Numerical Integration

Definite Integral

$$I(f) = \int_a^b f(x)\,dx$$

Where $f(x)$ can be:

- Analytical expression (e.g $\cos(x)$)
  → if simple, just compute $I(f)$ exactly.
- Set of discrete points:

  eg:

x:	0	5	10	15	20
$f(x)$:	0	3	8	20	30

  notice $x_i - x_{i-1} = 5 \equiv h$ (important) (book notation)

- A software function.

Applications:

- studying a physical law expressed as a derivative
- Acceleration → Velocity

  or

  Velocity → position
- Gyroscope rotation-rate (e.g °/s) to orientation (e.g °).
- As part of PID controller

Approaches to Approximating $I(f) = \int_a^b f(x)\,dx$

- **Basic**: ① Divide $[a, b]$ into subintervals — Rectangle Method
  ② Calculate (approximate) integral over subinterval — Midpoint Method
  ③ Add the results — Trapezoidal Method

All use straight lines to approximate integrand.
(i.e zero or 1st order poly)

- **Medium**: Approximate Integrand Using Higher-Order Polynomial:

Note: The polynomials pass through the data points.

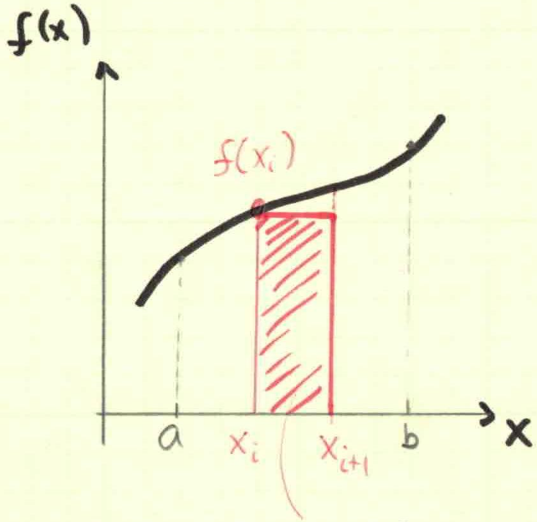
  - Quadratic: using 3 points — Simpson's 1/3 Method
  - Cubic: using 4 points — Simpson's 3/8 Method

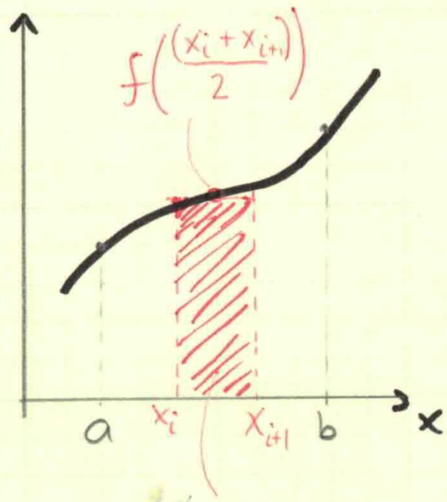
These polynomials can easily be integrated.

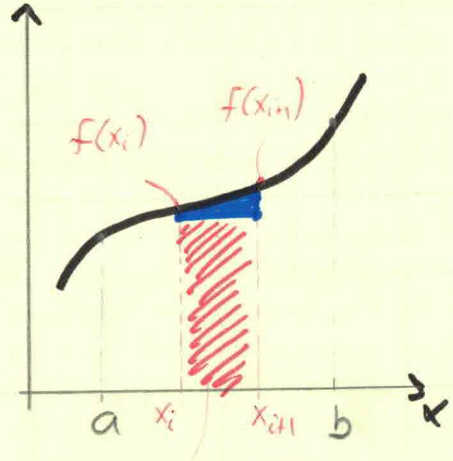
- **Fancy**: Approximate Integral using Weighted sum of $f(x)$ at different points (i.e Gauss points) — Gauss Quadrature Method
  - not equally spaces
  - don't include a or b (end points)

Details on Base Methods

- It takes $N+1$ points to define N intervals
- Say $h = x_{i+1} - x_i$ is uniform spacing







Subinterval Area : $f(x_i)(x_{i+1} - x_i)$

rectangle + **triangle**

$\underline{\underline{f(x_i)\cdot h}} + \underline{\underline{\left(\frac{1}{2}\right) h \left(f(x_{i+1}) - f(x_i)\right)}} = \frac{h}{2}\left(f(x_i) + \ldots f(x_{i+1})\right)$

$$I(f) \approx h \cdot \sum_{i=1}^{N} f(x_i)$$

Rectangle Method

$$I(f) \approx h \cdot \sum_{i=1}^{N} f\left(\frac{(x_i + x_{i+1})}{2}\right)$$

Midpoint Method

$$I(f) = \frac{h}{2} \sum_{i=1}^{N} \left[f(x_i) + f(x_{i+1})\right]$$

Trapezoidal Method