MATH 2116: Linear Algebra

Class Note: 03

Insitute of Information Technology (IIT), DU

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How to solve a system of linear equations

- Order the variables
- Write down the augmented matrix of the system
- Convert the matrix to **row echelon form**
- Check for consistency
- Convert the matrix to **reduced row echelon form**
- Write down the system corresponding to the reduced row echelon form
- Determine **leading (pivot)** and **free variables**
- Rewrite the system so that the leading variables are on the left while everything else is on the right
- Assign parameters to the free variables and write down the general solution in parametric form

Order the variables

The essential information of a linear system can be recorded compactly in a rectangular array called a **matrix**. Let a given system

$$\begin{cases} x - y - 2 = 0 \\ 2x - y - z - 3 = 0 \\ x + y + z - 6 = 0 \end{cases} \tag{1}$$

First we align the variables and constants of the equations

$$\begin{cases} x & - & y & & & = & 2 \\ 2x & - & y & - & z & = & 3 \\ x & + & y & + & z & = & 6 \end{cases}$$

Write down the augmented matrix

With the coefficients of each variable aligned in columns,

$$\mathbf{A} = \begin{bmatrix} 1 & -1 & 0 \\ 2 & -1 & -1 \\ 1 & 1 & 1 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 2 \\ 3 \\ 6 \end{bmatrix}$$

The matrix **A** is called the **coefficient matrix** and **b** is the column vector containing the constants of the linear system 1, and adding the column vector **b** to the right of **A** gives what we call an **augmented matrix**. That is,

$$\begin{bmatrix} \mathbf{A} & \mathbf{b} \end{bmatrix} \text{ or } \begin{bmatrix} \mathbf{A} & | & \mathbf{b} \end{bmatrix} = \left[\begin{array}{ccc|c} 1 & -1 & 0 & 2 \\ 2 & -1 & -1 & 3 \\ 1 & 1 & 1 & 6 \end{array}\right]$$

is called the **augmented matrix** of the system. So, by the notation $[\mathbf{A} \quad | \quad \mathbf{b}]$ or $[\mathbf{A} \quad \mathbf{b}]$ we meant that, the matrix **A** is enlarged or "augmented" by the extra column **b**.

Convert to row echelon form

Gaussian elimination

The goal of the Gaussian elimination is to convert the augmented matrix into **row echelon form**. A matrix is in **row echelon form** if:

1. All zero rows are at the bottom.
2. The first nonzero entry of a row is to the right of the first nonzero entry of the row above.
3. Below the first nonzero entry of a row, all entries are zero.

$$\begin{bmatrix} \boxed{*} & * & * & * & * \\ 0 & 0 & \boxed{*} & * & * \\ 0 & 0 & 0 & \boxed{*} & * \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{aligned} * &= \text{any number} \\ \boxed{*} &= \text{any nonzero number (pivot)} \end{aligned}$$

A **pivot** is the first nonzero entry of a row of a matrix in row echelon form.

Convert to row echelon form

Add -2 times the 1st equation to the 2nd equation:

$$\left[\begin{array}{ccc|c} 1 & -1 & 0 & 2 \\ 0 & 1 & -1 & -1 \\ 1 & 1 & 1 & 6 \end{array}\right]$$

Add -1 times the 1st equation to the 3rd equation

$$\left[\begin{array}{ccc|c} 1 & -1 & 0 & 2 \\ 0 & 1 & -1 & -1 \\ 0 & 2 & 1 & 4 \end{array}\right]$$

Add -2 times the 2nd equation to the 3rd equation

$$\left[\begin{array}{ccc|c} 1 & -1 & 0 & 2 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 3 & 6 \end{array}\right]$$

Check for consistency

There are **three possibilities**:

1. **Inconsistent system or no solution.**

$$\left[\begin{array}{ccc|c} \boxed{*} & * & * & * \\ 0 & \boxed{*} & * & * \\ 0 & 0 & 0 & \boxed{*} \end{array}\right]$$

$* =$ any number

$\boxed{*} =$ any nonzero number (pivot)

2. **Determinate system or unique solution.**

$$\left[\begin{array}{ccc|c} \boxed{*} & * & * & * \\ 0 & \boxed{*} & * & * \\ 0 & 0 & \boxed{*} & * \end{array}\right]$$

3. **Indeterminate system or infinitely many solutions.**

$$\left[\begin{array}{ccc|c} \boxed{*} & * & * & * \\ 0 & 0 & \boxed{*} & * \end{array}\right]$$

Check for consistency

The *row echelon form (ref)* is completed. And the augmented metrix becomes:

$$\left[\begin{array}{ccc|c} 1 & -1 & 0 & 2 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 3 & 6 \end{array}\right]$$

So it is a determinate system and has unique solution. we can solve the system by back substitution. However we can as well proceed with elementary operations to get the *reduced row echelon form (rref)*.

Convert to reduced row echelon form

Gauss-Jordan reduction

The goal of the Gauss-Jordan reduction is to convert the augmented matrix into **reduced row echelon form**. A matrix is in **reduced row echelon form** if it is in row echelon form, and in addition:

4. Each pivot is equal to 1.
5. Each pivot is the only nonzero entry in its column.

$$\begin{bmatrix} \boxed{1} & * & 0 & 0 & * \\ 0 & 0 & \boxed{1} & 0 & * \\ 0 & 0 & 0 & \boxed{1} & * \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{array}{l} * = \text{any number} \\ \boxed{1} = \text{pivot} \end{array}$$

We say that x_i is a **free variable** if its corresponding column in A is not a pivot column.

Convert to reduced row echelon form

Multiply the 3rd equation by $1/3$:

$$\left[\begin{array}{ccc|c} 1 & -1 & 0 & 2 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

Add the 3rd equation to the 2nd equation:

$$\left[\begin{array}{ccc|c} 1 & -1 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

Add the 2nd equation to the 1st equation

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

Determine leading (pivot) and free variables

Since the system is a determinate system and has unique solution. So there is no free variables.

$$\left[\begin{array}{ccc|c} \boxed{1} & 0 & 0 & 3 \\ 0 & \boxed{1} & 0 & 1 \\ 0 & 0 & \boxed{1} & 2 \end{array}\right]$$

Therefore, we get

$$\begin{cases} x & & & = & 3 \\ & y & & = & 1 \\ & & z & = & 2 \end{cases}$$

and **Solution:** $(x, y, z) = (3, 1, 2)$

Thanks

Thank you all.