Exercise 3

Proof: (The content in the second picture is the definition of weighted norm)

because and are non-negative functions. Besides, implies that for all . Since , must be true.

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For triangle inequality, we claim that

Therefore, the triangle inequality holds. (Wolfgang Walter Ordinary Differential Equations, P56)

Exercise 4

Proof: 1st sentence. **(It depends)**True if and only if is a subset or superset of . Suppose is not the subset or superset of . There exists some but . For any but , is not in or . If is a subset or superset of , then it’s obvious that is a vector space.

2nd sentence. True. For any element , both of them are in . Since is a vector space, . Similarly, we can show that . Therefore, is a subspace of V.

3rd sentence. True. Recall that real linear maps are of the form for some k. Denote the set containing all linear maps by . Since any real linear map is continuous, . For any where , . Besides, given , such that , . So is a subspace of .

4th sentence. False. Counterexample: Let . Let , let , . Choose . Then . . . Thus, . (Note that “given” means “for any” here)

5th sentence. True. Since is a norm, it will map anything to a non-negative number, therefore for all x. Furthermore, If , then . Since is a norm from V to , if and only if . This proves the first requirement for a norm. . For triangle inequality, we have the following:

Therefore, is a norm.

Exercise 6

Proof: i) Fix arbitrary . Given , . Thus, by choosing , whenever , we have . Since this is true for all x, f is continuous. It is trivial that . Note that our proof about the continuity of f doesn’t include the concrete value of a. So the inverse function is also continuous.

ii) Fix arbitrary . Given , . Thus, by choosing , whenever , we have . It is trivial that . Note that our proof about the continuity of f doesn’t include the concrete value of (as long as it is not zero). So the inverse function is also continuous.

Exercise 7

Solution: To illustrate, I only show the solution of the first one. The solution of other is based on the same ideas. Besides, the second one is JUST the example in Horst’s slides.

Fix . If , then . Therefore, the limit of is zero. If , trivially for all n; so . If , then as . But this means that . In conclusion, the limit of is

From the explicit form of f, or the graph of it, it’s obvious that f is uniformly continuous on any set in , where is a positive number. f is also uniformly continuous on any bounded subset of , where is a positive number.

Exercise 8

Proof: For the two problem above, the key is to construct a sequence of functions that satisfies the requirements. i) The sequence of functions can be , , for each .

To show that each is nowhere continuous, first fix some rational . For any , in the neighborhood , there is of course some irrational numbers, denote one of them by “y”. It follows that . Thus, it is not continuous at any rational points in . A similar argument holds for irrational points. Therefore, is a sequence of nowhere continuous functions that converges to the continuous function f.

ii) Since , g is uniformly continuous. This means given , there is a partition of for some , such that . To construct the function sequence, we will use the modified Van Der Waerden function (see Horst’s Slide 442) as the prototype. Denote the prototype-Van Der Waerden function by W. Choose some , such that attains its maximum (which is ). Denote the part of W on by (Part of W). Fix , choose some and a corresponding partition of subintervals . Note that the length of each of these subintervals is . For , if , then

on . If , then

on . The idea of constructing on is to use horizontal translation and stretch (i.e., ) and vertical translation (i.e., ) and vertical stretch (i.e., ). At each point , the constructed function has difference less than from the original function f. It’s easy to check that is continuous on each , and has the same value on the endpoints of “”s. So is continuous. Also, since the PW is nowhere differentiable, is nowhere differentiable. Furthermore, with some calculation one can check that uniformly. Therefore, following function sequence is what we want.

Exercise 9

Proof: Since and converges to f uniformly, there is some such that . , there is some such that . Choose some such that . Fix this , for any we have for any . Therefore, the function sequence converges uniformly to .

Exercise 10

Proof: i) Fix . Since is bounded and monotonic sequence, it converges. Because this is true for all , converges point-wisely to some function , where .

ii) Construct a function sequence such that for each n. First note that whenever , for all , because decreases to zero. Since the domain of is a closed interval , for each fixed n, attains its maximum at some point . The points where attains maximum(s) form a sequence , i.e., each term is a point where attains its maximum. Note that is bounded, so there is a subsequence converges to some point . Since , by continuity of (for each n) we know that converges to . Fix arbitrary . For large enough we can ensure that , so . Since for all “”s and , , it follows that whenever , . This means that . Since whenever , . This implies that as . Thus, converges uniformly to .