数学物理方法　期末考试

周子正　学号：1810334

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1　第一题

2　第二题

$$z=\frac{1-i\tan x}{1+i\tan x}=\frac{\cos x-i\sin x}{\cos x+i\sin x}=\frac{-i\left[\cos\left(\pi/2-x\right)+i\sin\left(\pi/2-x\right)\right]}{\exp\left(ix\right)}$$

$$=\exp\left(i\frac{3\pi}{2}\right)\exp\left(i\frac{\pi}{2}\right)\exp\left(-2ix\right)=\exp\left(-2ix\right)$$

$$=\cos 2x-i\sin 2x$$

3　第三题

$$1+i=\sqrt{2}\exp\left(i\frac{\pi}{4}\right)$$

$$1-i=\sqrt{2}\exp\left(-i\frac{\pi}{4}\right)$$

$$\left(1+i\right)^{1000}+\left(1-i\right)^{1000}=2^{500}\left[\exp\left(250\pi i\right)+\exp\left(-250\pi i\right)\right]=2^{501}$$

4　第四题

$$\because |\sqrt{3}+i|=|\sqrt{3}-i|$$

$$\therefore |\sqrt{3}+i|^n=|\sqrt{3}-i|^n$$

二者模长必然相同，若要求二者相等，只需要辐角满足

$$\arg\left(\sqrt{3}+i\right)^n=2k\pi+\arg\left(\sqrt{3}-i\right)^n,\quad k\in\mathbb{Z}$$

$$\therefore n\frac{\pi}{6}=-n\frac{\pi}{6}+2k\pi$$

$$\therefore n=6k,\quad k\in\mathbb{Z}$$

所以只需要 n 是整数且能被 6 整除即可。

5 第五题

平面上圆的一般方程可以写成

$$\alpha_1\left(x^2+y^2\right)+2\alpha_2 x+2\alpha_3 y+\alpha_4=0$$

其中 $\alpha_{1,2,3,4}\in\mathbb{R}$, 若 $\alpha_1=0$, 其退化为直线
若代入

$$x=\frac{z+\bar{z}}{2},y=\frac{z-\bar{z}}{2i}$$

设 $\beta=\alpha_2+i\alpha_3$ 有复数表示

$$\alpha_1 z\bar{z}+\bar{\beta}z+\beta\bar{z}+\alpha_4=0$$

对于此题，

$$w=\frac{z-i}{z+i}\Rightarrow z=-\frac{(w+1)\,i}{w-1},\bar{z}=\frac{(\bar{w}+1)\,i}{\bar{w}-1}$$

代入化简有

$$\gamma_1 w\bar{w}+\bar{\beta}_2 w+\beta_2\bar{w}+\gamma_4=0$$

其中

$$\begin{pmatrix}\gamma_1\\\gamma_2\\\gamma_3\\\gamma_4\end{pmatrix}=\begin{pmatrix}\alpha_1-2\alpha_3+\alpha_4\\\alpha_1-\alpha_4\\-2\alpha_2\\\alpha_1+2\alpha_3+\alpha_4\end{pmatrix}\qquad\beta_2=\gamma_2+i\gamma_3$$

因此广义上（直线是特殊的圆）看，w 将圆变成圆。特别的，对于此题的情况有：

$$\begin{pmatrix}\alpha_1&\alpha_2&\alpha_3&\alpha_4\end{pmatrix}=\begin{pmatrix}0&1/2&0&-a\end{pmatrix}\to\begin{pmatrix}\gamma_1&\gamma_2&\gamma_3&\gamma_4\end{pmatrix}=\begin{pmatrix}-a&a&-1&-a\end{pmatrix}$$

$$\begin{pmatrix}\alpha_1&\alpha_2&\alpha_3&\alpha_4\end{pmatrix}=\begin{pmatrix}0&0&1/2&-b\end{pmatrix}\to\begin{pmatrix}\gamma_1&\gamma_2&\gamma_3&\gamma_4\end{pmatrix}=\begin{pmatrix}-1-b&b&0&1-b\end{pmatrix}$$

显然当且仅当 $-a=0$,$-1-b=0$ 即 $a=0$,$b=1$ 时，退化为直线。

6 第六题

$$a=\sum_{k=0}^{\infty}\frac{\cos nz}{n!}$$

$$b=\sum_{k=0}^{\infty}\frac{\sin nz}{n!}$$

考虑做一对替换有：

$$c_+ \equiv a+bi=\sum_{k=0}^{\infty}\frac{\cos nz+i\sin nz}{n!}=\sum_{k=0}^{\infty}\frac{\exp\left(inz\right)}{n!}=\exp\left(\exp\left(iz\right)\right)$$

$$c_- \equiv a-bi=\sum_{k=0}^{\infty}\frac{\cos nz-i\sin nz}{n!}=\sum_{k=0}^{\infty}\frac{\exp\left(-inz\right)}{n!}=\exp\left(\exp\left(-iz\right)\right)$$

其中

$$\exp\left(\exp\left(iz\right)\right)=\exp\left(\cos z+i\sin z\right)=e^{\cos z}\left(\cos\sin z+\sin\sin z\right)$$

$$\exp\left(\exp\left(-iz\right)\right)=\exp\left(\cos z-i\sin z\right)=e^{\cos z}\left(\cos\sin z-\sin\sin z\right)$$

于是由替换的关系可以得到：

$$a=\frac{c_++c_-}{2}=e^{\cos z}\cos\sin z$$

$$b=\frac{c_+-c_-}{2i}=-ie^{\cos z}\sin\sin z$$

7 第七题

求

$$f\left(z\right)=\frac{1}{e^z-1}$$

在 $z=0$ 的洛朗级数，很明显此处不解析，考察 $n\in N^*$ 时，

$$\lim_{z\to 0}\frac{z^n}{e^z-1}=\begin{cases}1 & n=1\\ 0 & n>1\end{cases}$$

因此这是一阶极点，考虑求解析函数 $a(z)=zf(z)$ 在 $z=0$ 的泰勒展开，设其为

$$a\left(z\right)=\sum_{n=0}^{+\infty}a_nz^n$$

这不容易计算，考虑其倒数 $b(z)=1/a$，可以展开为

$$b\left(z\right)=\sum_{n=0}^{+\infty}\frac{z^n}{(n+1)!}$$

又因为恒等关系 $a(z)b(z)=1$

$$\left(\sum_{n=0}^{+\infty}a_nz^n\right)\cdot\left(\sum_{n=0}^{+\infty}\frac{z^n}{(n+1)!}\right)=1$$

比较 z 的各阶系数可以得到 $a_0=1,a_1=-\frac{1}{2}...$

在 $n \geq 1$ 时，其满足线性齐次方程组方程为：

$$\sum_{k=0}^{n} \frac{a_k}{(n-k+1)!} = 0$$

理论上可以解出任意的 a_n

我试图找到通项公式但是失败了，查阅资料发现这个数列与伯努利数相关，关系为 $a_n = B_n/n!$ 其被定义为

$$\frac{z}{e^z-1} = \sum_{n=0}^{\infty} B_n \frac{z^n}{n!}$$

递推关系为

$$B_m = [m=0] - \sum_{k=0}^{m-1} \left(\begin{array}{c} m \\ k \end{array} \right) \frac{B_k}{m-k+1}$$

$B_0 = 1$, 其中 $[m=0]$ 表示当 $m=0$ 时，取 1，其余取 0。

综上可得，

$$f(z) = \sum_{n=-1}^{+\infty} \frac{z^n}{(n+1)!} B_{n+1}$$

其前几项为

$$\frac{1}{e^x-1} = \frac{1}{z} - \frac{1}{2} + \frac{z}{12} - \frac{z^3}{720} + \cdots$$

8 第八题

由于所有极点都在围道内部，直接考察无穷远的留数，根据引理有：

$$\int_{|z|=200} f(z)\,dz = -2\pi i \operatorname*{Res}_{z\to\infty} f(z) = 2\pi i \operatorname*{Res}_{t\to 0} \left[f\left(\frac{1}{t}\right) \frac{1}{t^2} \right]$$

其中

$$f\left(\frac{1}{t}\right)\frac{1}{t^2} = \frac{1}{t^2} \prod_{k=1}^{100} \frac{1}{1-kt}$$

$$\because \lim_{t\to 0} t^2 \frac{1}{t^2} \prod_{k=1}^{100} \frac{1}{1-kt} = 1$$

$$\therefore \text{二阶极点}$$

所以可以计算出留数的值

$$\begin{aligned}\operatorname*{Res}_{t\to 0}\left[f\left(\frac{1}{t}\right)\frac{1}{t^2}\right] &= \lim_{t\to 0}\frac{d}{dt}\prod_{k=1}^{100}\frac{1}{1-kt} \\ &= \lim_{t\to 0}\sum_{n=1}^{100}\prod_{n=1\wedge k\neq n}^{100}\frac{n}{1-kt} \\ &= \sum_{n=1}^{100} n = 5050\end{aligned}$$

$$\therefore \int_{|z|=200} f(z)\,dz = 10100\pi i$$

9 第九题

考虑配对，设

$$\begin{aligned}I_1 &= \int_0^{2\pi} e^{\cos\theta}\cos(n\theta-\sin\theta)\,d\theta \\ I_2 &= \int_0^{2\pi} e^{\cos\theta}\sin(n\theta-\sin\theta)\,d\theta \\ I_1+iI_2 &= \int_0^{2\pi}\exp(\cos\theta - i\sin\theta)\exp(in\theta)d\theta \\ I_1-iI_2 &= \int_0^{2\pi}\exp(\cos\theta + i\sin\theta)\exp(-in\theta)d\theta\end{aligned}$$

对于 I_1+iI_2 设 $z=e^{i\theta}$, 有

$$I_1+iI_2 = \int_{|z|=1} e^{\frac{1}{z}} z^n \frac{dz}{zi} = 2\pi\operatorname*{Res}_{z\to 0} e^{\frac{1}{z}} z^{n-1}$$

包裹的奇点只在 $z=0$，我们计算其在 $z=0$ 洛朗展开为

$$e^{\frac{1}{z}} z^{n-1} = \sum_{k=-\infty}^{0}\frac{e^{n+k-1}}{(-k)!}$$

为求其留数，取 z^{-1} 项要求 $k=-n$ 所以有

$$\operatorname*{Res}_{z\to 0} e^{\frac{1}{z}} z^{n-1} = \frac{1}{n!}$$

对于 I_1-iI_2 设 $z=e^{-i\theta}$, 注意积分方向相反需要取出一个负号，化简有

$$I_1-iI_2 = \int_{|z|=1} e^{\frac{1}{z}} z^n \frac{dz}{zi} = I_1+iI_2 = \frac{2\pi}{n!}$$

于是要求的量即为

$$I_1 = \frac{2\pi}{n!}$$

10 第十题

直接考察积分，试图利用柯西公式

$$
\begin{aligned}
D_F(x-y) &= \int_{C_F} \frac{d^4p}{(2\pi)^4}\frac{ie^{-ip\cdot(x-y)}}{p^2-m^2} \\
&= \int_{C_F} \frac{d^3\vec{p}dp_0}{(2\pi)^4}\frac{ie^{i\vec{p}\cdot(\vec{x}-\vec{y})}e^{-ip_0(x_0-y_0)}}{p_0^2-(\vec{p}^2+m^2)} \\
&= \int \frac{d^3\vec{p}\left[e^{i\vec{p}\cdot(\vec{x}-\vec{y})}\right]}{(2\pi)^4}\int_{C_F}\frac{ie^{-ip_0(x_0-y_0)}dp_0}{p_0^2-E_{\vec{p}}^2}
\end{aligned}
$$

考察其中的

$$
\int_{C_F}\frac{ie^{-ip_0(x_0-y_0)}dp_0}{p_0^2-E_{\vec{p}}^2} = \int_{C_F}\frac{ie^{-ip_0(x_0-y_0)}dp_0}{(p_0-E_{\vec{p}})(p_0+E_{\vec{p}})}
$$

注意到在 C_F 的围道内只有一个一阶极点为

$$
\operatorname*{Res}_{p_0\to E_{\vec{p}}}\frac{ie^{-ip_0(x_0-y_0)}}{(p_0-E_{\vec{p}})(p_0+E_{\vec{p}})} = \frac{ie^{-ip_0(x_0-y_0)}}{2E_{\vec{p}}}
$$

考察其在无穷远的性质有

$$
\lim_{|p_0|\to\infty}\left|\frac{ip_0e^{-ip_0(x_0-y_0)}}{p_0^2-E_{\vec{p}}^2}\right| = 0
$$

实际上是一致收敛的

注意到 C_F 的围道方向为顺时针，因此有

$$
\int_{C_F}\frac{ie^{-ip_0(x_0-y_0)}dp_0}{p_0^2-E_{\vec{p}}^2} = 2\pi\frac{e^{-ip_0(x_0-y_0)}}{2E_{\vec{p}}}
$$

代入 Feynman 传播子表示可得到：

$$
D_F(x-y) = \int_{C_F}\frac{d^4p}{(2\pi)^4}\frac{ie^{-ip\cdot(x-y)}}{p^2-m^2} = \int\frac{d^3\vec{p}}{(2\pi)^3\,2E_{\vec{p}}}e^{-ip\cdot(x-y)}\bigg|_{p_0=E_{\vec{p}}}
$$

11 第十一题

以下都建立在 $D=2$ 上

引理 11.1. 导数定理

$$
f(x) = \mathcal{F}(g(k)) \Rightarrow \partial_\mu f(x) = \mathcal{F}(-ik_\mu g(k))
$$

证明.

$$\mathcal{F}^{-1}\left(\partial_\mu f\left(x\right)\right)=\frac{1}{2\pi}\int d^2x\exp\left(ik\cdot x\right)\partial_\mu f\left(x\right)$$

$$=-\frac{1}{2\pi}\int d^2x\left(ik_\mu\right)\exp\left(ik\cdot x\right)f\left(x\right)=-ik_\mu g\left(k\right)$$

$$\therefore\partial_\mu f\left(x\right)=\mathcal{F}\left(-ik_\mu g\left(k\right)\right)$$

同理，

$$\partial^\mu f\left(x\right)=\mathcal{F}\left(-ik^\mu g\left(k\right)\right)$$

□

引理 11.2. 卷积定理
如果定义

$$f*g\left(x\right)\equiv\int d^2\xi f\left(\xi\right)g\left(x-\xi\right)$$

则有

$$\mathcal{F}\left(f*g\left(x\right)\right)=2\pi\mathcal{F}\left(f\left(x\right)\right)\mathcal{F}\left(g\left(x\right)\right)$$

证明.

$$\begin{aligned}\mathcal{F}\left(f*g\left(x\right)\right)&=\frac{1}{2\pi}\int d^2xe^{-ik\cdot x}\int d^2\xi f\left(\xi\right)g\left(x-\xi\right)\\&=\int d^2\xi f\left(\xi\right)\left[\frac{1}{2\pi}\int d^2xe^{-ik\cdot x}g\left(x-\xi\right)\right]\\&=\mathcal{F}\left(g\left(x\right)\right)\int d^2\xi f\left(\xi\right)e^{-ik\cdot\xi}\\&=2\pi\mathcal{F}\left(f\left(x\right)\right)\mathcal{F}\left(g\left(x\right)\right)\end{aligned}$$

推论：对于三个函数的情形，只需要依次计算，即为

$$\mathcal{F}\left(f*g*h\left(x\right)\right)=4\pi^2\mathcal{F}\left(f\left(x\right)\right)\mathcal{F}\left(g\left(x\right)\right)\mathcal{F}\left(h\left(x\right)\right)$$

□

引理 11.3. 设对于 $f(x)$ 有 $f(x)=\mathcal{F}(g(k))$ 则，

$$\int d^2x\mathcal{F}\left(g\left(k\right)\right)=\left(2\pi\right)^2g\left(0\right)$$

证明.

$$\begin{aligned}\int d^2x\mathcal{F}\left(g\left(k\right)\right)&=\int d^2x\int d^2k\exp\left(-ik\cdot x\right)g\left(k\right)\\&=\int d^2kg\left(k\right)\int d^2x\exp\left(-ik\cdot x\right)\\&=\left(2\pi\right)^2\int d^2kg\left(k\right)\delta\left(-k\right)=\left(2\pi\right)^2g\left(0\right)\end{aligned}$$

□

我们考虑原始表达式

$$S=\int d^2x\left\{-\frac{1}{2}\partial_\mu\phi\left(x\right)\partial^\mu\phi\left(x\right)+\frac{1}{2}\lambda_{\mu v}\left(x\right)\left[\partial^\mu\phi\left(x\right)-\epsilon^{\mu\sigma}\partial_\sigma\phi\left(x\right)\right]\left[\partial^v\phi\left(x\right)-\epsilon^{v\rho}\partial_\rho\phi\left(x\right)\right]\right\}$$

先分别计算其中的积分

$$S_1=\int d^2x\left[\partial_\mu\phi\left(x\right)\partial^\mu\phi\left(x\right)\right]$$

$$S_2=\int d^2x\left\{\lambda_{\mu v}\left(x\right)\left[\partial^\mu\phi\left(x\right)-\epsilon^{\mu\sigma}\partial_\sigma\phi\left(x\right)\right]\left[\partial^v\phi\left(x\right)-\epsilon^{v\rho}\partial_\rho\phi\left(x\right)\right]\right\}$$

$$\because\phi\left(x\right)=\mathcal{F}\left(\phi\left(k\right)\right),\lambda_{\mu v}\left(x\right)=\mathcal{F}\left(\lambda_{\mu v}\left(k\right)\right)$$

换元，令

$$g_\mu=-ik_\mu\phi\left(k\right),g^\mu=-ik^\mu\phi\left(k\right)$$

利用导数定理 11.1

$$\because\mathcal{F}\left(-ik_\mu\phi\left(k\right)\right)=\partial_\mu\phi\left(x\right),\mathcal{F}\left(-ik^\mu\phi\left(k\right)\right)=\partial^\mu\phi\left(x\right)$$

利用卷积定理 11.2

$$S_1=\int d^2x\mathcal{F}\left(g_\mu\right)\mathcal{F}\left(g^\mu\right)=\frac{1}{2\pi}\int d^2x\mathcal{F}\left(g_\mu*g^\mu\right)$$

最后利用 11.3

$$\therefore S_1=\left(g_\mu*g^\mu\right)\big|_{k=0}=\int d^2\xi g_\mu\left(\xi\right)g^\mu\left(0-\xi\right)=\int d^2k\left(-ik_\mu\right)\phi\left(k\right)\left(-ik^\mu\right)\phi\left(k\right)$$

同理我们设

$$\eta^\mu=ik^\mu\phi\left(k\right)-\epsilon^{\mu\sigma}ik_\sigma\phi\left(k\right),\eta^\nu=ik^\nu\phi\left(k\right)-\epsilon^{v\rho}ik_\rho\phi\left(k\right)$$

有

$$S_2=\int d^2x\mathcal{F}\left(\lambda_{\mu v}\left(k\right)\right)\mathcal{F}\left(\eta^\mu\right)\mathcal{F}\left(\eta^\nu\right)=\frac{1}{4\pi^2}\int d^2x\mathcal{F}\left(\lambda_{\mu v}\left(k\right)*\eta^\mu*\eta^\nu\right)$$

$$S_2=\frac{1}{2\pi}\lambda_{\mu v}\left(k\right)*\eta^\mu*\eta^\nu\big|_{k=0}=\frac{1}{2\pi}\int d^2\xi\int d^2\zeta\lambda_{\mu v}\left(\zeta+\xi\right)\eta^\mu\left(0-\xi\right)\eta^\nu\left(0-\zeta\right)$$

即为

$$S_2=\frac{1}{2\pi}\int d^2k\int d^2k'\lambda_{\mu v}\left(-k-k'\right)\left[ik^\mu\phi\left(k\right)-\epsilon^{\mu\sigma}ik_\sigma\phi\left(k\right)\right]\left[ik^\nu\phi\left(k'\right)-\epsilon^{v\rho}ik_\rho\phi\left(k'\right)\right]$$

现在我们终于可以代回到原表达式

$$\begin{aligned}S_m=&-\frac{1}{2}\int d^2k\left(-ik_\mu\right)\left(ik^\mu\right)\phi(k)\phi(-k)\\&+\frac{1}{4\pi}\int d^2kd^2k'\lambda_{\mu\nu}\left(-k-k'\right)\left(ik^\mu-\epsilon^{\mu\sigma}ik_\sigma\right)\left(ik'^\nu-\epsilon^{\nu\rho}ik'_\rho\right)\phi(k)\phi\left(k'\right)\end{aligned}$$

这就是结果

12 第十二题

12.1

$$g_1(t) = \begin{cases} 2t/T & 0 \le t < T/2 \\ 2(1-t/T) & T/2 \le t \le T \end{cases}$$

根据周期函数

$$\mathcal{L}\left(g\left(t\right)\right) = \frac{1}{1-e^{-pT}} \int_0^T e^{-pt} g(t) dt$$

计算得

$$\int_0^{T/2} \exp\left(-pt\right) \frac{2t}{T} dt = \frac{2}{T}\left[\frac{1}{p^2} - e^{-Tp/2}\left(\frac{T}{2p} + \frac{1}{p^2}\right)\right]$$

$$-\int_{T/2}^{T} \exp\left(-pt\right) \frac{2t}{T} dt = \frac{2}{T} e^{-Tp/2}\left[\left(1-e^{-Tp/2}\right)\left(\frac{T}{2p} + \frac{1}{p^2}\right)\right]$$

$$\int_{T/2}^{T} 2\exp\left(-pt\right) dt = \frac{2}{p}\left(e^{-Tp/2} - e^{-Tp}\right)$$

代入化简得

$$\mathcal{L}\left(g_1\left(t\right)\right) = \frac{2}{Tp^2} \tan\mathrm{h}\left(\frac{Tp}{4}\right)$$

12.2

$$g_2\left(t\right) = \frac{2}{T}\left[\mathrm{tH}\left(\mathrm{t}\right) + 2\sum_{n=1}^{\infty}\left(-1\right)^n \left(t - \frac{1}{2}nT\right) H\left(t - \frac{1}{2}nT\right)\right]$$

$$\mathcal{L}\left(g_2\left(t\right)\right) = \frac{2}{Tp^2}\left[1 + 2\sum_{n=1}^{\infty}\left(-1\right)^n \exp\left(-\frac{1}{2}nT\right)\right]$$

如果 $g_1(t) = g_2(t)$，比照系数令 $x = Tp/4$ 换元有

$$\tanh x = 1 + 2\sum_{n=0}^{\infty}(-1)^n e^{-2nx}$$

这正是我们想证明的。因此只需证明 $g_1(t) = g_2(t)$.

下面证明 $g_1(t) = g_2(t)$

我们首先考察区间 $[0, T/2)$, 对于 $t \in [0, T/2]$ 由于全部的 $H(t - 1/2nT) = 0$ 因此二者显然相等。

然后考察区间 $[T/2, T)$, 对于 $t \in [0, T/2]$ 求和只保留到 $n = 1$，有

$$g_2 = \frac{2}{T}\left[t - 2\left(t - \frac{1}{2}T\right)\right] = 2 - \frac{2t}{T} = g_1$$

综上二者在 $t \in [0, T)$ 上相等。

一般地，对于 $n>0$ 情形，仅当 $t \geq 1/2nT$ 时,$H(t-1/2nT)=1$，其余情况为 0。如果我们只考察 t 非负半轴情况，求和可以写成：

$$g_2(t)=\frac{2}{T}\left[t+2\sum_{n=1}^{\left[\frac{2t}{T}\right]}(-1)^n\left(t-\frac{1}{2}nT\right)\right]$$

其中 $\left[\frac{2t}{T}\right]$ 表示括号内值向下取整。
接下来证明 $g_2(t)$ 也以 T 为周期，既 $g_2(t)=g_2(t+T)$

$$\begin{aligned}
g_2(t+T) &= \frac{2}{T}\left[\mathrm{t}+T+2\sum_{n=1}^{\left[\frac{2t}{T}+2\right]}(-1)^n\left(t-\frac{1}{2}nT+T\right)\right] \\
&= \frac{2}{T}\left[\mathrm{t}+T-(2t+T)+2t+2\sum_{n=3}^{\left[\frac{2t}{T}+2\right]}(-1)^n\left(t-\frac{1}{2}nT+T\right)\right] \\
&= \frac{2}{T}\left[t+2\sum_{n=1}^{\left[\frac{2t}{T}\right]}(-1)^n\left(t-\frac{1}{2}nT\right)\right] \\
&= g_2(t)
\end{aligned}$$

由于在 $[0,T)$ 区间相等并都具有以 T 为周期的周期性，因此 $g_1(t)=g_2(t)$，这正是我们需要的。于是

$$\tanh x=1+2\sum_{n=0}^{\infty}(-1)^n e^{-2nx}$$

得证

13 第十三题

以下的拉普拉斯方程的表达式都基于

$$\frac{1}{\sqrt{|g|}}\partial_\mu\left(g^{\mu v}\sqrt{|g|}\partial_v\psi\right)=0$$

在正交曲线坐标系下为

$$\sum_{\mu=1}^{3}\frac{1}{|h_1h_2h_3|}\partial_\mu\left(\frac{1}{h_\mu}\sqrt{|h_1h_2h_3|}\partial_\mu\psi\right)=0$$

13.1 球坐标

$$h_r=1,h_\theta=r,h_\varphi=r\sin\theta$$

拉普拉斯方程为：

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial\varphi^2}=0$$

首先，设

$$u(r,\theta,\varphi)=R(r)\mathrm{Y}(\theta,\varphi)$$

$$\frac{1}{R}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}R}{\mathrm{d}r}\right)=-\frac{1}{\mathrm{Y}\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\mathrm{Y}}{\partial\theta}\right)-\frac{1}{\mathrm{Y}\sin^2\theta}\frac{\partial^2\mathrm{Y}}{\partial\varphi^2}=l(l+1)$$

分解为两个方程

$$\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)-l(l+1)R\equiv 0$$

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2 Y}{\partial\varphi^2}+l(l+1)Y=0$$

进一步设

$$\mathrm{Y}(\theta,\varphi)=\Theta(\theta)\Phi(\varphi)$$

$$\frac{\sin\theta}{\Theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta\frac{\mathrm{d}\Theta}{\mathrm{d}\theta}\right)+l(l+1)\sin^2\theta=-\frac{1}{\Phi}\frac{\mathrm{d}^2\Phi}{\mathrm{d}\varphi^2}=\lambda$$

分解为两个常微分方程：

$$\Phi''+\lambda\Phi=0$$

$$\sin\theta\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta\frac{\mathrm{d}\Theta}{\mathrm{d}\theta}\right)+\left[l(l+1)\sin^2\theta-\lambda\right]\theta=0$$

13.2 柱坐标

$$h_\rho=1,h_\varphi=\rho,h_z=1$$

柱坐标的拉普拉斯方程表示为

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial u}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2 u}{\partial\varphi^2}+\frac{\partial^2 u}{\partial z^2}=0$$

设

$$u(\rho,\varphi,z)=R(\rho)\Phi(\varphi)Z(z)$$

代入得到

$$\Phi''+\lambda\Phi=0$$

以及

$$\frac{\rho^2}{R}\frac{\mathrm{d}^2R}{\mathrm{d}\rho^2}+\frac{\rho}{R}\frac{\mathrm{d}R}{\mathrm{d}\rho}+\rho^2\frac{Z''}{Z}=\lambda$$

分离有：

$$\frac{1}{R}\frac{\mathrm{d}^2R}{\mathrm{d}\rho^2}+\frac{1}{\rho R}\frac{\mathrm{d}R}{\mathrm{d}\rho}-\frac{m^2}{\rho^2}=-\frac{Z''}{Z}=-\mu$$

两个常微分方程

$$Z'' - \mu Z = 0$$

$$\frac{\mathrm{d}^2 R}{\mathrm{d}\rho^2} + \frac{1}{\rho}\frac{\mathrm{d}R}{\mathrm{d}\rho} + \left(\mu - \frac{m^2}{\rho^2}\right) R = 0$$

13.3 椭圆柱坐标

标准表示为

$$x = a\xi\eta, \quad y = a\sqrt{\left(\xi^2 - 1\right)\left(1 - \eta^2\right)}, \quad z = z$$

换元，设

$$\xi = \operatorname{ch} u, \quad \eta = \cos v$$

有

$$h_u^2 = h_v^2 = a^2 \left(\operatorname{ch}^2 u - \cos^2 v\right), \quad h_z = 1$$

拉普拉斯方程表示为

$$\nabla^2 \Phi = \frac{1}{a^2 \left(\operatorname{ch}^2 u - \cos^2 v\right)} \left\{\frac{\partial^2 \Phi}{\partial u^2} + \frac{\partial^2 \Phi}{\partial v^2}\right\} + \frac{\partial^2 \Phi}{\partial z^2} = 0$$

设

$$\Phi\left(u, v, z\right) = U\left(u\right) V\left(v\right) Z\left(z\right)$$

代入有

$$\frac{1}{a^2 \left(\operatorname{ch}^2 u - \cos^2 v\right)} \left\{\frac{1}{U}\frac{d^2 U}{du^2} + \frac{1}{V}\frac{d^2 V}{dv^2}\right\} + \frac{1}{Z}\frac{d^2 Z}{dz^2} = 0$$

分离出

$$\frac{d^2 Z}{dz^2} + \lambda Z = 0$$

另有

$$\frac{1}{U}\frac{d^2 U}{du^2} + \frac{1}{V}\frac{d^2 V}{dv^2} - \lambda a^2 \left(\operatorname{ch}^2 u - \cos^2 v\right) = 0$$

分离出

$$\frac{d^2 U}{du^2} - \left(\lambda a^2 ch^2 u + \mu\right) U = 0$$

$$\frac{d^2 V}{dv^2} + \left(\lambda a^2 \cos^2 v + \mu\right) V = 0$$

13.4 椭球坐标

$$x^2=\frac{\left(a^2+\lambda\right)\left(a^2+\mu\right)\left(a^2+v\right)}{\left(a^2-b^2\right)\left(a^2-c^2\right)}$$
$$y^2=\frac{\left(b^2+\lambda\right)\left(b^2+\mu\right)\left(b^2+v\right)}{\left(b^2-c^2\right)\left(b^2-a^2\right)}$$
$$z^2=\frac{\left(c^2+\lambda\right)\left(c^2+\mu\right)\left(c^2+v\right)}{\left(c^2-a^2\right)\left(c^2-b^2\right)}$$

λ,μ,ν 的变化范围为

$$\lambda>-c^2>\mu>-b^2>\nu>-a^2$$

若设

$$\varphi(\theta)=\left(a^2+\theta\right)\left(b^2+\theta\right)\left(c^2+\theta\right)$$

有

$$h_\lambda^2=\frac{(\lambda-\mu)(\lambda-\nu)}{4\varphi(\lambda)}$$
$$h_\mu^2=\frac{(\mu-\lambda)(\mu-\nu)}{4\varphi(\mu)}$$
$$h_\nu^2=\frac{(\nu-\lambda)(\nu-\mu)}{4\varphi(\nu)}$$

$$\begin{aligned}\nabla^2\Phi=\frac{2}{(\lambda-\mu)(\lambda-\nu)(\mu-\nu)}&\left[(\mu-\nu)\sqrt{\varphi(\lambda)}\frac{\partial}{\partial\lambda}\left(\sqrt{\varphi(\lambda)}\frac{\partial\Phi}{\partial\lambda}\right)\right.\\&+(\lambda-\nu)\sqrt{-\varphi(\mu)}\frac{\partial}{\partial\mu}\left(\sqrt{-\varphi(\mu)}\frac{\partial\Phi}{\partial\mu}\right)\\&\left.+(\lambda-\mu)\sqrt{\varphi(\nu)}\frac{\partial}{\partial\nu}\left(\sqrt{\varphi(\nu)}\frac{\partial\Phi}{\partial\nu}\right)\right]=0\end{aligned}$$

设 $\Phi=\Lambda(\lambda)M(\mu)N(\nu)$ 代入有

$$\begin{aligned}\frac{\mu-\nu}{\Lambda}\sqrt{\varphi(\lambda)}\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\sqrt{\varphi(\lambda)}\frac{\mathrm{d}\Lambda}{\mathrm{d}\lambda}\right)+\frac{\lambda-\nu}{M}\sqrt{-\varphi(\mu)}\frac{\mathrm{d}}{\mathrm{d}\mu}\left(\sqrt{-\varphi(\mu)}\frac{\mathrm{d}M}{\mathrm{d}\mu}\right)&\\+\frac{\lambda-\mu}{N}\sqrt{\varphi(\nu)}\frac{\mathrm{d}}{\mathrm{d}\nu}\left(\sqrt{\varphi(\nu)}\frac{\mathrm{d}N}{\mathrm{d}\nu}\right)=0&\end{aligned}$$

注意到有恒等式

$$(\mu-\nu)(K\lambda+C)+(\nu-\lambda)(K\mu+C)+(\lambda-\mu)(K\nu+C)\equiv 0$$

其中 K 和 C 为常数，比较系数得

$$4\sqrt{\varphi(\lambda)}\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\sqrt{\varphi(\lambda)}\frac{\mathrm{d}\Lambda}{\mathrm{d}\lambda}\right)=(K\lambda+C)\Lambda$$

设 $K=n(n+1)$, 分离变量的三个方程为:

$$4\sqrt{\varphi\left(\lambda\right)}\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\sqrt{\varphi\left(\lambda\right)}\frac{\mathrm{d}\Lambda}{\mathrm{d}\lambda}\right)=\left[\left(n\left(n+1\right)\right)\lambda+C\right]\Lambda$$

$$4\sqrt{\varphi\left(\mu\right)}\frac{\mathrm{d}}{\mathrm{d}\mu}\left(\sqrt{\varphi\left(\mu\right)}\frac{\mathrm{d}M}{\mathrm{d}\mu}\right)=\left[\left(n\left(n+1\right)\right)\mu+C\right]M$$

$$4\sqrt{\varphi\left(\nu\right)}\frac{\mathrm{d}}{\mathrm{d}\nu}\left(\sqrt{\varphi\left(\nu\right)}\frac{\mathrm{d}N}{\mathrm{d}\nu}\right)=\left[\left(n\left(n+1\right)\right)\nu+C\right]N$$

13.5 锥面坐标

13.6 抛物线柱坐标

14 第十四题