Externalities and Fairness

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Fair Division of Indivisible Goods

- There is a collection of m indivisible items, denoted by $\mathcal{M}$, to be fairly divided between n agents, denoted by $\mathcal{N}$.
- Each agent has a different additive utility function:

$$V_i : \mathcal{M} \to \mathbb{R}^+$$

	b_1	b_2	b_3	b_4	b_5
a_1	5	5	2	8	4
a_2	9	2	5	2	1
a_3	2	5	3	5	6
a_4	8	2	4	9	8
a_5	8	0	2	7	4

- But, how do we enforce a notion of fairness to this division?

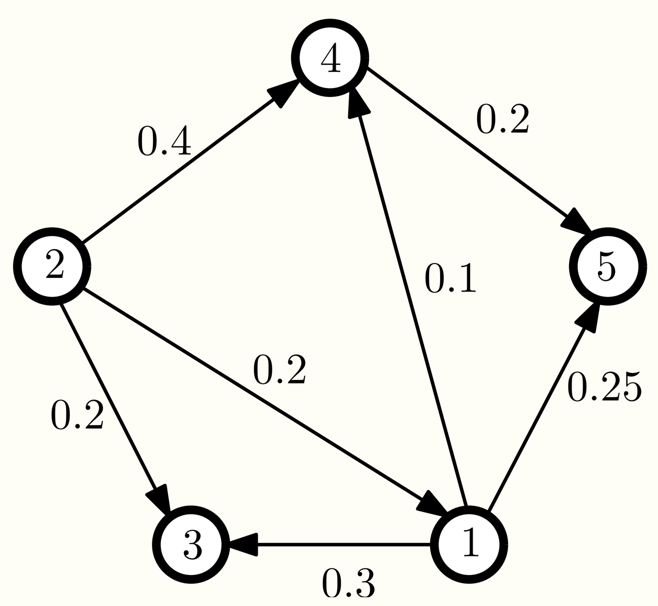
Definition

- **Maximin Share** of agent i ($\mathbf{MMS}_i$): The value of the minimum part in a partition P which maximizes $\min_{j=1}^{n} V_i(P_j)$.
- $P = \{P_1, P_2, \ldots, P_n\}$ is a partition of items.

- There are instances with no $\mathbf{MMS}_i$ guarantee.
- But there are algorithms to find a $3/4\mathbf{MMS}_i$ allocation.

Externalities

- We can consider externalities in fair division as the effect of allocating an item to agent i on other agents.
- We can capture externalities by a directed weighted graph.
- $w_{j,i}$: the influence of allocating an item to agent j on agent i.
- For some α, $\sum_{1 \le j \le n} w_{j,i} \le \frac{1}{\alpha}$.



Motivation

- How to fairly divide m employees with different skill sets into n teams in a company?
- Each employee has a different value for each team.
- The quality of each team affects other teams in different ways.
- For example, the Human Resources team has an almost equal effect on all teams. Or the different teams of a single product have huge impact on each other.
- We can find a provably fair division of the employees.

EMMS Fairness Criteria

- When we include externalities, **MMS** is no longer useful.
- Consider the case where $m < n$. In this case $\mathbf{MMS}_i = 0$, but if $w_{i,j} = 1/n$, we always have a non-zero guarantee.

- *I Cut, You Choose*, to divide fairly between two agents: ⋆

1. Ask one to split the items into two parts.
2. Let the other choose.

- **MMS** is a generalization of cut and choose for n agents.
- Each agent maximizes the utility in the worst allocation.
- We can extend the same idea to capture externalities.

Definition

- The worst allocation of a partition P:

$$\mathcal{W}_i(P) = \min_{\mathcal{A} \in \Omega_P} U_i(\mathcal{A})$$

- $\mathcal{A} : P \to [n]$ is an allocation of P.
- $U_i(\mathcal{A}) = \sum_{j=1}^{n} V_i(P_j) \cdot w_{\mathcal{A}(j),i}$ is the utility of agent i.
- Ω_P is the set of all possible allocations.

- Using the worst allocation function we just defined, we can extend Maximin Share to capture externalities:

Definition

- **Extended Maximin Share** of agent i ($\mathbf{EMMS}_i$): The utility of the worst allocation of a partition P which maximizes $\mathcal{W}_i(P)$.

- $w_1 \ge w_2 \ge \ldots \ge w_n$, sorted weights of incoming edges to i.
- $p_1 \le p_2 \le \ldots \le p_n$, sorted value of parts in a partition P.
- We want to maximize the utility of the worst allocation:

$$\mathbf{EMMS}_i = \max_P \mathcal{W}_i(P) = \max_P \sum_{j=1}^{n} w_j \cdot p_j$$

- Generalizes well-known partitions:
- Maximin partition: $w = [1, 0, \ldots, 0]$
- Minimax partition: $w = [\frac{1}{n-1}, \frac{1}{n-1}, \ldots, \frac{1}{n-1}, 0]$

Related Work

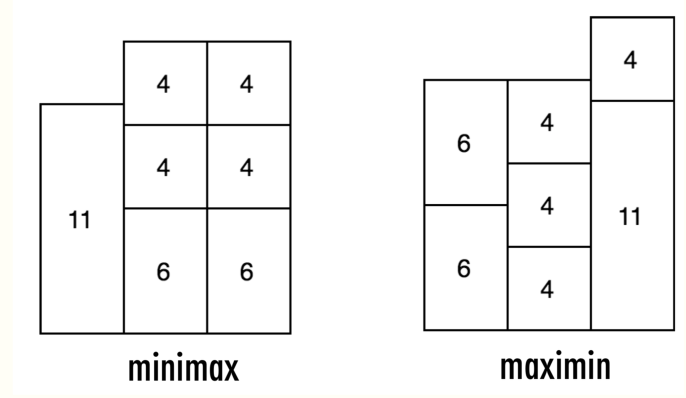
- The Combinatorial Assignment Problem: Approximate Competitive Equilibrium from Equal Incomes (BQGT'10)
- Fairness and Externalities (TE'16)
- Externalities in Cake Cutting (IJCAI'13)
- Truthful Cake Cutting Mechanisms with Externalities: Do not make them care for others too much! (IJCAI'15)

⋆ Spliddit: http://www.spliddit.org/

Computing EMMS

- Maximin is **NP**-hard ⇒ Computing $\mathbf{EMMS}_i$ is **NP**-hard.
- Maximin and minimax do not always give the optimal partition.

$$\begin{cases} w = [\frac{1}{2}, \frac{1}{2}, 0] : \text{minimax} > \text{maximin} \\ w = [1, 0, 0] : \text{minimax} < \text{maximin} \end{cases}$$

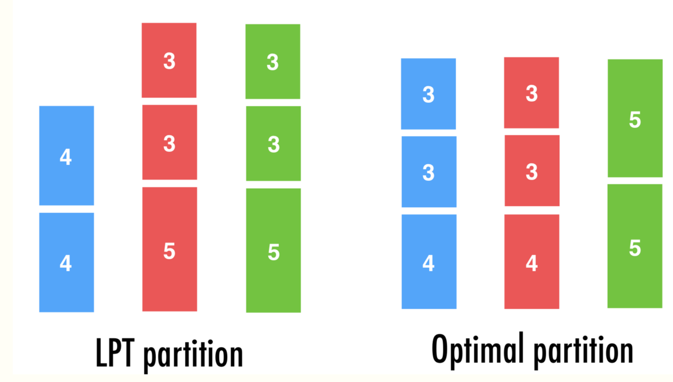


Algorithm: LPT

- Start with an empty partition.
- Repeatedly, add the maximum remaining item to the current minimum part.

- $\mathbf{LPT}_i$ is the partition returned.
- The main idea is to divide items into two groups:

$$\begin{cases} \textit{Huge} \text{ items:} V_i(b) > V_i(\mathcal{M})/n. \\ \textit{Small} \text{ items: remaining items.} \end{cases}$$



- If there is no huge item, $p_n \le 2p_1$.

Theorem

- Partition $\mathbf{LPT}_i$ gives a $1/2$-approximation of $\mathbf{EMMS}_i$.

Allocation

- Our main result is the following theorem:

Theorem

- Every instance of the fair allocation problem with externalities admits an $\alpha/2$-**EMMS** allocation.

- We use **Bundle Claiming** algorithm to prove this theorem.

Algorithm: BC

- For every agent a_i, set $\ell_i = 1$.
- Γ: a subset of $\mathcal{M}$ with the minimum size, such that for at least one agent a_j, $V_j(\Gamma) \ge 1/2 \cdot v_{j,\ell_j}$.
- Allocate Γ to a_j.
- For every remaining agent a_i, update ℓ_i according to the externalities of the satisfied agents.
- Repeat the process for the remaining agents.

- The challenging part of BC is the updating process.
- To update each ℓ_i, we define a mapping M_i which roughly maps the bundles allocated to the satisfied agents to the optimal partition O_i.