

Linear Programming Simplex Method

Already covered

- Introduction to linear programming
- Formulate to linear programming from the given problem.
- Solve linear programming problem using geometrical technique
- Convert LP problem to standard form
- Solve LP
- using simplex method.

Algebraic Solution of LPPs - Simplex Method

To solve an LPP algebraically, we first put it in the **standard form**.

- This means all decision variables are nonnegative
- all constraints (other than the nonnegativity restrictions) are **equations with nonnegative RHS.**

Tasks to convert standard form

- **Converting inequalities into equations**
 - Use slack/excess variable to do this
 - Make sure that right side doesn't contain negative sign.
 - Convert unrestricted variables to restricted variables.

Basic variables, Basic feasible Solutions

- Consider an LPP (in standard form) with m constraints and n decision variables.
 - We assume $m \leq n$.
 - We choose $n - m$ variables and set them equal to zero.
 - Thus we will be left with a system of m equations in m variables.
 - If this $m \times m$ square system has a unique solution, this solution is called a **basic solution**.
 - If further if it is feasible, it is called a **Basic Feasible Solution** (BFS).

Basic variables, Nonbasic Variables

- The $n-m$ variables set to zero are called **nonbasic** and the m variables which we are solving for are known as **basic variables**.
- Thus a basic solution is of the form
 $\mathbf{x} = (x_1, x_2, \dots, x_n)$
where $n-m$ “components” are zero and the remaining m components form the unique solution of the square system (formed by the m constraint equations).

Note that we may have a **maximum** of $\binom{n}{m}$ basic solutions.

Consider the LPP:

Maximize
$$z = 2x_1 + 3x_2$$

Subject to
$$x_1 + 3x_2 \leq 6$$

$$3x_1 + 2x_2 \leq 6$$

$$x_1, x_2 \geq 0$$

This is equivalent to the LPP (in standard form)

Maximize
$$z = 2x_1 + 3x_2$$

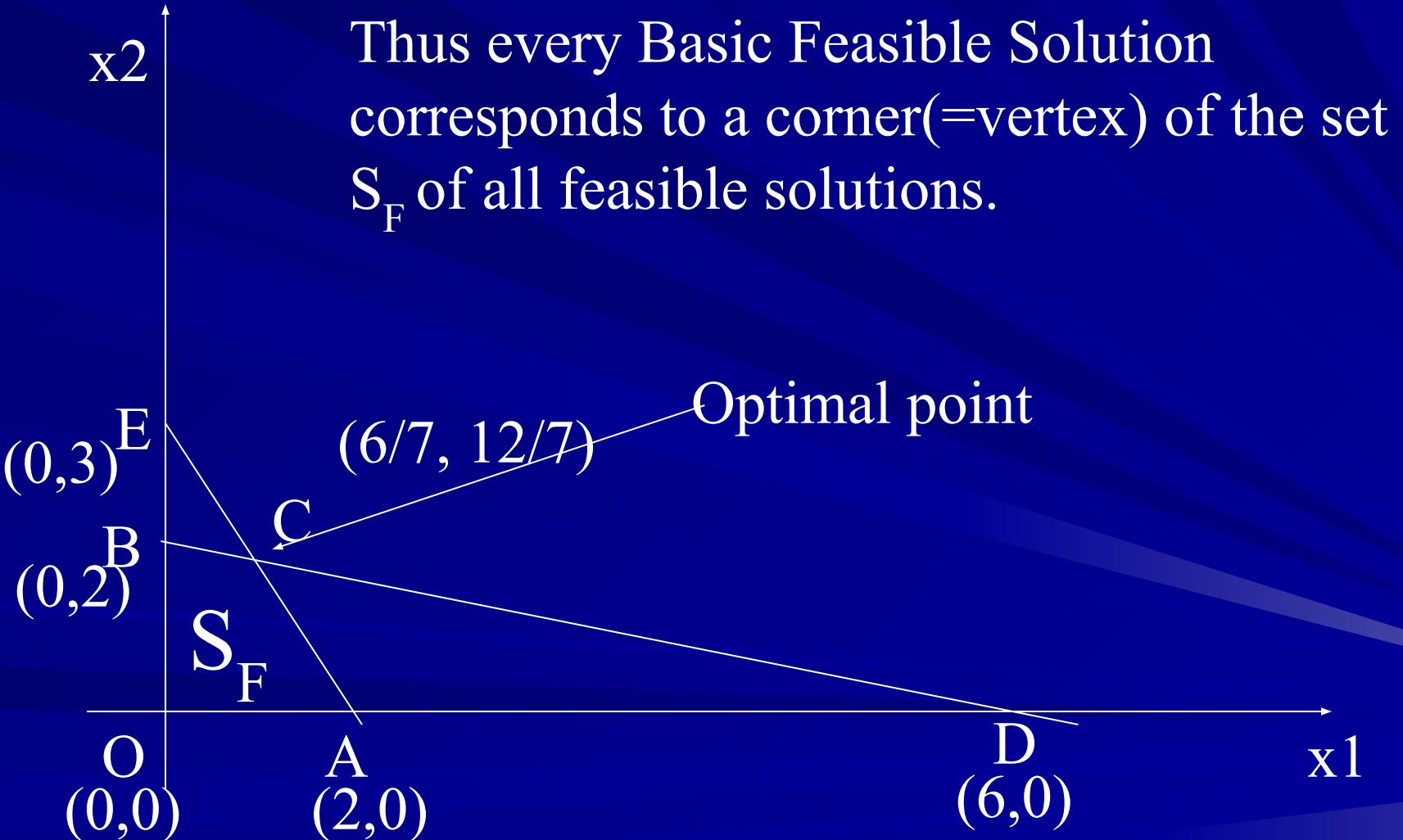
Subject to
$$x_1 + 3x_2 + s_1 = 6$$

$$3x_1 + 2x_2 + s_2 = 6$$

$$x_1, x_2, s_1, s_2 \geq 0$$

s_1, s_2 are slack variables.

Graphical solution of the above LPP



Nonbasic (zero) variables	Basic variables	Basic solution (x_1, x_2, s_1, s_2)	Assoc- iated corner point	Feasible?	Object-iv e value, z
(x_1, x_2)	(s_1, s_2)	$(0, 0, 6, 6)$	O	Yes	0
(x_1, s_1)	(x_2, s_2)	$(0, 2, 0, 2)$	B	Yes	6
(x_1, s_2)	(x_2, s_1)	$(0, 3, -3, 0)$	E	No	-
(x_2, s_1)	(x_1, s_2)	$(6, 0, 0, -12)$	D	No	-
(x_2, s_2)	(x_1, s_1)	$(2, 0, 4, 0)$	A	Yes	4
(s_1, s_2)	(x_1, x_2)	$(\frac{6}{7}, \frac{12}{7}, 0, 0)$	C	Yes	$48/7$ Optimal

The Simplex algorithm

Example Consider the LPP

Maximize $z = 8x_1 + 9x_2$

Subject to the constraints

$$2x_1 + 3x_2 \leq 50$$

$$2x_1 + 6x_2 \leq 80$$

$$3x_1 + 3x_2 \leq 70$$

$$x_1, x_2 \geq 0$$

Introducing slack variables, the LPP is same as

$$\text{Maximize } z = 8x_1 + 9x_2 + 0s_1 + 0s_2 + 0s_3$$

subject to the constraints

$$\begin{array}{ll} 2x_1 + 3x_2 + s_1 & = 50 \\ 2x_1 + 6x_2 + s_2 & = 80 \end{array} \quad (*)$$

$$3x_1 + 3x_2 + s_3 = 70$$

$$x_1, x_2, s_1, s_2, s_3 \geq 0$$

Basic Feasible Solution

- Let
 - x_1, x_2 as nonbasic variables
 - Then BFS can be got immediately as
 - $(0,0, 50,80,70)$.
- This is possible because in the LHS of the constraint equations (*) the coefficients of the basic variables s_1, s_2, s_3 form an Identity matrix (of size 3×3).

Simplex tableau (Starting tableau)

Basic	z	x_1	x_2	s_1	s_2	s_3	Solution
z	1	-8	-9	0	0	0	0
s_1	0	2	3	1	0	0	50
s_2	0	2	6	0	1	0	80
s_3	0	3	3	0	0	1	70

Closer Look (1)

- We note that the z-row is the objective function row.
- The remaining 3 rows are the basic variable rows.
- Each row corresponds to a basic variable;
 - the leftmost variable denotes the basic variable corresponding to that row.

Closer Look (2)

- In the objective function row,
 - the coefficients of the basic variables are zero.
- In each column corresponding to a basic variable,
 - basic variable has a non-zero coefficient, namely 1,
 - all the other basic variables have zero coefficients.

Closer Look (3)

- We see at present, the objective function, z , has value zero.
- We now seek to make
 - one of the nonbasic variables as basic (and so) one of the basic variables will become nonbasic (that is will drop down to zero)
 - the nonbasic variable that will become basic is chosen such that the objective function will “improve”:
 - in a maximization problem it will increase
 - in a minimization problem it will decrease.
- The nonbasic variable that will become basic is referred to as “**entering**” variable and the basic variable that will become nonbasic is referred to as “**leaving**” variable.

Criterion for “entering” variable: **(Optimality Condition)**

Choose that variable as the “entering” variable which has

the **most –ve** coefficient in the z-row in case it is a **maximization** problem
(as the variable which has the most +ve coefficient in the z-row in case it is a minimization problem).

(Break the ties arbitrarily.)

Criterion for “leaving” variable (Feasibility Condition)

Let b_i be the RHS of the i th row. Let a_{ij} be the coefficient of the entering variable x_j in the i th row. The following “minimum ratio test” decides the leaving variable:

Choose x_k as the leaving variable where k is given as that row index i for which the ratio

$$\left\{ \frac{b_i}{a_{ij}} : a_{ij} > 0 \right\} \text{ is least.}$$

(Break the ties arbitrarily.)

Pivot element

- The entering variable column is called the **pivot column**.
- The leaving variable row is called the **pivot row**.
- The coefficient in the intersection of the two is referred to as the **pivot element**.

Operations

- We apply *elementary row operations* to modify the simplex tableau so that the pivot column has **1** at the pivot element and **zero** in all other places.

The elementary Row operations are as follows:

- ◆ New pivot row = old pivot row \div pivot element
- ◆ New z row = old z row – (coefficient of the entering variable in old z row * New pivot row)
- ◆ Any other new row = corresponding old row – (old coefficient of the entering variable in that row * New pivot row)
- ◆ We shall also change the legend of the *new pivot row only* as the entering variable.

Simplex tableau (Starting tableau)

Basic	z	x_1	x_2	s_1	s_2	s_3	Solution
z	1	-8	-9	0	0	0	0
s_1	0	2	3	1	0	0	50
s_2	0	2	6	0	1	0	80
s_3	0	3	3	0	0	1	70

Leaves

Pivot element

Performing the elementary Row operations,
we get the new Simplex tableau below

Basic	z	x_1	x_2	s_1	s_2	s_3	Solution
z	1	-5	0	0	$3/2$	0	120
s_1	0	1	0	1	$-1/2$	0	10
x_2	0	$1/3$	1	0	$1/6$	0	$40/3$
s_3	0	2	0	0	$-1/2$	1	30

Leaves

↓ Enters

Pivot element

Performing the elementary Row operations,
we get the new Simplex tableau below

Basic	z	x_1	x_2	s_1	s_2	s_3	Solution
z	1	0	0	5	-1	0	170
x_1	0	1	0	1	-1/2	0	10
x_2	0	0	1	-1/3	1/3	0	10
s_3	0	0	0	-2	1/2	1	10

Leaves

↓ Enters

Pivot element

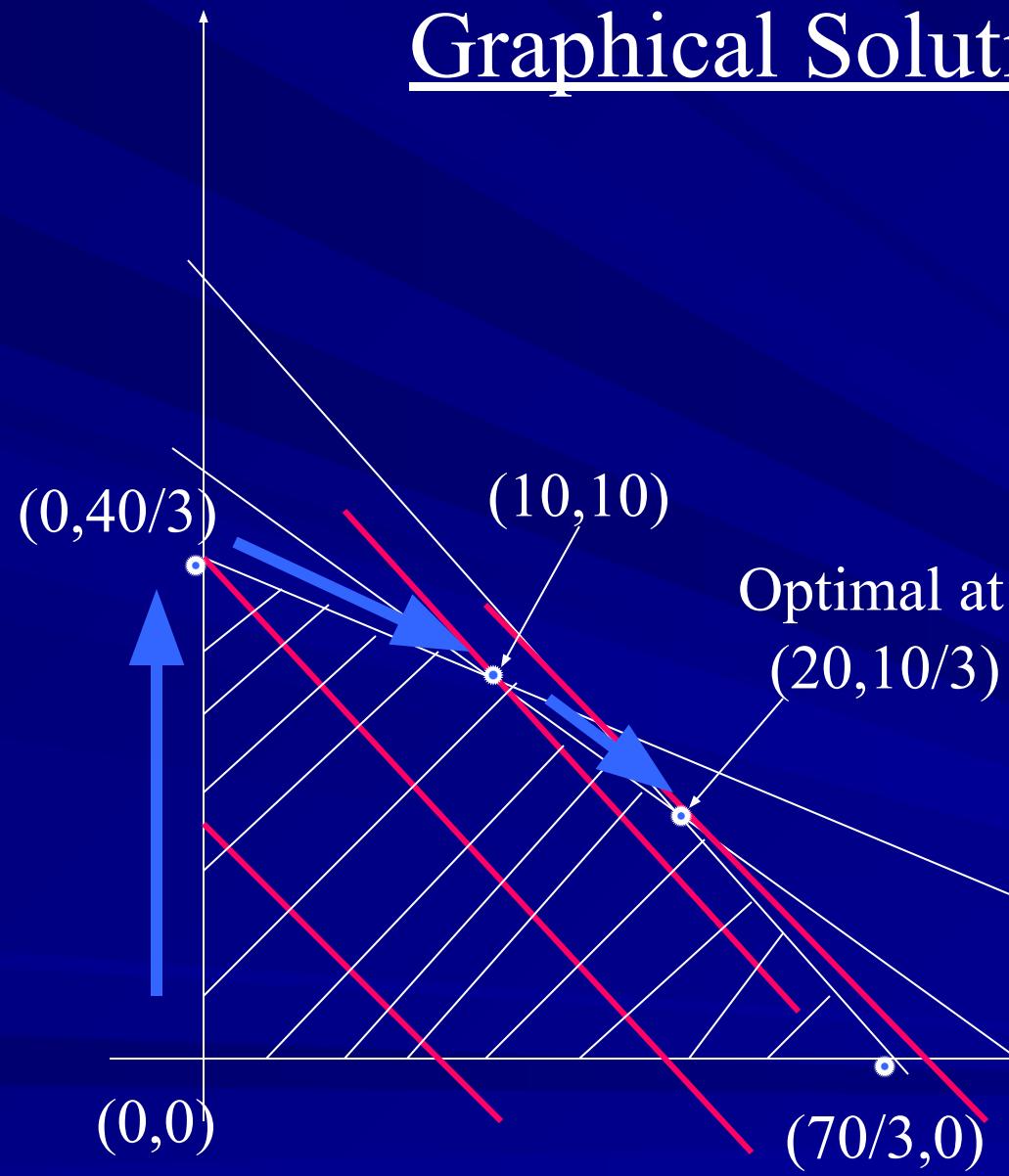
Performing the elementary Row operations,
we get the new Simplex tableau below.

This is optimal as all entries in z row are ≥ 0 .

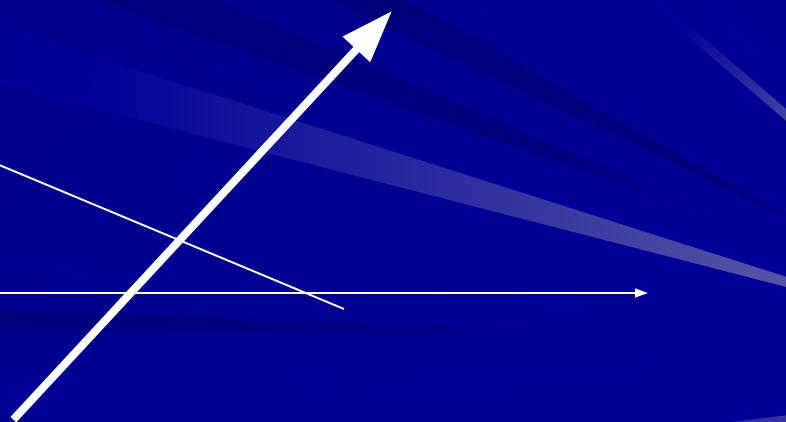
Basic	z	x_1	x_2	s_1	s_2	s_3	Solution
z	1	0	0	5	0	0	190
x_1	0	1	0	-1	0	0	20
x_2	0	0	1	1	0	0	10/3
s_2	0	0	0	-4	1	2	20

Optimum value = 190 at $x_1=20, x_2=10/3$

Graphical Solution of the problem



Direction of
increasing z



Example Consider the LPP

Maximize
$$z = 2x_1 + x_2 - 3x_3 + 5x_4$$

Subject to the constraints

$$x_1 + 2x_2 - 2x_3 + 4x_4 \leq 40$$

$$2x_1 - x_2 + x_3 + 2x_4 \leq 8$$

$$4x_1 - 2x_2 + x_3 - x_4 \leq 10$$

$$x_1, x_2, x_3, x_4 \geq 0$$

Introducing slack variables, the LPP becomes

Maximize

$$z = 2x_1 + x_2 - 3x_3 + 5x_4 + 0s_1 + 0s_2 + 0s_3$$

Subject to the constraints

$$x_1 + 2x_2 - 2x_3 + 4x_4 + s_1 = 40$$

$$2x_1 - x_2 + x_3 + 2x_4 + s_2 = 8$$

$$4x_1 - 2x_2 + x_3 - x_4 + s_3 = 10$$

$$x_1, x_2, x_3, x_4, s_1, s_2, s_3 \geq 0$$

Basic	z	x1	x2	x3	x4	s1	s2	s3	Sol.
z 1	-2	-1 3	-5 0 0	0 0	0				
s1 0	1 2	-2	4 1 0	0	0	40			
s2	0 2 -1 1	2 0 1 0	2						
s3	0 4 -2 1	-1 0 0 1	10						
z 1	3	-7/2	11/2	0 0	5/2	0	20		
s1 0	-3 4 -4	0 4	-2 0	24					
x4 0	1	-1/2	1/2	1 0	1/2 0	4			
s3 0	5	-5/2	3/2	0 0	1/2 1	14			
z 1	3/8 0	2	0 7/8 3/4	0 4 1					
x2 0		-3/4	1	-1	0 1/4 -1/2	0 6			
x4 0	5/8	0 0	1	1/8	1/4	0 7			
s3 0	25/8	0 -1	0 0	5/8 -3/4	1 29				

The last tableau is the optimal tableau as all entries in the objective function row are ≥ 0 and the LPP is a maximization problem.

Optimal Solution is

$$x_1 = 0, x_2 = 6, x_3 = 0, x_4 = 7$$

And the Optimal z (= Maximum z)

$$= 41$$

Solve the following LPP:

Maximize $z = x_1 + 2x_2 + 4x_3$

Subject to the constraints

$$3x_1 + x_2 + 5x_3 \leq 10$$

$$x_1 + 4x_2 + x_3 \leq 8$$

$$2x_1 + 2x_3 \leq 7$$

$$x_1, x_2, x_3 \geq 0$$

Introducing slack variables, the LPP becomes

Maximize

$$z = x_1 + 2x_2 + 4x_3 + 0s_1 + 0s_2 + 0s_3$$

Subject to the constraints

$$3x_1 + x_2 + 5x_3 + s_1 = 10$$

$$x_1 + 4x_2 + x_3 + s_2 = 8$$

$$2x_1 + 2x_3 + s_3 = 7$$

$$x_1, x_2, x_3, s_1, s_2, s_3 \geq 0$$

Basic	z	x1	x2	x3	s1	s2	s3	Sol.
z	1	-1	-2	-4	0	0	0	
s1	0	3	1	5	1	0	0	10
s2	0	1	4	1	0	1	0	8
s3	0	2	0	2	0	0	1	7
z	1	7/5	-6/5	0	4/5	0	0	8
x3	0	3/5	1/5	1	1/5	0	0	2
s2	0	2/5	19/5	0	-1/5	1	0	6
s3	0	4/5	-2/5	0	-2/5	0	1	3
z	1	29/19	0	0	14/19	6/19	0	188/19
x3	0	11/19	0	1	4/19	-1/19	0	32/19
x2	0	2/19	1	0	-1/19	5/19	0	30/19
s3	0	16/19	0	0	-8/19	2/19	1	69/19

The last tableau is the optimal tableau as all entries in the objective function row are ≥ 0 and the LPP is a maximization problem.

Optimal Solution is

$$x_1 = 0, x_2 = 30/19, x_3 = 32/19$$

And the Optimal z (= Maximum z)

$$= 188/19$$

Artificial Variable Techniques -

Big M-method

M method – basic idea

- If in a starting simplex tableau,
 - we don't have an identity submatrix (i.e. an obvious starting BFS),
 - then we introduce artificial variables to have a starting BFS.
 - There are two methods to find the starting BFS and solve the problem –
 - the Big M method
 - two-phase method

M method – when required

- Suppose a constraint equation i does not have a slack variable
 - there is no i th unit vector column in the LHS of the constraint equations. (This happens for example when the i th constraint in the original LPP is either \geq or $=$.)
- Then we augment the equation with an artificial variable R_i to form the i th unit vector column.

Artificial Variable

- As the artificial variable is extraneous to the given LPP,
 - we use a **feedback mechanism** in which the optimization process automatically attempts to force these variables to zero level.
 - This is achieved by **giving a large penalty** to the coefficient of the artificial variable in the objective function as follows:

Artificial variable objective coefficient
= - M in a maximization problem,
= M in a *minimization* problem
- where M is a very large positive number.

Consider the LPP:

$$\text{Minimize } z = 2x_1 + x_2$$

Subject to the constraints

$$3x_1 + x_2 \geq 9$$

$$x_1 + x_2 \geq 6$$

$$x_1, x_2 \geq 0$$

Putting this in the standard form, the LPP is:

Minimize $z = 2x_1 + x_2$

Subject to the constraints

$$3x_1 + x_2 - s_1 = 9$$

$$x_1 + x_2 - s_2 = 6$$

$$x_1, x_2, s_1, s_2 \geq 0$$

Here s_1, s_2 are surplus variables.

Note that we do NOT have a 2x2 identity submatrix in the LHS.

Introducing the artificial variables, R_1 , R_2 , the LPP is modified as follows:

$$\text{Minimize } z = 2x_1 + x_2 + MR_1 + MR_2$$

Subject to the constraints

$$3x_1 + x_2 - s_1 + R_1 = 9$$

$$x_1 + x_2 - s_2 + R_2 = 6$$

$$x_1, x_2, s_1, s_2, R_1, R_2 \geq 0$$

Note that we now have a 2x2 identity submatrix in the coefficient matrix of the constraint equations.

Now we solve the above LPP by the Simplex method.

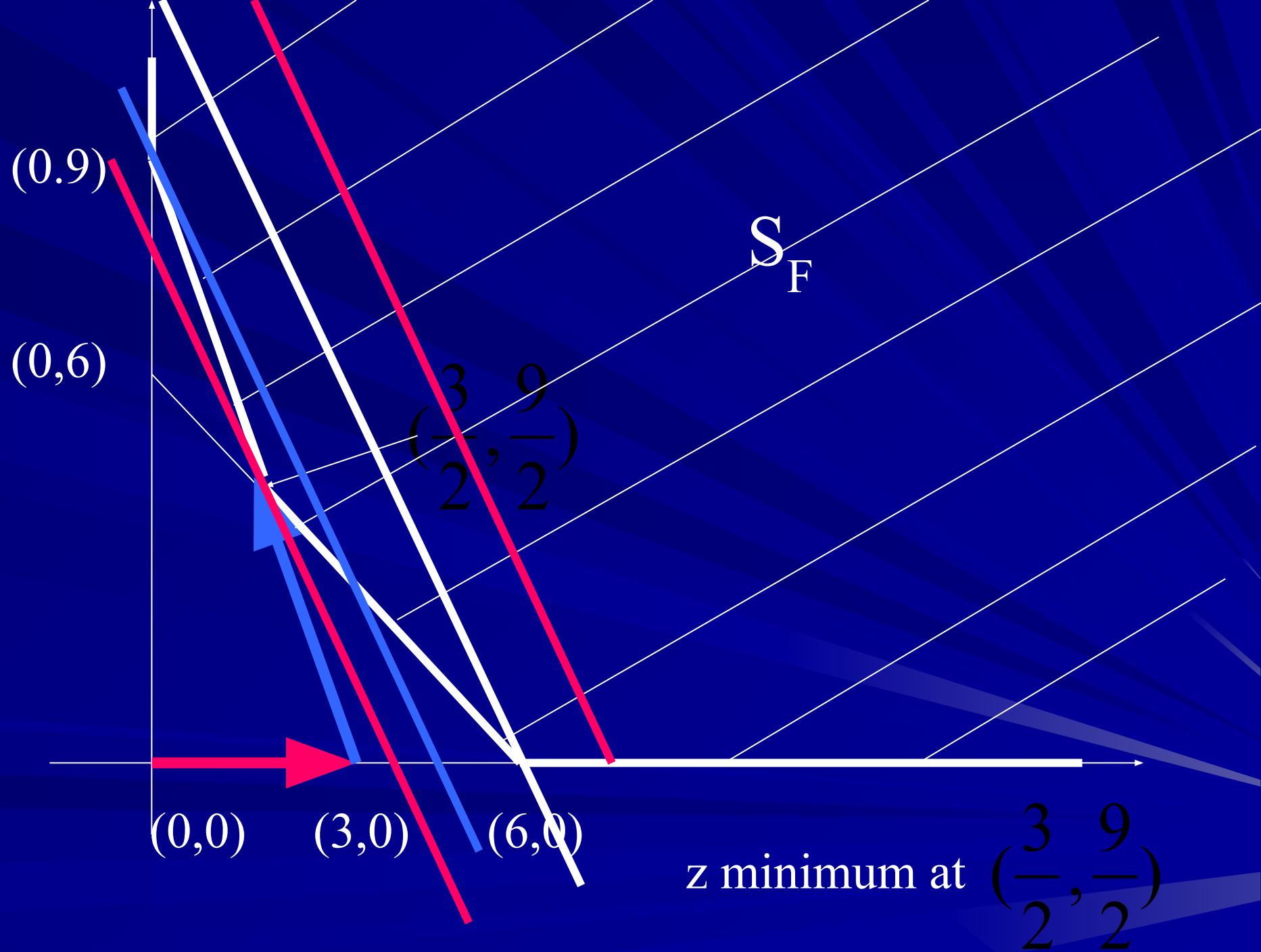
Basic	z	x1	x2	s1	s2	R1	R2	Sol.
	-2	+4M	-1	2M	-M	-M	0	15M
z 1	-2	-1	0	0	-M	-M	0	/
→R1 0	3	1	-1	0	1	0	9	
R2	0	1	1	0	-1	0	1	6
z 1	0	-1/3+	-2/3+	-M	2/3-	0	6+3M	
		2M/3	M/3		4M/3			
x1 0	1	1/3	-1/3	0	1/3	0	3	
→R2 0	0	2/3	1/3	-1	-1/3	1	3	
z 1	0	0	-1/2	-1/2	1/2-M	1/2-M	15/2	
x1 0		0	-1/2	1/2	1/2	-1/2	3/2	
x2 0	0	1	1/2	-3/2	-1/2	3/2	9/2	

Note that we have got the optimal solution to the given problem as

$$x_1 = \frac{3}{2}, \quad x_2 = \frac{9}{2}$$

Optimal $z = \text{Minimum } z = \frac{15}{2}$

It is illuminating to look at the graphical solution also.



Maximize $z = 2x_1 + 3x_2 - 5x_3$

Subject to the constraints

$$x_1 + x_2 + x_3 = 7$$

$$2x_1 - 5x_2 + x_3 \geq 10$$

$$x_1, x_2, x_3 \geq 0$$

Introducing surplus and artificial variables,
 s_2 , R_1 and R_2 , the LPP is modified as follows:

Maximize

$$z = 2x_1 + 3x_2 - 5x_3 - MR_1 - MR_2$$

Subject to the constraints

$$x_1 + x_2 + x_3 + R_1 = 7$$

$$2x_1 - 5x_2 + x_3 - s_2 + R_2 = 10$$

$$x_1, x_2, x_3, s_2, R_1, R_2 \geq 0$$

Now we solve the above LPP by the Simplex method.

Basic	z	x1	x2	x3	s2	R1	R2	Sol.
	-2	$-3M$	-3	$-4M$	5	$-2M$	M	0
z 1	-2	-3	5	0	M	M	0	-17M
R1 0	1	1	1	0	1	0	7	
R2	0	2	-5	1	-1	0	1	10
z 1	0	-8	6	-1	0	1	10	$-$
		$7M/2$	$M/2$	$M/2$		$3M/2$	$2M$	
R1 0	0	$7/2$	$1/2$	$1/2$	1	$-1/2$	2	
x1 0	1	$-5/2$	$1/2$	$-1/2$	0	$1/2$	5	
z 1	0	0	$50/7$	$1/7$	$16/7$	$+ -1/7$	$102/7$	
			M		$+M$			
x2 0	0	1	$1/7$	$1/7$	$2/7$	$-1/7$	$4/7$	
x1 0	1	0	$6/7$	$-1/7$	$5/7$	$1/7$	$45/7$	

The optimum (Maximum) value of

$$z = 102/7$$

and it occurs at

$$x_1 = 45/7, x_2 = 4/7, x_3 = 0$$

Remarks

- If in any iteration, there is a tie for entering variable between an artificial variable and other variable (decision, surplus or slack), we must prefer the non-artificial variable to enter the basis.
- If in any iteration, there is a tie for leaving variable between an artificial variable and other variable (decision, surplus or slack), we must prefer the *artificial* variable to leave the basis.

- If in the final optimal tableau, an artificial variable is present in the basis at a non-zero level, this means our original problem has *no feasible solution*.

$$\text{Maximize} \quad z = 5x_1 + 6x_2$$

Subject to the constraints

$$-2x_1 + 3x_2 = 3$$

$$x_1 + 2x_2 \leq 5$$

$$6x_1 + 7x_2 \leq 3$$

$$x_1, x_2 \geq 0$$

Introducing slack and artificial variables, s_2 , s_3 , and R_1 , the LPP is modified as follows:

Maximize
$$z = 5x_1 + 6x_2 - MR_1$$

Subject to the constraints

$$-2x_1 + 3x_2 + R_1 = 3$$

$$x_1 + 2x_2 + s_2 = 5$$

$$6x_1 + 7x_2 + s_3 = 3$$

$$x_1, x_2, R_1, s_2, s_3 \geq 0$$

Basic	z	x1	x2	R1	s2	s3	Sol
	-5+2M	-6-3M	0	0	0	0	-3M
z 1	-5	-6	M	0	0	0	
R1 0	-2	3	1	0	0	3	
s2	0	1	2	0	1	0	5
s3	0	6	7	0	0	1	3
z 1	1/7+	0	0	0	6/7+	18/7-	
	32M/7				3M/7	12M/7	
R1 0	-32/7	0	1	0	-3/7	12/7	
s2 0	-12/7	0	0	1	-2/7	29/7	
x2 0	6/7	1	0	0	1/7	3/7	

This is the optimal tableau. As R_1 is not zero, there is NO feasible solution.

$$\text{Minimize} \quad z = 4x_1 + 6x_2$$

Subject to the constraints

$$-2x_1 + 3x_2 = 3$$

$$4x_1 + 5x_2 \geq 10$$

$$4x_1 + 8x_2 \geq 5$$

$$x_1, x_2 \geq 0$$

Introducing the surplus and artificial variables,
 R_1, R_2 , the LPP is modified as follows:

Minimize $z = 4x_1 + 6x_2 + M R_1 + M R_2 + M R_3$

Subject to the constraints

$$-2x_1 + 3x_2 + R_1 = 3$$

$$4x_1 + 5x_2 - s_2 + R_2 = 10$$

$$4x_1 + 8x_2 - s_3 + R_3 = 5$$

$$x_1, x_2, s_2, s_3, R_1, R_2, R_3 \geq 0$$

Basic	z	x1	x2	s2	s3	R1	R2	R3	Sol.
		-4+6M	-6-16M	-M	-M	0	0	0	18M
z	1	-4	-6	0	0	-M	-M	-M	0
R1	0	-2	3	0	0	1	0	0	3
R2	0	4	5	-1	0	0	1	0	10
R3	0	4	8	0	-1	0	0	1	5
z	0	-1-2M	0	-M	-3/4	0	0	3/4	15/4
R1	0	-7/2	0	0	3/8	1	0	-3/8	9/8
R2	0	3/2	0	-1	5/8	0	1	-5/8	55/8
x2	0	1/2	1	0	-1/8	0	0	1/8	5/8

Basic	z	x1	x2	s2	s3	R1	R2	R3	Sol.
z	1	-1 -2M +M	0	-M	-3/4 0	0	0	3/4	15/4
R1	0	-7/2	0	0	3/8 1	0	0	-3/8	9/8
R2	0	3/2	0	-1	5/8	0	1	-5/8	55/8
x2	0	1/2	1	0	-1/8	0	0	1/8	5/8
z	1	-8 + 22M/3	0	-M	0 2	0 -M	6		
s3	0	-28/3	0	0	1 8/3	0 -1	3		
R2	0	22/3	0	-1	0 -5/3	1 0	0	5	
x2	0	-2/3	1	0	0 1/3	0 0	0	1	

Basic	z	x1	x2	s2	s3	R1	R2	R3	Sol.
z	1	-8 + 0	-M	0	2	0	-M	6	
		$22M/3$			$-8M/3$		$+5M$		
s3	0	$-28/3$	0	0	1	$8/3$	0	-1	3
R2	0	$22/3$	0	-1	0	$-5/3$	1	0	5
x2	0	$-2/3$	1	0	0	$1/3$	0	0	1
z	1	0 0	$-6/11$	0	$2/11$	$12/11$	-M		$\frac{126}{11}$
					$-M$	$-M$			
s3	0	0 0	$-14/11$	1	$6/11$	$14/11$	-1		$\frac{103}{11}$
x1	0	1 0	$-3/22$	0	$-5/22$	$3/22$	0		$\frac{15}{22}$
x2	0	0 1	$-1/11$	0	$2/11$	$1/11$	0		$\frac{16}{11}$

This is the optimal tableau.

The Optimal solution is:

$$x_1 = \frac{15}{22}, \quad x_2 = \frac{16}{11}$$

And Optimal z = Min z = $\frac{126}{11}$

$$\text{Minimize} \quad z = 5x_1 + 7x_2$$

Subject to the constraints

$$2x_1 + 3x_2 \geq 42$$

$$3x_1 + 4x_2 \geq 60$$

$$x_1 + x_2 \geq 18$$

$$x_1, x_2 \geq 0$$

$$\text{Ans: } x_1=12, x_2=6 \quad \text{Soln}(=\text{Min } z) = 102$$

Introducing the surplus and artificial variables, R_1 , R_2 , the LPP is modified as follows:

Minimize $z = 5x_1 + 7x_2 + MR_1 + MR_2 + MR_3$

Subject to the constraints

$$2x_1 + 3x_2 - s_1 + R_1 = 42$$

$$3x_1 + 4x_2 - s_2 + R_2 = 60$$

$$x_1 + x_2 - s_3 + R_3 = 18$$

$$x_1, x_2, s_1, s_2, s_3, R_1, R_2, R_3 \geq 0$$

Basic	z	x1	x2	s1	s2	s3	R1	R2	R3	Sol
		$-5+6M$	$-7+8M$	$-M$	$-M$	$-M$	0	0	0	$120M$
z	1	-5	✓	0	0	0	-M	-M	-M	0
R1	0	2	3	-1	0	0	1	0	0	42
R2	0	3	4	0	-1	0	0	1	0	60
R3	0	1	1	0	0	-1	0	0	1	18
z	1	$-\frac{1}{3} + \frac{2M}{3}$	0	$-\frac{7}{3} + \frac{5M}{3}$	$-M$	$-M$	$\frac{7}{3} - \frac{8M}{3}$	0	0	$98+8M$
x2	0	$2/3$	1	$-1/3$	0	0	$1/3$	0	0	14
R2	0	$1/3$	0	$4/3$	-1	0	$-4/3$	1	0	4
R3	0	$1/3$	0	$1/3$	0	-1	$-1/3$	0	1	4

Basic	z	x1	x2	s1	s2	s3	R1	R2	R3	Sol
		$\frac{1}{3} + \frac{2M}{3}$		$\frac{7}{3} + \frac{5M}{3}$			$\frac{7}{3} - \frac{8M}{3}$			
z	1	3	3	0	$\frac{-M}{3}$	$-M$	$\frac{3}{3}$	0	0	$98+8M$
x2	0	$2/3$	1	$-1/3$	0	0	$1/3$	0	0	14
R2	0	$1/3$	0	$4/3$	-1	0	$-4/3$	1	0	4
R3	0	$1/3$	0	$1/3$	0	-1	$-1/3$	0	1	4
z	1	$\frac{1}{4} + \frac{M}{4}$	0	0	$-\frac{7}{4} + \frac{M}{4}$	$-M$	$-M$	$\frac{7}{4} - \frac{5M}{4}$	0	$105+3M$
x2	0	$3/4$	1	0	$-1/4$	0	0	$1/4$	0	15
s1	0	$1/4$	0	1	$-3/4$	0	-1	$3/4$	0	3
R3	0	$1/4$	0	0	$1/4$	-1	0	$-1/4$	1	3

Basic	z	x1	x2	s1	s2	s3	R1	R2	R3	Sol
z	1	$\frac{1}{4} + \frac{M}{4}$	0	0	$-\frac{7}{4} + \frac{M}{4}$	-M	-M	$\frac{7}{4} - \frac{5M}{4}$	0	$105+3M$
x2	0	$3/4$	1	0	$-1/4$	0	0	$1/4$	0	15
s1	0	$1/4$	0	1	$-3/4$	0	-1	$3/4$	0	3
R3	0	1/4	0	0	$1/4$	-1	0	$-1/4$	1	3
z	1	0	0	0	-2	1	-M	$2-M$	$-1-M$	102
x2	0	0	1	0	-1	3	0	1	-3	6
s1	0	0	0	1	-1	1	-1	1	-1	0
x1	0	1	0	0	1	-4	0	-1	4	12

Basic	z	x_1	x_2	s_1	s_2	s_3	R_1	R_2	R_3	Sol
z	1	0	0	0	-2	1	$-M$	$2-M$	$-1-M$	102
x_2	0	0	1	0	-1	3	0	1	-3	6
s_1	0	0	0	1	-1	1	-1	1	-1	0
x_1	0	1	0	0	1	-4	0	-1	4	12
z	1	0	0	-1	-1	0	$1-M$	$1-M$	$2-M$	102
x_2	0	0	1	-3	2	0	3	-2	0	6
s_3	0	0	0	1	-1	1	-1	1	-1	0
x_1	0	1	0	4	-3	0	-4	3	0	12

Basic	z	x1	x2	s1	s2	s3	R1	R2	R3	Sol
z	1	0	0	-1	-1	0	1-M	1-M	2-M	102
x2	0	0	1	-3	2	0	3	-2	0	6
s3	0	0	0	1	-1	1	-1	1	-1	0
x1	0	1	0	4	-3	0	-4	3	0	12

This is the optimal tableau.

Optimal solution: $x_1=12, x_2=6$

Optimal $z = \text{Minimum } z = 102$