Numerical Solutions to Ordinary Differential Equations (ODEs)

Definitions:

- Differential Eqn: eqn that contains derivative of an unknown function
- ODE: a differential equation w/ one independent variable
- N^{th}- Order ODE: the highest-order derivative of the dependent variable is "N".
- Linear ODE: it is a Linear function of the dependent variable.
  - → can be non-linear function of the independent variable

Examples:

dependent variable

(A) $\frac{dy}{dx} + ax^2 + by = 0$

independent variable

- 1^{st} Order linear ODE
- independent variable: x

(B) $\frac{dy}{dx} + ayx + b\sqrt{y} = 0$

- 1^{st} Order non-linear ODE
- independent variable: x

(C) $\frac{d^2x}{dt^2} + g\cos x = 0$

- 2^{nd} Order non-linear ODE
- independent variable: t

(D) $\frac{dx}{dt} + 2x = 0$

- 1^{st} Order linear ODE
- independent variable: t

(E) $\frac{dy}{dx} = -1.2y + 7e^{(-0.3x)}$

- 1^{st} Order linear ODE
- independent variable: x

Example (E) Assume $y(x=0) = 3$

Exact Analytic Solution: $y(x) = \frac{70}{9} e^{(-0.3x)} - \frac{43}{9} e^{(-1.2x)}$

$\downarrow$

Satisfies original ODE.

Numerical Solution: a set of discrete points that approximates $y(x)$.

$\rightarrow$ assumes an interval (eg $x=a$ to $x=b$)

$\rightarrow$ assumes N subintervals.

(x_1, y_1)
(x_2, y_2)
$\vdots$
(x_{N+1}, y_{N+1})

To Solve, we must cast the ODE into the form:

$$\frac{dy}{dx} = f(x,y) \qquad \underline{\text{or}} \qquad \frac{dx}{dt} = f(t,x)$$

Example (E): already done.

(D): easy algebraic manipulation

(C): $\frac{d^2x}{dt^2} + g\cos x = 0$

define: $\vec{V} = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} x \\ x' \end{bmatrix} = \begin{bmatrix} x \\ \frac{dx}{dt} \end{bmatrix}$

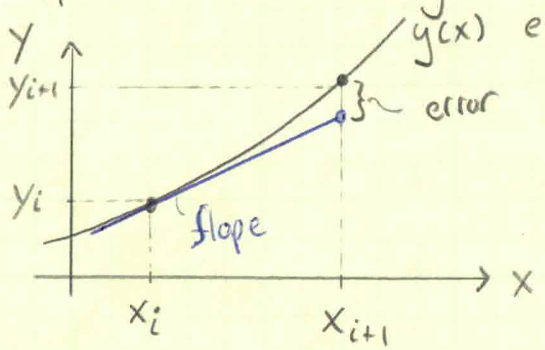
Summary: Can convert n^{th}-order ODE to a system of n 1^{st}-order ODE's.

then $\frac{dV_1}{dt} = V_2$ and $\frac{dV_2}{dt} = -g\cos(V_1)$

Relevant to HW4, prob 3.

Hence: $\underbrace{\frac{d\vec{V}}{dt} = \begin{bmatrix} \dot{V}_1 \\ \dot{V}_2 \end{bmatrix} = \begin{bmatrix} \frac{dV_1}{dt} \\ \frac{dV_2}{dt} \end{bmatrix}}_{\text{same thing in different notation}} = \begin{bmatrix} V_2 \\ -g\cos(V_1) \end{bmatrix}$

Approach to Solving Initial-Value ODE's $\frac{dy}{dx} = f(x, y)$



$h = x_{i+1} - x_i$

1) Start at known initial-value (x_1, y_1) known

2) Calculate the next value (x_2, y_2) using an estimate of the slope of $y(x)$ (i.e $\frac{dy}{dx}$) Key!

$$x_2 = x_1 + h$$

$$y_2 = y_1 + (\text{slope}) \cdot h$$

3) Repeat (2) until end of interval.

Different methods compute "slope" differently.

- Euler's Method: evalute f at (x_i, y_i) (aka: forward Euler's Method)
- Modified Euler's Method: average of slope at (x_i, y_i) and (x_{i+1}, y_{i+1}). eval func. f
- Runge Kutta Methods: weighted average of slope values (ie func f) of several points inside the interval.

Solving $y' = \frac{dy}{dx} = f(x, y)$

Euler's Method

$$y_{k+1} = y_k + \underbrace{f(x_k, y_k)}_{\text{WARNING: order of arguments depends on code implementation of function } f!} \overbrace{(x_{k+1} - x_k)}^{h \text{ (step size)}}$$

Modified Euler's Method

$$y_{k+1} = y_k + \frac{f(x_k, y_k) + f(x_{k+1}, \overbrace{y^{Eu}_{k+1}}^{\text{Solution using Euler's Method}})}{2} (x_{k+1} - x_k)$$