y_2 - \hat{y}_2$ are the error estimation of the first and second states, respectively. Then, the dynamics of system errors can be described as

$$\begin{aligned} D^q e_1 &= A_{11} e_1 \\ D^q e_2 &= A_{21} e_1 + A_{22} e_2 + f - v \end{aligned} \tag{23}$$

where $A_{11} = A_1 + P$ and matrix P, which has the same dimension as matrix A_1, has to be defined in such away that the matrix A_{11} be Hurwitz, i.e., $A_{11} < 0$. In other words, there exists a unique symmetric positive definite matrix $P_1 \in \mathbf{R}^{(n-1)\times(n-1)}$ for any symmetric positive definite matrix Q, which satisfies the following Lyapunov equation

$$P_1 A_{11} + A_{11}^{\mathrm{T}} P_1 = -Q. \tag{24}$$

Theorem 3 *The error estimation of the sliding observer would converge to zero, if the discontinuous input v in the observer system (22) is selected as*

$$v = \rho\, sign(e_2) = \begin{cases} -\rho & : e_2 < 0 \\ 0 & : e_2 = 0 \\ +\rho & : e_2 > 0 \end{cases} \tag{25}$$

where ρ is defined in Eq. (19), as the upper bound of $f(x, t)$.

Proof Assume the Lyapunov candidate function as

$$V = e_1^{\mathrm{T}} P_1 e_1 + e_2^2 \tag{26}$$

Then, based on Lemma. 1, the fractional derivative of this Lyapunov function would be as

$$\begin{aligned}
D^q V \leq\; & (D^q e_1)^{\mathrm{T}} P_1 e_1 + e_1^{\mathrm{T}} P_1 (D^q e_1) + 2e_2 D^q e_2 \\
&= e_1^{\mathrm{T}} A_{11}^{\mathrm{T}} P_1 e_1 + e_1^{\mathrm{T}} P_1 A_{11} e_1 + 2e_2(A_{21}e_1 \\
&\quad + A_{22}e_2 + f - v) \\
&= e_1^{\mathrm{T}} \left(A_{11}^{\mathrm{T}} P_1 + P_1 A_{11} \right) e_1 + 2e_2 A_{21} e_1 \\
&\quad + 2A_{22}e_2^2 + 2e_2 f - 2e_2 v \\
&= -e_1^{\mathrm{T}} Q e_1 + 2e_2 A_{21} e_1 + 2A_{22} e_2^2 \\
&\quad + 2e_2 f - 2e_2 \rho \ \mathrm{sign}(e_2) \\
&\leq -e_1^{\mathrm{T}} Q e_1 + 2e_2 A_{21} e_1 + 2A_{22} e_2^2 \\
&\quad + 2|e_2||f| - 2\rho |e_2| \\
&\leq -e_1^{\mathrm{T}} Q e_1 + 2e_2 A_{21} e_1 + 2A_{22} e_2^2
\end{aligned} \tag{27}$$

Let's consider $Q = A_{21}^{\mathrm{T}} Q_2^{-1} A_{21} + Q_1$, then Eq. 27 can be simplified as follows:

$$\begin{aligned}
D^q V \leq & - e_1^{\mathrm{T}} Q_1 e_1 - e_1^{\mathrm{T}} A_{21}^{\mathrm{T}} Q_2^{-1} A_{21} e_1 \\
& + 2e_2 A_{21} e_1 + 2A_{22} e_2^2
\end{aligned} \tag{28}$$

whenever Q_1 is a symmetric positive definite matrix. On the other hand, we have

$$\begin{aligned}
&(e_2 - Q_2^{-1} A_{21} e_1)^{\mathrm{T}} Q_2 (e_2 - Q_2^{-1} A_{21} e_1) \\
&\quad = e_2^{\mathrm{T}} Q_2 e_2 - e_1^{\mathrm{T}} A_{21}^{\mathrm{T}} e_2 - e_2^{\mathrm{T}} A_{21} e_1 \\
&\quad\ + e_1^{\mathrm{T}} A_{21}^{\mathrm{T}} Q_2^{-1} A_{21} e_1 \\
&\quad = e_2^{\mathrm{T}} Q_2 e_2 - 2e_2 A_{21} e_1 + e_1^{\mathrm{T}} A_{21}^{\mathrm{T}} Q_2^{-1} A_{21} e_1
\end{aligned} \tag{29}$$

Thus, we obtain

$$\begin{aligned}
D^q V \leq & - e_1^{\mathrm{T}} Q_1 e_1 - \tilde{e}^{\mathrm{T}} Q_2 \tilde{e} + Q_2 e_2^2 - 2e_2 A_{21} e_1 \\
& + 2e_2 A_{21} e_1 + 2A_{22} e_2^2 \\
= & - e_1^{\mathrm{T}} Q_1 e_1 - \tilde{e}^{\mathrm{T}} Q_2 \tilde{e} + Q_2 e_2^2 + 2A_{22} e_2^2
\end{aligned} \tag{30}$$

where $\tilde{e} = (e_2 - Q_2^{-1} A_{21} e_1)$. Then, choosing $Q_2 = -2A_{22} > 0$, concludes

$$D^q V \leq -e_1^{\mathrm{T}} Q_1 e_1 - \tilde{e}^{\mathrm{T}} Q_2 \tilde{e} \tag{31}$$

Furthermore, according to Lemma 3, we have the following inequalities:

$$\begin{aligned}
& -\left(\lambda_{\max}(P_1) \|e_1\|^2 + \|e_2\|^2 \right) \leq V \\
& \quad \leq + \left(\lambda_{\max}(P_1) \|e_1\|^2 + \|e_2\|^2 \right)
\end{aligned} \tag{32}$$

and,

$$D^q V \leq -\lambda_{\max}(Q_1) \|e_1\|^2 - \lambda_{\max}(Q_2) \|\tilde{e}\|^2 \tag{33}$$

Thus, according to Theorem 2, the theorem is proved. It means that e_1 and e_2 approach to zero. □

5 The proposed controller

A new dynamic sliding mode controller is introduced in this section. Then, the performance evaluation of the introduced controller is evaluated by comparison with the conventional FSMC method. The presented observer in (22) is utilized in both controllers to have a comprehensive comparison result. Finally, the effectiveness of the proposed method is shown theoretically and by some numerical results.

5.1 Conventional FSMC

Let us consider the sliding surface as follows:

$$\begin{aligned}
& s = \bar{\lambda} \hat{y}_1 + \lambda_n \hat{y}_2 \\
& \bar{\lambda} = [\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_{n-1}]
\end{aligned} \tag{34}$$

Then, using the presented observer in (22), we have

$$\begin{aligned}
D^q s = & \bar{\lambda} D^q \hat{y}_1 + \lambda_n D^q \hat{y}_2 \\
= & \bar{\lambda} \left(A_1 \hat{y}_1 + A_{12} \hat{y}_2 + A_{12} e_2 - P e_1 \right) \\
& + \lambda_n \left(A_{21} \hat{y}_1 + A_{22} \hat{y}_2 + u + v \right)
\end{aligned} \tag{35}$$

Theorem 4 *The motion trajectories of the system states would approach to zero as long as the sliding surface are zero. Actually*

$$\begin{aligned}
u = & - \frac{\bar{\lambda} \left(A_1 \hat{y}_1 + A_{12} \hat{y}_2 + A_{12} e_2 - P e_1 \right)}{\lambda_n} \\
& - \left(A_{21} \hat{y}_1 + A_{22} \hat{y}_2 + v \right) \\
& - \frac{k_1 sign(s) + k_2 s}{\lambda_n}
\end{aligned} \tag{36}$$

where $k_1 > 0$ and $k_2 > 0$ are constant coefficients.

Proof Assume the Lyapunov candidate function as: $V = \frac{1}{2}s^2$. Then, the derivative of this function would be as $D^q V \le sD^q s$. Hence, substituting Eq. (36) into (35), we have

$$D^q s = -k_1 \text{sign}(s) - k_2 s \tag{37}$$

Therefore, the Lyapunov candidate function can be dynamically simplified as:

$$D^q V \le -k_1|s| - k_2 s^2 \le -k_1|s| \tag{38}$$

Suppose, t_f would be the finite-time reaching to the sliding surface, i.e., $s(t_f) = 0$, then, one can easily show that: $t_f \le |s(0)|/k_1$ which means also reaching to sliding surface will be happened in finite time. □

5.2 The proposed FDSMC

According to the previous illustrations and Eqs. (18) and (22), the sliding surface can be determined as follows:

$$\begin{aligned} s &= \bar{\lambda}\hat{y}_1 + \lambda_n \hat{y}_2 + \lambda_{n+1} D^q \hat{y}_2 \\ \bar{\lambda} &= [\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_{n-1}] \end{aligned} \tag{39}$$

Then, time derivative of this surface can be calculated based on the observer Eq. (22) as

$$\begin{aligned} D^q s &= \bar{\lambda} D^q \hat{y}_1 + \lambda_n D^q \hat{y}_2 + \lambda_{n+1} D^{2q} \hat{y}_2 \\ &= \bar{\lambda}\left(A_1\hat{y}_1 + A_{12}\hat{y}_2 + A_{12}e_2 - Pe_1\right) \\ &\quad + \lambda_n\left(A_{21}\hat{y}_1 + A_{22}\hat{y}_2 + u + v\right) \\ &\quad + \lambda_{n+1}\left(A_{21}D^q\hat{y}_1 + A_{22}D^q\hat{y}_2 + D^q u + D^q v\right) \\ &= \bar{\lambda}\left(A_1\hat{y}_1 + A_{12}\hat{y}_2 + A_{12}e_2 - Pe_1\right) \\ &\quad + \lambda_n\left(A_{21}\hat{y}_1 + A_{22}\hat{y}_2 + u + u\right) \\ &\quad + \lambda_{n+1}A_{21}\left(A_1\hat{y}_1 + A_{12}\hat{y}_2 + A_{12}e_2 - Pe_1\right) \\ &\quad + \lambda_{n+1}A_{22}\left(A_{21}\hat{y}_1 + A_{22}\hat{y}_2 + u + v\right) \\ &\quad + \left(\lambda_{n+1}D^q u + \lambda_{n+1}D^q v\right) \\ &= \left(\bar{\lambda} + \lambda_{n+1}A_{21}\right)\left(A_1\hat{y}_1 + A_{12}\hat{y}_2 + A_{12}e_2 - Pe_1\right) \\ &\quad + \left(\lambda_n + \lambda_{n+1}A_{22}\right)\left(A_{21}\hat{y}_1 + A_{22}\hat{y}_2 + u + v\right) \\ &\quad + \left(\lambda_{n+1}D^q u + \lambda_{n+1}D^q v\right) \end{aligned} \tag{40}$$

which, based on Eq. (25), the derivative of input v can be written as follows:

$$D^q v = 0 \tag{41}$$

Hence, the time derivative of surface model would be simplified as

$$\begin{aligned} D^q s = &\left(\bar{\lambda} + \lambda_{n+1}A_{21}\right)\left(A_1\hat{y}_1 + A_{12}\hat{y}_2 + A_{12}e_2 - Pe_1\right) \\ &+ \left(\lambda_n + \lambda_{n+1}A_{22}\right)\left(A_{21}\hat{y}_1 + A_{22}\hat{y}_2 + u + v\right) \\ &+ \left(\lambda_{n+1}D^q u\right) \end{aligned} \tag{42}$$

Theorem 5 *The system states approach to zero as long as sliding surface set equal to zero. To achieve this goal, the feedback controller u can be defined dynamically as*

$$\begin{aligned} D^q u = &-\frac{\left(\bar{\lambda} + \lambda_{n+1}A_{21}\right)\left(A_1\hat{y}_1 + A_{12}\hat{y}_2 + A_{12}e_2 - Pe_1\right)}{\lambda_{n+1}} \\ &- \frac{\left(\lambda_n + \lambda_{n+1}A_{22}\right)\left(A_{21}\hat{y}_1 + A_{22}\hat{y}_2 + u + v\right)}{\lambda_{n+1}} \\ &- \frac{k_1 sign(s) + k_2 s}{\lambda_{n+1}} \end{aligned} \tag{43}$$

which $k_1 > 0$ and $k_2 > 0$ are constant positive numbers.

Proof Let's consider the Lyapunov candidate function as: $V = \frac{1}{2}s^2$. Then, its time derivative would be as $D^q V \le sD^q s$. Substituting $D^q u$ from Eq. (43) into (42), we obtain:

$$D^q s = -k_1 \text{ sign}(s) - k_2 s \tag{44}$$

Hence,

$$D^q V \le -k_1|s| - k_2 s^2 \le -k_1|s| \tag{45}$$

Assume that, t_f would be the finite-time reaching to the sliding surface, i.e., $s(t_f) = 0$, then, one can easily show that: $t_f \le |s(0)|/k_1$. □

6 Numerical results

Some numerical results are presented in this section to validate the performances of the theoretical discussions related to the introduced FDSMC method. The fractional-order Arneodo system presented in [48] can be described as

$$\begin{aligned} D^q x_1 &= x_2 \\ D^q x_2 &= x_3 \end{aligned}$$

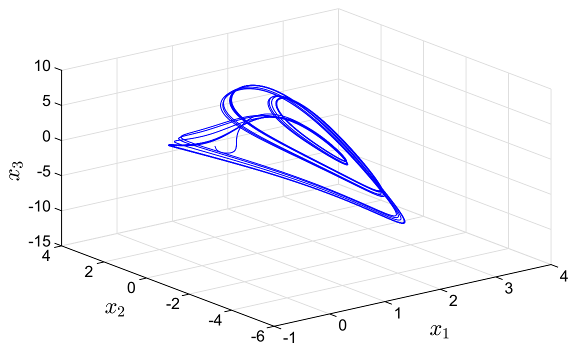

Fig. 2 Phase portrait of the Arneodo chaotic system for fractional order $q = 0.8$

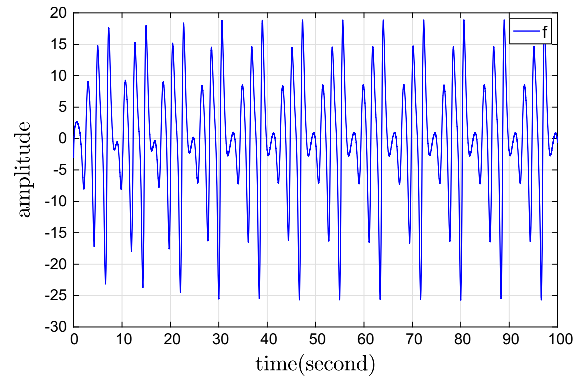

Fig. 3 Time series of function $f(x_1, x_2, x_3)$ for fractional order $q = 0.8$

$$D^q x_3 = f(x_1, x_2, x_3) + r$$
$$f(x_1, x_2, x_3) = -\mu_1 x_1 - \mu_2 x_2 - \mu_3 x_3 + \mu_4 x_1^3 + d \tag{46}$$

where x_i $i = 1, 2, 3$ stand for the state variables of the system and μ_1, μ_2, μ_3 and μ_4 denote the constant parameters of the system and d would be produced by uniformly probability distributed function (PDF) between zero and 0.2. The Arneodo system shows chaotic behavior if system parameters considered as $\mu_1 = -5.5, \mu_2 = 3.5, \mu_3 = -0.4$ and $\mu_4 = -1.0$ with fractional order $q = 0.8$. The motion trajectories of the chaotic system (46) are illustrated in Fig. 2, and the time series of function f is shown in Fig. 3, with the initial conditions $x_1(0) = 1, x_2(0) = 2$ and $x_3(0) = -2$. As these figures show, the state variables of the chaotic system (46) are bounded. Thus, f would be bounded. Considering Fig. 3, the upper bound of f can be selected as 20, i.e., $|f(x_1, x_2, x_3)| < \rho = 25$. All of the simulations are done by Matlab software with sampling time of 0.001 s.

Example 1 The proposed FDSMC design

The observer parameters are chosen as

$$P = \begin{bmatrix} -10 & 0 \\ 0 & -10 \end{bmatrix} \tag{47}$$

and,

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -5 & -6 & -7 \end{bmatrix} \tag{48}$$

and also the controller parameters are selected as:

$$\lambda = [1000, 300, 30, 1], \quad k_1 = 10, \quad k_2 = 5 \tag{49}$$

By the way, all of the initial conditions of the observer states are considered as zero and also $u(0) = 0$. The results obtained from FDSMC method in this example are depicted in Figs. 4, 5 and 6, with fractional order $q = 0.8$. As Figs. 4 and 5 show, the state variables of the system converge to zero. Moreover, system states are followed by the observer states. It is worth to mention that the controller is designed based on the observer states. As Fig. 5 shows, the tracking errors converge to zero. The q-order derivative of input control signal, obtained from Eq. (43), is depicted in Fig. 6a. The output of the q-order integrator of Fig. 6a, which is shown in Fig. 6b, is applied to the system. As Fig. 6b shows, the input control signal is smooth and has no chattering. Hence, the oscillations and the high amplitude of the q-order derivative of the input signal do not have considerable effect. In addition, the sliding surface is shown in Fig. 6c. It can be concluded that the sliding surface converges to zero in finite time, as well.

These simulations are repeated for fractional order $q = 0.9$. As we can see from the obtained results shown in Figs. 7, 8 and 9, the proposed method works well for different q values and that the results of $q = 0.8$ and $q = 0.9$ are similar.

Example 2 The proposed FSMC design

In order to present a valid comparison, the controller parameters are chosen as same as the previous example as follows:

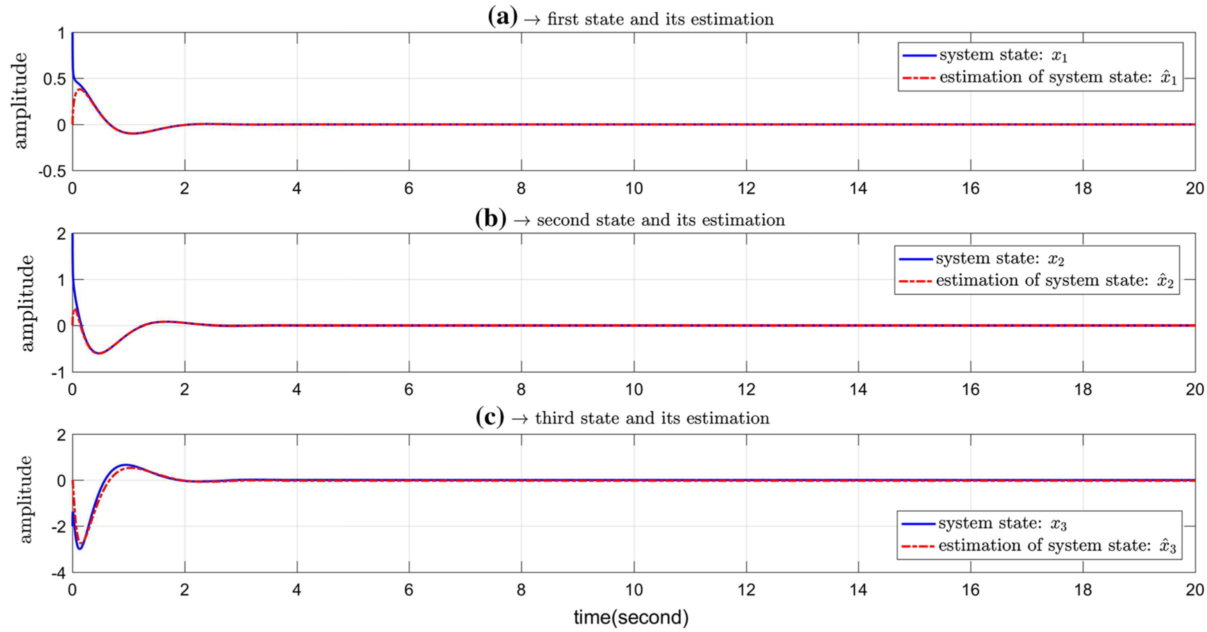

Fig. 4 Motion trajectories of the system state variables and their corresponding estimation obtained by FDSMC for fractional order $q = 0.8$

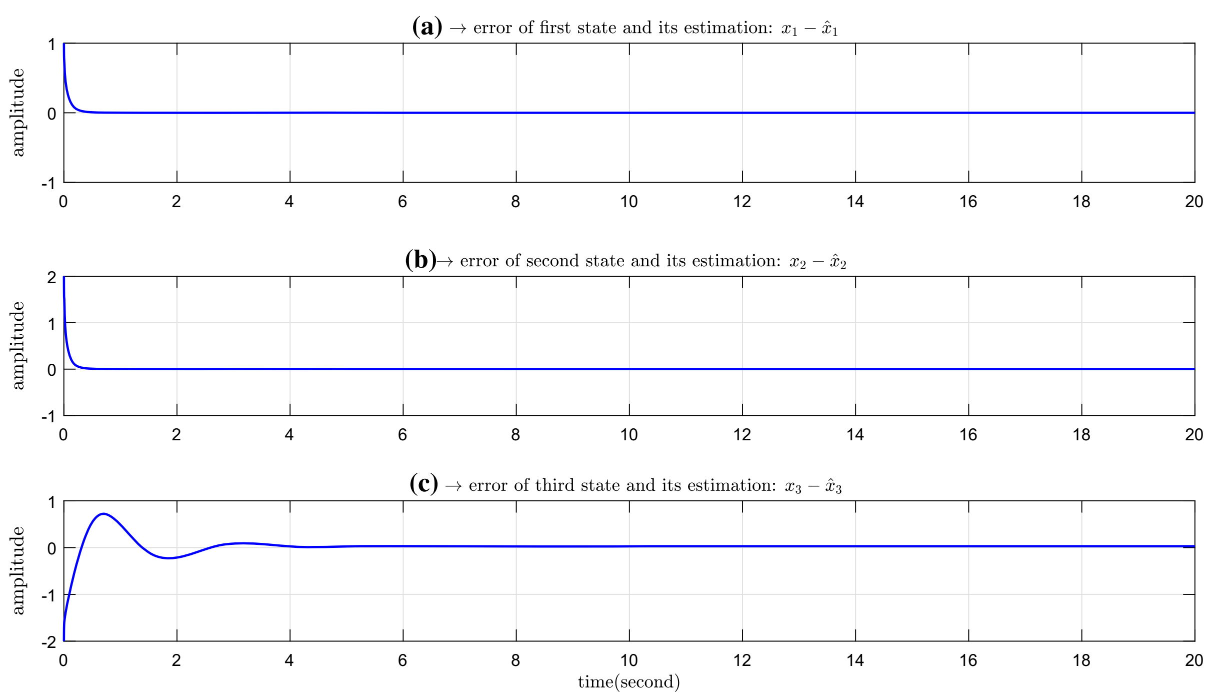

Fig. 5 Error between system state variables and their corresponding estimation obtained by FDSMC for fractional order $q = 0.8$

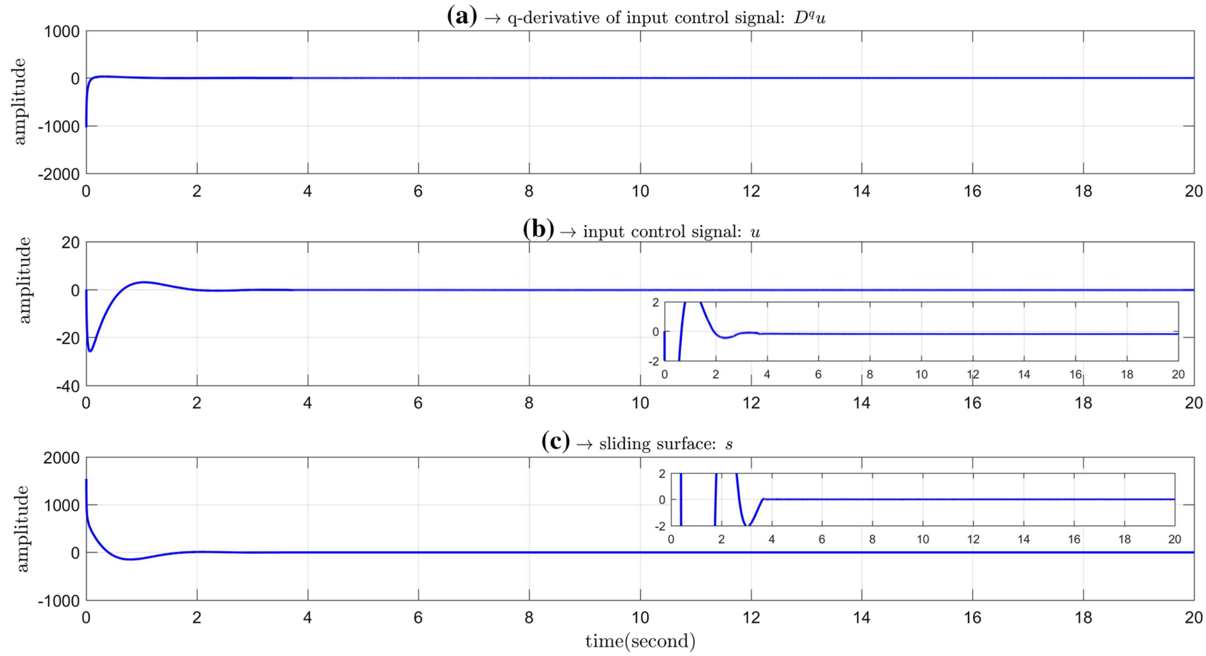

Fig. 6 Time series of **a** input control signal, **b** fractional q-order derivative of input signal of the system and **c** sliding surface, obtained by FDSMC for fractional order $q = 0.8$

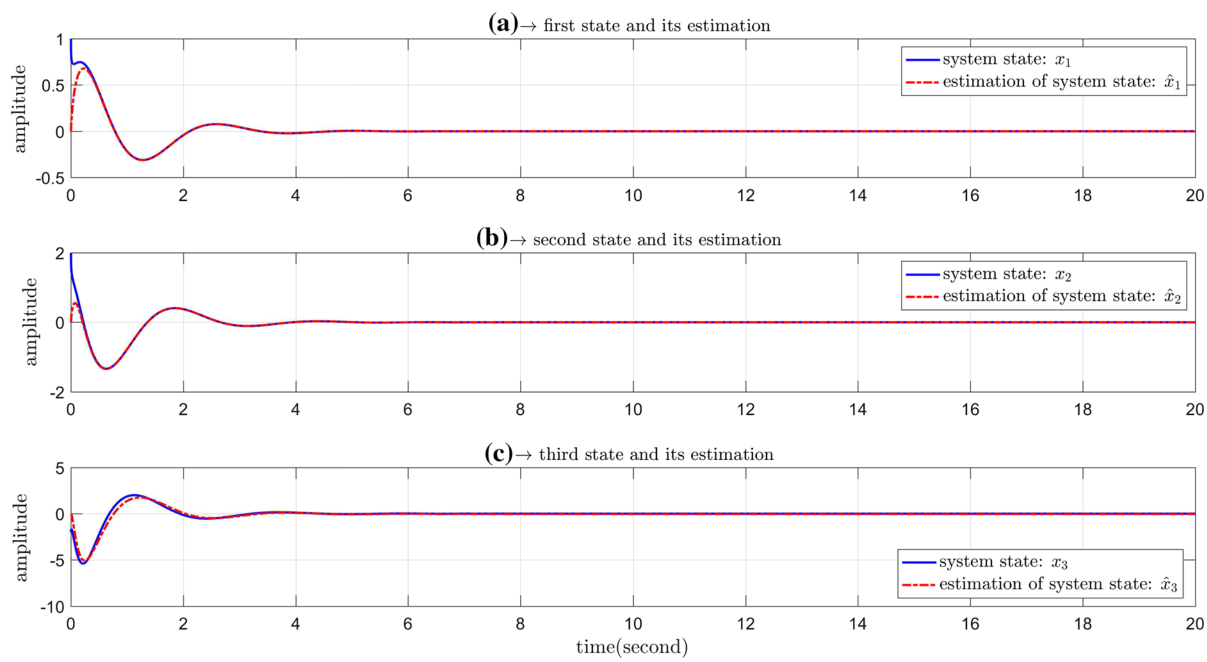

Fig. 7 Motion trajectories of the system state variables and their corresponding estimation obtained by FDSMC for fractional order $q = 0.9$

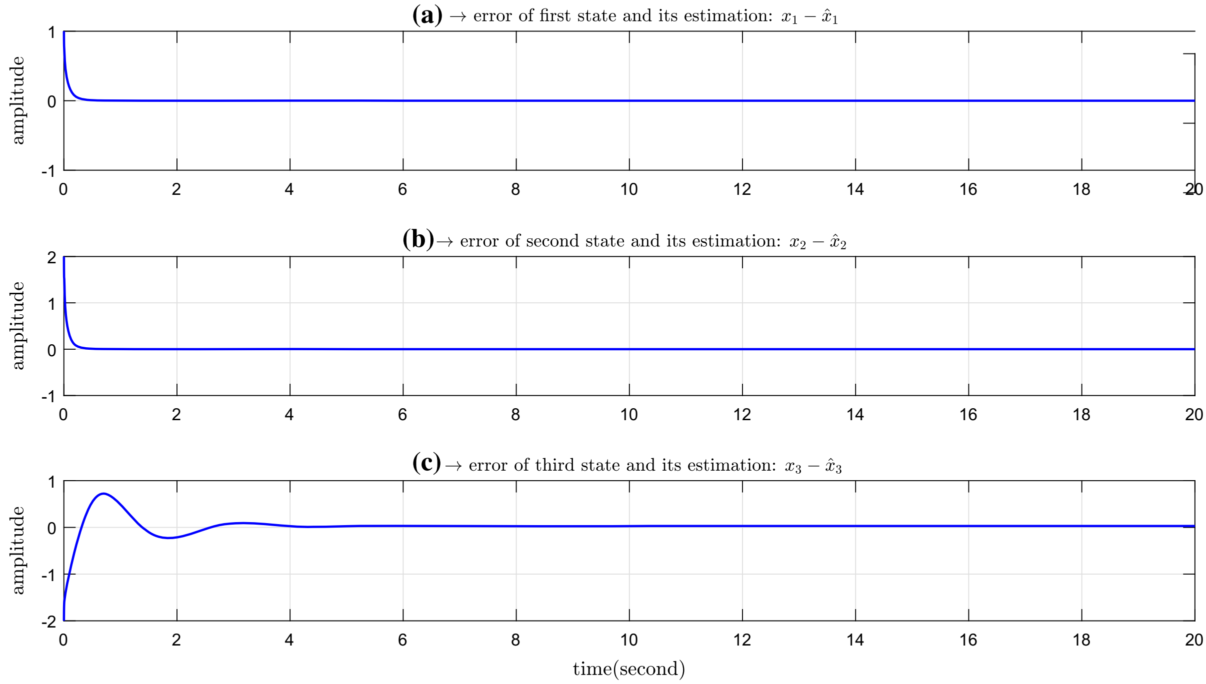

Fig. 8 Error between system state variables and their corresponding estimation obtained by FDSMC for fractional order $q = 0.9$

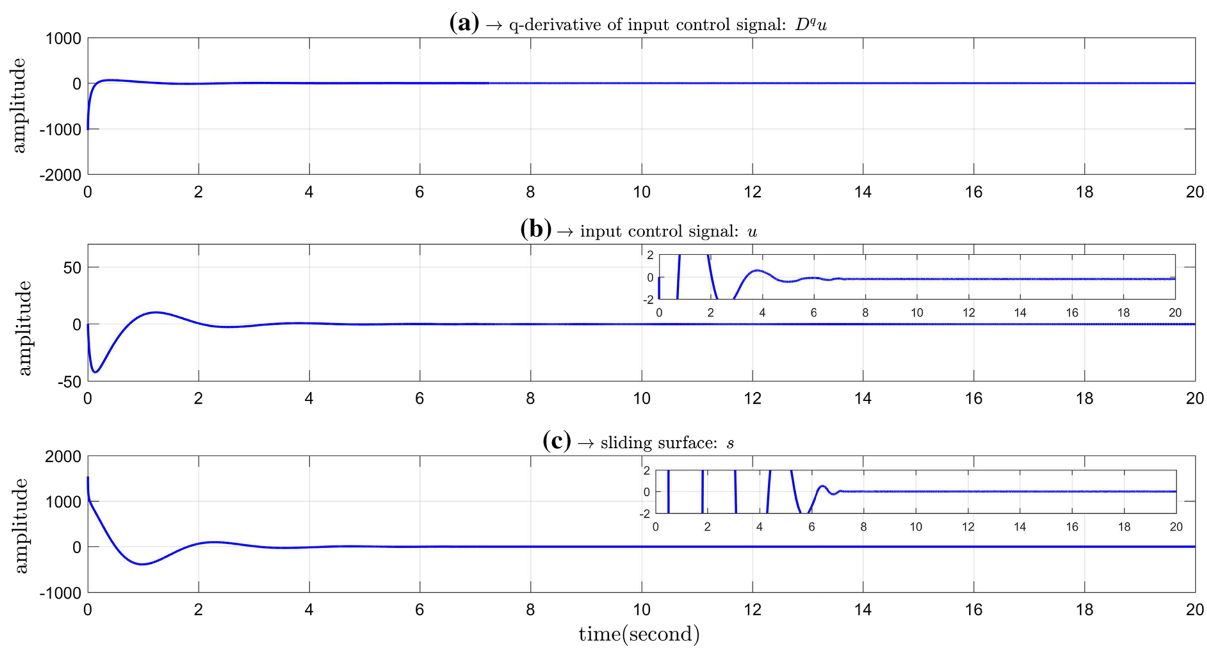

Fig. 9 Time series of **a** input control signal of the system, **b** fractional q-order derivative of input signal of the system and **c** sliding surface, obtained by FDSMC for fractional order $q = 0.9$

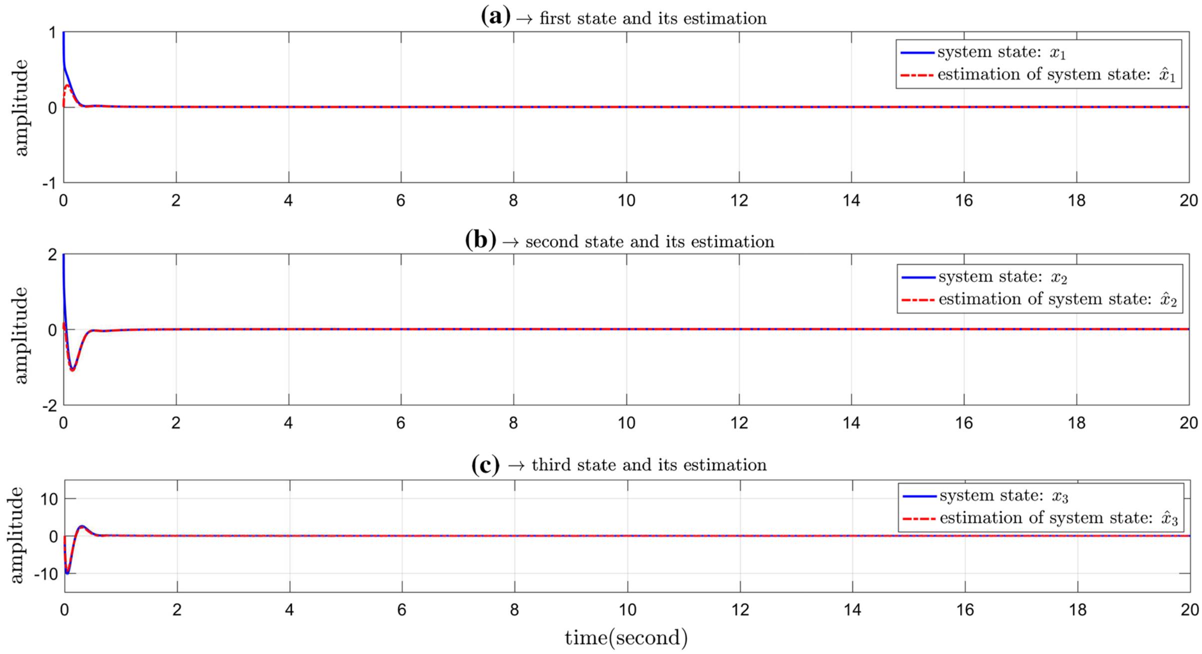

Fig. 10 Motion trajectories of the system state variables and their corresponding estimation obtained by FSMC for fractional order $q = 0.8$

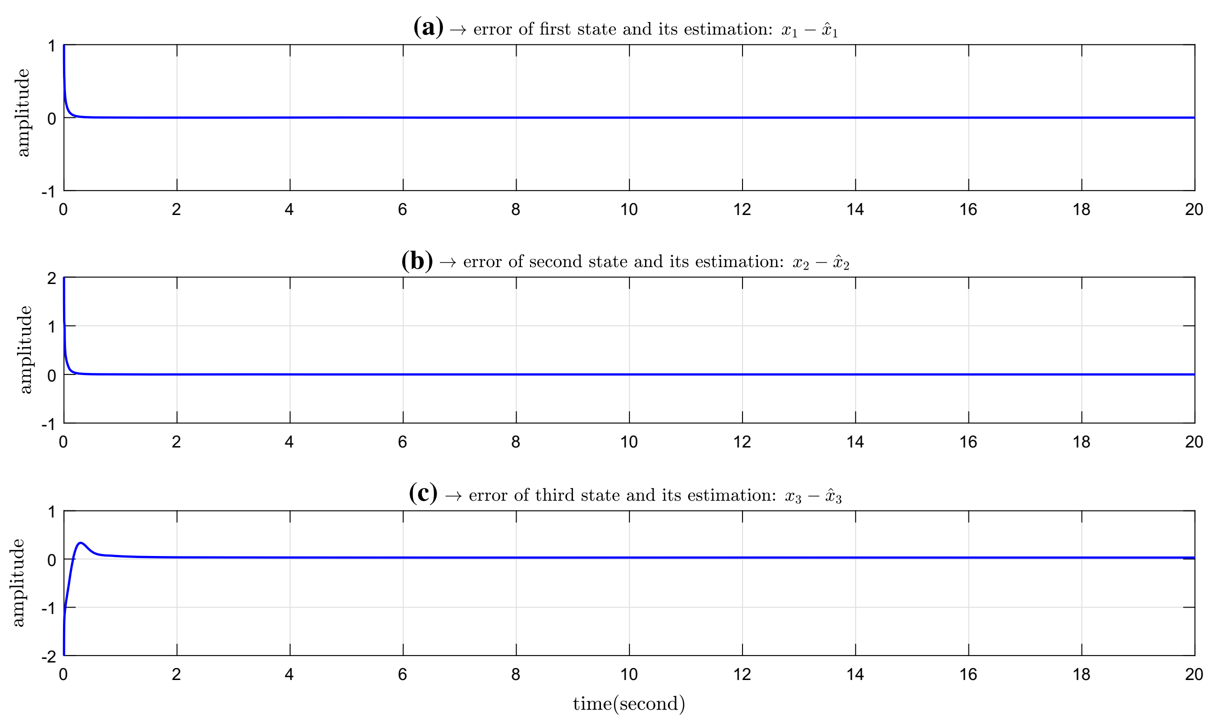

Fig. 11 Error between system state variables and their corresponding estimation obtained by FSMC for fractional order $q = 0.8$

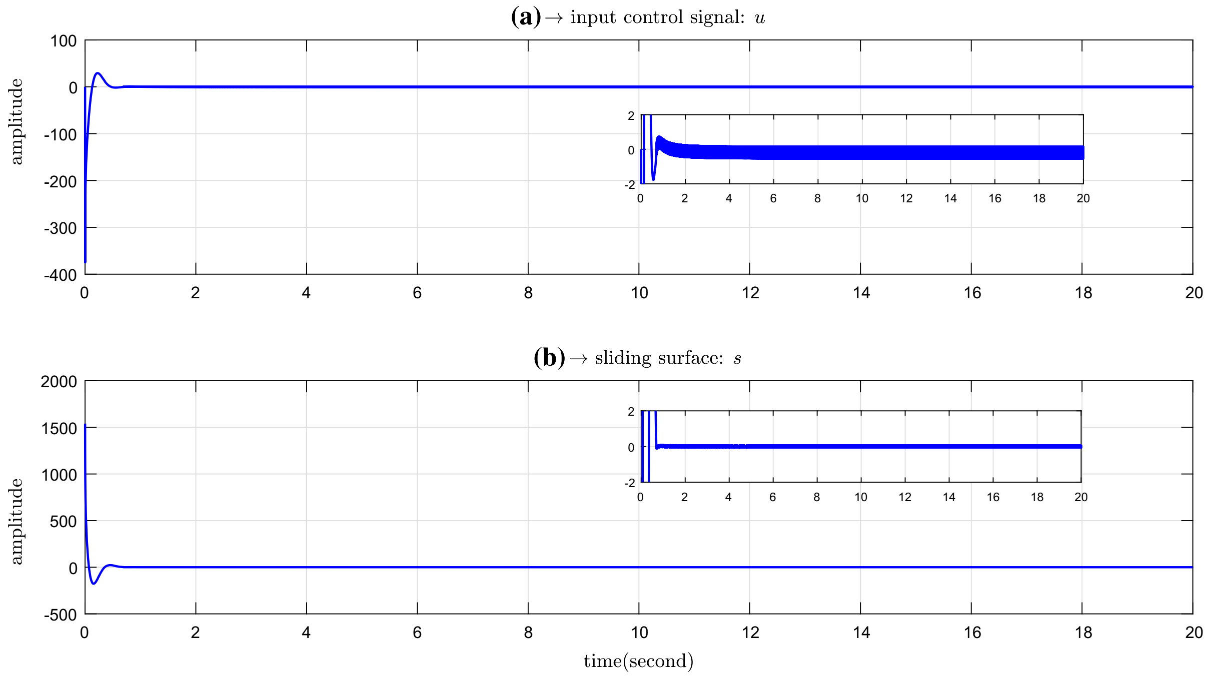

Fig. 12 Time series of **a** input control signal and **b** sliding surface, obtained by FSMC for fractional order $q = 0.8$

$$\lambda = [1000, 300, 30], \quad k_1 = 10, \quad k_2 = 5 \tag{50}$$

Moreover, the initial conditions of the observer states are set as zero as the previous example. The simulation results of this example are depicted in Figs. 10, 11 and 12. As Fig. 10 shows, the state variables of the system approach to zero and also the observer states track the system state variables. In addition, the system state variables errors approach to zero, which is shown in Fig. 11. Then, the input control signal is shown in Fig. 12a, where the chattering can be observed clearly. In addition, the sliding surface is shown in Fig. 12b, which converges to zero in finite time.

To have a valid comparison, the same observer is utilized either in FDSMC method and in FSMC method. Comparing the obtained results, the state variables of the investigated system approaches to zero in both control approaches. However, as can be seen from these results, the superior performance of the FDSMC method against the conventional FSMC can be observed in chattering elimination as it is shown in Figs. 6 and 12.

7 Conclusion

In this study, a novel method for the control of nonlinear fractional-order systems is introduced, which is constructed based on the dynamic sliding mode control (FDSMC) approach. The validity of the introduced method is proved by means of Lyapunov stability theory. Furthermore, some numerical simulations have presented to show the feasibility and efficiency of the introduced approach versus conventional sliding mode control (FSMC). A nonlinear observer is presented to solve the challenge of the FDSMC problem. The proposed approach produces substantially smoothed input signals, and also can afford of uncertainty in system model, at the same time maintaining the favorable features of the traditional SMC. Moreover, in the introduced approach, there is no need to use an uncertain bound, which is important in the practical applications. The simulation results using Arneodo chaotic system and their comparison show the effectiveness of the proposed controlling method.

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