CSE 102 - 01

14th Sept '22

Booklist - Discrete Mathematics - Kenneth H. Rosen

(7th / 8th edition)

CSE 102 - 02

15th Sept '22

= Logic and Proofs =

* logic is subjective

→ Quantifying qualitative ~~~~ entities / characters

* Machine doesn't understand ambiguous language

* Interaction / Recognition → mathematical expression of human logic

* → Step - 1 > Formation of logic

* Propositional logic

  * Proposition — A statement / sentence which is either true or false

* A question is not a statement (Proposition)

* If p / q is a TRUE statement

then $\neg p / \neg q$ (p' / q') [Not symbol]

$\bar{p} / \bar{q}$ is the FALSE statement

* Connective for multiple statements –

$\wedge$ – AND > $p \wedge q$ (p AND q)

$\vee$ – OR > $p \vee q$ (p OR q)

$\wedge$ – Both has to be correct (1/4 TRUE)

$\vee$ – Either or both can be correct (3/4 TRUE)

* 2^n combinations is possible

AND $(\wedge)$ = Conjunction

OR $(\vee)$ = Disjunction

* When either p or q can be TRUE but not both at the same time (2/4)

↳ Exclusive OR / XOR

↳ $p \oplus q$

* Implication

$$p \rightarrow q \quad [\, p \text{ implies } q \,]$$

One statement justifies / verifies the other.

$p \rightarrow q$ > If p is TRUE then q is TRUE

But if p is FALSE then q may be TRUE or FALSE.

* p implies q but $\bar{p}$ doesn't imply $\bar{q}$

(3/4 TRUE)

* p (T) and q (F) $\longrightarrow$ $\dfrac{p \rightarrow q}{\text{FALSE}}$

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

$p \rightarrow q$

" p is sufficient for q "

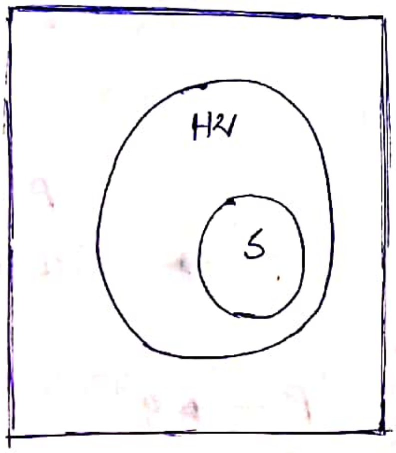
" q is necessary for p "

* Not chronological

Hard work $\rightarrow$ success

✓ Success $\rightarrow$ hard work

Hard work is a subset of success



" Success if sufficient for hard work. "

" Hard work is necessary for success "

When $p \rightarrow q$

* $\bar{p} \rightarrow \bar{q}$ > Inverse
* $q \rightarrow p$ > Converse
(Inverse, Converse: logical fallacy)

* $\bar{q} \rightarrow \bar{p}$ > Contrapositive $[p \rightarrow q]$

$p \longrightarrow q$

* p only if q

q if p

if p, then q

$q \longrightarrow p$

* q only if p

p if q

if q, then p

$p \rightarrow q \wedge q \rightarrow p$

$p \rightleftharpoons q$ (Biconditional)

p if and only if q

$\boxed{\text{p iff q}}$

p	q	$p \rightleftharpoons q$
T	T	T
T	F	F
F	T	F
F	F	T

Truth table of biconditional

21st Sept' 22
CSE 102 - 03

Equivalent

$p \equiv q$

p is equivalent to q

$p \to q \equiv \bar{p} \vee q$

p	q	$\bar{p}$	$p \to q$	$\bar{p} \vee q$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

Logical formula -

* proposition

a = "You can access internet from the campus"

c = "You are a CS student"

f = "You are a freshman".

* You can access the computer only if you are a CS student or you are not a freshman.

$$a \longrightarrow c \vee \bar{f} \quad [a \longrightarrow c \vee \neg f]$$

operator precedence –

1. $\neg$

2. $\vee$. $\wedge$

3. $\longrightarrow$

$\boxed{\text{Totology}}$ $\longrightarrow$ A statement that must always be true

Example – $a \vee \neg a$

$$x \longrightarrow x$$

$\boxed{\text{Contradiction}}$ $\longrightarrow$ A statement that's always ~~f~~ false

$$a \wedge \neg a$$

$\boxed{\text{Contingency}}$ ⟶ sometimes true, sometimes false

$a \wedge b$ $\quad a \vee b \vee c$

De Morgan's law -

$$\neg(x \wedge y) \equiv \neg x \vee \neg y$$

$$\neg(x \vee y) \equiv \neg x \wedge \neg y$$

* T or F dominates the variable or statement

Absorption laws -

$$p \vee (p \wedge q) \equiv p$$

$$p \wedge (p \vee q) \equiv q$$

$p \vee (p \wedge q)$ $\qquad$ $(p \vee p) \vee (p \wedge q)$

$= p + (pq)$

$= (p+p)(p+q)$

$= p(p+q)$

$= pp + pq$

$= p + pq$

$= p(1+q)$

$= p \cdot 1 = p$

p	q	$p \wedge q$	$p \vee (p \wedge q)$
T	T	T	T
T	F	F	T
F	T	F	F
F	F	F	F

$p \rightarrow q \equiv \neg p \vee q$

$p \rightarrow q \equiv \neg p \rightarrow \neg q$

Biconditional –

$p \rightleftharpoons q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$

$p \rightleftharpoons q \equiv \neg p \rightleftharpoons \neg q$

$p \rightleftharpoons q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$

$\neg (p \rightleftharpoons q) \equiv p \rightleftharpoons \neg q$

$$p \rightarrow q \equiv \neg p \vee q$$

$$p \rightarrow q \equiv \neg q \rightarrow \neg p$$

$$p \vee q \equiv \neg p \rightarrow q$$

$$p \wedge q \equiv \neg (p \rightarrow \neg q)$$

$$\neg (p \rightarrow q) \equiv p \wedge \neg q$$

$$(p \rightarrow q) \wedge (p \rightarrow r) \equiv p \rightarrow (q \wedge r)$$

$$(p \rightarrow r) \wedge (q \rightarrow r) \equiv (p \vee q) \rightarrow r$$

$$(p \rightarrow q) \vee (p \rightarrow r) \equiv p \rightarrow (q \vee r)$$

$$(p \rightarrow r) \vee (q \rightarrow r) \equiv (p \wedge q) \rightarrow r$$

$$p \wedge (p \vee q)$$

$$\equiv (p \wedge p) \vee (p \wedge q) \quad [DL]$$

$$\equiv p \vee (p \wedge q) \quad [Id.\ L] \quad \equiv (p \wedge T) \vee (p \wedge q) \quad [IL]$$

$$\equiv p \wedge (T \vee q) \quad [DL]$$

$$\equiv p \wedge T \quad [DL] \quad \equiv p$$

Predicate logic / First Order logic (FOL)

* ~~De~~ No direct truth value
* Depends on the value of the variable.

"$x > 3$"

" Computer x is corder attack "

Quantifier –

$\boxed{\forall}$ "For all" "universal quantifier"

$\boxed{\exists}$ "Existential quantifier" "There exists" / "There is a" / "there is some"

* $\forall x \quad x \cdot 1 = x$

For all x, $\quad x \cdot 1 = x$

* $\exists x \quad \frac{1}{x} = \frac{1}{5}$

There exists x, $\quad \frac{1}{x} = \frac{1}{5}$

* $\boxed{\text{Domain}}$ — Domain of discource

$$\forall x \underbrace{\neq 0}_{\text{Domain}} \cdot \frac{1}{x} \in \mathbb{R}$$

$P(x)$; $x+1 > x$

P, Q, R $\longrightarrow$ Predicate

x, y, z $\longrightarrow$ variable

$\forall x \quad P(x); \quad x+1 > x$

$Q(x) : x^2 > 0$

$\forall x \neq 0 \quad Q(x)$

$P(x) : x > 3$

$\exists x \; P(x)$

" There exists an x such that $P(x)$ is true "

$Q(x) : x = x+1$

$$\boxed{\neg \exists x \; Q(x) \equiv \forall x \; \neg Q(x)}$$

" There does not exist an x such that $Q(x)$ is true"

= " For all x $Q(x)$ is not true "

False

$\exists ! x$ " There exists a unique x"

" There is exactly one x"

$\forall x \; L(x, \text{ice cream}) \equiv \neg \quad \neg L(x, \text{ice cream})$

$\exists x \; L(x, \text{brocolli}) \equiv \neg \forall x \; \neg L(x, \text{brocolli})$

$\cancel{\neg \forall x \; L(x, \text{ice}}$

$$\neg \forall x \; L(x, \text{ice cream})$$

$$\equiv \exists x \; \neg L(x, \text{ice cream})$$

$$\neg \exists x \; L(x, \text{brocolli})$$

$$\equiv \forall x \; \neg L(x, \text{brocolli})$$

$L(x, y)$

Domain of x, y : All people

$$\forall x \forall y \; L(x, y) \equiv \forall y \forall x \; L(x, y)$$

$$\exists x \exists y \; L(x, y) \equiv \exists y \exists x \; L(x, y)$$

$$\forall x \left(\exists y \; L(x, y)\right) \not\equiv \exists y \forall x \; L(x, y)$$

$$\exists x \forall y \; L(x, y) \not\equiv \forall y \exists x \; L(x, y)$$

29th Sept' 22
CSE 102 - 04

Predicate logic -

$At(x, y)$

"x is at y"

("x is studying at y")

"Everyone who is at IIT is smart"

$Smart(x)$

Domain — All IITians

$\forall x \ Smart(x)$

Domain — All people

* $\boxed{\forall x \ At(x, IIT) \rightarrow Smart(x)}$

For all x, (x is at IIT) implies smart (x)

$\boxed{\forall x \ [At(x, IIT) \wedge Smart(x)]}$

For all x, x is at IIT and x is smart

* For any two logic statements to be equal, they have to be logically equivalent (that is, there truth table has to be equal).

" Some people who are at CSE are smart. "

Domain – $\boxed{\text{All people}}$

$$\boxed{\exists x \, [At(x, CSE) \land Smart(x)]}$$

" There exists x, x is at CSE and x is smart "

$$\boxed{\exists x \, [At(x, CSE) \longrightarrow Smart(x)]}$$

If the first condition of implication is false even then the entire statement is true

= CSE তে কেউ না পড়লেও there are some people who are smart

Intended meaning not expressed.

Ex Translate to English –

$\boxed{\text{Domain} - \mathbb{R}}$

a) $\forall x \, \exists y \, (x < y)$

" For all x, it is true that there is a y such that the following is true – $x < y$ "

$\longrightarrow$ "Every number has some numbers which are less ~~that~~ than itself."

* If the domain is All positive integers -

$\forall x \neq 1 \ \exists y \ (x < y)$

b) $\forall x \ \forall y \left[\left((x \geq 0) \land (y \geq 0) \right) \longrightarrow (xy \geq 0) \right]$

" The product of any two non-negative numbers is greater than or equal to zero."

$\longrightarrow$ " The product of any two non-negative numbers is always non-negative."

c) $\forall x \ \forall y \ \exists z \ (xy = z)$

" The product of any two real numbers is always real."

~~e) ∀~~

5. a) $V(\text{Sarah Smith}, \text{www.att.com})$

"Sarah Smith has visited www.att.com"

b) $\exists x\, V(x, \text{www.imdb.org})$.

"There are some students who have visited www.imdb.org."

c) $\exists y\, V(\text{José Orez}, y)$

"José Orez has visited some websites ~~y~~"

d) $\exists y\, (V(\text{Ashok Puri}, y) \wedge V(\text{Cindy Yoon}, y))$

"Ashok Puri and Cindy Yoon have visited ~~the~~ the same website ~~y~~"

e) $\exists y\, \forall z\, (y \neq (\text{David Belcher}) \wedge (V(\text{David Belcher}, z)) \rightarrow V(y, z)))$

~~"Some students except David Belcher has visited the website z"~~

~~" Some students If David Belcher did not visited the website z then some other students have visited the website z."~~

f) $\exists x \exists y \forall z \; ((x \neq y) \wedge (V(x,z)) \leftrightarrow V(y,z))$

" Some students have visited ~~the~~ some websites ~~z~~ if and only if ~~at~~ some other students have visited the ~~those that~~ same websites."

e) " If David Belcher visited ~~the~~ a set of websites ~~z~~ then some other student have also visited the ~~set web~~ same set of websites."

f) ~~A~~ A pair of student have visited the same set of websites.

12th Oct '22
CSE 102 - 05

Inference

* $\underbrace{\text{Premises}}_{\text{given / known}} \longrightarrow$ Conclusion

I) "Modus ponens"

* Original implication

$$\begin{array}{l} p \ (T) \\ p \rightarrow q \\ \hline \therefore \ q \ (T) \end{array}$$

II) "Modus tollens"

* Contrapositive

$$\begin{array}{l} p \rightarrow q \\ \neg q \rightarrow \neg p \end{array}$$

$$\begin{array}{l} \neg q \\ p \rightarrow q \\ \hline \therefore \ \neg p \end{array}$$

III)

$$\begin{array}{l} p \rightarrow q \\ q \rightarrow r \\ \hline \therefore \ p \rightarrow r \end{array}$$

(iv)

$$\begin{array}{l} p \vee q \\ \neg p \\ \hline \therefore q \end{array} \qquad \begin{array}{l} p \vee q \\ \neg q \\ \hline \therefore p \end{array}$$

(v)

$$\begin{array}{l} p \\ \hline \therefore p \vee q \end{array}$$

(vi)

$$\begin{array}{l} p \wedge q \\ \hline \therefore p \end{array}$$

(vii)

$$\begin{array}{l} p \\ q \\ \hline \therefore p \wedge q \end{array}$$

(viii)

$$\begin{array}{l} p \vee q \\ \neg p \vee r \\ \hline \therefore q \vee r \end{array} \longrightarrow$$ * Resolution

Premises

$\neg S(\ ) \wedge C(\ )$

" It is not sunny this afternoon and it is colder than yesterday." $M(\ ) \rightarrow S(\ )$

" We will go swimming only if it is sunny."

" If we do not go swimming then we will take a canoe trip" $\neg M(x) \rightarrow T(x)$

" If we take a canoe trip, then we will be home by sunset." $T(x) \rightarrow H(x)$

Conclusion

" We will be home by sunset."

$H(x)$

(i) $\neg S(x) \wedge C(x)$

$M(x) \rightarrow S(x)$

$\neg M(x) \rightarrow T(x)$

$T(x) \rightarrow H(x)$

(ii) $\neg S(x) \wedge C(x)$

$\therefore \neg S(x)$

(III) $\neg S(x)$

$$\frac{\begin{array}{c}\neg S(x) \\ M(x) \rightarrow S(x)\end{array}}{\therefore \neg M(x)}$$

(IV) $\neg M(x)$

$$\frac{\begin{array}{c}\neg M(x) \\ \neg M(x) \rightarrow T(x)\end{array}}{\therefore T(x)}$$

(V)

$$\frac{\begin{array}{c}T(x) \\ T(x) \rightarrow H(x)\end{array}}{\therefore H(x)}$$

$\neg$ Sunny $\wedge$ colder —(I)

Swimming $\rightarrow$ sunny —(II)

$\neg$ Swimming $\rightarrow$ Canoe —(III)

Canoe $\rightarrow$ Home —(IV)

From (I)

$\neg$ Sunny —— (V)

~~From (V)~~ and (II)

~~$\neg$ Swimmi~~

$\neg$ Sunny $\rightarrow$ $\neg$ swimming —— (VI)

From (V) and (VI)

$\neg$ Swimming —— (VII)

From (III) and (VIII)

Canoe —— (VIII)

From (IV) and (VIII)

Home

4. Predicate logic

* $P(x)$: " x is prime "

$P(2)$ ✓

$P(9)$ ✗

$P(13)$ ✓

* $\forall x \; P(x)$

$\exists x \; P(x)$

* generalization is the opposite of instantiation

(I)

Universal Instantiation

$$\frac{\forall x \; P(x)}{P(c)}$$

→ Any object in the domain

(II)

Universal generalization

$$\frac{P(c).}{\therefore \forall x \; P(x)}$$

logical fallacy

* → for any arbitrary c

(III)

Existential Instantiation

$$\frac{\exists x \; P(x)}{P(c)}$$

→ for some element c

(IV)

$$\frac{P(c)}{\exists x \; P(x)}$$

Existential generalization

Premises

$\exists x \neg R(x)$

" A student in this class has not read the book "

" Everyone in this class ~~pt~~ passed the first exam ".

$\forall x\, P(x)$

~~Conlu~~ Conclusion

" Someone who passed the first exam has not read the book .

$\exists x \left(P(x) \wedge \neg R(x) \right)$

(I)

$$\cancel{\frac{\exists x \neg R(x)}{\neg R(x)}}$$

(II)

$$\cancel{\frac{\neg R(x)}{\forall x \neg R(x)}}$$

(III)

$$\cancel{\frac{\begin{array}{c}\forall x\, P(x) \\ \forall x\, \neg R(x)\end{array}}{\quad}}$$

(II)

$$\cancel{\frac{\forall x\, P(x)}{P(c)}} \quad [\cancel{UG}\ UI]$$

$$\cancel{\frac{\begin{array}{c}\exists x\, P(x) \quad [EG] \\ \exists x\, \neg R(x)\end{array}}{\exists x \left(P(x) \wedge \neg R(x) \right)}}$$

$\Rightarrow$ Student (x)

Read (x)

Pass (x)

Statement -1

$$\exists x \left(\text{Student}(x) \land \neg \text{Read}(x) \right) \text{ — (i)}$$

Statement -2

$$\forall x \left(\text{Student}(x) \rightarrow \text{pass}(x) \right) \text{ — (ii)}$$

Conclusion

$$\exists x \left(\text{Pass}(x) \land \neg \text{Read}(x) \right) \text{ — (iii)}$$

$\Rightarrow$ Existential instantiation — (i)

Student (c) $\land$ $\neg$ Read (c) for some c — (iv)

Universal instantiation — (ii)

Student (c) $\rightarrow$ Pass (c) — (v)

From (IV)

$Student(c)$ —— (VI)

$\neg Read(c)$ ——→ (VII)

From (V) and (VI),

$Pass(c)$ —— (VIII)

From (VII) and (VIII),

$Pass(c) \wedge \neg Read(c)$ —— (IX)

Existential generalization from (IX),

$\exists x \left(Pass(x) \wedge \neg Read(x) \right)$ —— (X)

CamScanner

19th Oct' 22
CSE 102 - 06

Methods of Proof —

" Direct proof

(The sum of two even numbers is even).

Suppose, $x = 2m$, $y = 2n$

$$x + y = 2m + 2n = 2(m+n)$$

" Conventionally, $n - 2k$ even

$n = 2k+1 \longrightarrow$ odd

" Divisibility,

$\underline{(a|b)}$: $b = ak$ for some integer k.

$(b \div a)$

$\longrightarrow$ * For proof, replace b with ak to get the divident k.

// Simple divisibility facts —

I) If $a|b$ and $b|c$, then $a|c$.

II) If $a|b$, then $a|bc$ for all c.

III) If $a|b$, and $a|c$, then $a|sb+tc$ for for all s and t

IV) For all $c \neq 0$, $a|b$ if and only if $ca|cb$.

Proof of (i)

Given, $a|b$, let $= k_1$

$\Rightarrow a | bk_1$ $\therefore c = bk_2$

And $b|c$

$\Rightarrow b | bk_2$

$\Rightarrow ak_1 | ak_1 \cdot k_2$

$\Rightarrow k$

Proof of (i)

$a \mid b$

$\Rightarrow \quad b = ak$

$\Rightarrow \quad bc = ak \cdot c \quad \Rightarrow \quad bc = a\,(ck)$

$\cancel{a \mid akc}$

$\therefore \quad a \mid bc$

Proof of (ii)

$a \mid b \quad \Rightarrow \quad b = ak_1$

$b \mid c \quad \Rightarrow \quad c = bk_2$

$c = a k_1 k_2$

$\Rightarrow \quad a \mid c$

Proof of (iii)

$a \mid b \Rightarrow b = ak_1 , \quad a \mid c \Rightarrow c = ak_2$

$sb + tc = s \cdot ak_1 + ak_2 \cdot t$

$= a\,(sk_1 + tk_2)$

$\therefore \quad a \mid sb + tc$

// Contrapositive proof

$$p \longrightarrow q$$
$$\neg q \longrightarrow \neg p$$

4) If x is irrational, then $\sqrt{x}$ is irrational.

Contraposition

$$\sqrt{x} = a/b$$
$$x = a^2/b^2$$

// if and only if [An integer is even if and only if it's square is even]

↳ have to show both sides —

either — (i) Prove $p \rightarrow q$ and $q \rightarrow p$

or — (ii) Prove $p \rightarrow q$ and $\neg p \rightarrow \neg q$

Statement — If m is even, then m^2 is even.

$$m = 2k$$
$$m^2 = 4k^2 = 2(2k^2).$$

(half)

→ Proof the contrapositive –

"If m^2 is odd then m is odd".

// Proof by contradiction

$$\frac{\neg P \rightarrow F}{\therefore\ P}$$

Theorem : $\sqrt{2}$ is irrational

Suppose, $\sqrt{2}$ was rational

$\therefore\ \sqrt{2} = \frac{m}{n}$ [m, n doesn't have any common prime factors]

$\sqrt{2}\, n = m$

$2n^2 = m^2$

So, m is even.

So, we can assume $m = 2l$

$m^2 = 4l^2 \Rightarrow 2n^2 = 4l^2$

$n^2 = 2l^2$

So, n is even.

// Proof by cases

$$\begin{array}{c} p \vee q \\ p \rightarrow r \\ q \rightarrow r \\ \hline \therefore r \end{array}$$

" A nonzero number always has a positive square."

// – $\forall$ odd n, $\exists m$.
$n^2 = 8m + 1$

Idea 0 – Find counterexample
$\hookrightarrow$ examples verification

Idea 1 : Prove that $n^2 - 1$ is divisible by 8

Idea 2 : Consider $(2k+1)^2$

$$(2k+1)^2 = 4k^2 + 4k + 1 = 4(k^2 + k) + 1$$

If k is odd, then both k^2 and k are odd and so, $k^2 + k$ is even

If k is even, then both k^2 and k is even so, $k^2 + k$ is even.

$$\therefore (2k+1)^2 = 4(2m) + 1 = 8m + 1$$

$$\therefore n^2 = 8m + 1$$

20th Oct' 22
CSE 102 - 07

= Induction =

Fact : If m is odd and n is odd, then nm is odd.

Proposition : For an odd number m, m^k is odd for all non-negative integer k.

$$\forall k \in N \ \text{odd}(m^k)$$

Let $P(i)$ be the proposition that m^i is odd.

$$\forall k \in N \ P(k)$$

- $P(1)$ is true by definition
- $P(2)$ is true by $P(1)$ and the fact.

// The induction rule —

0 and (from n to $n+1$)

proves 0, 1, 2, 3 ....

$$P(0), \quad P(n) \longrightarrow P(n+1)$$

$$\forall m \in N, \quad P(m)$$

$$\forall r \neq 1, \quad 1 + r + r^2 + \dots + r^n = \frac{r^{n+1} - 1}{r - 1}$$

$LHS = 1$

$$RHS = \frac{r^{0+1} - 1}{r - 1} = \frac{r-1}{r-1} = 1$$

Induction step: Assume $P(n)$ for some $n = 0$ and prove $P(n+1)$.

$$\forall r \neq 1, \quad 1 + r + r^2 + \dots + r^{n+1} = \frac{r^{(n+1)+1} - 1}{r - 1}$$

Have $P(n)$ by assumption.

So let r be any number $\neq 1$, then from $P(n)$ we have

$$1 + r + r^2 + \cdots + r^n = \frac{r^{n+1} - 1}{r - 1}$$

// Induction step:

Assume $P(n)$ for some $n \geq 0$ and prove $P(n+1)$

Adding r^{n+1} to both sides,

$$1 + \cdots + r^n + r^{n+1} = \frac{r^{n+1} - 1}{r - 1} + r^{n+1}$$

$$= \frac{r^{n+1} - 1 + r^{n+1}(r-1)}{(r-1)}$$

$$= \left(r^{n+1} - 1 + r^{(n+1)+1} - r^{n+1}\right) / r-1$$

$$= \frac{r^{(n+1)+1} - 1}{r-1}$$

26th Oct '22

CSE 102 - 08

Proving an equality -

$$\forall n \geq 1, \quad 1^3 + 2^3 + \cdots + n^3 = \left(\frac{n(n+1)}{2}\right)^2$$

<u>Base case</u> $(n=1)$

$$L.H.S = 1^3 = 1 \qquad P(1)$$

$$R.H.S = \left(\frac{1(1+1)}{2}\right)^3 = 1 \qquad P(k) \longrightarrow P(k+1)$$

<u>Inductive case</u>

Assume it is true for $n=k$

$$1^3 + 2^3 + \cdots + k^3 = \left(\frac{k(k+1)}{2}\right)^2 \quad \cdots\cdots (1)$$

We have to show,

$$1^3 + 2^3 + \cdots + k^3 + (k+1)^3 = \left(\frac{(k+1)(k+2)}{2}\right)^3$$

Adding $(k+1)^3$ to both sides,

$$1^3 + 2^3 + 3^3 + \cdots + k^3 + (k+1)^3 = \left(\frac{k(k+1)}{2}\right)^2 + (k+1)^3$$

$$\Rightarrow (k+1)^2 \left(\frac{k^2}{4} + k + 1 \right)$$

$$\Rightarrow (k+1)^2 \left(\frac{k^2 + 4k + 4}{4} \right)$$

$$\Rightarrow \frac{(k+1)^2 (k^2 + 4k + 4)}{4}$$

$$\Rightarrow \frac{(k+1)^2 (k+2)^2}{4}$$

$$= \left(\frac{(k+1)(k+2)}{2} \right)^2 = R.H.S.$$

Proving a property

$\forall n \geq 1$, $2^{2n} - 1$ is divisible by 3.

<u>Base case</u> $(n=1)$

~~LHS~~ $2^{2 \cdot 1} - 1 = 4 - 1 = 3$

Induction step

~~Assume P(i)~~

[Assume $P(i)$ for some $i \geqslant 1$ and prove $P(i+1)$]

Assume $2^{2i}-1$ is divisible by 3, prove $2^{2(i+1)}-1$ is divisible by 3.

$$2^{2(i+1)}-1 = 2^{2i+2}-1$$

$$= 4\,2^{2i}-1 = 3\cdot 2^{2i} + 2^{2i}-1$$

Ⓑ $\forall n \geqslant 2$, n^3-n is divisible by 6.

Base case ($n=2$)

$2^3-2 = 8-6 = 6$ is divisible by 6.

Induction step

Let (k^3-k) is divisible by 6, it is required to prove that $(k+1)^3-(k+1)$ is divisible by 6.

$$(k+1)^3 - (k+1) = (k+1)\left((k+1)^2 - 1\right)$$
$$= (k+1)(k^2 + 2k + 1 - 1)$$
$$= (k+1)(k^2 + 2k)$$
$$= k^3 + k^2 + 2k^2 + 2k$$
$$= (k^3 - k) + 3k^2 + 3k$$
$$= \underline{(k^3 - k)} + 3(k^2 + k)$$

Here, $(k^3 - k)$ is divisible by 6. The sum of any integer and its square is an even number.

So, $3(k^2 + k)$ is also divisible by $(3 \times 2) = 6$.

Or,

$= k(k+1)(k+2)$

Product of three consecutive numbers.

$$= (k^3 - k) + 3\,\underline{(k^2 + k)}$$

[Divisible by 2 by case analysis]

Proving an Inequality —

$\forall n \geq 3, \ (2n+1) < 2^n$

Base case $(n=3)$

$$2n+1 = 2.3+1 = 7 < 2^n = 8$$

Induction step

Let $2i+1 < 2^i$, prove $2(i+1)+1 < 2^{(i+1)}$

$$2(i+1)+1 = 2i+1+2$$

$$< 2^i + 2 \quad [\text{by induction}]$$

$2^i > 2$

$2 < 2^i$

$$< 2^i + 2^i \quad [\text{since } i >= 3]$$

$$= 2^{(i+1)}$$

Strong Induction

// Prove $P(0)$

Then prove $P(n+1)$ assuming all of

$P(0),\ P(1),\ \cdots\ P(n)$

[Instead of just $P(n)$]

$$P(0) \land P(1) \land \cdots\ P(n) \longrightarrow P(n+1)$$

Claim – Every integer > 1 is a product of primes.

* Every number ≥ 2 is a product of primes.

Base $n=2$

Inductive assumption :

$2 \leq k \leq n$

$P(n) \longrightarrow P(n+1)$

Proof (SI)

→ Base case

→ Suppose the claim is true for all $2 \leq l < n$.

→ Consider an integer n.

→ If n is prime, then done

→ So, $n = k\,m$

$n+1$

prime

Composite

$p \cdot q$

$$\left.\begin{array}{l} 2 \le p \le n \\ 2 \le q \le n \end{array}\right\} \rightarrow P(n+1)$$