10. Let (f_n) be a sequence of functions in $C([a,b])$, and (f_n) converges to some function f uniformly. Prove that if $f \neq 0$ on $[a,b]$, then $\left(\frac{1}{f_n}\right)$ converges to $\frac{1}{f}$ uniformly

Proof: Since $f \neq 0$ and (f_n) converges to f uniformly, there is some $M \in N$ such that $\forall n > M_1, f_n \neq 0$. $\forall \varepsilon > 0$, there is some $M_2 \geq M_1$ such that $\forall n > M_2, \forall x \in [a,b], |f_n(x) - f(x)| < \varepsilon \cdot 0.5(\min f)^2$. Choose some $M_3 \geq M_1$ such that $\forall n > M_3, \min f_n \geq 0.5 \min f$. Fix this ε, for any $n > M_3$ we have $\left|\frac{1}{f(x)} - \frac{1}{f_n(x)}\right| = \left|\frac{f_n(x)-f(x)}{f(x)f_n(x)}\right| < \frac{\varepsilon \cdot 0.5(\min f)^2}{0.5\min f \cdot \min f} = \varepsilon$ for any $x \in [a,b]$. Therefore, the function sequence $(\frac{1}{f_n})$ converges uniformly to $\frac{1}{f}$.

8. Prove that if (a_k) is a sequence such that $a_k \geq 0$ and $\sum a_k$ diverges, then the series $\sum \frac{a_k}{1+a_k}$ also diverges (Spivak *Calculus*, P463)

Proof: If the sequence (a_k) is not bounded, then there is some subsequence (a_{n_k}) such that $\lim a_{n_k} = +\infty$. This means $\lim \frac{a_{n_k}}{1+a_{n_k}} = 1$. It follows that for some $M \in N$, $\sum \frac{a_k}{1+a_k} \geq \sum \frac{a_{n_k}}{1+a_{n_k}} = \sum \left(1 - \frac{1}{1+a_{n_k}}\right) \geq \sum_{k \geq M} \frac{a_{n_k}}{1+a_{n_k}} > \sum_{k \geq M} \frac{1}{2} = +\infty$. Therefore, the series diverges. If the sequence (a_k) is bounded, we investigate the ratio of a_k and $\frac{a_k}{1+a_k}$: $\frac{\frac{a_k}{1+a_k}}{a_k} = \frac{1}{1+a_k} \geq \frac{1}{1+\sup(a_k)}$. This means for any n, $\sum_{k=0}^{n} \frac{a_k}{1+a_k} \geq \frac{1}{1+\sup(a_k)} \cdot \sum_{k=0}^{n} a_k$. Note that $\frac{1}{1+\sup(a_k)}$ is nothing but a positive number. Thus, $\sum \frac{a_k}{1+a_k}$ diverges.