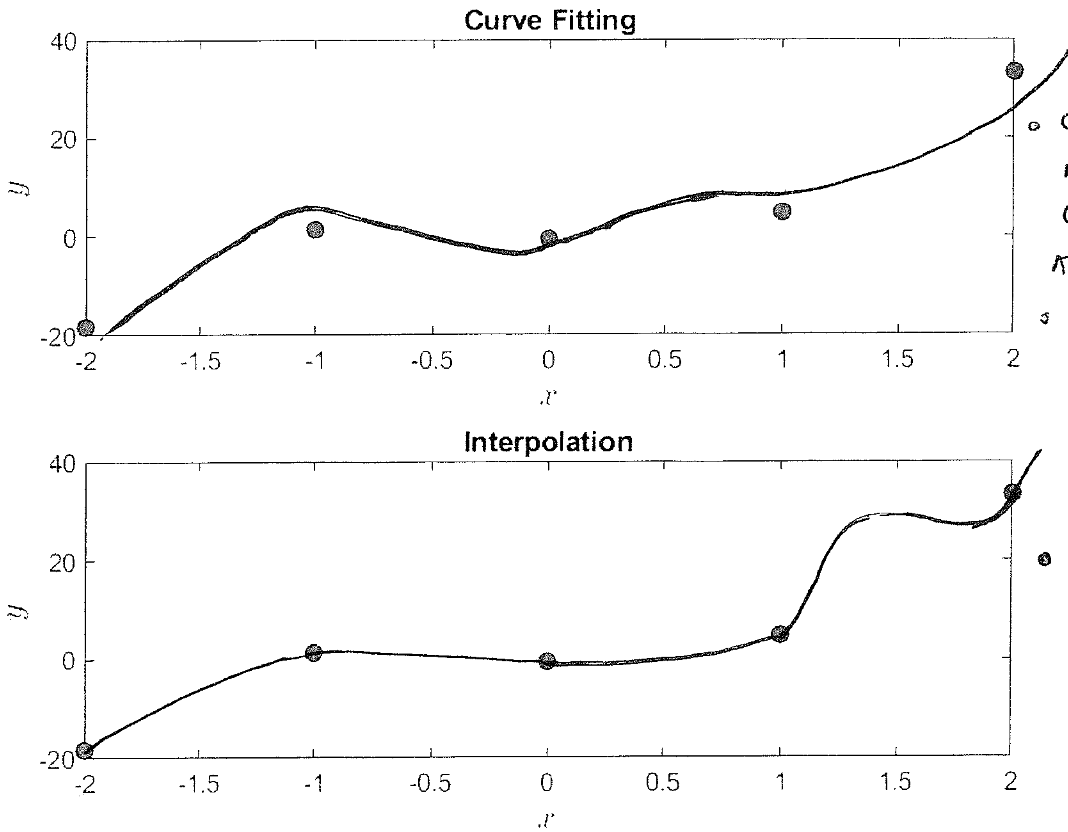


- CURVE DOES NOT NECESSARILY GO THROUGH ALL DATA POINTS
- CLOSE AS POSSIBLE IN A GLOBAL SENSE

- CURVE GOES THROUGH ALL POINTS GIVEN

Figure 1: Interpolation vs. curve fitting.

Mathematical Modeling and Simulation | **University of Minnesota**
AEM 3103 | **April 13, 2023**

Practice Problems for Exam #2

1. What is the difference between curve fitting and interpolation? Answer this question by showing what the difference between these to approaches is using the points given in Figure 1

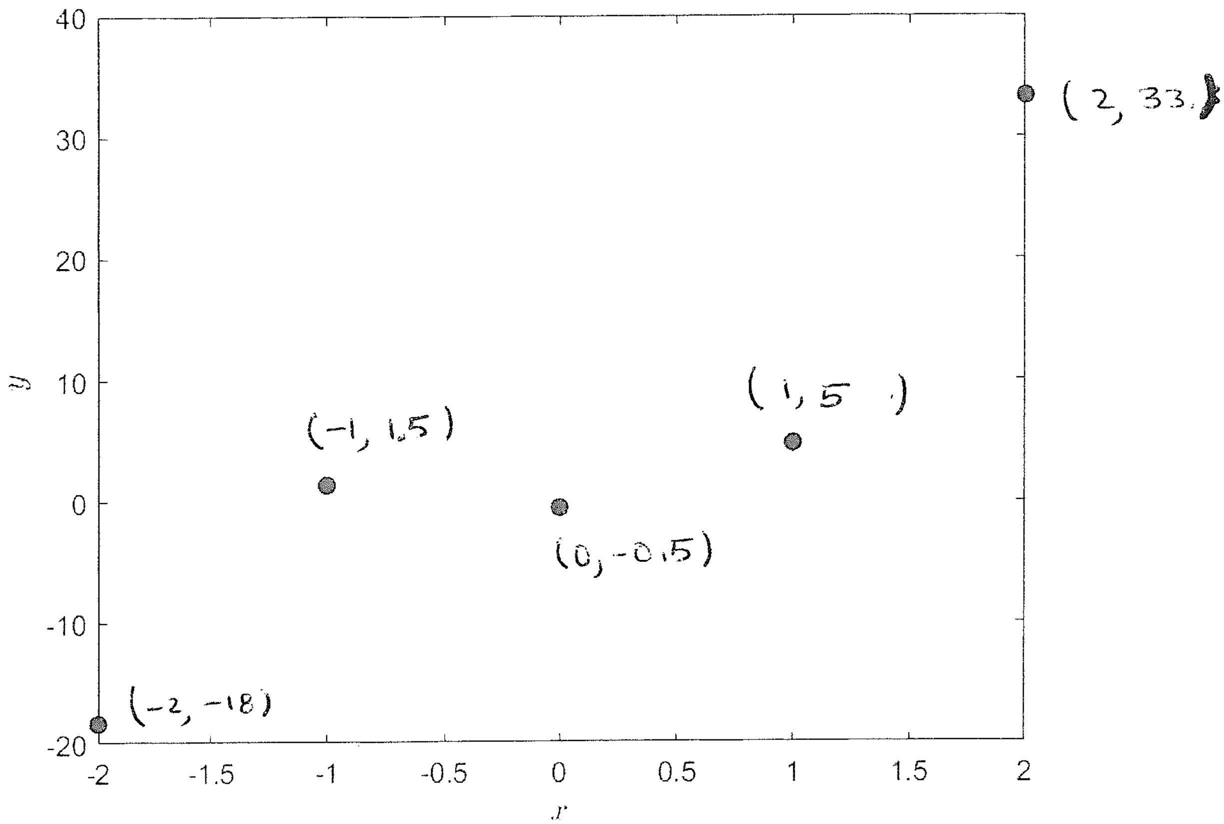


Figure 2: Least squares curve fitting.

2. *Least Squares Curve Fitting*: For the data shown in Figure 2, determine the coefficient of a third order polynomial that fits the given data. Note, it will be difficult to do this problem by hand, so show clearly how your would set up in MATLAB to solve.

$y = a_3x^3 + a_2x^2 + a_1x + a_0$ NEED TO DETERMINE a_i

$$E = \sum_{i=1}^{5} \left(y_i - [a_3x_i^3 + a_2x_i^2 + a_1x_i + a_0]\right)^2$$

SET UP FOUR EQUATIONS FOR SOLVING FOR a_j FOR $j = 0, 1, 2, 3$

$\frac{\partial E}{\partial a_i} = 0 \longrightarrow$ EQUATION FOR a_i

THIS WILL LEAD TO A SYSTEM OF EQUATIONS FOR a_i WHICH LOOKS AS FOLLOWS:

$$\underbrace{\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix}}_{\underline{Z}} = \underbrace{\begin{bmatrix} x_1^3 & x_1^2 & x_1 & 1 \\ x_2^3 & x_2^2 & x_2 & 1 \\ x_3^3 & x_3^2 & x_3 & 1 \\ x_4^3 & x_4^2 & x_4 & 1 \\ x_5^3 & x_5^2 & x_5 & 1 \end{bmatrix}}_{[M]} \begin{bmatrix} a_3 \\ a_2 \\ a_1 \\ a_0 \end{bmatrix}$$

$$\begin{bmatrix} a_3 \\ a_2 \\ a_1 \\ a_0 \end{bmatrix} = \text{pinv}([M]) \cdot \underline{Z}$$

$$= \begin{bmatrix} (-2)^3 & (-2)^2 & -2 & 1 \\ (-1)^3 & (-1)^2 & -1 & 1 \\ 0 & 0 & 0 & 0 \\ 5^3 & 5^2 & 5 & 1 \\ 2^3 & 2^2 & 2 & 1 \end{bmatrix} \begin{bmatrix} -18 \\ 1.5 \\ -0.5 \\ 5 \\ 33 \end{bmatrix}$$

$$= \begin{bmatrix} 3.7 \\ 1.8 \\ -1.9 \\ 0.7 \end{bmatrix}$$

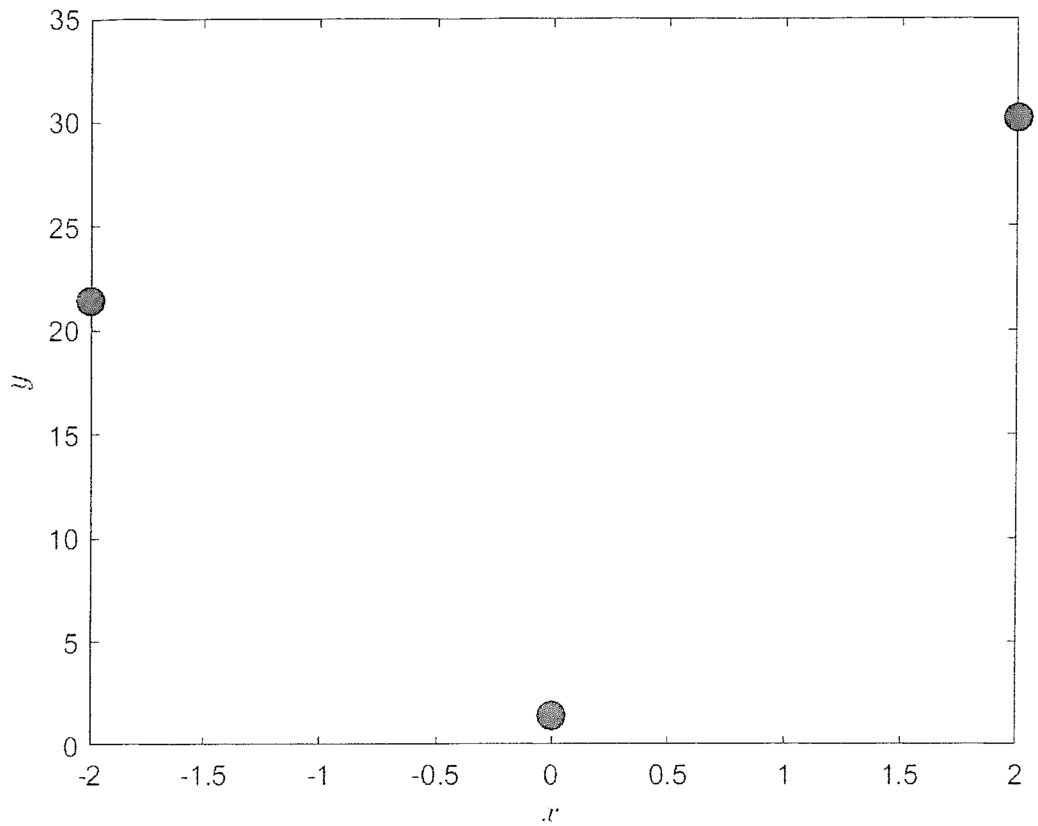


Figure 3: Lagrangian interpolation.

3. *Interpolation*: For the data shown in Figure 3, use Lagrangian interpolation to determine the value of $f(x)$ at $x = 1$.

$$p(x) = a_1(x-x_2)(x-x_3) + a_2(x-x_1)(x-x_3) + a_3(x-x_1)(x-x_2)$$

$$= a_1 x(x-2) + a_2(x+2)(x-2) + a_3(x+2)(x)$$

@ $x = -2$, $p(x) = 8a_1 = 21 \longrightarrow a_1 = \frac{21}{8}$

@ $x = 0$, $p(x) = -4a_2 = 1.5 \longrightarrow a_2 = -\frac{3}{8}$

@ $x = 2$, $p(x) = 8a_3 = 30 \longrightarrow a_3 = \frac{30}{8}$

$$p(x) = \frac{1}{8}\left[21x(x-2) - 3(x-2)(x+2) + 30x(x+2)\right]$$

Thus, @ $x = 1$

$$p(1) = \frac{1}{8}\left[(21 \times -1) - 3(-1)(3) + 30 \times 3\right]$$

$$= \frac{1}{8}\left[-21 + 9 + 90\right]$$

$$\approx \underline{\underline{9.75}}$$

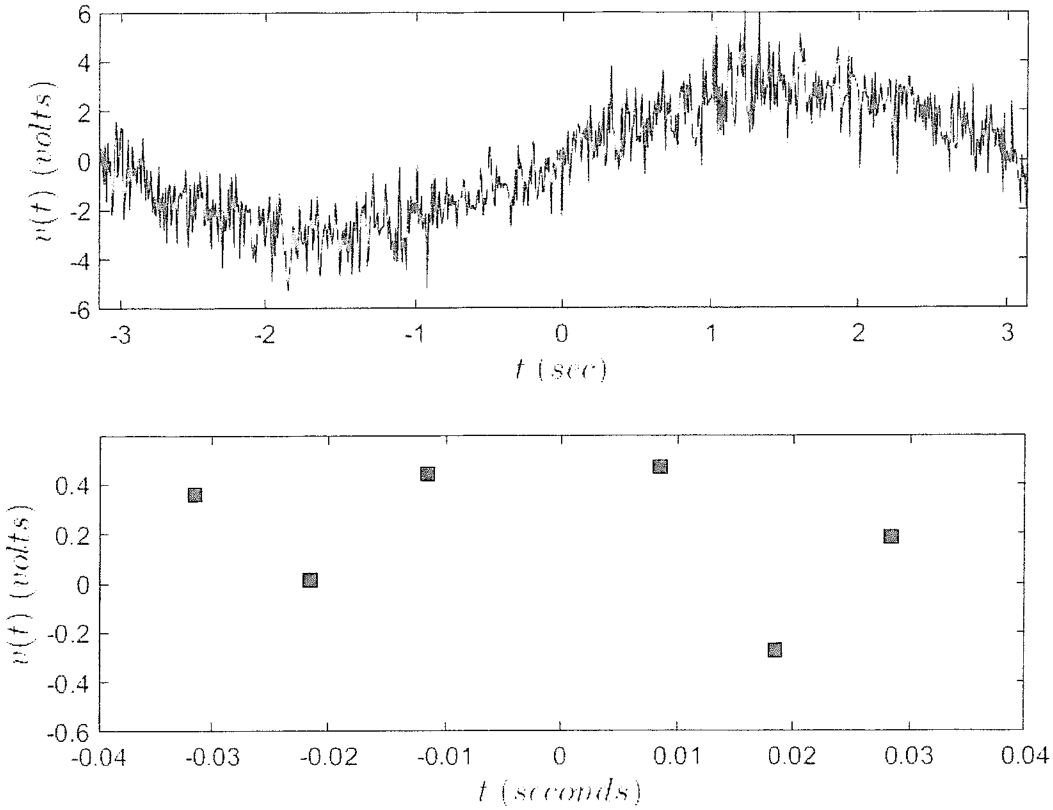


Figure 4: Time history of voltage $v(t)$ accorss an electronic component.

4. Numerical Differentiation: The uppper plot shown in in Figure 4 is experimental data collected by measuring the voltage across some electronic component. The lower figure is a zoom-in around $t = 0$ seconds. Explain how you would use a forward difference to generate an accurate estimate of the derivative of $v(t)$. Then calculate and report the value of your estimate for the derivative.

YOU MIGHT BE TEMPTED TO USE A FORWARD DIFFERENCE, BACKWARD DIFFERENCE OR CENTRAL DIFFERENCE TO CALCULATE THE DERIVATIVE. FOR EXAMPLE IF WE DO A CENTRAL DIFFERENCE (TWO POINT) WE GET

$$\dot{v}(t) = \frac{0.43 - 0.48}{2 \times 0.01} = \frac{-0.05}{0.02} \approx -2.5 \leftarrow$$

THIS, HOWEVER, IS A POOR ESTIMATE DUE TO NOISE BECAUSE NOTE

$$V(t) \approx 3\sin\left(\frac{\pi}{3}t\right)$$

$$\dot{V}(t) = \pi\cos\left(\frac{\pi}{3}t\right) \text{ WHICH @ } t=0 \quad \dot{V} = \pi = 3.14159\ldots$$

SO YOUR BEST BET IS TO:

1. FILTER $V(t)$ AND THEN APPLY NUMERICAL DIFF. ~~TECHNI~~ TECHNIQUES

2. FIT A CURVE TO $V(t)$ AND DIFFERENTIATE THE FIT.

A LOW TECH VERSION OF A BLEND OF 1 & 2 IS AS FOLLOWS:

$$\dot{V}(t) \approx \frac{\text{RISE}}{\text{RUN}} = \frac{6.3}{2} \approx 3.15$$

5. *Numerical Integration*: Evaluate the definite integral below using (i) The rectangular, (ii) Trapezoidal and (3) Simpson's 3/8 rule using a step size $h = 0.3142$

$$\int_0^{\frac{\pi}{2}} \frac{dx}{\sqrt{\cos x + 2}}$$

$x_1 = 0 \qquad x_2 = 0.3142 \qquad x_3 = 0.6182 \qquad x_4 = 0.9426$

$x_5 = 1.250 \qquad x_6 = 1.570$

$f(x_1) = \frac{1}{\sqrt{3}} \qquad f(x_2) = 0.5821 \qquad f(x_3) = 0.5961$

$f(x_4) = 0.6211 \qquad f(x_5) = 0.6571 \qquad f(x_6) - 0.7071$

i) $I(f) = h \cdot \sum_{i=1}^{5} = (0.3142)\left(\frac{1}{\sqrt{3}} + 0.5821 + 0.5961 + 0.6211 + 0.6571\right)$

[NOTE: YOU CAN ALSO GO FROM $i = 2:6$

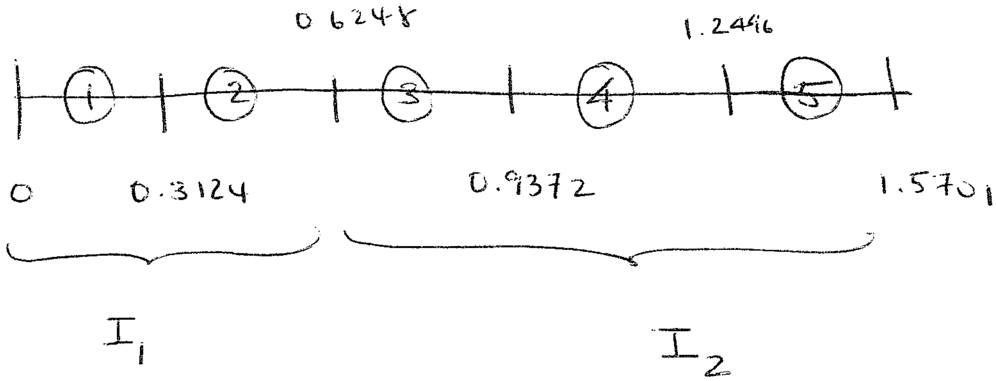
$= 0.9479$

ii) $I(f) = \frac{h}{2}\left[f(x_1) + f(x_2)\right] + \frac{h}{2}\left[f(x_2) + f(x_3)\right] + \ldots$

$= \frac{h}{2}\left[f(x_1) + 2\sum_{i=2}^{5} f(x_i) + f(x_6)\right]$

$= 0.9679$

iii)



$$I_1 = \frac{h}{8}\left[f(0) + 4f(0.3124) + f(0.6248)\right] = 0.3647$$

$$I_2 = \frac{3h}{8}\left[f(0.6248) + 4\left[f(0.9372) + f(1.2496)\right] + f(1.5701)\right] = 0.6020$$

$$I = I_1 + I_2 = \underline{\underline{0.9667}}$$

6. What is the advantage of the Runge-Kutta method over the Euler method when it comes to solving ODEs numerically? What is the advantage of Euler's method over the Runge-Kutta approach?

FOR THE SAME STEP SIZE, RK GENERATES MORE ACCURATE RESULTS THAN EULER.

EULER IS LESS COMPLEX AND LESS INTENSIVE ON COMPUTER RESOURCES.

7. *Solving ODEs numerically*: Consider the following ordinary differential equation:

$$\begin{aligned} y'' &= \frac{1}{2}(x + y + y' + 2) \\ y(0) &= 0 \\ y'(0) &= 0 \end{aligned}$$

Solve this differential equation numerically using Euler's method for $0 \leq x \leq 5$ and a step size $h = 1$. Show your work clearly and report the value of y and y' at each of the six point in the interval $0 \leq x \leq 5$.

Define Intermediate Variable $\underline{V}$

$$\underline{V} = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} y' \\ y \end{bmatrix} \qquad \underline{V}' = \begin{bmatrix} y'' \\ y' \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{1}{2}(x + V_2 + V_1 + 2) \\ V_2 \end{bmatrix}}_{f(x,\underline{V})}$$

$$\underline{V}(2) = \underset{[0,0]^T}{\underline{V}(1)} + h \cdot f(x, \underset{[3,0]^T}{\underline{V}(1)}) = [3, 0]^T$$

$$\underline{V}(3) = \underset{[3,0]^T}{\underline{V}(2)} + h\, f(x, \underset{[7,3]^T}{\underline{V}(2)}) = [10, 3]^T$$

$$\underline{V}(4) = \underset{[10,3]^T}{\underline{V}(3)} + h\, f(x, \underset{[18,10]^T}{\underline{V}(3)}) = [28, 13]^T$$

$$\underline{V}(5) = \underset{[28,13]^T}{\underline{V}(4)} + h f(x, \underset{[47,28]^T}{\underline{V}(4)}) = [75, 41]^T$$

8. *Solving ODEs numerically*: Consider the following ordinary differential equation:

$$y' = x + y \quad = f(x,y)$$
$$y(0) = 0$$

Solve this differential equation numerically using the Runge-Kutta method for $0 \le x \le 5$ and a step size $h = 1$. Show your work clearly and report the value of y and y' at each of the six point in the interval $0 \le x \le 5$.

STEP 1 for $k = 2$

$k_1 = f(0,0) = 0$

$k_2 = f(0.5, 0) = 0.5$

$k_3 = f(0.5, \frac{1}{2} 0.5) = 0.75$

$k_4 = f(1, 0.75) = 1.75$

$y_2 = y_1 + 1 \cdot (k_1 + 2k_2 + 2k_3 + k_4) = 0.7083$

REPEAT FOR $k = 3, 4, 5$

K	X	Y	K1	K2	K3	K4	y(k+1)
1.0000	0	0	0	0.5000	0.7500	1.7500	0.7083
2.0000	1.0000	0.7083	1.7083	3.0625	3.7396	6.4479	4.3351
3.0000	2.0000	4.3351	6.3351	10.0026	11.8364	19.1714	15.8658
4.0000	3.0000	15.8658	18.8658	28.7987	33.7652	53.6310	48.8032
5.0000	4.0000	48.8032	52.8032	79.7049	93.1557	146.9589	139.7171

SOLUTION