

数学物理方法 期末考试

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1 第一题

2 第二题

$$\begin{aligned} z &= \frac{1 - i \tan x}{1 + i \tan x} = \frac{\cos x - i \sin x}{\cos x + i \sin x} = \frac{-i [\cos(\pi/2 - x) + i \sin(\pi/2 - x)]}{\exp(ix)} \\ &= \exp\left(i\frac{3\pi}{2}\right) \exp\left(i\frac{\pi}{2}\right) \exp(-2ix) = \exp(-2ix) \\ &= \cos 2x - i \sin 2x \end{aligned}$$

3 第三题

$$\begin{aligned} 1 + i &= \sqrt{2} \exp\left(i\frac{\pi}{4}\right) \\ 1 - i &= \sqrt{2} \exp\left(-i\frac{\pi}{4}\right) \\ (1 + i)^{1000} + (1 - i)^{1000} &= 2^{500} [\exp(250\pi i) + \exp(-250\pi i)] = 2^{501} \end{aligned}$$

4 第四题

$$\begin{aligned} \because |\sqrt{3} + i| &= |\sqrt{3} - i| \\ \therefore |\sqrt{3} + i|^n &= |\sqrt{3} - i|^n \end{aligned}$$

二者模长必然相同, 若要求二者相等, 只需要辐角满足

$$\arg(\sqrt{3} + i)^n = 2k\pi + \arg(\sqrt{3} - i)^n, \quad k \in \mathbb{Z}$$

$$\begin{aligned} \therefore n\frac{\pi}{6} &= -n\frac{\pi}{6} + 2k\pi \\ \therefore n &= 6k, \quad k \in \mathbb{Z} \end{aligned}$$

所以只需要 n 是整数且能被 6 整除即可。

5 第五题

将 $z = x + yi$ 代入 ω 展开有:

$$\omega = \frac{x^2 + y^2 - 1}{x^2 + (y+1)^2} - \frac{2x}{x^2 + (y+1)^2}i$$

6 第六题

$$a = \sum_{k=0}^{\infty} \frac{\cos nz}{n!}$$

$$b = \sum_{k=0}^{\infty} \frac{\sin nz}{n!}$$

考虑做一对替换有:

$$c_+ \equiv a + bi = \sum_{k=0}^{\infty} \frac{\cos nz + i \sin nz}{n!} = \sum_{k=0}^{\infty} \frac{\exp(inz)}{n!} = \exp(\exp(iz))$$

$$c_- \equiv a - bi = \sum_{k=0}^{\infty} \frac{\cos nz - i \sin nz}{n!} = \sum_{k=0}^{\infty} \frac{\exp(-inz)}{n!} = \exp(\exp(-iz))$$

其中

$$\exp(\exp(iz)) = \exp(\cos z + i \sin z) = e^{\cos z} (\cos \sin z + i \sin \sin z)$$

$$\exp(\exp(-iz)) = \exp(\cos z - i \sin z) = e^{\cos z} (\cos \sin z - i \sin \sin z)$$

于是由替换的关系可以得到:

$$a = \frac{c_+ + c_-}{2} = e^{\cos z} \cos \sin z$$

$$b = \frac{c_+ - c_-}{2i} = -ie^{\cos z} \sin \sin z$$

7 第七题

求

$$f(z) = \frac{1}{e^z - 1}$$

在 $z = 0$ 的洛朗级数, 很明显此处不解析, 考察 $n \in N^*$ 时,

$$\lim_{z \rightarrow 0} \frac{z^n}{e^z - 1} = \begin{cases} 1 & n = 1 \\ 0 & n > 1 \end{cases}$$

因此这是一阶极点, 考虑求解析函数 $a(z) = zf(z)$ 在 $z = 0$ 的泰勒展开, 设其为

$$a(z) = \sum_{n=0}^{+\infty} a_n z^n$$

这不容易计算, 考虑其倒数 $b(z) = 1/a$, 可以展开为

$$b(z) = \sum_{n=0}^{+\infty} \frac{z^n}{(n+1)!}$$

又因为恒等关系 $a(z)b(z) = 1$

$$\left(\sum_{n=0}^{+\infty} a_n z^n \right) \cdot \left(\sum_{n=0}^{+\infty} \frac{z^n}{(n+1)!} \right) = 1$$

比较 z 的各阶系数可以得到 $a_0 = 1, a_1 = -\frac{1}{2} \dots$

在 $n \geq 1$ 时, 其满足线性齐次方程组方程为:

$$\sum_{k=0}^n \frac{a_k}{(n-k+1)!} = 0$$

理论上可以解出任意的 a_n

我试图找到通项公式但是失败了, 查阅资料发现这个数列与伯努利数相关, 关系为 $a_n = B_n/n!$ 其被定义为

$$\frac{z}{e^z - 1} = \sum_{n=0}^{\infty} B_n \frac{z^n}{n!}$$

递推关系为

$$B_m = [m=0] - \sum_{k=0}^{m-1} \binom{m}{k} \frac{B_k}{m-k+1}$$

$B_0 = 1$, 其中 $[m=0]$ 表示当 $m=0$ 时, 取 1, 其余取 0。

综上可得,

$$f(z) = \sum_{n=-1}^{+\infty} \frac{z^n}{(n+1)!} B_{n+1}$$

8 第八题

由于所有极点都在围道内部, 直接考察无穷远的留数, 根据引理有:

$$\int_{|z|=200} f(z) dz = -2\pi i \operatorname{Res}_{z \rightarrow \infty} f(z) = 2\pi i \operatorname{Res}_{t \rightarrow 0} \left[f\left(\frac{1}{t}\right) \frac{1}{t^2} \right]$$

其中

$$f\left(\frac{1}{t}\right) \frac{1}{t^2} = \frac{1}{t^2} \prod_{k=1}^{100} \frac{1}{1-kt}$$

$$\therefore \lim_{t \rightarrow 0} t^2 \frac{1}{t^2} \prod_{k=1}^{100} \frac{1}{1-kt} = 1$$

\therefore 二阶极点

所以可以计算出留数的值

$$\begin{aligned} \operatorname{Res}_{t \rightarrow 0} \left[f \left(\frac{1}{t} \right) \frac{1}{t^2} \right] &= \lim_{t \rightarrow 0} \frac{d}{dt} \prod_{k=1}^{100} \frac{1}{1-kt} \\ &= \lim_{t \rightarrow 0} \sum_{n=1}^{100} \prod_{n=1 \wedge k \neq n}^{100} \frac{n}{1-kt} \\ &= \sum_{n=1}^{100} n = 5050 \end{aligned}$$

$$\therefore \int_{|z|=200} f(z) dz = 10100\pi i$$

9 第九题

10 第十题

直接考察积分，试图利用柯西公式

$$\begin{aligned} D_F(x-y) &= \int_{C_F} \frac{d^4 p}{(2\pi)^4} \frac{ie^{-ip \cdot (x-y)}}{p^2 - m^2} \\ &= \int_{C_F} \frac{d^3 \vec{p} dp_0}{(2\pi)^4} \frac{ie^{i\vec{p} \cdot (\vec{x}-\vec{y})} e^{-ip_0(x_0-y_0)}}{p_0^2 - (\vec{p}^2 + m^2)} \\ &= \int \frac{d^3 \vec{p} [e^{i\vec{p} \cdot (\vec{x}-\vec{y})}]}{(2\pi)^4} \int_{C_F} \frac{ie^{-ip_0(x_0-y_0)} dp_0}{p_0^2 - E_{\vec{p}}^2} \end{aligned}$$

考察其中的

$$\int_{C_F} \frac{ie^{-ip_0(x_0-y_0)} dp_0}{p_0^2 - E_{\vec{p}}^2} = \int_{C_F} \frac{ie^{-ip_0(x_0-y_0)} dp_0}{(p_0 - E_{\vec{p}})(p_0 + E_{\vec{p}})}$$

注意到在 C_F 的围道内只有一个一阶极点为

$$\operatorname{Res}_{p_0 \rightarrow E_{\vec{p}}} \frac{ie^{-ip_0(x_0-y_0)}}{(p_0 - E_{\vec{p}})(p_0 + E_{\vec{p}})} = \frac{ie^{-ip_0(x_0-y_0)}}{2E_{\vec{p}}}$$

考察其在无穷远的性质有

$$\lim_{|p_0| \rightarrow \infty} \left| \frac{ip_0 e^{-ip_0(x_0-y_0)}}{p_0^2 - E_{\vec{p}}^2} \right| = 0$$

实际上是一致收敛的

注意到 C_F 的围道方向为顺时针, 因此有

$$\int_{C_F} \frac{ie^{-ip_0(x_0-y_0)} dp_0}{p_0^2 - E_{\vec{p}}^2} = 2\pi \frac{e^{-ip_0(x_0-y_0)}}{2E_{\vec{p}}}$$

代入 Feynman 传播子表示可得到:

$$D_F(x-y) = \int_{C_F} \frac{d^4 p}{(2\pi)^4} \frac{ie^{-ip \cdot (x-y)}}{p^2 - m^2} = \int \frac{d^3 \vec{p}}{(2\pi)^3} \frac{e^{-ip \cdot (x-y)}}{2E_{\vec{p}}} \Big|_{p_0=E_{\vec{p}}}$$

11 第十一题

$$S = \int d^2 x \left\{ -\frac{1}{2} \partial_\mu \phi(x) \partial^\mu \phi(x) + \frac{1}{2} \lambda_{\mu\nu}(x) [\partial^\mu \phi(x) - \epsilon^{\mu\sigma} \partial_\sigma \phi(x)] [\partial^\nu \phi(x) - \epsilon^{v\rho} \partial_\rho \phi(x)] \right\}$$

先分别计算其中的积分

$$S_1 = \int d^2 x [\partial_\mu \phi(x) \partial^\mu \phi(x)]$$

$$S_2 = \int d^2 x \{ \lambda_{\mu\nu}(x) [\partial^\mu \phi(x) - \epsilon^{\mu\sigma} \partial_\sigma \phi(x)] [\partial^\nu \phi(x) - \epsilon^{v\rho} \partial_\rho \phi(x)] \}$$

$$\because \phi(x) = \mathcal{F}(\phi(k)), \lambda_{\mu\nu}(x) = \mathcal{F}(\lambda_{\mu\nu}(k))$$

换元, 令

$$g_\mu = -ik_\mu \phi(k), g^\mu = -ik^\mu \phi(k)$$

导数定理

$$\because \mathcal{F}(-ik_\mu \phi(k)) = \partial_\mu \phi(x), \mathcal{F}(-ik^\mu \phi(k)) = \partial^\mu \phi(x)$$

卷积定理

$$S_1 = \int d^2 x \mathcal{F}(g_\mu) \mathcal{F}(g^\mu) = \frac{1}{2\pi} \int d^2 x \mathcal{F}(g_\mu * g^\mu)$$

$$\therefore S_1 = (g_\mu * g^\mu)|_{k=0} = \int d^2 \xi g_\mu(\xi) g^\mu(0-\xi) = \int d^2 k (-ik_\mu) \phi(k) (-ik^\mu) \phi(k)$$

同理我们设

$$\eta^\mu = ik^\mu \phi(k) - \epsilon^{\mu\sigma} ik_\sigma \phi(k), \eta^\nu = ik^\nu \phi(k) - \epsilon^{v\rho} ik_\rho \phi(k)$$

有

$$S_2 = \int d^2 x \mathcal{F}(\lambda_{\mu\nu}(k)) \mathcal{F}(\eta^\mu) \mathcal{F}(\eta^\nu) = \frac{1}{4\pi^2} \int d^2 x \mathcal{F}(\lambda_{\mu\nu}(k) * \eta^\mu * \eta^\nu)$$

$$S_2 = \frac{1}{2\pi} \lambda_{\mu\nu}(k) * \eta^\mu * \eta^\nu|_{k=0} = \frac{1}{2\pi} \int d^2\xi \int d^2\zeta \lambda_{\mu\nu}(\zeta + \xi) \eta^\mu(0 - \xi) \eta^\nu(0 - \zeta)$$

即为

$$S_2 = \frac{1}{2\pi} \int d^2k \int d^2k' \lambda_{\mu\nu}(-k - k') [ik^\mu \phi(k) - \epsilon^{\mu\sigma} ik_\sigma \phi(k)] [ik'^\nu \phi(k') - \epsilon^{\nu\rho} ik'_\rho \phi(k')]$$

现在我们终于可以代回到原表达式

$$\begin{aligned} S_m = & -\frac{1}{2} \int d^2k (-ik_\mu) (ik^\mu) \phi(k) \phi(-k) \\ & + \frac{1}{4\pi} \int d^2k d^2k' \lambda_{\mu\nu}(-k - k') (ik^\mu - \epsilon^{\mu\sigma} ik_\sigma) (ik'^\nu - \epsilon^{\nu\rho} ik'_\rho) \phi(k) \phi(k') \end{aligned}$$

这就是结果

12 第十二题

12.1

$$g_1(t) = \begin{cases} 2t/T & 0 \leq t < T/2 \\ 2(1 - t/T) & T/2 \leq t \leq T \end{cases}$$

根据周期函数

$$\mathcal{L}(g(t)) = \frac{1}{1 - e^{-pT}} \int_0^T e^{-pt} g(t) dt$$

计算得

$$\begin{aligned} \int_0^{T/2} \exp(-pt) \frac{2t}{T} dt &= \frac{2}{T} \left[\frac{1}{p^2} - e^{-Tp/2} \left(\frac{T}{2p} + \frac{1}{p^2} \right) \right] \\ - \int_{T/2}^T \exp(-pt) \frac{2t}{T} dt &= \frac{2}{T} e^{-Tp/2} \left[(1 - e^{-Tp/2}) \left(\frac{T}{2p} + \frac{1}{p^2} \right) \right] \\ \int_{T/2}^T 2 \exp(-pt) dt &= \frac{2}{p} (e^{-Tp/2} - e^{-Tp}) \end{aligned}$$

代入化简得

$$\mathcal{L}(g_1(t)) = \frac{2}{Tp^2} \tanh\left(\frac{Tp}{4}\right)$$

12.2

$$\begin{aligned} g_2(t) &= \frac{2}{T} \left[\text{th}(t) + 2 \sum_{n=1}^{\infty} (-1)^n \left(t - \frac{1}{2}nT \right) H\left(t - \frac{1}{2}nT\right) \right] \\ \mathcal{L}(g_2(t)) &= \frac{2}{Tp^2} \left[1 + 2 \sum_{n=1}^{\infty} (-1)^n \exp\left(-\frac{1}{2}nT\right) \right] \end{aligned}$$

如果 $g_1(t) = g_2(t)$, 比照系数令 $x = Tp/4$ 换元有

$$\tanh x = 1 + 2 \sum_{n=0}^{\infty} (-1)^n e^{-2nx}$$

这正是我们想证明的。因此只需证明 $g_1(t) = g_2(t)$.

下面证明 $g_1(t) = g_2(t)$

我们首先考察区间 $[0, T/2]$, 对于 $t \in [0, T/2]$ 由于全部的 $H(t - 1/2nT) = 0$ 因此二者显然相等。

然后考察区间 $[T/2, T]$, 对于 $t \in [0, T/2]$ 求和只保留到 $n = 1$, 有

$$g_2 = \frac{2}{T} \left[t - 2 \left(t - \frac{1}{2}T \right) \right] = 2 - \frac{2t}{T} = g_1$$

综上二者在 $t \in [0, T]$ 上相等。

一般地, 对于 $n > 0$ 情形, 仅当 $t \geq 1/2nT$ 时, $H(t - 1/2nT) = 1$, 其余情况为 0。如果我们只考察 t 非负半轴情况, 求和可以写成:

$$g_2(t) = \frac{2}{T} \left[t + 2 \sum_{n=1}^{\left[\frac{2t}{T} \right]} (-1)^n \left(t - \frac{1}{2}nT \right) \right]$$

其中 $\left[\frac{2t}{T} \right]$ 表示括号内值向下取整。

接下来证明 $g_2(t)$ 也以 T 为周期, 既 $g_2(t) = g_2(t + T)$

$$\begin{aligned} g_2(t + T) &= \frac{2}{T} \left[t + T + 2 \sum_{n=1}^{\left[\frac{2t}{T} + 2 \right]} (-1)^n \left(t - \frac{1}{2}nT + T \right) \right] \\ &= \frac{2}{T} \left[t + T - (2t + T) + 2t + 2 \sum_{n=3}^{\left[\frac{2t}{T} + 2 \right]} (-1)^n \left(t - \frac{1}{2}nT + T \right) \right] \\ &= \frac{2}{T} \left[t + 2 \sum_{n=1}^{\left[\frac{2t}{T} \right]} (-1)^n \left(t - \frac{1}{2}nT \right) \right] \\ &= g_2(t) \end{aligned}$$

由于在 $[0, T]$ 区间相等并都具有以 T 为周期的周期性, 因此 $g_1(t) = g_2(t)$, 这正是我们需要的。于是

$$\tanh x = 1 + 2 \sum_{n=0}^{\infty} (-1)^n e^{-2nx}$$

得证

13 第十三题

14 第十四题