Exercise 3

Proof: (The content in the second picture is the definition of weighted norm)

$||u|| \geq 0$ because $|u(\cdot)|$ and $p(\cdot)$ are non-negative functions. Besides, $||u|| = 0$ implies that $|u(z)|p(z) = 0$ for all $z \in [a,b]$. Since $p > 0$, $u = 0$ must be true.

$||\lambda u|| = \sup\{|\lambda u(z)|p(z):z \in [a,b]\} = \sup\{|\lambda||u(z)|p(z):z \in [a,b]\} = |\lambda| \sup\{|u(z)|p(z):z \in [a,b]\} = |\lambda|||u||$.

For triangle inequality, we claim that

$$\sup\{|f(z)+g(z)|p(z):z \in [a,b]\} \leq \sup\{|f(z)|p(z)+|g(z)|p(z):z \in [a,b]\}$$
$$\leq \sup\{|f(z)|p(z):z \in [a,b]\} + \sup\{|g(z)|p(z):z \in [a,b]\}$$

Therefore, the triangle inequality $||f+g|| \leq ||f|| + ||g||$ holds. (Wolfgang Walter Ordinary Differential Equations, P56)

Exercise 4

Proof: 1st sentence. **(It depends)** True if and only if V_1 is a subset or superset of V_2. Suppose V_1 is not the subset or superset of V_2. There exists some $v_1 \in V_1$ but $v_1 \notin V_2$. For any $v_2 \in V_2$ but $v_2 \notin V_1$, $v_1 + v_2$ is not in V_1 or V_2. If V_1 is a subset or superset of V_2, then it's obvious that $V_1 \cup V_2$ is a vector space.

2nd sentence. True. For any element $v_1, v_2 \in V_1 \cap V_2$, both of them are in V_1. Since V_1 is a vector space, $\alpha v_1 + \beta v_2 \in V_1$. Similarly, we can show that $\alpha v_1 + \beta v_2 \in V_2$. Therefore, $V_1 \cap V_2$ is a subspace of V.

3rd sentence. True. Recall that real linear maps are of the form $L:R \to R$, $L(x) = kx$ for some k. Denote the set containing all linear maps by $\mathcal{L}(\mathcal{R})$. Since any real linear map is continuous, $\mathcal{L}(\Re) \subseteq C(R)$. For any $L \in L(R)$ where $L(x) = kx$, $\lambda L = \lambda k \cdot x \in L(R)$. Besides, given $L_1, L_2 \in L(R)$, such that $L_1(x) = k_1 x$, $L_2(x) = k_2 x$, $(L_1 + L_2)(x) = (k_1 + k_2)x \in L(R)$. So $\mathcal{L}(\mathcal{R})$ is a subspace of $C(R)$.

4th sentence. False. Counterexample: Let $n \geq 2$. Let $(x_1,x_2,\dots,x_n) \in \Re^n$, let $||(x_1,x_2,\dots,x_n)|\quad|_1 = \sum_{i=1}^{n} |x_i|$, $||(x_1,x_2,\dots,x_n)|\quad|_2 = \max_{1\leq i\leq n} |x_i|$. Choose $x = (10,6,0,\dots,0)$, $y = (10,1,0,\dots,0)$. Then $||x+y|| = ||(20,7,0,\dots,0)|| = \sqrt{27 \times 20} \approx 23.237$. $||x|| = ||(10,6,0,\dots,0)|| = \sqrt{16 \times 10} \approx 12.649$. $||y|| = ||(10,1,0,\dots,0)|| = \sqrt{11 \times 10} \approx 10.488$. Thus, $||x|| + ||y|| < ||x+y||$. (Note that "given" means "for any" here)

5th sentence. True. Since $||\cdot|\quad|_2$ is a norm, it will map anything to a non-negative number, therefore $||x|| \geq 0$ for all x. Furthermore, If $||x|| = 0$, then $||x||_1 = 0$. Since $||\cdot|\quad|_1$ is a norm from V to $\Re$, $||x||_1 = 0$ if and only if $x = 0$. This proves the first requirement for a norm. $||\lambda x|| = ||||\lambda x||_1|| = |||\lambda| \cdot ||x||_1||_2 = |\lambda| \cdot ||x||$. For triangle inequality, we have the following:

$$||x+y|| = ||||x+y||_1||_2 \leq ||||x||_1 + ||y||_1||_2 \leq ||||x||_1||_2 + ||||y||_1||_2 = ||x|| + ||y||$$

Therefore, $||\cdot|| := ||\cdot|\quad|_2 \circ ||\cdot|\quad|_1$ is a norm.

Exercise 6

Proof: i) Fix arbitrary $x \in V$. Given $\varepsilon > 0$, $||f(x) - f(y)|| = ||(a+x) - (a+y)|| = ||x-y||$. Thus, by choosing $\delta = 0.5\,\varepsilon$, whenever $||x-y|| < \delta$, we have

$||f(x) - f(y)|| < \varepsilon$. Since this is true for all x, f is continuous. It is trivial that $f^{-1}(y) = y - a$. Note that our proof about the continuity of f doesn't include the concrete value of a. So the inverse function is also continuous.

ii) Fix arbitrary $x \in V$. Given $\varepsilon > 0$, $||g(x) - g(y)|| = ||(\lambda x) - (\lambda y)|| = |\lambda|||x - y||$. Thus, by choosing $\delta = \frac{1}{2\lambda}\varepsilon$, whenever $||x - y|| < \delta$, we have $||f(x) - f(y)|| < \varepsilon$. It is trivial that $f^{-1}(y) = \frac{1}{\lambda}y$. Note that our proof about the continuity of f doesn't include the concrete value of λ (as long as it is not zero). So the inverse function is also continuous.

Exercise 7

Solution: To illustrate, I only show the solution of the first one. The solution of other is based on the same ideas. Besides, the second one is JUST the example in Horst's slides.

Fix $x \in R$. If $x \in (-1,1)$, then $\lim |x|^n = 0$. Therefore, the limit of $(f_n(x))$ is zero. If $|x| = 1$, trivially $f_n(x) = \frac{1}{2}$ for all n; so $\lim f_n(x) = \frac{1}{2}$. If $|x| > 1$, then $|x|^n \to \infty$ as $n \to \infty$. But this means that $\lim \frac{|x|^n}{1+|x|^n} = 1$. In conclusion, the limit of (f_n) is

$$f{:}R \to R,\ f(x) = \begin{cases} 0 & |x| < 1 \\ \frac{1}{2} & |x| = 1 \\ 1 & |x| > 1 \end{cases}$$

From the explicit form of f, or the graph of it, it's obvious that f is uniformly continuous on any set in $[-1 + \delta, 1 - \delta]$, where δ is a positive number. f is also uniformly continuous on any bounded subset of $(-\infty, -1 - \delta] \cup [1 + \delta, +\infty)$, where δ is a positive number.

Exercise 8

Proof: For the two problem above, the key is to construct a sequence of functions that satisfies the requirements. i) The sequence of functions can be (f_n), $f_n{:}[a,b] \to \Re$,

$$f_n(x) = \begin{cases} f(x) + \frac{1}{n}, & x \text{ is raitonal} \\ f(x) - \frac{1}{n}, & x \text{ is irrational} \end{cases} \quad \text{for each } n \in N.$$

To show that each f_n is nowhere continuous, first fix some rational $x \in [a,b]$. For any $\delta > 0$, in the neighborhood $(x - \delta, x + \delta) \cap [a,b]$, there is of course some irrational numbers, denote one of them by "y". It follows that $|f_n(x) - f(y)| = \frac{2}{n}$. Thus, it is not continuous at any rational points in [a,b]. A similar argument holds for irrational points. Therefore, (f_n) is a sequence of nowhere continuous functions that converges to the continuous function f.

ii) Since $g \in C([a,b])$, g is uniformly continuous. This means given $\varepsilon > 0$, there is a partition of $[a,b]{:}I_1 = [a, a + \delta(\varepsilon)], I_2 = [a + \delta(\varepsilon), a + 2\delta(\varepsilon)], \ldots, I_m = [b - \delta(\varepsilon), b]$

for some $\delta(\varepsilon) > 0$, such that $\forall k \in \{1,2,...,m\}$, $\forall x,y \in I_k, |f(x) - f(y)| < \frac{1}{2}\varepsilon$. To construct the function sequence, we will use the modified Van Der Waerden function (see Horst's Slide 442) as the prototype. Denote the prototype-Van Der Waerden function by W. Choose some $x_0 \in [0,1]$, such that $W(x_0)$ attains its maximum (which is $\sum 0.5 \cdot 10^{-n} = 5/9$). Denote the part of W on $[0,x_0)$ by PW (Part of W). Fix $n \in N$, choose some $\delta\left(\frac{1}{n}\right) > 0$ and a corresponding partition of subintervals $\{I_1,I_2,...,I_m\}$. Note that the length of each of these subintervals is $\delta(\frac{1}{n})$. For I_k, $k \in \{1,2,...,m\}$, if $f\left(a + (k-1)\delta\left(\frac{1}{n}\right)\right) \leq f\left(a + k\delta\left(\frac{1}{n}\right)\right)$, then

$$f_n(x) = \left|f\left(a + (k-1)\delta\left(\frac{1}{n}\right)\right) - f\left(a + k\delta\left(\frac{1}{n}\right)\right) - \frac{1}{2n}\right| \cdot PW\left(\frac{\delta\left(\frac{1}{n}\right)}{x_0}x + a + (k-1)\delta\left(\frac{1}{n}\right)\right)$$
$$+ f\left(a + (k-1)\delta\left(\frac{1}{n}\right)\right) - \frac{1}{4n}$$

on I_k. If $f\left(a + (k-1)\delta\left(\frac{1}{n}\right)\right) > f\left(a + k\delta\left(\frac{1}{n}\right)\right)$, then

$$f_n(x) = -\left|f\left(a + (k-1)\delta\left(\frac{1}{n}\right)\right) - f\left(a + k\delta\left(\frac{1}{n}\right)\right) + \frac{1}{2n}\right| \cdot PW\left(\frac{\delta\left(\frac{1}{n}\right)}{x_0}x + a + (k-1)\delta\left(\frac{1}{n}\right)\right)$$
$$+ f\left(a + (k-1)\delta\left(\frac{1}{n}\right)\right) + \frac{1}{4n}$$

on I_k. The idea of constructing f_n on I_k is to use horizontal translation and stretch (i.e., $\left(\frac{\delta\left(\frac{1}{n}\right)}{x_0}x + a + (k-1)\delta\left(\frac{1}{n}\right)\right)$) and vertical translation (i.e., $+ f\left(a + (k-1)\delta\left(\frac{1}{n}\right)\right) \pm \frac{1}{4n}$) and vertical stretch (i.e., $\left|f\left(a + (k-1)\delta\left(\frac{1}{n}\right)\right) - f\left(a + k\delta\left(\frac{1}{n}\right)\right) \pm \frac{1}{2n}\right|$). At each point $x \in [a,b]$, the constructed function has difference less than $\frac{2}{n}$ from the original function f. It's easy to check that f_n is continuous on each I_k, and has the same value on the endpoints of "I_k"s. So f_n is continuous. Also, since the PW is nowhere differentiable, f_n is nowhere differentiable. Furthermore, with some calculation one can check that $|f_n - f| \to 0$ uniformly. Therefore, following function sequence (f_n) is what we want.

Exercise 9

Proof: Since $f \neq 0$ and (f_n) converges to f uniformly, there is some $M \in N$ such that $\forall n > M_1, f_n \neq 0$. $\forall \varepsilon > 0$, there is some $M_2 \geq M_1$ such that $\forall n > M_2, \forall x \in [a,b], |f_n(x) - f(x)| < \varepsilon \cdot 0.5(\min f)^2$. Choose some $M_3 \geq M_1$ such that $\forall n > M_3, \min f_n \geq 0.5 \min f$. Fix this ε, for any $n > M_3$ we have $\left|\frac{1}{f(x)} - \frac{1}{f_n(x)}\right| = \left|\frac{f_n(x) - f(x)}{f(x) f_n(x)}\right| < \frac{\varepsilon \cdot 0.5(\min f)^2}{0.5\min f \cdot \min f} = \varepsilon$ for any $x \in [a,b]$. Therefore, the function sequence $(\frac{1}{f_n})$ converges uniformly to $\frac{1}{f}$.

Exercise 10

Proof: i) Fix $x \in [a,b]$. Since $(f(x))$ is bounded and monotonic sequence, it converges. Because this is true for all $x \in [a,b]$, (f_n) converges point-wisely to some function f, where $f(x) := \lim f_n(x)$.

ii) Construct a function sequence (g_n) such that $g_n = |f_n - f|$ for each n. First note that whenever $m > n$, $g_m(x) \leq g_n(x)$ for all $x \in [a,b]$, because $|f_n(x) - f(x)|$ decreases to zero. Since the domain of (g_n) is a closed interval [a,b], for each fixed n, g_n attains its maximum at some point $x_n \in [a,b]$. The points where (g_n) attains maximum(s) form a sequence (x_n), i.e., each term x_n is a point where g_n attains its maximum. Note that (x_n) is bounded, so there is a subsequence (x_{n_k}) converges to some point $y \in [a,b]$. Since $\lim_{n\to\infty} g_n(y) = 0$, by continuity of g_n (for each n) we know that $g_n(x_{n_k})$ converges to $g_n(y)$. Fix arbitrary $\varepsilon > 0$. For large enough n_k we can ensure that $\left|g_n(x_{n_k}) - g_n(y)\right| < \varepsilon$, so $\left|g_n(x_{n_k}) - 0\right| \leq \left|g_n(x_{n_k}) - g_n(y)\right| + |g_n(y) - 0| < 2\,\varepsilon$. Since for all "$x_{n_k}$"s and $m > n$, $g_m(x_{n_k}) \leq g_n(x_{n_k})$, it follows that whenever $n_k > n$, $\left|g_{n_k}(x_{n_k}) - 0\right| \leq \left|g_n(x_{n_k}) - 0\right| < 2\varepsilon$. This means that $\lim(\max g_{n_k}) = 0$. Since $\max g_m \leq \max g_n$ whenever $m > n$, $\lim(\max g_n) = 0$. This implies that $\max_{x\in[a,b]} |f_n(x) - f(x)| \to 0$ as $n \to 0$. Thus, (f_n) converges uniformly to f.