Hypothesis Testing

Mahbub Latif, PhD

April 2025

Plan

- Introduction to hypothesis testing
- Test concerning the mean of a normal population
- Test equality of means of two normal populations

Introduction to hypothesis testing

Introduction

- Suppose that a construction firm has just purchased a large supply of cables that have been guaranteed to have an average breaking strength of at least 7,000 pounds per square inch
- To verify this claim, the firm has decided to take a random sample of 10 of these cables to determine their breaking strengths
- They will then use the result of this experiment to ascertain whether or not they accept the cable manufacturer’s hypothesis that the population mean is at least 7,000 pounds per square inch

Introduction

- A statistical hypothesis is a statement about a set of parameters of a population distribution
  - It is called a hypothesis because it is not known whether or not it is true
  - A hypothesis is tested based on a sample drawn from the population of interest
- A primary problem is to develop a procedure for determining whether or not the values of a random sample from this population are consistent with the hypothesis
  - If the random sample is deemed consistent with the hypothesis under consideration, we say that the hypothesis has been "accepted"; otherwise, we say that it has been "rejected."

Introduction

- In accepting a given hypothesis, we are not actually claiming that it is **true** but rather, we are saying that the resulting data appear to be consistent with it
- Consider a particular normally distributed population with an unknown mean value θ and known variance 1.
- The statement $\theta < 1$ is a statistical hypothesis that we want to test based on a sample of size 10 from $N(\theta, 1)$ population
  - Suppose the average sample value is 1.25 $\rightarrow$ not unlikely, sample is consistent with the hypothesis $\rightarrow$ "accept the hypothesis"
  - Suppose the average sample value is 3.0 $\rightarrow$ seems unlikely, sample is inconsistent with the hypothesis $\rightarrow$ "reject the hypothesis"

Testing for gender bias in selecting managers

- A large supermarket chain periodically selects employees to receive management training
- A group of women employees recently claimed that the company selects males at a disproportionately high rate for such training
- The company denied this claim, i.e., employees are selected randomly for the training
  - How could the women employees statistically back up their assertion?
- Suppose 9 out of 10 employees selected for the most recent training were male
  - Would it (9 out of 10 were male) be really unusual if employees were selected randomly?

Significance Levels

Null hypothesis

- Suppose the population is distributed as $F_{\boldsymbol{\theta}}$, the population is completely specified by the parameter vector $\boldsymbol{\theta}$
- The hypothesis of interest is known as "null hypothesis", which is denoted by H_0
  - A null hypothesis is called a simple null hypothesis if it takes only a single value of the parameter; otherwise it is known as a composite hypothesis
    - $H_0 : \theta = 1 \ \rightarrow$ a simple null hypothesis
    - $H_0 : \theta > 1 \ \rightarrow$ a composite null hypothesis
- The null hypothesis is tested against another hypothesis, which is known as the alternative hypothesis (denoted by H_1)
  - Rejection of null hypothesis $\rightarrow$ acceptance of alternative hypothesis

Critical region

- A specific null hypothesis H_0 is tested on the basis of a random sample $X_1, \ldots, X_n$ drawn from the population $F_{\boldsymbol{\theta}}$
- A test for H_0 is developed by defining a region C, known as the critical region, so that
  - Accept H_0 if $(X_1, \ldots, X_n) \notin C$
  - Reject H_0 if $(X_1, \ldots, X_n) \in C$
- For example, if the population is $N(\theta, 1)$ and $H_0 : \theta = 1$, the critical region is defined as

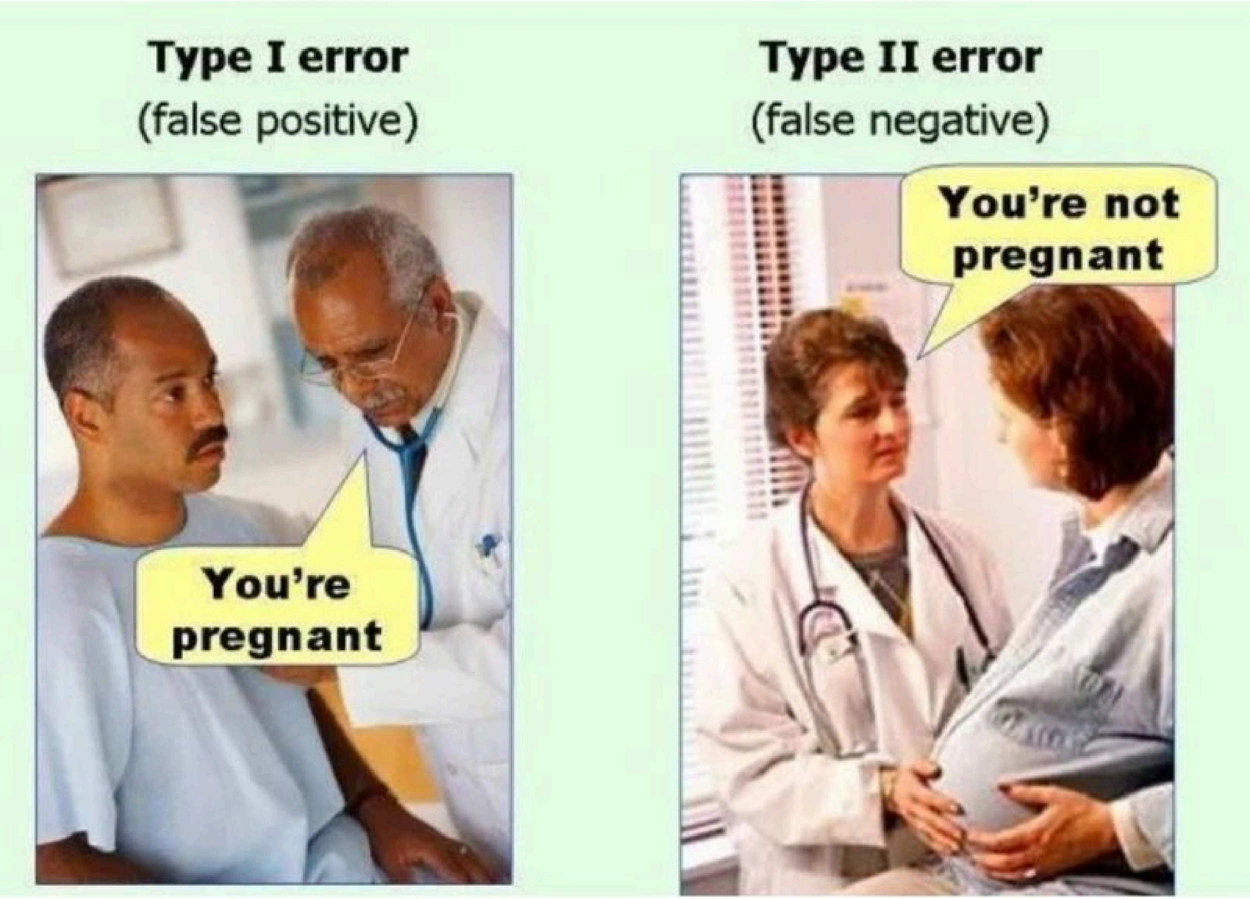
$$C = \left\{ (X_1, \ldots, X_n) : \left| \sum_{i=1}^{n} (X_i/n) - 1 \right| > \frac{1.96}{\sqrt{n}} \right\}$$

Level of significance

- Two types of error can occur in developing a hypothesis test for a null hypothesis H_0
  - Type I error occurs if the test incorrectly rejects H_0 when it is indeed correct (False positive)
  - Type II error occurs if the test incorrectly accepts H_0 when it is false (False negative)
- The probability of type I error is known as the level of significance, which is denoted by α

$$\alpha = P\left(\text{reject } H_0 \,\middle|\, H_0 \text{ is true}\right)$$

  - One approach of the hypothesis test is to fix a level of significance in advance, so the test has the property that probability of occurring type I error can never be greater than α
  - Commonly chosen values: $\alpha = 0.1, 0.05, 0.005$


Type I error
(false positive)
You're pregnant
Type II error
(false negative)
You're not pregnant

Test regarding the mean of a normal population

Test regarding the mean of a normal population

- Let $X_1, \ldots, X_n$ be a random sample from a normal distribution with mean μ (unknown) and variance σ^2 (known)

- We are interested in testing hypotheses

$$H_0 : \mu = \mu_0 \;\; against \;\; H_1 : \mu \neq \mu_0$$

  - μ_0 is known

- Since $\bar{X}$ is a point estimator of μ, it is reasonable to accept the null hypothesis $H_0 : \mu = \mu_0$ if $\bar{X}$ is not far from μ_0, so the critical region for the test would be of the form

$$C = \left\{(X_1, \ldots, X_n) : \left|\bar{X} - \mu_0\right| > c\right\}$$

  - $c \; \rightarrow$ a suitably chosen value

Test regarding the mean of a normal population

- For a given value of the level of significance α, the value of c can be determined
  - The critical value c must satisfy

$$P\left(\left|\bar{X}-\mu_0\right| > c \,\middle|\, H_0 \text{ is true}\right) = \alpha$$

- Sampling distribution of $\bar{X}$

$$\bar{X} \sim N\left(\mu, \sigma^2/n\right) \;\text{ and if } H_0 \text{ is true} \rightarrow \;\bar{X} \sim N\left(\mu_0, \sigma^2/n\right)$$

- So, the critical value c must satisfy

$$P\left(\left|\frac{\bar{X}-\mu_0}{\sigma/\sqrt{n}}\right| > \frac{c\sqrt{n}}{\sigma}\right) = \alpha \Rightarrow P\left(|Z| > \frac{c\sqrt{n}}{\sigma}\right) = \alpha$$

Test regarding the mean of a normal population

- It can be shown that

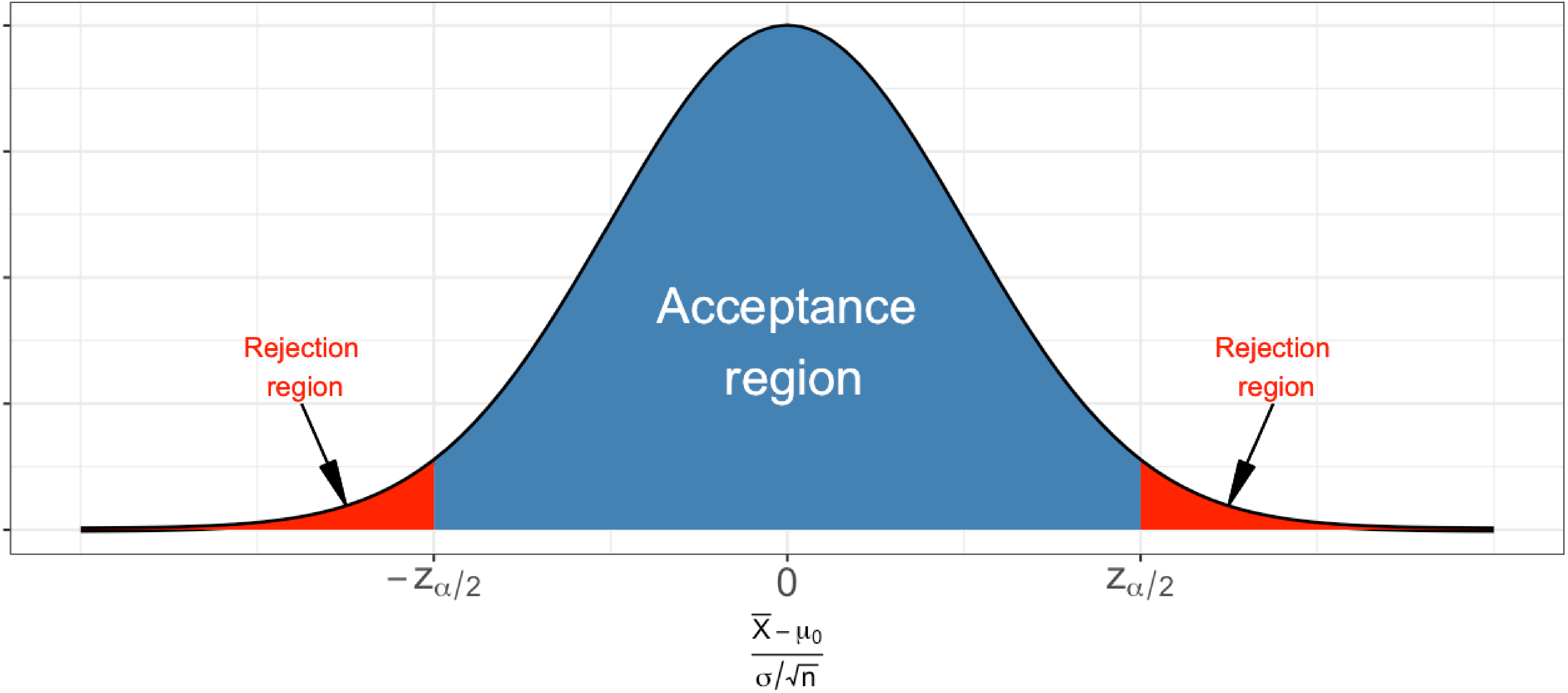
$$P\left(|Z| > \frac{c\sqrt{n}}{\sigma}\right) = \alpha \Rightarrow 2P\left(Z > \frac{c\sqrt{n}}{\sigma}\right) = \alpha$$

$$P\left(Z > \frac{c\sqrt{n}}{\sigma}\right) = \alpha/2 \Rightarrow \frac{c\sqrt{n}}{\sigma} = z_{\alpha/2} \Rightarrow c = \frac{\sigma z_{\alpha/2}}{\sqrt{n}}$$

- **Decision**: Reject $H_0 : \mu = \mu_0$ if

$$|\bar{X} - \mu_0| > \frac{\sigma z_{\alpha/2}}{\sqrt{n}} \Rightarrow \left|\frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}}\right| > z_{\alpha/2}$$

  - Otherwise, don't reject H_0


Acceptance
region
Rejection
region
Rejection
region
$-z_{\alpha/2}$
0
$z_{\alpha/2}$
$\frac{\overline{X} - \mu_0}{\sigma/\sqrt{n}}$

Test regarding the mean of a normal population

- Population $N(\mu, \sigma^2)$, where σ is known
  - Sample $X_1, \dots, X_n$
- Hypotheses

$$H_0 : \mu = \mu_0 \;\; versus \;\; H_1 : \mu \neq \mu_0$$

- Test statistic

$$Z = \frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}}$$

  - $Z \sim N(0,1)$ under H_0

- Decision rules at α level of significance
  - Reject H_0 at α level of significance if

$$|Z| > z_{\alpha/2}$$

  - Don't reject H_0 at α level of significance if

$$|Z| \leq z_{\alpha/2}$$

Example 8.3a

- It is known that if a signal of value μ is sent from location A, then the value received at location B is normally distributed with mean μ and standard deviation 2.
- There is reason for the people at location B to suspect that the signal value $\mu = 8$ will be sent today
- Test this hypothesis if the same signal value is independently sent five times and the average value received at location B is $\bar{X} = 9.5$.

Example 8.3a

- It is known that if a signal of value μ is sent from location A, then the value received at location B is normally distributed with mean μ and standard deviation 2.
- There is reason for the people at location B to suspect that the signal value $\mu = 8$ will be sent today
- Test this hypothesis if the same signal value is independently sent five times and the average value received at location B is $\bar{X} = 9.5$.

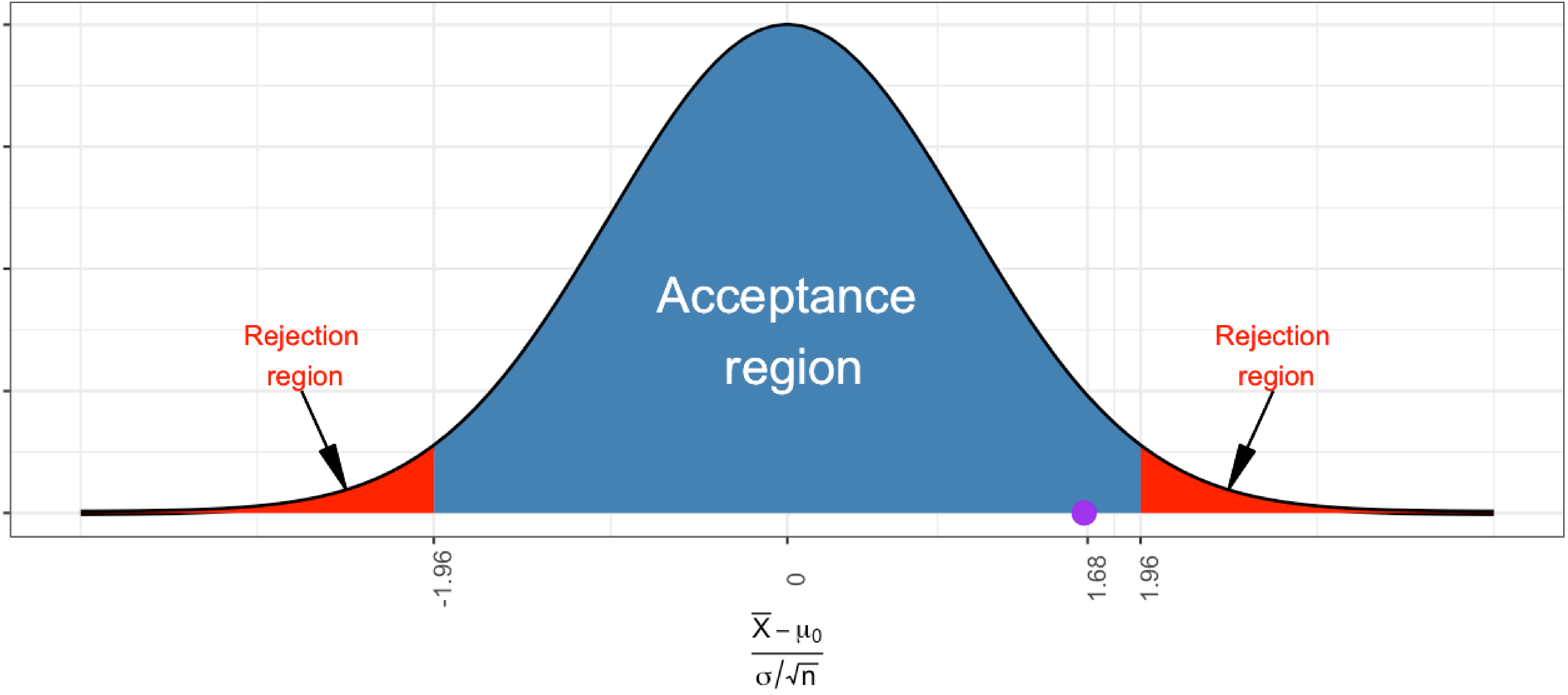
- Hypotheses

$$H_0 : \mu = 8 \;\; vs \;\; H_1 : \mu \neq 8$$

- Test statistic

$$Z = \frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}} = \frac{9.5 - 8}{2/\sqrt{5}} = 1.68$$

- Level of significance $\alpha = .05$
- Since, $|Z| = 1.68 < z_{.025} = 1.96$
  - We cannot reject H_0 at 5% level of significance


Acceptance
region
Rejection
region
Rejection
region
-1.96
0
1.68
1.96
$\frac{\overline{X} - \mu_0}{\sigma/\sqrt{n}}$

Example 8.3a

- It is known that if a signal of value μ is sent from location A, then the value received at location B is normally distributed with mean μ and standard deviation 2.
- There is reason for the people at location B to suspect that the signal value $\mu = 8$ will be sent today
- Test this hypothesis if the same signal value is independently sent five times and the average value received at location B is $\bar{X} = 9.5$.

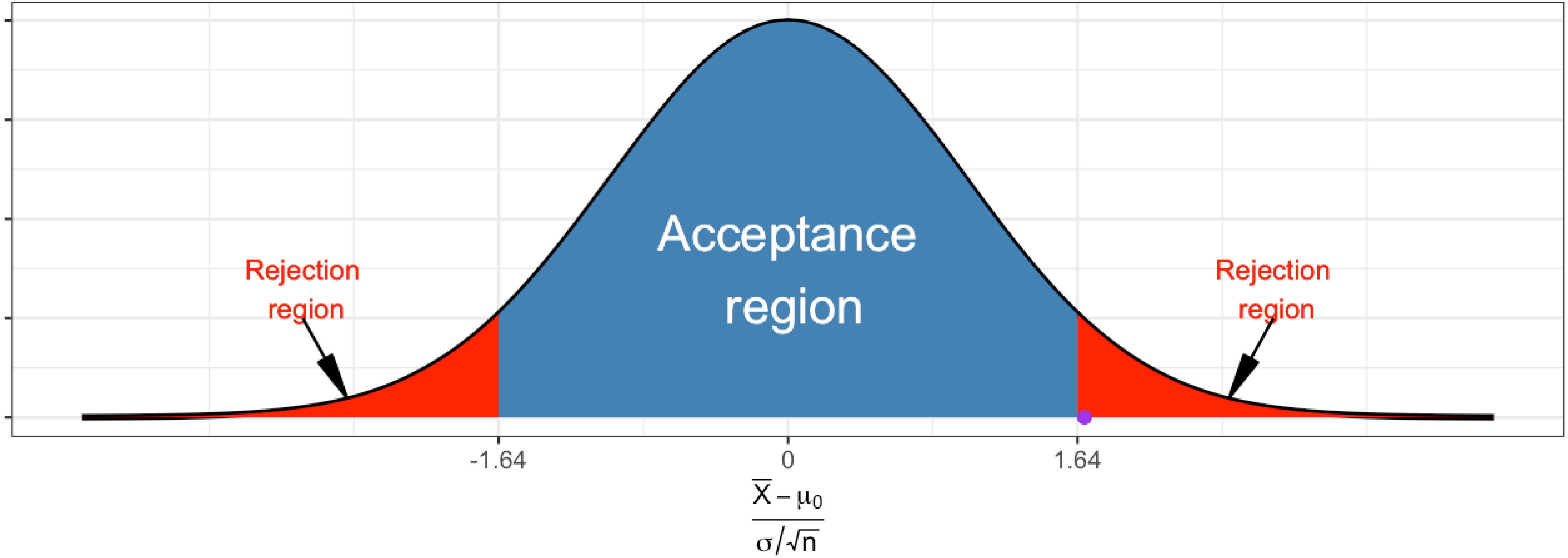
- Hypotheses

$$H_0 : \mu = 8 \;\; vs \;\; H_1 : \mu \neq 8$$

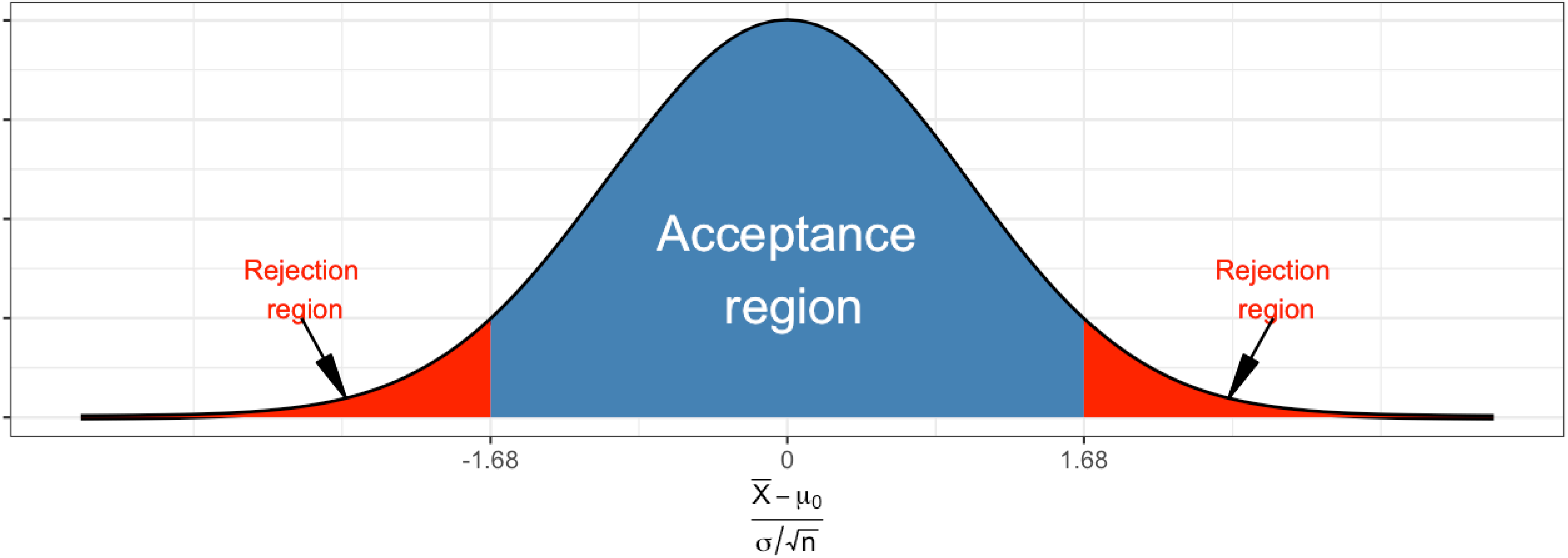
- Test statistic

$$Z = \frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}} = \frac{9.5 - 8}{2/\sqrt{5}} = 1.68$$

- Level of significance $\alpha = .10$
- Since, $|Z| = 1.68 > z_{.05} = 1.64$
  - We can reject H_0 at 10% level of significance



- Is it possible to reject the H_0 at lower that 10% of level of significance when the calculated value of the test statistic is 1.68?



- Is it possible to reject the H_0 at a lower than 10% of level of significance when the calculated value of the test statistic is 1.68?
- p-value, $p = 2P(|Z| > 1.68) = 0.093 \rightarrow$ smallest significance level to reject H_0

Example 8.3a

- It is known that if a signal of value μ is sent from location A, then the value received at location B is normally distributed with mean μ and standard deviation 2.
- There is reason for the people at location B to suspect that the signal value $\mu = 8$ will be sent today
- Test this hypothesis if the same signal value is independently sent five times and the average value received at location B is $\bar{X} = 9.5$.
- Test the null hypothesis $H_0 : \mu = 8$ when the sample mean obtained from five observations is 8.5
- Test the null hypothesis $H_0 : \mu = 8$ when the sample mean obtained from five observations is 11.5

Probability of Type II error

- The probability of type II error is the probability of accepting (not rejecting) the null hypothesis, $H_0 : \mu = \mu_0$, when the null hypothesis is not true, i.e. $\mu \neq \mu_0$

- It is denoted by $\beta(\mu)$ and is defined as

$$\begin{aligned} \beta(\mu) &= P\big(\text{accepting } H_0 \,|\, H_0 \text{ false}\big) \\ &= P\big(\text{accepting } H_0 \,|\, \mu \neq \mu_0\big) \end{aligned}$$

  - $\beta(\mu) \rightarrow$ operating characteristic (OC) curve
  - $\beta(\mu) \rightarrow$ a function of the true mean μ

- General notation

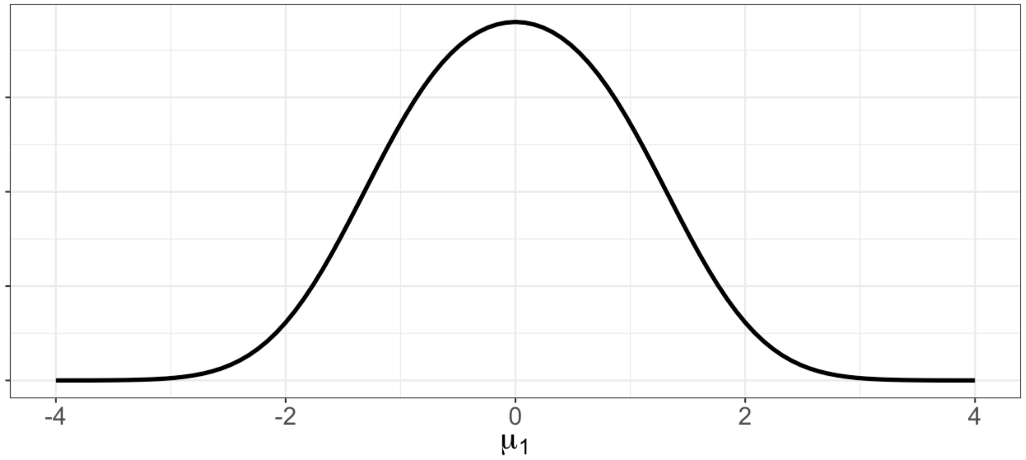
$$\beta(\mu_1) = P\big(\text{accepting } H_0 \,|\, \mu = \mu_1 \neq \mu_0\big)$$

$$
\begin{aligned}
\beta(\mu_1) &= P\big(\text{accepting } H_0 \mid \mu = \mu_1 \neq \mu_0\big) \\
&= P\left(-z_{\alpha/2} < \frac{\bar{X}-\mu_0}{\sigma/\sqrt{n}} < z_{\alpha/2} \,\middle|\, \mu = \mu_1\right) \\
&= P\left(-z_{\alpha/2} - \frac{\mu_1}{\sigma/\sqrt{n}} < \frac{\bar{X}-\mu_0-\mu_1}{\sigma/\sqrt{n}} < z_{\alpha/2} - \frac{\mu_1}{\sigma/\sqrt{n}}\right) \\
&= P\left(-z_{\alpha/2} - \frac{\mu_1}{\sigma/\sqrt{n}} + \frac{\mu_0}{\sigma/\sqrt{n}} < \frac{\bar{X}-\mu_1}{\sigma/\sqrt{n}} < z_{\alpha/2} - \frac{\mu_1}{\sigma/\sqrt{n}} + \frac{\mu_0}{\sigma/\sqrt{n}}\right) \\
&= P\left(\frac{\mu_0-\mu_1}{\sigma/\sqrt{n}} - z_{\alpha/2} < \frac{\bar{X}-\mu_1}{\sigma/\sqrt{n}} < \frac{\mu_0-\mu_1}{\sigma/\sqrt{n}} + z_{\alpha/2}\right) \\
&= \Phi\left(\frac{\mu_0-\mu_1}{\sigma/\sqrt{n}} + z_{\alpha/2}\right) - \Phi\left(\frac{\mu_0-\mu_1}{\sigma/\sqrt{n}} - z_{\alpha/2}\right)
\end{aligned}
$$

- Population $N(\mu, 3^2)$

- The null and alternative hypotheses

$$H_0 : \mu = 0 \;\; vs \;\; \mu = \mu_1 \neq 0$$



- The OC curve

1 - β = p(rejecting H I H is false)

$$\beta(\mu_1) = \Phi\left(\frac{\mu_0 - \mu_1}{\sigma/\sqrt{n}} + z_{\alpha/2}\right) - \Phi\left(\frac{\mu_0 - \mu_1}{\sigma/\sqrt{n}} - z_{\alpha/2}\right)$$

- $\left[1 - \beta(\mu)\right] \rightarrow$ the power function, which is the probability of rejecting H_0 when μ is the true value

- Calculate the power of test $H_1 : \mu_0 = 8$, for $n = 5$, $H_1 : \mu = 10$, $\alpha = .05$, and $\sigma = 2$

One-sided tests

- The null and alternative hypotheses

$$H_0 : \mu = \mu_0 \;\; vs \;\; H_1 : \mu > \mu_0$$

- The critical region

$$C = \left\{(X_1, \ldots, X_n) : \bar{X} - \mu_0 > c\right\}$$

- Under $H_0 : \mu = \mu_0$

$$\bar{X} \sim N\left(\mu_0, \sigma^2/n\right)$$

- The probability of type I error

$$\alpha = P\left(\bar{X} - \mu_0 > c \,|\, \mu = \mu_0\right)$$
$$= P\left(Z > \frac{c\sqrt{n}}{\sigma}\right)$$
$$\Rightarrow \quad c = \frac{\sigma z_\alpha}{\sqrt{n}}$$

- Reject H_0 if

$$\bar{X} - \mu_0 > \frac{\sigma z_\alpha}{\sqrt{n}} \quad \Rightarrow \quad \frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}} > z_\alpha$$

One-sided test regarding the mean of a normal population

- Population $N(\mu, \sigma^2)$, where σ is known
  - Sample $X_1, \ldots, X_n$
- Hypotheses

$$H_0 : \mu = \mu_0 \;\; versus \;\; H_1 : \mu > \mu_0$$

- Test statistic

$$Z = \frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}}$$

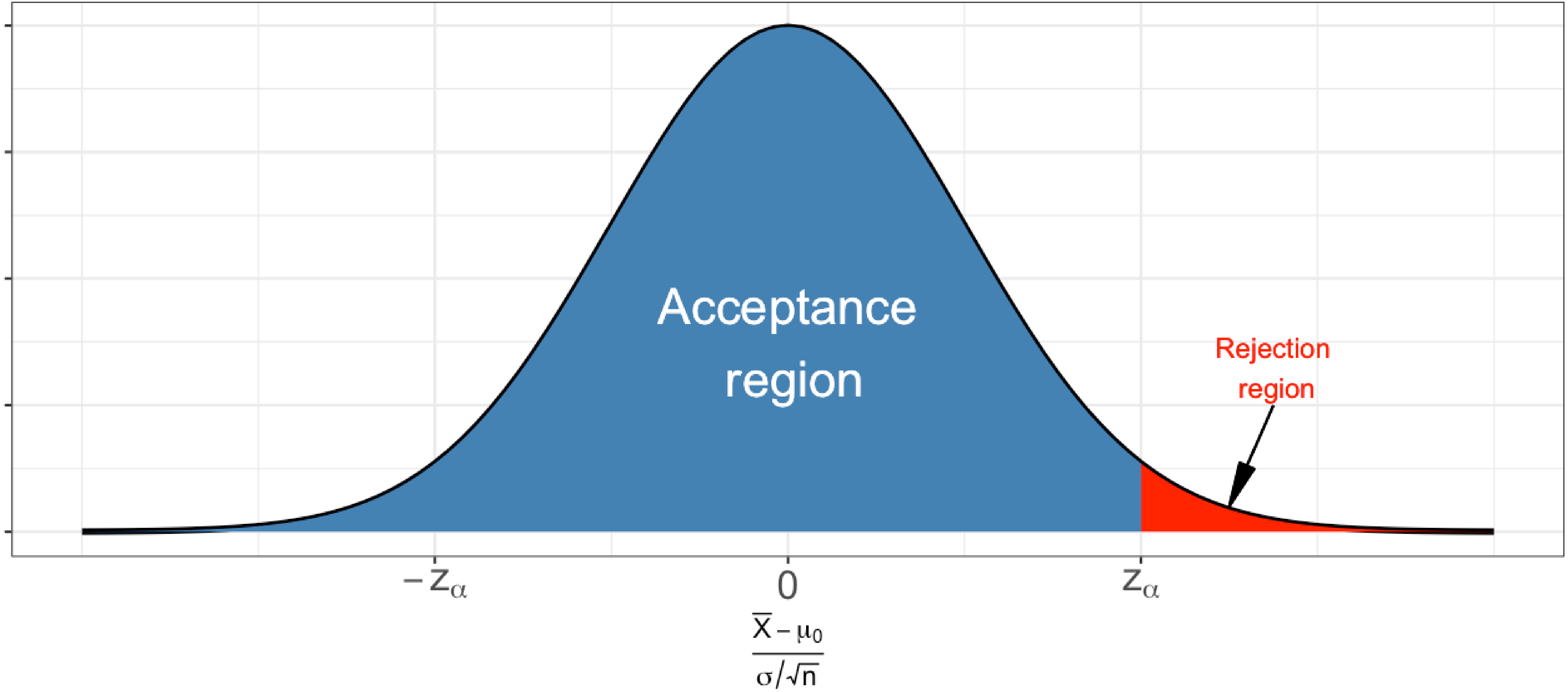
  - $Z \sim N(0, 1)$ under H_0

- Decision rules at α level of significance
  - Reject H_0 at α level of significance if

$$Z > z_\alpha$$

  - Don't reject H_0 at α level of significance if

$$Z \leq z_\alpha$$


Acceptance
region
Rejection
region
$-z_\alpha$
0
z_α
$\frac{\overline{X}-\mu_0}{\sigma/\sqrt{n}}$

Example 8.3a

- It is known that if a signal of value is normally distributed with mean μ and standard deviation 2.

- There is reason for the people at location B to suspect that the signal value $\mu = 8$ will be sent today

- It is known in advance that the value is at least as large as 8

- Test this hypothesis if the same signal value is independently sent five times and the average value received at location B is $\bar{X} = 9.5$.

- Hypotheses

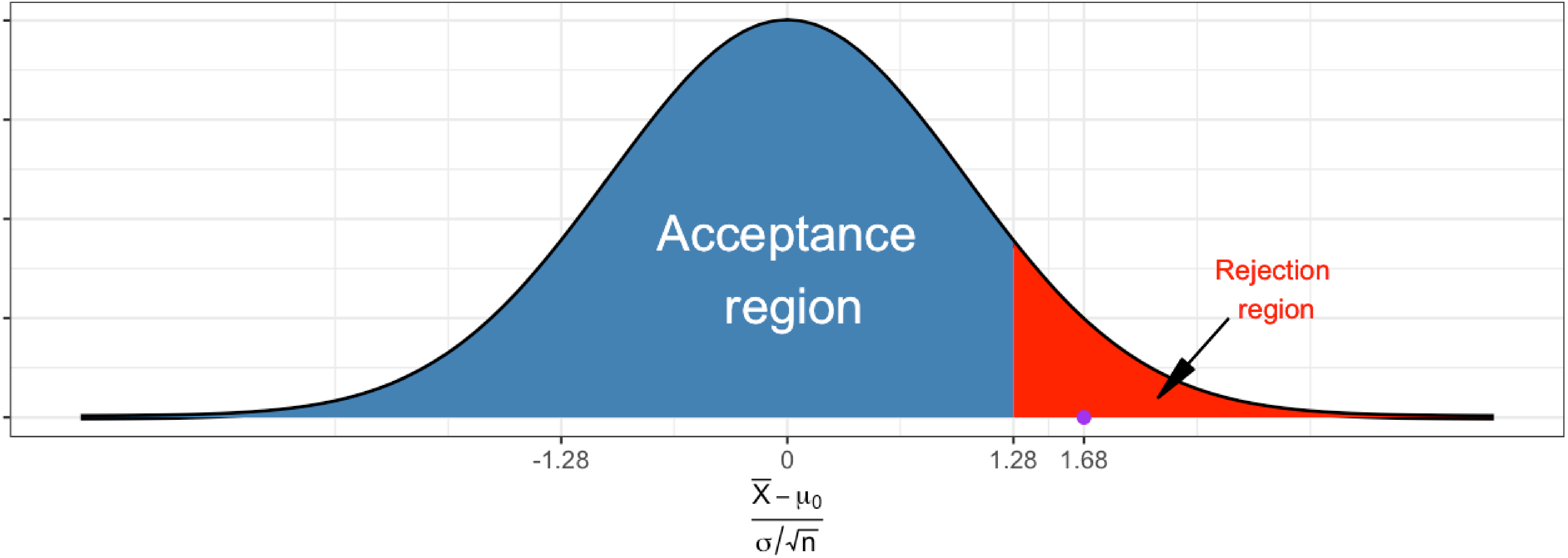
$$H_0 : \mu = 8 \;\; vs \;\; H_1 : \mu > 8$$

- Test statistic

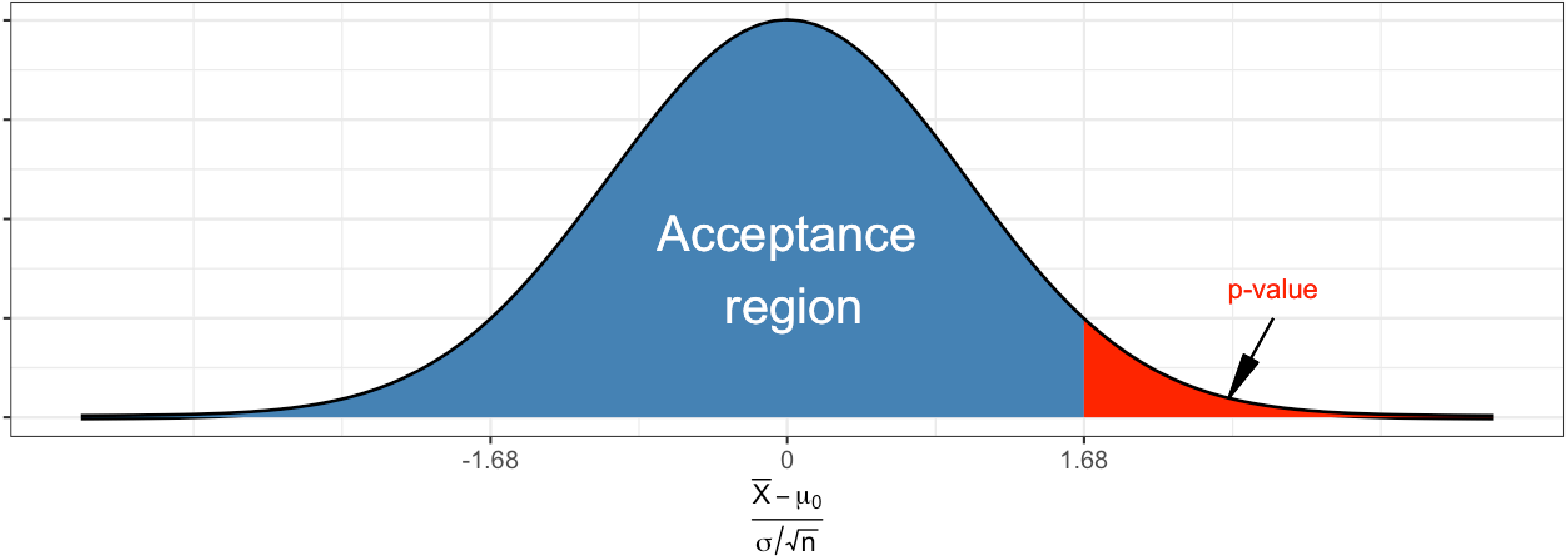
$$Z = \frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}} = \frac{9.5 - 8}{2/\sqrt{5}} = 1.68$$

- Level of significance $\alpha = .10$

- Since, $|Z| = 1.68 > z_{.10} = 1.28$
    - We can reject H_0 at 10% level of significance



- What would be the p-value of the test?



- What is the p-value of the test
- p-value, $p = P(Z > 1.68) = 0.046 \;\rightarrow$ smallest significance level to reject H_0

Power of test (one-sided alternative)

- The OC curve for one-sided test

$$
\begin{aligned}
\beta(\mu_1) &= P\big(\text{accepting } H_0 \mid \mu \neq \mu_0\big) = P\left(\frac{\bar{X}-\mu_0}{\sigma/\sqrt{n}} < z_\alpha \,\middle|\, \mu = \mu_1\right) \\
&= P\left(\frac{\bar{X}-\mu_0-\mu_1}{\sigma/\sqrt{n}} < z_\alpha - \frac{\mu_1}{\sigma/\sqrt{n}}\right) = P\left(\frac{\bar{X}-\mu_1}{\sigma/\sqrt{n}} < z_\alpha - \frac{\mu_1}{\sigma/\sqrt{n}} + \frac{\mu_0}{\sigma/\sqrt{n}}\right) \\
&= P\left(Z < z_\alpha + \frac{\mu_0-\mu_1}{\sigma/\sqrt{n}}\right) = \Phi\left(z_\alpha + \frac{\mu_0-\mu_1}{\sigma/\sqrt{n}}\right)
\end{aligned}
$$

- Power of the test

$$
1-\beta(\mu_1) = 1-\Phi\left(z_\alpha + \frac{\mu_0-\mu_1}{\sigma/\sqrt{n}}\right)
$$

Example 8.3f

- All cigarettes have an average nicotine content of at least 1.6 mg per cigarette.
- A firm that produces cigarettes claims that the average nicotine content of a cigarette is less than 1.6 mg.
- To test this claim, a sample of 20 of the firm’s cigarettes was analyzed.
- If the standard deviation of a cigarette’s nicotine content is .8 mg, what conclusions can be drawn, at the 5 percent level of significance, if the average nicotine content of the 20 cigarettes is 1.54?

Confidence interval and hypothesis test

- There is a direct analogy between confidence interval estimation and hypothesis testing
- The $100(1-\alpha)$ percent confidence interval for μ

$$\bar{X} \pm z_{\alpha/2}\frac{\sigma}{\sqrt{n}} \;\Rightarrow\; P\left\{\mu \in \left(\bar{X} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}}, \bar{X} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right)\right\} = 1-\alpha$$

  - Under $H_0 : \mu = \mu_0$, $P\left\{\mu_0 \in \left(\bar{X} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}}, \bar{X} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right)\right\} = 1-\alpha$

- To test $H_0 : \mu = \mu_0$ against $H_1 : \mu \neq \mu_0$ at α level of significance, the critical region will be

$$\mu_0 \notin \left(\bar{X} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}}, \bar{X} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right)$$

Summary

- Hypothesis testing about the mean of a normal population (μ) when population variance (σ^2) is known

H_0	H_1	**Test stat, z**	**Reject H_0 if**	**p-value**
$\mu = \mu_0$	$\mu \neq \mu_0$	$\sqrt{n}(\bar{X} - \mu_0)/\sigma$	$\lvert z \rvert > z_{\alpha/2}$	$2P(Z \geq \lvert z \rvert)$
$\mu = \mu_0$	$\mu > \mu_0$	$\sqrt{n}(\bar{X} - \mu_0)/\sigma$	$z > z_\alpha$	$P(Z \geq z)$
$\mu = \mu_0$	$\mu < \mu_0$	$\sqrt{n}(\bar{X} - \mu_0)/\sigma$	$z < -z_\alpha$	$P(Z \leq z)$

Hypothesis test regarding normal mean when the population variance is unknown

t-test

- Let $X_1, \ldots, X_n$ be a random sample from $N(\mu, \sigma^2)$
  - μ and σ^2 are unknown
- The null and alternative hypotheses

$$H_0 : \mu = \mu_0 \;\; vs \;\; H_1 : \mu \neq \mu_0$$

- Point estimates
  - $\hat{\mu} = \bar{X}$
  - $\hat{\sigma}^2 = S^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X})^2$

- Under H_0, $\bar{X} \sim N(\mu_0, \sigma^2/n)$

$$\frac{\bar{X} - \mu_0}{S/\sqrt{n}} \sim t_{n-1}$$

- Critical region

$$C = \{(X_1, \ldots, X_n) : |\bar{X} - \mu_0| > c\}$$

t-test

- Level of significance

$$\alpha = P\big(|\bar{X} - \mu_0| > c \,\big|\, \mu = \mu_0\big)$$
$$= P\left(\left|\frac{\bar{X} - \mu_0}{S/\sqrt{n}}\right| > \frac{c\sqrt{n}}{S}\right)$$

- It can be shown that

$$t_{n-1,\alpha/2} = \frac{c\sqrt{n}}{S} \;\Rightarrow\; c = \frac{t_{n-1,\alpha/2}S}{\sqrt{n}}$$

- Reject H_0 if

$$|\bar{X} - \mu_0| > c = \frac{t_{n-1,\alpha/2}S}{\sqrt{n}}$$
$$|t| = \left|\frac{\bar{X} - \mu_0}{S/\sqrt{n}}\right| > t_{n-1,\alpha/2}$$

- Cannot reject H_0 if

$$|t| = \left|\frac{\bar{X} - \mu_0}{S/\sqrt{n}}\right| \le t_{n-1,\alpha/2}$$

- $\text{p-value} = 2P\big(T_{n-1} > |t|\big)$

One-sided t-test

- The null and alternative hypotheses

$$H_0 : \mu = \mu_0 \;\; vs \;\; H_1 : \mu > \mu_0$$

- Reject H_0 if

$$t = \frac{\bar{X} - \mu_0}{S/\sqrt{n}} > t_{n-1,\alpha}$$

  - Otherwise cannot reject H_0

- $\text{p-value} = P\big(T_{n-1} > t\big)$

- The null and alternative hypotheses

$$H_0 : \mu = \mu_0 \;\; vs \;\; H_1 : \mu < \mu_0$$

- Reject H_0 if

$$t = \frac{\bar{X} - \mu_0}{S/\sqrt{n}} < -t_{n-1,\alpha}$$

  - Otherwise cannot reject H_0

- $\text{p-value} = P\big(T_{n-1} < t\big)$

Summary

- Hypothesis testing about the mean of a normal population (μ) when population variance (σ^2) is unknown

- Point estimates

  - $\hat{\mu} = \bar{X}$ and $\hat{\sigma}^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$

H_0	H_1	**Test stat, t**	**Reject H_0 if**	**p-value**
$\mu = \mu_0$	$\mu \neq \mu_0$	$\sqrt{n}(\bar{X} - \mu_0)/S$	$\lvert T \rvert > t_{n-1,\alpha/2}$	$2P(T_{n-1} \geq \lvert t \rvert)$
$\mu = \mu_0$	$\mu > \mu_0$	$\sqrt{n}(\bar{X} - \mu_0)/S$	$T > t_{n-1,\alpha}$	$P(T_{n-1} \geq t)$
$\mu = \mu_0$	$\mu < \mu_0$	$\sqrt{n}(\bar{X} - \mu_0)/S$	$T < -t_{n-1,\alpha}$	$P(T_{n-1} \leq t)$

Testing equality of means of two normal populations

Case I: Variances are known

- Samples are drawn from two populations
  - $X_1, \ldots, X_n$ from $N(\mu_x, \sigma_x^2)$
  - $Y_1, \ldots, Y_m$ from $N(\mu_y, \sigma_y^2)$
  - σ_x^2 and σ_y^2 are known
- The null and alternative hypotheses

$$H_0 : \mu_x = \mu_y \ vs \ H_1 : \mu_x \neq \mu_y$$

$$H_0 : \mu_x - \mu_y = 0 \ vs \ H_1 : \mu_x - \mu_y \neq 0$$

- Point estimates
  - $\bar{X} = \frac{1}{n} \sum_{i=1}^{n} X_i$
  - $\bar{Y} = \frac{1}{m} \sum_{i=1}^{n} Y_i$
- Decision rules
  - Reject H_0 if $|\bar{X} - \bar{Y}| > c$
  - Accept H_0 if $|\bar{X} - \bar{Y}| \leq c$

Case I: Variances are known

- Sampling distribution of $\bar{X} - \bar{Y}$

$$\bar{X} - \bar{Y} \sim N\left(\mu_x - \mu_y, \frac{\sigma_x^2}{n} + \frac{\sigma_y^2}{m}\right)$$

$$\bar{X} - \bar{Y} \sim N\left(0, \frac{\sigma_x^2}{n} + \frac{\sigma_y^2}{m}\right) \text{ Under } H_0$$

- The test statistic

$$Z = \frac{\bar{X} - \bar{Y}}{\sqrt{\frac{\sigma_x^2}{n} + \frac{\sigma_y^2}{m}}} \sim N(0,1) \text{ under } H_0$$

- Decision rules to test $H_0 : \mu_x = \mu_y$ at α level of significance
  - Reject H_0 if

$$|Z| = \frac{|\bar{X} - \bar{Y}|}{\sqrt{\frac{\sigma_x^2}{n} + \frac{\sigma_y^2}{m}}} > z_{\alpha/2}$$

  - Do not reject (Accept) H_0 if

$$|Z| = \frac{|\bar{X} - \bar{Y}|}{\sqrt{\frac{\sigma_x^2}{n} + \frac{\sigma_y^2}{m}}} \leq z_{\alpha/2}$$

Example 8.4a

- To compare two new tire production methods, a tire manufacturer produces a sample of 10 tires using the first method and a sample of 8 using the second
- The first set will be road tested at location A and the second at location B.
- It is known from the past experience that the lifetime of a tire that is road tested at one of these locations is normally distributed with a mean life due to the tire but with a variance due to the location.
- It is known that the lifetimes of tires tested at location A are normal with a standard deviation equal to 4,000 kilometers, whereas those tested at location B are normal with a standard deviation of 6,000 kilometers.
- To test the hypothesis that there is no appreciable difference in the mean life of tires produced by either method, what conclusion should be drawn at the 5 percent level of significance.

Example 8.4a

- Tyre tested at location A

```
##  [1] 61.1 58.2 62.3 64.0 59.7 66.2 57.8 61.4 62.2 63.6
```

- Tyre tested at location B

```
## [1] 62.2 56.6 66.4 56.2 57.4 58.4 57.6 65.4
```

- Sample means
    - Tyre A: 61.65
    - Tyre B: 60.03

Case II: Variances are unknown and assumed equal

- Samples are drawn from two populations
  - $X_1, \ldots, X_n$ from $N(\mu_x, \sigma_x^2)$
  - $Y_1, \ldots, Y_m$ from $N(\mu_y, \sigma_y^2)$
  - $\sigma_x^2 = \sigma_y^2 = \sigma^2$
- The null and alternative hypotheses

$$H_0 : \mu_x = \mu_y \ vs \ H_1 : \mu_x \neq \mu_y$$

- Point estimates
  - $\bar{X} = \frac{1}{n} \sum_{i=1}^{n} X_i$
  - $\bar{Y} = \frac{1}{m} \sum_{i=1}^{n} Y_i$
  - $\hat{\sigma}^2 = S_p^2 = \frac{(n-1)S_x^2 + (m-1)S_y^2}{n+m-2}$
  - $S_x^2 = \frac{1}{n-1} \sum (X_i - \bar{X})^2$
  - $S_y^2 = \frac{1}{m-1} \sum (Y_i - \bar{Y})^2$

Case II: Variances are unknown and assumed equal

- The test statistic

$$T = \frac{\bar{X} - \bar{Y}}{S_p\sqrt{\frac{1}{n} + \frac{1}{m}}} \sim t_{n+m-2} \text{ under } H_0$$

- Decision rules to test $H_0 : \mu_x = \mu_y$ at α level of significance **Reject H_0 if**

$$|T| = \frac{|\bar{X} - \bar{Y}|}{S_p\sqrt{\frac{1}{n} + \frac{1}{m}}} > t_{n+m-2,\alpha/2}$$

  - Otherwise, **do not reject** (accept) H_0
  - $\text{p-value} = 2P\big(T_{n+m-2} \geq |t|\big)$

Variances are unknown and assumed to be unequal

- Samples are drawn from two populations
  - $X_1, \ldots, X_n$ from $N(\mu_x, \sigma_x^2)$
  - $Y_1, \ldots, Y_m$ from $N(\mu_y, \sigma_y^2)$
- The null and alternative hypotheses

$$H_0 : \mu_x = \mu_y \; vs \; H_1 : \mu_x \neq \mu_y$$

- Test statistic

$$Z = \frac{\bar{X} - \bar{Y}}{\sqrt{\frac{S_x^2}{n} + \frac{S_y^2}{m}}} \sim N(0,1) \text{ under } H_0$$

  - m and n are large!
- Reject H_0 at α level of significance if

$$|Z| > z_{\alpha/2}$$

Example 8.4b

- Twenty-two volunteers at a cold research institute caught a cold after having been exposed to various cold viruses.
- A random selection of 10 volunteers was given tablets containing 1 gram of vitamin C.
- The control group of the other 12 volunteers was given placebo tablets that looked and tasted exactly the same as the vitamin C tablets.
- The length of time the cold lasted was then recorded.
- Treated with Vitamin C (10 subjects)

```
##  [1] 5.5 6.0 7.0 6.0 7.5 6.0 7.5 5.5 7.0 6.5
```

- Treated without Vitamin C (12 subject)

```
## 6.5 6 8.5 7 6.5 8 7.5 6.5 7.5 6
##  8.5 7
```

- Is there any significant effect of Vitamin C on the time the cold lasted?

The paired t-test

- Suppose the interest is to determine whether the installation of a certain antipollution device will affect a car's mileage
- An experiment is planned, and n car's are selected for the experiment
  - $X_i \rightarrow$ gas consumption of ith car before installation of the device
  - $Y_i \rightarrow$ gas consumption of ith car after installation of the device

The paired t-test

- Let us define $W_i = X_i - Y_i$, the difference of mileage before and after installation of the device
  - Assume $W_i \sim N(\mu_w, \sigma_w^2)$
- To test the null hypothesis that there is no effect of the device on car's mileage

$$H_0 : \mu_w = 0 \;\; vs \;\; H_1 : \mu_w \neq 0$$

The paired t-test

- Test statistic

$$Z = \frac{\bar{W}}{S_w/\sqrt{n}} \sim N(0,1) \ \text{ under } H_0$$

  - $\bar{W} = \sum_{i=1}^{n} W_i/n$
  - $S_w^2 = \frac{1}{n-1}\sum(W_i - \bar{W})$

- Reject H_0 at α level of significance

$$|Z| > z_{\alpha/2}$$

- Otherwise, we cannot reject H_0
- $\text{p-value} = 2P(Z > |z|)$

Example 8.4d

- An industrial safety program was recently instituted in the computer chip industry.
- The average weekly loss in labor hours due to accidents in 10 similar plants before and after the program is available.
- Determine, at the 5 percent significance level, whether the safety program has been proven effective.
- Let W be the difference in weekly loss before and after introducing the safety program
  - Assume $W \sim N(\mu_w, \sigma_w^2)$
- The null and alternative hypothesis

$$H_0 : \mu_w = 0 \ \ vs \ \ H_1 : \mu_w \neq 0$$

Example 8.4d

Plant	Before	After	W = After - Before
1	30.5	23.0	-7.5
2	18.5	21.0	2.5
3	24.5	22.0	-2.5
4	32.0	28.5	-3.5
5	16.0	14.5	-1.5
6	15.0	15.5	0.5
7	23.5	24.5	1.0
8	25.5	21.0	-4.5
9	28.0	23.5	-4.5
10	18.0	16.5	-1.5

- Test statistic

$$Z = \frac{\bar{W}}{S_w/\sqrt{n}} = \frac{-2.15}{3/\sqrt{10}} = -2.2659$$

- Since

$$|Z| = 2.2659 > z_{\alpha/2} = z_{.025} = 1.96$$

we can reject the H_0, i.e., the safety program has a significant effect on labor-hours loss.

- What is the p-value of the test?

Hypothesis tests concerning the variance of a normal population

Hypothesis tests concerning the variance of a normal population

- Let $X_1, \ldots, X_n$ be a random sample from a normal distribution with mean μ and variance σ^2

- Suppose we want to test

$$H_0 : \sigma^2 = \sigma_0^2 \;\; vs \;\; H_1 : \sigma^2 \neq \sigma_0^2$$

- The point estimate of σ^2

$$S^2 = \frac{1}{n-1} \sum_i (X_i - \bar{X})^2$$

- The sampling distribution of S^2

$$\frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1} \quad \Rightarrow \quad \text{under } H_0, \;\; \frac{(n-1)S^2}{\sigma_0^2} \sim \chi^2_{n-1}$$

Hypothesis tests concerning the variance of a normal population

- The chi-square distribution is not symmetric, so $\chi^2_{n-1,\alpha/2} \neq -\chi^2_{n-1,1-\alpha/2}$
- Acceptance region at α level of significance

$$\chi^2_{n-1,1-\alpha/2} < \frac{(n-1)S^2}{\sigma_0^2} \leq \chi^2_{n-1,\alpha/2}$$

- Rejection region

$$\frac{(n-1)S^2}{\sigma_0^2} > \chi^2_{n-1,\alpha/2} \quad or \quad \frac{(n-1)S^2}{\sigma_0^2} \leq \chi^2_{n-1,1-\alpha/2}$$

- The p-value of the test

$$p = 2\min\left\{P\left(\chi^2_{n-1} < c\right), 1 - P\left(\chi^2_{n-1} < c\right)\right\}$$

Testing for the equality of variances of two normal populations

- Two independent samples from two populations
  - Let $X_1, \dots, X_n$ be a random sample from $N(\mu_x, \sigma_x^2)$
  - Let $Y_1, \dots, Y_m$ be a random sample from $N(\mu_y, \sigma_y^2)$
- The null and alternative hypotheses

$$H_0 : \sigma_x^2 = \sigma_y^2 \ \ vs \ \ H_0 : \sigma_x^2 \neq \sigma_y^2$$

- Point estimates of σ_x^2 and σ_y^2

$$S_x^2 = \frac{1}{n-1} \sum_i (X_i - \bar{X})^2 \ \ and \ \ S_y^2 = \frac{1}{m-1} \sum_i (Y_i - \bar{Y})^2$$

F-distribution

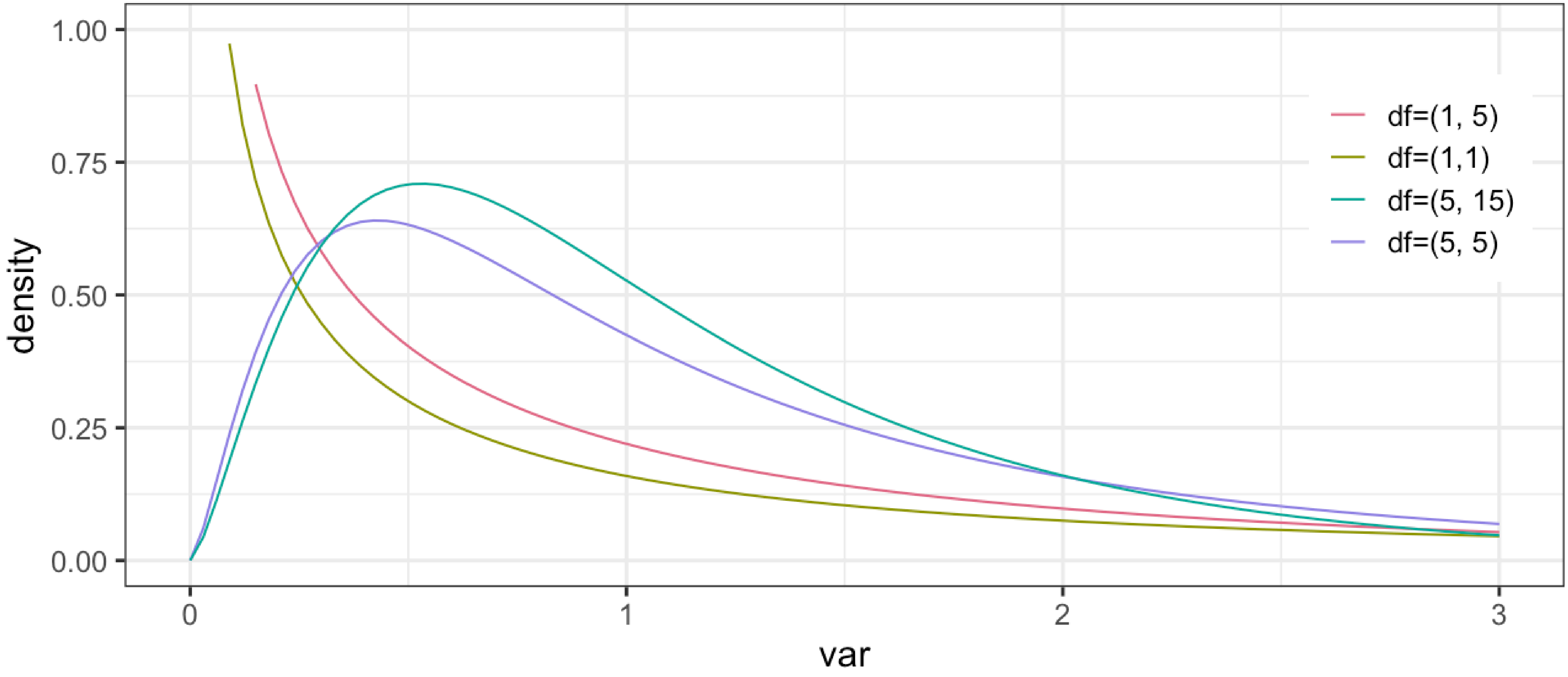
- Let X and Y follow chi-square distributions with n and m degrees of freedom, and X and Y are independent

$$X \sim \chi_n^2 \quad and \quad Y \sim \chi_m^2$$

- It can be shown that

$$F = \frac{X/n}{Y/m} \sim F_{n,m}$$

  - The statistic F follows an F-distribution with degrees of freedom n and m
  - F-distributed random variable can take only positive values



1.00
0.75
0.50
0.25
0.00
density
0
1
2
3
var
df=(1, 5)
df=(1,1)
df=(5, 15)
df=(5, 5)

Testing for the equality of variances of two normal populations

- Sampling distributions of point estimates S_x^2 and S_y^2

$$C_1 = \frac{(n-1)S_x^2}{\sigma_x^2} \sim \chi^2_{n-1} \;\; and \;\; C_2 = \frac{(m-1)S_y^2}{\sigma_y^2} \sim \chi^2_{m-1}$$

- By the definition of F-distribution

$$\frac{C_1/(n-1)}{C_2/(m-1)} = \frac{S_x^2/\sigma_x^2}{S_y^2/\sigma_y^2} \sim F_{n-1,m-1}$$

- Under the $H_0 : \sigma_x^2 = \sigma_y^2$

$$\frac{S_x^2}{S_y^2} \sim F_{n-1,m-1} \;\Rightarrow\; P\left(F_{n-1,m-1,1-\alpha/2} \leq \frac{S_x^2}{S_y^2} \leq F_{n-1,m-1,\alpha/2}\right) = 1-\alpha$$

Testing for the equality of variances of two normal populations

- At α level of significance
  - Reject the $H_0 : \sigma_x^2 = \sigma_y^2$ against $H_0 : \sigma_x^2 \neq \sigma_y^2$ if

$$\left\{ \frac{S_x^2}{S_y^2} < F_{n-1,m-1,1-\alpha/2} \right\} \;\; or \;\; \left\{ \frac{S_x^2}{S_y^2} > F_{n-1,m-1,\alpha/2} \right\}$$

  - Cannot reject the H_0 if

$$F_{n-1,m-1,1-\alpha/2} \leq \frac{S_x^2}{S_y^2} \leq F_{n-1,m-1,\alpha/2}$$

- The p-value of the test

$$p = 2 \times \min \left\{ P\big(F_{n-1,m-1} < S_x^2/S_y^2\big), 1 - P\big(F_{n-1,m-1} < S_x^2/S_y^2\big) \right\}$$

Example 8.5b

- There are two different choices of a catalyst to stimulate a certain chemical process.
- To test whether the variance of the yield is the same no matter which catalyst is used, a sample of 10 batches is produced using the first catalyst, and 12 using the second.
- If the resulting data are $S_1^2 = .14$ and $S^2 = .28$, can we reject the hypothesis of equal variance at the 5 percent level?

Hypothesis tests in the Bernoulli population

Hypothesis tests in the Bernoulli population

- Consider a production process that manufactures items that can be classified either as acceptable or defective
- Assumptions
  - The probability that a product will be defective is p, which remains constant
  - Items are produced independently, i.e., the status (acceptable or defective) of an item does not depend on the status of other items

Hypothesis tests in the Bernoulli population

- Consider a sample of n items and let X be the number of defects in a sample of n items
  - X follows a binomial distribution with parameters n and p, i.e., $X \sim B(n, p)$
- The hypotheses of interest

$$H_0 : p \leq p_0 \quad vs \quad H_1 : p > p_0$$

  - For a large value of X, we will reject the null hypothesis

Approximate test for p

- The number of defects out of n items

$$X \sim B(n, p)$$

- The hypotheses of interest

$$H_0 : p \leq p_0 \;\; vs \;\; H_1 : p > p_0$$

- The parameter n is considered large if

$$np > 5 \;\; and \;\; n(1-p) > 5$$

- If n is large (i.e. approximately)

$$X \sim N\big(np, np(1-p)\big)$$

  - Under $H_0 : p = p_0$

$$X \sim N\big(np_0, np_0(1-p_0)\big)$$

  - The Z-statistic

$$Z = \frac{X - np_0}{\sqrt{np_0(1-p_0)}} \sim N(0,1)$$

- A large value of the Z-statistic leads to rejecting the null hypothesis

Approximate test for p

- The null and alternative hypotheses

$$H_0 : p \le p_0 \;\; against \;\; H_1 : p > p_0$$

- The Z-statistic under $H_0 : p = p_0$

$$Z = \frac{X - np_0}{\sqrt{np_0(1-p_0)}} \sim N(0,1)$$

- Reject the H_0 at α level of significance if

$$Z > z_\alpha$$

  - Otherwise, we cannot reject the null hypothesis

- The approximate p-value of the test

$$p = P(Z > z)$$

Exact tests for p

- The number of defects out of n items

$$X \sim B(n, p)$$

- The hypotheses of interest

$$H_0 : p \leq p_0 \;\; vs \;\; H_1 : p > p_0$$

  - Under $H_0 : p = p_0$

$$X \sim B(n, p_0)$$

- A large value of X leads to rejecting

$$H_0 : p \leq p_0 \;\; against \;\; H_1 : p > p_0$$

- Suppose, x is the observed number of defects in n items

- The p-value of the test

$$\begin{aligned} p &= P(X > x \,|\, p = p_0) \\ &= \sum_{i=x}^{n} \binom{n}{i} p_0^i (1 - p_0)^{n-i} \end{aligned}$$

  - Reject H_0 at α level of significance if $p \leq \alpha$

Example 8.6a

- A computer chip manufacturer claims that up to 2 percent of its chips are defective
- An electronics company, impressed with this claim, has purchased a large number of such chips.
- To determine if the manufacturer's claim can be taken literally, the company has decided to test a sample of 300 of these chips.
- If 10 of these 300 chips are found to be defective, should the manufacturer’s claim be rejected?

Example 8.6a

- A computer chip manufacturer claims that up to 2 percent of its chips are defective
  - To determine if the manufacturer's claim can be taken literally, the company has decided to test a sample of 300 of these chips.
  - If 10 of these 300 chips are found to be defective, should the manufacturer's claim be rejected?
- **Given**:
  - Let X be the number of defective chips in 300 chips and $X \sim B(300, p)$
  - The null and alternative hypothesis $H_0 : p \leq 0.02$ against $H_1 : p > 0.02$
  - Observed number of defective chips is 10

Example 8.6a

- The number of defective chips in 300 chips $X \sim B(300, p)$

$$H_0 : p \leq 0.02 \;\; against \;\; H_1 : p > 0.02$$

- Under H_0

$$X \sim B(300, 0.02)$$

- The observed number of defective chips is 10

Exact test

- The p-value of the test

$$\begin{aligned} p &= P(X \geq 10) \\ &= \sum_{i=10}^{300} \binom{300}{i} 0.02^i (1 - 0.02)^{300-i} \\ &= 0.0818 \end{aligned}$$

Example 8.6a

- The null and alternative hypotheses

$$H_0 : p \leq 0.02 \quad vs \quad H_1 : p > 0.02$$

- Under H_0

$$X \sim B(300, 0.02)$$

- The observed number of defects: $x = 10$
- For a large n, $X \sim N\big(np, np(1-p)\big)$

Approximate test

- n is large because
  - $np_0 = (300)(0.02) = 6 > 5$
  - $n(1 - p_0) = 294 > 5$
- Under $H_0 : p \leq 0.02$, $X \sim N\big(6, 5.88\big)$
- The test statistic

$$Z = \frac{10 - 6}{\sqrt{5.88}} = 1.443$$

Example 8.6a

Approximate test

- The test statistic

$$Z = \frac{10 - 6}{\sqrt{5.88}} = 1.44$$

- The p-value of the test

$$p = P(Z > 1.44) = 0.0749$$

Exact test

- The p-value of the test

$$\begin{aligned} p &= P(X \geq 10) \\ &= \sum_{i=10}^{300} \binom{300}{i} 0.02^i (1 - 0.02)^{300-i} \\ &= 0.0818 \end{aligned}$$

Hypothesis tests in the Bernoulli population

- Suppose $X \sim B(n, p)$
- Construct approximate and exact tests for

$$H_0 : p \geq p_0 \;\; against \;\; p < p_0$$

Hypothesis tests in the Bernoulli population

- Suppose $X \sim B(n, p)$
- Construct an approximate test for

$$H_0 : p = p_0 \;\; against \;\; p \neq p_0$$

- For a large n, under H_0

$$X \sim N\big(np_0, np_0(1 - p_0)\big)$$

Approximate test

- Test statistic

$$Z = \frac{X - np_0}{\sqrt{np_0(1 - p_0)}} \sim N(0, 1)$$

- Reject H_0 at α level of significance if

$$|Z| > z_{\alpha/2}$$

- p-value of the test

$$p = 2 \times P\big(|Z| > z\big)$$

Hypothesis tests in the Bernoulli population

- Suppose $X \sim B(n, p)$
- Construct exact test for

$$H_0 : p = p_0 \;\; against \;\; p \neq p_0$$

- Under H_0

$$X \sim B(n, p_0)$$

Exact test

- Suppose, x is the observed value of X
- Reject H_0 at α level of significance if

$$P(X \geq x) \leq \alpha/2 \;\; or \;\; P(X \geq x) \leq \alpha/2$$

- p-value of the test

$$p = 2 \times \min \left\{ P(X \geq x), P(X \leq x) \right\}$$

Example 8.6c

- Historical data indicate that 4 percent of the components produced at a certain manufacturing facility are defective.
- A particularly acrimonious labor dispute has recently been concluded, and management is curious whether it will result in any change in this figure of 4 percent.
- If a random sample of 500 items indicated 16 defectives (3.2 percent), is this significant evidence, at the 5 percent level of significance, to conclude that a change has occurred?

Testing equality of parameters of two Bernoulli distributions

- Suppose there are two methods, namely Method I and II, for producing a certain type of transistor
  - $p_1 \;\; \rightarrow$ probability that a transistor produced by Method I is defective
  - $p_2 \;\; \rightarrow$ probability that a transistor produced by Method II is defective
- Hypothesis of interest $H_0 : p_1 = p_2$ against $H_1 : p_1 \neq p_2$
  - Test the hypothesis based on samples of n_1 and n_2 transistors of Method I and II, respectively

Testing equality of parameters of two Bernoulli distributions

- Let X_1 be the number of defectives out of n_1 transistors produced by Method I
- Let X_2 be the number of defectives out of n_2 transistors produced by Method II
  - X_1 and X_2 are independent and

$$X_1 \sim B(n_1, p_1) \;\; and \;\; X_2 \sim B(n_2, p_2)$$

- Under $H_0 : p_1 = p_2 = p$

$$X_1 + X_2 \sim B(n_1 + n_2, p)$$

Testing equality of parameters of two Bernoulli distributions

- Under $H_0 : p_1 = p_2 = p$

$$X_1 + X_2 \sim B(n_1 + n_2, p)$$

- Reject H_0 at α level of significance if

$$P(X_1 \geq x_1 \mid X_1 + X_2 = k) = \sum_{i=x_1}^{k} P(X_1 = i \mid X_1 + X_2 = k) \leq \alpha/2$$

$$P(X_1 \leq x_1 \mid X_1 + X_2 = k) = \sum_{i=x_1}^{k} P(X_1 = i \mid X_1 + X_2 = k) \leq \alpha/2$$

- This test procedure is called the Fisher-Irwin test

Testing equality of parameters of two Bernoulli distributions

- Let us consider two independent variables X and Y with the following distributional assumptions

$$X \sim B(n,p), Y \sim B(m,p),\ X+Y \sim B(n+m,p)$$

- The distribution of X given $X+Y$

$$P(X=x \,|\, X+Y=k) = \frac{P(X=x, X+Y=k)}{P(X+Y=k)}$$

$$= \frac{P(X=x, Y=k-x)}{P(X+Y=k)}$$

$$= \frac{P(X=x)\,P(Y=k-x)}{P(X+Y=k)} = \frac{\binom{n}{x}\binom{m}{k-x}}{\binom{n+m}{k}}$$

Testing equality of parameters of two Bernoulli distributions

- Reject H_0 at α level of significance if

$$P(X_1 \geq x_1 \,|\, X_1 + X_2 = k) = \sum_{i=x_1}^{k} P(X_1 = i \,|\, X_1 + X_2 = k) \leq \alpha/2$$

$$P(X_1 \leq x_1 \,|\, X_1 + X_2 = k) = \sum_{i=x_1}^{k} P(X_1 = i \,|\, X_1 + X_2 = k) \leq \alpha/2$$

where

$$P(X_1 = i \,|\, X_1 + X_2 = k) = \frac{\binom{n_1}{i}\binom{n_2}{k-i}}{\binom{n_1+n_2}{k}}$$

Approximate test for equality of two population proportions

- $X \sim B(n, p_1) \;\; and \;\; Y \sim B(m, p_2)$
- $H_0 : p_1 = p_2 = p \;\Rightarrow\; H_0 : p_1 - p_2 = 0$
- $\hat{p}_1 = X/n \sim N\big(p_1, p_1(1 - p_1)/n\big)$ and $\hat{p}_2 = Y/m \sim N\big(p_2, p_2(1 - p_2)/m\big)$
- $\hat{p}_1 - \hat{p}_2 \sim N\big(p_1 - p_2, p_1(1 - p_1)/n + p_2(1 - p_2)/m\big)$
- Under H_0, $\hat{p}_1 - \hat{p}_2 \sim N\big(0, p(1 - p)/n + p(1 - p)/m\big)$, where $\hat{p} = (X + Y)/(m + n)$
- Test statistic

$$Z = \frac{\hat{p}_1 - \hat{p}_2}{\sqrt{\hat{p}(1 - \hat{p})\left(\frac{1}{n} + \frac{1}{m}\right)}} \sim N(0, 1)$$

Problems: Chapter 8

- 2, 3, 4, 6, 7, 9, 11, 13, 14, 20, 21, 25, 27, 28, 29, 33, 39, 40, 43, 46, 47, 48, 51, 52, 53, 55, 58, 59, 65, 66, 68, 70