Answer for RC4

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VV186 - Honors Mathmatics II

Exercise

3. Let $f\colon [0, +\infty) \to \mathbb{R}$ be a continuous function such that $\lim\limits_{x\to\infty} f(x)$ exists and is finite. Show that f is uniformly continuous on $[0, +\infty)$.

Solution

3.Proof:

Fix $\varepsilon > 0$, there is some $N \in \mathbb{N}$ such that $\forall x > N$, $|f(x) - \lim_{x\to\infty} f(x)| < \varepsilon$. Furthermore, f is uniformly continuous on $[0, N]$, i.e., there is some $\delta > 0$ such that $\forall x, y \in [0, N], |x - y| < \delta \Rightarrow |f(x) - f(y)| < \frac{1}{2}\varepsilon$. Note that this δ also works for $[N, +\infty)$ (because $\forall x > N, |f(x) - \lim_{x\to\infty} f(x)| < \varepsilon$.

Therefore, f is uniformly continuous.

Exercises

4. Let $f : [a, b] \to \mathbb{R}$ be a continuous function with $b > a$. Given $\varepsilon > 0$, show that there is a polygonal function g such that $|f(x) - g(x)| < \varepsilon$ for all $x \in [a, b]$.

Note: A polygonal function is a function formed by a finite number of line segments. Of course, a polygonal function is continuous.

- Try to write a complete proof!

Solution

Proof: Since f's domain is a closed interval (or you can say bounded closed set), and f is continuous, f is uniformly continuous on $[a,b]$. Thus, given an $\varepsilon > 0$, there is a partition: $\left[a, a+\frac{b-a}{n}\right] =: I_1, \left[a+\frac{b-a}{n}, a+\frac{2(b-a)}{n}\right] =: I_2, \ldots, \left[b-\frac{b-a}{n}, b\right] =: I_m$ such that for any $x, y \in I_i$ where $i \in \{1,2,\ldots,m\}$, $|f(x)-f(y)| < \frac{1}{2}\varepsilon$. Fix this ε. Note that our polygonal function g is a continuous function whose graph is composed by line segments that connect $f(a), f\left(a+\frac{b-a}{n}\right), \ldots, f\left(b-\frac{b-a}{n}\right), f(b)$. For any $x, y \in I_i$ and any $i \in \{1,2,\ldots,m\}$, $|g(x)-g(y)| \leq \left|g\left(a+(i-1)\frac{b-a}{n}\right) - g\left(a+i\frac{b-a}{n}\right)\right| = \left|f\left(a+(i-1)\frac{b-a}{n}\right) - f\left(a+i\frac{b-a}{n}\right)\right| < \frac{1}{2}\varepsilon$. Now, for any fixed $x \in [a,b]$, x is in some interval I_i. It follows that $|f(x)-g(x)| = \left|f(x) - f\left(a+i\frac{b-a}{n}\right) + f\left(a+i\frac{b-a}{n}\right) - g(x)\right| = \left|f(x) - f\left(a+i\frac{b-a}{n}\right) + g\left(a+i\frac{b-a}{n}\right) - g(x)\right| \leq \left|f(x) - f\left(a+i\frac{b-a}{n}\right)\right| + \left|g\left(a+i\frac{b-a}{n}\right) - g(x)\right| < \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon.$

Exercises

5. The function $f(x)$ is defined on interval I. Proof that $f(x)$ is uniformly continuous if and only if: for any sequence $x_n', x_n'' \subset I$, if $\lim_{n\to\infty}(x_n' - x_n'') = 0$, then $\lim_{n\to\infty}(f(x_n') - f(x_n'')) = 0$.

Solution

($\Rightarrow$): Suppose $f(x)$ is uniformly continuous on I, then $\forall \varepsilon > 0$, $\exists \delta(\varepsilon) > 0, \forall x', x'' \in I, |x' - x''| < \delta, |f(x') - f(x'')| < \varepsilon$. Since $\lim_{n\to\infty}(x'_n - x''_n) = 0$,we have for the $\delta > 0$ above, $\exists N > 0, \forall n > N, |x'_n - x''_n| < \delta$, using the uniform continuty of f,

$$|f(x'_n) - f(x''_n)| < \varepsilon$$

so

$$\lim_{n\to\infty}(f(x'_n) - f(x''_n)) = 0$$

Solution

($\Leftarrow$): Proof by contradiction: suppose that $f(x)$ is not uniformly continuous on I, then

$$\exists \varepsilon_0 > 0, \forall \delta > 0, \exists x', x'', |x' - x''| < \delta, |f(x') - f(x'')| \geq \varepsilon_0$$

Let $\delta_1 = 1, \exists x_1', x_1'' \in I, |x_1' - x_1''| < 1, |f(x_1') - f(x_1'')| \geq \varepsilon_0$

Let $\delta_2 = \frac{1}{2}, \exists x_2', x_2'' \in I, |x_2' - x_2''| < \frac{1}{2}, |f(x_2') - f(x_2'')| \geq \varepsilon_0$

...

Let $\delta_n = \frac{1}{n}, \exists x_n', x_n'' \in I, |x_n' - x_n''| < \frac{1}{n}, |f(x_n') - f(x_n'')| \geq \varepsilon_0$

...

So $\lim_{n\to\infty}(x_n' - x_n'') = 0$, but

$$\lim_{n\to\infty}(f(x_n') - f(x_n'')) \neq 0$$

, contradiction.

Exercise

6.Let $f : (0, +\infty) \to \mathbb{R}$ be a continuous function such that $f(x^2) = f(x)$. Please show that f is a constant function on $(0, +\infty)$, i.e.,

$$\exists M \in \mathbb{R}, \forall x \in domf, f(x) = M$$

(SJTU Math textbook, P70)

Solutions

Proof: Suppose $\exists x, y \in (0, +\infty)$ with $f(x) \neq f(y)$. Suppose $f(x) = a$, $f(y) = b$.

Take $f(x) = f\left(x^{\frac{1}{2}}\right) = f\left(x^{\frac{1}{4}}\right) = \cdots = f\left(x^{\frac{1}{2^n}}\right) = \cdots$. Since f is continuous and the sequence $\left(x^{\frac{1}{2^n}}\right)$ converges to 1, $f(1) = f(x) = a$. Similarly, $f(1) = f(y) = b$.

However, this leads to a contradiction to the definition of "function", since a function cannot take two distinct values at a same point.

Exercises

8. Let $f : \Omega \to \mathbb{R}$ be a real function that satisfies Lipschitz condition, that is, there is a constant $M > 0$ such that for all x and y in the domain of f, $|f(x) - f(y)| \leq M|x - y|$.

(i) Show that f is uniformly continuous

(ii) Now Let $\Omega =: [a, +\infty)$, where $a > 0$. Show that $f(x)/x$ is uniformly continuous

Solution

Proof: i) Given an $\varepsilon > 0$, choose $\delta = \frac{\varepsilon}{M}$. Then $\forall x, y \in \Omega$ with $|x - y| < \delta$, we have: $|f(x) - f(y)| \leq M \cdot |x - y| < M \cdot \frac{\varepsilon}{M} = \varepsilon$. It follows that f is uniformly continuous.

ii) Let $g: [a, +\infty) \to \mathbb{R}$, $g(x) = \frac{f(x)}{x}$. Note that $\forall y \geq a$, since $|f(y)| \leq |f(a)| + M \cdot |y - a|$, $\frac{|f(y)|}{|y|} \leq \frac{|f(a)|}{y} + M \cdot \frac{|y-a|}{y} \leq \frac{|f(a)|}{a} + M \cdot \frac{|y-a|}{y}$. Since $\frac{|f(a)|}{a}$ is nothing but a fixed non-negative number and $\frac{|y-a|}{y} = \frac{y-a}{y} \leq 1$, there is some $C \geq 1$ such that $\frac{|f(y)|}{|y|} \leq CM$.

Solution

Fix arbitrary $\varepsilon > 0$. $\forall x, y \geq a$, $|g(x) - g(y)| = \left|\frac{f(x)}{x} - \frac{f(y)}{y}\right| = \frac{|yf(x)-xf(y)|}{|xy|} =$
$\frac{|yf(x)-yf(y)+yf(y)-xf(y)|}{|xy|} \leq \frac{|yf(x)-yf(y)|+|yf(y)-xf(y)|}{|xy|} \leq \frac{|y||f(x)-f(y)|}{|xy|} + \frac{|y-x||f(y)|}{|xy|} =$
$\frac{|f(x)-f(y)|}{|x|} + \frac{|x-y|}{|x|} \cdot \frac{|f(y)|}{|y|} \leq \left(\frac{M|x-y|}{a} + \frac{1}{a^2}|x - y|CM\right)$. It follows that $|g(x) - g(y)| \leq$
$(\frac{M}{a} + \frac{CM}{a^2})|x - y|$. Therefore, g satisfies Lipschitz condition. By i), g is uniformly continuous.