

数学物理方法 期末考试

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1 第一题

2 第二题

$$\begin{aligned} z &= \frac{1 - i \tan x}{1 + i \tan x} = \frac{\cos x - i \sin x}{\cos x + i \sin x} = \frac{-i [\cos(\pi/2 - x) + i \sin(\pi/2 - x)]}{\exp(ix)} \\ &= \exp\left(i\frac{3\pi}{2}\right) \exp\left(i\frac{\pi}{2}\right) \exp(-2ix) = \exp(-2ix) \\ &= \cos 2x - i \sin 2x \end{aligned}$$

3 第三题

$$\begin{aligned} 1 + i &= \sqrt{2} \exp\left(i\frac{\pi}{4}\right) \\ 1 - i &= \sqrt{2} \exp\left(-i\frac{\pi}{4}\right) \\ (1 + i)^{1000} + (1 - i)^{1000} &= 2^{500} [\exp(250\pi i) + \exp(-250\pi i)] = 2^{501} \end{aligned}$$

4 第四题

$$\begin{aligned} \because |\sqrt{3} + i| &= |\sqrt{3} - i| \\ \therefore |\sqrt{3} + i|^n &= |\sqrt{3} - i|^n \end{aligned}$$

二者模长必然相同, 若要求二者相等, 只需要辐角满足

$$\arg(\sqrt{3} + i)^n = 2k\pi + \arg(\sqrt{3} - i)^n, \quad k \in \mathbb{Z}$$

$$\begin{aligned} \therefore n\frac{\pi}{6} &= -n\frac{\pi}{6} + 2k\pi \\ \therefore n &= 6k, \quad k \in \mathbb{Z} \end{aligned}$$

所以只需要 n 是整数且能被 6 整除即可。

5 第五题

平面上圆的一般方程可以写成

$$\alpha_1(x^2 + y^2) + 2\alpha_2x + 2\alpha_3y + \alpha_4 = 0$$

其中 $\alpha_{1,2,3,4} \in \mathbb{R}$, 若 $\alpha_1 = 0$, 其退化为直线

若代入

$$x = \frac{z + \bar{z}}{2}, y = \frac{z - \bar{z}}{2i}$$

设 $\beta = \alpha_2 + i\alpha_3$ 有复数表示

$$\alpha_1 z \bar{z} + \bar{\beta} z + \beta \bar{z} + \alpha_4 = 0$$

对于此题,

$$w = \frac{z - i}{z + i} \Rightarrow z = -\frac{(w + 1)i}{w - 1}, \bar{z} = \frac{(\bar{w} + 1)i}{\bar{w} - 1}$$

代入化简有

$$\gamma_1 w \bar{w} + \bar{\beta}_2 w + \beta_2 \bar{w} + \gamma_4 = 0$$

其中

$$\begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \\ \gamma_4 \end{pmatrix} = \begin{pmatrix} \alpha_1 - 2\alpha_3 + \alpha_4 \\ \alpha_1 - \alpha_4 \\ -2\alpha_2 \\ \alpha_1 + 2\alpha_3 + \alpha_4 \end{pmatrix} \quad \beta_2 = \gamma_2 + i\gamma_3$$

因此广义上(直线是特殊的圆)看, w 将圆变成圆。特别的, 对于此题的情况有:

$$\begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \end{pmatrix} = \begin{pmatrix} 0 & 1/2 & 0 & -a \end{pmatrix} \rightarrow \begin{pmatrix} \gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 \end{pmatrix} = \begin{pmatrix} -a & a & -1 & -a \end{pmatrix}$$

$$\begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1/2 & -b \end{pmatrix} \rightarrow \begin{pmatrix} \gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 \end{pmatrix} = \begin{pmatrix} -1 - b & b & 0 & 1 - b \end{pmatrix}$$

显然当且仅当 $-a = 0, -1 - b = 0$ 即 $a = 0, b = 1$ 时, 退化为直线。

6 第六题

$$a = \sum_{k=0}^{\infty} \frac{\cos kz}{k!}$$

$$b = \sum_{k=0}^{\infty} \frac{\sin kz}{k!}$$

考虑做一对替换有:

$$c_+ \equiv a + bi = \sum_{k=0}^{\infty} \frac{\cos kz + i \sin kz}{k!} = \sum_{k=0}^{\infty} \frac{\exp(ikz)}{k!} = \exp(\exp(iz))$$

$$c_- \equiv a - bi = \sum_{k=0}^{\infty} \frac{\cos kz - i \sin kz}{k!} = \sum_{k=0}^{\infty} \frac{\exp(-ikz)}{k!} = \exp(\exp(-iz))$$

其中

$$\exp(\exp(iz)) = \exp(\cos z + i \sin z) = e^{\cos z} (\cos \sin z + i \sin \sin z)$$

$$\exp(\exp(-iz)) = \exp(\cos z - i \sin z) = e^{\cos z} (\cos \sin z - i \sin \sin z)$$

于是由替换的关系可以得到:

$$a = \frac{c_+ + c_-}{2} = e^{\cos z} \cos \sin z$$

$$b = \frac{c_+ - c_-}{2i} = -ie^{\cos z} \sin \sin z$$

7 第七题

求

$$f(z) = \frac{1}{e^z - 1}$$

在 $z = 0$ 的洛朗级数, 很明显此处不解析, 考察 $n \in N^*$ 时,

$$\lim_{z \rightarrow 0} \frac{z^n}{e^z - 1} = \begin{cases} 1 & n = 1 \\ 0 & n > 1 \end{cases}$$

因此这是一阶极点, 考虑求解析函数 $a(z) = zf(z)$ 在 $z = 0$ 的泰勒展开, 设其为

$$a(z) = \sum_{n=0}^{+\infty} a_n z^n$$

这不容易计算, 考虑其倒数 $b(z) = 1/a$, 可以展开为

$$b(z) = \sum_{n=0}^{+\infty} \frac{z^n}{(n+1)!}$$

又因为恒等关系 $a(z)b(z) = 1$

$$\left(\sum_{n=0}^{+\infty} a_n z^n \right) \cdot \left(\sum_{n=0}^{+\infty} \frac{z^n}{(n+1)!} \right) = 1$$

比较 z 的各阶系数可以得到 $a_0 = 1, a_1 = -\frac{1}{2} \dots$

在 $n \geq 1$ 时, 其满足线性齐次方程组方程为:

$$\sum_{k=0}^n \frac{a_k}{(n-k+1)!} = 0$$

理论上可以解出任意的 a_n

我试图找到通项公式但是失败了, 查阅资料发现这个数列与伯努利数相关, 关系为 $a_n = B_n/n!$ 其被定义为

$$\frac{z}{e^z - 1} = \sum_{n=0}^{\infty} B_n \frac{z^n}{n!}$$

递推关系为

$$B_m = [m=0] - \sum_{k=0}^{m-1} \binom{m}{k} \frac{B_k}{m-k+1}$$

$B_0 = 1$, 其中 $[m=0]$ 表示当 $m=0$ 时, 取 1, 其余取 0。

综上可得,

$$f(z) = \sum_{n=-1}^{+\infty} \frac{z^n}{(n+1)!} B_{n+1}$$

8 第八题

由于所有极点都在围道内部, 直接考察无穷远的留数, 根据引理有:

$$\int_{|z|=200} f(z) dz = -2\pi i \operatorname{Res}_{z \rightarrow \infty} f(z) = 2\pi i \operatorname{Res}_{t \rightarrow 0} \left[f\left(\frac{1}{t}\right) \frac{1}{t^2} \right]$$

其中

$$f\left(\frac{1}{t}\right) \frac{1}{t^2} = \frac{1}{t^2} \prod_{k=1}^{100} \frac{1}{1-kt}$$

$$\because \lim_{t \rightarrow 0} t^2 \frac{1}{t^2} \prod_{k=1}^{100} \frac{1}{1-kt} = 1$$

\therefore 二阶极点

所以可以计算出留数的值

$$\begin{aligned} \operatorname{Res}_{t \rightarrow 0} \left[f\left(\frac{1}{t}\right) \frac{1}{t^2} \right] &= \lim_{t \rightarrow 0} \frac{d}{dt} \prod_{k=1}^{100} \frac{1}{1-kt} \\ &= \lim_{t \rightarrow 0} \sum_{n=1}^{100} \prod_{n=1 \wedge k \neq n}^{100} \frac{n}{1-kt} \\ &= \sum_{n=1}^{100} n = 5050 \end{aligned}$$

$$\therefore \int_{|z|=200} f(z) dz = 10100\pi i$$

9 第九题

考虑配对, 设

$$\begin{aligned} I_1 &= \int_0^{2\pi} e^{\cos \theta} \cos(n\theta - \sin \theta) d\theta \\ I_2 &= \int_0^{2\pi} e^{\cos \theta} \sin(n\theta - \sin \theta) d\theta \\ I_1 + iI_2 &= \int_0^{2\pi} \exp(\cos \theta - i \sin \theta) \exp(in\theta) d\theta \\ I_1 - iI_2 &= \int_0^{2\pi} \exp(\cos \theta + i \sin \theta) \exp(-in\theta) d\theta \end{aligned}$$

对于 $I_1 + iI_2$ 设 $z = e^{i\theta}$, 有

$$I_1 + iI_2 = \int_{|z|=1} e^{\frac{1}{z}} z^n \frac{dz}{zi} = 2\pi \operatorname{Res}_{z \rightarrow 0} e^{\frac{1}{z}} z^{n-1}$$

包裹的奇点只在 $z = 0$, 我们计算其在 $z = 0$ 洛朗展开为

$$e^{\frac{1}{z}} z^{n-1} = \sum_{k=-\infty}^0 \frac{e^{n+k-1}}{(-k)!}$$

为求其留数, 取 z^{-1} 项要求 $k = -n$ 所以有

$$\operatorname{Res}_{z \rightarrow 0} e^{\frac{1}{z}} z^{n-1} = \frac{1}{n!}$$

对于 $I_1 - iI_2$ 设 $z = e^{-i\theta}$, 注意积分方向相反需要取出一个负号, 化简有

$$I_1 - iI_2 = \int_{|z|=1} e^{\frac{1}{z}} z^n \frac{dz}{zi} = I_1 + iI_2 = \frac{2\pi}{n!}$$

于是要求的量即为

$$I_1 = \frac{2\pi}{n!}$$

10 第十题

直接考察积分, 试图利用柯西公式

$$\begin{aligned} D_F(x-y) &= \int_{C_F} \frac{d^4 p}{(2\pi)^4} \frac{ie^{-ip \cdot (x-y)}}{p^2 - m^2} \\ &= \int_{C_F} \frac{d^3 \vec{p} dp_0}{(2\pi)^4} \frac{ie^{i\vec{p} \cdot (\vec{x}-\vec{y})} e^{-ip_0(x_0-y_0)}}{p_0^2 - (\vec{p}^2 + m^2)} \\ &= \int \frac{d^3 \vec{p} [e^{i\vec{p} \cdot (\vec{x}-\vec{y})}]}{(2\pi)^4} \int_{C_F} \frac{ie^{-ip_0(x_0-y_0)} dp_0}{p_0^2 - E_{\vec{p}}^2} \end{aligned}$$

考察其中的

$$\int_{C_F} \frac{ie^{-ip_0(x_0-y_0)} dp_0}{p_0^2 - E_{\vec{p}}^2} = \int_{C_F} \frac{ie^{-ip_0(x_0-y_0)} dp_0}{(p_0 - E_{\vec{p}})(p_0 + E_{\vec{p}})}$$

注意到在 C_F 的围道内只有一个一阶极点为

$$\text{Res}_{p_0 \rightarrow E_{\vec{p}}} \frac{ie^{-ip_0(x_0-y_0)}}{(p_0 - E_{\vec{p}})(p_0 + E_{\vec{p}})} = \frac{ie^{-ip_0(x_0-y_0)}}{2E_{\vec{p}}}$$

考察其在无穷远的性质有

$$\lim_{|p_0| \rightarrow \infty} \left| \frac{ip_0 e^{-ip_0(x_0-y_0)}}{p_0^2 - E_{\vec{p}}^2} \right| = 0$$

实际上是一致收敛的

注意到 C_F 的围道方向为顺时针, 因此有

$$\int_{C_F} \frac{ie^{-ip_0(x_0-y_0)} dp_0}{p_0^2 - E_{\vec{p}}^2} = 2\pi \frac{e^{-ip_0(x_0-y_0)}}{2E_{\vec{p}}}$$

代入 Feynman 传播子表示可得到:

$$D_F(x-y) = \int_{C_F} \frac{d^4 p}{(2\pi)^4} \frac{ie^{-ip \cdot (x-y)}}{p^2 - m^2} = \int \frac{d^3 \vec{p}}{(2\pi)^3 2E_{\vec{p}}} e^{-ip \cdot (x-y)} \Big|_{p_0=E_{\vec{p}}}$$

11 第十一题

以下都建立在 $D=2$ 上

引理 11.1. 导数定理

$$f(x) = \mathcal{F}(g(k)) \Rightarrow \partial_\mu f(x) = \mathcal{F}(-ik_\mu g(k))$$

证明.

$$\begin{aligned} \mathcal{F}^{-1}(\partial_\mu f(x)) &= \frac{1}{2\pi} \int d^2 x \exp(ik \cdot x) \partial_\mu f(x) \\ &= -\frac{1}{2\pi} \int d^2 x (ik_\mu) \exp(ik \cdot x) f(x) = -ik_\mu g(k) \\ &\therefore \partial_\mu f(x) = \mathcal{F}(-ik_\mu g(k)) \end{aligned}$$

同理,

$$\partial^\mu f(x) = \mathcal{F}(-ik^\mu g(k))$$

□

引理 11.2. 卷积定理

如果定义

$$f * g(x) \equiv \int d^2\xi f(\xi)g(x - \xi)$$

则有

$$\mathcal{F}(f * g(x)) = 2\pi\mathcal{F}(f(x))\mathcal{F}(g(x))$$

证明.

$$\begin{aligned}\mathcal{F}(f * g(x)) &= \frac{1}{2\pi} \int d^2x e^{-ik \cdot x} \int d^2\xi f(\xi)g(x - \xi) \\ &= \int d^2\xi f(\xi) \left[\frac{1}{2\pi} \int d^2x e^{-ik \cdot x} g(x - \xi) \right] \\ &= \mathcal{F}(g(x)) \int d^2\xi f(\xi) e^{-ik \cdot \xi} \\ &= 2\pi\mathcal{F}(f(x))\mathcal{F}(g(x))\end{aligned}$$

推论：对于三个函数的情形，只需要依次计算，即为

$$\mathcal{F}(f * g * h(x)) = 4\pi^2\mathcal{F}(f(x))\mathcal{F}(g(x))\mathcal{F}(h(x))$$

□

引理 11.3. 设对于 $f(x)$ 有 $f(x) = \mathcal{F}(g(k))$ 则，

$$\int d^2x \mathcal{F}(g(k)) = (2\pi)^2 g(0)$$

证明.

$$\begin{aligned}\int d^2x \mathcal{F}(g(k)) &= \int d^2x \int d^2k \exp(-ik \cdot x) g(k) \\ &= \int d^2k g(k) \int d^2x \exp(-ik \cdot x) \\ &= (2\pi)^2 \int d^2k g(k) \delta(-k) = (2\pi)^2 g(0)\end{aligned}$$

□

我们考虑原始表达式

$$S = \int d^2x \left\{ -\frac{1}{2} \partial_\mu \phi(x) \partial^\mu \phi(x) + \frac{1}{2} \lambda_{\mu\nu}(x) [\partial^\mu \phi(x) - \epsilon^{\mu\sigma} \partial_\sigma \phi(x)] [\partial^\nu \phi(x) - \epsilon^{\nu\rho} \partial_\rho \phi(x)] \right\}$$

先分别计算其中的积分

$$S_1 = \int d^2x [\partial_\mu \phi(x) \partial^\mu \phi(x)]$$

$$S_2 = \int d^2x \{ \lambda_{\mu\nu}(x) [\partial^\mu \phi(x) - \epsilon^{\mu\sigma} \partial_\sigma \phi(x)] [\partial^\nu \phi(x) - \epsilon^{\nu\rho} \partial_\rho \phi(x)] \}$$

$$\because \phi(x) = \mathcal{F}(\phi(k)), \lambda_{\mu\nu}(x) = \mathcal{F}(\lambda_{\mu\nu}(k))$$

换元, 令

$$g_\mu = -ik_\mu \phi(k), g^\mu = -ik^\mu \phi(k)$$

利用导数定理 11.1

$$\because \mathcal{F}(-ik_\mu \phi(k)) = \partial_\mu \phi(x), \mathcal{F}(-ik^\mu \phi(k)) = \partial^\mu \phi(x)$$

利用卷积定理 11.2

$$S_1 = \int d^2x \mathcal{F}(g_\mu) \mathcal{F}(g^\mu) = \frac{1}{2\pi} \int d^2x \mathcal{F}(g_\mu * g^\mu)$$

最后利用 11.3

$$\therefore S_1 = (g_\mu * g^\mu)|_{k=0} = \int d^2\xi g_\mu(\xi) g^\mu(0 - \xi) = \int d^2k (-ik_\mu) \phi(k) (-ik^\mu) \phi(k)$$

同理我们设

$$\eta^\mu = ik^\mu \phi(k) - \epsilon^{\mu\sigma} ik_\sigma \phi(k), \eta^\nu = ik^\nu \phi(k) - \epsilon^{\nu\rho} ik_\rho \phi(k)$$

有

$$S_2 = \int d^2x \mathcal{F}(\lambda_{\mu\nu}(k)) \mathcal{F}(\eta^\mu) \mathcal{F}(\eta^\nu) = \frac{1}{4\pi^2} \int d^2x \mathcal{F}(\lambda_{\mu\nu}(k) * \eta^\mu * \eta^\nu)$$

$$S_2 = \frac{1}{2\pi} \lambda_{\mu\nu}(k) * \eta^\mu * \eta^\nu|_{k=0} = \frac{1}{2\pi} \int d^2\xi \int d^2\zeta \lambda_{\mu\nu}(\zeta + \xi) \eta^\mu(0 - \xi) \eta^\nu(0 - \zeta)$$

即为

$$S_2 = \frac{1}{2\pi} \int d^2k \int d^2k' \lambda_{\mu\nu}(-k - k') [ik^\mu \phi(k) - \epsilon^{\mu\sigma} ik_\sigma \phi(k)] [ik'^\nu \phi(k') - \epsilon^{\nu\rho} ik'_\rho \phi(k')]$$

现在我们终于可以代回到原表达式

$$\begin{aligned} S_m = & -\frac{1}{2} \int d^2k (-ik_\mu) (ik^\mu) \phi(k) \phi(-k) \\ & + \frac{1}{4\pi} \int d^2k d^2k' \lambda_{\mu\nu}(-k - k') (ik^\mu - \epsilon^{\mu\sigma} ik_\sigma) (ik'^\nu - \epsilon^{\nu\rho} ik'_\rho) \phi(k) \phi(k') \end{aligned}$$

这就是结果

12 第十二题

12.1

$$g_1(t) = \begin{cases} 2t/T & 0 \leq t < T/2 \\ 2(1-t/T) & T/2 \leq t \leq T \end{cases}$$

根据周期函数

$$\mathcal{L}(g(t)) = \frac{1}{1 - e^{-pT}} \int_0^T e^{-pt} g(t) dt$$

计算得

$$\begin{aligned} \int_0^{T/2} \exp(-pt) \frac{2t}{T} dt &= \frac{2}{T} \left[\frac{1}{p^2} - e^{-Tp/2} \left(\frac{T}{2p} + \frac{1}{p^2} \right) \right] \\ - \int_{T/2}^T \exp(-pt) \frac{2t}{T} dt &= \frac{2}{T} e^{-Tp/2} \left[(1 - e^{-Tp/2}) \left(\frac{T}{2p} + \frac{1}{p^2} \right) \right] \\ \int_{T/2}^T 2 \exp(-pt) dt &= \frac{2}{p} (e^{-Tp/2} - e^{-Tp}) \end{aligned}$$

代入化简得

$$\mathcal{L}(g_1(t)) = \frac{2}{Tp^2} \tanh\left(\frac{Tp}{4}\right)$$

12.2

$$\begin{aligned} g_2(t) &= \frac{2}{T} \left[\text{th}(t) + 2 \sum_{n=1}^{\infty} (-1)^n \left(t - \frac{1}{2}nT \right) H\left(t - \frac{1}{2}nT\right) \right] \\ \mathcal{L}(g_2(t)) &= \frac{2}{Tp^2} \left[1 + 2 \sum_{n=1}^{\infty} (-1)^n \exp\left(-\frac{1}{2}nT\right) \right] \end{aligned}$$

如果 $g_1(t) = g_2(t)$, 比照系数令 $x = Tp/4$ 换元有

$$\tanh x = 1 + 2 \sum_{n=0}^{\infty} (-1)^n e^{-2nx}$$

这正是我们想证明的。因此只需证明 $g_1(t) = g_2(t)$.

下面证明 $g_1(t) = g_2(t)$

我们首先考察区间 $[0, T/2)$, 对于 $t \in [0, T/2]$ 由于全部的 $H(t - 1/2nT) = 0$ 因此二者显然相等。

然后考察区间 $[T/2, T)$, 对于 $t \in [0, T/2]$ 求和只保留到 $n = 1$, 有

$$g_2 = \frac{2}{T} \left[t - 2 \left(t - \frac{1}{2}T \right) \right] = 2 - \frac{2t}{T} = g_1$$

综上二者在 $t \in [0, T)$ 上相等。

一般地, 对于 $n > 0$ 情形, 仅当 $t \geq 1/2nT$ 时, $H(t - 1/2nT) = 1$, 其余情况为 0。如果我们只考察 t 非负半轴情况, 求和可以写成:

$$g_2(t) = \frac{2}{T} \left[t + 2 \sum_{n=1}^{\left[\frac{2t}{T}\right]} (-1)^n \left(t - \frac{1}{2}nT \right) \right]$$

其中 $\left[\frac{2t}{T}\right]$ 表示括号内值向下取整。

接下来证明 $g_2(t)$ 也以 T 为周期, 既 $g_2(t) = g_2(t+T)$

$$\begin{aligned} g_2(t+T) &= \frac{2}{T} \left[t+T + 2 \sum_{n=1}^{\left[\frac{2t}{T}+2\right]} (-1)^n \left(t - \frac{1}{2}nT + T\right) \right] \\ &= \frac{2}{T} \left[t+T - (2t+T) + 2t + 2 \sum_{n=3}^{\left[\frac{2t}{T}+2\right]} (-1)^n \left(t - \frac{1}{2}nT + T\right) \right] \\ &= \frac{2}{T} \left[t + 2 \sum_{n=1}^{\left[\frac{2t}{T}\right]} (-1)^n \left(t - \frac{1}{2}nT\right) \right] \\ &= g_2(t) \end{aligned}$$

由于在 $[0, T)$ 区间相等并都具有以 T 为周期的周期性, 因此 $g_1(t) = g_2(t)$, 这正是我们需要的。于是

$$\tanh x = 1 + 2 \sum_{n=0}^{\infty} (-1)^n e^{-2nx}$$

得证

13 第十三题

以下的拉普拉斯方程的表达式都基于

$$\frac{1}{\sqrt{|g|}} \partial_\mu \left(g^{\mu\nu} \sqrt{|g|} \partial_\nu \psi \right) = 0$$

在正交曲线坐标系下为

$$\sum_{\mu=1}^3 \frac{1}{|h_1 h_2 h_3|} \partial_\mu \left(\frac{1}{h_\mu} \sqrt{|h_1 h_2 h_3|} \partial_\mu \psi \right) = 0$$

13.1 球坐标

$$h_r = 1, h_\theta = r, h_\varphi = r \sin \theta$$

拉普拉斯方程为:

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial u}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial u}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 u}{\partial \varphi^2} = 0$$

首先, 设

$$u(r, \theta, \varphi) = R(r)Y(\theta, \varphi)$$

$$\frac{1}{R} \frac{d}{dr} \left(r^2 \frac{dR}{dr} \right) = -\frac{1}{Y \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial Y}{\partial \theta} \right) - \frac{1}{Y \sin^2 \theta} \frac{\partial^2 Y}{\partial \varphi^2} = l(l+1)$$

分解为两个方程

$$\begin{aligned} \frac{d}{dr} \left(r^2 \frac{dR}{dr} \right) - l(l+1)R &\equiv 0 \\ \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial Y}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2 Y}{\partial \varphi^2} + l(l+1)Y &= 0 \end{aligned}$$

进一步设

$$\begin{aligned} Y(\theta, \varphi) &= \Theta(\theta)\Phi(\varphi) \\ \frac{\sin \theta}{\Theta} \frac{d}{d\theta} \left(\sin \theta \frac{d\Theta}{d\theta} \right) + l(l+1) \sin^2 \theta &= -\frac{1}{\Phi} \frac{d^2 \Phi}{d\varphi^2} = \lambda \end{aligned}$$

分解为两个常微分方程:

$$\begin{aligned} \Phi'' + \lambda \Phi &= 0 \\ \sin \theta \frac{d}{d\theta} \left(\sin \theta \frac{d\Theta}{d\theta} \right) + [l(l+1) \sin^2 \theta - \lambda] \Theta &= 0 \end{aligned}$$

13.2 柱坐标

$$h_\rho = 1, h_\varphi = \rho, h_z = 1$$

柱坐标的拉普拉斯方程表示为

$$\frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial u}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 u}{\partial \varphi^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

设

$$u(\rho, \varphi, z) = R(\rho)\Phi(\varphi)Z(z)$$

代入得到

$$\Phi'' + \lambda \Phi = 0$$

以及

$$\frac{\rho^2}{R} \frac{d^2 R}{d\rho^2} + \frac{\rho}{R} \frac{dR}{d\rho} + \rho^2 \frac{Z''}{Z} = \lambda$$

分离有:

$$\frac{1}{R} \frac{d^2 R}{d\rho^2} + \frac{1}{\rho R} \frac{dR}{d\rho} - \frac{m^2}{\rho^2} = -\frac{Z''}{Z} = -\mu$$

两个常微分方程

$$\begin{aligned} Z'' - \mu Z &= 0 \\ \frac{d^2 R}{d\rho^2} + \frac{1}{\rho} \frac{dR}{d\rho} + \left(\mu - \frac{m^2}{\rho^2} \right) R &= 0 \end{aligned}$$

13.3 椭圆柱坐标

标准表示为

$$x = a\xi\eta, \quad y = a\sqrt{(\xi^2 - 1)(1 - \eta^2)}, \quad z = z$$

换元, 设

$$\xi = \operatorname{ch} u, \quad \eta = \cos v$$

有

$$h_u^2 = h_v^2 = a^2 (\operatorname{ch}^2 u - \cos^2 v), \quad h_z = 1$$

拉普拉斯方程表示为

$$\nabla^2 \Phi = \frac{1}{a^2 (\operatorname{ch}^2 u - \cos^2 v)} \left\{ \frac{\partial^2 \Phi}{\partial u^2} + \frac{\partial^2 \Phi}{\partial v^2} \right\} + \frac{\partial^2 \Phi}{\partial z^2} = 0$$

设

$$\Phi(u, v, z) = U(u) V(v) Z(z)$$

代入有

$$\frac{1}{a^2 (\operatorname{ch}^2 u - \cos^2 v)} \left\{ \frac{1}{U} \frac{d^2 U}{du^2} + \frac{1}{V} \frac{d^2 V}{dv^2} \right\} + \frac{1}{Z} \frac{d^2 Z}{dz^2} = 0$$

分离出

$$\frac{d^2 Z}{dz^2} + \lambda Z = 0$$

另有

$$\frac{1}{U} \frac{d^2 U}{du^2} + \frac{1}{V} \frac{d^2 V}{dv^2} - \lambda a^2 (\operatorname{ch}^2 u - \cos^2 v) = 0$$

分离出

$$\begin{aligned} \frac{d^2 U}{du^2} - (\lambda a^2 \operatorname{ch}^2 u + \mu) U &= 0 \\ \frac{d^2 V}{dv^2} + (\lambda a^2 \cos^2 v + \mu) V &= 0 \end{aligned}$$

13.4 椭球坐标

$$\begin{aligned} x^2 &= \frac{(a^2 + \lambda)(a^2 + \mu)(a^2 + v)}{(a^2 - b^2)(a^2 - c^2)} \\ y^2 &= \frac{(b^2 + \lambda)(b^2 + \mu)(b^2 + v)}{(b^2 - c^2)(b^2 - a^2)} \\ z^2 &= \frac{(c^2 + \lambda)(c^2 + \mu)(c^2 + v)}{(c^2 - a^2)(c^2 - b^2)} \end{aligned}$$

λ, μ, ν 的变化范围为

$$\lambda > -c^2 > \mu > -b^2 > \nu > -a^2$$

若设

$$\varphi(\theta) = (a^2 + \theta)(b^2 + \theta)(c^2 + \theta)$$

有

$$h_\lambda^2 = \frac{(\lambda - \mu)(\lambda - \nu)}{4\varphi(\lambda)}$$

$$h_\mu^2 = \frac{(\mu - \lambda)(\mu - \nu)}{4\varphi(\mu)}$$

$$h_\nu^2 = \frac{(\nu - \lambda)(\nu - \mu)}{4\varphi(\nu)}$$

$$\begin{aligned} \nabla^2 \Phi = & \frac{2}{(\lambda - \mu)(\lambda - \nu)(\mu - \nu)} \left[(\mu - \nu)\sqrt{\varphi(\lambda)} \frac{\partial}{\partial \lambda} \left(\sqrt{\varphi(\lambda)} \frac{\partial \Phi}{\partial \lambda} \right) \right. \\ & + (\lambda - \nu)\sqrt{-\varphi(\mu)} \frac{\partial}{\partial \mu} \left(\sqrt{-\varphi(\mu)} \frac{\partial \Phi}{\partial \mu} \right) \\ & \left. + (\lambda - \mu)\sqrt{\varphi(\nu)} \frac{\partial}{\partial \nu} \left(\sqrt{\varphi(\nu)} \frac{\partial \Phi}{\partial \nu} \right) \right] = 0 \end{aligned}$$

设 $\Phi = \Lambda(\lambda)M(\mu)N(\nu)$ 代入有

$$\begin{aligned} & \frac{\mu - \nu}{\Lambda} \sqrt{\varphi(\lambda)} \frac{d}{d\lambda} \left(\sqrt{\varphi(\lambda)} \frac{d\Lambda}{d\lambda} \right) + \frac{\lambda - \nu}{M} \sqrt{-\varphi(\mu)} \frac{d}{d\mu} \left(\sqrt{-\varphi(\mu)} \frac{dM}{d\mu} \right) \\ & + \frac{\lambda - \mu}{N} \sqrt{\varphi(\nu)} \frac{d}{d\nu} \left(\sqrt{\varphi(\nu)} \frac{dN}{d\nu} \right) = 0 \end{aligned}$$

注意到有恒等式

$$(\mu - \nu)(K\lambda + C) + (\nu - \lambda)(K\mu + C) + (\lambda - \mu)(K\nu + C) \equiv 0$$

其中 K 和 C 为常数, 比较系数得

$$4\sqrt{\varphi(\lambda)} \frac{d}{d\lambda} \left(\sqrt{\varphi(\lambda)} \frac{d\Lambda}{d\lambda} \right) = (K\lambda + C)\Lambda$$

设 $K = n(n+1)$, 分离变量的三个方程为:

$$4\sqrt{\varphi(\lambda)} \frac{d}{d\lambda} \left(\sqrt{\varphi(\lambda)} \frac{d\Lambda}{d\lambda} \right) = [(n(n+1))\lambda + C]\Lambda$$

$$4\sqrt{\varphi(\mu)} \frac{d}{d\mu} \left(\sqrt{\varphi(\mu)} \frac{dM}{d\mu} \right) = [(n(n+1))\mu + C]M$$

$$4\sqrt{\varphi(\nu)} \frac{d}{d\nu} \left(\sqrt{\varphi(\nu)} \frac{dN}{d\nu} \right) = [(n(n+1))\nu + C]N$$

13.5 锥面坐标

13.6 抛物线柱坐标

14 第十四题