MATH 2116: Linear Algebra

Class Note: 02

Institute of Infornation Technology (IIT), DU

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Linear Algebra

Key Terms

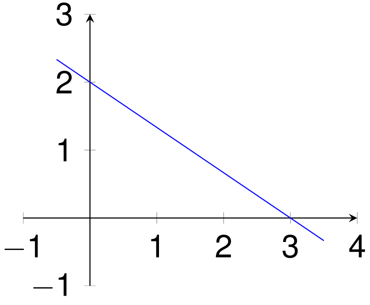
- Scalars: A **scalar** is a single number.
- Vectors: A **vector** is a specifically ordered one-dimensional array of values or simply ordered list of numbers.
- Matrices: A **matrix** is specifically ordered two-dimensional array of values.
- Linear Combination: We add vectors to get $\mathbf{v} + \mathbf{w}$. We multiply them by numbers c and d to get $c\mathbf{v}$ and $d\mathbf{w}$. Combining those two operations (adding $c\mathbf{v}$ to $d\mathbf{w}$) gives the **linear combination** $c\mathbf{v} + d\mathbf{w}$.

Linear Combination

The heart of Linear Algebra is in two operations – both with vectors. We can add two vectors (**vector addition**) and scale any vector (**scalar multiplication**). These two are the basic **vector operations**. We add two vectors $\mathbf{u}$ and $\mathbf{v}$ to get $\mathbf{u} + \mathbf{v}$. We multiply them by scalars/numbers c and d to get $c\mathbf{u}$ and $d\mathbf{v}$. Combining those two operations (adding $c\mathbf{v}$ to $d\mathbf{w}$) gives the **linear combination** $c\mathbf{v} + d\mathbf{w}$.

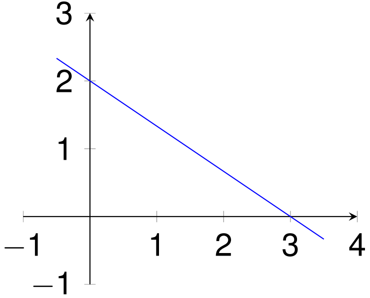
Linear Equation

Example 1: The equation $2x + 3y = 6$ is called linear because its solution set is a straight line in $\mathbb{R}^2$. A solution of the equation is a pair of numbers $(\alpha, \beta) \in \mathbb{R}^2$ such that $2\alpha + 3\beta = 6$.



$2x + 3y = 6$

Linear Equation



For example, $(3, 0)$ and $(0, 2)$ are solutions. Alternatively, we can write the first solution as $x = 3$, $y = 0$.

General equation of a line: $ax + by = c$, where x, y are variables and a, b, c are constants (except for the case $a = b = 0$).

Linear Equation

Definition: A linear equation in variables $x_1, x_2, \cdots, x_n$ is an equation of the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b \tag{1}$$

where $a_1, \cdots, a_n$, and b are constants.

A solution of the equation 1 is an array of numbers $(\gamma_1, \gamma_2, \cdots, \gamma_n) \in \mathbb{R}^n$ such that

$$a_1\gamma_1 + a_2\gamma_2 + \cdots + a_n\gamma_n = b$$

The set of all such solutions of a linear equation is called the **solution set**.

Linear Equation

Homogeneous/Non-homogeneous linear equation

If $b = 0$, then the equation 1 is called a **homogeneous linear equation** and if $b \neq 0$, then the equation 1 is called a **non-homogeneous linear equation**.

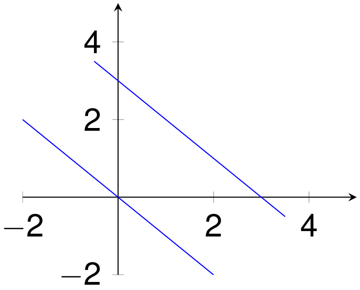
Non-degenerate/degenerate linear equation

The general linear equation 1 is also called **Non-degenerate linear equation**. A linear equation is said to be **degenerate linear equation** if it has the form $0x_1 + 0x_2 + \cdots + 0x_n = b$. The solution of such a degenerate linear equation is as follows:

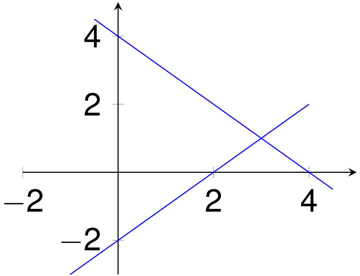
1. No solution, if the constant $b \neq 0$.
2. Every vector $\mathbf{u}$ is a solution, if the constant $b = 0$.

System of Linear Equations

Example 2: The linear system $\left.\begin{aligned} x + y &= 3 \\ x + y &= 0 \end{aligned}\right\}$ has no solution.

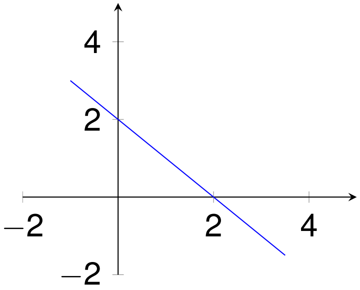


Example 3: The linear system $\left.\begin{array}{r} x+y=4 \\ x-y=2 \end{array}\right\}$ has only one solution.



System of Linear Equations

Example 4: The linear system $\left.\begin{aligned} x + y &= 2 \\ 3x + 3y &= 6 \end{aligned}\right\}$ has infinitely many solutions.



System of Linear Equations

A general form of a **System of Linear Equations** (or a set of m **Simultaneous Linear Equations** or **Linear System**) is as following

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \cdots\cdots\cdots\cdot & \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}\right\} \tag{2}$$

Here $x_1, x_2, \cdots, x_n$ are variables and a_{ij}, b_i are constants.
A *solution* of the system is a common solution of all equations in the system. A system of linear equations can have **one solution**, **infinitely many solutions**, or **no solution** at all.

System of Linear Equations

Consistent/Inconsistent System

A system of linear equations is called **consistent** if it has at least one solution and **inconsistent** if it has no solution.

Determinate/Indeterminate System

A consistent system is called **determinate** if it has a unique solution and **indeterminate** if it has more than one solution. An indeterminate system of linear equations always has an infinite number of solutions.

System of Linear Equations

Homogeneous/Non-homogeneous System

If all the b_i of the system 2 are zero, then the system 2 is called a **homogeneous system of linear equations** and if at least one b_i of the system 2 is not zero, then the system 2 is called a **non-homogeneous system of linear equations**.

Equivalent System

Two systems of linear equations are called **equivalent** if every solution of the first system is a solution of the second and conversely (vice versa).

Solution of homogeneous SLE

A system of linear equations of the following form is called **homogeneous system of linear equations**

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= 0 \\ \cdots\cdots\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= 0 \end{aligned}\right\} \tag{3}$$

Every homogeneous system of linear equations is consistent, since $x_1 = 0,\ x_2 = 0,\ \cdots,\ x_n = 0$ is always a solution of the system. This solution is called **trivial (or zero) solution**. If the other solutions exist, they a called the **non-trivial (or non-zero) solutions**.

Solution of homogeneous SLE

Thus the above homogeneous system can always be reduced to an equivalent homogeneous system in **echelon form**:

$$
\left.\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1r}x_r + \cdots + a_{1n}x_n &= 0 \\
c_{22}x_2 + c_{23}x_3 + \cdots + c_{2r}x_r + \cdots + c_{2n}x_n &= 0 \\
d_{33}x_3 + \cdots + d_{3r}x_r + \cdots + d_{3n}x_n &= 0 \\
\cdots\cdots\cdots \\
k_{rr}x_r + \cdots + k_{rn}x_n &= 0
\end{aligned}\right\}
$$

Hence we have the following two possibilities:

1. If $r = n$, then the system has only the **trivial (or zero) solution**.
2. If $r < n$, then the system also has an infinite number of **non-trivial (or non-zero) solutions**.

Solution of non-homogeneous SLE

A system of linear equations of the following form is called **non-homogeneous system of linear equations** if at least one b_i of the system is not equal to zero.

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \cdots\cdots\cdots & \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}\right\} \tag{4}$$

Solution of non-homogeneous SLE

If we reduced the system 4 to a simpler equivalent system in **echelon form**:

$$\left.\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1r}x_r + \cdots + a_{1n}x_n &= b_1 \\
c_{22}x_2 + c_{23}x_3 + \cdots + c_{2r}x_r + \cdots + c_{2n}x_n &= \bar{b}_2 \\
d_{33}x_3 + \cdots + d_{3r}x_r + \cdots + d_{3n}x_n &= \bar{b}_3 \\
\cdots\cdots\cdots\cdot & \\
k_{rr}x_r + \cdots + k_{rn}x_n &= \bar{b}_r \\
0 &= \bar{b}_{r+1} \\
\cdots & \\
0 &= \bar{b}_m
\end{aligned}\right\}$$

Solution of non-homogeneous SLE

where $r \leq n$. We see that there are three possible cases:

1. **Inconsistent system or no solution**, if $r < n$ and one of the numbers $\bar{b}_{r+1}, \cdots, \bar{b}_m$ is not zero.
2. **Determinate system or precisely one solution**, If $r = n$ and $\bar{b}_{r+1}, \cdots, \bar{b}_m$ if present are zero.
3. **Indeterminate system or infinitely many solutions**, If $r < n$ and $\bar{b}_{r+1}, \cdots, \bar{b}_m$ if present, are zero. Then any of these solutions is obtained by choosing values for the unknowns $x_{r+1}, \cdots, x_n$. The unknowns $x_{r+1}, \cdots, x_n$ are also called **free variables**.

Four Ways of Writing a Linear System I

1. As a **system of equations**:

$$\begin{cases} 2x_1 & + & 3x_2 & - & 2x_3 & = & 7 \\ x_1 & - & x_2 & - & 3x_3 & = & 5 \end{cases}$$

2. As an **augmented matrix**:

$$\left[\begin{array}{ccc|c} 2 & 3 & -2 & 7 \\ 1 & -1 & -3 & 5 \end{array}\right]$$

3. As a **vector equation** $(x_1\mathbf{v_1} + x_2\mathbf{v_2} + \cdots + x_n\mathbf{v_n} = \mathbf{b})$:

$$x_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ -1 \end{bmatrix} + x_3 \begin{bmatrix} -2 \\ -3 \end{bmatrix} = \begin{bmatrix} 7 \\ 5 \end{bmatrix}$$

Four Ways of Writing a Linear System II

4. As a **matrix equation** ($\mathbf{A}x = \mathbf{b}$):

$$\begin{bmatrix} 2 & 3 & -2 \\ 1 & -1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 5 \end{bmatrix}$$

In particular, *all four have the same solution set*.

Solving systems of linear equations

Row Reduction

Row reduction / Elementary row operations / Gaussian elimination involves a successive "Whiting away" of the variables in order to isolate their values. This method is based on the following **three elementary operations** which alter the form of the equations, but not the solutions:

1. Interchange a pair of equations.
2. Multiplying an equation by non-zero number.
3. Adding a multiple of one equation to another eq.

Theorem (i) Applying elementary operations to a SLE does not change the solution set of the system. **(ii)** Any elementary operation can be undone by another elementary operation

Gaussian elimination

The goal of the Gaussian elimination is to convert the augmented matrix into **row echelon form**. A matrix is in **row echelon form** if:

1. All zero rows are at the bottom.
2. The first nonzero entry of a row is to the right of the first nonzero entry of the row above.
3. Below the first nonzero entry of a row, all entries are zero.

$$\begin{bmatrix} \boxed{*} & * & * & * & * \\ 0 & 0 & \boxed{*} & * & * \\ 0 & 0 & 0 & \boxed{*} & * \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{aligned} * &= \text{any number} \\ \boxed{*} &= \text{any nonzero number (pivot)} \end{aligned}$$

A **pivot** is the first nonzero entry of a row of a matrix in row echelon form.

Solving systems of linear equations

Gauss-Jordan reduction

The goal of the Gauss-Jordan reduction is to convert the augmented matrix into **reduced row echelon form**. A matrix is in **reduced row echelon form** if it is in row echelon form, and in addition:

4. Each pivot is equal to 1.
5. Each pivot is the only nonzero entry in its column.

$$\begin{bmatrix} \boxed{1} & * & 0 & 0 & * \\ 0 & 0 & \boxed{1} & 0 & * \\ 0 & 0 & 0 & \boxed{1} & * \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{aligned} * &= \text{any number} \\ \boxed{1} &= \text{pivot} \end{aligned}$$

We say that x_i is a **free variable** if its corresponding column in A is not a pivot column.

Number of Solutions

There are **three possibilities** for the reduced row echelon form of the augmented matrix of a linear system.

1. **The last column is a pivot column.** In this case, the system is inconsistent. There are zero solutions. For example, the matrix
$$\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]$$
comes from a linear system with no solutions.
2. **Every column except the last column is a pivot column.** In this case, the system has a unique solution. For example, the matrix
$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \end{array}\right]$$
tells us that the unique solution is $(x, y, z) = (a, b, c)$.

Number of Solutions

3. **The last column is not a pivot column, and some other column is not a pivot column either.** In this case, the system has infinitely many solutions, corresponding to the infinitely many possible values of the free variable(s). For example, in the system corresponding to the matrix

$$\left[\begin{array}{cccc|c} 1 & -2 & 0 & 3 & 1 \\ 0 & 0 & 1 & 4 & -1 \end{array}\right]$$

any values for x_2 and x_4 yield a solution to the system of equations.

How to solve a system of linear equations

- Order the variables
- Write down the augmented matrix of the system
- Convert the matrix to **row echelon form**
- Check for consistency
- Convert the matrix to **reduced row echelon form**
- Write down the system corresponding to the reduced row echelon form
- Determine **leading (pivot)** and **free variables**
- Rewrite the system so that the leading variables are on the left while everything else is on the right
- Assign parameters to the free variables and write down the general solution in parametric form

Try to solve the following SLE

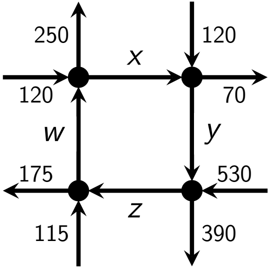
1. $$\begin{cases} x & - & y & & & = & 2 \\ 2x & - & y & - & z & = & 3 \\ x & + & y & + & z & = & 6 \end{cases} \quad \text{Sol.: } \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$$

2. $$\begin{cases} x & + & 2y & + & 3z & = & 6 \\ 2x & + & 5y & + & 2z & = & 4 \\ 6x & - & 3y & + & z & = & 2 \end{cases} \quad \text{Sol.: } \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$$

3. $$\begin{cases} x & + & 3y & + & 5z & = & 4 \\ x & + & 2y & - & 3z & = & 5 \\ 2x & + & 5y & + & 2z & = & 8 \end{cases} \quad \text{Sol.: No sol.}$$

Try to solve the following SLE

4. The following diagram represents traffic flow around the town square. The streets are all one way, and the numbers and arrows indi- cate the number of cars per hour flowing along each street, as measured by sensors underneath the roads.



There are no sensors underneath some of the streets, so we do not know how much traffic is flowing around the square itself. What are the values of x, y, z, w?

Try to solve the following SLE

5. $\begin{cases} 2x + 10y = -1 \\ 3x + 15y = 2 \end{cases}$

6. $\begin{cases} 2x + y + 12z = 1 \\ x + 2y + 9z = -1 \end{cases}$

7. $\left[\begin{array}{cccc|c} 1 & 0 & 0 & 3 & 2 \\ 0 & 0 & 1 & 4 & -1 \end{array}\right]$

Thanks

Thank you all.