Review VII(Slides 362 - 443)

Series & More about Func. Seq

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VV186 - Honors Mathmatics II

Series

Let (a_n) be a sequence in a normed vector space $(V, ||\cdot||)$. We define $s_n := \sum_{k=0}^{n} a_k$ as the **n-th partial sum** of (a_n). We say that (a_n) is **summable** with sum $s \in V$ if $\lim_{n\to\infty} s_n = s$. We use $\sum_{k=0}^{\infty} a_k$, or a_k to denote s as well as the "procedure of summing the sequence (a_n)", and call this notation **infinite series**.

Comment. While the definition of series is in general vector space, we will focus on real series and real function series later on.

Cauchy Criterion

Generally, a closed form sum of a sequence is hard to find. Instead, we will mostly focus on whether the series converges. The starting point will be the **Cauchy Criterion**(Slides 380):

Let $\sum a_k$ be a sequence in a complete vector space $(V, ||\cdot||)$ Then

$$\begin{aligned}\sum a_k \text{ converges} &\Leftrightarrow (s_n) \text{ converges}\\ &\Leftrightarrow (s_n) \text{ is Cauchy}\\ &\Leftrightarrow \forall \varepsilon > 0, \exists N \in \mathbb{N} \text{ such that } \forall m, n > N, ||s_m - s_n|| < \varepsilon\\ &\Leftrightarrow \forall \varepsilon > 0, \exists N \in \mathbb{N} \text{ such that } \forall m > n > N, ||\sum_{k=n+1}^{m} a_k|| < \varepsilon\end{aligned}$$

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Two important colloaries are:

- If (a_n) is summable, $a_n \to 0$ as $n \to \infty$ (and its contraposition)
- If (a_n) is summable, $\sum_{k=n}^{\infty} a_k \to 0$ as $n \to \infty$

Tests for Convergence

A number of tests (for series of positive real numbers) are given throughout the slides:

- 1. The Comparison Test (P393, 3.5.15)
- 2. The Root Test (P402, 3.5.22)
- 3. The Root Test in Limits Form (P406, 3.5.26)
- 4. The Ratio Test (P408, 3.5.28)
- 5. The Ratio Test in Limits Form (P411, 3.5.30)
- 6. The Ratio Comparison Test (P412, 3.5.31)
- 7. Raabe's Test (P413, 3.5.32)

Procedure of Determining Convergence

We rank the "usefulness" of all these tests as follows:

Cauchy Criteria
$>$ Comparison Test
$>$ Ratio Test (in Limits)
$>$ Root Test (in Limits)
$>$ Ratio Comparison Test/Raabe's Test...

When you are asked to determine whether a series converges, it's recommended to use the tests in this order. Thus if you have a hard time memorizing all the tests, do first memorize the more "important" tests.

Exercises

1. Please determine whether the following series converge or not!

- $$\sum_{n=0}^{\infty} \frac{4n(n+2)!}{(2n)!}$$
- $$\sum_{n=1}^{\infty} \frac{\sin(n\theta)}{n^2}, \text{ where } \theta \text{ is fixed}$$
- $$\sum_{n=0}^{\infty} \frac{(2n)!}{4^n(n+1)!n!} \text{ (Hint: This appears in the slides!)}$$
- $$\sum_{n=2}^{\infty} \frac{1}{(\ln n)^{(\ln n)}}$$

Exercises

2. Prove the **limit comparison test**:

For two positive series $\sum a_n$ and $\sum b_n$, if

$$0 < \lim_{n \to \infty} \frac{a_n}{b_n} < \infty$$

then a_n and b_n both converges or diverges.

Exercises

2. Prove the **limit comparison test**:

For two positive series $\sum a_n$ and $\sum b_n$, if

$$0 < \lim_{n\to\infty} \frac{a_n}{b_n} < \infty$$

then a_n and b_n both converges or diverges.

3. Use this result! Prove that if a positive series a_n diverges, then

$$\sum \frac{a_n}{1 + a_n}$$

also diverges.

Absolute and Conditionally Convergence

- A series Pan is called **absolutely convergent** if $\sum ||a_n||$ converges.
- If $\sum a_n$ converges while $\sum ||a_n||$ doesn't, than it's called **conditionally convergent**.
- In a complete vector space (which is the case in our cases), absolutely convergent implies convergent.

To test for conditionally convergence, we have the following theorem: Let $\sum \alpha_k$ be a complex series whose partial sum are bounded but need not converge. Let (a_k) be a decreasing convergent sequence with limit zero, then the series $\sum \alpha_k a_k$ converges (Slide 418)

Comment. With this result, it is easy to see that $\sum_{k=1}^{\infty} \frac{(-1)^k}{k}$ converges

Exercises

4. Let $\sum a_k$ be an absolutely convergent real series. Then for any rearrangement $b_j = a_{k_j}, j : \mathbb{N} \to \mathbb{N}$ bijective, $\sum b_j = \sum a_k$. (Slides 417)

Power Series

Of all function series, one useful kind is the **power series**, which is the infinite sum of monomials.

$$\sum_{k=0}^{\infty} a_k z^k \text{ or simply } \sum a_k z_k$$

We call this *formal* as we are yet to find whether the series converge or not for given z.

We can add and multiply two power series:

- $\sum a_n z^n + \sum b_n z^n = \sum (a_n + b_n) z^n$
- $\sum a_n z^n \cdot \sum b_n z^n = \sum (a * b)_n z^n$

Why convolution?

Raduis of Convergence

Let $\sum a_k z^k$ be a complex power series. Then there exists a unique number $\rho \in (0, +\infty)$ such that

i) the power series $\sum a_k z^k$ is absolutely convergent at $z_0 \in \mathbb{C}$ if $|z_0| < \rho$;

ii) the power series diverges at $z_0 \in \mathbb{C}$ if $|z_0| > \rho$

Hadamard's formula:

$$\rho = \frac{1}{\overline{\lim\limits_{k\to\infty}} \sqrt[k]{|a_k|}}$$

where ρ is called the radius of convergence, if we informally write $1/\infty = 0, 1/0 = \infty$.

Remarks

- For a complex power series, the set of z which the series converge will always be a circle. For a real power series, the set will be a line segment and the radius of convergence is one half of the length.
- We can't say much if we have $|z| = \rho$. The series may converge or diverge or conditionally converge.

Do check for the boundary!

3.6.6. Example. The formal power series $\sum_{k=0}^{\infty} \frac{(-1)^k}{k} z^k$ has radius of convergence $\varrho = 1$. The series converges for $z_0 = 1$ and diverges for $z_0 = -1$. Other values of z_0 with $|z_0| = 1$ can be checked individually.

Exercise

5. Decide for the following real power series, on which interval would it converge?

- $$\sum_{n=1}^{\infty} \frac{(x-1)^n}{2^n n}$$
- $$\sum_{n=1}^{\infty} \frac{x^{2n}}{n-3^{2n}}$$

Differentiability of Power Series

The power series $\sum a_k z^k$ with radius of convergence ρ defines a differentiable function $f\colon B_\rho(0) \to \mathbb{C}$. Furthermore,

$$f'(z_0) = \sum k a_k z_0^{k-1}$$

Remarks:

1. This means that we can differentiate a power series "term by term" inside the radius of convergence.
2. Recursively apply this theorem to see that any power series is infinitely differentiable inside its radius of convergence. In fact, for a function to be expressable as a power series (which we call it **analytic**) is stronger than being infinitely differentiable. (You will learn more about this in Vv286!)

Compare and Contrast

Proof (continued).

This is enough to show that f is differentiable at z_0 and $f'(z_0) = g(z_0)$. Fix $\varepsilon > 0$ and choose some δ such that $|z_0 + h| < r$ if $h \in B_\delta(z_0)$.

$$\frac{f(z_0+h) - f(z_0)}{h} - g(z_0) = \left(\frac{S_N(z_0+h) - S_N(z_0)}{h} - S_N'(z_0)\right) + (S_N'(z_0) - g(z_0)) + \frac{E_N(z_0+h) - E_N(z_0)}{h}$$

Proof (continued).

Then for $|h| < \delta$ we have

$$\begin{aligned} |f(x+h) - f(x)| &\le |f(x+h) - f_n(x+h)| + |f_n(x+h) - f_n(x)| \\ &\quad + |f_n(x) - f(x)| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

4.

Proof: Let $\langle x_n \rangle$ be such a sequence and let $\langle x_{n_k} \rangle$, $\langle x_{n_j} \rangle$ be two subsequences of it with different limits a, b. Let $\varepsilon = \frac{1}{3}\rho(a, b)$. For any $N \in \mathbb{N}$, choose some $n_k > N$ and $n_j > N$, such that $\rho(x_{n_k}, a) < \varepsilon$ and $\rho(x_{n_j}, b) < \varepsilon$. If $\rho(x_{n_k}, x_{n_j}) < \varepsilon$ then

$$\rho(a, b) \le \rho(a, x_{n_k}) + \rho(x_{n_k}, x_{n_j}) + \rho(x_{n_j}, b) < 3\varepsilon = \rho(a, b),$$

Exercise

6. You've learnt about Taylor polynomials of a function in your assignments (see Exercise 7.2). If we let $n \to \infty$ in $T_n(x; x_0)$, it becomes a *Taylor Series*. Please use this to find a series representation of

$$f(x) = \ln(1 + x)$$

around $x = 0$ and determine its radius of convergence. (You will learn more about Taylor series in the last part of the course)

Convergence of Continuous Functions

Uniform convergence is stronger than pointwise convergence as it preserves crucial properties of functions such as continuity:

3.4.3. Theorem. Let $[a, b] \subset \mathbb{R}$ be a closed interval. Let (f_n) be a sequence of continuous functions defined on $[a, b]$ such that $f_n(x)$ converges to some $f(x) \in \mathbb{R}$ as $n \to \infty$ for every $x \in [a, b]$. If the sequence (f_n) converges uniformly to the thereby defined function $f\colon [a, b] \to \mathbb{R}$, then f is continuous.

Using this, we can prove that $C([a, b])$ is complete:

3.4.4. Theorem. Let $[a, b] \subset \mathbb{R}$ be a closed interval and $C([a, b])$ the vector space of continuous functions on $[a, b]$, endowed with the metric

$$\varrho(f, g) = \|f - g\|_\infty = \sup_{x \in [a,b]} |f(x) - g(x)|.$$

Then the metric space $(C([a, b]), \varrho)$ is complete, i.e., every Cauchy sequence in the space converges.

Exercise

7. Let $f_n(x)$ be an sequence of functions in $C^{\infty}(a, b)$, and

$$||f_n{}'(x) - f_m{}'(x)|| \leq A||f_n(x) - f_m(x)||$$

If $f_n \to f$ as $n \to \infty$ uniformly, prove that $f_n{}' \to f'$ uniformly.
(Adapted from vv286 homework)

Reference

- Exercises from 2019–Vv186 TA-Zhang Leyang.
- Exercises from 2020-Vv186 TA-Xia Yuxuan.
- Mathematical Analysis II. *School of Mathematical Sciences, ECNU*,version 5. Beijing: High Education Press, 2019.5 print.

End

Thanks!