University of Cyprus
MAI613 - Research Methodologies and Professional Practices in AI
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Exercises on Linear Algebra

Exercises

Solve Exercises 2.1, 2.3, 2.7, 2.9, 2.10, 2.14, 2.20 from the book:

- M.P. Deisenroth, A. A. Faisal, and C. S. Ong, "Mathematics for Machine Learning," Cambridge University Press, 2020. https://mml-book.github.io/book/mml-book.pdf

Note: These exercises are intended for self-assessment purposes.

Solutions:

2.1 We consider $(\mathbb{R}\backslash\{-1\}, \star)$, where

$$a \star b := ab + a + b, \qquad a, b \in \mathbb{R}\backslash\{-1\} \tag{2.1}$$

a. Show that $(\mathbb{R}\backslash\{-1\}, \star)$ is an Abelian group.
b. Solve

$$3 \star x \star x = 15$$

in the Abelian group $(\mathbb{R}\backslash\{-1\}, \star)$, where $\star$ is defined in (2.1).

a. First, we show that $\mathbb{R}\backslash\{-1\}$ is closed under $\star$: For all $a, b \in \mathbb{R}\backslash\{-1\}$:

$$a \star b = ab + a + b + 1 - 1 = \underbrace{(a+1)}_{\neq 0}\underbrace{(b+1)}_{\neq 0} - 1 \neq -1$$

$$\Rightarrow a \star b \in \mathbb{R}\backslash\{-1\}$$

Next, we show the group axioms

- **Associativity:** For all $a, b, c \in \mathbb{R}\backslash\{-1\}$:

$$\begin{aligned}(a \star b) \star c &= (ab + a + b) \star c \\ &= (ab + a + b)c + (ab + a + b) + c \\ &= abc + ac + bc + ab + a + b + c \\ &= a(bc + b + c) + a + (bc + b + c) \\ &= a \star (bc + b + c) \\ &= a \star (b \star c)\end{aligned}$$

- **Commutativity:**

$$\forall a, b \in \mathbb{R}\backslash\{-1\} : a \star b = ab + a + b = ba + b + a = b \star a$$

- **Neutral Element:** $n = 0$ is the neutral element since

$$\forall a \in \mathbb{R}\backslash\{-1\} : a \star 0 = a = 0 \star a$$

- **Inverse Element:** We need to find $\bar{a}$, such that $a \star \bar{a} = 0 = \bar{a} \star a$.

$$\begin{aligned}\bar{a} \star a = 0 &\iff \bar{a}a + a + \bar{a} = 0 \\ &\iff \bar{a}(a+1) = -a \\ &\overset{a \neq -1}{\iff} \bar{a} = -\frac{a}{a+1} = -1 + \frac{1}{a+1} \neq -1 \in \mathbb{R}\backslash\{-1\}\end{aligned}$$

b.

$$\begin{aligned} 3 \star x \star x = 15 &\iff 3 \star (x^2 + x + x) = 15 \\ &\iff 3x^2 + 6x + 3 + x^2 + 2x = 15 \\ &\iff 4x^2 + 8x - 12 = 0 \\ &\iff (x-1)(x+3) = 0 \\ &\iff x \in \{-3, 1\} \end{aligned}$$

2.3 Consider the set $\mathcal{G}$ of 3×3 matrices defined as follows:

$$\mathcal{G} = \left\{ \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} \in \mathbb{R}^{3\times 3} \;\middle|\; x, y, z \in \mathbb{R} \right\}$$

We define $\cdot$ as the standard matrix multiplication.
Is $(\mathcal{G}, \cdot)$ a group? If yes, is it Abelian? Justify your answer.

- Closure: Let a, b, c, x, y and z be in $\mathbb{R}$ and let us define $\boldsymbol{A}$ and $\boldsymbol{B}$ in $\mathcal{G}$ as

$$\boldsymbol{A} = \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix}, \qquad \boldsymbol{B} = \begin{bmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix}.$$

Then,

$$\boldsymbol{A} \cdot \boldsymbol{B} = \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & a+x & c+xb+z \\ 0 & 1 & b+y \\ 0 & 0 & 1 \end{bmatrix}.$$

Since $a + x$, $b + y$ and $c + xb + z$ are in $\mathbb{R}$ we have $\boldsymbol{A} \cdot \boldsymbol{B} \in \mathcal{G}$. Thus, $\mathcal{G}$ is closed under matrix multiplication.

- Associativity: Let α, β and γ be in $\mathbb{R}$ and let $\boldsymbol{C}$ in $\mathcal{G}$ be defined as

$$\boldsymbol{C} = \begin{bmatrix} 1 & \alpha & \gamma \\ 0 & 1 & \beta \\ 0 & 0 & 1 \end{bmatrix}.$$

It holds that

$$\begin{aligned} (\boldsymbol{A} \cdot \boldsymbol{B}) \cdot \boldsymbol{C} &= \begin{bmatrix} 1 & a+x & c+xb+z \\ 0 & 1 & b+y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & \alpha & \gamma \\ 0 & 1 & \beta \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & \alpha+a+x & \gamma+\alpha\beta+x\beta+c+xb+z \\ 0 & 1 & \beta+b+y \\ 0 & 0 & 1 \end{bmatrix}. \end{aligned}$$

Similarly,

$$\boldsymbol{A}\cdot(\boldsymbol{B}\cdot\boldsymbol{C}) = \begin{bmatrix}1 & x & z\\0 & 1 & y\\0 & 0 & 1\end{bmatrix}\cdot\begin{bmatrix}1 & \alpha+a & \gamma+\alpha\beta+c\\0 & 1 & \beta+b\\0 & 0 & 1\end{bmatrix}$$
$$= \begin{bmatrix}1 & \alpha+a+x & \gamma+\alpha\beta+x\beta+c+xb+z\\0 & 1 & \beta+b+y\\0 & 0 & 1\end{bmatrix} = (\boldsymbol{A}\cdot\boldsymbol{B})\cdot\boldsymbol{C}\,.$$

Therefore, $\cdot$ is associative.

- Neutral element: For all $\boldsymbol{A}$ in $\mathcal{G}$, we have: $\boldsymbol{I}_3\cdot\boldsymbol{A} = \boldsymbol{A} = \boldsymbol{A}\cdot\boldsymbol{I}_3$ and thus $\boldsymbol{I}_3$ is the neutral element.
- Non-commutativity: We show that $\cdot$ is not commutative. Consider the matrices $\boldsymbol{X}, \boldsymbol{Y}\in\mathcal{G}$, where

$$\boldsymbol{X} = \begin{bmatrix}1 & 1 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}, \qquad \boldsymbol{Y} = \begin{bmatrix}1 & 0 & 0\\0 & 1 & 1\\0 & 0 & 1\end{bmatrix}. \tag{2.5}$$

Then,

$$\boldsymbol{X}\cdot\boldsymbol{Y} = \begin{bmatrix}1 & 1 & 1\\0 & 1 & 1\\0 & 0 & 1\end{bmatrix},$$
$$\boldsymbol{Y}\cdot\boldsymbol{X} = \begin{bmatrix}1 & 1 & 0\\0 & 1 & 1\\0 & 0 & 1\end{bmatrix} \neq \boldsymbol{X}\cdot\boldsymbol{Y}\,.$$

Therefore, $\cdot$ is not commutative.

- Inverse element: Let us look for a right inverse $\boldsymbol{A}_r^{-1}$ of $\boldsymbol{A}$. Such a matrix should satisfy $\boldsymbol{A}\boldsymbol{A}_r^{-1} = \boldsymbol{I}_3$. We thus solve the linear system $[\boldsymbol{A}|\boldsymbol{I}_3]$ that we transform into $[\boldsymbol{I}_3|\boldsymbol{A}_r^{-1}]$:

$$\left[\begin{array}{ccc|ccc}1 & x & z & 1 & 0 & 0\\0 & 1 & y & 0 & 1 & 0\\0 & 0 & 1 & 0 & 0 & 1\end{array}\right]\begin{array}{l}-zR_3\\-yR_3\\ \end{array}$$
$$\rightsquigarrow \left[\begin{array}{ccc|ccc}1 & x & 0 & 1 & 0 & -z\\0 & 1 & 0 & 0 & 1 & -y\\0 & 0 & 1 & 0 & 0 & 1\end{array}\right]\begin{array}{l}-xR_2\\ \\ \end{array}$$
$$\rightsquigarrow \left[\begin{array}{ccc|ccc}1 & 0 & 0 & 1 & -x & xy-z\\0 & 1 & 0 & 0 & 1 & -y\\0 & 0 & 1 & 0 & 0 & 1\end{array}\right].$$

Therefore, we obtain the right inverse

$$\boldsymbol{A}_r^{-1} = \begin{bmatrix}1 & -x & xy-z\\0 & 1 & -y\\0 & 0 & 1\end{bmatrix}\in\mathcal{G}$$

Because of the uniqueness of the inverse element, if a left inverse $\boldsymbol{A}_l^{-1}$ exists, then it is equal to the right inverse. But as $\cdot$ is not commutative, we need to check manually that we also have $\boldsymbol{A}\boldsymbol{A}_r^{-1} = \boldsymbol{I}_3$, which we do next:

$$\boldsymbol{A}_r^{-1}\boldsymbol{A} = \begin{bmatrix} 1 & -x & -z+xy \\ 0 & 1 & -y \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & x-x & z-xy-z+xy \\ 0 & 1 & y+y \\ 0 & 0 & 1 \end{bmatrix} = \boldsymbol{I}_3\,.$$

Thus, every element of $\mathcal{G}$ has an inverse. Overall, $(\mathcal{G}, \cdot)$ is a non-Abelian group.

2.7 Find all solutions in $\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3$ of the equation system $\boldsymbol{A}\boldsymbol{x} = 12\boldsymbol{x}$, where

$$\boldsymbol{A} = \begin{bmatrix} 6 & 4 & 3 \\ 6 & 0 & 9 \\ 0 & 8 & 0 \end{bmatrix}$$

and $\sum_{i=1}^{3} x_i = 1$.

We start by rephrasing the problem into solving a homogeneous system of linear equations. Let $\boldsymbol{x}$ be in $\mathbb{R}^3$. We notice that $\boldsymbol{A}\boldsymbol{x} = 12\boldsymbol{x}$ is equivalent to $(\boldsymbol{A} - 12\boldsymbol{I})\boldsymbol{x} = \boldsymbol{0}$, which can be rewritten as the homogeneous system $\tilde{\boldsymbol{A}}\boldsymbol{x} = \boldsymbol{0}$, where we define

$$\tilde{\boldsymbol{A}} = \begin{bmatrix} -6 & 4 & 3 \\ 6 & -12 & 9 \\ 0 & 8 & -12 \end{bmatrix}.$$

The constraint $\sum_{i=1}^{3} x_i = 1$ can be transcribed as a fourth equation, which leads us to consider the following linear system, which we bring to reduced row echelon form:

$$\left[\begin{array}{ccc|c} -6 & 4 & 3 & 0 \\ 6 & -12 & 9 & 0 \\ 0 & 8 & -12 & 0 \\ 1 & 1 & 1 & 1 \end{array}\right] \begin{array}{l} +R_2 \\ \cdot\frac{1}{3} \\ \cdot\frac{1}{4} \\ \\ \end{array}$$

$$\rightsquigarrow \left[\begin{array}{ccc|c} 0 & -8 & 12 & 0 \\ 2 & -4 & 3 & 0 \\ 0 & 2 & -3 & 0 \\ 1 & 1 & 1 & 1 \end{array}\right] \begin{array}{l} +4R_3 \\ +2R_3 \\ \\ \\ \end{array}$$

$$\rightsquigarrow \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 2 & 0 & -3 & 0 \\ 0 & 2 & -3 & 0 \\ 1 & 1 & 1 & 1 \end{array}\right] \begin{array}{l} \\ \cdot\frac{1}{2} \\ \cdot\frac{1}{2} \\ \\ \end{array}$$

$$\rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & -\frac{3}{2} & 0 \\ 0 & 1 & -\frac{3}{2} & 0 \\ 1 & 1 & 1 & 1 \end{array}\right] \begin{array}{l} \\ \\ -R_1 - R_2 \end{array}$$

$$\rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & -\frac{3}{2} & 0 \\ 0 & 1 & -\frac{3}{2} & 0 \\ 0 & 0 & 4 & 1 \end{array}\right] \begin{array}{l} +(\frac{3}{8})R_3 \\ +(\frac{3}{8})R_3 \\ \cdot\frac{1}{4} \end{array} \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{3}{8} \\ 0 & 1 & 0 & \frac{3}{8} \\ 0 & 0 & 1 & \frac{1}{4} \end{array}\right]$$

Therefore, we obtain the unique solution

$$\boldsymbol{x} = \begin{bmatrix} \frac{3}{8} \\ \frac{3}{8} \\ \frac{1}{4} \end{bmatrix} = \frac{1}{8}\begin{bmatrix} 3 \\ 3 \\ 2 \end{bmatrix}.$$

2.9 Which of the following sets are subspaces of $\mathbb{R}^3$?

a. $A = \{(\lambda, \lambda + \mu^3, \lambda - \mu^3) \mid \lambda, \mu \in \mathbb{R}\}$
b. $B = \{(\lambda^2, -\lambda^2, 0) \mid \lambda \in \mathbb{R}\}$
c. Let γ be in $\mathbb{R}$.
$C = \{(\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3 \mid \xi_1 - 2\xi_2 + 3\xi_3 = \gamma\}$
d. $D = \{(\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3 \mid \xi_2 \in \mathbb{Z}\}$

As a reminder: Let V be a vector space. $U \subseteq V$ is a subspace if

1. $U \neq \emptyset$. In particular, $\mathbf{0} \in U$.
2. $\forall \boldsymbol{a}, \boldsymbol{b} \in U : \boldsymbol{a} + \boldsymbol{b} \in U$ Closure with respect to the inner operation
3. $\forall \boldsymbol{a} \in U, \lambda \in \mathbb{R} : \lambda \boldsymbol{a} \in U$ Closure with respect to the outer operation

The standard vector space properties (Abelian group, distributivity, associativity and neutral element) do not have to be shown because they are inherited from the vector space $(\mathbb{R}^3, +, \cdot)$.
Let us now have a look at the sets A, B, C, D.

a. 1. We have that $(0, 0, 0) \in A$ for $\lambda = 0 = \mu$.
2. Let $a = (\lambda_1, \lambda_1 + \mu_1^3, \lambda_1 - \mu_1^3)$ and $b = (\lambda_2, \lambda_2 + \mu_2^3, \lambda_2 - \mu_2^3)$ be two elements of A, where $\lambda_1, \mu_1, \lambda_2, \mu_2 \in \mathbb{R}$. Then,

$$\begin{aligned} a + b &= (\lambda_1, \lambda_1 + \mu_1^3, \lambda_1 - \mu_1^3) + (\lambda_2, \lambda_2 + \mu_2^3, \lambda_2 - \mu_2^3) \\ &= (\lambda_1 + \lambda_2, \lambda_1 + \mu_1^3 + \lambda_2 + \mu_2^3, \lambda_1 - \mu_1^3 + \lambda_2 - \mu_2^3) \\ &= (\lambda_1 + \lambda_2, (\lambda_1 + \lambda_2) + (\mu_1^3 + \mu_2^3), (\lambda_1 + \lambda_2) - (\mu_1^3 + \mu_2^3)) , \end{aligned}$$

which belongs to A.
3. Let α be in $\mathbb{R}$. Then,

$$\alpha(\lambda, \lambda + \mu^3, \lambda - \mu^3) = (\alpha\lambda, \alpha\lambda + \alpha\mu^3, \alpha\lambda - \alpha\mu^3) \in A .$$

Therefore, A is a subspace of $\mathbb{R}^3$.

b. The vector $(1, -1, 0)$ belongs to B, but $(-1) \cdot (1, -1, 0) = (-1, 1, 0)$ does not. Thus, B is not closed under scalar multiplication and is not a subspace of $\mathbb{R}^3$.

c. Let $\boldsymbol{A} \in \mathbb{R}^{1\times 3}$ be defined as $\boldsymbol{A} = [1, -2, 3]$. The set C can be written as:

$$C = \{\boldsymbol{x} \in \mathbb{R}^3 \mid \boldsymbol{A}\boldsymbol{x} = \gamma\} .$$

We can first notice that $\mathbf{0}$ belongs to B only if $\gamma = 0$ since $\boldsymbol{A}\mathbf{=0}$.
Let thus consider $\gamma = 0$ and ask whether C is a subspace of $\mathbb{R}^3$. Let $\boldsymbol{x}$ and $\boldsymbol{y}$ be in C. We know that $\boldsymbol{A}\boldsymbol{x} = \mathbf{0}$ and $\boldsymbol{A}\boldsymbol{y} = \mathbf{0}$, so that

$$\boldsymbol{A}(\boldsymbol{x} + \boldsymbol{y}) = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{A}\boldsymbol{y} = \mathbf{0} + \mathbf{0} = \mathbf{0} .$$

Therefore, $\boldsymbol{x} + \boldsymbol{y}$ belongs to C. Let λ be in $\mathbb{R}$. Similarly,

$$\boldsymbol{A}(\lambda\boldsymbol{x}) = \lambda(\boldsymbol{A}\boldsymbol{x}) = \lambda\mathbf{0} = \mathbf{0}$$

Therefore, C is closed under scalar multiplication, and thus is a subspace of $\mathbb{R}^3$ if (and only if) $\gamma = 0$.

d. The vector $(0, 1, 0)$ belongs to D but $\pi(0, 1, 0)$ does not and thus D is not a subspace of $\mathbb{R}^3$.

2.10 Are the following sets of vectors linearly independent?

a.

$$\boldsymbol{x}_1 = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} , \quad \boldsymbol{x}_2 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} , \quad \boldsymbol{x}_3 = \begin{bmatrix} 3 \\ -3 \\ 8 \end{bmatrix}$$

To determine whether these vectors are linearly independent, we check if the $\mathbf{0}$-vector can be non-trivially represented as a linear combination of $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_3$. Therefore, we try to solve the homogeneous linear equation system $\sum_{i=1}^{3} \lambda_i \boldsymbol{x}_i = \mathbf{0}$ for $\lambda_i \in \mathbb{R}$. We use Gaussian elimination to solve $\boldsymbol{A}\boldsymbol{x} = \mathbf{0}$ with

$$\boldsymbol{A} = \begin{bmatrix} 2 & 1 & 3 \\ -1 & 1 & -3 \\ 3 & -2 & 8 \end{bmatrix} ,$$

which leads to the reduced row echelon form

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} .$$

This means that $\boldsymbol{A}$ is rank deficient/singular and, therefore, the three vectors are linearly dependent. For example, with $\lambda_1 = 2, \lambda_2 = -1, \lambda_3 = -1$ we have a non-trivial linear combination $\sum_{i=1}^{3} \lambda_i \boldsymbol{x}_i = \mathbf{0}$.

b.

$$\boldsymbol{x}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} , \quad \boldsymbol{x}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} , \quad \boldsymbol{x}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

Here, we are looking at the distribution of 0s in the vectors. $\boldsymbol{x}_1$ is the only vector whose third component is non-zero. Therefore, λ_1 must be 0. Similarly, λ_2 must be 0 because of the second component (already conditioning on $\lambda_1 = 0$). And finally, $\lambda_3 = 0$ as well. Therefore, the three vectors are linearly independent.
An alternative solution, using Gaussian elimination, is possible and would lead to the same conclusion.

2.14 Consider two subspaces U_1 and U_2, where U_1 is spanned by the columns of $\boldsymbol{A}_1$ and U_2 is spanned by the columns of $\boldsymbol{A}_2$ with

$$\boldsymbol{A}_1 = \begin{bmatrix} 1 & 0 & 1 \\ 1 & -2 & -1 \\ 2 & 1 & 3 \\ 1 & 0 & 1 \end{bmatrix}, \quad \boldsymbol{A}_2 = \begin{bmatrix} 3 & -3 & 0 \\ 1 & 2 & 3 \\ 7 & -5 & 2 \\ 3 & -1 & 2 \end{bmatrix}.$$

a. Determine the dimension of U_1, U_2

We start by noting that $U_1, U_2 \subseteq \mathbb{R}^4$ since we are interested in the space spanned by the columns of the corresponding matrices. Looking at $\boldsymbol{A}_1$, we see that $-\boldsymbol{d}_1 + \boldsymbol{d}_3 = \boldsymbol{d}_2$, where $\boldsymbol{d}_i$ are the columns of $\boldsymbol{A}_1$. This means that the second column can be expressed as a linear combination of $\boldsymbol{d}_1$ and $\boldsymbol{d}_3$. $\boldsymbol{d}_1$ and $\boldsymbol{d}_3$ are linearly independent, i.e., $\dim(U_1) = 2$.

Similarly, for $\boldsymbol{A}_2$, we see that the third column is the sum of the first two columns, and again we arrive at $\dim(U_2) = 2$.

Alternatively, we can use Gaussian elimination to determine a set of linearly independent columns in both matrices.

b. Determine bases of U_1 and U_2

A basis B of U_1 is given by the first two columns of $\boldsymbol{A}_1$ (any pair of columns would be fine), which are independent. A basis C of U_2 is given by the second and third columns of $\boldsymbol{A}_2$ (again, any pair of columns would be a basis), such that

$$B = \left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} \right\}, \quad C = \left\{ \begin{bmatrix} -3 \\ 2 \\ -5 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 2 \\ 2 \end{bmatrix} \right\}$$

c. Determine a basis of $U_1 \cap U_2$

Let us call $\boldsymbol{b}_1, \boldsymbol{b}_2, \boldsymbol{c}_1$ and $\boldsymbol{c}_2$ the vectors of the bases B and C such that $B = \{\boldsymbol{b}_1, \boldsymbol{b}_2\}$ and $C = \{\boldsymbol{c}_1, \boldsymbol{c}_2\}$. Let $\boldsymbol{x}$ be in $\mathbb{R}^4$. Then,

$$\begin{aligned}
\boldsymbol{x} \in U_1 \cap U_2 &\iff \exists \lambda_1, \lambda_2, \lambda_3, \lambda_4 \in \mathbb{R} \colon (\boldsymbol{x} = \lambda_1 \boldsymbol{b}_1 + \lambda_2 \boldsymbol{b}_2) \wedge (\boldsymbol{x} = \lambda_3 \boldsymbol{c}_1 + \lambda_4 \boldsymbol{c}_2) \\
&\iff \exists \lambda_1, \lambda_2, \lambda_3, \lambda_4 \in \mathbb{R} \colon (\boldsymbol{x} = \lambda_1 \boldsymbol{b}_1 + \lambda_2 \boldsymbol{b}_2) \\
&\qquad \wedge (\lambda_1 \boldsymbol{b}_1 + \lambda_2 \boldsymbol{b}_2 = \lambda_3 \boldsymbol{c}_1 + \lambda_4 \boldsymbol{c}_2) \\
&\iff \exists \lambda_1, \lambda_2, \lambda_3, \lambda_4 \in \mathbb{R} \colon (\boldsymbol{x} = \lambda_1 \boldsymbol{b}_1 + \lambda_2 \boldsymbol{b}_2) \\
&\qquad \wedge (\lambda_1 \boldsymbol{b}_1 + \lambda_2 \boldsymbol{b}_2 - \lambda_3 \boldsymbol{c}_1 - \lambda_4 \boldsymbol{c}_2 = \boldsymbol{0})
\end{aligned}$$

Let $\boldsymbol{\lambda} := [\lambda_1, \lambda_2, \lambda_3, \lambda_4]^\top$. The last equation of the system can be written as the linear system $\boldsymbol{A}\boldsymbol{\lambda} = \boldsymbol{0}$, where we define the matrix $\boldsymbol{A}$ as the concatenation of the column vectors $\boldsymbol{b}_1, \boldsymbol{b}_2, -\boldsymbol{c}_1$ and $-\boldsymbol{c}_2$.

$$\boldsymbol{A} = \begin{bmatrix} 1 & 0 & 3 & 0 \\ 1 & -2 & -2 & -3 \\ 2 & 1 & 5 & -2 \\ 1 & 0 & 1 & -2 \end{bmatrix}.$$

We solve this homogeneous linear system using Gaussian elimination.

$$\begin{aligned}
&\left[\begin{array}{cccc} 1 & 0 & 3 & 0 \\ 1 & -2 & -2 & -3 \\ 2 & 1 & 5 & -2 \\ 1 & 0 & 1 & -2 \end{array}\right] \begin{array}{l} \\ -R_1 \\ -2R_1 \\ -R_1 \end{array} \\
\rightsquigarrow &\left[\begin{array}{cccc} 1 & 0 & 3 & 0 \\ 0 & -2 & -5 & -3 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & -2 & -2 \end{array}\right] \begin{array}{l} \\ +2R_3 \\ \text{swap with } R_2 \\ \cdot(-\frac{1}{2}) \end{array} \\
\rightsquigarrow &\left[\begin{array}{cccc} 1 & 0 & 3 & 0 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & -7 & -7 \\ 0 & 0 & 1 & 1 \end{array}\right] \begin{array}{l} -3R_4 \\ +R_4 \\ +7R_4 \\ \text{swap with } R_3 \end{array} \quad \rightsquigarrow \left[\begin{array}{cccc} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]
\end{aligned}$$

From the reduced row echelon form we find that the set

$$S := \operatorname{span}[\begin{bmatrix} -3 \\ -1 \\ 1 \\ -1 \end{bmatrix}]$$

describes the solution space of the system of equations in $\boldsymbol{\lambda}$.

We can now resume our equivalence derivation and replace the homogeneous system with its solution space. It holds

$$\begin{aligned}
\boldsymbol{x} \in U_1 \cap U_2 &\iff \exists \lambda_1, \lambda_2, \lambda_3, \lambda_4, \alpha \in \mathbb{R} \colon (\boldsymbol{x} = \lambda_1 \boldsymbol{b}_1 + \lambda_2 \boldsymbol{b}_2) \\
&\qquad \wedge ([\lambda_1, \lambda_2, \lambda_3, \lambda_4]^\top = \alpha[-3, -1, 1, -1]^\top) \\
&\iff \exists \alpha \in \mathbb{R} \colon \boldsymbol{x} = -3\alpha \boldsymbol{b}_1 - \alpha \boldsymbol{b}_2 \\
&\iff \exists \alpha \in \mathbb{R} \colon \boldsymbol{x} = \alpha[-3, -1, -7, -3]^\top
\end{aligned}$$

Finally,

$$U_1 \cap U_2 = \operatorname{span}[\begin{bmatrix} -3 \\ -1 \\ -7 \\ -3 \end{bmatrix}] .$$

Alternatively, we could have expressed the solutions of $\boldsymbol{x}$ in terms of $\boldsymbol{b}_1$ and $\boldsymbol{c}_2$ with the condition on $\boldsymbol{\lambda}$ being $\exists \alpha \in \mathbb{R}\colon (\lambda_3 = \alpha) \wedge (\lambda_4 = -\alpha)$ to obtain $[3, 1, 7, 3]^\top$.

2.20 Let us consider $\boldsymbol{b}_1, \boldsymbol{b}_2, \boldsymbol{b}'_1, \boldsymbol{b}'_2$, 4 vectors of $\mathbb{R}^2$ expressed in the standard basis of $\mathbb{R}^2$ as

$$\boldsymbol{b}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad \boldsymbol{b}_2 = \begin{bmatrix} -1 \\ -1 \end{bmatrix}, \quad \boldsymbol{b}'_1 = \begin{bmatrix} 2 \\ -2 \end{bmatrix}, \quad \boldsymbol{b}'_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

and let us define two ordered bases $B = (\boldsymbol{b}_1, \boldsymbol{b}_2)$ and $B' = (\boldsymbol{b}'_1, \boldsymbol{b}'_2)$ of $\mathbb{R}^2$.

a. Show that B and B' are two bases of $\mathbb{R}^2$ and draw those basis vectors.
The vectors $\boldsymbol{b}_1$ and $\boldsymbol{b}_2$ are clearly linearly independent and so are $\boldsymbol{b}'_1$ and $\boldsymbol{b}'_2$.

b. Compute the matrix $\boldsymbol{P}_1$ that performs a basis change from B' to B.
We need to express the vector $\boldsymbol{b}'_1$ (and $\boldsymbol{b}'_2$) in terms of the vectors $\boldsymbol{b}_1$ and $\boldsymbol{b}_2$. In other words, we want to find the real coefficients λ_1 and λ_2 such that $\boldsymbol{b}'_1 = \lambda_1 \boldsymbol{b}_1 + \lambda_2 \boldsymbol{b}_2$. In order to do that, we will solve the linear equation system

$$\left[\begin{array}{cc|c} \boldsymbol{b}_1 & \boldsymbol{b}_2 & \boldsymbol{b}'_1 \end{array}\right]$$

i.e.,

$$\left[\begin{array}{cc|c} 2 & -1 & 2 \\ 1 & -1 & -2 \end{array}\right]$$

and which results in the reduced row echelon form

$$\left[\begin{array}{cc|c} 1 & 0 & 4 \\ 0 & 1 & 6 \end{array}\right].$$

This gives us $\boldsymbol{b}'_1 = 4\boldsymbol{b}_1 + 6\boldsymbol{b}_2$.
Similarly for $\boldsymbol{b}'_2$, Gaussian elimination gives us $\boldsymbol{b}'_2 = -1\boldsymbol{b}_2$.
Thus, the matrix that performs a basis change from B' to B is given as

$$\boldsymbol{P}_1 = \begin{bmatrix} 4 & 0 \\ 6 & -1 \end{bmatrix}.$$

c. We consider $\boldsymbol{c}_1, \boldsymbol{c}_2, \boldsymbol{c}_3$, three vectors of $\mathbb{R}^3$ defined in the standard basis of $\mathbb{R}$ as

$$\boldsymbol{c}_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \quad \boldsymbol{c}_2 = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}, \quad \boldsymbol{c}_3 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

and we define $C = (\boldsymbol{c}_1, \boldsymbol{c}_2, \boldsymbol{c}_3)$.

(i) Show that C is a basis of $\mathbb{R}^3$, e.g., by using determinants (see Section 4.1).
We have:

$$\det(\boldsymbol{c}_1, \boldsymbol{c}_2, \boldsymbol{c}_3) = \begin{vmatrix} 1 & 0 & 1 \\ 2 & -1 & 0 \\ -1 & 2 & -1 \end{vmatrix} = 4 \neq 0$$

Therefore, C is regular, and the columns of C are linearly independent, i.e., they form a basis of $\mathbb{R}^3$.

(ii) Let us call $C' = (\boldsymbol{c}'_1, \boldsymbol{c}'_2, \boldsymbol{c}'_3)$ the standard basis of $\mathbb{R}^3$. Determine the matrix $\boldsymbol{P}_2$ that performs the basis change from C to C'.
In order to write the matrix that performs a basis change from C to C', we need to express the vectors of C in terms of those of C'. But as C' is the standard basis, it is straightforward that $\boldsymbol{c}_1 = 1\boldsymbol{c}'_1 + 2\boldsymbol{c}'_2 - 1\boldsymbol{c}'_3$ for example. Therefore,

$$\boldsymbol{P}_2 := \begin{bmatrix} 1 & 0 & 1 \\ 2 & -1 & 0 \\ -1 & 2 & -1 \end{bmatrix}.$$

simply contains the column vectors of C (this would not be the case if C' was not the standard basis).

d. We consider a homomorphism $\Phi : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$, such that

$$\begin{aligned} \Phi(\boldsymbol{b}_1 + \boldsymbol{b}_2) &= \boldsymbol{c}_2 + \boldsymbol{c}_3 \\ \Phi(\boldsymbol{b}_1 - \boldsymbol{b}_2) &= 2\boldsymbol{c}_1 - \boldsymbol{c}_2 + 3\boldsymbol{c}_3 \end{aligned}$$

where $B = (\boldsymbol{b}_1, \boldsymbol{b}_2)$ and $C = (\boldsymbol{c}_1, \boldsymbol{c}_2, \boldsymbol{c}_3)$ are ordered bases of $\mathbb{R}^2$ and $\mathbb{R}^3$, respectively.
Determine the transformation matrix $\boldsymbol{A}_\Phi$ of Φ with respect to the ordered bases B and C.

Adding and subtracting both equations gives us

$$\begin{cases} \Phi(\boldsymbol{b}_1+\boldsymbol{b}_2)+\Phi(\boldsymbol{b}_1-\boldsymbol{b}_2) &= 2\boldsymbol{c}_1+4\boldsymbol{c}_3 \\ \Phi(\boldsymbol{b}_1+\boldsymbol{b}_2)-\Phi(\boldsymbol{b}_1-\boldsymbol{b}_2) &= -2\boldsymbol{c}_1+2\boldsymbol{c}_2-2\boldsymbol{c}_3 \end{cases}$$

As Φ is linear, we obtain

$$\begin{cases} \Phi(2\boldsymbol{b}_1) &= 2\boldsymbol{c}_1+4\boldsymbol{c}_3 \\ \Phi(2\boldsymbol{b}_2) &= -2\boldsymbol{c}_1+2\boldsymbol{c}_2-2\boldsymbol{c}_3 \end{cases}$$

And by linearity of Φ again, the system of equations gives us

$$\begin{cases} \Phi(\boldsymbol{b}_1) &= \boldsymbol{c}_1+2\boldsymbol{c}_3 \\ \Phi(\boldsymbol{b}_2) &= -\boldsymbol{c}_1+\boldsymbol{c}_2-\boldsymbol{c}_3 \end{cases}.$$

Therefore, the transformation matrix of $\boldsymbol{A}_\Phi$ with respect to the bases B and C is

$$\boldsymbol{A}_\Phi = \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 2 & -1 \end{bmatrix}.$$

e. Determine $\boldsymbol{A}'$, the transformation matrix of Φ with respect to the bases $\boldsymbol{B}'$ and $\boldsymbol{C}'$.
We have:

$$\boldsymbol{A}' = \boldsymbol{P}_2\boldsymbol{A}\boldsymbol{P}_1 = \begin{bmatrix} 1 & 0 & 1 \\ 2 & -1 & 0 \\ -1 & 2 & -1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 4 & 0 \\ 6 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ -10 & 3 \\ 12 & -4 \end{bmatrix}.$$

f. Let us consider the vector $\boldsymbol{x} \in \mathbb{R}^2$ whose coordinates in B' are $[2,3]^\top$. In other words, $\boldsymbol{x} = 2\boldsymbol{b}'_1 + 3\boldsymbol{b}'_2$.

(i) Calculate the coordinates of $\boldsymbol{x}$ in B.
By definition of $\boldsymbol{P}_1$, $\boldsymbol{x}$ can be written in B as

$$\boldsymbol{P}_1\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 6 & -1 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 8 \\ 9 \end{bmatrix}$$

(ii) Based on that, compute the coordinates of $\Phi(\boldsymbol{x})$ expressed in C.
Using the transformation matrix $\boldsymbol{A}$ of Φ with respect to the bases B and C, we get the coordinates of $\Phi(\boldsymbol{x})$ in C with

$$\boldsymbol{A}\begin{bmatrix} 8 \\ 9 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 8 \\ 9 \end{bmatrix} = \begin{bmatrix} -1 \\ 9 \\ 7 \end{bmatrix}$$

(iii) Then, write $\Phi(\boldsymbol{x})$ in terms of $\boldsymbol{c}'_1, \boldsymbol{c}'_2, \boldsymbol{c}'_3$.
Going back to the basis C' thanks to the matrix $\boldsymbol{P}_2$ gives us the expression of $\Phi(\boldsymbol{x})$ in C'

$$\boldsymbol{P}_2\begin{bmatrix} -1 \\ 9 \\ 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 2 & -1 & 0 \\ -1 & 2 & -1 \end{bmatrix}\begin{bmatrix} -1 \\ 9 \\ 7 \end{bmatrix} = \begin{bmatrix} 6 \\ -11 \\ 12 \end{bmatrix}$$

In other words, $\Phi(\boldsymbol{x}) = 6\boldsymbol{c}_1' - 11\boldsymbol{c}_2' + 12\boldsymbol{c}_3'$.

(iv) Use the representation of $\boldsymbol{x}$ in B' and the matrix $\boldsymbol{A}'$ to find this result directly.
We can calculate $\Phi(\boldsymbol{x})$ in C directly:

$$\boldsymbol{A}' \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ -10 & 3 \\ 12 & -4 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 6 \\ -11 \\ 12 \end{bmatrix}$$