Linear Programming
Simplex Method

Already covered

- Introduction to linear programming
- Formulate to linear programming from the given problem.
- Solve linear programming problem using geometrical technique
- Convert LP problem to standard form
- Solve LP
- using simplex method.

Algebraic Solution of LPPs - Simplex Method

To solve an LPP algebraically, we first put it in the **standard form**.

- This means all decision variables are nonnegative
- all constraints (other than the nonnegativity restrictions) are **equations with nonnegative RHS.**

Tasks to convert standard form

- **Converting inequalities into equations**
  - **Use slack/excess variable to do this**
  - **Make sure that right side doesn't contain negative sign.**
  - **Convert unrestricted variables to restricted variables.**

Basic variables, Basic feasible Solutions

- Consider an LPP (in standard form) with m constraints and n decision variables.
  - We assume $m \leq n$.
  - We choose $n - m$ variables and set them equal to zero.
  - Thus we will be left with a system of m equations in m variables.
  - If this $m \times m$ square system has a unique solution, this solution is called a **basic solution**.
  - If further if it is feasible, it is called a **Basic Feasible Solution** (BFS).

Basic variables, Nonbasic Variables

- The n-m variables set to zero are called nonbasic and the m variables which we are solving for are known as basic variables.
- Thus a basic solution is of the form

$$\boldsymbol{x} = (x1, x2, \ldots, xn)$$

where n-m “components” are zero and the remaining m components form the unique solution of the square system (formed by the m constraint equations).

Note that we may have a ***maximum*** of $\binom{n}{m}$ basic solutions.

Consider the LPP:

Maximize $\boxed{z = 2x_1 + 3x_2}$

Subject to

$$\begin{aligned} x_1 + 3x_2 &\le 6 \\ 3x_1 + 2x_2 &\le 6 \\ x_1, x_2 &\ge 0 \end{aligned}$$

This is equivalent to the LPP (in standard form)

Maximize $z = 2x_1 + 3x_2$

Subject to

$$x_1 + 3x_2 + s_1 = 6$$

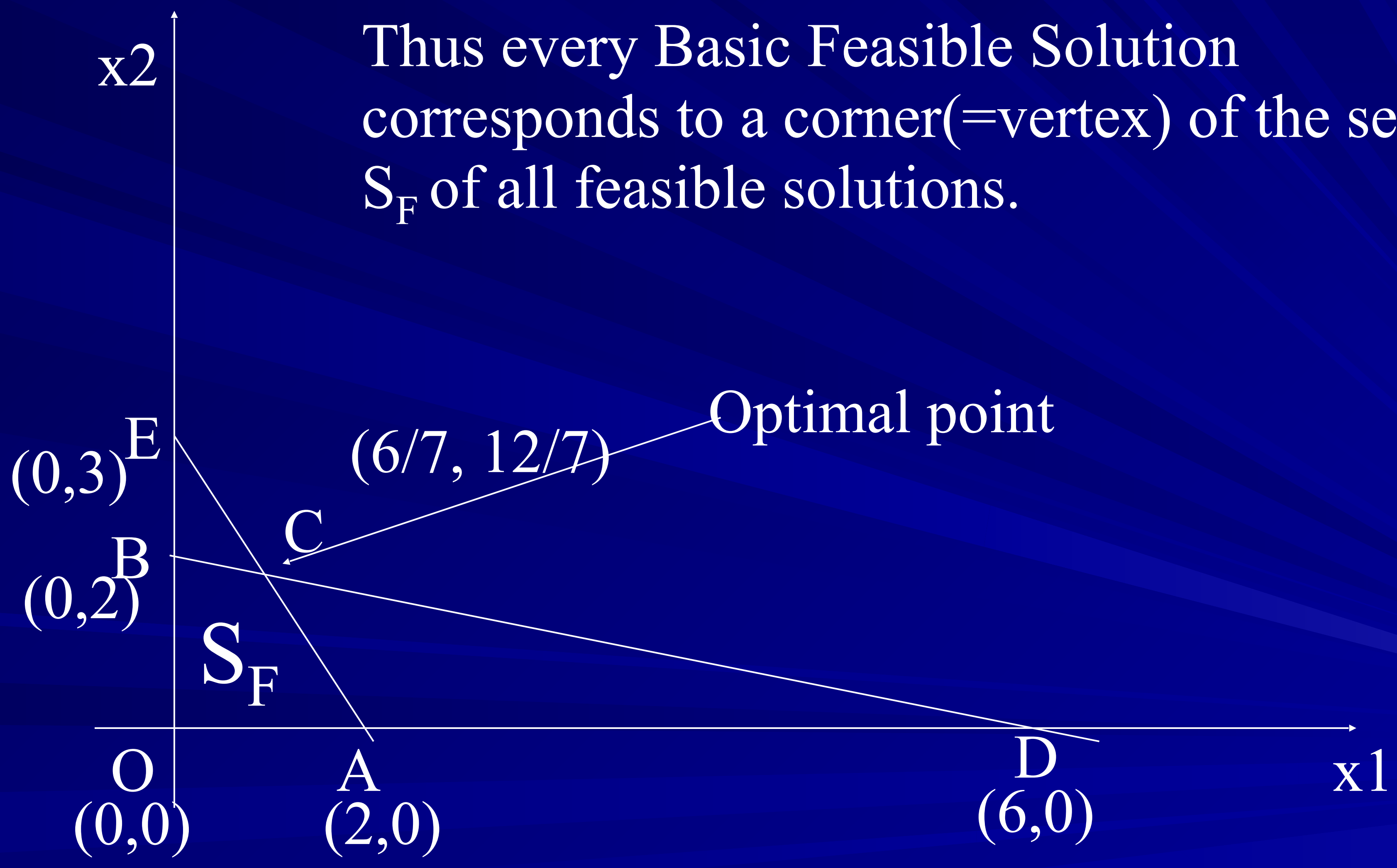
$$3x_1 + 2x_2 + s_2 = 6$$

$$x_1, x_2, s_1, s_2 \geq 0$$

s_1, s_2 are slack variables.

Graphical solution of the above LPP

Thus every Basic Feasible Solution corresponds to a corner(=vertex) of the set S_F of all feasible solutions.



Nonbasic (zero) variables	Basic variables	Basic solution (x_1, x_2, s_1, s_2)	Assoc-iated corner point	Feasible?	Object-ive value, z
(x_1, x_2)	(s_1, s_2)	(0,0,6,6)	O	Yes	0
(x_1, s_1)	(x_2, s_2)	(0,2,0,2)	B	Yes	6
(x_1, s_2)	(x_2, s_1)	(0,3,-3,0)	E	No	-
(x_2, s_1)	(x_1, s_2)	(6,0,0, -12)	D	No	-
(x_2, s_2)	(x_1, s_1)	(2,0,4,0)	A	Yes	4
(s_1, s_2)	(x_1, x_2)	$(\frac{6}{7}, \frac{12}{7}, 0, 0)$	C	Yes	48/7 Optimal

The Simplex algorithm

Example Consider the LPP

Maximize $z = 8x_1 + 9x_2$

Subject to the constraints

$$
\begin{aligned}
2x_1 + 3x_2 &\le 50 \\
2x_1 + 6x_2 &\le 80 \\
3x_1 + 3x_2 &\le 70 \\
x_1, x_2 &\ge 0
\end{aligned}
$$

Introducing slack variables, the LPP is same as

Maximize $z = 8x_1 + 9x_2 + 0s_1 + 0s_2 + 0s_3$

subject to the constraints

$$\begin{aligned} 2x_1 + 3x_2 + s_1 \qquad\qquad &= 50 \\ 2x_1 + 6x_2 \qquad + s_2 \qquad &= 80 \\ 3x_1 + 3x_2 \qquad\qquad + s_3 &= 70 \\ x_1, x_2, s_1, s_2, s_3 &\geq 0 \end{aligned} \qquad (*)$$

Basic Feasible Solution

- Let
  - $x1$, $x2$ as nonbasic variables
  - Then BFS can be got immediately as
    - (0,0, 50,80,70).
- This is possible because in the LHS of the constraint equations (*) the coefficients of the basic variables $s1$, $s2$, $s3$ form an Identity matrix (of size 3 x 3).

Simplex tableau (Starting tableau)

Basic	z	x_1	x_2	s_1	s_2	s_3	Solution
z	1	-8	-9	0	0	0	0
s_1	0	2	3	1	0	0	50
s_2	0	2	6	0	1	0	80
s_3	0	3	3	0	0	1	70

Closer Look (1)

- We note that the *z*-row is the objective function row.
- The remaining 3 rows are the basic variable rows.
- Each row corresponds to a basic variable;
  – the leftmost variable denotes the basic variable corresponding to that row.

Closer Look (2)

- In the objective function row,
  - the coefficients of the basic variables are zero.
- In each column corresponding to a basic variable,
  - basic variable has a non-zero coefficient, namely 1,
  - all the other basic variables have zero coefficients.

Closer Look (3)

- We see at present, the objective function, z, has value zero.
- We now seek to make
  - one of the nonbasic variables as basic (and so) one of the basic variables will become nonbasic (that is will drop down to zero)
  - the nonbasic variable that will become basic is chosen such that the objective function will “improve”:
    - in a maximization problem it will increase
    - in a minimization problem it will decrease.
- The nonbasic variable that will become basic is referred to as “entering” variable and the basic variable that will become nonbasic is referred to as “leaving” variable.

Criterion for "entering" variable:
(**Optimality Condition**)

Choose that variable as the "entering" variable which has

the **most –ve** coefficient in the z-row in case it is a **maximization** problem

(as the variable which has the most +ve coefficient in the z-row in case it is a minimization problem).

(Break the ties arbitrarily.)

Criterion for "leaving" variable (**Feasibilty Condition**)

Let b_i be the RHS of the ith row. Let a_{ij} be the coefficient of the entering variable x_j in the ith row. The following "minimum ratio test" decides the leaving variable:
Choose x_k as the leaving variable where k is given as that row index i for which the ratio

$$\left\{ \frac{b_i}{a_{ij}} : a_{ij} > 0 \right\}$$

is least.

(Break the ties arbitrarily.)

Pivot element

- The entering variable column is called the **pivot column**.
- The leaving variable row is called the **pivot row**.
- The coefficient in the intersection of the two is referred to as the **pivot element**.

Operations

- We apply ***elementary row operations*** to modify the simplex tableau so that the pivot column has **1** at the pivot element and **zero** in all other places.

The elementary Row operations are as follows:

- New pivot row = old pivot row ÷ pivot element
- New z row = old z row – (coefficient of the entering variable in old z row * **New** pivot row)
- Any other new row = corresponding old row – (old coefficient of the entering variable in that row * **New** pivot row)
- We shall also change the legend of the *new pivot row only* as the entering variable.

Simplex tableau (Starting tableau)

↓Enters

Basic	z	x_1	x_2	s_1	s_2	s_3	Solution
z	1	-8	-9	0	0	0	0
s_1	0	2	3	1	0	0	50
s_2	0	2	6	0	1	0	80
s_3	0	3	3	0	0	1	70

Leaves

Pivot element

Performing the elementary Row operations, we get the new Simplex tableau below

↓ Enters

Basic	z	x_1	x_2	s_1	s_2	s_3	Solution
z	1	-5	0	0	3/2	0	120
s_1	0	1	0	1	-1/2	0	10
x_2	0	1/3	1	0	1/6	0	40/3
s_3	0	2	0	0	-1/2	1	30

Pivot element

Leaves

Performing the elementary Row operations, we get the new Simplex tableau below

↓ Enters

Basic	z	x_1	x_2	s_1	s_2	s_3	Solution
z	1	0	0	5	-1	0	170
x_1	0	1	0	1	-1/2	0	10
x_2	0	0	1	-1/3	1/3	0	10
s_3	0	0	0	-2	1/2	1	10

Pivot element

Leaves

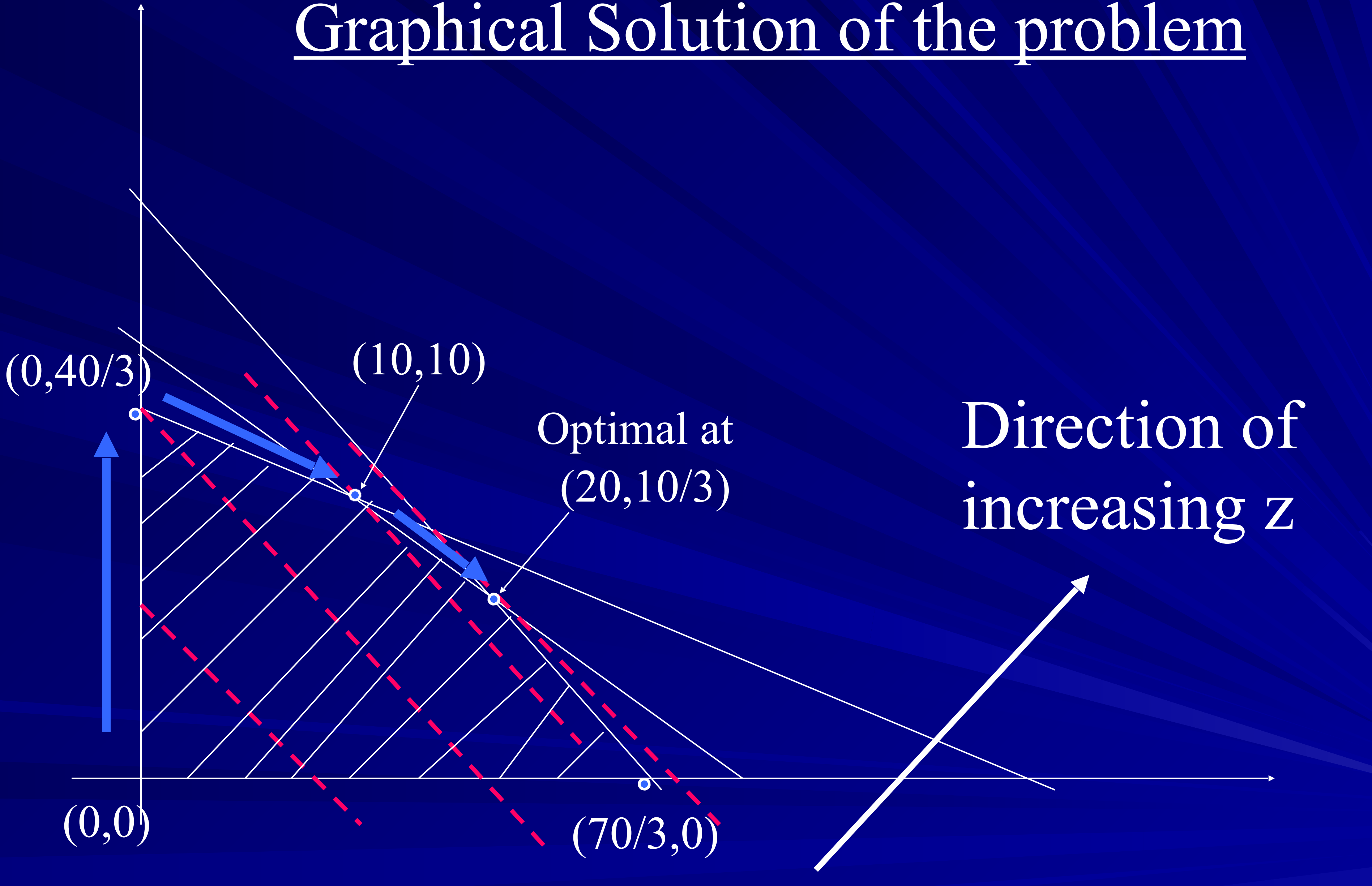
Performing the elementary Row operations, we get the new Simplex tableau below.

This is optimal as all entries in z row are ≥ 0.

Basic	z	x_1	x_2	s_1	s_2	s_3	Solution
z	1	0	0	5	0	0	190
x_1	0	1	0	-1	0	0	20
x_2	0	0	1	1	0	0	10/3
s_2	0	0	0	-4	1	2	20

Optimum value = 190 at x_1=20, x_2=10/3

Graphical Solution of the problem



Example Consider the LPP

Maximize $z = 2x_1 + x_2 - 3x_3 + 5x_4$

Subject to the constraints

$$
\begin{aligned}
x_1 + 2x_2 - 2x_3 + 4x_4 &\le 40 \\
2x_1 - x_2 + x_3 + 2x_4 &\le 8 \\
4x_1 - 2x_2 + x_3 - x_4 &\le 10 \\
x_1, x_2, x_3, x_4 &\ge 0
\end{aligned}
$$

Introducing slack variables, the LPP becomes

Maximize

$$z = 2x_1 + x_2 - 3x_3 + 5x_4 + 0s_1 + 0s_2 + 0s_3$$

Subject to the constraints

$$\begin{aligned}
x_1 + 2x_2 - 2x_3 + 4x_4 + s_1 \qquad &= 40 \\
2x_1 - x_2 + x_3 + 2x_4 \quad + s_2 \quad &= 8 \\
4x_1 - 2x_2 + x_3 - x_4 \qquad + s_3 &= 10 \\
x_1, x_2, x_3, x_4, s_1, s_2, s_3 \geq 0 &
\end{aligned}$$

Basic	z	x1	x2	x3	x4	s1	s2	s3	Sol.
z	1	-2	-1	3	-5	0	0	0	0
s1	0	1	2	-2	4	1	0	0	40
s2	0	2	-1	1	$\boxed{2}$	0	1	0	8
s3	0	4	-2	1	-1	0	0	1	10
z	1	3	-7/2	11/2	0	0	5/2	0	20
s1	0	-3	$\boxed{4}$	-4	0	1	-2	0	24
x4	0	1	-1/2	1/2	1	0	1/2	0	4
s3	0	5	-5/2	3/2	0	0	1/2	1	14
z	1	3/8	0	2	0	7/8	3/4	0	41
x2	0	-3/4	1	-1	0	1/4	-1/2	0	6
x4	0	5/8	0	0	1	1/8	1/4	0	7
s3	0	25/8	0	-1	0	5/8	-3/4	1	29

The last tableau is the optimal tableau as all entries in the objective function row are ≥ 0 and the LPP is a maximization problem.

Optimal Solution is

$x_1 = 0, x_2 = 6, x_3 = 0, x_4 = 7$

And the Optimal z (= Maximum z)

$= 41$

Solve the following LPP:

Maximize $z = x_1 + 2x_2 + 4x_3$

Subject to the constraints

$$
\begin{aligned}
3x_1 + x_2 + 5x_3 &\le 10 \\
x_1 + 4x_2 + x_3 &\le 8 \\
2x_1 + 2x_3 &\le 7 \\
x_1, x_2, x_3 &\ge 0
\end{aligned}
$$

Introducing slack variables, the LPP becomes

Maximize

$$z = x_1 + 2x_2 + 4x_3 + 0s_1 + 0s_2 + 0s_3$$

Subject to the constraints

$$\begin{aligned}
3x_1 + x_2 + 5x_3 + s_1 &= 10 \\
x_1 + 4x_2 + x_3 + s_2 &= 8 \\
2x_1 + 2x_3 + s_3 &= 7 \\
x_1, x_2, x_3, s_1, s_2, s_3 &\geq 0
\end{aligned}$$

Basic	z	x1	x2	x3	s1	s2	s3	Sol.
z	1	-1	-2	-4	0	0	0	0
s1	0	3	1	5	1	0	0	10
s2	0	1	4	1	0	1	0	8
s3	0	2	0	2	0	0	1	7
z	1	7/5	-6/5	0	4/5	0	0	8
x3	0	3/5	1/5	1	1/5	0	0	2
s2	0	2/5	19/5	0	-1/5	1	0	6
s3	0	4/5	-2/5	0	-2/5	0	1	3
z	1	29/19	0	0	14/19	6/19	0	188/19
x3	0	11/19	0	1	4/19	-1/19	0	32/19
x2	0	2/19	1	0	-1/19	5/19	0	30/19
s3	0	16/19	0	0	-8/19	2/19	1	69/19

The last tableau is the optimal tableau as all entries in the objective function row are ≥ 0 and the LPP is a maximization problem.

Optimal Solution is

$x_1 = 0, x_2 = 30/19, x_3 = 32/19$

And the Optimal z (= Maximum z)

$= 188/19$

Artificial Variable Techniques –

Big M-method

M method – basic idea

- If in a starting simplex tableau,
  - we don't have an identity submatrix (i.e. an obvious starting BFS),
    - then we introduce artificial variables to have a starting BFS.
    - There are two methods to find the starting BFS and solve the problem –
      - the Big M method
      - two-phase method

M method – when required

- Suppose a constraint equation i does not have a slack variable
  - there is no ith unit vector column in the LHS of the constraint equations. (This happens for example when the ith constraint in the original LPP is either ≥ or = .)
- Then we augment the equation with an artificial variable R_i to form the ith unit vector column.

Artificial Variable

- As the artificial variable is extraneous to the given LPP,
  - we use a feedback mechanism in which the optimization process automatically attempts to force these variables to zero level.
  - This is achieved by giving a large penalty to the coefficient of the artificial variable in the objective function as follows:

    Artificial variable objective coefficient

    = - M in a maximization problem,

    = M in a *minimization* problem
  - where M is a very large positive number.

Consider the LPP:

Minimize $z = 2x_1 + x_2$

Subject to the constraints

$$3x_1 + x_2 \geq 9$$

$$x_1 + x_2 \geq 6$$

$$x_1, x_2 \geq 0$$

Putting this in the standard form, the LPP is:

Minimize $z = 2x_1 + x_2$

Subject to the constraints

$$\begin{aligned} 3x_1 + x_2 - s_1 \quad &= 9 \\ x_1 + x_2 \quad - s_2 &= 6 \\ x_1, x_2, s_1, s_2 &\geq 0 \end{aligned}$$

Here s_1, s_2 are surplus variables.

Note that we do NOT have a 2x2 identity submatrix in the LHS.

Introducing the artificial variables, R_1, R_2, the LPP is modified as follows:

Minimize $z = 2x_1 + x_2 + MR_1 + MR_2$

Subject to the constraints

$$
\begin{aligned}
3x_1 + x_2 - s_1 \quad + R_1 \quad &= 9 \\
x_1 + x_2 \quad - s_2 \quad + R_2 &= 6 \\
x_1, x_2, s_1, s_2, R_1, R_2 &\geq 0
\end{aligned}
$$

Note that we now have a 2x2 identity submatrix in the coefficient matrix of the constraint equations.

Now we solve the above LPP by the Simplex method.

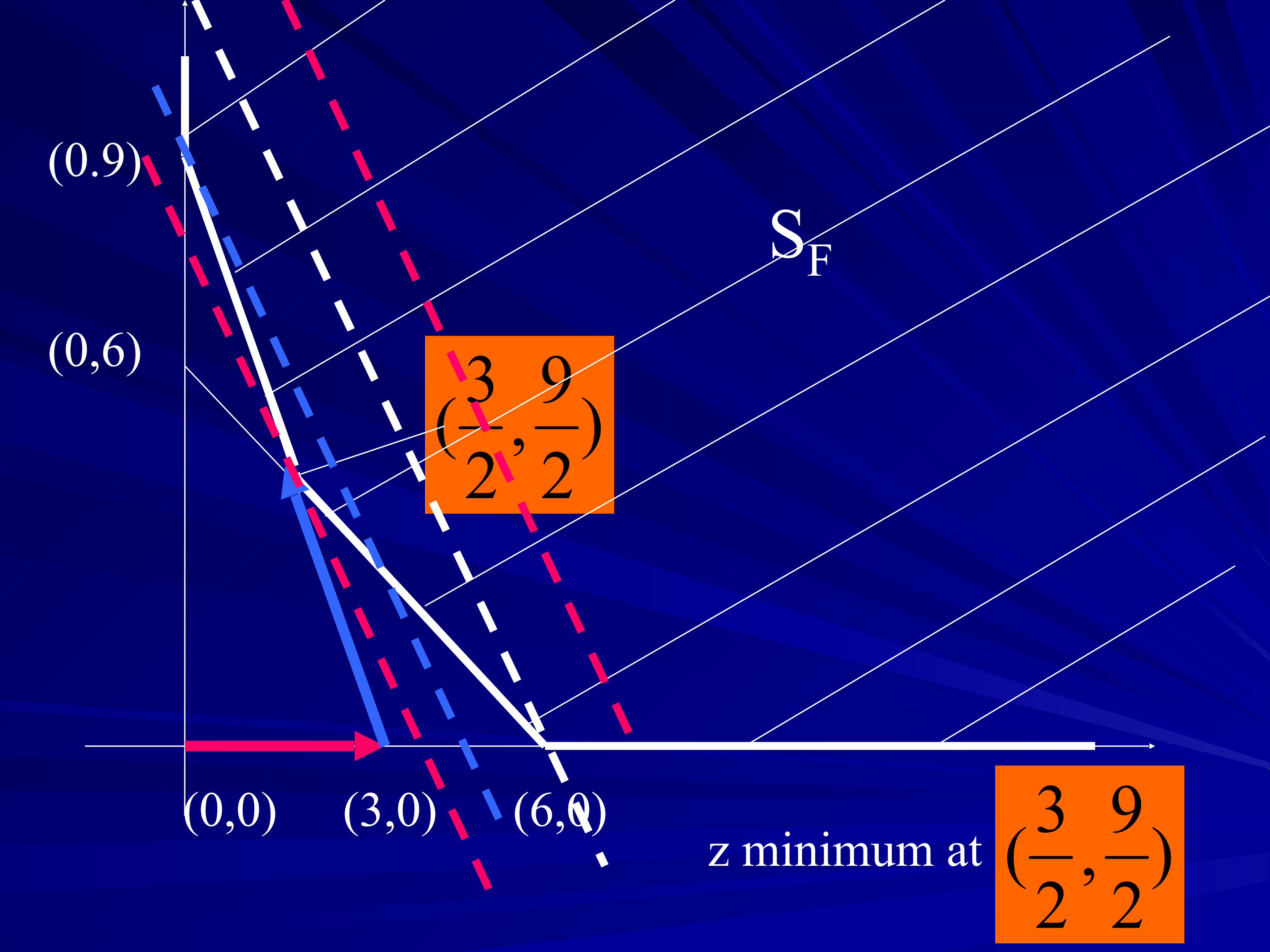
Basic	z	x1	x2	s1	s2	R1	R2	Sol.
z	1	-2+4M $\cancel{-2}$	-1+2M $\cancel{-1}$	-M $\cancel{0}$	-M $\cancel{0}$	0 $\cancel{-M}$	0 $\cancel{-M}$	15M $\cancel{0}$
R1	0	$\boxed{3}$	1	-1	0	1	0	9
R2	0	1	1	0	-1	0	1	6
z	1	0	-1/3+ 2M/3	-2/3+ M/3	-M	2/3- 4M/3	0	6+3M
x1	0	1	1/3	-1/3	0	1/3	0	3
R2	0	0	$\boxed{2/3}$	1/3	-1	-1/3	1	3
z	1	0	0	-1/2	-1/2	1/2–M	1/2-M	15/2
x1	0	1	0	-1/2	1/2	1/2	-1/2	3/2
x2	0	0	1	1/2	-3/2	-1/2	3/2	9/2

Note that we have got the optimal solution to the given problem as

$$\boxed{x_1 = \frac{3}{2},\ x_2 = \frac{9}{2}}$$

Optimal z = Minimum $\boxed{z = \frac{15}{2}}$

It is illuminating to look at the graphical solution also.


(0.9)
(0,6)
S_F
$(\frac{3}{2},\frac{9}{2})$
(0,0)
(3,0)
(6,0)
z minimum at $(\frac{3}{2},\frac{9}{2})$

Maximize $z = 2x_1 + 3x_2 - 5x_3$

Subject to the constraints

$$x_1 + x_2 + x_3 = 7$$
$$2x_1 - 5x_2 + x_3 \geq 10$$
$$x_1, x_2, x_3 \geq 0$$

Introducing surplus and artificial variables, s_2, R_1 and R_2, the LPP is modified as follows:

Maximize

$$z = 2x_1 + 3x_2 - 5x_3 - MR_1 - MR_2$$

Subject to the constraints

$$\begin{aligned} x_1 + x_2 + x_3 + R_1 &= 7 \\ 2x_1 - 5x_2 + x_3 - s_2 + R_2 &= 10 \\ x_1, x_2, x_3, s_2, R_1, R_2 &\geq 0 \end{aligned}$$

Now we solve the above LPP by the Simplex method.

Basic	z	x1	x2	x3	s2	R1	R2	Sol.
z	1	-2-3M $\cancel{-2}$	-3+4M $\cancel{-3}$	5-2M $\cancel{5}$	M $\cancel{0}$	0 $\cancel{M}$	0 $\cancel{M}$	-17M $\cancel{0}$
R1	0	1	1	1	0	1	0	7
R2	0	$\boxed{2}$	-5	1	-1	0	1	10
z	1	0	-8 - 7M/2	6 - M/2	-1 - M/2	0	1 + 3M/2	10 - 2M
R1	0	0	$\boxed{7/2}$	1/2	1/2	1	-1/2	2
x1	0	1	-5/2	1/2	-1/2	0	1/2	5
z	1	0	0	50/7	1/7	16/7 + M	-1/7 +M	102/7
x2	0	0	1	1/7	1/7	2/7	-1/7	4/7
x1	0	1	0	6/7	-1/7	5/7	1/7	45/7

The optimum (Maximum) value of

$z = 102/7$

and it occurs at

$x_1 = 45/7, x_2 = 4/7, x_3 = 0$

Remarks

- If in any iteration, there is a tie for entering variable between an artificial variable and other variable (decision, surplus or slack), we must prefer the <u>non-artificial</u> variable to enter the basis.
- If in any iteration, there is a tie for leaving variable between an artificial variable and other variable (decision, surplus or slack), we must prefer the *artificial* variable to leave the basis.

- If in the final optimal tableau, an artificial variable is present in the basis at a non-zero level, this means our original problem has *no feasible solution*.

Maximize $z = 5x_1 + 6x_2$

Subject to the constraints

$$
\begin{aligned}
-2x_1 + 3x_2 &= 3 \\
x_1 + 2x_2 &\le 5 \\
6x_1 + 7x_2 &\le 3 \\
x_1, x_2 &\ge 0
\end{aligned}
$$

Introducing slack and artificial variables, s_2, s_3, and R_1, the LPP is modified as follows:

Maximize $z = 5x_1 + 6x_2 - MR_1$

Subject to the constraints

$$
\begin{aligned}
-2x_1 + 3x_2 + R_1 \qquad\qquad &= 3 \\
x_1 + 2x_2 \qquad + s_2 \qquad &= 5 \\
6x_1 + 7x_2 \qquad\qquad + s_3 &= 3 \\
x_1, x_2, R_1, s_2, s_3 \geq 0 &
\end{aligned}
$$

Basic	z	x1	x2	R1	s2	s3	Sol
z	1	-5+2M ~~-5~~	-6-3M ~~-6~~	0 ~~M~~	0 ~~0~~	0 ~~0~~	-3M ~~0~~
R1	0	-2	3	1	0	0	3
s2	0	1	2	0	1	0	5
s3	0	6	$\boxed{7}$	0	0	1	3
z	1	1/7+ 32M/7	0	0	0	6/7+ 3M/7	18/7- 12M/7
R1	0	-32/7	0	1	0	-3/7	12/7
s2	0	-12/7	0	0	1	-2/7	29/7
x2	0	6/7	1	0	0	1/7	3/7

This is the optimal tableau. As R_1 is not zero, there is NO feasible solution

Minimize $z = 4x_1 + 6x_2$

Subject to the constraints

$$
\begin{aligned}
-2x_1 + 3x_2 &= 3 \\
4x_1 + 5x_2 &\geq 10 \\
4x_1 + 8x_2 &\geq 5 \\
x_1, x_2 &\geq 0
\end{aligned}
$$

Introducing the surplus and artificial variables, R_1, R_2, the LPP is modified as follows:

Minimize $z = 4x_1 + 6x_2 + M\,R_1 + M\,R_2 + M\,R_3$

Subject to the constraints

$$
\begin{aligned}
-2x_1 + 3x_2 \qquad\qquad + R_1 \qquad\qquad &= 3 \\
4x_1 + 5x_2 \quad - s_2 \quad + R_2 \quad &= 10 \\
4x_1 + 8x_2 \qquad - s_3 \quad + R_3 &= 5 \\
x_1, x_2, s_2, s_3, R_1, R_2, R_3 &\geq 0
\end{aligned}
$$

Basic	z	x1	x2	s2	s3	R1	R2	R3	Sol.
z	1	-4+6M ~~-4~~	-6+16M ~~-6~~	-M ~~0~~	-M ~~0~~	0 ~~-M~~	0 ~~-M~~	0 ~~-M~~	18M ~~0~~
R1	0	-2	3	0	0	1	0	0	3
R2	0	4	5	-1	0	0	1	0	10
R3	0	4	$\boxed{8}$	0	-1	0	0	1	5
z	0	-1-2M	0	-M	-3/4 +M	0	0	3/4 -2M	15/4 +8M
R1	0	-7/2	0	0	3/8	1	0	-3/8	9/8
R2	0	3/2	0	-1	5/8	0	1	-5/8	55/8
x2	0	1/2	1	0	-1/8	0	0	1/8	5/8

Basic	z	x1	x2	s2	s3	R1	R2	R3	Sol.
z	1	-1-2M	0	-M	-3/4 +M	0	0	3/4 -2M	15/4 +8M
R1	0	-7/2	0	0	$\boxed{3/8}$	1	0	-3/8	9/8
R2	0	3/2	0	-1	5/8	0	1	-5/8	55/8
x2	0	1/2	1	0	-1/8	0	0	1/8	5/8
z	1	-8 + 22M/3	0	-M	0	2 -8M/3	0	-M	6 +5M
s3	0	-28/3	0	0	1	8/3	0	-1	3
R2	0	22/3	0	-1	0	-5/3	1	0	5
x2	0	-2/3	1	0	0	1/3	0	0	1

Basic	z	x1	x2	s2	s3	R1	R2	R3	Sol.
z	1	-8 + 22M/3	0	-M	0	2 -8M/3	0	-M	6 +5M
s3	0	-28/3	0	0	1	8/3	0	-1	3
R2	0	$\boxed{22/3}$	0	-1	0	-5/3	1	0	5
x2	0	-2/3	1	0	0	1/3	0	0	1
z	1	0	0	-6/11	0	2/11 - M	12/11 -M	-M	$\frac{126}{11}$
s3	0	0	0	-14/11	1	6/11	14/11	-1	$\frac{103}{11}$
x1	0	1	0	-3/22	0	-5/22	3/22	0	$\frac{15}{22}$
x2	0	0	1	-1/11	0	2/11	1/11	0	$\frac{16}{11}$

This is the optimal tableau.

The Optimal solution is:

$$\boxed{x_1 = \frac{15}{22}, \; x_2 = \frac{16}{11}}$$

And Optimal z = Min z = $\boxed{\frac{126}{11}}$

Minimize $$z = 5x_1 + 7x_2$$

Subject to the constraints

$$\begin{aligned} 2x_1 + 3x_2 &\geq 42 \\ 3x_1 + 4x_2 &\geq 60 \\ x_1 + x_2 &\geq 18 \\ x_1, x_2 &\geq 0 \end{aligned}$$

Ans: x_1=12, x_2=6 Soln(= Min z) = 102

Introducing the surplus and artificial variables, R_1, R_2, the LPP is modified as follows:

Minimize $z = 5x_1 + 7x_2 + MR_1 + MR_2 + MR_3$

Subject to the constraints

$$\begin{aligned}
2x_1 + 3x_2 - s_1 \quad + R_1 \quad &= 42 \\
3x_1 + 4x_2 \quad - s_2 \quad + R_2 &= 60 \\
x_1 + x_2 \quad - s_3 \quad + R_3 &= 18 \\
x_1, x_2, s_1, s_2, s_3, R_1, R_2, R_3 &\geq 0
\end{aligned}$$

Basic	z	x1	x2	s1	s2	s3	R1	R2	R3	Sol
		-5+6M	-7+8M	-M	-M	-M	0	0	0	120M
z	1	$\cancel{-5}$	$\cancel{-7}$	$\cancel{0}$	$\cancel{0}$	$\cancel{0}$	$\cancel{-M}$	$\cancel{-M}$	$\cancel{-M}$	$\cancel{0}$
R1	0	2	$\boxed{3}$	-1	0	0	1	0	0	42
R2	0	3	4	0	-1	0	0	1	0	60
R3	0	1	1	0	0	-1	0	0	1	18
z	1	$-\frac{1}{3}+\frac{2M}{3}$	0	$-\frac{7}{3}+\frac{5M}{3}$	-M	-M	$\frac{7}{3}-\frac{8M}{3}$	0	0	98+8M
x2	0	2/3	1	-1/3	0	0	1/3	0	0	14
R2	0	1/3	0	4/3	-1	0	-4/3	1	0	4
R3	0	1/3	0	1/3	0	-1	-1/3	0	1	4

Basic	z	x1	x2	s1	s2	s3	R1	R2	R3	Sol
z	1	$-\frac{1}{3}+\frac{2M}{3}$	0	$-\frac{7}{3}+\frac{5M}{3}$	-M	-M	$\frac{7}{3}-\frac{8M}{3}$	0	0	98+8M
x2	0	2/3	1	-1/3	0	0	1/3	0	0	14
R2	0	1/3	0	$\boxed{4/3}$	-1	0	-4/3	1	0	4
R3	0	1/3	0	1/3	0	-1	-1/3	0	1	4
z	1	$\frac{1}{4}+\frac{M}{4}$	0	0	$-\frac{7}{4}+\frac{M}{4}$	-M	- M	$\frac{7}{4}-\frac{5M}{4}$	0	105+3M
x2	0	3/4	1	0	-1/4	0	0	1/4	0	15
s1	0	1/4	0	1	-3/4	0	-1	3/4	0	3
R3	0	1/4	0	0	1/4	-1	0	-1/4	1	3

Basic	z	x1	x2	s1	s2	s3	R1	R2	R3	Sol
z	1	$\frac{1}{4}+\frac{M}{4}$	0	0	$-\frac{7}{4}+\frac{M}{4}$	-M	-M	$\frac{7}{4}-\frac{5M}{4}$	0	105+3M
x2	0	3/4	1	0	-1/4	0	0	1/4	0	15
s1	0	1/4	0	1	-3/4	0	-1	3/4	0	3
R3	0	$\boxed{1/4}$	0	0	1/4	-1	0	-1/4	1	3
z	1	0	0	0	-2	1	-M	2-M	-1-M	102
x2	0	0	1	0	-1	3	0	1	-3	6
s1	0	0	0	1	-1	1	-1	1	-1	0
x1	0	1	0	0	1	-4	0	-1	4	12

Basic	z	x1	x2	s1	s2	s3	R1	R2	R3	Sol
z	1	0	0	0	-2	1	-M	2-M	-1-M	102
x2	0	0	1	0	-1	3	0	1	-3	6
s1	0	0	0	1	-1	$\boxed{1}$	-1	1	-1	0
x1	0	1	0	0	1	-4	0	-1	4	12
z	1	0	0	-1	-1	0	1-M	1-M	2-M	102
x2	0	0	1	-3	2	0	3	-2	0	6
s3	0	0	0	1	-1	1	-1	1	-1	0
x1	0	1	0	4	-3	0	-4	3	0	12

Basic	z	x1	x2	s1	s2	s3	R1	R2	R3	Sol
z	1	0	0	-1	-1	0	1-M	1-M	2-M	102
x2	0	0	1	-3	2	0	3	-2	0	6
s3	0	0	0	1	-1	1	-1	1	-1	0
x1	0	1	0	4	-3	0	-4	3	0	12

This is the optimal tableau.

Optimal solution: $x_1=12$, $x_2=6$

Optimal z = Minimum z = 102