Answer for RC3

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VV186 - Honors Mathmatics II

Exercise

2. Prove that $\lim_{n\to\infty} \sqrt[n]{n} = 1$.

Some results:

- $\lim_{n\to\infty} \sqrt[n]{a} = 1, a > 0$
- $\lim_{n\to\infty} \frac{1}{n^\alpha} = 0, \alpha \in (0, +\infty)$

Solution

2.

We consider the case for $n \geq 2$. Since $\sqrt[n]{n} > 1$ for all $n \geq 2$, we can choose a non-negative sequence, let say $(b_n)_{n\geq 2}$ such that

$$1 + b_n = \sqrt[n]{n}$$

From the fact that

$$(1 + b_n)^n = \sum_{k=0}^{n} \binom{n}{k} 1^k \cdot b_n^{n-k}$$

and another fact that $(1 + b_n)^n = n$, we get the following inequality

$$n = (1 + b_n)^n \geq \binom{n}{2} b_n^2 = \frac{n(n-1)}{2} b_n^2$$

Solution

2.(*continue*)

After some algebraic works, one can see

$$b_n \leq \sqrt{\frac{2}{n-1}}$$

But this means

$$0 \leq \sqrt[n]{n} - 1 \leq \sqrt{\frac{2}{n-1}}$$

By taking the limit, we have

$$0 \leq \lim_{n\to\infty} (\sqrt[n]{n} - 1) \leq \lim_{n\to\infty} \sqrt{\frac{2}{n-1}} = 0$$

Solution

2.(*continue*)

Finally, by Squeeze theorem, we get

$$\lim_{n\to\infty}(\sqrt[n]{n}-1)=\lim_{n\to\infty}\sqrt[n]{n}-1=0$$

We conclude

$$\lim_{n\to\infty}\sqrt[n]{n}=1 \quad \square$$

Exercise

3. A sequence is defined as

$$(S_n)_{n\in\mathbb{N}}, S_1 = \sqrt{2}, S_2 = \sqrt{2+\sqrt{2}}, S_3 = \sqrt{2+\sqrt{2+\sqrt{2}}}, \ldots$$

Please prove that it is convergent and calculate the limit of (S_n) as $n \to \infty$.

Solution

3.

Obviously S_n is increasing, now we use induction to prove that S_n is bounded. Let $A(n) : S_n < 2$.

Base case: $S_1 = \sqrt{2} < 2$, so $A(1)$ is valid.

Inductive case: Suppose that $A(n)$ is true, namely $S_n < 2$, we have $S_{n+1} = \sqrt{2 + S_n} < \sqrt{2 + 2} = 2$, therefore $A(n+1)$ is true.

Since S_n is monotonic and bounded, so it converges, say $\lim_{n\to\infty} S_n = s$. Considering the recursive definition

$$S_{n+1}^2 = S_n + 2$$

, we take the limit from both side and get $s^2 = 2 + s$. Since s should be positive, so $s = 2$. And we conclude

$$\lim_{n\to\infty} \sqrt{2 + \sqrt{2 + \cdots + \sqrt{2}}} = 2$$

Exercise

4. Prove that every Cauchy sequence has at most one accumulation point. (A Former Midterm Question)

Tips:

- You should work on an abstract metric space, using ρ instead of $|\cdot|$.
- Try to prove this without using proof by contradiction!

Solution

4.

Proof: Let $\langle x_n \rangle$ be such a sequence and let $\langle x_{n_k} \rangle$, $\langle x_{n_j} \rangle$ be two subsequences of it with different limits a, b. Let $\varepsilon = \frac{1}{3}\rho(a, b)$. For any $N \in \mathbb{N}$, choose some $n_k > N$ and $n_j > N$, such that $\rho(x_{n_k}, a) < \varepsilon$ and $\rho(x_{n_j}, b) < \varepsilon$. If $\rho(x_{n_k}, x_{n_j}) < \varepsilon$ then

$$\rho(a, b) \leq \rho(a, x_{n_k}) + \rho(x_{n_k}, x_{n_j}) + \rho(x_{n_j}, b) < 3\varepsilon = \rho(a, b),$$

which implies that $\rho(a, b) < 0$, a contradiction.

Without using proof by contradiction: try to prove $a = b$.

Exercise

5. Let (a_n) be a real sequence that converges to $L \in \mathbb{R}$. Prove that the sequence $(\frac{\sum_{i=1}^{n} a_i}{n})$ is convergent. Furthermore $\lim_{n\to\infty} (\frac{\sum_{i=1}^{n} a_i}{n}) = L$.

Solution

Proof: $\left|\frac{a_1+a_2+\cdots+a_n}{n} - L\right| = \left|\frac{(a_1-L)+(a_2-L)+\cdots+(a_n-L)}{n}\right| \le \frac{1}{n}(\sum_{i=1}^{n}|a_i - L|)$. Given an $\varepsilon > 0$, we choose some $N \in \mathbb{N}$ such that for all $n > N$, $|a_n - L| < \frac{1}{2}\varepsilon$. For any such n, we have

$$\frac{1}{n}(\sum_{i=1}^{n}(a_i - L)) = \frac{1}{n}(\sum_{i=1}^{N}|a_i - L|) + \frac{1}{n}(\sum_{i=N+1}^{n}|a_i - L|)$$

$$< \frac{1}{n}(\sum_{i=1}^{N}|a_i - L|) + \frac{1}{n}\left(\sum_{i=N+1}^{n}\frac{1}{2}\varepsilon\right)$$

$$= \frac{1}{n}(\sum_{i=1}^{N}|a_i - L|) + \frac{n-N}{n}\cdot\frac{1}{2}\varepsilon$$

$$\le \frac{1}{n}(\sum_{i=1}^{N}|a_i - L|) + \frac{1}{2}\varepsilon$$

Note that $\sum_{i=1}^{N}|a_i - L|$ is just a fixed non-negative value, since N is fixed.

Therefore, by choosing n large enough, we can ensure that $\frac{1}{n}(\sum_{i=1}^{N}|a_i - L|) < \frac{1}{2}\varepsilon$.

But this means

$$\forall\varepsilon > 0, \exists M \in \mathbb{N} \text{ such that } \forall n > N, \left|\frac{a_1+a_2+\cdots+a_n}{n} - L\right| < \varepsilon$$

It follows that the sequence $\left(\frac{\sum_{i=1}^{n}a_i}{n}\right)$ converges to L.

Exercise

6. Let $(a_n), (b_n)$ be two real sequences. Furthermore, assume that $a_n < b_n$ for all n, $[a_{n+1}, b_{n+1}] \subseteq [a_n, b_n]$, $\lim(a_n - b_n) = 0$. Prove that there is an unique $m \in [a_n, b_n]$ for all n, such that

$$\lim a_n = \lim b_n = m$$

Solution

6.

Proof :

We first show that $(a_n), (b_n)$ are convergent. Notice that $[a_{n+1}, b_{n+1}] \subseteq [a_n, b_n]$ means that (a_n) is increasing, while (b_n) is decreasing. Furthermore, both (a_n) and (b_n) are bounded by $[a_0, b_0]$. Therefore, (a_n) and (b_n) are convergent.

Next we show that $\lim a_n = \lim b_n$. Suppose $\lim b_n = L$, where L is a unique number since a sequence has precisely one limit. Then we know

$$\lim a_n = \lim[(a_n - b_n) + b_n] = \lim(a_n - b_n) + \lim b_n = 0 + L = L$$

Moreover, since $\begin{cases} L \geq a_n \\ L \leq b_n \end{cases}$, we know that $L \in [a_n, b_n]$. $\square$

Exercise

7. Let (a_n) be a sequence that $a_n = \frac{1}{\sqrt{n^2+1}} + \cdots + \frac{1}{\sqrt{n^2+n}}$. Calculate the limit of (a_n).

Solution

7.

We first consider two sequence $(b_n)_{n\geq 1}, (c_n)_{n\geq 1}$ given by

$$\begin{cases} b_n := \frac{1}{\sqrt{n^2}} + \cdots + \frac{1}{\sqrt{n^2}} \\ c_n := \frac{1}{\sqrt{n^2+n}} + \cdots + \frac{1}{\sqrt{n^2+n}} \end{cases}$$

Clearly, $\begin{cases} (b_n) \leq (a_n) \\ (c_n) \geq (a_n) \end{cases}$ for all $n \geq 1$.

Then one can easily find out the limit for both (b_n) and (c_n), which can be calculated as

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{\sqrt{n^2}} = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{n} = \lim_{n\to\infty} \frac{n}{n} = \lim_{n\to\infty} 1 = 1$$

Solution

7.(*continue*)

$$\lim_{n\to\infty} c_n = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{\sqrt{n^2+n}} = \lim_{n\to\infty} \frac{n}{\sqrt{n^2+n}} = \lim_{n\to\infty} \frac{1}{\sqrt{1+\frac{1}{n}}} = 1$$

By Squeeze Theorem, we know

$$\lim_{n\to\infty} b_n \leq \lim_{n\to\infty} a_n \leq \lim_{n\to\infty} c_n$$

, namely

$$1 \leq \lim_{n\to\infty} a_n \leq 1$$

We conclude that $\lim_{n\to\infty} a_n = 1$ $\square$