Let A be a matrix taken from the set

$$\left\{ \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \right\}$$

Let

$$f_1 = ax + by, f_2 = cx + dy.$$

We want to determine a, b, c, d such that

$$\begin{pmatrix} Af_1(x,y) \\ Af_2(x,y) \end{pmatrix} = A \begin{pmatrix} f_1(x,y) \\ f_2(x,y) \end{pmatrix}$$

where

$$Af_i(x,y) = f_i(a_{1,1}x + a_{1,2}y, a_{2,1}x + a_{2,2}y)$$

and

$$A \begin{pmatrix} f_1(x,y) \\ f_2(x,y) \end{pmatrix} = \begin{pmatrix} a_{1,1}f_1(x,y) + a_{1,2}f_2(x,y) \\ a_{2,1}f_1(x,y) + a_{2,2}f_2(x,y) \end{pmatrix}.$$

For $A = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, we have the following equation

$$\begin{pmatrix} \frac{1}{\sqrt{2}}(a+b)x + \frac{1}{\sqrt{2}}(a-b)y \\ \frac{1}{\sqrt{2}}(c+d)x + \frac{1}{\sqrt{2}}(c-d)y \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}}(a+c)x + \frac{1}{\sqrt{2}}(b+d)y \\ \frac{1}{\sqrt{2}}(a-c)x + \frac{1}{\sqrt{2}}(b-d)y \end{pmatrix}$$

We can write the previous equation as a matrix

$$\begin{pmatrix} 0 & 1 & -1 & 0 \\ 1 & -2 & 0 & -1 \\ -1 & 0 & 2 & 1 \\ 0 & -1 & 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \mathbf{0}$$

The linear equation system has nontrivial solution since

$$\det \begin{pmatrix} 0 & 1 & -1 & 0 \\ 1 & -2 & 0 & -1 \\ -1 & 0 & 2 & 1 \\ 0 & -1 & 1 & 0 \end{pmatrix} = 0.$$

For example, we take $a = 2, b = 1, c = 1, d = 0$. Then we have

$$f_1 = 2x + y, f_2 = x.$$

We have that

$$\begin{pmatrix} Af_1(x,y) \\ Af_2(x,y) \end{pmatrix} = \begin{pmatrix} \frac{3}{\sqrt{2}}x + \frac{1}{\sqrt{2}}y \\ \frac{1}{\sqrt{2}}x + \frac{1}{\sqrt{2}}y \end{pmatrix} \tag{1}$$

and

$$A \begin{pmatrix} f_1(x,y) \\ f_2(x,y) \end{pmatrix} = \begin{pmatrix} \frac{3}{\sqrt{2}}x + \frac{1}{\sqrt{2}}y \\ \frac{1}{\sqrt{2}}x + \frac{1}{\sqrt{2}}y \end{pmatrix} \tag{2}$$

We have (1)=(2).