2025

APPLIED MATHEMATICS

For General Foundation Program Leading to Calculus

APPLIED MATHEMATICS (LEVEL – II)

CLFS MATHEMATICS TEAM

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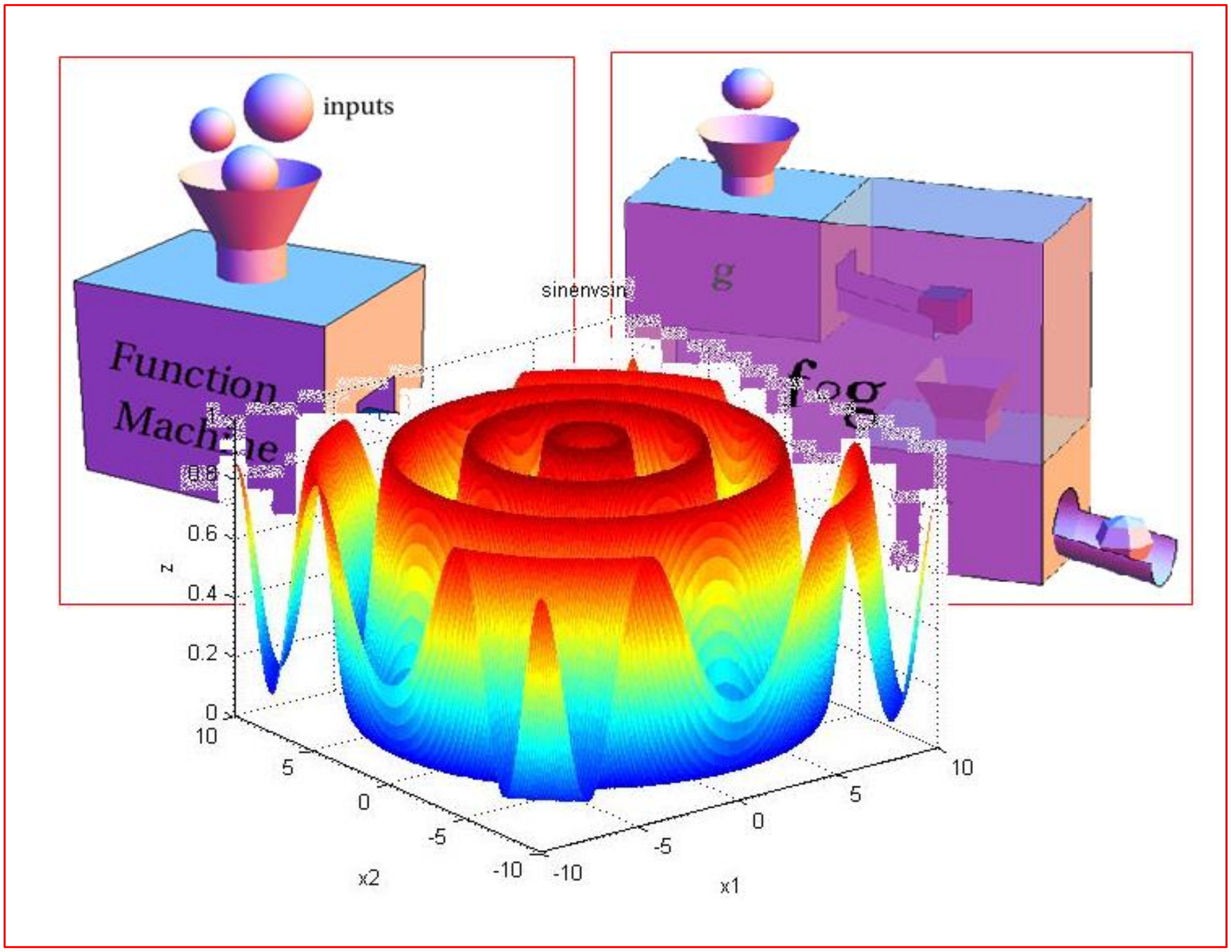
CHAPTER – 1 FUNCTIONS AND THEIR GRAPHS

OBJECTIVES: Upon successful completion of this topic, students will be able to

- Describe the function and its graph.
- Use three types of symmetry of an equation to sketch its graph.



What is a Function?

A function $\boldsymbol{f}$ is a **rule** that is applied to each element $\boldsymbol{x} \in \boldsymbol{A}$, exactly one element $\boldsymbol{y} \in \boldsymbol{B}$, this relation is called "a function from $\boldsymbol{A}$ to $\boldsymbol{B}$", denoted by $\boldsymbol{f}: \boldsymbol{A} \longrightarrow \boldsymbol{B}$. We also say that $\boldsymbol{y} = \boldsymbol{f}(\boldsymbol{x})$, read as ($\boldsymbol{y}$ *is the function of* $\boldsymbol{x}$). The Venn diagram further explains the understanding of function. In below figures the first (Figure 2) is the correct representation for being a function but the second (Figure 1) is not the correct way of expression a function.

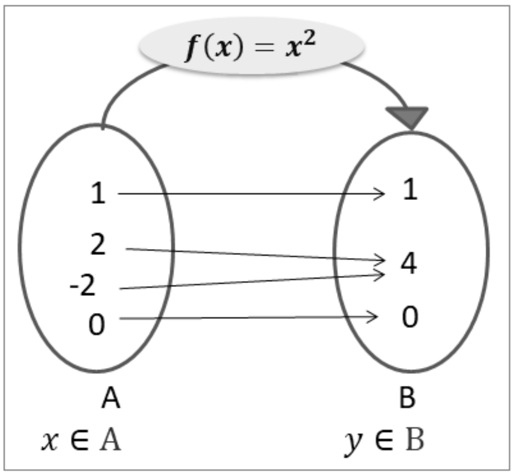


Figure 1: Yes, a function

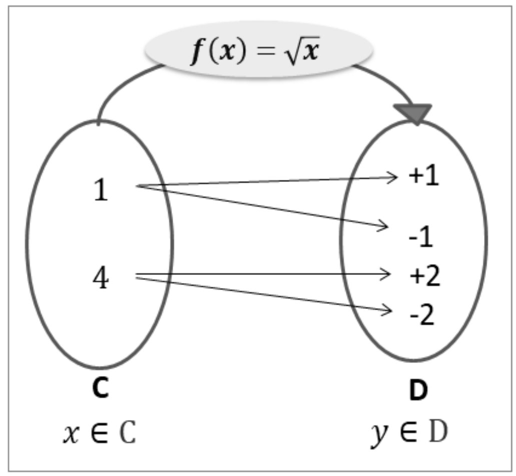


Figure 2: No, not a function

The ordered pairs $(1,1), (2,4), (-2,4), (0,0)$ in (Figure 1) is a correct relation because each element in set **A** has been used only once. On the other hand, the ordered pairs $(1,1), (1,-1), (4,2), (4,-2)$ in (Figure 2) is not a correct relation because some elements in set **B** are used more than once.

Functions in Real Life

a) Height of a human is the **function** of his age.

b) Area of the Rectangle, is the **function** of length and breadth.

c) Bacteria count in an experiment is a **function** of time.

d) Area of a circle is the **function** of its radius.

Functions as a Machine

Below are the example of function and its real-life example as shown in (Figure 3).

Figure 3: Real life example of a Function
Source: (Breville Slow Juicer, 2018)

Domain of a Function

The set of all possible input values, which produce a defined value from a certain function, see (Table 1) for some basic formulas of algebraic functions with examples.

Table 1: Domain of Functions

Expression	**Domain**	**Description**	**Example**
$x^n : n \in Z^+$	$x \in \mathbb{R}$	Polynomial functions	$x, x^2, x^3 + 1, 3x^4 + 2x^2 - 1$
$\frac{1}{x}$	$x \neq 0$	Rational functions	$\frac{1}{x+2}, \frac{x^2+1}{2x-5}, \frac{3}{x^2+2x+1}$
$\sqrt{x}$	$x \geq 0$	Radical functions	$\sqrt{x+2}, \sqrt{x-\frac{1}{2}}, \sqrt{x^2-6}$
$\frac{1}{\sqrt{x}}$	$x > 0$	Irrational functions	$\frac{1}{\sqrt{x+2}}, \frac{3x}{\sqrt{x^2-4}}, \frac{x^3}{\sqrt{3x+6}}$

Range of a function

The set of all possible output values is called the range of the function. For exam the set of values of $\boldsymbol{y} = \boldsymbol{f}(\boldsymbol{x})$ is called the ***range*** of a function. The figure below seen from (Pluspng.com, 2018) clarifies the relation between the domain and range of a function. To find the range of any function we use the same rule as per domain but for $\boldsymbol{x} = \boldsymbol{f}(\boldsymbol{y})$. Algebraically five steps are followed to find the **range** of a function.

- Let the function be $\boldsymbol{y} = \boldsymbol{f}(\boldsymbol{x})$.
- Express the function as $\boldsymbol{x} = \boldsymbol{f}(\boldsymbol{y})$.
- Find all values of $\boldsymbol{y}$ where $\boldsymbol{f}(\boldsymbol{y})$ is defined.
- Eliminate the values of $\boldsymbol{y}$ for which $\boldsymbol{f}(\boldsymbol{y})$ undefined.
- Write the **range** in interval form or in inequality form.

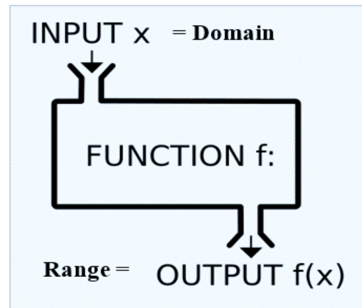
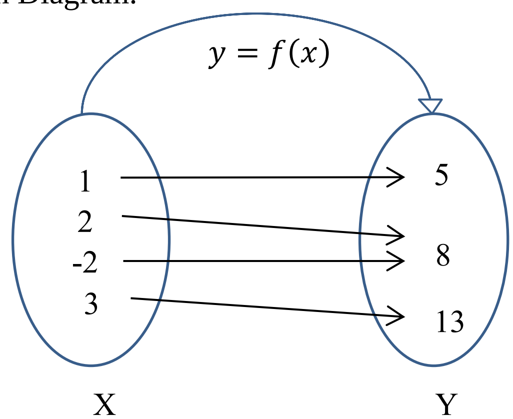


Figure 4: Domain-Function-Range

Example – 1: Find the **Range** and **Domain** in the given Venn Diagram.

Solution: From the given figure: $f : x \to y$.
Domain of $f = x = \{1, 2, -2, 3\}$

Range of $f = y = \{5, 8, 13\}$



Example – 2: Determine the **Domain** and **Range** of the function, $f: A \longrightarrow B$ where

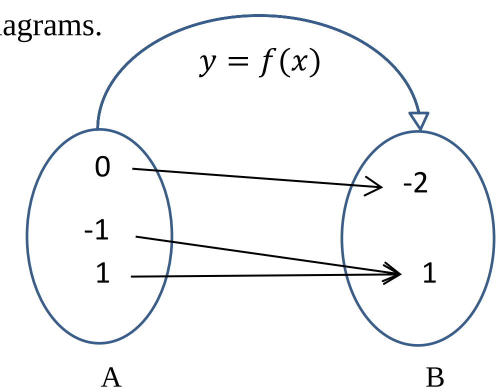
$f(x) = 3x^2 - 2$ and $A = \{0, -1, 1\}$. Also draw its Venn Diagrams.

Solution: Domain of $f(x) = A = \{-1, 0, 1\}$

Range of $f(x) = \{f(0), f(-1), f(1)\}$

$= \{-2, 1\}$

Range of $f(x) = B$



Example – 3: Evaluate the **Domain** and **Range** of the **function**, $f(x) = \frac{5}{x-6}$.

Solution: Formula: **domain** for type: $\frac{1}{x}: x \neq 0: x \in$ R.
So, $Dom\ f = x - 6 \neq 0$

$= x \neq 6$

$Dom\ f = \{x: x \in \text{R}: x \neq 6\}$ ----- Answer

For the $\boldsymbol{Range = y = f(x)}$**,** we follow the five steps that are defined in the previous section.

Let the function be	$y = \dfrac{5}{x-6}$
From above we need x	$y(x-6) = 5$
	$xy - 6y = 5$
	$xy = 6y + 5$
	$x = \dfrac{6y+5}{y}$
Use same rules as of domain	$y \neq 0$ but $y \in$ R for $6y + 5$
Range $f = \{y \in \text{R}: \text{y} \neq 0\}$	or $y \in (-\infty, 0) \cup (0, \infty)$ ------ Answer

Analyzing the Functions

By analysing the function, we mean, in what order (or steps) an algebraic expression can be solved. In this discussion we write a function in two ways, function in words and function in notation.

Example – 1: Express the function $\boldsymbol{f(x) = x^2 + 7}$ in **words**?

Solution: Function in **words = square**, and then **add** 7. --- Answer

Example – 2: Express the function "square root, multiply 8 and then divide by 10" in **notation**?

Solution: Function in **notation=** $\frac{(\sqrt{x} \times 8)}{10} = \frac{8\sqrt{x}}{10} = \frac{4\sqrt{x}}{5}$. ---- Answer

Evaluating the Functions

Finding the value of a function at the given value of x, is called the evaluation of s function. There can be a single value function or a piecewise function, which are explained in the following examples respectively.

Example – 1: Evaluate the function: $f(x) = \frac{x}{6} - 4$, at $x = -3$?

Solution: Putting the value of $\boldsymbol{x}$ in the given function, we get

$$f(-3) = \frac{(-3)}{6} - 4$$

$$f(-3) = -\frac{1}{2} - 4 = -\frac{9}{2} = -4.5 \text{ ---- Answer}$$

Example – 2: Evaluate the following piecewise function at the indicated values

$$f(x) = \begin{cases} 2x^3 - 5, if\ x < 2 \\ 4x + 1, if\ \ x \geq 2 \end{cases}$$

whenever $f(0)$, $f(2)$, $f(4)$, $f(-2)$?

Solution: In piecewise functions, we compare the given value of x with conditions exist in a problem. We use the same function which matches the conditions.

- Since $0 < 2$ so, we use the first function $f(0) = 2(0)^3 - 5 = 0 - 5 = -5$ --- Answer.
- Since $2 = 2$ so, we use the second function $f(2) = 4(2) - 5 = 8 - 5 = 3$ --- Answer.
- Since $4 > 2$ so, we use the second function $f(4) = 4(4) - 5 = 16 - 5 = 11$ --- Answer.
- Since $-2 < 2$ so, we use the first function $f(-2) = 2(-2)^3 - 5 = -16 - 5 = -21$---Ans.

Algebra of Functions

For the given **two** or **more than two functions** we can add, subtract, multiply and divide just as simple algebraic expressions. Let f and g be functions with domains A and B then

1. Sum of functions: $(f + g)(x) = f(x) + g(x)$ $\quad$ $Domain\ x \in A \cap B$
2. Difference of functions: $(f - g)(x) = f(x) - g(x)$ $\quad$ $Domain\ x \in A \cap B$
3. Product of functions: $(f \times g)(x) = f(x) \times g(x)$ $\quad$ $Domain\ x \in A \cap B$
4. Division of functions: $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}, \quad g(x) \neq 0$ $\quad$ $Domain\ x \in A \cap B$

Example-1: $Find\ f + g, f - g, f.g\ and\ \frac{f}{g}; g \neq 0$, whenever $f(x) = x^2 + 2x$ and $g(x) = 3x^2 - 1$.

Solution:

- $f + g = (x^2 + 2x) + (3x^2 - 1)$

$= x^2 + 2x + 3x^2 - 1$

$= 4x^2 + 2x - 1$ ------------- **Answer**

1. $f - g = (x^2 + 2x) - (3x^2 - 1)$

$= x^2 + 2x - 3x^2 + 1$

$= -x^2 + 2x + 1$ --------------- **Answer**

2. $f.g = (x^2 + 2x).(3x^2 - 1)$, (multiply by **FOIL** method)

$= 3x^4 - x^2 + 6x^3 - 2x$

$= 3x^4 + 6x^3 - x^2 - 2x$ ------------- **Answer**

3. $\frac{f}{g} = \frac{x^2+2x}{3x^2-1} = \frac{x(x+2)}{(3x+1)(3x-1)}$ --------------- **Answer**

Exercise – 1.1

Which of the following pair of relations shows "**a function**" or "**not a function**"?

1. $(2,3),(4,7),(5,7),(7,3)$

2. $(3,5),(4,-7),(3,7),(-7,-3)$

3. $(12,3),(-4,7),(-12,7),(4,3)$

4. $(7,-3),(-7,7),(5,7),(6,3)$

5. $(1,3),(-4,7),(-1,7),(4,3)$

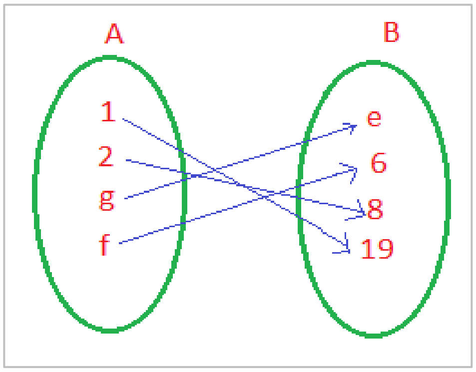
6. $(1,3),(-4,7),(-2,7),(1,3)$

7. $(1,1),(3,3),(4,4),(5,5)$

8. $(-1,1),(-3,3),(-1,4),(-5,-5)$

Determine whether the following sets of Venn diagrams represent the properties for being a **function** or **not a function** (see Figure 5) below. State the **reason** of your answer.

9.



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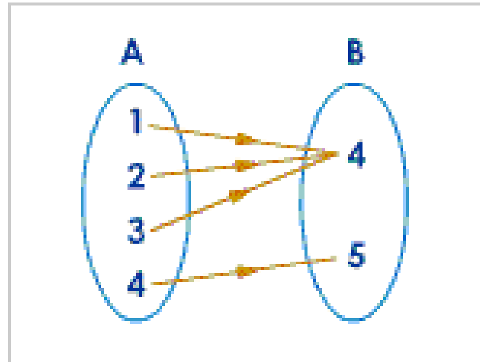
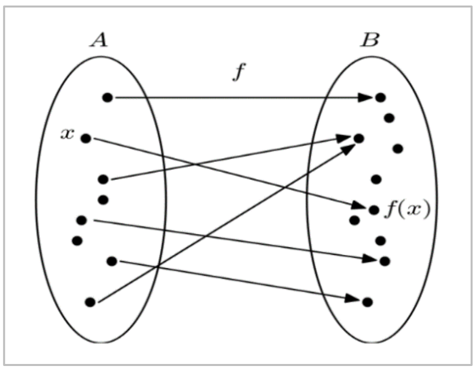


11.



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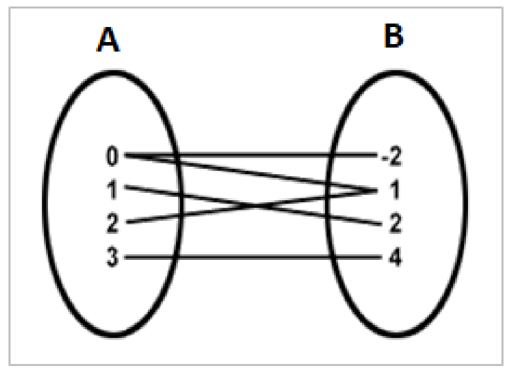


Figure 5: The relation is function or not a function
Source: Figure 1 (Swiftutors.com, n.d.), Figure 2 (Winpossible, n.d.),
Figure 3 (Commons, Wikimedia, n.d.), Figure 4 (Relations of functions, n.d.)

Express the following rules in function **notation**?

13. Subtract 1, and then square root.

14. Square root, and then Subtract 1.

15. Take the Square root, add 7, and then multiply by $\frac{1}{5}$.

16. Add 15, then divide 4.

Express the following functions in **words**?

17. $x^3 + 3$

18. $\sqrt{(a-b)^2}$

19. $(x-5)^2 + 4$

20. $\frac{x-1}{10}$

Evaluate the functions at the **indicated values.**

21. $h(x) = |4x - 1|$ Find the value of: $h(-3), h(0), h\left(\frac{1}{2}\right), h(3)$

22. $g(t) = t^2 + t$ Find the values of: $g(-1), g(a), g(-4), g(\frac{1}{3})$

23. $f(x) = \frac{x-1}{1+x}$ Find the values of: $f(b-1), f(11), f(-9), f(\frac{1}{2})$

24. $h(x) = \sqrt{x-10}$ Find the values of: $h(-10), h(10), h(131), h(3.50)$

25. $f\ (x) = 2|x-2|$ Find the values of $f(2), f(-2), f\left(-\frac{1}{2}\right), f(0), f(2x).$

Use the function to **Evaluate** and **Simplify** the indicated expressions.

26. $S(x) = 2x - 3;\ S(2x),\ S(\frac{2}{x})$

27. $t(x) = 6x - 18;\ \frac{t(x)}{3},\ t(\frac{x}{3})$

28. $u(x) = x + 4;\ u(x^2),\ (u(x))^2.$

29. $g(x) = \frac{x-1}{10};\ g(g(x))$

Evaluate the piecewise functions at the indicated values.

30. $f(x) = \begin{cases} x - 1 & \text{if } x \geq 0 \\ 2x^2 & \text{if } x < 0 \end{cases}$

$f(-5), f(2), f\left(-\frac{1}{4}\right), f(0)$

31. $k(x) = \begin{cases} 5x & \text{if } x = 0 \\ x^2 - 1 & \text{if } x < 0 \\ 3x - 8 & \text{if x} > 1 \end{cases}$

$k(-2), k(3), k\left(-\frac{1}{3}\right), k(0)$

32. $r(x) = \begin{cases} x & \text{if } 0 \leq x \leq 2 \\ x^2 - x & \text{if } x < -1 \\ x - 3 & \text{if x} > 2 \end{cases}$

$r(2.1), r(-2), r(-4), r(0)$

33. $g(x) = \begin{cases} 5 - x & \text{if } x = 0 \\ x^2 + 2x - 1 & \text{if } 1 < x < 3 \\ 5x - 1 & \text{if x} \geq 3 \end{cases}$

$g(2), g(3), g(4), g(0)$

Find the **value** of $\boldsymbol{S+H}$, S$-\boldsymbol{H}$, $\boldsymbol{S.H}$ and $\frac{\boldsymbol{S}}{\boldsymbol{H}}$ for the given functions, also evaluate $(\boldsymbol{S}+\boldsymbol{H})(-\mathbf{3})$, $(\boldsymbol{S}-\boldsymbol{H})(\mathbf{2})$, $(\boldsymbol{S}\times\boldsymbol{H})(\mathbf{0})$ and $\left(\frac{\boldsymbol{S}}{\boldsymbol{H}}\right)(\frac{\mathbf{1}}{\mathbf{3}})$.

34. $S(x) = 2x - 3;\quad H(x) = -x + 2$

35. $S(x) = x^2 - 4x - 3;\quad H(x) = -7x^2 + 2x$

36. $S(x) = \frac{x+3}{15};\quad H(x) = \frac{-x^2-3x}{3x}$

37. $S(x) = \sqrt{x-1};\quad H(x) = \sqrt{4(x-1)^2}$

Determine and **simplify**: $\boldsymbol{f+g}, \boldsymbol{f-g}, \boldsymbol{f \times g}$ and $\boldsymbol{\frac{f}{g}}$, for the following functions.

38. $f=-2x+4$ and $g=-4+2x$

39. $f=x+4$ and $g=x^2-16$

40. $f=s^2-2s+1$ and $g=2s-2$

41. $f=\frac{x^2+6x+9}{(x+3)^2}$ and $g=\frac{3}{x-3}$

42. $f=\frac{x^2-2x}{x-2}$ and $g=\frac{4-2x}{-6}$

43. $f=t^3-t$ and $g=2t$

44. $f=\sqrt{49(x-6)^2}$ and $g=14x^2-504$

45. $f=\frac{8t+1}{2-t^2}$ and $g=\frac{11}{3t}$

Find: $\boldsymbol{S(a)}, \boldsymbol{S(a+t)}$, and the **difference** quotient $\frac{\boldsymbol{S(a+t)-S(a)}}{\boldsymbol{t}}$, where $t \neq 0$.

46. $S(x)=x-1$

47. $S(x)=3x+4$

48. $S(x)=2x+1$

49. $S(x)=x^2$

50. $\frac{x}{3}$

51. $S(x)=x^2-3$

Evaluate the **Domain** of the following

46. $h(x)=x^3-1$

47. $k(x)=\frac{x}{2x+6}$

48. $m(x)=\frac{3x}{(x-4)^2}$

49. $r(x)=\sqrt{7-4x}$

50. $p(x)=\sqrt{2x-7}$

51. $f(x)=\frac{4}{\sqrt{5-x}}$

52. $\mathrm{t}(x)=x^3+x^2-x+3$

53. $\mathrm{g}(x)=\frac{1}{x^3-8}$

54. $v(x)=\frac{4x-3}{3x+8}$

55. $q(x)=\frac{x-5}{\sqrt{x^2-1}}$

Application of Functions in Geometry and in Business

56. Production Amount: The amount $\boldsymbol{D}$ in dollars of making $\boldsymbol{t}$ feet of a certain cloth are given by the function, $\boldsymbol{D(t)=1100+5t+0.03t^2+0.002t^3}$.
Calculate $\boldsymbol{D(100)}$ and $\boldsymbol{D(25)}$.

Figure 6: Cloth on Sale

57. Area of a Sphere (Tutorvista, 2016) **(**Figure 3**):** The area of a sphere is given by a function $\boldsymbol{A(r) = 4\pi r^2}$**,** where '$r$' is radius of a sphere. Find $\boldsymbol{A(2)}$ and $\boldsymbol{A(3)}$?

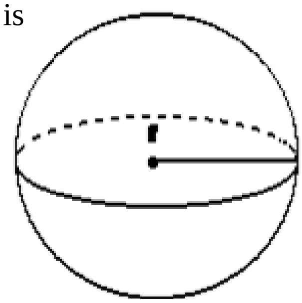


Figure 7: Area of a Sphere

58. Income Tax: In some countries, the income tax $\boldsymbol{T}$ (in dollars) is estimated by the following functions.

$$T(r) = \begin{cases} 0.01r & \text{if } r \le 5000 \\ 0.05r & \text{if } 5000 < r \le 20000 \\ 500 + 0.06r & \text{if r} > 20000 \end{cases}$$

Find $T(\mathbf{900}), \boldsymbol{T}(\mathbf{14000})$ and $\boldsymbol{T}(\mathbf{25,000})$.

Figure 8: Tax Collection

59. Speed Cameras: The cameras which are installed on the highways are used to monitor the traffic violations such as speed ($\boldsymbol{v}$ **km**) limit consequently helps to reduce the road accidents. In case, a driver crosses a certain speed limit then he/she has to pay the penalty by the following rules, where speed is represented by $\boldsymbol{v}$ and the amount by **A** (in Riyals Omani).

Figure 9: Speed Cameras

$$\mathrm{A}(v) = \begin{cases} 0 & if\ v < 140 \\ 8.6 + 0.01\, v & if\ 140 \le v < 150 \\ 9 + 0.04\, v & if\ 140 \le v < 150 \\ 15 + 0.195\, v & if\ 150 \le v < 180 \\ 50 + 0.28\, v & if\ v \ge 180 \end{cases}$$

Find how much speeding **fine** would be imposed or not if someone drive with the **speed** of 130 km, 150 km, 175 km,185 km and 160 km.

Remark: Note that the information in above question is **not real**. It is just an example to show the application of piecewise functions in the real life.

Composition of Functions

The operation of **two** functions $\boldsymbol{f}$ and $\boldsymbol{g}$ in such a way that generate a new function $\boldsymbol{h}$. Mathematically this type of operation is denoted by $\boldsymbol{f}\big(\boldsymbol{g}(\boldsymbol{x})\big) = \boldsymbol{h}(\boldsymbol{x})$. Symbolically we write $(\boldsymbol{f} \circ \boldsymbol{g})(\boldsymbol{x}) = \boldsymbol{h}(\boldsymbol{x})$, where '∘' means 'composition'. We read this as: $'f$ **composition** g'. In the study of composition of functions, one function is merged into another function and the result is a new function. we can compose two or more functions to create a new function. Two functions $\boldsymbol{f}$ *and* $\boldsymbol{g}$ are given (see Figure 10) then the composition between them is $\boldsymbol{f} \circ \boldsymbol{g}$ is defined by $(\boldsymbol{f} \circ \boldsymbol{g})(\boldsymbol{x}) = \boldsymbol{f}(\boldsymbol{g}(\boldsymbol{x}))$.

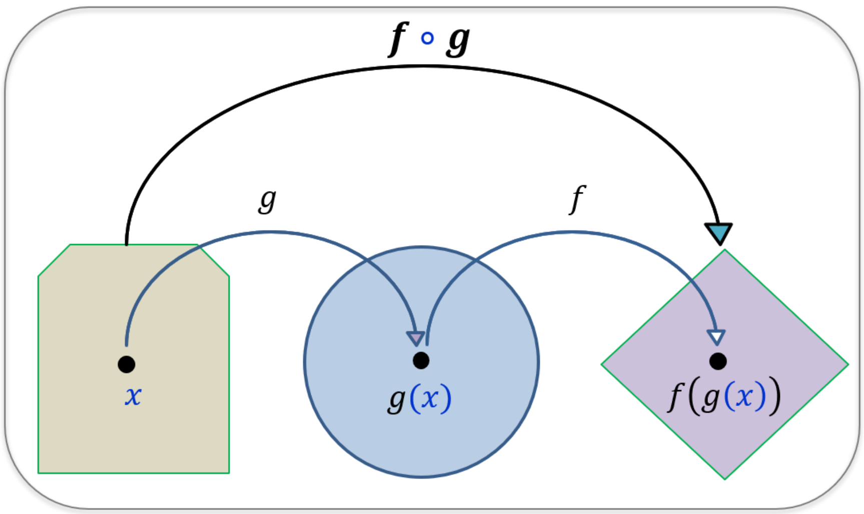


Figure 10: Composition of two functions

Example-1: Use $\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{3x - 5}$ and $\boldsymbol{g}(\boldsymbol{x}) = \boldsymbol{2 - x^2}$ to evaluate the expression.

(a). $f(g(0))$ (b). $g\big(f(0)\big)$

Solution: $\boldsymbol{f}\big(\boldsymbol{g}(\boldsymbol{0})\big) =?$

(a). $f\big(g(x)\big) = f(2 - x^2)$, but $x = 0$

$f\big(g(0)\big) = f(2 - (0)^2)$

$= f(2 - 0) = f(2)$, this means put $x = 2$ in $f(x) = 3x - 5$.

$= 3(2) - 5 = 6 - 5$

$f\big(g(0)\big) = 1$ ----------- **Answer**

SOLUTION: $\boldsymbol{g}\big(\boldsymbol{f}(\boldsymbol{0})\big) =?$

(b). $g\big(f(x)\big) = g(3x - 5)$, but $x = 0$

$g\big(f(0)\big) = g(3(0) - 5)$

$= g(0 - 5) = g(-5)$, this means put $x = -5$ in $g(x) = 2 - x^2$.

$= 2 - (-5)^2$

$= 2 - 25g \Longrightarrow \big(f(0)\big) = -23$ ---------- **Answer**

One – to – One Functions

A function with domain $\boldsymbol{A} = \{\boldsymbol{x_1}, \boldsymbol{x_2}, \boldsymbol{x_3}, \cdots, \boldsymbol{x_n}\}$, is called One−to−One function if **no two elements** of $\boldsymbol{A}$ have the **same** images in $\boldsymbol{B}$. Algebraically we write that

$$f(x_1) \neq f(x_2) \text{ whenever } x_1 \neq x_2.$$

It can also be defined as; a function is said to be one-to-one function if **no two different** elements of $\boldsymbol{A}$ have the **same** images in $\boldsymbol{B}$. Mathematically we say that

$$f(x_1) = f(x_2) \text{ whenever if } x_1 = x_2.$$

The Venn diagrams in (**Error! Reference source not found.** and **Error! Reference source not found.**), further explains the properties of one–one functions.

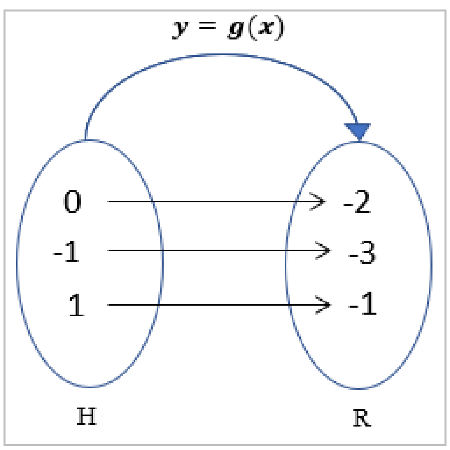


Figure 11: one-to-one function from H to R

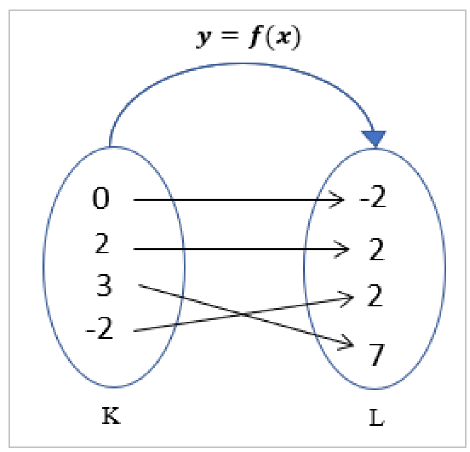


Figure 12: not one-to-one function K to L

To understand the definition of one-one function, we see that in (Figure 11) for example $\{0, -1\} \in H$ produces $\{-2, -3\} \in R$, such that $-2 \neq -3$ whenever $0 \neq -1$, hence "g" is one-to-one function. On the other hand, it is clear from (Figure 12) that $\{2, 3\} \in K$ produces $\{2, 2\} \in L$, such that $2 = 2$ but $2 \neq 3$, the function "f" is not one-to-one function.

Example-1: Determine whether the function $\boldsymbol{f(x) = 3x - 2}$ is One-to-One or **not**?

Solution: Suppose, $f(x_1) = f(x_2)$, then we have

$3x_1 - 2 = 3x_2 - 2$ (Cancel 2, from both sides)

$3x_1 = 3x_2$ (Dividing both sides by 3)

$x_1 = x_2$

Hence this function is $\boldsymbol{One - to - One}$ function.

Example-2: Determine whether the function $\boldsymbol{f(x) = 3x^2 - 2}$ is $\boldsymbol{One - to - One}$ or **not**?

Solution: Suppose, $f(x_1) = f(x_2)$, then we have

$3x_1^2 - 2 = 3x_2^2 - 2$ (Cancel 2, from both sides)

$3x_1^2 = 3x_2^2$ (Dividing both sides by 3)

$x_1^2 = x_2^2$

$x_1^2 - x_2^2 = 0$ (On factorizing we get)

$$(x_1 + x_2)(x_1 - x_2) = 0$$

This implies, $(x_1 + x_2) = 0$ OR $(x_1 - x_2) = 0$

$$x_1 = -x_2 \text{ OR } x_1 = x_2 .$$

Since this function does not produce a unique output $(x_1 = x_2)$, and hence this function is **NOT** $\boldsymbol{One{-}to{-}One}$ function.

The Horizontal Line Test

A curve in the coordinate plane is the "**graph** of a one-to-one **function**", if and only if "No Horizontal Line" intersects the curve more than once. The following diagram clarifies the meaning of horizontal line test in recognizing the graph of a one-to-one function.

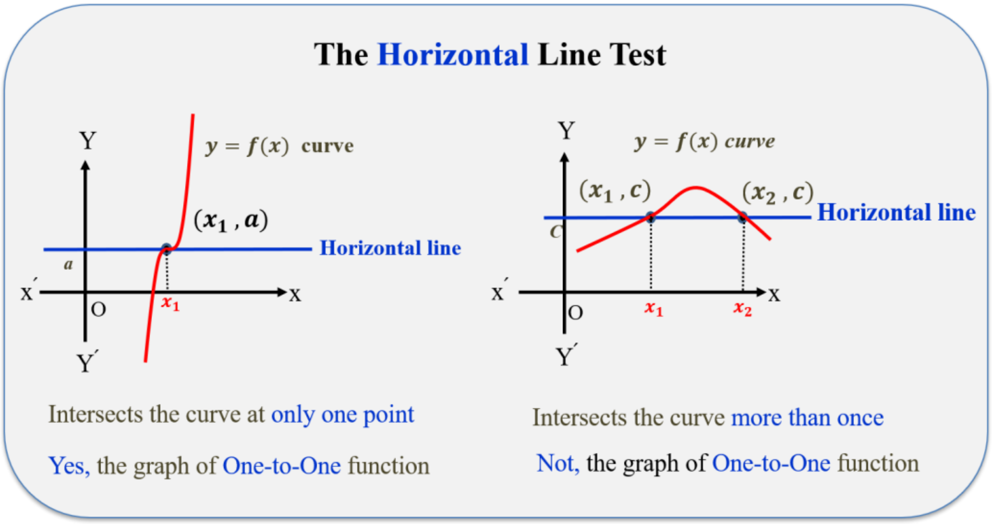


Figure 13: Identifying the graphs of one-one functions

Example-2: Identify whether the function $f(x) = 3x^2 - 2$ is One-to-One or **not**, by using the Graphing Calculator and Horizontal Line Test.

Solution: First, we use the GRAPHFREE to draw the graph of the given function and using the Horizontal line to see whether the curve is intersected only once or more than once.

In (**Error! Reference source not found.**), we see that the Horizontal Line intersects the graph of a given function at more than once. So, by Horizontal line test we understand that this is not the graph of a one-to-one function.

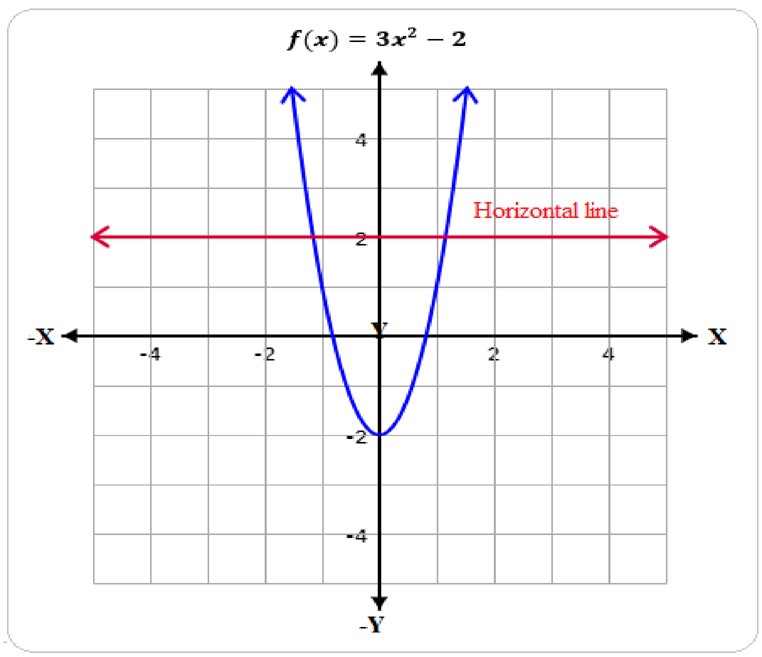


Figure 14: Not one-one function

The Inverse of a Function

Let f be a function with **domain** A and **range** B. Then the "Inverse function $f^{-1}(x)$ *has domain* B *and range* A *defined as*;

$$y = f(x) \Leftrightarrow f^{-1}(y) = x \text{ for all } x \in A \text{ and } y \in B$$

The Inverse Function Property: Two functions are said to be inverse to each other if they satisfy the following properties.

$f^{-1}(f(x)) = x$: for every $x \in A$ if and only if $f(f^{-1}(x)) = x$: for every $x \in B$.

For further intuition one can study the (Figure 15).

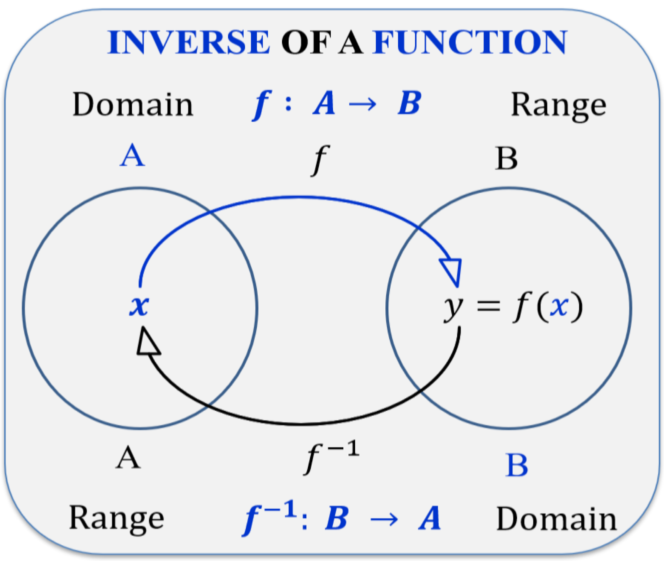


Figure 15: The Inverse of one-one function

Example-1: Find the **Inverse** of the function $\boldsymbol{f(x) = 2x - 3}$?

Solution: Here we need to find $\boldsymbol{f^{-1}(x)}$.The rule is: $y = f(x) \Leftrightarrow f^{-1}(y) = x$.
Given that $\boldsymbol{f(x) = 2x - 3}$, therefore we can write

$$y = 2x - 3, \quad \text{(find '}x\text{' from here we get)}$$

$$x = \frac{y+3}{2}$$

$$f^{-1}(y) = \frac{y+3}{2}, \quad \text{(replacing } y \text{ by } x\text{, we get)}$$

$$f^{-1}(x) = \frac{x+3}{2} \text{ ------ Answer}$$

Example-2: Show that the following functions $\boldsymbol{f}$ and $\boldsymbol{g}$ are **inverses** to each other.

$$\text{where } f(x) = 2x - 3 \text{ and } g(x) = \frac{\boldsymbol{x+3}}{\boldsymbol{2}}.$$

Proof: Two **functions** are **inverses** to each other if the following property holds.

$$f\big(g(x)\big) = x = g(f(x)).$$

Therefore, we consider

$$f\big(g(x)\big) = f(\tfrac{x+3}{2})$$
$$f\big(g(x)\big) = 2\left(\frac{x+3}{2}\right) - 3$$
$$= (x+3) - 3$$
$$f\big(g(x)\big) = x \text{ ----- } (1)$$

Now consider,

$$g\big(f(x)\big) = g(2x-3)$$
$$= \frac{(2x-3)+3}{2}$$
$$= \frac{2x-3+3}{2}$$
$$= \frac{2x}{2}$$
$$g\big(f(x)\big) = x \text{ ------ } (2)$$

It is obvious from (1) and (2) that $f\big(g(x)\big) = x = g(f(x))$. Thus $\boldsymbol{f}$ and $\boldsymbol{g}$ are inverses to each other.

Graphical representation of the Inverse functions

Graphically two inverse functions must intersect at one point. For both functions the intercepts and all points interchange to each other such that (x, y) becomes (y, x). For example, by using GRAPHFREE one can test the functions: $f(t) = 2t^3 - 1$ and $g(x) = \sqrt[3]{\frac{t+1}{2}}$. Both graphs must interest at a certain point see (Figure 16).

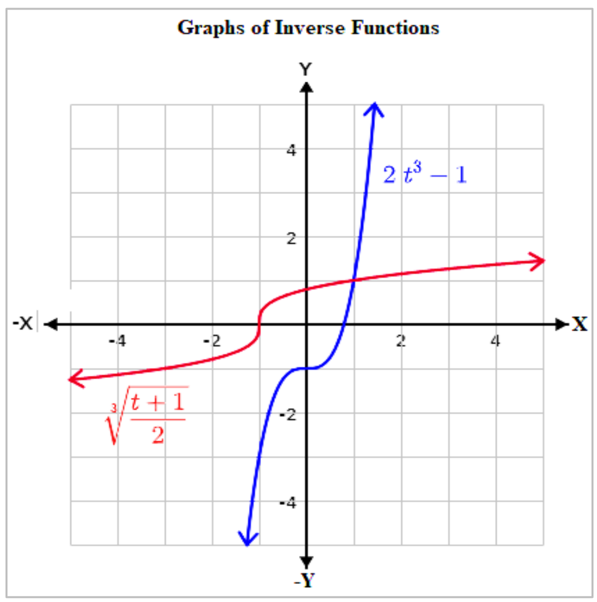


Figure 16: Graphs of Inverse functions

Exercise –1.2

1. Use the functions $\boldsymbol{f(x) = 3x - 4}$ and $\boldsymbol{g(x) = x^2 + 5}$ to determine the following **compositions**.

(a). $f(g(x))$ *(b).* $g(f(x))$

(c). $f(g(0))$ *(d).* $g(f(0))$

(e). $f(g(-2))$ *(f).* $g(f(-2))$

(g). $(f \circ g)_{(4)}$ *(h).* $(g \circ f)_{(-4)}$

(i). $(f \circ g)_{(\frac{1}{2})}$ *(j).* $(g \circ f)_{(-\frac{1}{3})}$

(k). $f(g(x+2))$ *(l).* $g(f(3-x))$

2. Use the functions $\boldsymbol{f(x) = \frac{3x+5}{2-x}}$ and $\boldsymbol{g(x) = x^2 + 2}$ to determine the **compositions**.

(a). $f(g(x))$ *(b).* $g(f(x))$

(c). $f(g(5))$ *(d).* $g(f(-3))$

(e). $f(g(0))$ *(f).* $g(f(1))$

(g). $f(g(3x))$ *(h).* $g(f\left(\frac{1}{x}\right))$

3. **Verify** whether the following functions are $\boldsymbol{one - to - one}$?

(a). $f(x) = \frac{3x+5}{2-x}$ *(b).* $g(x) = x^2 + 2$

(c). $t(x) = 4x^3 - 7$ *(d).* $h(t) = 2x^4 + 3$

(e). $d(x) = -5x + 3$ *(f).* $k(t) = 2t^2 - 5$

4. **Determine** the **Inverse** of the following functions?

(a). $f(x) = \frac{3x+5}{2-x}$ *(b).* $g(x) = x^2 + 2$

(c). $t(x) = x^2 - 2x + 1$ *(d).* $h(t) = 2x^4 + 6$

(e). $d(x) = -5x + 3$ *(f).* $k(t) = 2t^2 - 4$

5. **Show** that the following functions are **Inverses** to each other?

(a). $f(x) = \frac{x+3}{4}$ and $g(x) = 4x - 3$ *(b).* $g(x) = x^2 + 2$ and $m(x) = \sqrt{x-2}$

(c). $r(t) = 2t^3 - 1$ and $k(t) = \sqrt[3]{\frac{t+1}{2}}$ *(d).* $h(x) = \frac{3x-5}{x+1}$ and $s(x) = \frac{x+5}{3-x}$

(e). $g(x) = \frac{3-x}{5}$ and $d(x) = -5x + 3$ *(f).* $k(t) = 2t^2 - 4$ and $f(x) = \frac{t}{2} + 8$

6. Given that $\boldsymbol{f(x) = \sqrt{\frac{3x^2-5}{3}}}$, then find $\boldsymbol{f^{-1}(x)}$?

7. Given that $\boldsymbol{g(x) = \frac{3}{x+7}}$, then find $\boldsymbol{g^{-1}(x)}$?

8. Given that $\boldsymbol{k(x) = -12x + 6}$, then find $\boldsymbol{k^{-1}(x)}$?

9. Using the concept of **Horizontal test**, determine whether the following graphs are the graphs of $\boldsymbol{one - to - one}$ functions? see (Figure 17) and (Figure 18).

(a).

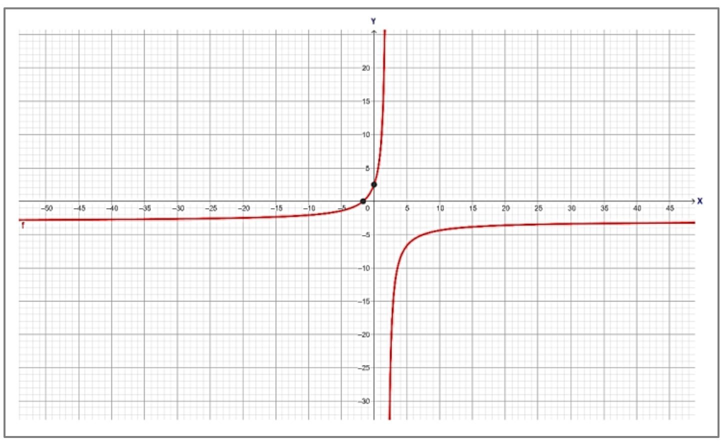


Figure 17: one-one or not

(b).

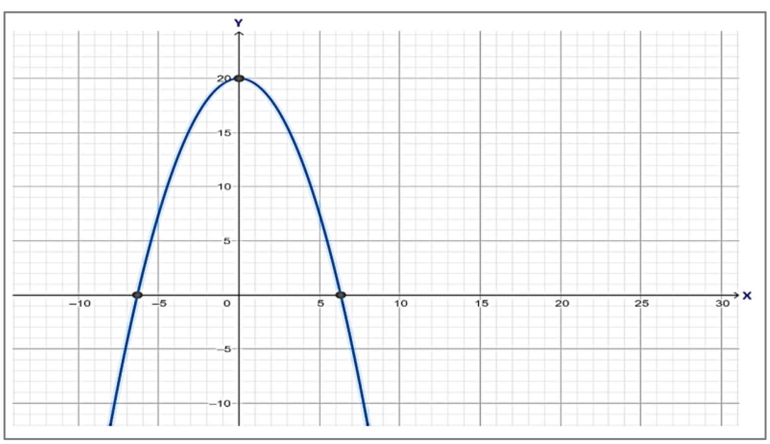


Figure 18: one-one or not

Graph of a Function in a Coordinate Plane

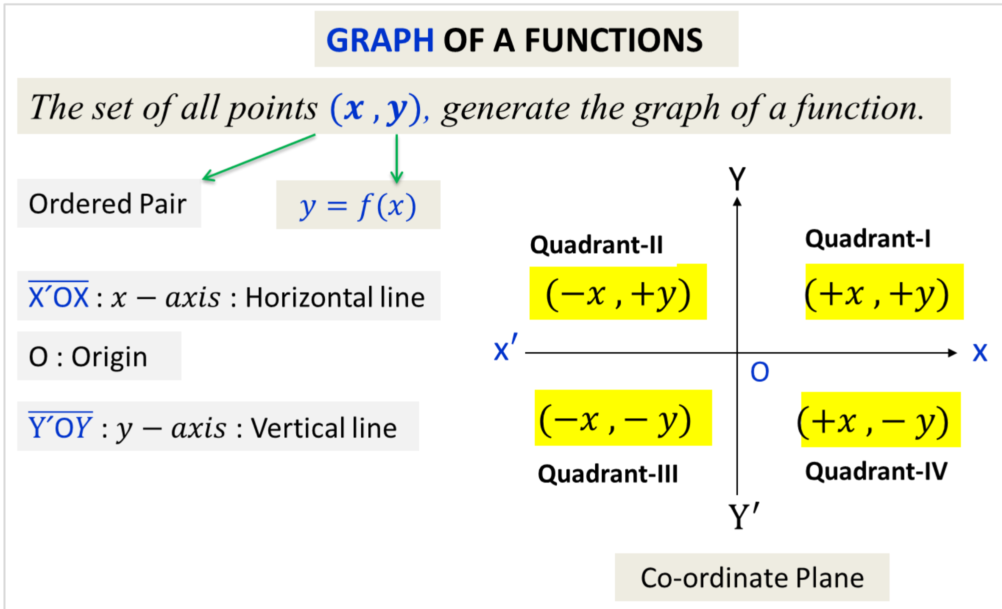


Figure 19: A Coordinate Plane

Example – 1: Plot the points in the coordinate plane $A(4, 6), B(-4, -6), C(-4, 6), D(4, -6)$.

Solution: Points are placed according to signs of an ordered pair $(\boldsymbol{x}, \boldsymbol{y})$. For this purpose one may refer to (Figure 19).

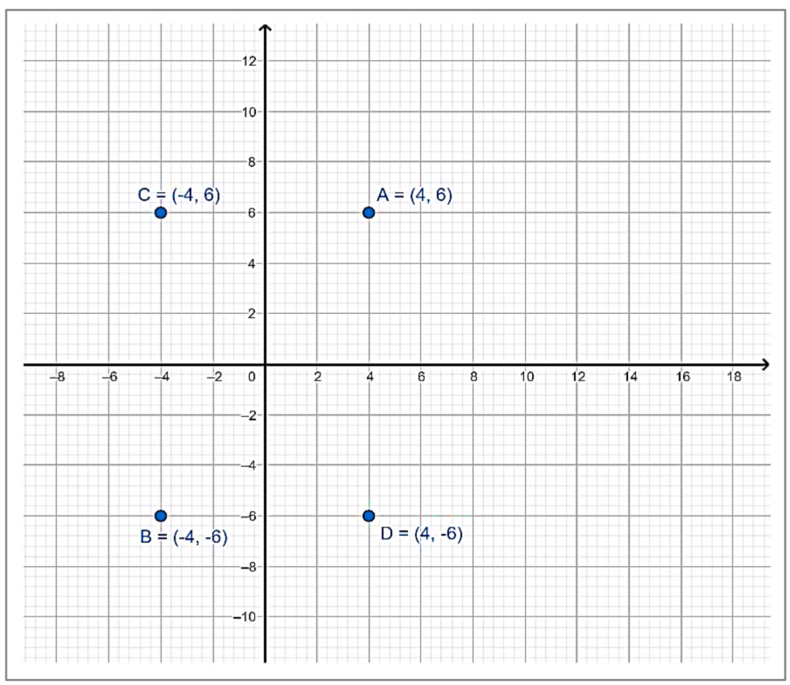


Figure 20: Plotting the points in the Plane

Example – 1: Draw the graph of $\boldsymbol{f(x) = x^2 - 3}$ by making table values?

Solution: Since, $y = f(x)$ so, putting different values for $\boldsymbol{x}$ we get the values of $\boldsymbol{y}$.

$\boldsymbol{x}$	$\boldsymbol{y = f(x)}$
-2	1
-1	-2
0	-3
1	-2
2	1
3	6
$\vdots$	$\vdots$

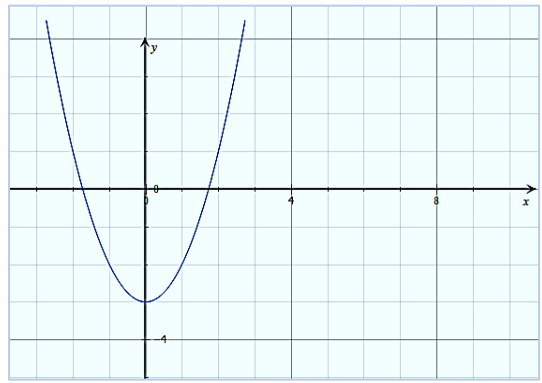


Figure 21: Graph of a Quadratic function

The Vertical Line Test

A curve in the coordinate plane is the "**graph** of a **function**", if and only if "No Vertical Line" intersects the curve more than once.
The diagram in (Figure 22) clarifies the meaning of vertical line test in recognizing for being the graph of a function or not.

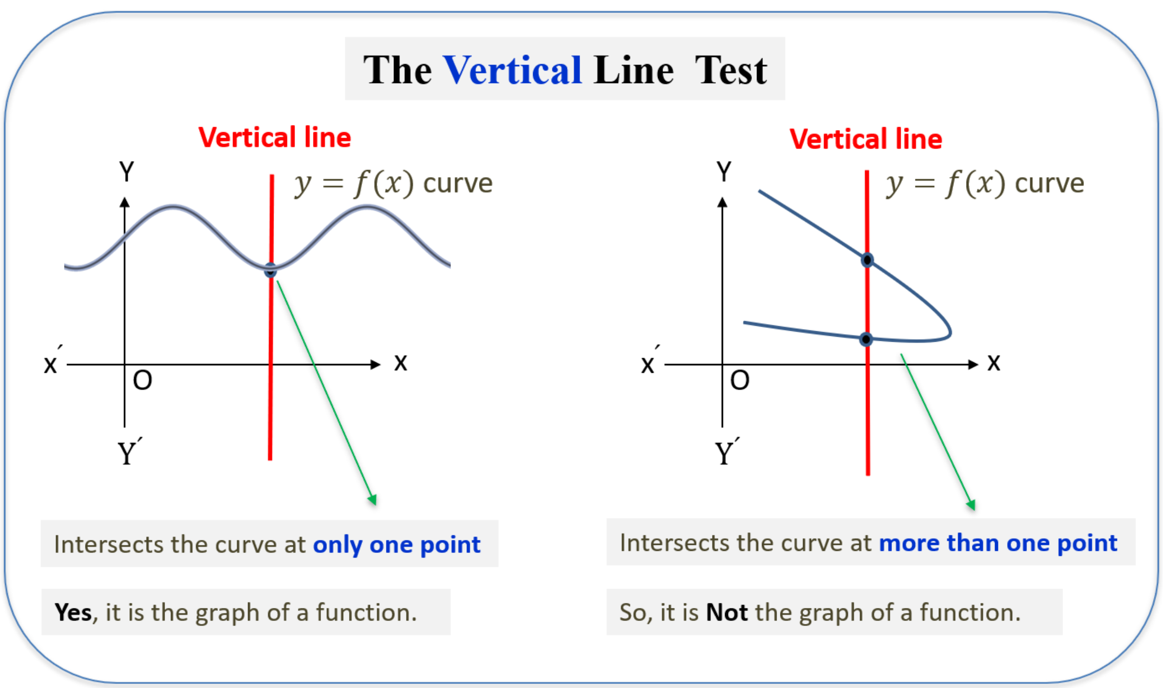


Figure 22: The Vertical Line Test

Standard (or Parent) Functions their Graphs, Domains, and Ranges

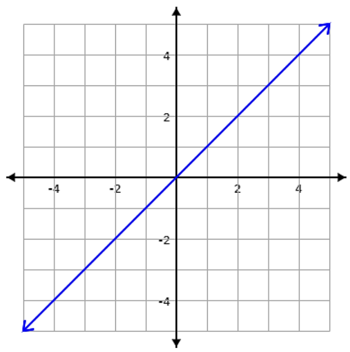
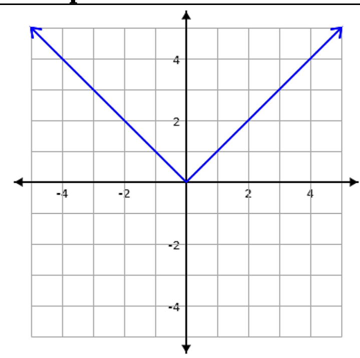
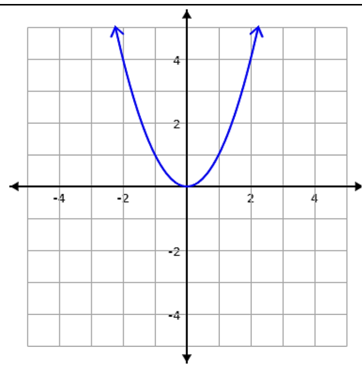
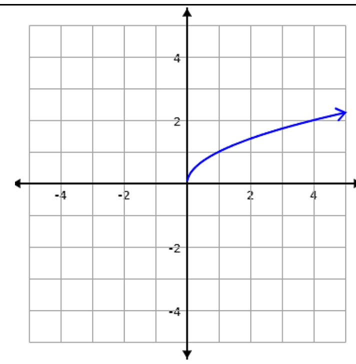
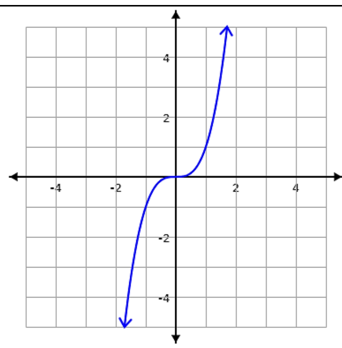
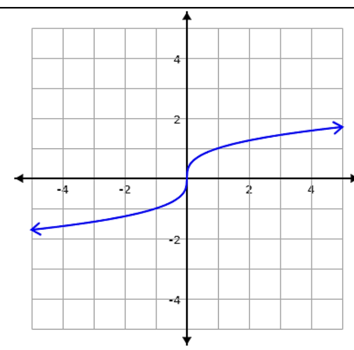
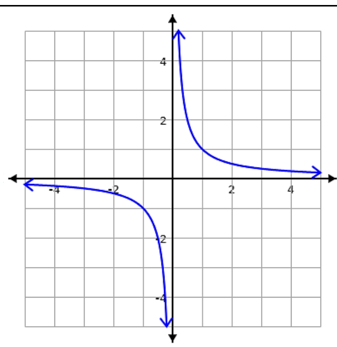
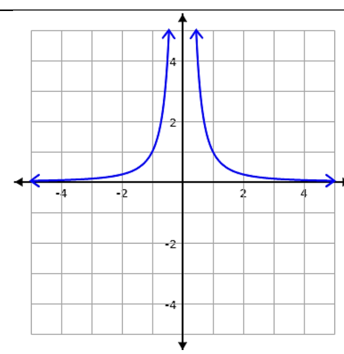
Standard Functions	Graphs	Standard Functions	Graphs
$y = f(x) = x$ Linear and Odd function Domain: $(-\infty, \infty)$ Range: $(-\infty, \infty)$		$y = f(x) = \lvert x \rvert$ Absolute value and Even function Domain: $(-\infty, \infty)$ Range: $(0, \infty)$	
$y = f(x) = x^2$ Quadratic and Even function Domain: $(-\infty, \infty)$ Range: $[0, \infty)$		$y = f(x) = \sqrt{x}$ Radical, and neither Odd nor Even function Domain: $[0, \infty)$ Range: $(0, \infty)$	
$y = f(x) = x^3$ Cubic and Odd function Domain: $(-\infty, \infty)$ Range: $[0(-\infty, \infty)$		$y = f(x) = \sqrt[3]{x}$ Cube root and Odd function Domain: $(-\infty, \infty)$ Range: $(-\infty, \infty)$	
$y = f(x) = \dfrac{1}{x}$ Rational (Inverse) and Odd function Domain: $(-\infty, 0) \cup (0, \infty)$ Range: $(-\infty, 0) \cup (0, \infty)$		$y = f(x) = \dfrac{1}{x^2}$ Rational (Inverse) and Even function Domain: $(-\infty, 0) \cup (0, \infty)$ Range: $(0, \infty)$	

Figure 23: Standard functions and their Graphs

Transformation of Graphs

Normally the graph transformation or shifting takes place by applying the following rules.

a) $y = f(x)$: Original Graph of curve or any graph

b) $y = f(x) + c$: Vertical Shift, **c** units upward

c) $y = f(x) - c$: Vertical Shift, **c** units downward

d) $y = f(x - c)$: Horizontal Shift, **c** units to the right

e) $y = f(x + c)$: Horizontal Shift, **c** units to the left

f) $y = -f(x)$: Reflect about $\boldsymbol{x - axis}$

g) $y = f(-x)$: Reflect about $\boldsymbol{y - axis}$

h) $y = cf(x)$: c > 1 : Vertically stretch away from $\boldsymbol{x - axis}$ by a factor **c**

i) $y = cf(x)$: 0 < c < 1 : Vertically shrink toward from $\boldsymbol{x - axis}$ by a factor **c**

j) $y = f(cx)$: c > 1 : Horizontally shrink the graph by a factor **1/c**

k) $y = f(cx)$: 0 < c < 1 : Horizontally stretch the graph by a factor **1/c**

The graphs shown in (Figure 24) shows the **horizontal shrink** and **horizontal stretch**, the second one shows the **vertical stretch** and **vertical shrink**, the third graph shows the reflection about $\boldsymbol{y - axis}$ while the fourth graph shows the reflection about $\boldsymbol{x - axis}$, chose from (Tsishchanka, n.d.) and (Hisema01, 2014).

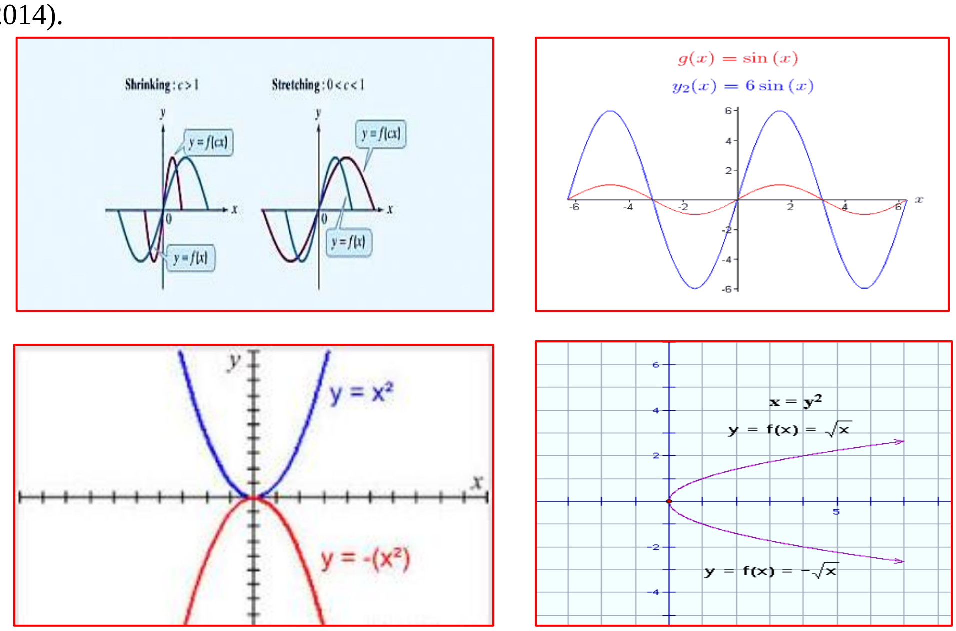


Figure 24: Graph Transformations

Example – 1: Draw the graph of the following functions by using the $\boldsymbol{f(x) = x^2}$.

a) $\boldsymbol{y = x^2 + 2}$

b) $\boldsymbol{y = x^2 - 2}$

Solution: Vertical shift of 2 units

a) Vertical shift 2 units upward

b) Vertical shift 2 units downward

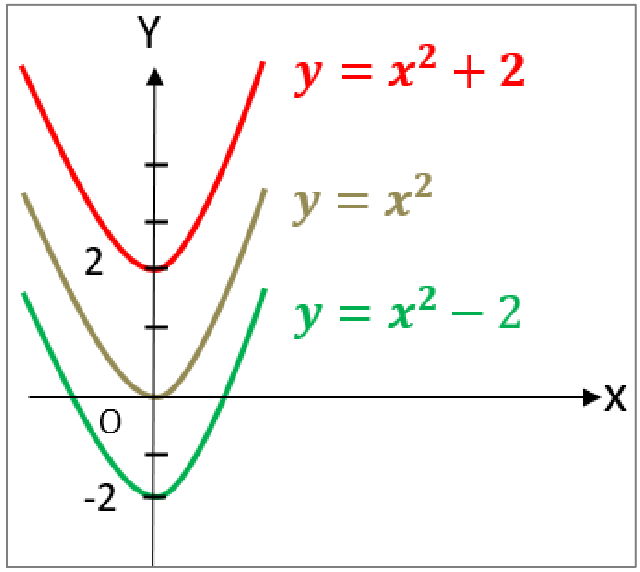


Figure 25: Vertical transformation of a graph

Example – 2: Draw the graph of $y = -x^2$, (Use the graph of $f(x) = \boldsymbol{x^2}$)**?**

Solution: Here negative sign shows the **Reflection** and it is a **vertical reflection**.

The $\boldsymbol{y - axis}$ works like a **mirror** (Reflection) as shown in (Figure 26).

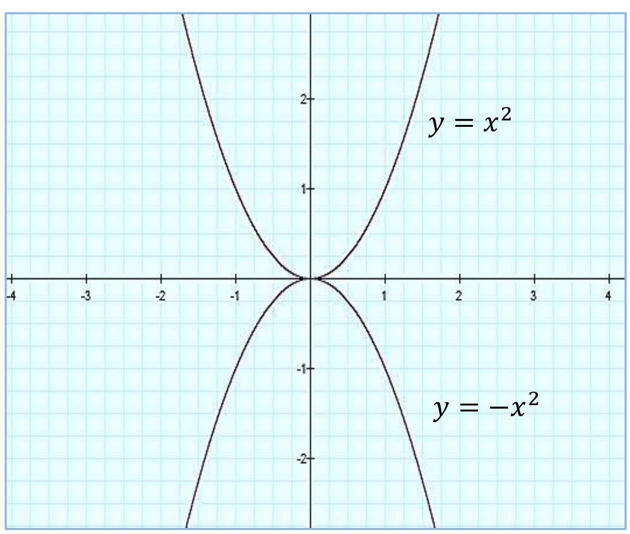


Figure 26: Reflection of a function
Source: (Andy, 2009)

Exercise – 1.3

Match the following functions with its **graph**.

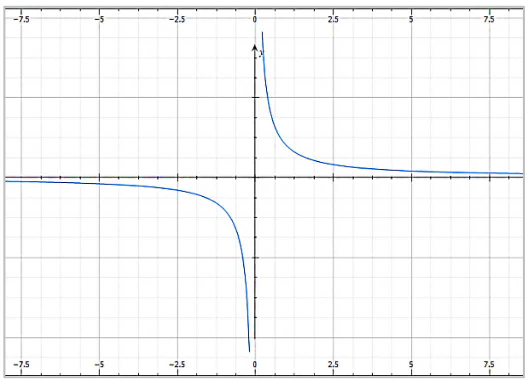
a) $S(x) = x^2$

b) $h(r) = r^3$

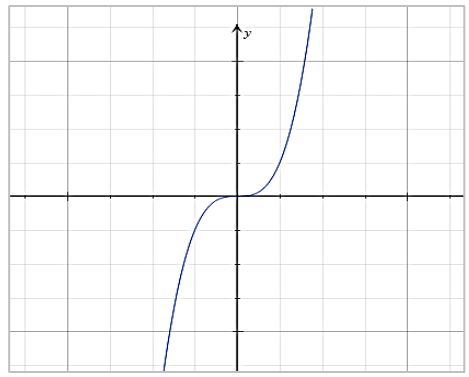
c) $A(t) = \sqrt{t}$

d) $r(x) = \frac{1}{x}$

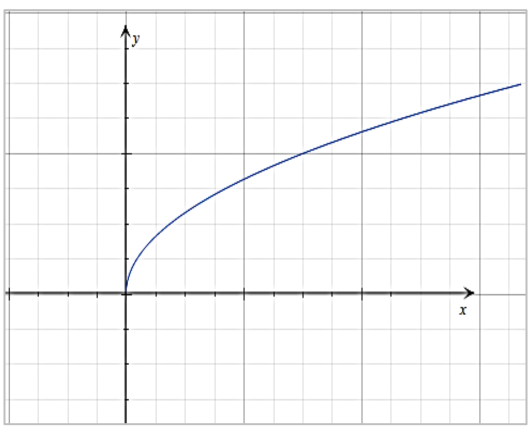
1.



2.



3.



4.

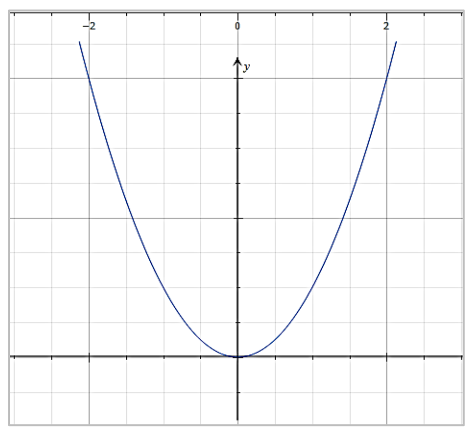


Figure 27: Graphs of different Functions
Source: (Andy, 2009)

Draw the **graph** of following by preparing values **in the table**.

5. $f(x) = -4$
6. $n(x) = 3x - 4$
7. $g(x) = x^2 + 3$
8. $k(x) = (x-2)^2$
9. $r(x) = |x+3|$
10. $t(r) = \frac{1}{r+1}$

11. $h(r) = \sqrt{r+2}$

12. $w(t) = -\sqrt{t+1}$

13. $s(t) = t^3 - 7$

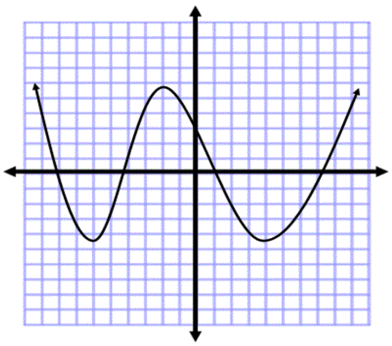
14. $t(x) = -x - 5$

15. $\mathrm{h}(x) = 4x - 5$

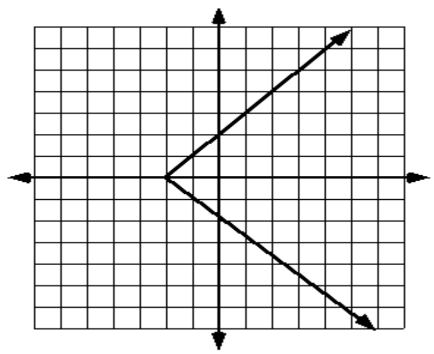
16. $h(s) = 8 - s^2$

Use the concept of **vertical Line test** to identify whether the curve is the graph of the function.

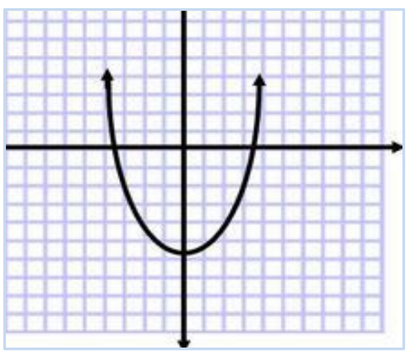
17.



18.



19.



20.

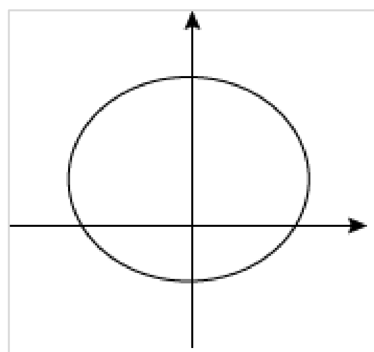


Figure 28: Different Graphs for being a Function or Not a Function
Source: **(Google, n.d.)**

Match the **function** upon its graph given.

21. $y = |x| - 1$

22. $y = -x^2$

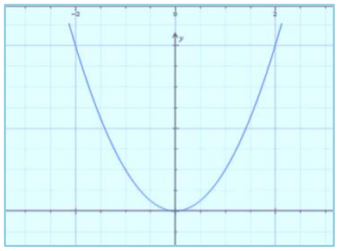
23. $y = (x-1)^2$

24. $y = |x| + 2$

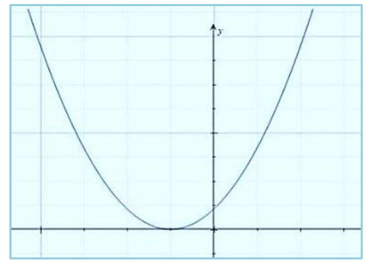
25. $y = (x+1)^2$

26. $y = x^2$

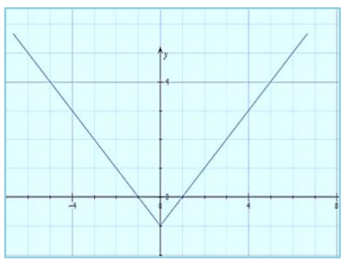
a)



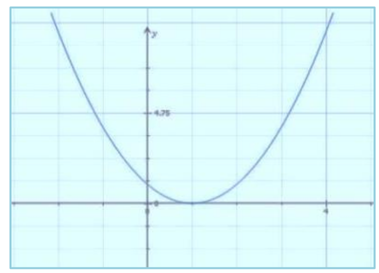
b)



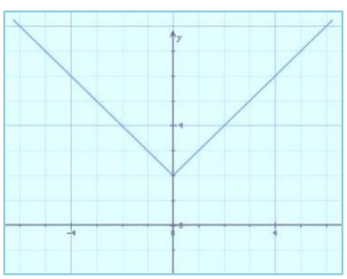
c)



d)



e)



f)

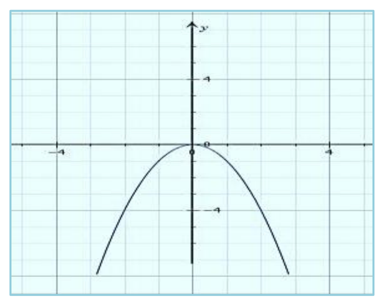


Figure 29: Different Graphs of Functions

Source: **(Andy, 2009)**

Draw the **graph** by using the **standard** (parent) **graphs** and **transformations**.

27. $s(x) = \sqrt{x}$-2

28. $h(t) = \sqrt{t-3}$

29. $d(x) = -x^3$

30. $r(x) = x^3 + 4$

31. $r(t) = |t+4| - 3$

32. $g(x) = \sqrt[3]{x} + 4$

Write the **equation** of the following **graph** according to the information given.

33. $f(x) = \sqrt{x}$; shift upward 2 units

34. $f(x) = x^2$; shift 3 units to right

35. $f(x) = x^3$; shift 1 unit to the left

36. $f(x) = x^3$; 3 units to the right, and then up 5 units.

37. $f(x) = |x|$; shift down 4 units, then 2 units to the right.

38. $f(x) = \sqrt[3]{x}$; reflect about $x - axis$.

39. $f(x) = \sqrt[3]{x}$; reflect about $y - axis$.

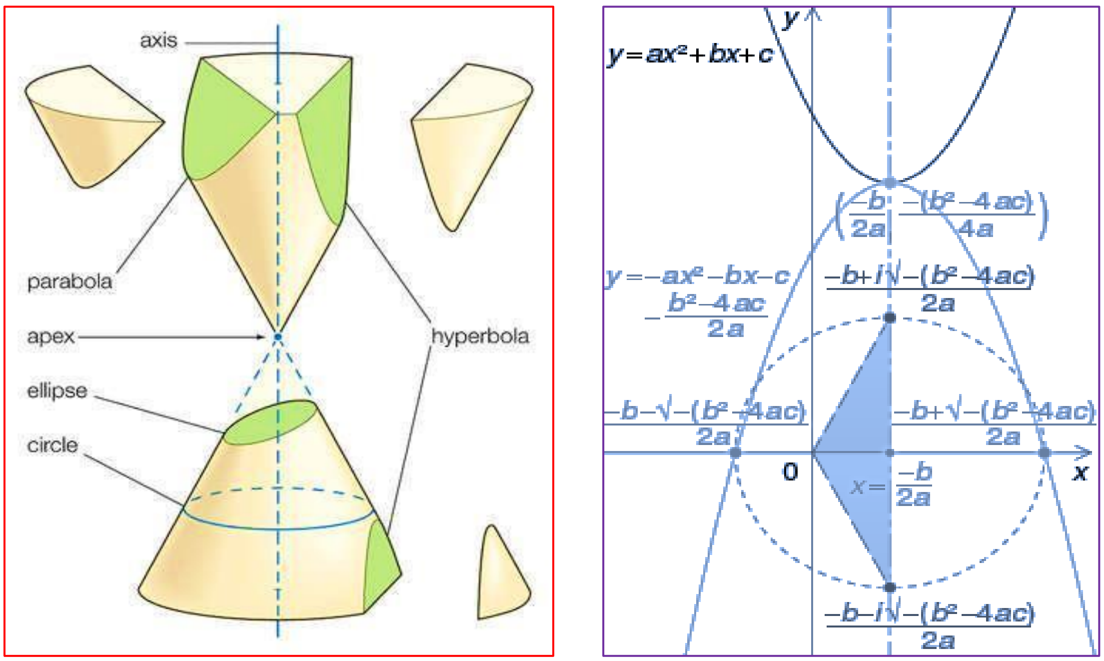
40. $k(x) = \frac{1}{x}$; shift 4 units to the left, then shift 5 units downward.

41. Draw the **graphs** of above questions (**33 – 40**) by using the graph paper?

CHAPTER – 2 THE QUADRATIC FUNCTION AND ITS APPLICATIONS

OBJECTIVES: Upon successful completion of this topic, students will be able to

- ***Find the zeros and the maximum or minimum of a quadratic function.***
- ***Find Solution of related problems, including those arising from real world applications.***
- ***Drawing the graphs of a quadratic function.***
- ***Solve Quadratic Inequalities***



(a) Parabola in Cone *(b) Quadratic Equation*

Figure 30: Parabola and Quadratic Equation

Figure (a) Source: (Pinterest, n.d.) and Figure (b) Source: (Commons, n.d.)

What is a Quadratic Function?

General Form of a Quadratic Function: $\boldsymbol{f(x) = ax^2 + bx + c}$, where $\boldsymbol{a \neq 0}$ and $\boldsymbol{x}$ =variable.

Quadratic Equation: $ax^2 + bx + c = 0$, is a general form of quadratic equation, where $a \neq 0$ and x =variable

Quadratic Formula: If $ax^2 + bx + c = 0$is a quadratic equation, the solution of quadratic equation "$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$"

Example: The following are few examples of quadratic functions. $y = 7x^2 - 5x + 10$, $y = x^2 - 3$, $y = -\frac{1}{5}x^2 + 3x$

Graph: The graph of a quadratic function is known as Parabola.

$y = f(x) = ax^2 + bx + c$ is a parabola.

1. Parabola open upward if $a > 0$ and Parabola open downward If $a < 0$
2. The Vertex is the turning point of the parabola.

   Vertex = $\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$
3. y – intercept = c
4. If $a < 0$, Maximum Value of the parabola = $f\left(-\frac{b}{2a}\right)$

   if $a > 0$, Minimum Value of the parabola = $f\left(-\frac{b}{2a}\right)$
5. The axis of the symmetry is the vertical line that intersects the parabola at the Vertex. This can be found by: $\boldsymbol{x = -\frac{b}{2a}}$.
6. Graph of Quadratic function (parabola) is symmetric about vertical line.
7. Domain =$(-\infty, \infty)$ or set of all real no.
8. Range = $\left[f\left(\frac{-b}{2a}\right), \infty\right)$, when $a > 0$, $\left(-\infty, f\left(\frac{-b}{2a}\right)\right]$, when$a < 0$
9. Total Revenue =(Price)× (Quantity), implies $r = p \times q$

   where p =price per unit when $q$units are demanded (per week) by consumers.

Vertex of a Quadratic Function (or a Parabola)

The graph produced by a quadratic function is known as Parabola. Each parabola has an extreme point from where the shape of parabola opens in upward direction or in downward direction. In fact, vertex is a distant point (x, y) on or away from the origin. In below (Figure 31) the point $A(0,0), B(5,2), C(-3,-2), D(5,-2)$ and $E(-5,2)$ are the vertices of these parabolas.

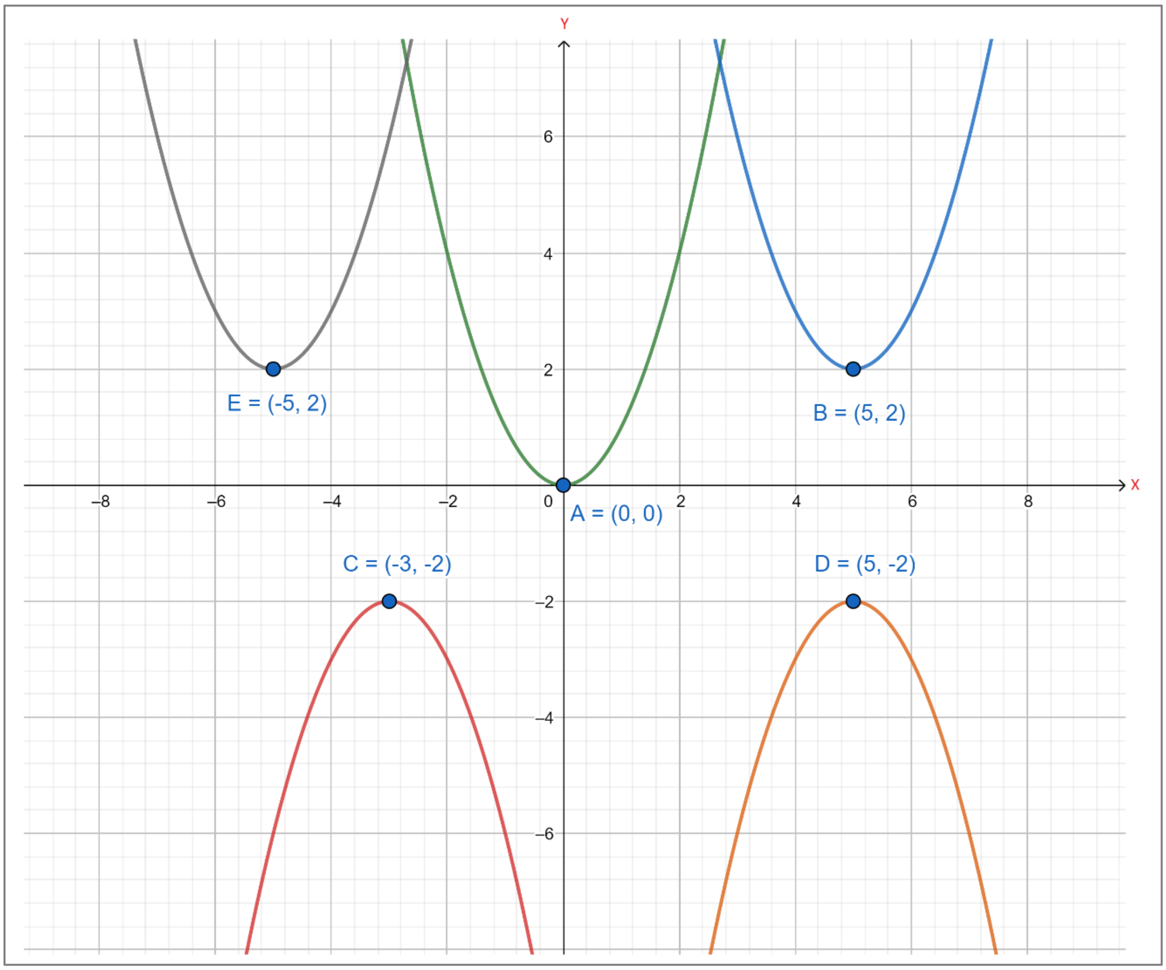


Figure 31: Vertex of a Parabola

The vertex of a Parabola can be found by the formula: $Vertex = (x, y) = \left(-\frac{b}{2a}, f\left(-\frac{b}{2z}\right)\right)$.

Example – 1. Determine the vertex of $g(x) = x^2 + 4x + 5$.

Solution: in the given function $a = 1, b = 4$ and $c = 5$. Here $a > 0$, so the parabola opens upward.

$$x = -\frac{b}{2a} = -\frac{4}{2(1)} = -\frac{4}{2} = -2$$

$$y = g\left(-\frac{b}{2a}\right) = g(-2) = (-2)^2 + 4(-2) + 5 = 1 \text{ So, vertex} = (-2,1).$$

The $x-$ and $y-intercept(s)$ of a Parabola

Geometrically intercept actually means, where a graph of a certain function intersects the $x - axis$ and $y - axis$. See for example in the following graphs (Figure 32) x-intercepts= $\{-3,1\}$ and y-intercept:$y = \{-3\}$.

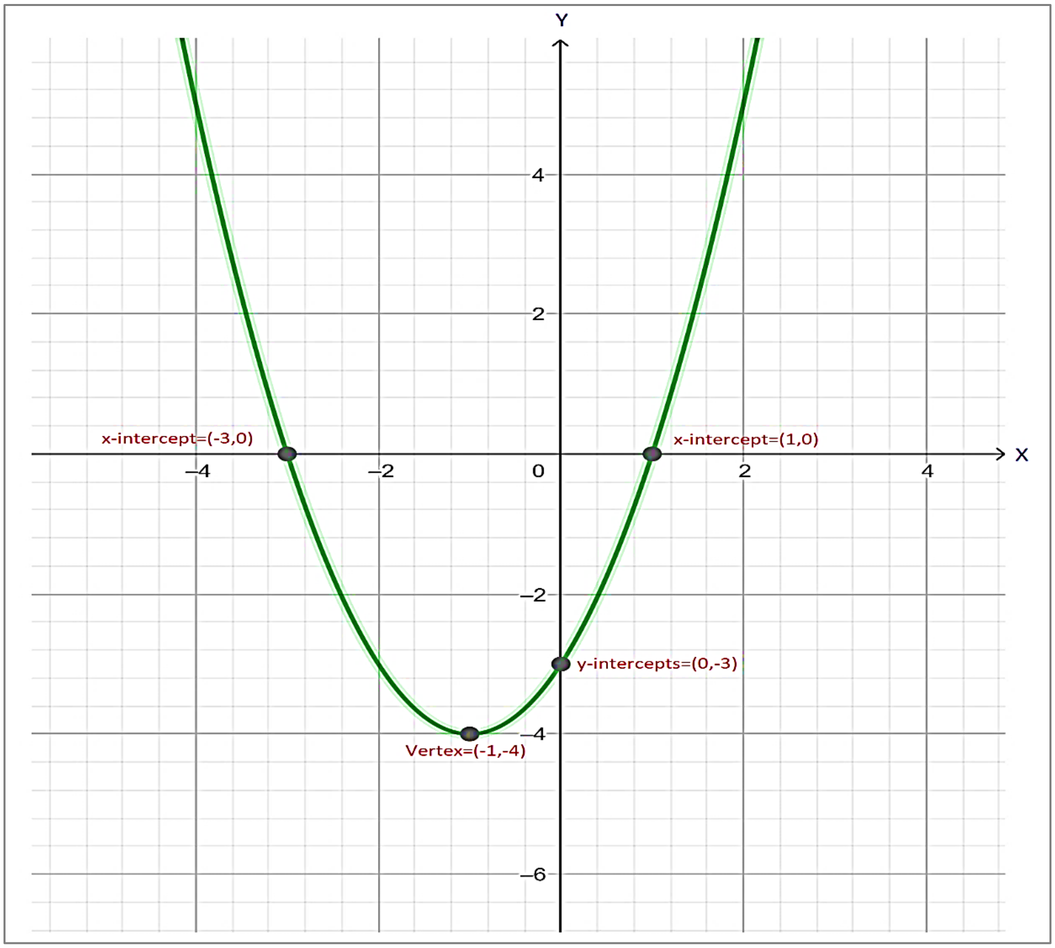


Figure 32: Intercept(s) of a Parabola

Example – 2. Determine x & y −intercept $g(x) = -x^2 + 7x$

Solution: For x −intercept: $g(x) = 0$

$$-x^2 + 7x = 0$$

$$x(-x + 7) = 0 \Rightarrow x = 0, x = 7$$

$$y - Intercept: g(0) = -(0)^2 + 7(0) = 0$$

Thus x −intercept = $(0,0)$ and $(7,0)$, y −intercept =$(0,0)$.

The Maxima or Minima of a Parabola

The maxima or minima of a quadratic function (or Parabola) depends on the coefficient of x^2 in a standard quadratic function ($f(x) = ax^2 + bx + c$). The value of $y = f(x)$ represents the value of a maxima or minima. Formula to find the maximum or minimum value of a quadratic functions is given by:

$$y = ax^2 + bx + c = \begin{cases} If\ \boldsymbol{a} > \mathbf{0}\ then\ the\ function\ has\ a\ \boldsymbol{Minimum}\ value\ at\ x = -\dfrac{b}{2a} \\ If\ \boldsymbol{a} < \mathbf{0}\ then\ the\ function\ has\ a\ \boldsymbol{Maximum}\ value\ at\ x = -\dfrac{b}{2a} \end{cases}$$

Note that the coefficient of $x^2 = a$, coefficient of $x = b$, and $y = f\left(-\frac{b}{2a}\right)$.

Example – 1. Find maximum or minimum value of $f(x) = x^2 - 2x + 6$ and write $f(x)$ has a maximum or minimum value.

Solution: Here $a = 1, b = -2$ and $c = 6$

Since $a = 1 > 0$, the function has the minimum value.

minimum value $= f\left(-\frac{b}{2a}\right)$

$$= f\left(-\frac{(-2)}{2(1)}\right) = f\left(\frac{2}{2}\right) = f(1) = (1)^2 - 2(1) + 6 = 1 - 2 + 6 = 5.$$

Domain and Range of a Quadratic Function

Domain: The domain of a function is the set of all real values of x, that gives a well-defined value $y \in \mathbb{R}$. The domain of a quadratic function is the set of all real numbers.

$$Dom\ f = x \in \mathbb{R} = (-\infty, \infty)$$

Range: The range $y \in$ R of a function is the set of all real values of y, that is produced by plugging a real number into x. The range of a quadratic function depends on the coefficient of x^2 in a standard quadratic function ($f(x) = ax^2 + bx + c$). Formula to find the range of a quadratic functions is given by:

$$Range\ f = \begin{cases} \left[f\left(-\dfrac{b}{2a}\right), \infty\right)\ if\ a > 0 \\ \left(-\infty, f\left(-\dfrac{b}{2a}\right)\right]\ if\ a < 0 \end{cases}$$

Note that the coefficient of $x^2 = a$, coefficient of $x = b$.

Example – 1: The graph of quadratic function $y = x^2$, is given below. Find the **domain** and **range** of the function?

Solution: Domain $f = x \in \mathbb{R} = x \in (-\infty, \infty)$ ---- Answer

Range $f = y \geq 0 = y \in [0, \infty)$ ---- Answer

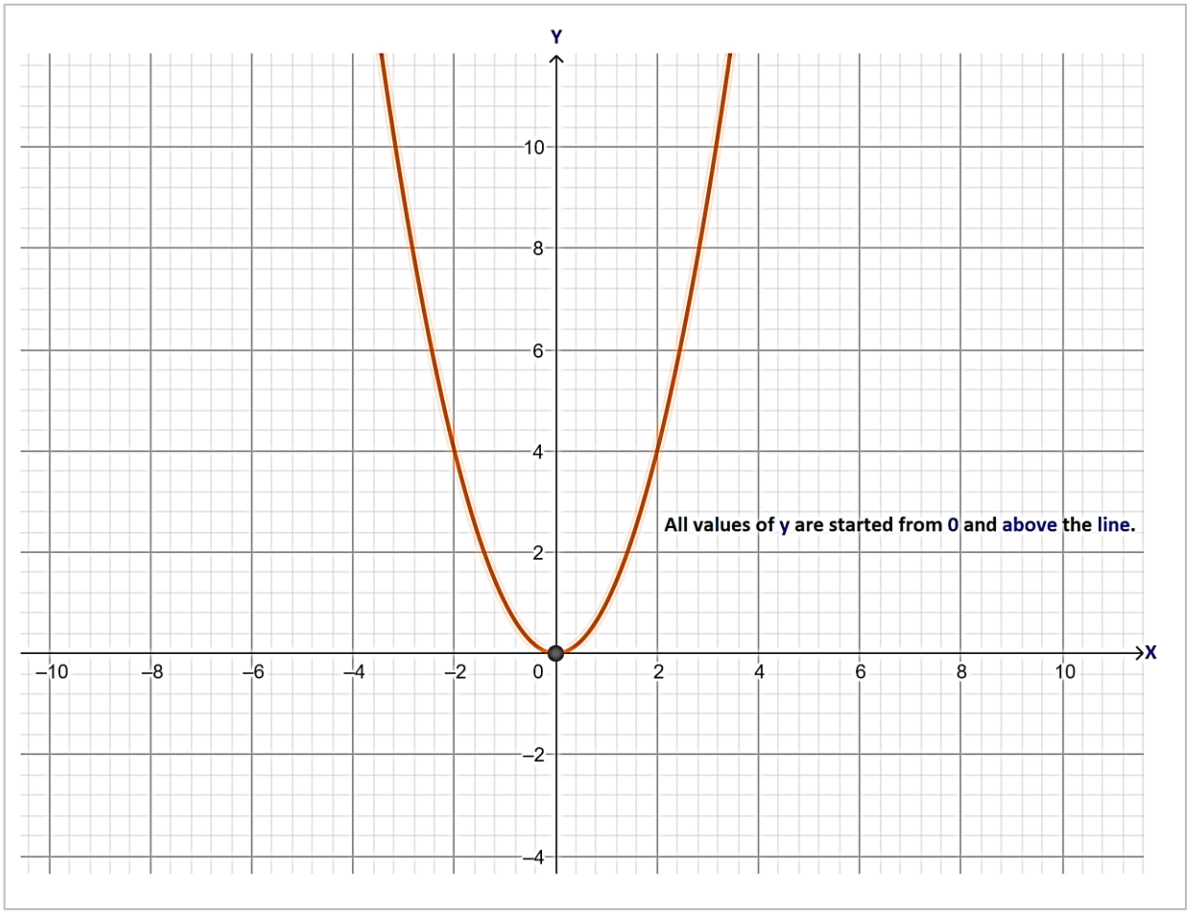


Figure 33: Dom and Range of a Quadratic function

Example – 2: Find the **domain** and **range** of the function, $y = g(x) = -x^2 + 5$, by studying the graph of the function.

Solution: Domain $f = x \in (-\infty, \infty)$ and Range $f = y \leq 5 = [-\infty, 5]$.

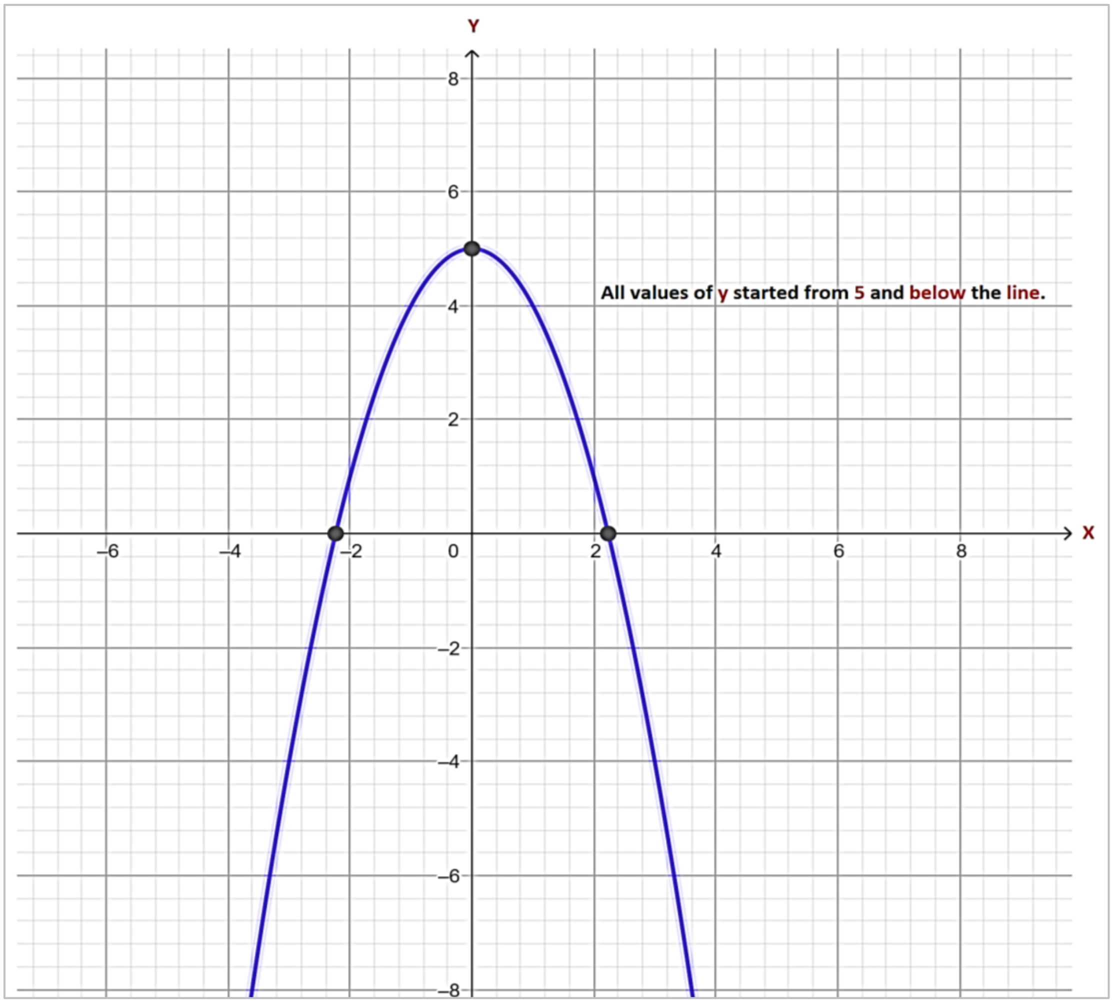


Figure 34: Dom and Range of a Quadratic function

Example – 3: Find the **vertex**, **domain,** and **range** for parabola.

$$g(x) = -3x^2 + 3x - 7$$

Solution: In the given question $a = -3, b = 3$and $c = -7$, and $a < 0$, so the graph of quadratic function (parabola) opens downward

$x = -\frac{b}{2a} = -\frac{3}{2(-3)} = \frac{3}{6} = \frac{1}{2}$

$$y = g\left(-\frac{b}{2a}\right) = g\left(\frac{1}{2}\right) = -3\left(\frac{1}{2}\right)^2 + 3\left(\frac{1}{2}\right) - 7$$

$$= -3\left(\frac{1}{4}\right) + \frac{3}{2} - 7$$

$$= -\frac{3}{4} + \frac{6}{4} - \frac{28}{4} = -\frac{19}{4}$$

Thus, vertex $= \left(\frac{1}{2}, -\frac{19}{4}\right)$, and Domain $f = (-\infty, \infty)$, Range= $\left(-\infty, -\frac{19}{4}\right)$ ----- Answer

Application to a Quadratic Function

Quadratic function has been widely applied in different filed of sciences. Most frequently in business analysis, in finding the height of a projectile etc.

Calculating the Height of a Ball

Example – 1: The height,s, of a ball is thrown vertically upward is given by $s(t) = -16t^2 + 85t + 32$**,** where s = height (in meters), t = elapsed time (in seconds). Determine the **maximum height** and **time** (in second) to reach in maximum height?

Solution: $s(t) = -16t^2 + 85t + 32$.

In given quadratic function $a = -16 < 0$, so parabola open downward and $s(t)$ has a maximum value that can find at the vertex where
$t = -\frac{b}{2a} = -\frac{85}{2(-16)} \approx 2.7$ Sec. when $t = 2.7$,
Then $s(t) = -16(2.7)^2 + 85(2.7) + 22 = 134.86$ meters.

Example – 2: Rahma's quilt **4 ft.**$\times$ **5 ft**. she has **10 sq. ft**. of fabric to create a border around the quilt. What will be the width of Rahma's quilt so she can use all the fabric? (Border of quilt must be same on all sides of quilt).

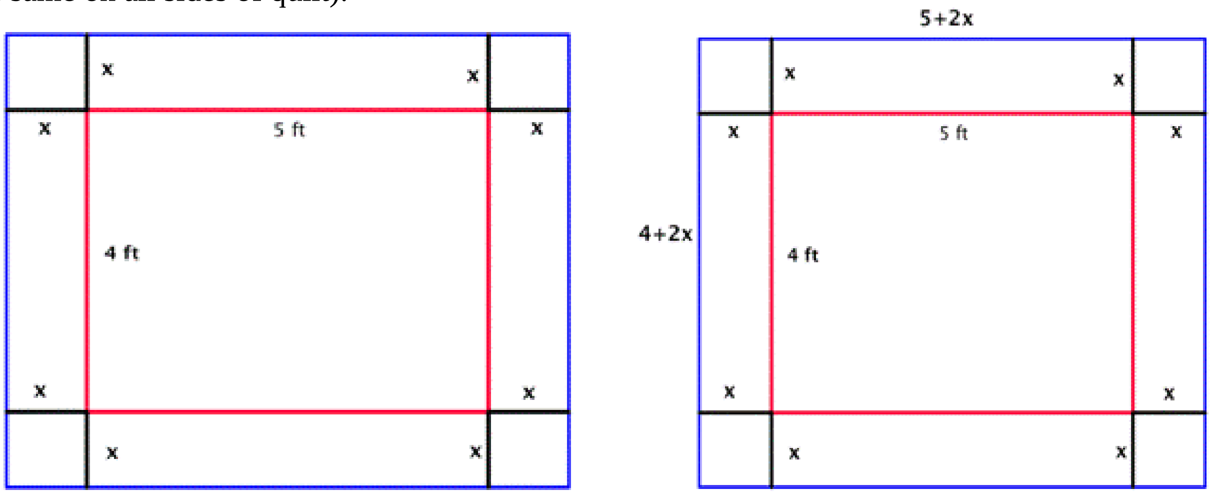


Figure 35: Quilt as a Rectangle
Source: (Montereyinstitute, n.d.)

Solution: Length of quilt (with border of width x added) = $5 + 2x$,

Width of quilt (with border of width x added) = $4 + 2x$.

AREA OF BORDER =$(4 + 2x)(5 + 2x) - (4)(5)$

$$10 = (4 + 2x)(5 + 2x) - 20, 10 = [20 + 10x + 8x + 4x^2] - 20$$

$$10 = 4x^2 + 18x$$

$$4x^2 + 18x - 10 = 0$$

By the quadratic formula, here $a = 4$, $b = 18$and $c = -10$

x= $\frac{-18\pm\sqrt{(18)^2-4(4)(-10)}}{2(4)} = \frac{-18\pm\sqrt{484}}{8} = \frac{-18\pm 22}{8}$

$=\frac{-18+22}{8} = \frac{4}{8} = 0.5$ or $=\frac{-18-22}{8} = \frac{-40}{8} = -5$

The width of the border = 0.5 ft.

Example – 3: Determine the two numbers whose **product** is a **maximum** and **sum** is 78.

Solution: Let x, y be two numbers whose sum is 78.

So $x + y = 78$, $y = 78 - x$.

Their product is then $p(x) = x\,(78 - x) = 78x - x^2$
Since $a = -1 < 0, p(x)$ has a maximum value that occurs at the vertex where
$x = -\frac{b}{2a} = -\frac{78}{2(-1)} = 39$ and $y = 78 - x = 78 - 39 = 39$.
Thus, two numbers are 39 and 39.

Calculating the Revenue

Example – 4: If p is the price (in dollars) per unit when q units are demanded (per day).
The demand function for a manufacturer's product is $p = f(q) = 400 - 5q$, find the **level of production** that maximizes the manufacturer's total revenue and determine this **revenue**.

Solution: If $r = pq$, then $r = (400 - 5q)q = 400q - 5q^2$

$$r = 400q - 5q^2$$

In the given quadratic function $a = -5, b = 400, c = 0$, and $a < 0$ so the graph of the function is a parabola that opens downward, and r is maximum at the vertex (q, r).

$q = -\frac{b}{2a} = -\frac{400}{2(-5)} = 40$.
$r = 400(40) - 5(40)^2 = 8000$.

So, the maximum revenue =$8000, and Production level = 40 units.

Solving the Quadratic Inequalities

The **quadratic Inequalities** are the functions of degree 2. These types of functions involve symbols called **inequalities.** They are the symbols known as: less than ($<$), less than or equal to ($\leq$), greater than ($>$) and greater than or equal to ($\geq$). This means we can state that:

$$\boldsymbol{ax^2} + \mathbf{bx} + \mathbf{c} \neq \mathbf{0}, \text{ (Quadratic Inequality)}$$

This leads us to write: $\boldsymbol{ax^2} + \mathbf{bx} + \mathbf{c} \begin{Bmatrix} < \\ \leq \\ > \\ \geq \end{Bmatrix} \mathbf{0}$. Note that in problems we can have only one inequality.

Steps for solving a Quadratic Inequality

Step:1 Turn inequality into an equation.

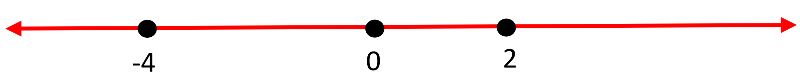
Step:2 Find solution.

Step:3 Make a number line and check each solution and interval.

Example – 5: Solve the quadratic inequality: $\boldsymbol{x^2 + 2x - 8 \geq 0.}$

Solution: Step.1: Consider: $x^2 + 2x - 8 = 0$
Step.2: Factorization $(x + 4)(x - 2) = 0$
$x = -4$ or $x = 2$

Step:3



☑Check-I, $x = -10$ ☒ Check-II, $x = 0$ ☑ Check-III, $x = 10$.

So, the solution is $x \in (-\infty, -4] \cup [2, \infty)$.

EXERCISE – 2.1

Determine the **Vertex** of the following parabolas.

(1). $g(x) = -3x^2 + 6x + 5$

(2). $g(x) = 4x^2 - 4x + 1$

(3). $f(x) = 2x^2 - 12x + 26$

(4). $f(t) = -7t^2 - 2t + 9$

(5). $g(x) = 1 - 5x - x^2$

(6). $f(x) = x^2 - 7x + 1$

Determine $\boldsymbol{x} - and\ \boldsymbol{y} -$ **intercepts** of the parabolas.

(7). $f(x) = 3x^2 + x - 5$

(8). $f(x) = -4x^2 - 15x + 3$

(9). $g(x) = 2x^2 + 8x$

(10). $g(x) = -x^2 + 6x - 5$

(11). $h(x) = 5(x-1)^2 - 20$

(12). $g(a) = (a+4)^2 - 3$

Decide whether the following function has a **maximum** or **minimum** value, then determine maximum or minimum value of each function.

(13). $h(x) = -64 + 24x - 2x^2$

(14). $f(x) = x^2 + 8x + 5$

(15). $g(x) = -x^2 + 6x - 15$

(16). $f(x) = 4x^2 + 16x$

(17). $g(x) = x^2 - 8$

(18). $g(r) = 25r^2 + 10r + 1$

Determine the **Domain** and **Range** of the Quadratic functions.

(19). $h(s) = s^2 + 6s + 7$

(20). $R(t) = -3t^2 + 6t + 5$

(21). $P(s) = -3s^2 + 6s - 2$

(22). $g(x) = -x^2 - 4x + 11$

(23). $f(x) = 2s^2 - 6s + 2$

(24). $k(t) = -t^2 + 4t - 5$

Find the **axis of Symmetry** of following Parabolas.

(25). $h(x) = x^2 - 6x + 7$

(26). $R(t) = -2t^2 + 6t - 1$

(27). $P(x) = -3x^2 + 7x - 2$

(28). $g(x) = -2x^2 - 8x + 3$

(29). Find any **two number** whose **product** is a **maximum** and **addition** is **24**.

(30). Find any **two number** whose **product** is a **maximum** and **addition** is **72**.

(31). **HEIGHT OF BALL**: Suppose that the height $\boldsymbol{S}$ of a ball is thrown vertically upward is $S(t) = -16t^2 + 16t + 4$, where $\boldsymbol{S} =$ height (in meters), and $\boldsymbol{t} =$ elapsed time (in seconds). What is the **maximum height** of the ball and how much **time** (in second) ball does take to reach the maximum height?

(32). **REVENUE**: If $\boldsymbol{p}$ =price (in dollars) per unit and $\boldsymbol{q}$ =units are demanded (per day). The demand function for a manufacturer's product is $\boldsymbol{p = f(q) = 1800 - 3q}$, Determine the **level of production** that **maximizes** the manufacturer's total **revenue** and determine maximum revenue.

Solve the following **Quadratic Inequalities**.

(33).	$\mathbf{-x^2 + x + 2 \le 0}$	(34).	$\mathbf{(x-5)(x+4) \ge 0}$
(35).	$\mathbf{x^2 + 5x + 6 > 0}$	(36).	$\mathrm{x}^2 - \mathrm{x} - 6 < 0$
(37).	$\mathbf{(x+7)(x-3) \le 0}$	(38).	$\mathrm{x}^2 - 6\mathrm{x} + 5 \ge -2$
(39).	$\boldsymbol{-2x^2 + 4x + 6 \le 0}$	(40).	$-x^2 + 2x - 3 \ge 0$

CHAPTER – 3 EXPONENTIAL AND LOGARITHMIC FUNCTIONS

OBJECTIVES: Upon successful completion of this course, students will be able to

- Define and manipulate exponential and Logarithmic functions and solve problems arising from real life applications.
- Understand the inverse relationship between exponents and logarithm functions and use this relationship to solve related problem.

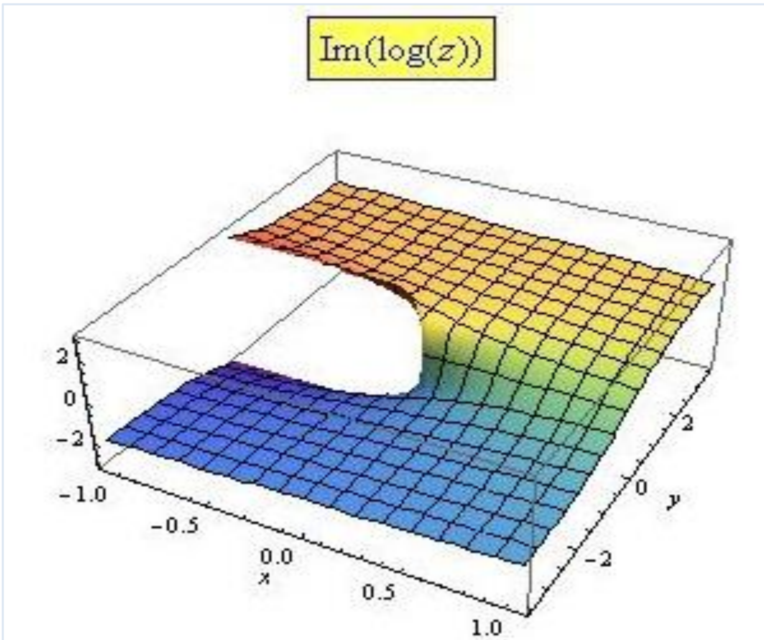


Figure 36: Exponential and Logarithmic Functions
Source: (Wolfram Mathematica, n.d.)

Important Formulae of Exponential and Logarithmic Functions

1. Exponential Function with base a: $f(x) = a^x$ such that $a > 0,\ a \neq 1$
2. Natural Exponential Function with base e: $f(x) = e^x$
3. Logarithmic Function: $log_a x = y \Leftrightarrow a^y = x$
4. Properties of Logarithms:

   a) $loa_a 1 = 0$ b) $log_a a = 1$ c) $log_a a^x = x$ d) $a^{log_a x} = x$

5. Common Logarithm : $log\, x = log_{10} x$
6. Natural Logarithm: $ln\, x = log_e x$ and by definition, $ln\, x = y \Leftrightarrow e^y = y$
7. Properties of Natural Logarithms:

   a) $ln\, 1 = 0$ b) $ln\, e = 1$ c) $ln\, e^x = x$ d) $e^{ln\, x} = x$

8. Laws of Logarithms:

   a) $log_a(A \times B) = log_a A + log_a B$

   b) $log_a \left(\frac{A}{B}\right) = log_a A - log_a B$

   c) $log_a A^c = c\, log_a A$

9. Change of Base Formula: $log_b x = \frac{log\, x}{log\, b} = \frac{lnx}{lnb}$
10. $log_b x = \frac{1}{log_x b}$
11. Formula of Simple Interest: $S.I = Prt$
12. Amount using the Simple Interest: $A(t) = P(1 + rt)$
13. Amount using the Compound Interest: $A(t) = P\left(1 + \frac{r}{n}\right)^{nt}$
14. Amount using the Continuously Compound Interest: $A(t) = Pe^{rt}$
15. Exponential Growth (Relative Growth Rate): $\mathrm{n(t) = n_{\circ}e^{rt}}$
16. Radioactive decay Model: $m(t) = m_0 e^{-rt}$; where $r = \frac{ln\, 2}{h}$
17. Newton's Law of Cooling: $T(t) = T_s + D_0 e^{-kt}$, where $D_0 = |T_{\circ} - T_s|$.

Exponential Functions

Definition: If 'x' is a real number, then the exponential function with base 'a' is defined by

$$f(x) = a^x, where\ a > 0\ and\ a \neq 1\ \forall\ x \in \mathbb{R}$$

Examples: $f(x) = 4^x$, is the ***exponential*** function with **base** 4.

2) The "base" of $f(x) = 5^x$ is 5

3) Is that $f(x) = (-3)^x$ an exponential function?

Evaluating Exponential Functions

Examples – 1: Evaluate $f(x) = 4^x$ at $f(2), f\left(-\frac{1}{2}\right), f(\sqrt{3})$

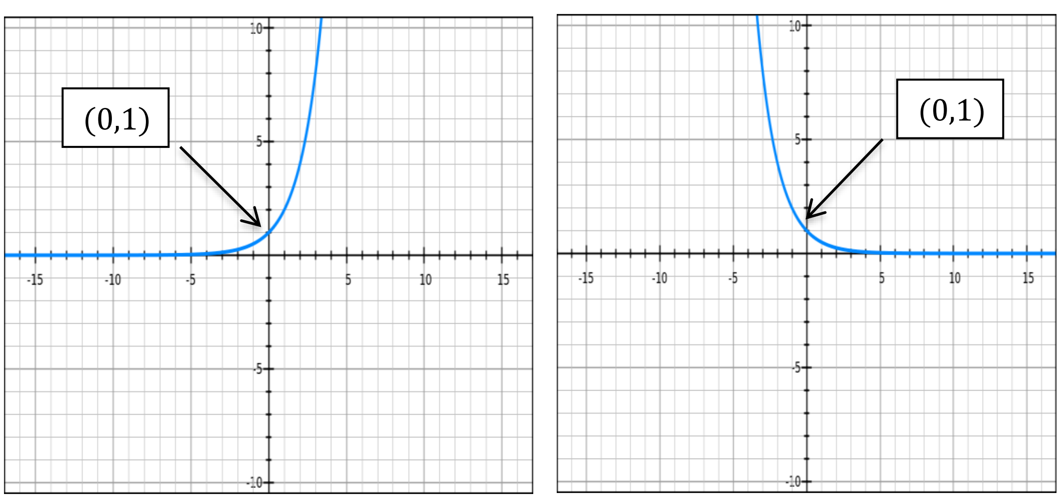
SOLUTIONS

$f(2) = 4^2 = 16$ ---- Answer

$f\left(-\frac{1}{2}\right) = 4^{-\frac{1}{2}} = 0.5$ ---- Answer

$f(\sqrt{3}) = 4^{\sqrt{3}} = 11.03$ ---- Answer

The Graph of Exponential Functions



$f(x) = a^x, where\ a > 1\ and\ a \neq 1$

$f(x) = a^x, where\ 0 < a < 1\ and\ a \neq 1$

Figure 37: Graph of Exponential Functions
Source: (Andy, 2009)

Important facts about Exponential Functions

Some facts about the exponential function are given in the following points, See (Fun, Math is, n.d.).

a) It is greater than 0, and never crosses the $\boldsymbol{x-axis}$.

b) It always intersects the $\boldsymbol{y-axis}$ at $y = 1$, or in other words it passes through $(0, 1)$.

c) For $x = 1, f(x) = a$, means it passes through $(1, a)$.

d) **Domain of exponential function**: It can use all values of x. This means domain is the set of Real Numbers $(x \in \mathbb{R})$.

e) **Range of exponential function:** It can produce only positive reals numbers. So, the range is the set of all positive real numbers $(y \in \mathbb{R}^+)$.

Example – 2: Draw the the graphs of $f(x) = 2^x$ *and* $g(x) = \left(\frac{1}{2}\right)^x$ on one set of axes by using the values from the table below.

$\boldsymbol{x}$	$\boldsymbol{f(x) = 2^x}$	$\boldsymbol{g(x) = \left(\frac{1}{2}\right)^x}$
⋮	⋮	⋮
-3	0.1	8
-2	0.2	4
-1	0.5	2
0	1	1
1	2	0.5
2	4	0.2
3	8	0.1
⋮	⋮	⋮

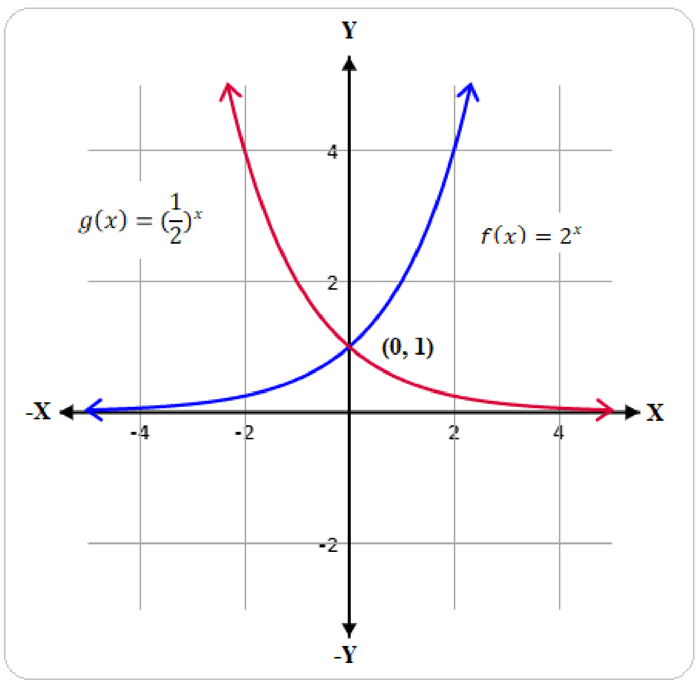


Figure 38: Graph of two Exponential Functions on the same axes

Exponential Function from Graphs

Example – 1: Determine the exponential function $f(x) = a^x$ whose graph is shown in (Figure 39).

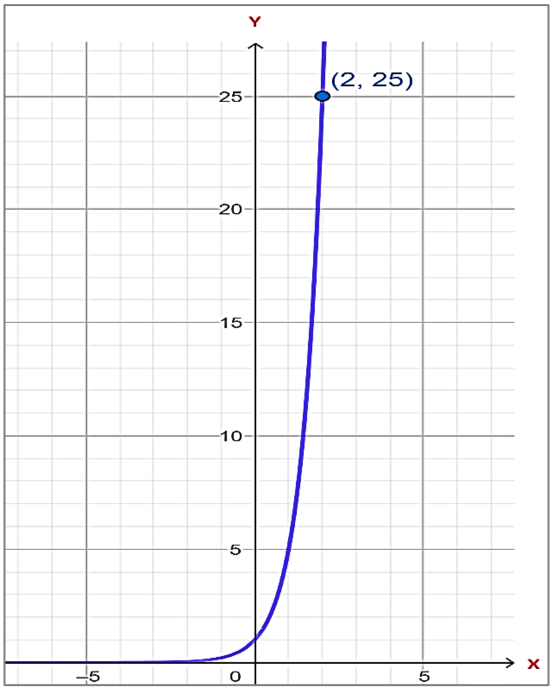


Figure 39: Function from a graph

Solution: Here we need the function of the form $f(x) = a^x$, $where\ a > 1\ and\ a \neq 1$. So we need to find the value of $'a'$, because 'x' is variable.

From the given graph we can see that: $\big(x, f(x)\big) = (2, 25)$

$$x = 2 \text{ and } y = 25$$

Since $$f(x) = a^x$$

Therefore, $$y = a^x$$

$$a^2 = 25$$

$$\sqrt{a^2} = \sqrt{25}$$

$$a = 5$$

Thus, the required function is, $f(x) = 5^x$ ---- Answer

Simple Interest (S.I)

The simple interest is calculated by the formula: $\boldsymbol{S.I = PRT}$

where **P**: Principal amount, **R**: Rate of interest (in %), **T**: Time duration.

Amount Due at the end of the time period, $\boldsymbol{A = P + S.I}$
OR
Amount Due at the end of the time period, $\boldsymbol{A = P\,(1 + rt)}$**.**

Compound Interest (C.I)

The formula for calculating the compound interest is given by:

$$\boldsymbol{A(t) = P\left(1 + \frac{r}{n}\right)^{nt}}$$

OR

$$\boldsymbol{A = P + C.I}$$

where $A\,(t) = amount\ after\ 't'\ years$
$P = \ principal\ amount$
$r = \ interest\ rate$
$n = number\ of\ times\ interest\ is\ compounded\ per\ year$
$t = \ number\ of\ years.$

Table 2: Compounding Table

Compounding term	$\boldsymbol{n}$
Annual	1
Semi annual	2
Quarterly	4
Monthly	12
Daily	365

The value of "$\boldsymbol{n}$" is chosen from above (Table 2) according to the term used in a certain problem.

Example – 1: A sum of OMR 1000 is invested at an interest rate of 12 % per year. Find the amount in the account after 3 years if interest is compounded annually, semiannually, quarterly, monthly, and daily.

Solution: Given data

$$P = 1000$$

$$r = 12\% = \frac{12}{100} = 0.12$$

$$t = 3$$

The required equation is $A(t) = P\left(1 + \frac{r}{n}\right)^{nt}$

$$A(3) = 1000\left(1 + \frac{0.12}{n}\right)^{n\times 3}$$

If interest compounded **annually** then, $n = 1$ (Table 2),

$$A(3) = 1000\left(1 + \frac{0.12}{1}\right)^{1\times 3}$$

$A(3) = 1404.93$ ---- Answer

If interest compounded **semiannually** then, $n = 2$ (Table 2)

Then,

$$A(3) = 1000\left(1+\frac{0.12}{2}\right)^{2\times 3}$$

$$A(3) = 1418.52 \text{ ---- Answer}$$

If interest compounded **quarterly** then, $n = 4$ (Table 2),

$$A(3) = 1000\left(1+\frac{0.12}{4}\right)^{4\times 3}$$

$$A(3) = 1452.76 \text{ ---- Answer}$$

If interest compounded **monthly** then $n = 12$ (Table 2),

$$A(3) = 1000\left(1+\frac{0.12}{12}\right)^{12\times 3}$$

$$A(3) = 1430.77 \text{ ---- Answer}$$

If interest compounded daily then $n = 365$ (Table 2),

$$A(3) = 1000\left(1+\frac{0.12}{365}\right)^{365\times 3}$$

$$A(3) = 1433.24 \text{ ---- Answer}$$

Exercise – 3.1

Use calculator to evaluate the function at the indicated values. [source: (James Stewart, 2012)]

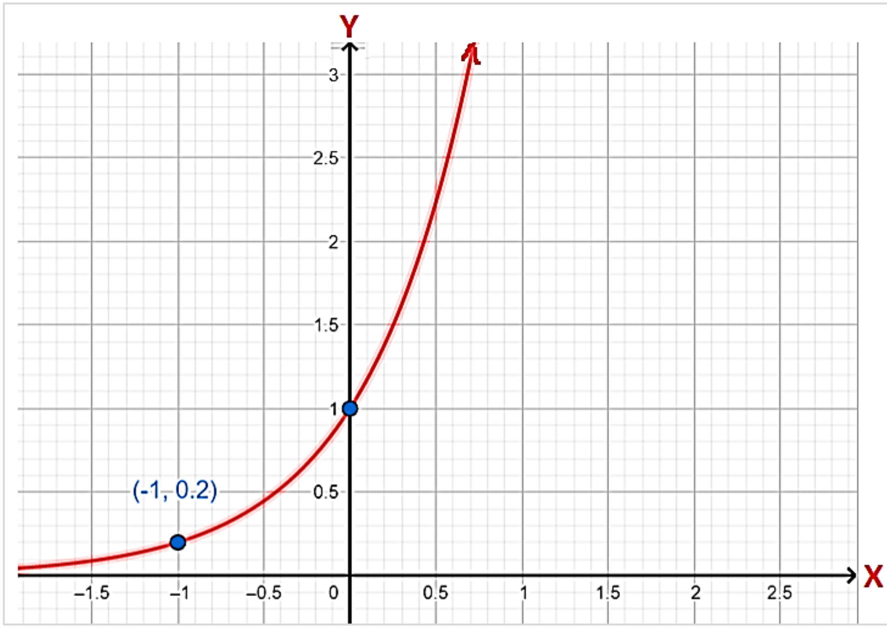
1. $f(x) = 4^x$ at $f(0.5), f\left(\frac{1}{3}\right), f(\sqrt{2}), f(-\pi)$
2. $f(x) = 3^{x+1}$ at $f(-1.5), f\left(-\frac{5}{4}\right), f(\sqrt{3}), f(e)$
3. $g(x) = \left(\frac{3}{4}\right)^{2x}$ *at* $g(0.7), g\left(\frac{\sqrt{7}}{2}\right), g\left(\frac{1}{\pi}\right), , g\left(\frac{2}{3}\right)$

Graph both functions on one set of axes

4. $f(x) = 3^x$ *and* $g(x) = \left(\frac{1}{3}\right)^x$
5. $f(x) = \left(\frac{2}{3}\right)^x$ *and* $g(x) = \left(\frac{4}{3}\right)^x$
6. $f(x) = 5^x$ *and* $g(x) = 7^x$

Find the exponential function $f(x) = a^x$ whose graphs are given:

7.



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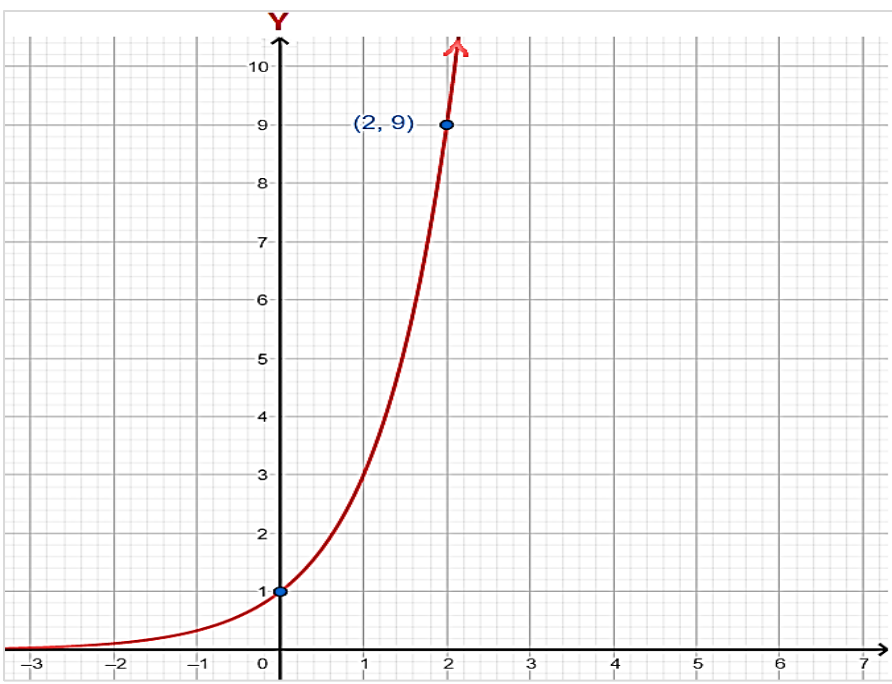

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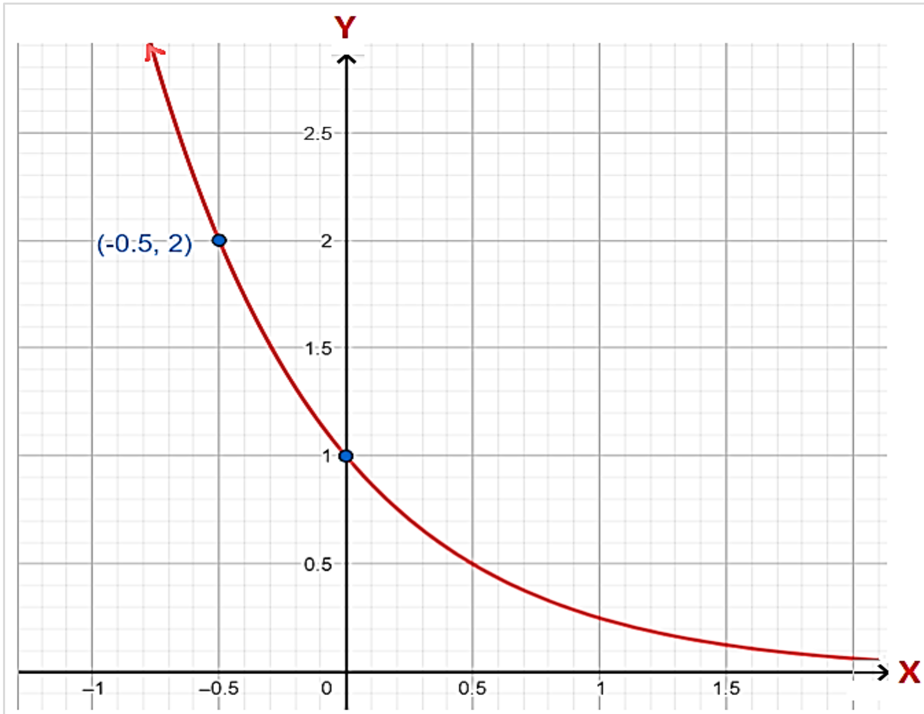

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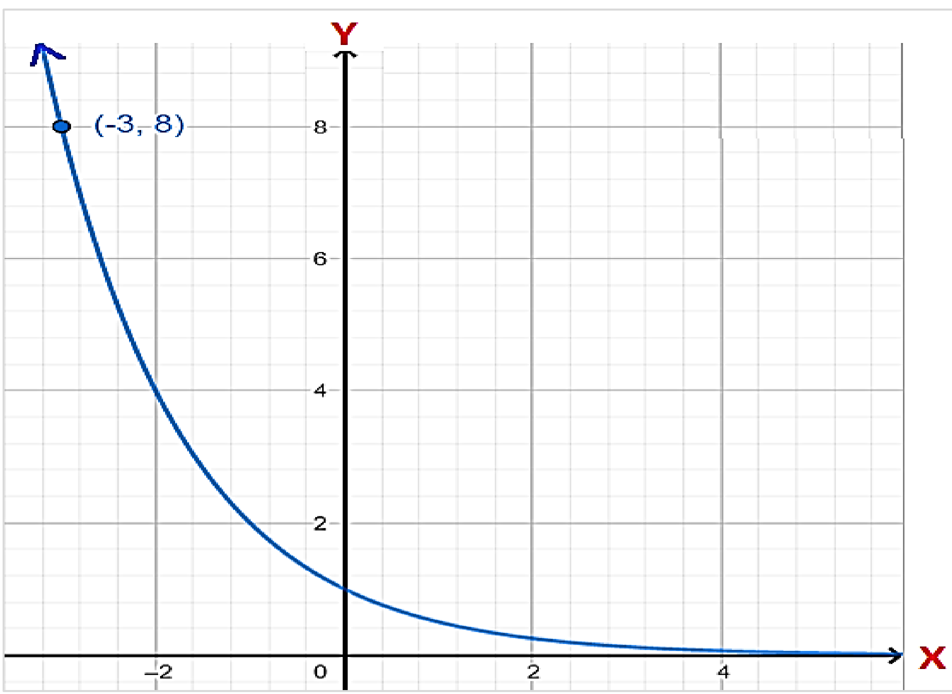


Problems on Interest (Application of exponential functions in Business)

11. A sum of \$ 25000 becomes \$ 27250 at the end of 3 years when calculated at simple interest. Find the rate of interest?
12. Find the present worth of \$ 78000 due in 4 years at 5% interest per year?
13. A certain principal amounts to \$15000 in 2.5 years and to 16500 in 4 years at the same rate of interest. Find the rate of interest?
14. Find the compound interest on \$ 3000 at 5% for 2 years, compounded annually?
15. If 2000 Riyals Omani is deposited at an "interest rate" of 3% per year, semi-annually compounded, what amount will depositors get after following number of years

    a) 5 years   b) 10 years   c) 15 years

16. If OMR 3500 is deposited at the 2.5% rate of interest per year, what will be the deposited amount for following calculated compounding

    a) annually   b) semi-annually   c) quarterly

17. Ahmed deposited 3500 Riyals Omani to bank Sohar, "compounded quarterly", What money will he get after 5 years for the following interest rate?

    a) 2.5%   b) 3.75%   c) 4%

18. John invests 4000 Riyals Omani compounded yearly in bank that pays compound interest at the rate 5% per year. Calculate the amount that John has in the bank after 3 years.
19. After 80 years of 5.8% interest compounded monthly, an account has 102,393.44. What was the original deposit amount?
20. Rahman has \$700 in a saving account that earns 10% interest per year. The interest is not compounded. How much will she have in 3 years?
21. Iman has \$470 in a saving account. The interest rate is 5% per year and is not compound. How much interest will he earn in 6 years?
22. A sum of \$5000 is deposited into a bank account and the annual interest rate is 3%. How much is the simple interest and compounded interest after 4 years? What is the difference in simple interest and compound interest?
23. Tina deposits \$90,000 into an account that pays simple interest at the rate 3% per year. Sam deposits \$90,000 into an account that also pays 3% interest per year but it is

compounded annually. Find how much interest Tina and Sam got during first three years then decide who earns more interest?

The Natural Exponential Function

The Natural Exponential Function: $\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{e}^{\boldsymbol{x}}$ [Source: (MathisFun, 2017)].
where "$\boldsymbol{e}$" is an "$Euler's\ Number$" $= \mathbf{2.718281828459}$ (and more ...). It is an Irrational number ($e \in Q'$).

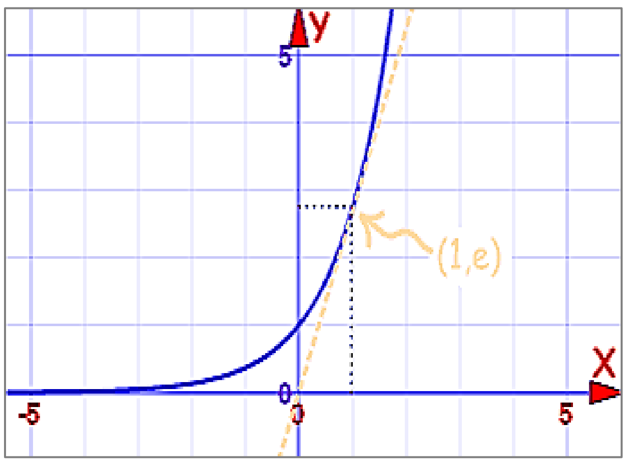


Figure 40: Graph of Natural Exponential function

At the point $(1, e)$, the slope of the line is 'e' and the line is tangent to the curve.

Evaluating the Natural Exponential Function

Example – 1: Evaluate the expression $f(x) = 3e^{-0.75x}$, for $f(\sqrt{2})$ and $f(-3)$.
Solution: Putting $x = 2$ in the given function: $f(\sqrt{2}) = 3e^{-0.75\times\sqrt{2}}$. Using the calculator in the following manner.

3	*shift*	*ln*	(	$-\mathbf{0.75} \times \sqrt{\mathbf{2}}$	)

$f(\sqrt{2}) \approx 1.03868$ ---- Answer
Similarly: $f(-3) = 3e^{-0.75\times-3} \approx 28.4632$ ---- Answer

Graph of the Exponential Function

Example – 1 (James Stewart, 2012): Draw the graph of $f(x) = 3e^{0.5x}$?

Solution: Preparing the table of values by using Calculator we get

x	-3	-2	-1	0	1	2	3
$f(x) = 3e^{0.5x}$	0.67	1.10	1.82	3	4.95	8.15	13.45

Now by plotting all the point on the graph paper from the above table, we obtain the required graph.

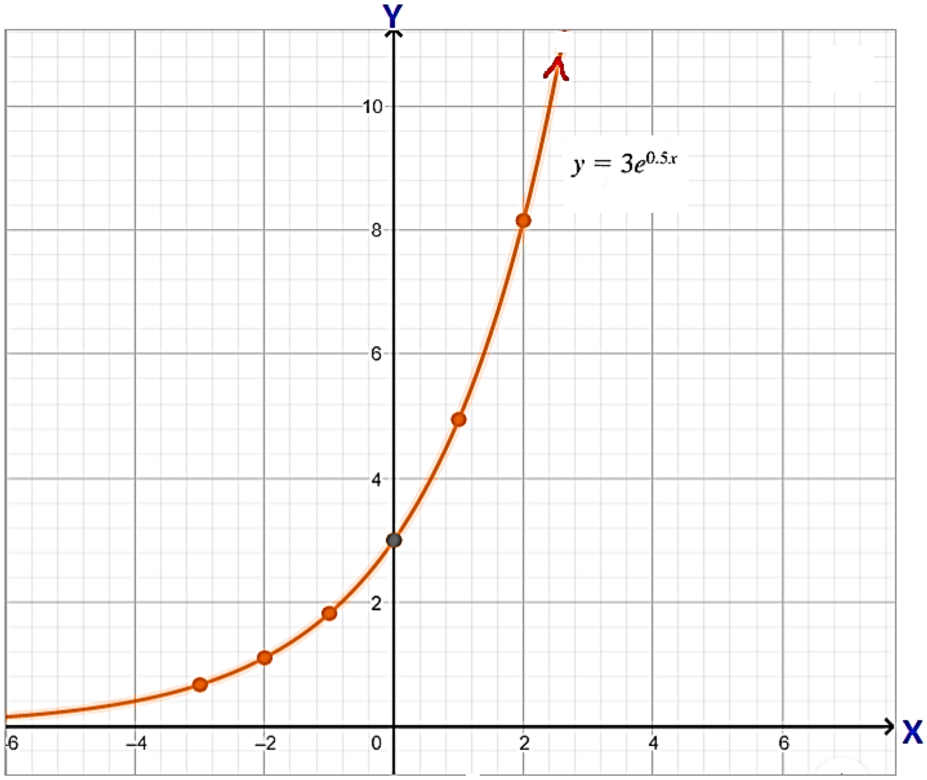


Figure 41: Graph of Exponential Function

Example – 2: Draw the graph of $f(x) = e^{-x}$, by using the transformation techniques.

Solution: Thinking about the graph of $f(x) = e^{x}$ and reflect in the $y - axis$ to obtain the graph of $f(x) = e^{-x}$.
The Reflecting properties are given by:

a) To graph $y = -f(x)$, reflect the graph of $y = f(x)$ about $x - axis$.
b) To graph $y = f(-x)$, reflect the graph of $y = f(x)$ about $y - axis$.

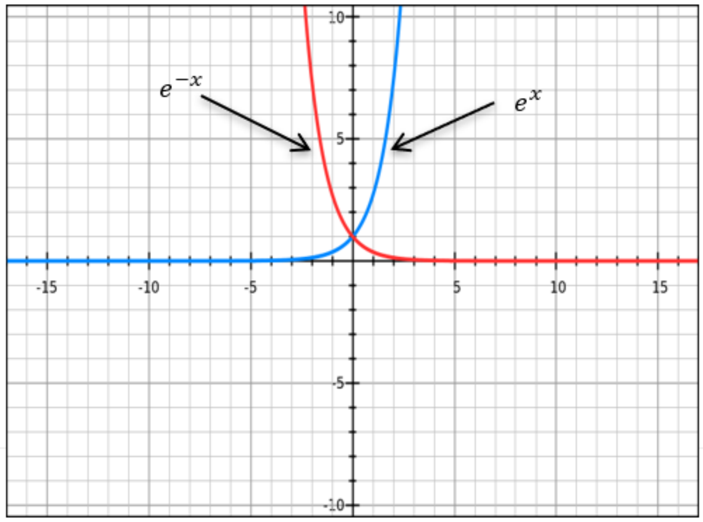


Figure 42: Graph of Exponential Function

The Continuous Compound Interest (C.C.I)

The interest which is free from time bound. It means the deposited amount is Continuously increasing with a certain rate of interest. The continuously compound interest is calculated by

$$A(t) = Pe^{rt}$$

Example – 1: A sum of $\$\mathbf{1200}$ amount is deposited at the rate of $\mathbf{7.25}\%$ per year compounded continuously for **6** years? Calculate the present value of the account? How much interest is earned in this duration?

Solution: Given that the principal value: $P = \ \$\,1200$
Rate of interest; $r = 7.25\% = \frac{7.25}{100} = 0.0725$
Time duration: $t = 6\ yrs.$
Using the relation: $A(t) = Pe^{rt}$, putting the values we get

$$A(after\ 6\ years) = 1200e^{0.0725\times 6} = 1200e^{0.435}$$

$$A(after\ 6\ years) = \$1853.96 \text{ ---- Answer}$$

The Compound interest: $C.I = A - P$, putting the values we have

$$C.I = 1853.96 - 1200 = \$653.96 \text{ ---- Answer}$$

Exponential Inequalities

The general form of exponential inequality is given by

$$a^{f(x)} > a^{g(x)} = \begin{cases} f(x) > g(x) \ \ if\ a > 1 \\ f(x) < g(x) \ \ if\ 0 < a < 1 \end{cases}$$

OR

$$a^{f(x)} < a^{g(x)} = \begin{cases} f(x) < g(x) \ \ if\ a > 1 \\ f(x) > g(x) \ \ if\ 0 < a < 1 \end{cases}$$

Note that a question involves only one sign (<, >, ≤, ≥).

Example – 1: Solve the exponential inequality function: $\mathbf{3^{3x+7} < 3^{2x-1}}$**?**

Solution: Here the base: $a = 3 > 1$, therefore the sign (<) will not change.

$$3x + 7 < 2x - 1$$

$$x < -8 \text{ ---- Answer}$$

Example – 2. Solve the exponential inequality function: $\mathbf{(0.3)^{5x-x^2-8} \leq 9}$**?**

Solution: Since $9 = 3^2 = \left(\left(\frac{1}{3}\right)^{-1}\right)^2 = \left(\frac{1}{3}\right)^{-2}$

Here $a = \frac{1}{3}$, that is $0 < \frac{1}{3} < 1$**,** then

$$5x - x^2 - 8 \geq -2$$

$$-x^2 + 5x - 6 \geq 0$$

$$x^2 - 5x + 6 \leq 0$$

$$(x - 2)(x - 3) \leq 0 \text{ ---- (1)}$$

Then consider, $(x - 2) = 0$ or $(x - 3) = 0$

$$x = \{2, 3\}$$

Now testing a number on the left of 2, another number on the right side of 3 and one number between 2 and 3. We find that $2 \leq x \leq 3$ or $x \in [2,3]$ ---- Answer

Note: For testing the intervals, one may refer to Chapter 2.

Example – 3. Solve $(0.5)^{x^2+1} < (0.25)^{x+2}$?

Solution: Making the bases equal we get: $(0.5)^{x^2+1} < (0.5)^{2(x+2)}$.
Here $a = 0.5 < 1$, therefore the sign (<), will change.

$$x^2 + 1 > 2(x + 2)$$

$$x^2 + 1 > 2x + 4$$
$$x^2 - 2x - 3 > 0$$
$$x^2 - 3x + x - 3 > 0$$
$$(x - 3)(x + 1) > 0$$
$$x > 3 \text{ or } x > -1$$

The final solution is $-1 < x < 3 = (-1, 3)$ ---- Answer.

Exercise – 3.2

Evaluate the function (by using calculator) [source: (James Stewart, 2012)].

(1). Evaluate $h(x) = 2e^{-3x}$ for $h(1), h(\sqrt{2}), h(-3)$ and $h(\frac{1}{2})$?

(2). Complete the following table of values:

x	-2	-1	-0.5	0	1	2
$f(x) = 3e^x$						

Problems on Interest (Application to natural exponential functions)

(3). Find the compound interest on \$ **3000** at **5**% for **2** years, compounded continuously?

(4). If **2000** Riyals Omani is deposited at an interest rate of **2.7**% per year, continuously compounded, what amount will depositors get after **15** years?

(5). A sum of amount is deposited in a certain business company which was compounded continuously for **12** years with interest rate of **4.5%**. The present value of the account is \$ **5000**. Calculate the invested money?

Solve the exponential **inequalities** given below

(6). $5^{6x+9} < 5^{2x+1}$

(7). $\left(\frac{1}{3}\right)^{x^2+1} < \left(\frac{1}{9}\right)^{x+2}$

(8). $3^{-2x+2} > 3^{5x-5}$

(9). $\left(\frac{1}{3}\right)^{-2x+1} \leq \left(\frac{1}{3}\right)^{x-11}$

(10). $2^{3x-9} > 2^{4x-6}$

Logarithmic Function

Let 'a' be a positive number real number but $a \neq 1$. The **logarithmic function** with base $\boldsymbol{a}$

denoted by $\log_a$, is defined by: $\boldsymbol{log_a x = y \Leftrightarrow a^y = x}$

$\boxed{\log_a x = y}$ → Logarithmic form

$\boxed{a^y = x}$ → Exponential form

Note: The exponential and logarithmic functions are **inverse** to each other. This implies we can write

60. The **exponential** function is the **inverse** of **logarithmic** function.

61. The **logarithmic** function is the **inverse** of **exponential** function.

The inverse property of both the functions can also be seen in their graphs (Figure 43).

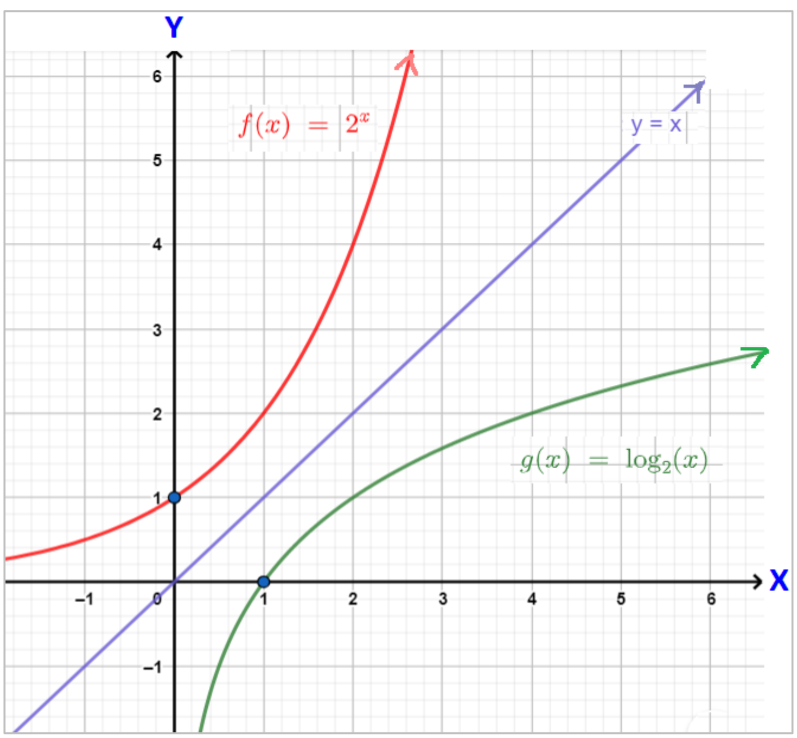


Figure 43: The Inverse relation between Exponential and Logarithmic functions

Mathematically, it can be proved that 2^x is the inverse of $\log_2 x$ see the following example.

Example 1: Show that $f(x) = 2^x$ is the **inverse** of $\mathrm{g(x)} = \log_2 x$?

Proof: The main definition and properties of inverse functions is given on page 17 as follow:

$$f\big(g(x)\big) = x = g(f(x))$$

This means we need to prove that: $f(\log_2 x) = x = g(2^x)$. Therefore,

$$f(\log_2 x) = 2^{\log_2 x} = x \text{ ---- } (1)$$

Now $$g(2^x) = \log_2 2^x = x \text{ ---- } (2)$$

It is clear from (1) and (2) that: $f\big(g(x)\big) = g\big(f(x)\big) = x$. Hence f and g are inverses to each other.

Note: Exponential form is the **inverse** of Logarithmic form and Logarithmic form is the **inverse** of Exponential form.

Properties of Logarithmic Function

Property	a) $\log_a 1 = 0$	b) $\log_a a = 1$	c) $\log_a a^x = x$	d) $a^{\log_a x} = x$
Example	$\log_2 1 = 0$	$\log_3 3 = 1$	$\log_3 3^4 = 4$	$3^{\log_3 4} = 4$

Example 1: The exponential form of $\log_2 32 = 5$ is $2^5 = 32$.

Example 2: The logarithmic form of $16^{\frac{1}{2}} = 4$ is $\log_{16} 4 = \frac{1}{2}$.

Graphs of Logarithmic Function

The graph of logarithmic functions intersects at $(1, 0)$ on $x - axis$.

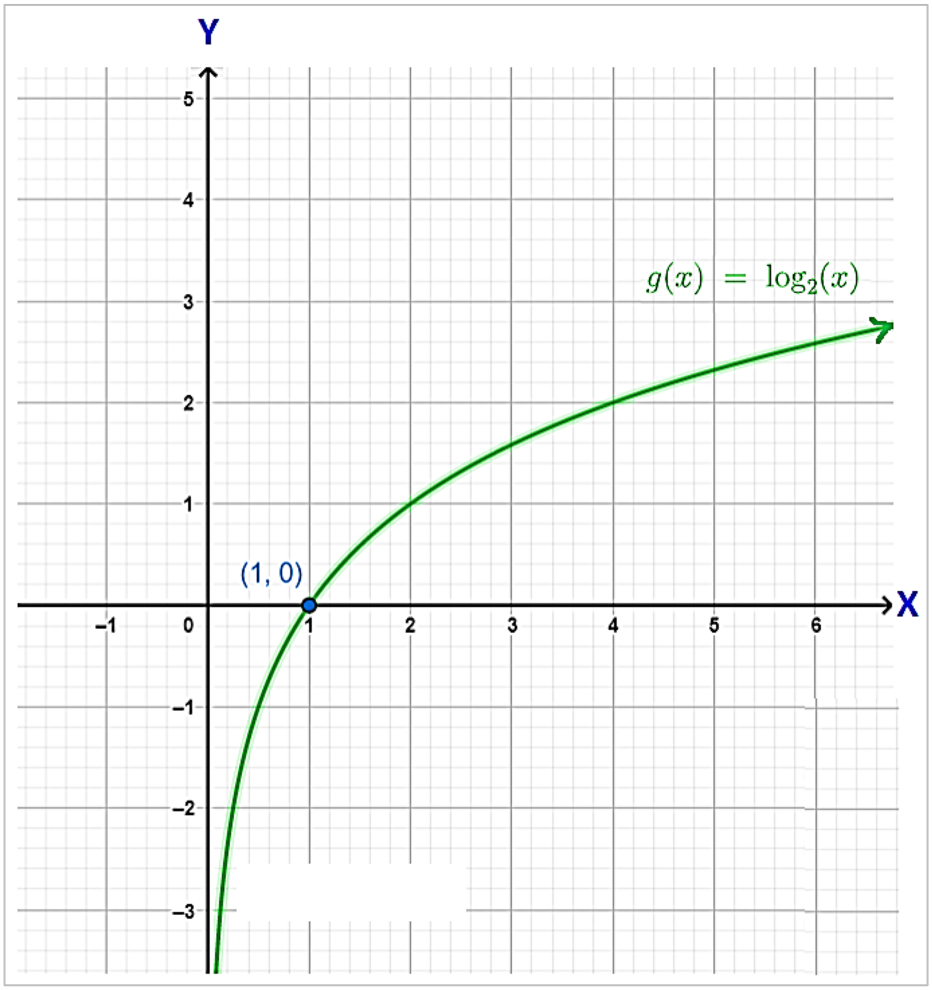


Figure 44: Graph of Logarithmic Functions

Example – 1 (James Stewart, 2012): Draw the graph $f(x) = \log_2 x$, by using table of values?

Solution: Preparing the table of values and then plotting these values in the coordinate plane.

x	2^3	2^2	2	1	2^{-1}	2^{-2}	2^{-3}	2^{-4}	$\cdots$
$f(x) = \log_2 x$	3	2	1	0	-1	-2	-3	-4	$\cdots$

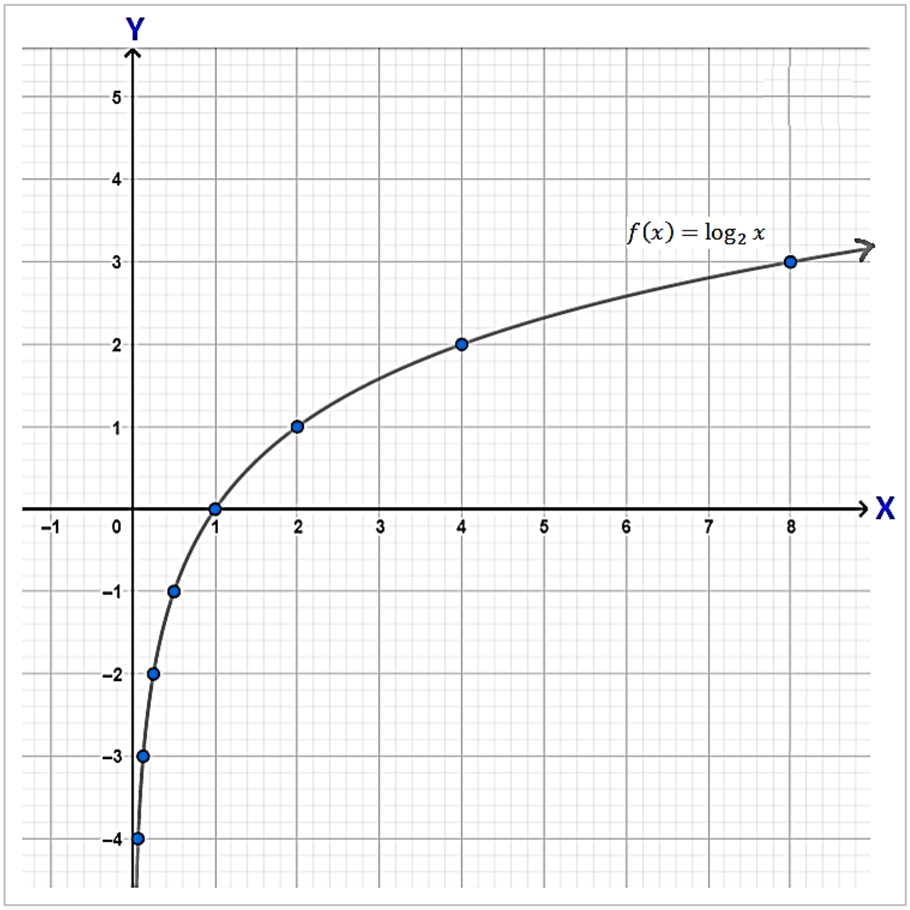


Figure 45: Graph of Logarithmic function

Reflective Graphs of Logarithmic Functions

The Reflecting properties are given by:

a) To graph $y = -f(x)$, reflect the graph of $y = f(x)$ about $x - axis$.
b) To graph $y = f(-x)$, reflect the graph of $y = f(x)$ about $y - axis$

Example – 1. Draw the graph of following function using transformation technique

(1). $g(x) = -\log_2 x$

(2). $h(x) = \log_2(-x)$

Solution: (a) The graph of $\boldsymbol{f(x) = \log_2 x}$, which reflects about $x -\ axis$ is the graph of $\boldsymbol{g(x) = -\log_2 x}$ (See Figure 46).

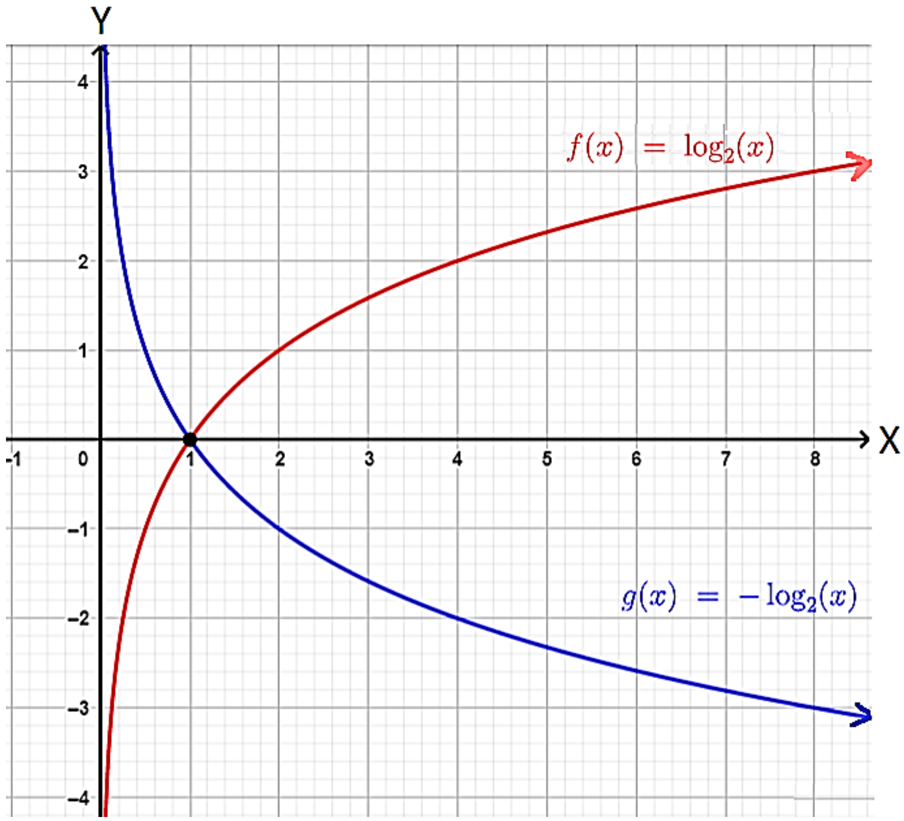


Figure 46: Reflection along x-axis

a) The graph of $\boldsymbol{f(x) = \log_2 x}$, which reflects about $y-\ axis$ is the graph of $\boldsymbol{h(x) = \log_2(-x)}$ (See Figure 47).

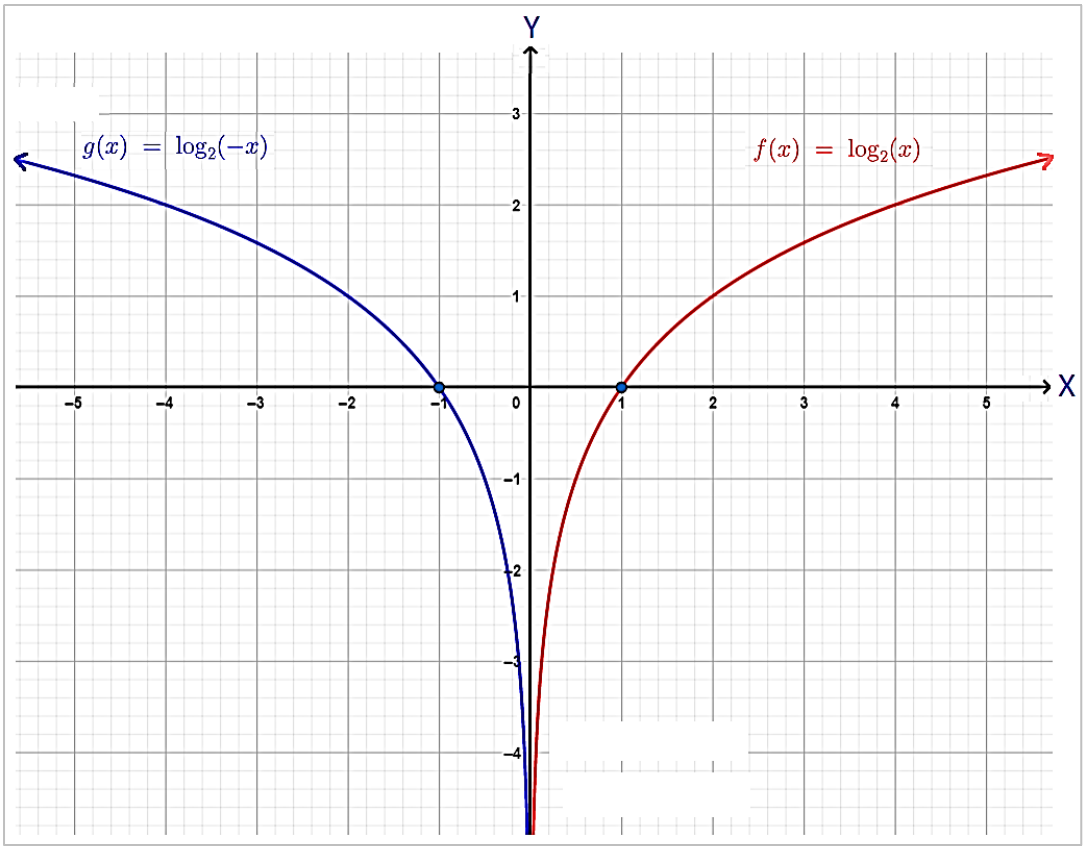


Figure 47: Reflection along y-axis

Types of Logarithms

There are mainly two kinds of logarithms. One is called the common logarithm and the other one is known as natural logarithm.

Common Logarithm

The logarithm with **base 10** is called the **common** logarithm. Mathematically it is defined by: $\boldsymbol{log x = y \Leftrightarrow x = 10^{y}}$. It is customary to write the common logarithm as: $\boldsymbol{log x = \log_{10} x}$. This function holds the following properties.

Properties of Common Logarithms

Property	a) $\log 1 = 0$	b) $\log 10 = 1$	c) $\log 10^x = x$	d) $10^{\log x} = x$

Natural Logarithm

The logarithm with **base** '$\boldsymbol{e}$' is called the **Natural** logarithm. Mathematically it is defined by: $\boldsymbol{lnx = y \Leftrightarrow x = e^{\,y}}$. It is customary to write the common logarithm as: $\boldsymbol{lnx = \log_e x}$. This function holds the following properties.

Properties of Natural Logarithms

Property	a) $ln = 0$	b) $lne = 1$	c) $ln\, e^x = x$	d) $e^{lnx} = x$

Some examples of Logarithmic functions

Example – 1: Evaluate $\log_4 64$?
Solution: Since $log_4 64 = log_4 4^3$
$= 3$ ---- Answer (by using property that: $\log_a a^x = x$)
Example – 2: Evaluate $\log_{10}\sqrt{10}$
Solution: Since $\log_{10}\sqrt{10} = \log_{10} 10^{\frac{1}{2}}$
$= \frac{1}{2}$ ---- Answer (by using property that: $\log_a a^x = x$)
Example – 3: Evaluate $\log_3\left(\frac{1}{27}\right)$?
Solution: Since $\log_3\left(\frac{1}{27}\right) = \log_3\left(\frac{1}{3^3}\right)$
$= \log_3 3^{-3}$
$= -3$ ---- Answer
Example – 4: Evaluate $\log_3 1$?

Solution: $\log_3 1 = 0$ ---- Answer (by using property that: $\log_a 1 = 0$).

Example – 5: Evaluate $e^{ln\pi}$?

Solution: Using the property that: $e^{ln\,x} = x$
$e^{ln\pi} = \pi$ ---- Answer
Example – 6: Evaluate 10^{log5}?

Solution: Using the property that: $10^{logx} = x$,
$10^{log5} = 5$ ---- Answer
Example – 7: Find 'x' $\boldsymbol{\log_5 x = 4}$**?**

Solution: We know that: $\boldsymbol{\log_a x = y \Leftrightarrow a^y = x}$
Then, $\log_5 x = 4 \Leftrightarrow 5^4 = x$
$x = 625$ ---- Answer
Example – 8: Calculate' $\log_2 16 = x$?
Solution: The exponential form of $log_2 16 = x$ is $2^x = 16$.

Therefore, we write $2^x = 2^4$

$x = 4$ ---- Answer (same bases are cancelled).

Example – 9: Determine 'x' $\log_x 6 = \frac{1}{2}$?

Solution: The exponential form of $\log_x 6 = \frac{1}{2}$ is $x^{\frac{1}{2}} = 6$.

Therefore, we can write $\sqrt{x} = 6$

$$\left(\sqrt{x}\right)^2 = 6^2$$

$x = 36$ ---- Answer

Exercise – 3.3

Complete the following **tables** as required**.**

(1).

	Logarithmic Form	Exponential Form
i.		$4^3 = 64$
ii.	$\log_4 2 = \frac{1}{2}$	
iii.		$4^{\frac{3}{2}} = 8$
iv.	$\log_4\left(\frac{1}{16}\right) = -2$	
v.	$\log_4\left(\frac{1}{2}\right) = -\frac{1}{2}$	
vi.		$4^{-\frac{5}{2}} = \frac{1}{32}$
vii.	$\log_8 8 = 1$	
viii.	$\log_8 64 = 2$	
ix.		$8^{\frac{2}{3}} = 4$
x.		$8^3 = 512$
xi.	$\log_8\left(\frac{1}{8}\right) = -1$	
xii.		$8^{-2} = \frac{1}{64}$

Write the following logarithmic expressions in **exponential form**.

(2). $\log_5 25 = 2$

(3). $\log_5 1 = 0$

(4). $\log_2 \frac{1}{8} = -3$

(5). $\log_8 4 = \frac{2}{3}$

(6). $\ln(x+1) = 2$

(7). $\ln y = 5$

Express the following exponential expressions in **logarithmic form**.

(8). $81^{\frac{1}{2}} = 9$

(9). $10^{-4} = 0.0001$

(10). $2^{-3} = \frac{1}{8}$

(11). $\boldsymbol{e^{0.5x} = t}$

(12). $e^x = 2$

(13). $e^3 = y$

Evaluate the following expressions by using the **properties** of **logarithms** (No Calculator).

(14). $\log_3 9$

(15). $\log_5 0.2$

(16). $\log_5 125$

(17). $ln\left(\frac{1}{e}\right)$

(18). lne^4

(19). $2^{\log_2 32}$

Find the value of '$\boldsymbol{x}$' in the following logarithmic expressions.

(20). $\log_2 x = 5$

(21). $\log_3 243 = x$

(22). $\log_4 x = 2$

(23). $\log_{10} 0.1 = x$

(24). $\log_x 1000 = 3$

(25). $\log_x 16 = 4$

(26). $\log_x 3 = \frac{1}{3}$

(27). $\log_x 2 = \frac{1}{3}$

Determine by using calculator. **Round off** the answer up to **four decimal** places.

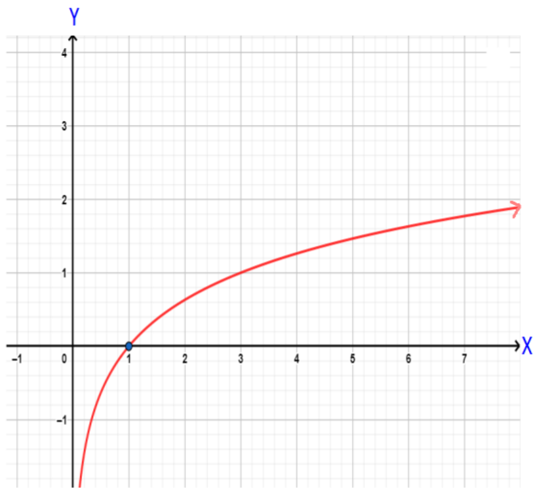
(28). $\log 50$

(29). $\ln 25.3$

(30). **Match** the following **logarithmic functions** with its **graphs**.

a) $f(x) = \log_3 x$	b) $g(x) = -\log_3 x$
c) $k(x) = -\log_3(-x)$	d) $h(x) = \log_3(-x)$

e)


Y
X
4
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-1
-1
0
1
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5
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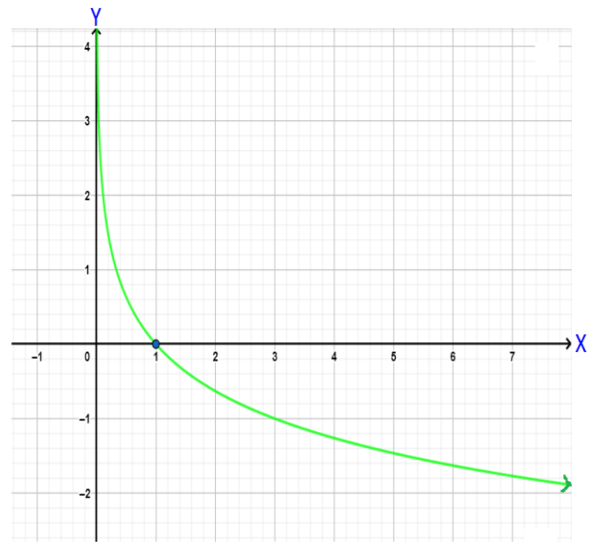
7

f)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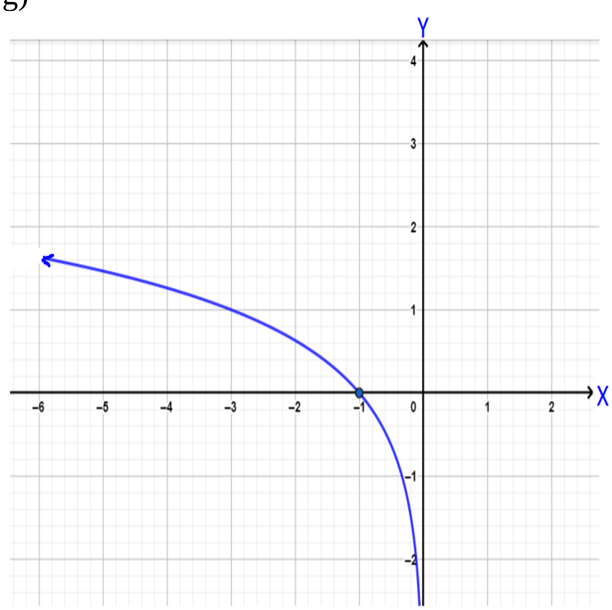
g)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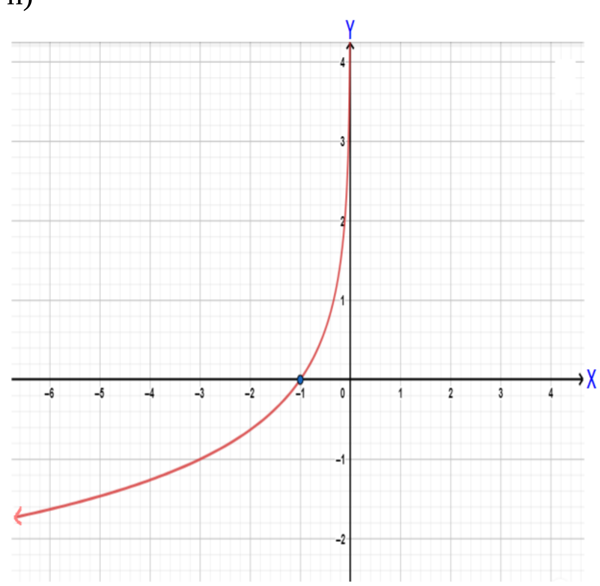
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4

The laws of Logarithms

There are many rules in Logarithms. The expressions involving logarithms can be re-written in different ways. For the time being we will learn the addition and subtraction between the two logarithmic expressions however it is restricted for two expressions.

The addition of two or more logarithmic expressions

When two or more than two logarithmic expressions are required to add. The bases of logarithms must be same then sum converted into multiplication. Mathematically we write

The First law: $\boldsymbol{\log_a A + \log_a B = \log_a(A \times B)}$.

For example: We can write, $\log_{10} 5 + \log_{10} 4 = \log_{10}(5 \times 4) = \log_{10} 20 \approx 1.3010$.

The subtraction of two or more logarithmic expressions

The Second Law: $\boldsymbol{\log_a A - \log_a B = \log_a \left(\frac{A}{B}\right)}$.

For example: We can write, $ln12 - ln6 = \ln\left(\frac{12}{6}\right) = ln2 \approx 0.6931$.

The exponent of logarithmic expressions

The Third Law: $\boldsymbol{\log_a A^C = C \log_a A}$

For example: We can write, $\log_2 2^3 = 3\log_2 2 = 3 \times 1 = 3$.

Changing the Bases in logarithmic expressions

Sometimes we need to change the bases of logarithmic expressions to simplify it easily. The change in bases occurred mostly when bases are other than '10' or 'e'. The following formula is used to apply this operation.

a) Change of Base formula: $\boldsymbol{\log_a x = \frac{logx}{loga} = \frac{lnx}{lnx}}$.
b) Exchange of a base and argument (the number): $\log_a x = \frac{1}{\log_x a}$.

Example: Calculate $\log_5 125$, by using the change of base formula?

Solution: $\log_5 125 = \frac{log125}{log5} = 3$ ---- Answer (changing the bases into 10: $\log_a x = \frac{logx}{loga}$).

Example – 2: Calculate $\log_{49} 7$, by using the change of base formula?

Solution: using the exchange of base and the number: $\log_a x = \frac{1}{\log_x a}$. Here $x = 7$ and $a = 49$.
Therefore, we can write: $\log_{49} 7 \ = \frac{1}{\log_7 49} = \frac{1}{\log_7 7^2} = \frac{1}{2}$ ---- Answer

---

Logarithmic Inequalities

Logarithmic Inequalities are inequalities in which one (or both) sides involve a logarithm. Like exponential inequalities they are useful in analyzing situations involving repeated multiplication, such as in the calculating the interest and exponential decay.

Solving the Logarithmic Inequalities

When the bases on both sides of an inequality are same. Then we have the following rule.

$$log_a f(x) > log_a g(x) = \begin{cases} f(x) > g(x): a > 1 \\ \\ f(x) < g(x): 0 < a < 1 \end{cases}$$

and

$$log_a f(x) < log_a g(x) = \begin{cases} f(x) < g(x): a > 1 \\ \\ f(x) > g(x): 0 < a < 1 \end{cases}$$

Also, for all logarithms with base $a > 0$ but not 1, $\boldsymbol{f(x) > 0}$ and $\boldsymbol{g(x) > 0}$, then choose the most restrictive inequality or interval.

The following examples will make the concept clearer about the solution of logarithmic inequalities.

Example – 1: Solve the logarithmic Inequality: $log_2(2x + 6) > log_2(3x)$?

Solution: Since Base $= 2 > 1$, then the sign ($>$) will not change. Cancelling log_2, on both sides

we write,
$2x + 6 > 3x$
$2x - 3x > -6$
$-x > -6$, (dividing both sides by -1, will change the sign again)
$x < 6$.

Thus, the logarithmic inequality is $x < 6$. Further consider: $2x + 6 > 0$ and $3x > 0$, we obtain: $x > -3$ and $x > 0$. Now drawing the graphs of: $x < 6$, $x > -3$ and $x > 0$, we have

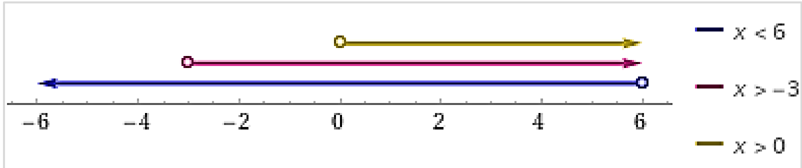


Figure 48: Graph of Logarithmic Inequality

Hence, the most restrictive solution (see Figure 48) is $\mathbf{0 < x < 6 = [0, 6]}$ ---- Answer.

Example – 4: Solve the logarithmic Inequality: $log_{\frac{1}{3}}(8x + 1) > log_{\frac{1}{3}}(2x + 7)$?

Solution: Base= $\frac{1}{3} < 1$, then the sign (>) will change. Cancelling $log_{\frac{1}{3}}$, on both sides we get,

$$8x + 1 < 2x + 7$$

$$8x - 2x < 7 - 1$$
$$6x < 6$$
$$x < 1$$

Thus, the final solution to this logarithmic inequality is $x < 1$. Further consider: $8x + 1 > 0$ and $2x + 7 > 0$. we obtain: $x > -\frac{1}{8}$ and $x > -\frac{7}{2}$. Now drawing the graphs of: $x < 1$, $x > -\frac{1}{8}$ and $x > -\frac{7}{2}$, we have

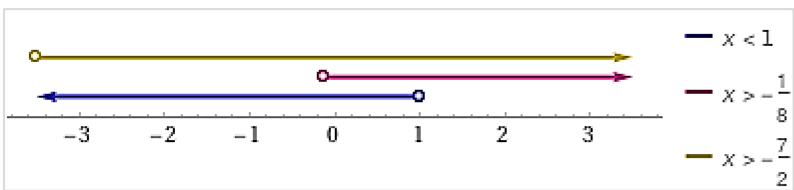


Figure 49: The Graph of Logarithmic Inequality

Hence, the most restrictive solution (see Figure 49) is $\mathbf{0 < x < -\frac{1}{8} = \left[0, -\frac{1}{8}\right]}$ ---- Answer.

Exercise – 3.4

Evaluate the expression **without** using **calculator**. Rather use the **laws** of logarithms.

(1). $\log_4 \sqrt{16}$

(2). $log\left(\frac{1}{\sqrt{1000}}\right)$

(3). $log_4 192 - \log_4 3$

(4). $log_{12} 9 + \log_{12} 16$

(5). $log_2 8^{33}$

(6). $ln\left(\ln e^{e^{200}}\right)$

(7). $3 \log_{10} 1000$

(8). $e^{ln(1000)}$

Use **Laws** of **Logarithm** to **expand** the following expressions

(9). $\log_2 \left(x\ (x-1)\ \right)$ **(10).** $\log_2(AB^2C^3)$

(11). $\log_3(x\sqrt{y})$ **(12).** $\log_2 \frac{(AB)^2}{C^3}$

(13). $ln\left(x\sqrt{\frac{y}{z}}\right)$ **(14).** $\log\left(\frac{x^4y^5}{z^2}\right)$

(15). $\log\sqrt{\frac{x-1}{x+1}}$ **(16).** $ln\sqrt{x} + lny - lnz^4$

Use **Laws** of **Logarithms** to **Combine** the following expressions

(17). $\log_2 x - \log_2 y^2 + \log_2 z^3$ **(18).** $\log_2 t^2 + 2\log_2(xy)$

(19). $\log_3 y^2 - 5\log_3 x - \log_3 k$ **(20).** $\log_4(xy)^3 - \log_4 x + 3\log_4 b$

(21). $3\log_3 x^4 + \log_3(x+4)^2$ **(22).** $\log_5(x^2-1) - \log_5(x+1)^3$

Evaluate by using "**Change** of **base** and calculator". Round off the answer up to **four decimal** places.

(23). $\boldsymbol{log_4 125}$ **(24).** $\boldsymbol{log_8 5}$ **(25).** $\boldsymbol{log_7 2.61}$

(26). $\boldsymbol{log_{0.5} 12.5}$ **(27).** $log_6 55$ **(28).** $log_{2.5} 2.7$

Solve the logarithmic **Inequalities**.

(29). $log_3(5x+3) < log_3(6x)$ **(30).** $log_{\frac{1}{2}}(2x+2) > log_{\frac{1}{2}}(x+6)$

(31). $log_{0.4}(x+3) > log_{0.4}(6x-4)$ **(32).** $log_{\frac{1}{3}}(5x-3) < log_{\frac{1}{3}}(6x+2)$

(33). $log_4(x-1) < log_4(x+4)$ **(34).** $log_{\frac{1}{4}}(3x-11) > log_{\frac{1}{4}}(3x+1)$

Exponential and Logarithmic Equations

The following steps are followed to solve the Exponential and Logarithmic Equations.

Step 1: Arrange the exponential (or logarithmic) statement on any side.

Step 2: Take the logarithm on both sides, and then use laws of logarithm.

Step 3: Solve the equation for a variable exist in the equation.

Example – 1: Solve $10^x = 4$?

Solution: Applying 'Log' of base 10, on both sides, we get

$$log10^x = log4$$

$$x\ log10\ = log4$$ (Using property of common logarithms.)

$$x \times 1 = 0.6021 \approx 0.6021$$ ---- Answer

Example – 2: Solve $\boldsymbol{e^{-x} = 7}$?

Solution: Applying 'ln' on both sides (base 'e')

$$lne^{-x} = \ln 7$$

$$-x = 1.9459$$ (Using property of natural logarithms)

$$x \approx -1.9459$$ -------- Answer

Example – 3: Solve $4 + 3^{5x}\ = 8$?

Solution: On simplifying we have: $3^{5x} = 8 - 4 = 4$

$$3^{5x} = 4$$

$$log3^{5x} = log4$$ (Taking $\log_3$ on both sides)

$$5xlog_3 3 = \log_3 4$$

$$5x(1) = \frac{log4}{log3} \approx 1.2619 \qquad (\because log_3 3 = 1)$$

$$x = \frac{1.2619}{5} \approx 0.2538$$ ---- Answer

Example – 4: Solve $e^{3-5x} = 16$?

Solution: Applying Natural logarithm on both sides

$$lne^{3-5x} = ln16$$

$$3 - 5x = 2.7726$$

$$-5x = 2.7726 - 3 = -0.2274$$

$$x = \frac{-0.2274}{-5} = 0.04548$$ ---- Answer

Example – 5: Solve $5^x = 4^{x+1}$?

Solution: Applying $\log_5$ on both side we get

$$\log_5 5^x = \log_5 4^{x+1}$$

$$x log_5 5 = (x+1)\log_5 4$$

$$x = (x+1)\frac{log4}{log5}$$

$$x(0.1386) = 0.8614$$

$$x = \frac{0.8614}{0.1386} \approx 6.2150$$ ---- Answer

Example – 6: Solve $logx = -2$?
Solution: Writing in exponential form: $x = 10^{-2} = \frac{1}{100}$ ---- Answer
Example – 7: Solve $\log_2 3 + \log_2 x = \log_2 5 + \log_2(x-2)$?
Solution: Combining both sides using by laws of logarithm we get

$$\log_2 3x = \log_2 5(x-2)$$

$$3x = 5(x-2)$$
$$3x = 5x - 10$$
$$3x - 5x = 10$$
$$-2x = 10$$
$$x = \frac{10}{-2} = -5$$ ---- Answer

Example – 8: Solve $\log_3(x+15) - \log_3(x-1) = 2$?

Solution: Combining the logarithms, we have

$$\log_3 \frac{x+15}{x-1} = 2$$

$$\frac{x+15}{x-1} = 3^2 = 9$$ (Using definition of logarithm)

$$x + 15 = 9(x-1)$$

$$x + 15 = 9x - 9$$

$$-8x = -24,$$

$$x = \frac{-24}{-8} = 3$$ ---- Answer

Exercise – 3.5

Solve the equation. Adjust your answer up to **four** decimals.

(1). $4(1+10^{5x})=9$

(2). $2^{3x+1}=3^{x-2}$

(3). $2e^{12x}=17$

(5). $e^{1-6t}=2$

(4). $7^{x+4}=2$

(6). $\frac{5^{2x}}{3}=\frac{4}{5}$

Solve the logarithmic equation for 'x'?

(7). $2\log_b 4+\log_b 5=\log_b 10+\log_b x$

(8). $\log_b 30-\log_b 5^2=\log_b x$

(9). $\log_9(x-5)+\log_9(x+3)=1$

(10). $\log_5(x+1)-\log_5(x-1)=2$

(11). $2logx=log2+log(3x-4)$

(12). $\ln(x+2)-\ln(x-6)=2$

(13). $\log_3(x+2)-\log_3(x-5)=2$

(14). $\log_4(x+2)+\log_4(5)=2$

(15). $\ln(x-1)+\ln(x+1)=2$

Modeling with Exponential and Logarithmic Function

Both exponential and logarithmic functions are widely used to model the mathematical problems. They are used in the modelling of biological problems, business analysis, physics, chemistry and many more.

Exponential Growth (Relative Growth Rate)

A population that experiences exponential growth increases according to the model

$$\boldsymbol{n(t)=n_\circ e^{rt}}$$

where $n(t)=population\ at\ time\ 't'$

$n_\circ=initial\ size\ of\ the\ population$

$r = realtive\ rate\ of\ growth\ (\%)$

$t = time$

Example – 1: (Science-Tech, 2010)**:** The initial bacterium count in a culture is 700. A biologist later makes a sample count of bacteria in the culture and finds that relative rate of growth is 50% per hour.

a) Find a function that models the number of bacteria ‘n’ after ‘t’ hours?
b) What is the estimated count ‘n’ after 12 hours?
c) When will bacteria count to 100000?

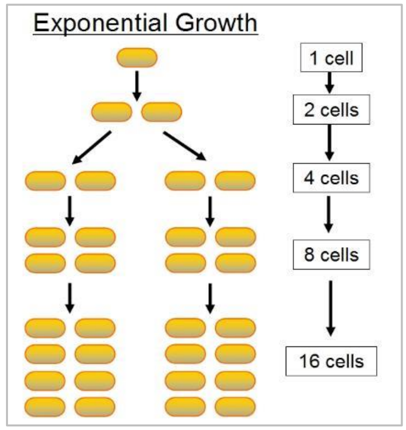


Figure 50: Exponential Growth

Solutions: Here the initial population: $n_0 = 700$

Relative growth rate: $r = 50\% = \frac{50}{100} = 0.5$

a) The required equation: $n(t) = n_0 e^{rt}$

$n(t) = 700e^{0.5\,t}$ ---- Answer

b) when $t = 12$, then

$n(12) = 700e^{0.5\times 12}$

= 282400.1554 ---- Answer

c) If $n(t) = 100000$ (1 million)

$n(t) = 700e^{0.5\,t}$

$100000 = 700e^{0.5t}$

$\frac{100000}{700} = e^{0.5t}$

$142.86 = e^{0.5t}$

$ln142.86 = lne^{0.5t}$

$4.9619 = 0.5t$

$t = \frac{4.9619}{0.5} \approx 9.9237$ ---- Answer

Radio Active Decay Model

The radioactive decay is defined by: $\boldsymbol{m(t) = m_\circ e^{-rt}}$, where $m(t)$ is the mass after 't' years, $m_\circ$ is the initial size of mass and t = time in years. The rate of decay can be calculated by: $r = \frac{ln2}{h}$, where $\boldsymbol{h}$ is the $half - life$ an element or substance.

Example (Wikipedia, 2018)**:** Polonium-210 has a half-life of 140 days. Suppose a sample of this substance has a mass of 500 mg. Then

a) Find a function $m(t) = m_0 e^{-rt}$ that models the mass of the mass remaining after 't' days?
b) Find the mass remaining after 2 days?
c) How long will it take for the sample to decay to a mass of 100 mg?

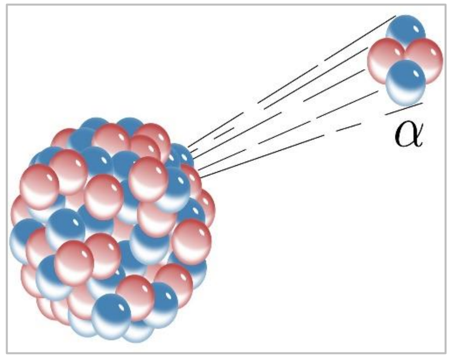


Figure 51: Polonium atomic mass

Solutions: Here the initial mass: m_0 =500 mg

The half-life of polonium: $h = 140$ days

Rate of decay: $r = \frac{ln2}{140} = 0.0049$.

a) the required equation is $m(t) = m_0 e^{-rt}$

$$m(t) = 500\ e^{-0.0049\,t}$$---- Answer

b) when time: $t = 2,\ \ m(t) = 500\ e^{-0.0049\,t}$

$m(2) = 500\ e^{-0.0049\,\times 2} = 495.12$ ---- Answer

c) If mass at some time is $m(t) = 100$, then $t =?$

$100 = 500\ e^{-0.0049\,t}$

$\frac{100}{500} = e^{-0.0049\,t}$

0.2= $e^{-0.0049\,t}$

$ln\ 0.2 = ln e^{-0.0049\,t}$

$-1.6094 = -0.0049 \times t$

$t = \frac{-1.6094}{-0.0049} = 328.4567$ days ---- Answer

Newton's Law of Cooling

If D_0 is the **initial** temperature difference between an **object** and its **surroundings**, and if its surroundings have temperature T_s, then the temperature of the object at time 't' is modelled by the function"

$$T(t) = T_s + D_0 e^{-kt}$$

and

$$D_\circ = |T_\circ - T_s|$$

where 'k' is a positive constant that depends on the type of object.

Example – 1: (Hero, n.d.)**:** A cup of coffee has temperature of $300^0F$and is placed in a room that has a temperature of 80^0F. After 20 minutes the temperature of the coffee is 170^0F.

a) Find a function that models the temperature of the coffee at time 't'?
b) Find the temperature of the coffee after 25 minutes?
c) When will the coffee have cooled to 120^0F?

Figure 52: Newton's Law of Cooling

Solution: Given data

$T_s = 80^0$, $D_0 = 300 - 80 = 220^0$,

a) The required function is $T(t) = T_s + D_0 e^{-kt}$

$$T(t) = 80 + 220e^{-kt} \text{ ---- Answer}$$

b) We need to find the constant 'k'. it is given that after 20 minutes the temperature of the coffee is 170^0F. when $t = 20, T(20) = 170$ then

$$80 + 220e^{-20k} = 170$$
$$220e^{-20k} = 170 - 80 = 90$$
$$e^{-20k} = \frac{90}{220} = 0.409$$
$$lne^{-20k} = ln0.409$$
$$-20k = -0.894$$
$$k = \frac{-0.894}{-20} = 0.045$$

Now using the value of $t = 25$ and $k = 0.045$ in part (a), we get

$$T(t) = 80 + 220e^{-kt}$$
$$T(25) = 80 + 220e^{-0.04\times 25}$$
$$T = 160.9^\circ F \text{ ---- Answer}$$

c) Given that $T(t) = 120^\circ F$, then

$$T(t) = 80 + 220e^{-kt}$$

$$120 = 80 + 220e^{-0.04\,t}$$

$$120 - 80 = 220e^{-0.04\,t}$$

$$40 = 220e^{-0.04\,t}$$

$$\frac{40}{220} = e^{-0.04\,t}$$

$$ln\left(\frac{40}{220}\right) = lne^{-0.04\,t}$$

$$-0.04\,t = -1.7148$$

$t = \frac{-1.7148}{-0.04} \approx$ 42.86 minutes ---- Answer

Exercise – 3.6

NOTE: The following Questions (1 – 6) are chosen from the book of Pre – Calculus for Calculus authored by (James Stewart, 2012).

(1). The relative growth rate of the population of a certain species of fish is 1.8% per year. It is found that the population in 2010 was 10 million.

a) Find an exponential function that models the populations 't' years after 2010
b) Estimate the fish population in the year 2020?

(2). The fox population in a certain region has a relative growth rate of 5% per year. It is estimated that the population in the year 2000 was 15000. then

a) Find a function $n(t) = n_0e^{rt}$ that models the population 't' year after 2000
b) Estimate the fox population in the year 2025?
c) After how many years will the fox population reach 50,000?

(3). The half-life of radium-226 is 1600 years. Suppose we have 19 mg of sample

a) Find a function $m(t) = m_0e^{-rt}$ that models the mass of the mass remaining after 't' years?
b) How much of the sample will remain after 3000 years?
c) How long will it take for the sample to decay to a mass of 15mg?

(4). The half-life of cesium-137 is 25 years. Suppose that we have 5 mg of sample

a) Find a function $m(t) = m_0e^{-rt}$ that models the mass of the mass remaining after 't' years?
b) How much of the sample will remain after 100 years?

c) After how long will only **3** gm of the sample remain?

(5). A hot bowl of soup is served at a dinner party. It starts to cool according to Newton's Law of Cooling, so its temperature at time '**t**' is given by

$$T(t) = 60 + 140e^{-0.04t}$$

where t is measured in minutes and T is measured in ^{0}**F**.

a) What is the initial temperature of the soup?
b) What is the temperature after **15** minutes?
f) After how long will the temperature be **90^{0}F**?

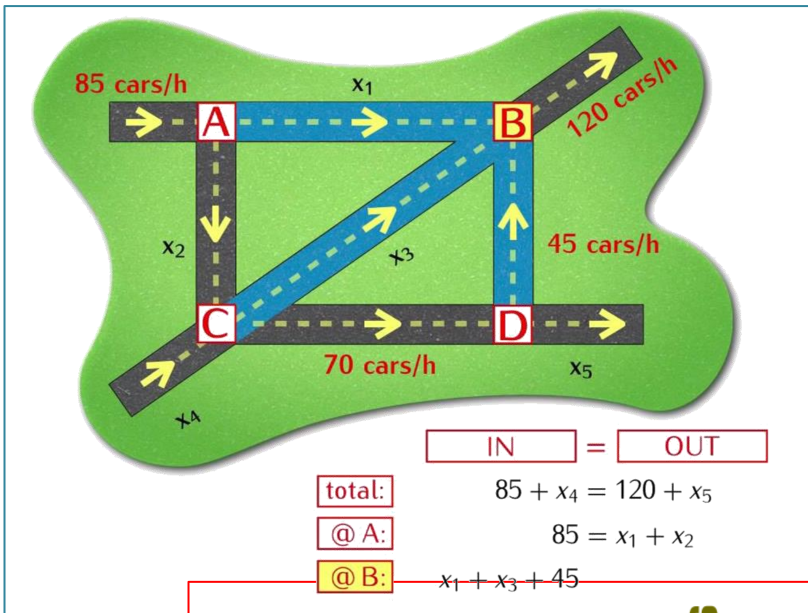
(6). Newton's Law of cooling is used in homicide investigations to determine the time of death. The normal body temperature is $98.6^{0}F$. Immediately following the death, the body begins to cool. It has been determined experimentally that the constant in Newton's law of cooling is approximately $k = 0.1947$, assuming that time is measured in hours. Suppose that the temperature of the surrounding is **60^{0}F**.

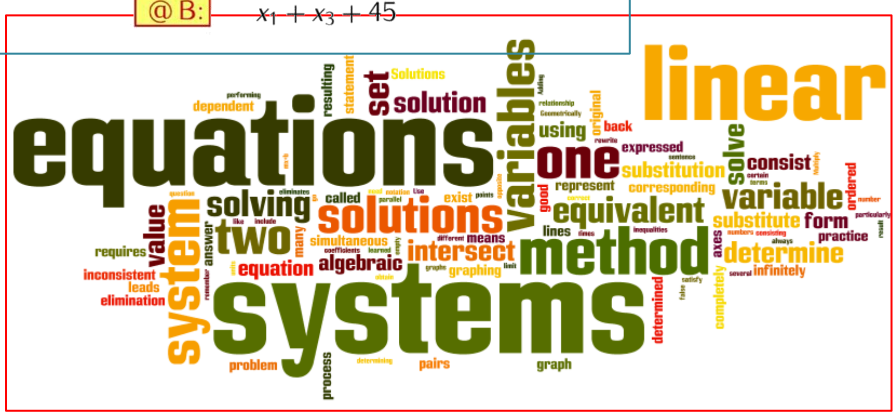
a) Find a function **T(t)** that models the temperature t hours after death
b) If the Temperature of the body is now **72^{0}F**, how long ago was the time of death?

CHAPTER – 4 THE SYSTEM OF LINEAR EQUATIONS AND LINEAR INEQUALITIES IN TWO VARIABLES

OBJECTIVES: Upon successful completion of this course, students will be able to

- Solve two variables' equations and inequalities and sketch their graph.
- Interpret a series of three simultaneous inequalities of two variables, display them graphically and determine the solution set.





System of Linear Equations in Two Variables

General Form: Two linear equations together form a system of two linear equations.

$$ax + by = p$$
$$cx + dy = q$$

where x and y are variables and in each equation any one variable can be the zero but each equation must have at least one variable in it.

Note: The system of equations is called linear if all variables have the power one.

Example: The system of linear equation in two variables

$$3x - y = 7$$
$$2x + 3y = 1$$

Solving System of Linear Equations in Two Variables

Solution of system is a value of x & y which can satisfy both equation in same time.

Possible solution of system of linear equation in two variables are

a) No solution
b) One Solution
c) Infinite solutions

Methods for Solving the System of linear equation

There can be many other methods to solve the system of linear equation such as comparison method, equating the coefficients of equations, solving by matrix method and many others. However, on this level of study we will discuss two methods which are called as substitution method and graphical method.

Substitution Method

a) Solve any one equation for any one of the variables.
b) Variable value substitute in other equation and solve for another variable.

If each value of a variable satisfies both equation in same time, they are called the solution of system of linear equations.

Example – 1: Find the solution of following system. $\begin{cases} 3x - y = 7 \\ 2x + 3y = 1 \end{cases}$

Solution: $3x - y = 7$ -------- (1)

$2x + 3y = 1$ -------- (2)

Solving equation (1) for $y = 3x - 7$ ----- (3)

Now, substituting (3) in (2) we get

$$2x + 3(3x - 7) = 1$$

$$2x + 9x - 21 = 1$$

$$11x = 22$$

$$x = 2$$

Then $y = 3(2) - 7 = 6 - 7 = -1$

Answer: $x = 2$, $y = -1$

Graphical Method

The coordinates of the points where the graph of each equation intersect is called the solution of system of linear equation in two variables.

Example 2. Find the solution of system (By Graphical Method)

$y = 2x + 4$ ---- (1)

$y = 3x + 2$ ----- (2)

Solution: Here we need to find the intercepts of the above equations and then plot in the graph paper. The point of intersection would be the solution of the system of equations.

From (1): If $y = 0$, then $x - interept = -2$

From (1): If $x = 0$, then $y - interept = 4$

&

From (2): If $y = 0$, then $x - interept = -\frac{2}{3} \approx -0.667$

From (2): If $x = 0$, then $y - interept = 2$

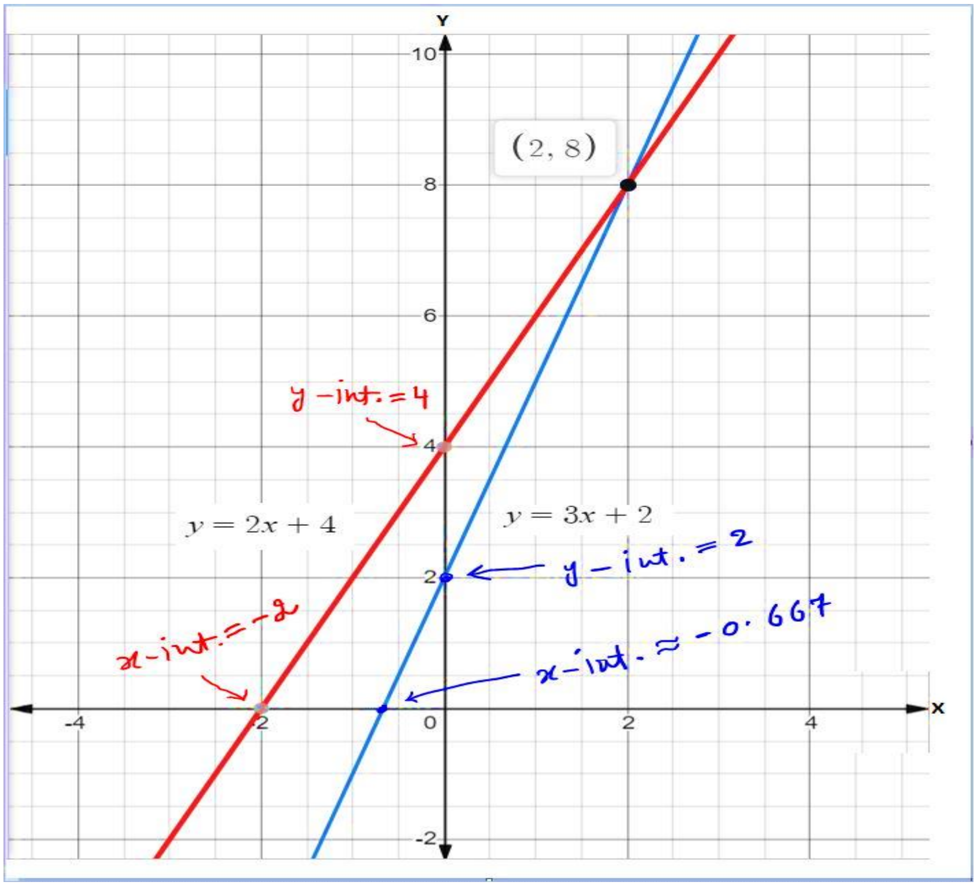


Figure 53: System of Linear Equations

Thus, the solution $(\boldsymbol{x}, \boldsymbol{y}) = (\mathbf{2}, \mathbf{8})$

Exercise – 4.1

Find the **solution** by using the **Substitution** method.

(1). $\boldsymbol{x - y = 1}$
$\boldsymbol{4x + 3y = 18}$

(2). $\boldsymbol{3x + y = 1}$
$\boldsymbol{5x + 2y = 1}$

(3). $\boldsymbol{2x + y = 7}$
$\boldsymbol{x + 2y = 2}$

(4). $2x - 3y = -2$
$y + 4x = 24$

(5). $\boldsymbol{7x - 28y = 5}$
$\boldsymbol{3x + y = 12}$

(6). $5x + 2y = -9$
$-y + 5x = 27$

Find the **solution** by **Graphical** Method.

(7). $2x - y = 4$
$3x + y = 6$

(8). $5x + 6y = 2$
$x - 4y = 5$

(9). $2x + y = 2$
$x - y = 4$

(10). $2x + 3y = 9$
$x - y = 2$

Linear Inequality in Two Variables

Standard Form: In linear inequalities we come across the following formats.

$ax + by < c$, $ax + by \geq c$, $ax + by > c$ and $ax + by \leq c$ where a, b, c = constant not both $a$and $b \neq 0$

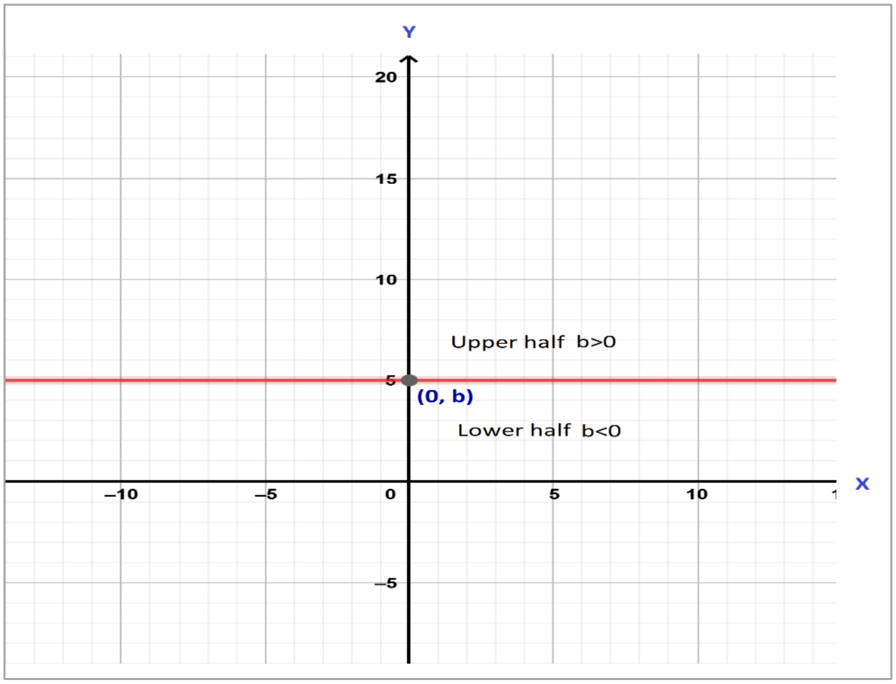
Steps to Solve Linear Inequality

A linear inequality splits the coordinate plane into two halves by a borderline where one half represents the solutions of the inequality. The borderline is dashed for > and < and solid for ≤ and ≥. The half-plane that is a solution to the inequality is generally shaded

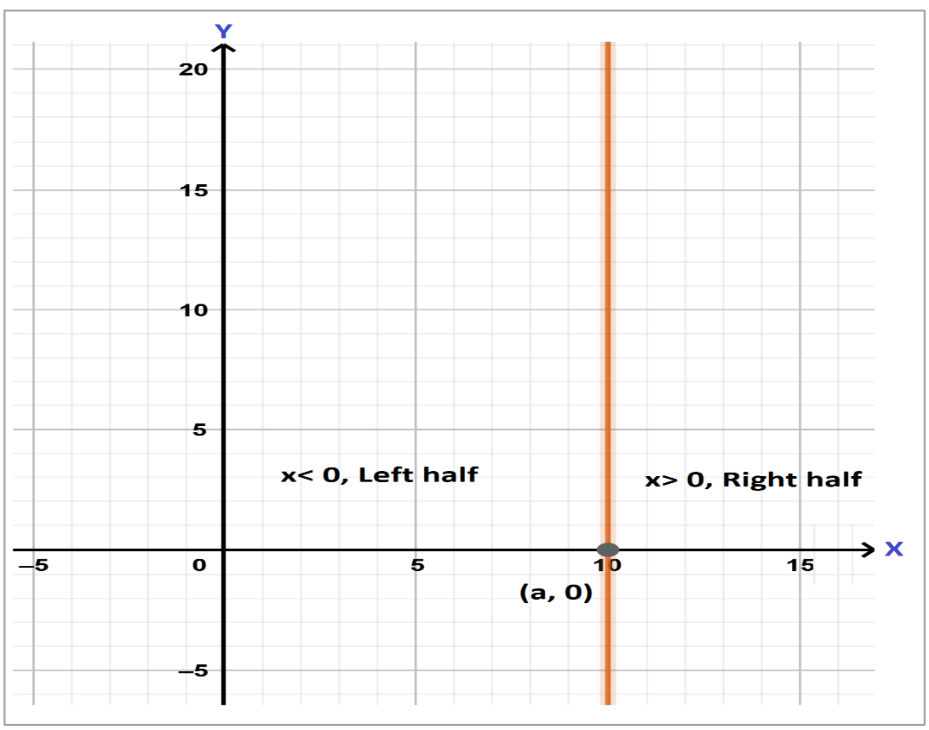
the graph of non-vertical line $y = mx + b$ separates the plane into three distinct parts

(1) The line itself involving of all points (x, y) whose coordinate satisfy the equation $y = mx + b$.
(2) The region above the line, involving of all point (x, y) whose, coordinate satisfy the inequality: $y > mx + b$
(3) The region below the line, involving of all point (x, y) whose coordinate satisfy the inequality: $y < mx + b$

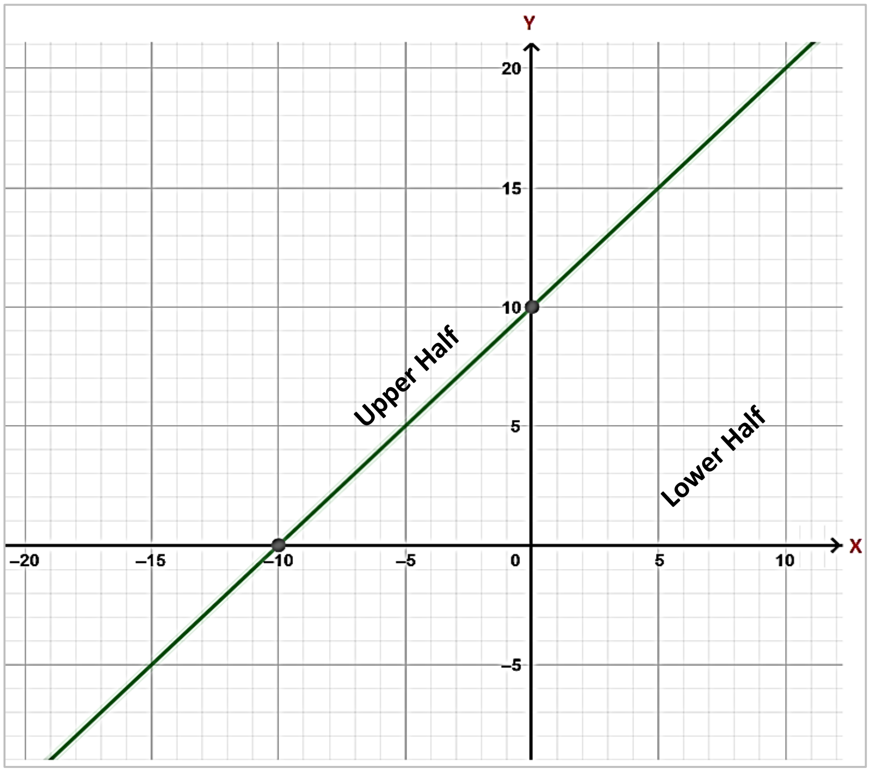
Horizontal Line Graph



Vertical Line Graph

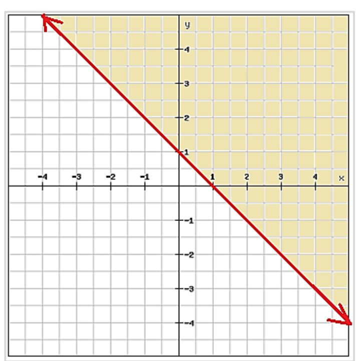
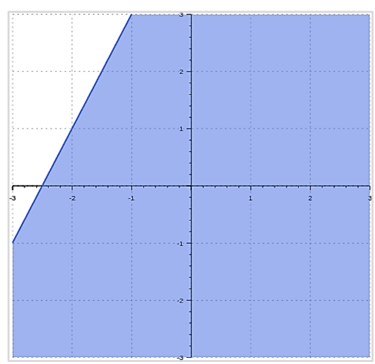


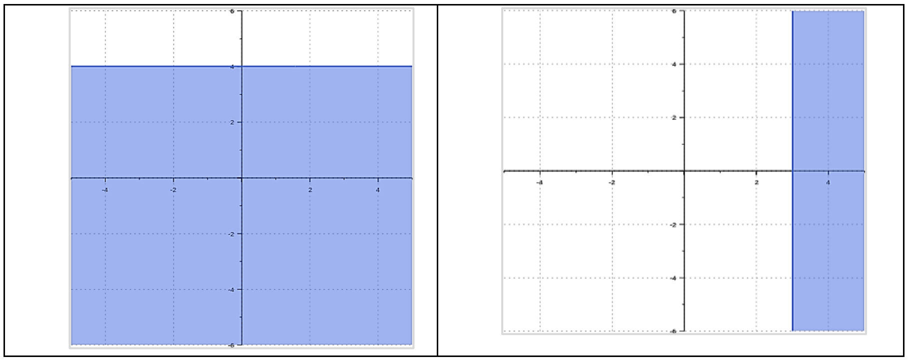
Non-Vertical Line Graph



Example – 1. Solve the following inequalities graphically.

Solution: By using GeoGebra Program we can try these graphs and observe the region which gets shaded, because the shaded region shows the solution of an inequality.

(1). $y \geq -x + 1$ 	(2). $y < 2x + 5$ 
(3). $y \leq 4$	(4). $x \geq 3$



The System of Linear Inequalities

System of linear equation contains two or more linear equations. The solution of a linear system is the ordered pair that is a solution to all equations in the system. after solving each inequality, find the shared region. that shared region is called the solution of the system of inequalities.

Example – 1. Solve the system of inequalities: $\begin{cases} y \geq -2 + 10 \\ y \geq x - 2 \end{cases}$

Solution: To solve the inequalities $y \geq -2x + 10$ and $y \geq x - 2$, we need to find the common region, and the common region would be the solution of the given system of linear inequalities.

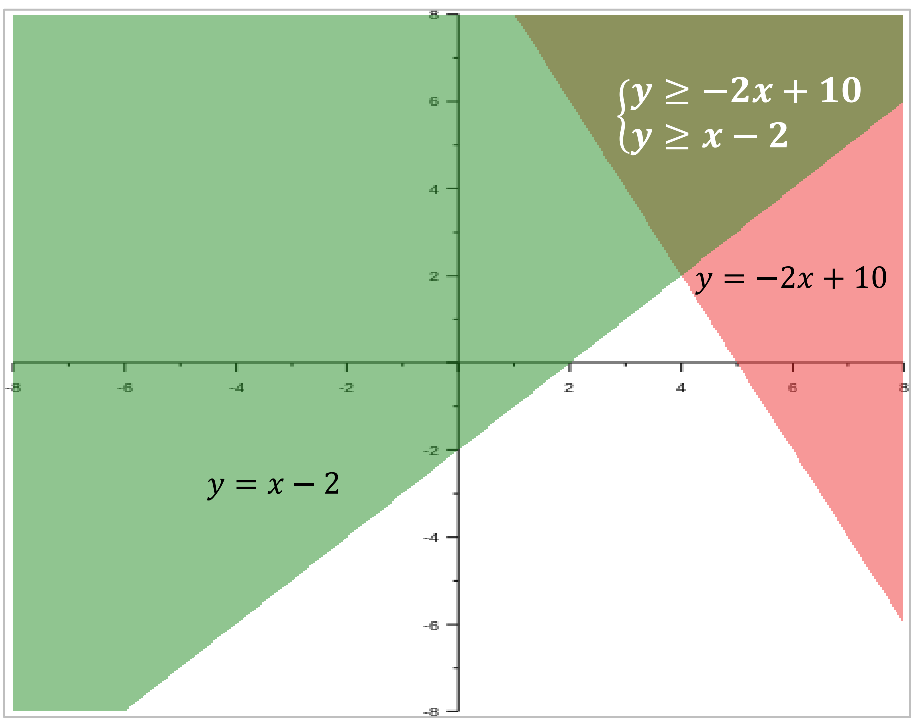


Figure 54: System of Linear Inequalities
Source: (Andy, 2009)

Example-2. (Andy, 2009)**:** Solve the following system of inequalities: $\begin{cases} x + 3y \leq 12 - - - (1) \\ x + y \leq 8 - - - (2) \\ x \geq 4 - - - (3) \end{cases}$

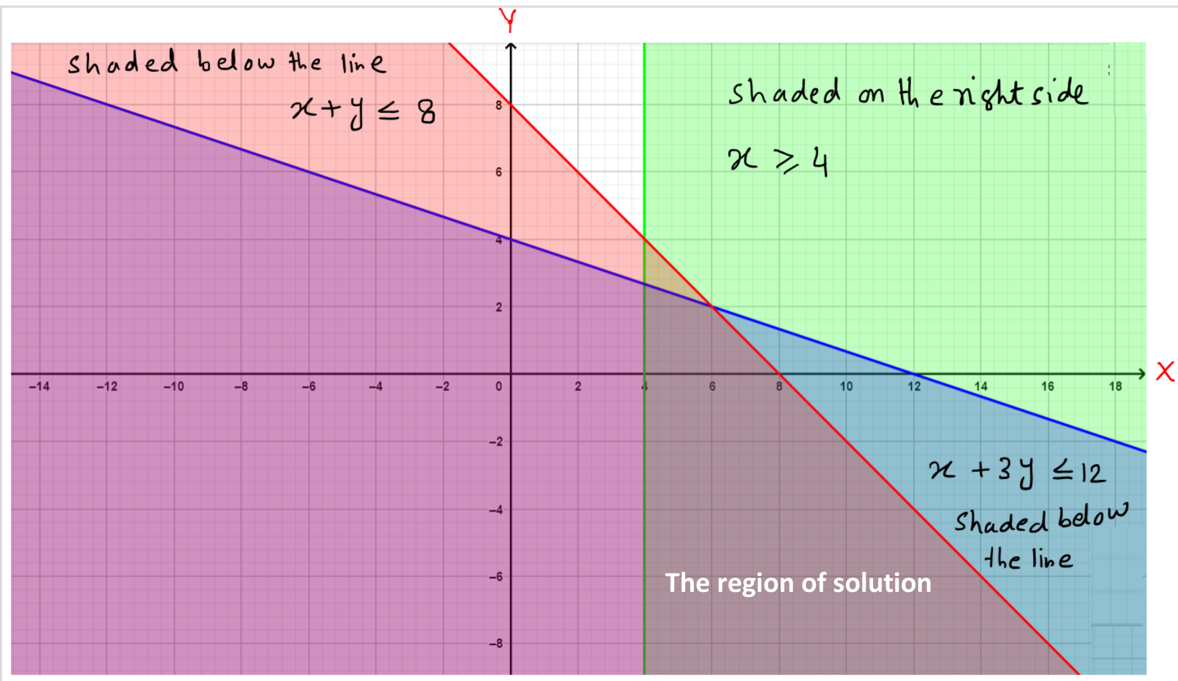
Solution: To draw the graphs, we need to find the intercepts for each inequality and then plot on the graph paper and finally shade the region according to sign used in an inequality.

For (1): the $x - intercept = 12$ & the $y - intercept = 4$.

For (2): the $x - intercept = 8$ & the $y - intercept = 8$.

For (3): the $x - intercept = 4$.
From (1): $y \leq \frac{12-x}{3}$, means shade the lower region of the same line.
From (2): $y \leq 8 - x$, means shade the lower region of the same line.
From (3): $x \geq 4$, means shade the right region of the same line.

The common region of all three inequalities shows the solution for this system of inequalities.



Exercise – 4.2

Solve the following **Inequalities** and **draw** their graphs?

(1). $3x + 4y > 2$

(2). $3x + 5y \geq 12$

(3). $x - y < 4$

(4). $-3y + 5x > 10$

(5). $y > -4$

(6). $x < -5$

(7). $y \leq -4$

(8). $x \geq -5$

(9). $5 - x > 2 + 3y$

(10). $-9y < 3x + 6$

Solve the given system of Linear **inequalities** by graphical method?

(11). $\begin{cases} 2x + 3y > -6 \\ 3x - y < 6 \end{cases}$

(12). $\begin{cases} x - 3y > -4 \\ x + y \geq -4 \end{cases}$

(13). $\begin{cases} x - y > 4 \\ x < 2 \\ y > -54 \end{cases}$

(14). $\begin{cases} y < 2x + 4 \\ x \geq \_2 \\ y < 1 \end{cases}$

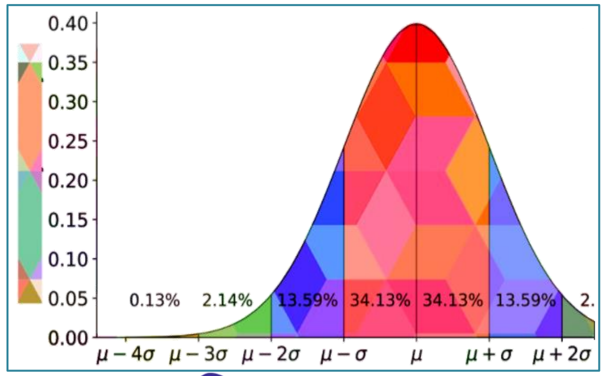
(15). $\begin{cases} -5x + 4y > 2 \\ x - 2y < -5 \end{cases}$

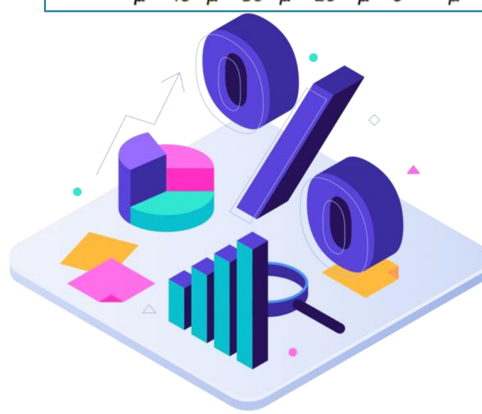
(16). $\begin{cases} y - 3x > 4 \\ y \leq 4 \\ x > -3 \end{cases}$

CHAPTER – 5 INTRODUCTION TO STATISTICS AND PROBABILITY

OBJECTIVES: Upon successful completion of this course, students will be able to

- Define the basic concepts of descriptive statistics, mean, median, mode and summarize data into tables and simple graphs (bar charts, histogram, and pie chart).
- Define and formulate basic probability concepts, permutations and combinations.





Introduction to Statistics

Statistics: The branch of mathematics dealing with the study of the collecting data, organizing the data, analysing the data, Interpretation and presentation of data is called statistics.

Data: The numerical figures obtained from any field of study are known as data. Data is actually the useful information we obtained about something.

Types of Data

1. **Un – Grouped Data:** Numerical facts which are obtained on the first hand and recorded as they stand are known as un – grouped data. For example: Ten Players of a football team have the following Scores: $\{5, 4, 3, 5, 2, 1, 0, 3, 5, 4\}$.

2. **Grouped Data:** When the data have gone through some Statistical process such as the data may be classified into certain groups, rows or columns known as grouped data. For example:

Marks (x)	10 – 20	20 – 30	30 – 40	40 – 50	50 – 60	60 – 70	70 – 80
Number of Students (f)	5	3	4	3	3	7	9

Important statistical Terms

Class: In above table there are 7 classes, each class has two limits. For example: In the above table, the class (10 – 20), 10 is Lower limit & 20 is Upper Limit**.**

Class limit: $\boldsymbol{l} = Upper\ limit - \ Lower\ limit$.

For example**:** In the above table, the class (20 – 30), the class interval: $l\ =\ 30 - 20\ = 10$.

Mid – Value ($\boldsymbol{x}$) $= \frac{\text{Upper limit+ Lower limit}}{\mathbf{2}} = \frac{10+20}{2} = \frac{30}{2} = 15.$

Note: Class interval must be **same** for all classes.

Frequency: The rate at which the something is repeated, means the number of repetitions of a same thing. For example, in the above table $f\ = 5$ means that there 5 students who obtained marks from 10 to 20.

Measure Of Central Tendency (Mean, Median & Mode)

In real-life situations, Measure of Central Tendency is helpful to describe data from a single number that is most representative of the entire collection of numbers. Such a number is called a measure of central tendency.
The most commonly used measures are mean, median and mode.
Mean of the data: The average result of some data is called the mean or average of the data. There are two formulas used to find the mean of some data depending on the type of data.

Mean

The most common understanding of mean (or average) = $\frac{\text{Sum of all data}}{\text{Number of data}}$. Mathematically we write as: $\overline{x} = \frac{x_1+x_2+.....x_n}{n}$. Therefore, arithmetic Mean for ungrouped data is given by:

- Mean: $\overline{x} = \frac{\Sigma x}{n}$, for ungrouped data.
- Mean: $\overline{x} = \frac{\Sigma xf}{\Sigma f}$, for grouped data.

Median

The middle number 'n' of the data is known as median of the same data. For the sake of correct median, the ordering (ascending/descending order) the data is must. If n is odd, the middle number after being ordered is the median of the data but if n is even, the median is the average of the two middle numbers.

To find the median of any set of data, arrange the data in ascending or descending order.

- If the number of entries (n) is odd, then Median $= \left(\frac{n+1}{2}\right)^{th} entry.$
- If the number of entries (n) is even, then Median $= \frac{1}{2}\left(\left(\frac{n}{2}\right)^{th} entry + \left(\frac{n}{2}+1\right)^{th} entry\right).$

Mode

The most frequent value in the data is called the **mode** of the data. It is possible to have more than one mode and it is also possible to have no mode or every number is mode (if all numbers are different). We can also define as under; the highest frequency number of the data is known as mode of the data.

Exercise – 5.1

(1). Determine the **mean** of the following data**:** 23, 26, 76, 10, 12, and 13.

(2). Determine the **mean** of the following data**:** 23.4, 54.6, 6, 77, 89.6, 90, 122, and 213.

(3). The marks obtained by 20 students of Class IX of a School in an English paper consisting of 100 marks are presented in the table below. Determine the mean of the marks obtained by the students.

Marks (x)	10	20	36	40	50	56	60	70
Students' Number (f)	1	1	3	4	3	2	4	2

(4). Find the **mean** from the below table.

CLASS INTERVAL	NO. OF STUDENTS (f)	$CLASS\ MARK\ (x)$
$10-26$	2	17.5
$26-40$	3	32.5
$40-55$	7	47.5
$55-70$	6	62.5
$70-85$	6	77.5
$85-100$	6	92.5
TOTAL		

(5). The distribution of daily wages of 40 workers of factory are as following. Determine the mean daily wages.

DAILY WAGES (IN DOLLAR)	100 – 140	140 – 180	180 – 220	220 – 260	260 – 300
NO. OF WORKER (f)	12	10	8	6	4

(6). The daily expenditure on food of 25 house hold locality is as following. Find the mean expenditure.

Expenditure (in dollars)	100 – 120	120 – 140	140 – 160	160 – 180
Number of houses holds (f)	6	7	10	2

(7). If the mean weight of 25 **students** is 53 kg in a class, and the weight of a **teacher** is included the mean weight increases by 2.5 kg. Find the **weight** of a **teacher**?

(8). Determine the **mean** of the first 10 even natural numbers.

(9). Determine the value of $\boldsymbol{x}$, if the arithmetic **mean** of numbers $10, 8, 0, x$ and 4 is $\mathbf{6}$.

(10). Find the **median** of the following data: 3, 7, 0, 1, 9, 2, 11?

(11). Find the **median** of the following data: 12, 15, 67, 55, 23, 45, 45, 3, 13,45?

(12). The following data is arranged in ascending order and the **median** is 20. Find the value of $\boldsymbol{x}$ in the given data: $10, 15, 17, 21, 22, x, 25, 29, 34$ and 37?

(13). The numbers $50, 42, 2x + 10, 2x - 8, 12, 11, 8, 6$ are written in descending order and their **median** is 25, find x.

(14). In 10 cricket matches wickets taken by a bowler are given below. Determine the **mode** of following data: 2, 3, 4, 9, 0, 2, 1, 3, 2, 6?

(15). Determine the **mode** of 23, 25, 23, 45, 23, 41, 25, 23 and 46

(16). Find the **mode** of 123, 132, 145, 176, 180, and 120

(17). Find the **mode** of 13, 15, 15, 13, 16, 18, 15, 13, 17, 14 and 14.

Measure of Dispersion (Variance & Standard Deviation)

There are two types of dispersions, one is called Variance and the other one is called the Standard deviation.

Standard Deviation of the Data

The standard deviation measures how concentrated the data is around the mean (or how far the data tends to be from the mean value). It is a summary measure of the differences of each observation from the mean. The symbol σ (sigma) is often used to represent the standard deviation of a population.

Standard deviation = $\boldsymbol{S.D.(\sigma)} = \sqrt{\frac{\sum(x-\bar{x})^2}{n}}$

where, $\bar{x}$ is mean or average of data, Σ (sigma) is Greek letter that means summation. Basically, a large standard deviation indicates that the data points are far from the mean, and a small standard deviation indicates that they are clustered closely around the mean.

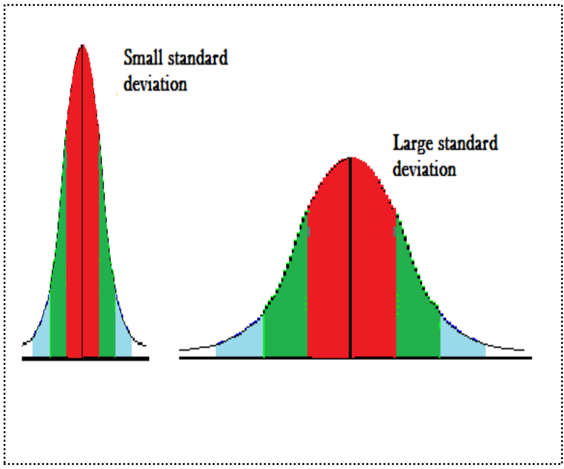


Figure 55: Standard Deviation

Variance in the data

Variance shows that how much the data is varying from the mean of the data $\boldsymbol{Variance(V)} = \frac{\sum(x-\bar{x})^2}{n}$

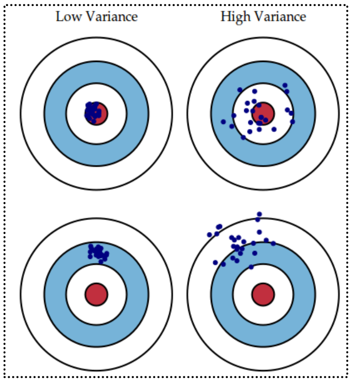


Figure 56: Variance in the data

For example, each of the three populations {0,0,14,14}, {0,6,8,14}, and {6,6,8,8} has a mean of 7. Here the standard deviations are 7, 5, and 1 respectively. The third population has a much smaller standard deviation than the other two because its values are all close to 7. Taken from (Kiddle, n.d.).

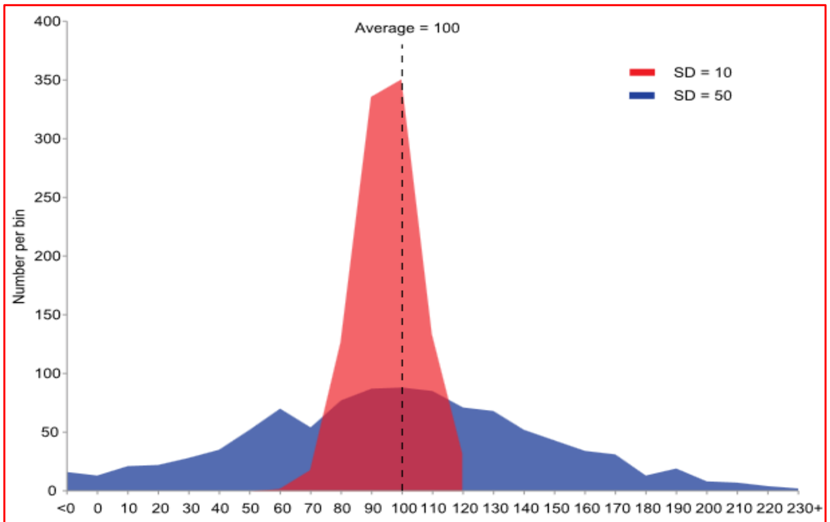


Figure 57: Standard Deviation

Source: (Thompson, n.d.)

Application of Dispersion in Real life

Climatic Variation: As a simple example, consider the average daily maximum temperatures for two cities, one inland and one on the coast. It is helpful to understand that the range of daily maximum temperatures for cities near the coast is smaller than for cities inland. Thus, while these two cities may each have the same average maximum temperature, the standard deviation of the daily maximum temperature for the coastal city will be less than that of the inland city as, on a particular day, the actual maximum temperature is more likely to be farther from the average maximum temperature for the inland city than for the coastal one. Source: (Kiddle, n.d.)

Example-1: Find the **standard deviation** of the given data: 35, 41, 39, 49, 43, 40, 47.

The mean of these 7 items is: $\overline{x} = \frac{\sum x}{n} = \frac{35+41+39+.....+47}{7} = \frac{294}{7} = 42$.

Data, x	$x - \overline{x} = x - 42$	$(x - \overline{x}\,)^2$
35	-7	49
41	-1	1
39	-3	9
49	7	49

43	1	1
40	-2	4
47	5	25
Σx = 294	$\Sigma(x-\overline{x})$ = 0	$\Sigma(x-\overline{x})^2$ = 138

$$V = \frac{\sum(x-\overline{x})^2}{n} = \frac{138}{7} = 19.714$$
$$\sigma = \sqrt{V} = \sqrt{19.714} = 4.44$$

Example:2 A class sat tests in Applied Mathematics. Their results are given below. *Mark, x*:

45 72 63 59 78 64 51 67

Find the **mean** and **standard deviation** for each test.

Solution: Here mean: $\overline{x} = \frac{45+42+\dots\dots\dots+67}{8} = \frac{499}{8} = 62.375$

$\Sigma x^2 = 45^2 + 72^2 + \cdots + 67^2 = 31929.$

So, S. D = $\sqrt{\frac{\sum x^2}{n} - (\overline{x})^2} = \sqrt{\frac{31929}{8} - 62.375^2} = \sqrt{100.48} = 10$ ---- Answer

---

Exercise – 5.2

(1). Determine the **mean** ($\overline{x}$), **variance** (V) and **Standard deviation** (σ) of the following data.

a) 2, 10, 8, 4, 25, 3, 17.
b) 9, 15, 6, 3, 11, 23, 45, 8, 40.
c) 2, 2, 2, 2, 2, 2.
d) 84, 75, 60, 45, 20, 98, 100

(2). The different heights of a type of material are 600 mm, 450 mm ,160 mm 410 mm and 320 mm. Calculate the **variance** and **standard deviation** between their heights.

(3). Find the **standard deviation** of the first 10 natural number?

(4). A booklet has 12 pages with the following numbers of words: 271, 354, 296, 301, 333, 326, 285, 298, 327, 314, 287 and 316. Find the **standard deviation** number of words per page?

(5). The height (in cm) of students of a class is given by $\{163, 158, 167, 174, 148\}$. Find the **standard deviation** and **variance**.

(6). The height (in meter) of the nine palm trees is given by $\{9, 8, 10, 12, 11, 13, 7, 9, 10\}$. Find the **variance** between their heights.

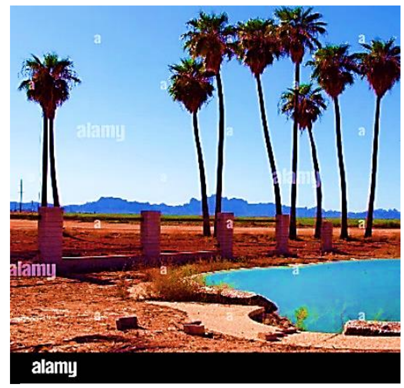


Figure 58: Variance among palm trees

Graphical Representation of Data

Data can be represented in many ways but here we shall focus on three basic methods namely pie chart, bar graph and histogram.

Pie Chart

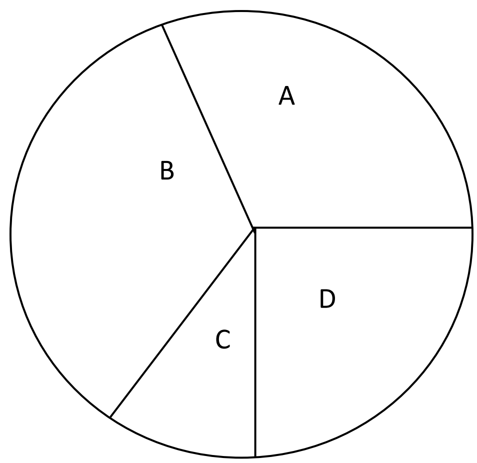
A Pie Chart is a circular chart which is divided into sectors. It is drawn on the circle. Since the degree of the circle is $\mathbf{360°}$**,** each data is to be converted into degree first".

Example – 1: Draw a pie chart for given data.

CONTINENT	AREA (millions of km^2)
A	30
B	25
C	5
D	20

Solution: First, we need to calculate the angle or sector of each continent and then use the information to draw a pie graph.

CONTINENT	AREA (millions of km^2)	Fractional	Central angle
A	30	$\frac{30}{80}=\frac{3}{8}$	$\frac{3}{8}\times 360° = 135°$
B	25	$\frac{25}{80}=\frac{5}{16}$	$\frac{5}{16}\times 360° = 112.5°$
C	5	$\frac{5}{80}=\frac{1}{16}$	$\frac{1}{16}\times 360° = 22.5°$
D	20	$\frac{20}{80}=\frac{1}{4}$	$\frac{1}{4}\times 360° = 90°$
Total	80	Sum of angles $=\frac{16}{16}$	360°



Bar Graph

A Bar Graph is a pictorial representation of the numerical data by a number of bars (Rectangles) of uniform width erected horizontally or vertically with equal spacing between them. Each bar represents only one value of the numerical data and so there are as many bars as the number of values in the numerical data. The height or length of a bar indicates on a suitable scale the corresponding value of the numerical data.

(1). The width of all the bars and the gap between two bars must be uniform.
(2). Bars can be either horizontal or vertical.

Example – 2:

Day	Monday	Tuesday	Wednesday	Thursday	Friday	Saturday	Sunday
Number of T.V	**300**	**400**	**150**	**250**	**100**	**350**	**200**

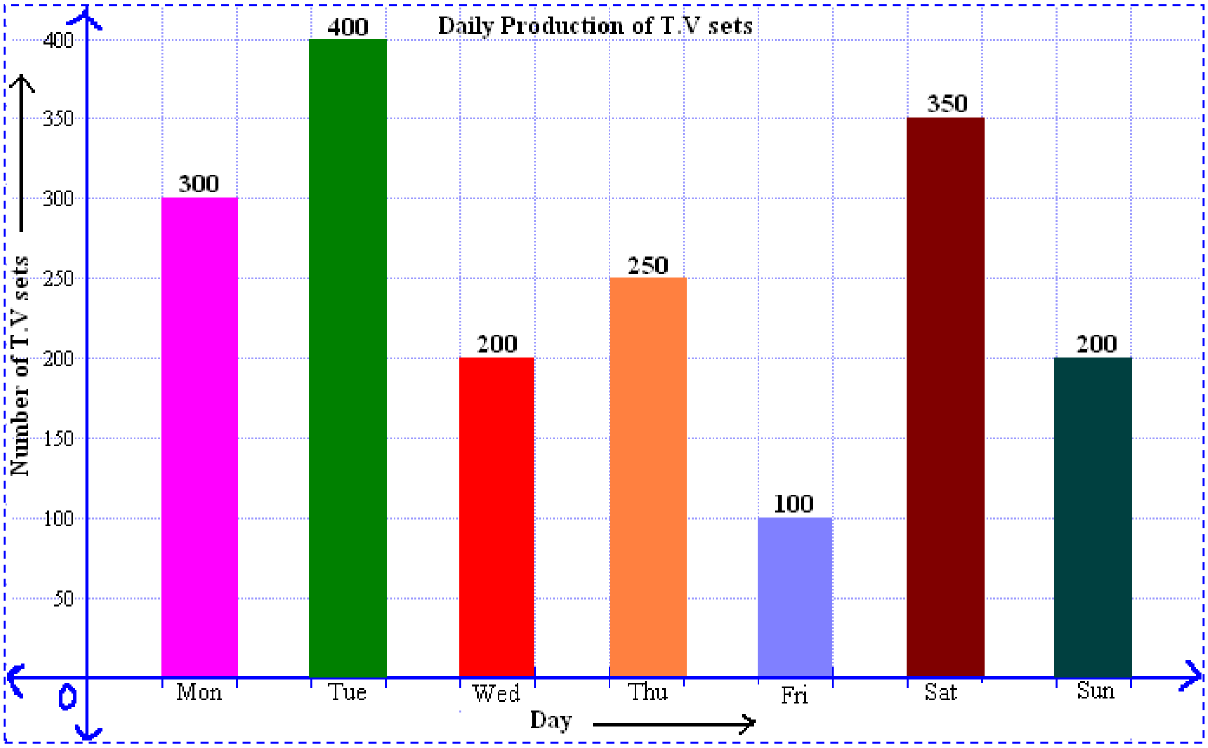


Figure 59: Bar Graph of TV weekly sale
Source: (Gettyimages, n.d.)

Histogram

In a histogram, the area is very important, there should be continuous frequency distribution of data. If the given data is not continuous then we have to first convert it into the continuous form. Here we have represented class intervals on the x axis. The heights of the bars show the frequency. Also, there is no gap between the bars.

Example: 3

CLASS INTERVAL	FREQUENCY
0 – 10	2
10 – 20	10
20 – 30	21
30 – 40	19
40 – 50	7
50 – 60	1
Total	60

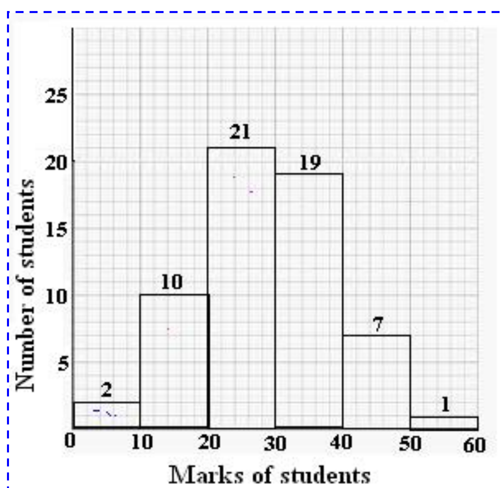


Figure 60: Histogram Source: (Govilkar, n.d.)

Exercise – 5.3

(1). Represent the following data by a Bar diagram

Boys	***P***	***Q***	***R***	***S***
No. of Books with them	8	14	9	5

(2). Draw a Bar Diagram for the below given information.

Month	September	October	November	December
Average Attendance of a class	34	56	20	42

a) In which month the average attendance is minimum?
b) In which month it was less than 25?

(3). Draw a histogram for the given data.

Portfolio Mark	25 – 30	30 – 35	35 – 40	40 – 45	45 – 50
Number of students	5	2	8	5	10

(4). 50% of students go to school-by-school bus, 25% go by walk, 10% go by bicycle and the rest of them go by car. Draw a pie chart

(5). The total area of six continents Africa, Asia, Australia, Europe, North America, and South America is 134 million square kilometers. The area percentages of each continent are shown in the given Pie Chart.

a. Find the area of Asia.
b. Find the area of Europe.
c. How much bigger is Asia than Australia?

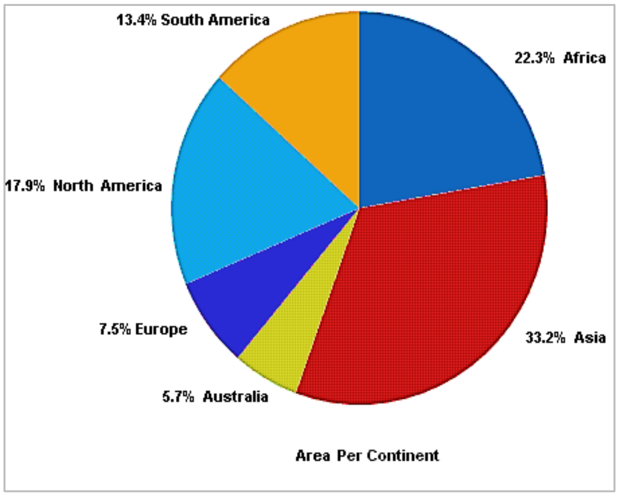


Figure 61: Pie Chart, source: (Dendane, n.d.)

(6). Ahmed earns RO 1400 monthly. His expenses are given in the below table.

Expenses	House Rent	Food items	Cloths	Studies	Savings
Amount (OMR)	300	200	150	250	500

Use a pie chart to represent the data.

(7). The heights distribution of 30 people is shown in the given Histogram.

a) Find the number of people having heights between 169.5 cm and 179.5 cm?
b) How many people have heights more than 189.5 cm?
c) Find the number of people having heights more than 169.5 cm?
d) Find the percentage of people having heights between 149.5 and 179.5 cm.

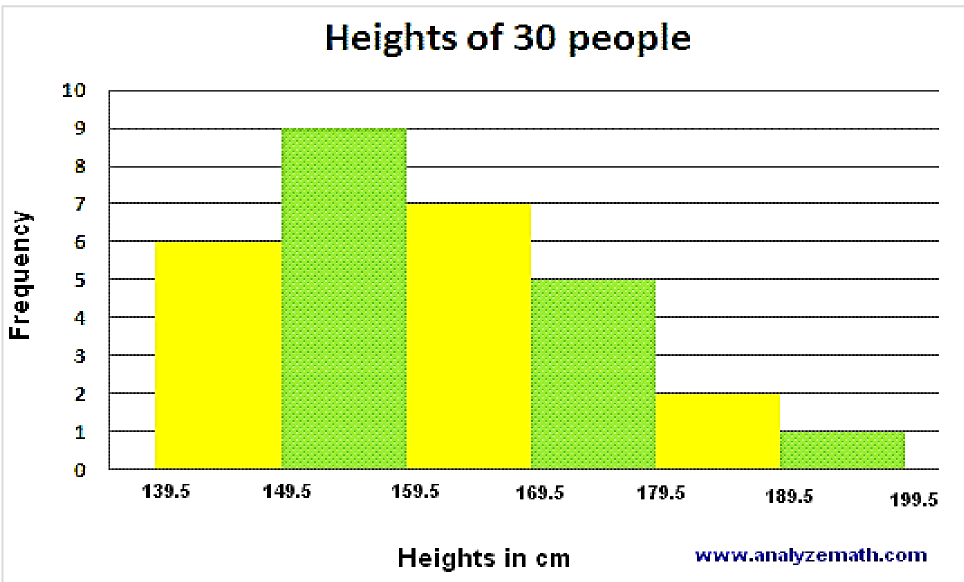


Figure 62: : Histogram, source: (Analyzemath, n.d.)

(8). Essay grade data of an English Class is given below. Represent the data by a **Bar graph** and **Pie chart**.

Grade	**A**	**B**	**C**	**D**	**F**
Frequency	4	7	9	3	2

Introduction to Probability

Probability is a branch of mathematics that deals with calculating the possibility or chances of a given event's occurrence. It is expressed as a number from 0 to 1. An event with probability 1, can be considered as a certainty. An event with a probability 0 can be considered impossibility.
For example, when we toss a coin, probability of getting either head or tail is 1. Because This is the only possibility, assuming the coin lands flat. Also, the probability of getting "heads" is 0.5, because the toss is equally as likely to result in "tails". The probability that the coin will land flat without either side facing up is 0.

Basic Definitions in Probability

Random experiment: Random experiment is one whose results depend on chance that the result cannot be predicted, for example tossing a coin, throwing a die, swiping cards etc.

Trial: Performing a random experiment is called a trial.

Outcomes: The results of a random experiment are called its outcomes. For example, if we toss two coins, the possible outcomes can be $\mathbf{S = \{HH, HT, TH, TT\} = 4.}$ **S**, means the sample space.

Event: An outcome or a combination of outcomes of a random experiment is called an event. For example, in a random experiment of tossing a coin, getting a **Head** or **Tail** is an event.

Probability Formulae

The following rule is used for a theoretical probability or classical probability.

$$P(E) = \frac{n}{m} = \frac{n(E)}{m(S)} = \frac{\textit{No. of outcomes favourable to E}}{\textit{Total No. of Elements in the Sample}}$$

Note that the sum of all probabilities is 1, thus the mathematical form becomes

$$P(E) + P(not\ E) = 1$$

Solved Examples of Probability problems

Example – 1: If a coin is tossed once, find the probability of getting a head and a tail.
Solution: There are two possible outcomes, a head and a Tail.

Let E, be the event of 'getting a Head'.

The number of Outcomes favourable to E, (getting a head) is 1.

$$P\ (E)\ =\ P\ (getting\ a\ Head)\ =\ \frac{No.\ of\ outcomes\ favourable\ to\ E}{Total\ No.\ of\ trials} = \frac{1}{2}$$

In the same way, $P(getting\ a\ Tail)\ =\ \frac{1}{2}$.

Example – 2: A box contains a **red** pen, a **blue** pen and a **yellow** pen, all the pen being of the same size. Mona takes out a pen from the box without looking into it. What is the **probability** that she takes out the is

a) Blue pen b) Red pen c) Yellow pen d) Black pen

Solution: The number of possible outcomes is 3.

a) The number of outcomes favourable to the event "getting a blue pen" is 1.
b) The number of outcomes favourable to the event "getting a Red pen" is 1.
c) The number of outcomes favourable to the event "getting a Yellow pen" is 1.
d) The number of outcomes favourable to the event "getting a Black pen" is 0.

So, P(Blue) = 1/3, P(Red) =1/3 and P (Y) =1/3

There is no black pen in the box. So, probability of getting black pen is impossible means P(Black) = 0.

Example – 3: If a **die** throws once, (i) what is the **probability** of getting a number >4?

(ii) What is the **probability** of getting a number $\leq$ 4.

Solution: (i) Let E, be the event 'getting a number greater than 4'.

The number of possible outcomes is six: 1, 2, 3, 4, 5 and 6.

The outcomes favourable to E are 5 and 6.

Therefore, the number of outcomes favourable to E is 2.

So, P (E) = P (number > 4) =2/6 =1/3.

(ii) Let F be the event 'getting a number $\leq$ 4'

Number of possible outcomes = 6

Outcomes favourable to the event F, are 1, 2, 3, and 4.

So, the number of outcomes favourable to F is 4.

Therefore, P (F) = 4/6 =2/3.

Example – 4: Two players, Hajar and Aisha, play a tennis match. The probability of Hajar winning the match is 0.61. Find the probability of Aisha winning the match.

Solution: The probability of Hajar winning the match = P(H) = 0.61.

The probability of Aisha winning the match = P(A) = 1 – P(H)= 1 – 0.61 = 0.39.

Example – 5: Saif and Abdullah are friends. Find the probability that both will have (i) different birthdays (ii) the same birthday.

Solution: (i) P (Abdullah's birthday is different from Saif's birthday) =364 / 365

(ii) P (Saif and Abdullah have the same birthday)

= 1 – P (both having different birthdays) = 1 – 364 / 365 = 1 / 365.

Example – 6: In the Foundation Program Math class, out of 50 students, 35of them are girls and 15 are boys. The class teacher wants to select one student as a class leader. She writes the name of each student on a separate card, the cards being identical. She puts identical cards with students' name in a bag and draws one card from the bag.

What is the probability that the name written on the card is (i) the name of a girl? (ii) the name of a boy?

Solution: (i) P (card with name of a Girl) =35/50 = 7 /10

(ii) P (card with name of a Boy) = 15 / 50 = 3/ 10

Or P (Boy) = 1 – P (not Boy) = 1 – P (Girl) = 1- 7/10 = 3/10.

Example – 7: Juhaina tosses two different coins simultaneously, find the probability that she gets *at* least one head?

Solution: Let H be 'head' and T be 'tail'. When two coins are tossed simultaneously, the possible outcomes are (H, H), (H, T), (T, H), (T, T).

The outcomes favourable to the event E, 'at least one head' are (H, H), (H, T) and (T, H).

So, the number of outcomes favourable to E = 3

Therefore, P (E) = $\frac{3}{4}$

The probability (Juhaina gets at least one head) = $\frac{3}{4}$.

The Probability of an Event by Tree Diagram

Tree diagram (MathisFun, 2017): A tree diagram is simply a way of representing a sequence of events. A tree diagram is used to calculate the combined probabilities of two independent events.

- The probability of each event is mentioned on the branch.
- The outcome is mentioned at the end of the branch.

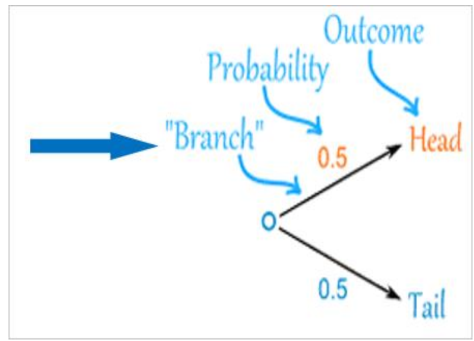


Figure 63: Probability by Tree Diagram

Examples of solving the Probability by tree diagram

Example – 8 (Models-hierarchies-and-systems, n.d.)**:** Flip **a coin** and then rolling **a die**. What is the **probability** of getting **a Head** and **a 4**? Show the event by using a **tree diagram**.

SOLUTION: First list the outcomes of "4" and "not 4".

Each branch signifies a possible outcome, and the fractions show the probability.

For each pair of branches, the sum of the probabilities is 1.

Therefore, P (H, 4) = $\frac{1}{6}$ *of* $\frac{1}{2} = \frac{1}{6} \times \frac{1}{2} = \frac{1}{12}$

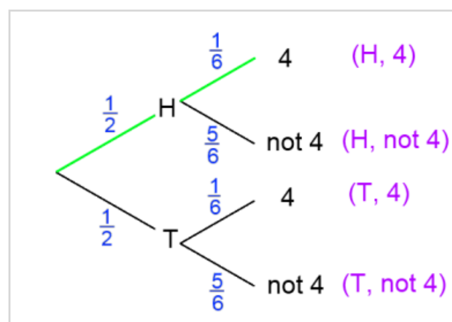


Figure 64: Probability by Tree diagram

Example – 9 (Azuan, n.d.)**:** The below tree diagram shows the tossing of two coins.
To calculate the overall probabilities, follow the below mentioned points.

- Multiply probabilities along the branches.
- Add probabilities down columns

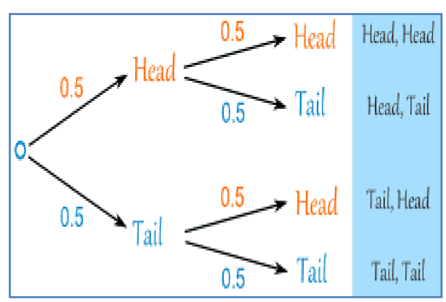


Figure 65: Tree diagram

Probability (Head, Head) = 0.5 × 0.5 = 0.25

Total probability = 0.25+0.25+0.25+0.25 = 1

The probability (getting at least one Head) =

P (H, H) + P (H, T) + P (T, H) = 0.25+0.25+0.25 = 0.75

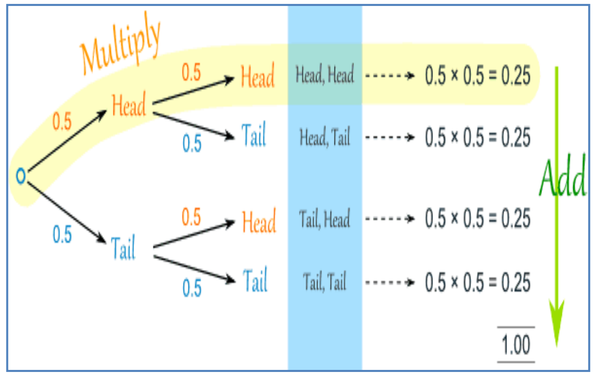
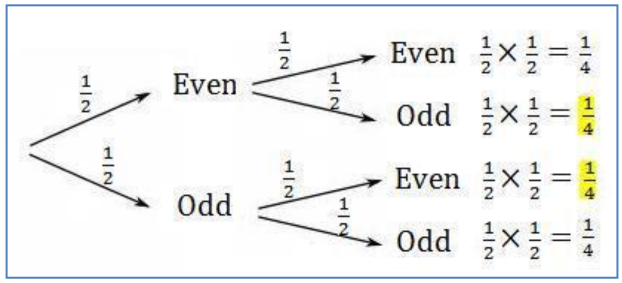


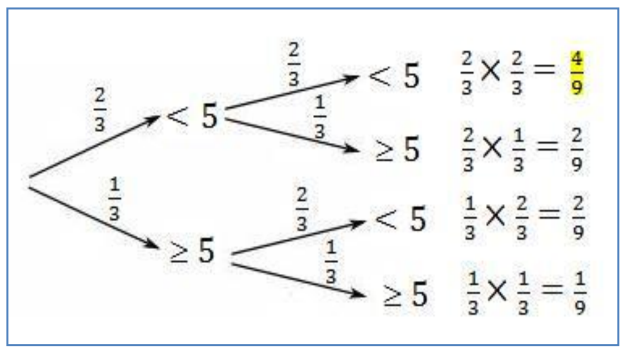
Figure 66: Sum of Probabilities

Example – 10 (Azuan, n.d.)**:** Two dice are thrown together. Find the probability that one number is even and the other is odd (use Tree Diagram).



P (one number is even and the other is odd) $=$ $P(E,O) + P(O,E) = \frac{1}{4}+\frac{1}{4}=\frac{2}{4}=\frac{1}{2}$

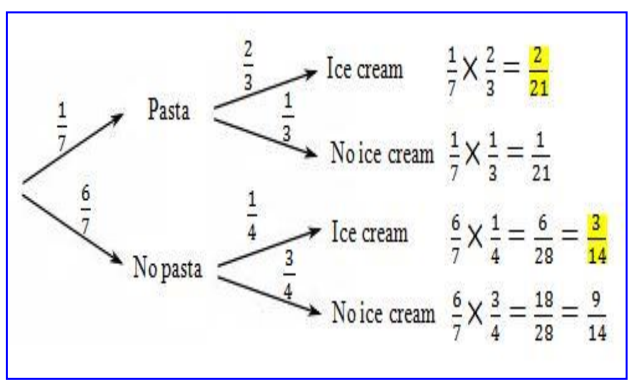
Example – 11: If two dice are thrown together, find the probability of both numbers less than five. (Use the below tree diagram).



P (both numbers are less than five) = $\frac{2}{3}\times\frac{2}{3}=\frac{4}{9}$

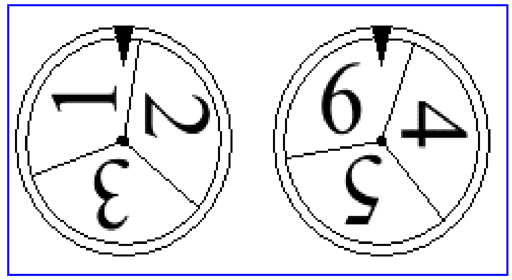
Example – 12: Aliya's favorite food is pasta with ice cream for dessert. Aliya's Mother cooks' pasta once a week.

If she cooks pasta, P (Aliya gets ice cream) = $\frac{2}{3}$, If she does not cook pasta, P (getting ice cream) = $\frac{1}{4}$. What is the probability that Alia gets ice cream for dessert?



Solution: P (Aliya gets ice cream) = $\frac{2}{21}+\frac{3}{14}=\frac{4}{14}+\frac{9}{42}=\frac{13}{42}$

Example – 13 (Onlinemathlearning, n.d.)**:** Ayesha spins 2 spinners; one of them is labeled as 1, 2 and 3, and the other one as 4, 5 and 6.



i) Draw a tree diagram.
ii) P (the spinners stop at “2” and “5”)
iii) P (the spinners do not stop at “2” and “5”)
iv) P (the first spinner does not stop at “1”)

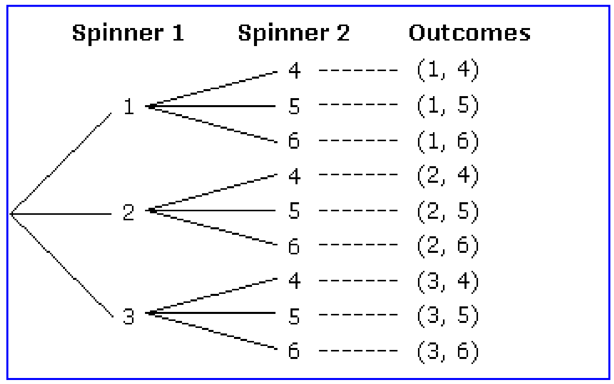
Solution:

i) Tree diagram

ii) P (spinners stop at 2 and 5) = $\frac{1}{9}$

iii) P (spinners do not stop at 2 and 5) = $1 - \frac{1}{9} = \frac{8}{9}$

P (the first spinner does not stop at “1”) =1-P (first spinner stops at 1) = $1 - \frac{1}{3} = \frac{2}{3}$



Exercise – 5.4

(1) Fill in the blanks.

(i) Probability of an event ‘E’ + Probability of the event ‘not E’ is _______.

(ii) The probability of an event that never happen is ________.

(iii) The sum of probabilities of all events is_______.

(iv) The probability of an event is between the numbers ___ and ___.

(2) If P (E) = 0.05, then find P (not E)?

(3) A bag contains mango flavoured toffies only. Maryam takes out one toffee without looking into the bag. What is the probability of getting?

(i). A mango flavoured toffee?

(ii).A mint flavoured toffee?

(4) If a die is thrown once, find the probability of getting

(i). a prime number

(ii). a number between 2 and 5

(iii). an odd number.

(5) A child has a die whose six faces show the letters as given in the figure The die is thrown once. What is the probability of getting ***A***?

Figure 67: Probability of occurring 'A'

(6) Find the probability of getting **a Head** first and then **a Tail** when a coin is tossed **twice**
(Use tree diagram).

(7) Bag A has **3 red balls** and **4 blue balls**. Bag B consists of **5 red balls** and **3 blue balls**. A ball is taken from each bag in turn. Find the **missing probabilities** in the given tree diagram.

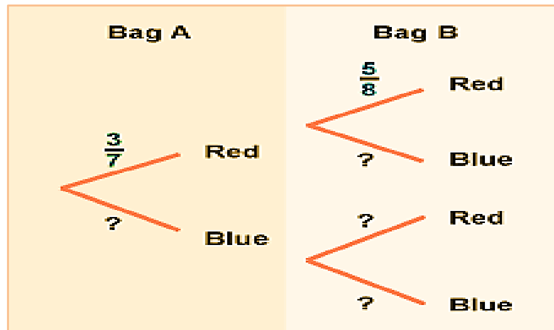


Figure 68: Simple Tree diagram

Permutation and Combination

Permutations and combinations are the basic ways of counting from a given set, to form subsets. If the **order does not matter** then we have a **combination**, if the **order** do **matter**, then we have a **permutation**.

A combination focuses on the selection of objects without regard to the order in which they are selected. A permutation, in contrast, focuses on the arrangement of objects about the order in which they are arranged.

My fruit salad is a combination of apples, grapes, and bananas. Here order does not matter. We can take fruits in any order, that is "bananas, grapes and apples" or "grapes, apples and bananas" and so on, the result will be the same fruit salad.

The combination to the number lock is 472. Now we must consider the order [see (MathisFun, 2017)].

Note: A Permutation is an **ordered** Combination.

Types of Permutation and Combination

(1) with repetition
(2) without repetition

Here we are discussing the permutation and combination without repetition only.
The number of **permutations** of ***n*** objects taken ***r*** at a time, is determined by the following formula:

$$\boldsymbol{P(n,r) = \frac{n!}{(n-r)!}}$$

The number of **Combinations** of ***n*** objects taken ***r*** at a time, is determined by the following formula:

$$\boldsymbol{C(n,r) = \frac{n!}{r!\,(n-r)!}}$$

where $n! = n(n-1)(n-2)(n-3)\cdots 3\times 2\times 1$. The operator '$x!$'read as factorial of a number. For example: $6! = 6\times 5\times 4\times 3\times 2\times 1 = 720$. Note that $0! = 1,\ 1! = 1$.
We cannot directly multiply or divide two factorials. Such as $3!\times 2! \neq 6!$, similarly $\frac{10!}{5!} \neq 2!$. In fact, $3!\times 2! = 3\times 2\times 2\times 1 = 12$, and $\frac{10!}{5!} = \frac{10\times 9\times 8\times 7\times 6\times 5!}{5!} = 10\times 9\times 8\times 7\times 6 = 30240$.

Applications to Permutation and Combination in Real Life

Example – 1 (Gettyimages, n.d.)**:** Let a Judge is observing a cat beauty contest. There are **50** cats, and you must pick the **prettiest**, the **2nd prettiest**, and the **3rd prettiest**. In how many ways can Judge do this?

Figure 69: Decision by Combination

Solution: Note that **order** matters here, so this is a **permutation problem**. There are 50 ways to pick the prettiest cat, 49 ways to pick the 2nd prettiest, and 48 ways to pick the 3rd prettiest. So, there are $50\times 49\times 48 = 117600$ ways of picking these cats.
In general, the number of ways to **permute** (arrange) ***r*** *items* from a set of ***n*** items

$$P(n,r) = \frac{n!}{(n-r)!}$$

$$P(50,3) = \frac{50!}{(50-3)!} = 117600.$$ ---- Answer

Example – 2: You want to buy **3 parrots** from a pet store that has **50 parrots**. In how many ways can you do this?

Solution: Note that **order doesn't** matter here, so this is a **combination problem**. (Mathematically, we are asking for how many subsets of size 3 there are from a set of size 50).

One can do this anyway by picking your 1st parrot, your 2nd parrot and your 3rd parrot. Then you'd get the same answer as above: $50 \times 49 \times 48 = 117600$. But you could have chosen these parrots in **any order**. So, you must divide this result by the number of ways these 3 parrots can ordered. And that is $3 \times 2 \times 1 = 6$. So, your final answer is $\frac{\mathbf{50\times49\times48}}{\mathbf{3\times2\times1}} = \mathbf{19600}$.

In general, the number of ways to choose ***r*** *items* from a set of n *item*

$$C(P, r) = \frac{n!}{r!\,(n-r)!}$$

$C(50,3) = \frac{50!}{3!(50-3)!} = \frac{50!}{3!47!} = \frac{50\times49\times48\times47!}{3!47!} = \frac{50\times49\times48}{3\times2\times1} = 19600$. ---- Answer

---

Key words for Permutation and Combination:

The following key words are helpful in deciding whether to apply the rule of permutation or combination. Sometimes does not work then we must read the problem and understand the scenario and then proceed for solution.

For Permutation: The order of the items or data matters.

- Arrangement, Line up, President/Chief/Head/In charge, First, second, third means selecting the position, ordered, words can be made/formed.

For Combination: The order of the items or data does not matter.

- Group, choose, team, committee, select, pick, combination, unique.

Exercise – 5.5

Determine the following values of **Permutation** or **Combination**.

(1). ${}_6P_3$

(2). ${}_{95}P_1$

(3). ${}_6P_6$

(4). ${}_9P_4$

(5). ${}_6P_3 \times {}_4P_3$

(6). ${}_{99}P_5 \div {}_{99}P_3$

(7). ${}_6C_4$

(8). ${}_{100}C_{100}$

(9). ${}_{1,000,001}C_1$

(10). ${}_5C_3 \times {}_4C_2$

(11). In how many ways can a coach choose three Swimmers from among five swimmers? (Certifikid, n.d.)

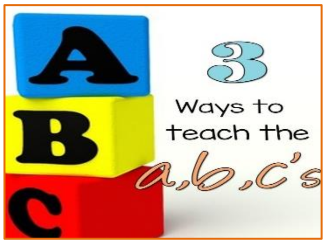
(12). How many 3-**digit** numbers can be made by using the digits **1,2,3,4** and **5** without repetitions**?**

(13). How many **6** letter words can be made by using the letters in the word “TEMPERATURE” without repetitions?

(14). In how many ways a committee of **4** students can be selected from **12** students?

(15). A group of **3** boys and **4** girls is to be formed from a class of **10** boys and **12** girls. How many different groups can be formed from the class?

(16). How many ways are there to choose three alphabets? from **A, B, C, D and E** such that repetition is not allowed.
(Laoblogger, n.d.)



(17). A password has **five** different letters of the alphabet, in which each letter is used only once. How many ways are there to make the password?

(18). How many **4** letter words can be formed (with or without meaning) from the letters of the word, 'LOGARITHMS', if repetition of letters is not allowed?

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APPLIED MATHEMATICS (LEVEL – II)

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