Advanced Computer Contest Preparation

Lecture 14

# DYNAMIC PROGRAMMING

# Sample Problem: Mouse Journey (CCC '12 S5)

- You are a mouse in a $R \times C$ grid of cages
  - For this version: $1 \leq R, C \leq 1{,}000$
- You start at the top-left corner $(1,1)$
- You may move to the cage directly right or below
- There are cages that contain cats; you cannot enter these cages
- How many paths are there from $(1,1)$ to $(R,C)$?

# Mouse Journey (CCC ’12 S5)

# Solutions?

- Brute force?
  - Generate all possible paths?
  - What is the runtime of this algorithm?
  - Between $O(2^N)$ and $O(4^N)$
- Greedy?
  - Not an optimization problem
  - How can we easily compare two squares?
  - We cannot use greedy

# Dynamic Programming

- *Dynamic programming (DP)* is a problem solving technique
- DP can solve optimization problems
  - The “best” answer
  - For example, maximum/minimum answer
- DP can solve counting problems
  - “What are the number of ways…”
- Correct DP *should always* compute the correct answer

# Subproblems

- Suppose we are given a problem
- A *subproblem* is another instance of the problem, but with smaller constraints

- Example: what is the $N^{th}$ Fibonacci number?
- The problem is: find $fib(N)$
- The subproblems are: find $fib(i)$ $for$ $i = 1$ $to$ $N$

# Optimal Substructure

- *Optimal substructure* is the property where a problem's solution depends on its subproblems' solutions
- The relationship between a problem's solution and its subproblems' solutions must be found
- Eventually, subproblems can be easily solved
  - These subproblems are the *base cases*

# Optimal Substructure

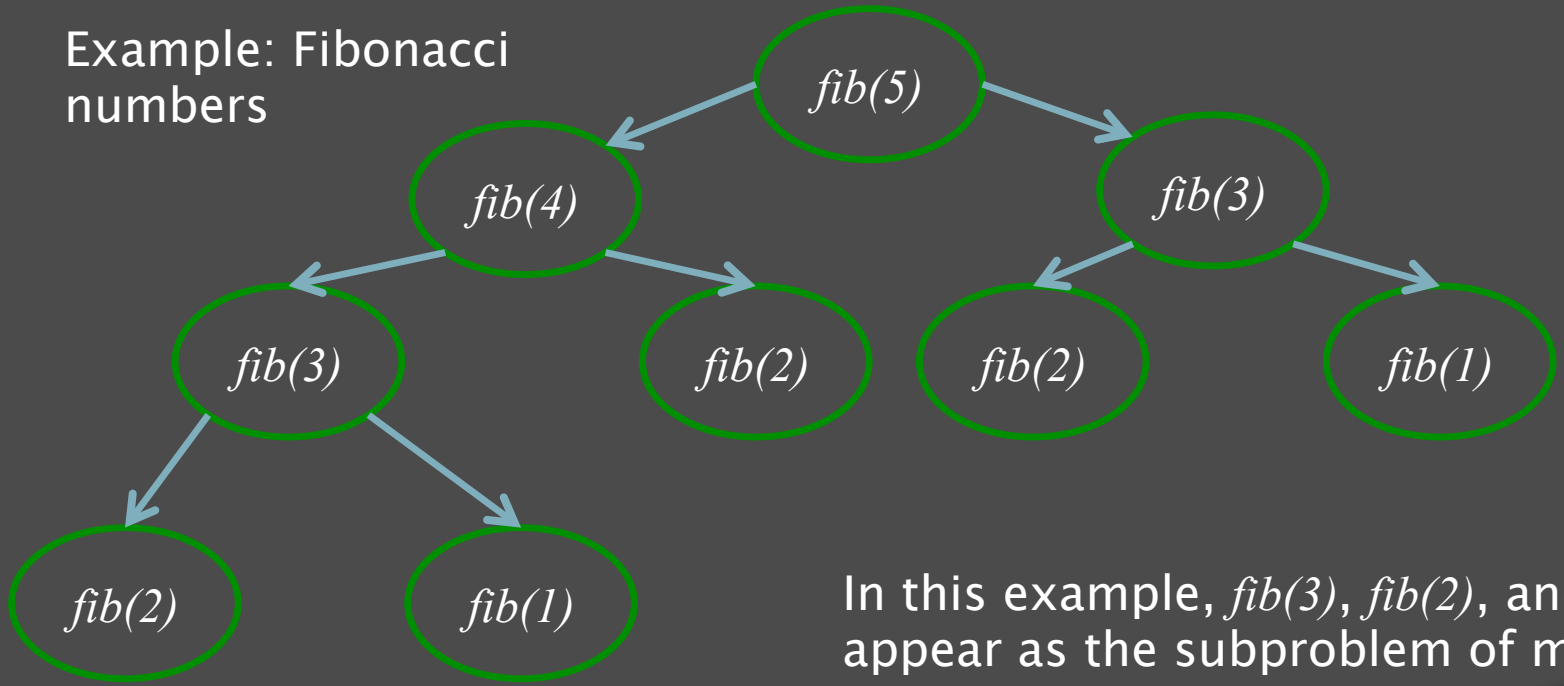
- Example: Fibonacci numbers
- The dependent subproblems of *fib(i)* are *fib(i-1)* and *fib(i-2)*
- The exact relationship between *fib(i)* and its subproblems is *fib(i) = fib(i-1)+fib(i-2)*
- The base cases are *fib(1) = 1, fib(2) = 1*

# Overlapping Subproblems

- *Overlapping subproblems* occur when the same subproblem might occur more than once
- In other words, more than one problem might depend on the same subproblem to compute their solutions
- If we form a graph of the subproblem dependencies, we will form a DAG

# Overlapping Subproblems

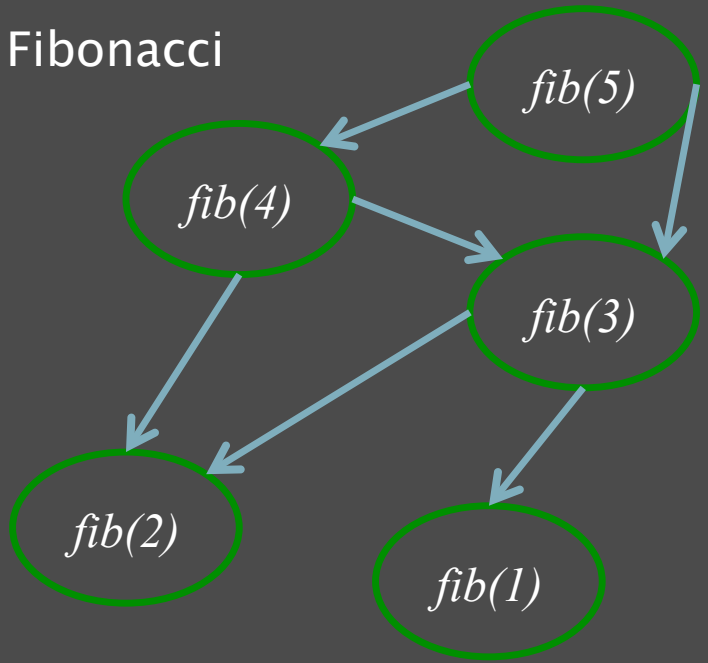
Example: Fibonacci numbers




In this example, *fib(3)*, *fib(2)*, and *fib(1)* appear as the subproblem of more than one problem. Therefore, the subproblems overlap.

# Overlapping Subproblems

Example: Fibonacci numbers




2 problems directly depend on *fib(3)*: *fib(5)* and *fib(4)*

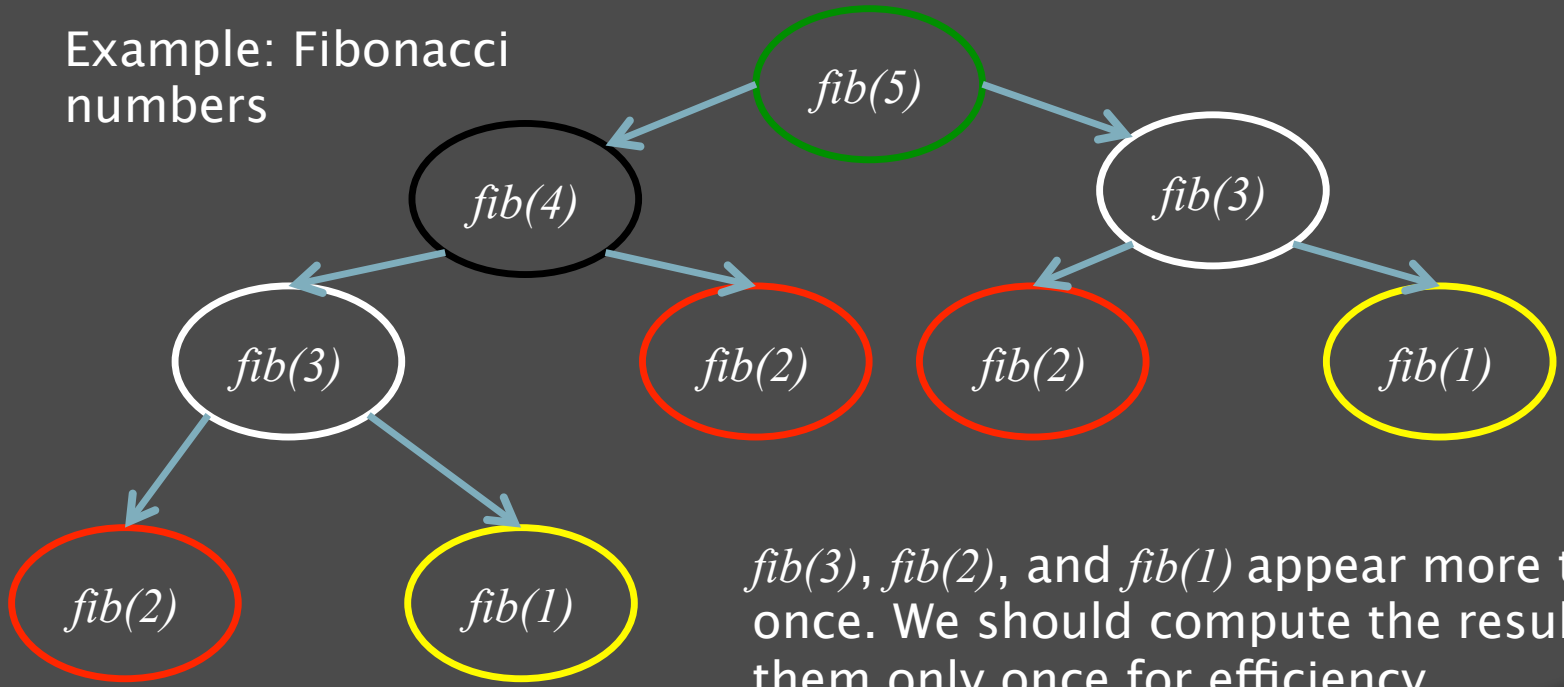
Note that the graph of the problem dependencies is a DAG

# Memoization

- *Memoization* is the storing of computed solutions of subproblems
- The solution of each subproblem is computed exactly once and stored in some data structure
- When the solution for the subproblem is needed, we look it up in the data structure instead of re-computing it
- Also known as *tabling*, data structure is known as *lookup table*

# Memoization

Example: Fibonacci numbers




*fib(3)*, *fib(2)*, and *fib(1)* appear more than once. We should compute the result of them only once for efficiency.

# DP State

- The most common data structure used to store solutions to subproblems is the array
- The constraints of a subproblem are collectively called the *DP state*
- The array must be able to quickly locate an answer given the state
- Normally, we will use a $N$-dimensional array, one dimension for each variable
- The size of the $i^{th}$ dimension is the number of possible values of the $i^{th}$ variable

# DP State

- Example 1: Fibonacci numbers
  - We only have 1 variable, $N$, which corresponds to the $N^{th}$ Fibonacci number
  - $1 \leq N \leq 1{,}000$
  - The DP array is `dp[1005]`
- Example 2:
  - We have 3 variables: $x$, $y$, and $z$
  - $1 \leq x \leq 1{,}000$, $1 \leq y \leq 100$, $z$ is a `bool`
  - The DP array is `dp[1005][105][2]`

# Dynamic Programming – Summary

- In order to use DP, the problem must have overlapping subproblems and the optimal substructure property
  - Subproblems occur more than once
  - Problem's solution depends on its subproblems' solutions
- Relationship between problem's solution and its subproblems' solutions must be found
  - Most challenging step!
- Solution to each subproblem is stored in a data structure for future reference so that it doesn't need to be computed again

# Two DP Directions

- DP can be done iteratively or recursively
- Iterative DP starts from the base cases and works upward to the final problem
  - Typically, there is a for loop for every variable
  - Order of for loops matters; ensure that all dependent subproblems for any problem are already computed
- Recursive DP starts from the final problem, works towards the base cases, and works back to the final problem

# DP Analysis

- The time complexity of a DP algorithm is generally:
  - The number of required states

    multiplied by

  - The time complexity to compute the solution for a state
    - This time complexity is at least the number of subproblems that the current state depends on

# DP Analysis Example

- Example 1: Fibonacci numbers
  - Find the $N^{th}$ Fibonacci number
  - Number of states: $O(N)$
  - Time complexity to solve one problem: $O(1)$
  - Final time complexity: $O(N)$
- Example 2:
  - Number of states: $O(XYZ)$
  - Time complexity to solve one problem: $O(X^2Y)$
  - Final time complexity: $O(X^3Y^2Z)$

# Back to Mouse Journey (CCC '12 S5)

- You are a mouse in a $R \times C$ grid of cages
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- How many paths are there from $(1,1)$ to $(R,C)$?

# Solution – Dynamic Programming

- Can DP be used?
  - This problem is a *counting problem*
  - Does the problem have the optimal substructure property?
  - Does the problem have overlapping subproblems?

# Solution – Dynamic Programming

- What is the problem we are trying to solve?
  - What is the number of paths to cell $(R,C)$?
- What are the subproblems?
  - What is the number of paths to cell $(i,j)$ *for* $i = 1$ *to* $R$ and *for* $j = 1$ *to* $C$?

# Solution – Dynamic Programming

- For a problem $p(i,j)$, what are the dependent subproblems?
  - Note that we can only move right or down
  - Thus, we must have came from the cell above or to the left
  - The solution to $p(i,j)$ depends on subproblems $p(i-1,j)$ and $p(i,j-1)$
- What are the base cases?
  - $p(i,j) = 0$ if $i = 0$ or $j = 0$ or if cell $(i,j)$ contains a cat
  - $p(1,1) = 1$

# Solution – Dynamic Programming

- For a problem $p(i,j)$, how is it related to its subproblems?
  - $p(i,j) = p(i-1,j) + p(i,j-1)$
  - The number of paths to cell $(i,j)$ is equal to the sum of the number of paths to cell $(i-1,j)$ and cell $(i,j-1)$

# Solution – Dynamic Programming

- What is the DP state format?
  - There are two variables, $R$ and $C$
  - Each of them can be in the range $[1,1000]$
  - Therefore, the DP array is `dp[1005][1005]`

# DP Table - Visualization

| | | | | | |
|---|---|---|---|---|---|
| 1  | 1 | 1 | 1 | 1 | 1 |
| 1 | 0  | 1 | 2 | 3 | 4 |
| 1 | 1 | 2 | 4 | 0  | 4 |
| 1 | 0  | 2 | 6 | 6 | 10 |
| 1 | 1 | 3 | 9 | 15 | 25  |

# Pseudocode – Iterative

```
bool cat[R+5][C+5];  //true if cat is in cell
int dp[R+5][C+5]; //number of paths, all initially 0

//process input here

dp[1][1] = 1;
for (row 1 to R){
       for (col 1 to C){
              if (cat[row][col]) continue;
              dp[row][col] = dp[row-1][col] + dp[row][col-1];
       }
}
print dp[R][C];
```

# Pseudocode – Recursive

```
bool cat[R+5][C+5];  //true if cat is in cell
int dp[R+5][C+5]; //number of paths, all initially 0

//process input here

int solve(int r, int c){
       if (dp[r][c] != 0) return dp[r][c];
       if (r == 0 || c == 0 || cat[r][c]) return 0;
       if (r == 1 && c == 1) dp[r][c] = 1;
       else dp[r][c] = solve(r-1,c) + solve(r,c-1);
       return dp[r][c];
}

print solve(R,C);
```

# Mouse Journey Analysis

- How many states are there?
  - $O(RC)$
- How many subproblems does each state depend on?
  - $O(1)$
- What is the time complexity needed to compute the solution to a problem?
  - $O(1)$
- Therefore, the final time complexity is $O(RC)$

# THANK YOU!