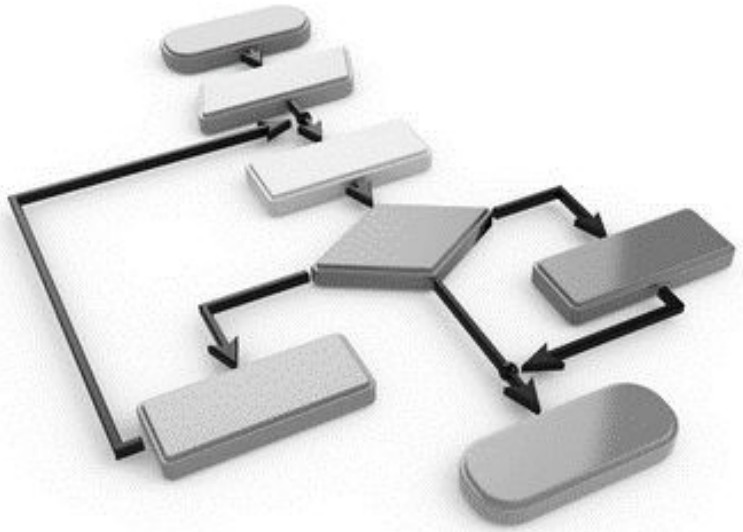


Dynamic Programming: MCM

CSE 301: Combinatorial Optimization

Matrix-chain Multiplication

- Input: a sequence (chain) $\langle A_1, A_2, \ldots, A_n \rangle$ of n matrices
- Aim: compute the product $A_1 \cdot A_2 \cdot \ldots \cdot A_n$
- A product of matrices is fully parenthesized if
  - It is either a single matrix
  - Or, the product of two fully parenthesized matrix products surrounded by a pair of parentheses.

    $\triangleright (A_i (A_{i+1} A_{i+2} \ldots A_j))$

    $\triangleright ((A_i A_{i+1} A_{i+2} \ldots A_{j-1}) A_j)$

    $\triangleright ((A_i A_{i+1} A_{i+2} \ldots A_k)(A_{k+1} A_{k+2} \ldots A_j))$ for $i \le k < j$
  - All parenthesizations yield the same product; matrix product is associative

Matrix-chain Multiplication

- Input: $\langle A_1, A_2, A_3, A_4 \rangle$
- 5 distinct ways of full parenthesization

  $(A_1(A_2(A_3A_4)))$

  $(A_1((A_2A_3)A_4))$

  $((A_1A_2)(A_3A_4))$

  $((A_1(A_2A_3))A_4)$

  $(((A_1A_2)A_3)A_4)$

- The way we parenthesize a chain of matrices can have a dramatic effect on the cost of computing the product

Matrix-chain Multiplication

Cost of Multiplying two Matrices

Matrix has two attributes

- rows[A]: # of rows
- cols[A]: # of columns

# of scalar mult-adds in $C \leftarrow AB$ is

rows[A]×cols[B]×cols[A]

A: (p×q)
B: (q×r)
} C=A·B is p×r.

# of mult-adds is p×r×q

MATRIX-MULTIPLY(A, B)

```
if cols[A]≠rows[B] then
    error("incompatible dimensions")
for i ←1 to rows[A] do
    for j←1 to cols[B] do
        C[i,j] ← 0
        for k←1 to cols[A] do
            C[i,j]←
C[i,j]+A[i,k]·B[k,j]
return C
```

Matrix-chain Multiplication

Input: a chain $\langle A_1, A_2, \ldots, A_n \rangle$ of n matrices, A_i is a $p_{i-1} \times p_i$ matrix

Aim: fully parenthesize the product $A_1 \cdot A_2 \cdot \ldots \cdot A_n$ such that the number of scalar mult-adds are minimized.

- Ex.: $\langle A_1, A_2, A_3 \rangle$ where A_1: 10×100; A_2: 100×5; A_3: 5×50

$((\underbrace{A_1 A_2}_{10\times5}) \underbrace{A_3}_{5\times50})$:	$\underbrace{10 \times 100 \times 5}_{A_1 A_2}$	+	$\underbrace{10 \times 5 \times 50}_{(A_1 A_2) A_3}$	=7500

$(\underbrace{A_1}_{10\times100} (\underbrace{A_2 A_3}_{100\times50}))$:	$\underbrace{100\times5\times50}_{A_2 A_3}$	+	$\underbrace{10\times100\times50}_{A_1 (A_2 A_3)}$	=75000

$\Rightarrow$ First parenthesization yields 10 times faster computation.

Number of Parenthesizations

- Brute force approach: exhaustively check all parenthesizations
- P(n): # of parenthesizations of a sequence of n matrices
- We can split sequence between kth and (k+1)st matrices for any k=1, 2, … , n–1, then parenthesize the two resulting sequences independently, i.e.,

$$(A_1 A_2 A_3 \ldots A_k)(A_{k+1} A_{k+2} \ldots A_n)$$

- We obtain the recurrence

$$P(1) = 1 \text{ and } P(n) = \sum_{k=1}^{n-1} P(k)P(n-k)$$

Number of Parenthesizations

- The recurrence generates the sequence of Catalan Numbers
- Solution is $P(n) = C(n-1)$ where

$$C(n) = \frac{1}{n+1}\binom{2n}{n} = \Omega(4^n/n^{3/2})$$

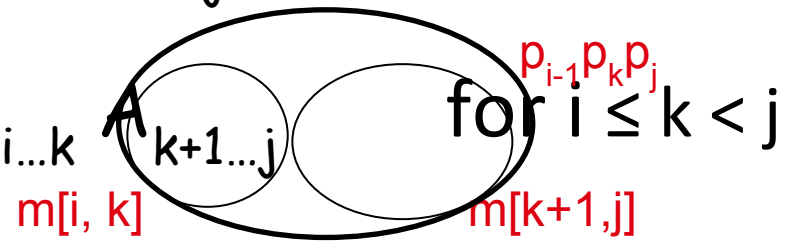
- The number of solutions is exponential in n
- Therefore, brute force approach is a poor strategy

Establishing the Recurrence

Consider the subproblem of parenthesizing

$A_{i \ldots j} = A_i A_{i+1} \cdots A_j$ for $1 \le i \le j \le n$

$= A_{i \ldots k} A_{k+1 \ldots j}$ for $i \le k < j$



Assume that the optimal parenthesization splits the product $A_i A_{i+1} \cdots A_j$ at k ($i \le k < j$)

$m[i, j] =$ $\underbrace{m[i, k]}_{\text{min \# of scalar multiplications to compute } A_{i \ldots k}}$ + $\underbrace{m[k+1, j]}_{\text{min \# of multiplications to compute } A_{k+1 \ldots j}}$ + $\underbrace{\quad}_{\text{\# of multiplications to compute } A_{i \ldots k} A_{k+1 \ldots j}}$

$p_{i-1} p_k p_j$

The Recurrence relation

$$m[i, j] = \begin{cases} 0 & \text{if } i = j \\ \min_{i \le k < j} \{m[i, k] + m[k+1, j] + p_{i-1} p_k p_j\} & \text{if } i < j \end{cases}$$

Recursive Matrix-chain

Recursive matrix-chain order

RMC(p, i, j)

if $i = j$ **then**
  return 0

$m[i, j] \leftarrow \infty$

for $k \leftarrow i$ **to** $j - 1$ **do**

  $q \leftarrow \text{RMC}(p, i, k) + \text{RMC}(p, k+1, j) + p_{i-1} p_k p_j$

  if $q < m[i, j]$ **then**

    $m[i, j] \leftarrow q$

return $m[i, j]$

Running Time of Recursive Matrix-chain

$$T(1) \geq 1$$

$$T(n) \geq 1 + \sum_{k=1}^{n-1} \left(T(k) + T(n-k) + 1 \right) \text{ for } n > 1$$

- For $i = 1, 2, \ldots, n$ each term $T(i)$ appears twice
  - Once as $T(k)$, and once as $T(n-k)$
- Collect $n-1$ 1's in the summation together with the front 1

$$T(n) \geq 2 \sum_{i=1}^{n-1} T(i) + n$$

- Prove via substitution that T(n) is $\Omega(2^n)$

Running Time of Recursive Matrix-chain

- Try to show that $T(n) \geq 2^{n-1}$ (by induction)

Base case: $T(1) \geq 1 = 2^0 = 2^{1-1}$ for $n = 1$

IH: $T(i) \geq 2^{i-1}$ for all $i = 1, 2, \ldots, n-1$ and $n \geq 2$

$$T(n) \geq 2\sum_{i=1}^{n-1} 2^{i-1} + n$$

$$= 2\sum_{i=0}^{n-2} 2^i + n = 2(2^{n-1} - 1) + n$$

$$= 2^{n-1} + (2^{n-1} - 2 + n)$$

$$\Rightarrow T(n) \geq 2^{n-1}$$ Q.E.D.

Elements of Dynamic Programming

When should we look for a DP solution to an optimization problem?

Two key ingredients for the problem

- Optimal substructure
- Overlapping subproblems

Elements of Dynamic Programming

Optimal Substructure

- A problem exhibits optimal substructure
  - if an optimal solution to a problem contains within it optimal solutions to subproblems
- Example: matrix-chain-multiplication

  Optimal parenthesization of $A_1 A_2 \ldots A_n$ that splits the product between A_k and A_{k+1},

  contains within it optimal soln's to the problems of parenthesizing $A_1 A_2 \ldots A_k$ and $A_{k+1} A_{k+2} \ldots A_n$

$$m[i, j] = \begin{cases} 0 & \text{if } i = j \\ \min_{i \le k < j} \{m[i, k] + m[k+1, j] + p_{i-1} p_k p_j\} & \text{if } i < j \end{cases}$$

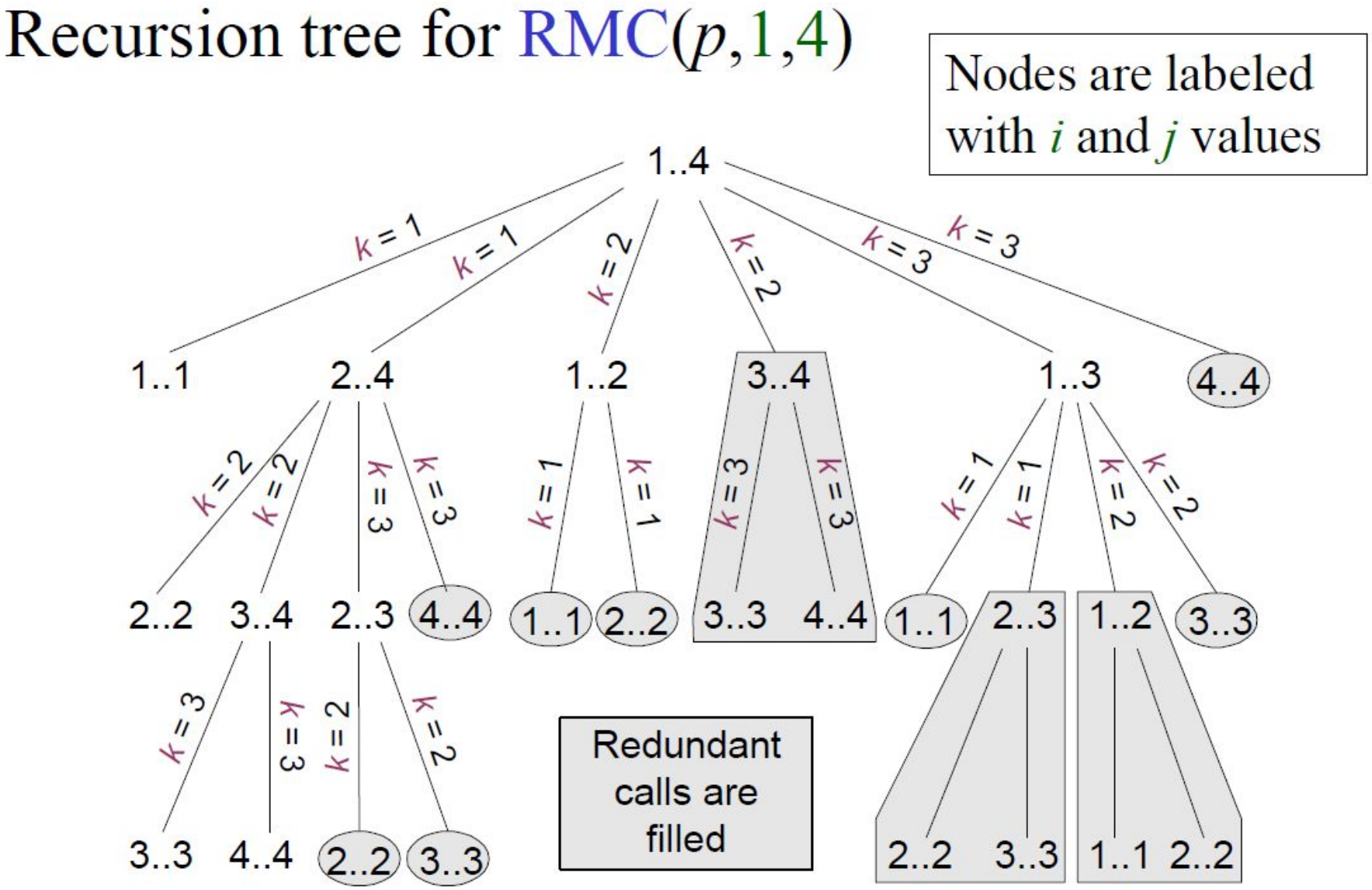
Elements of Dynamic Programming

Overlapping Subproblems

- Total number of distinct subproblems should be polynomial in the input size
- When a recursive algorithm revisits the same problem over and over again

  we say that the optimization problem has overlapping subproblems

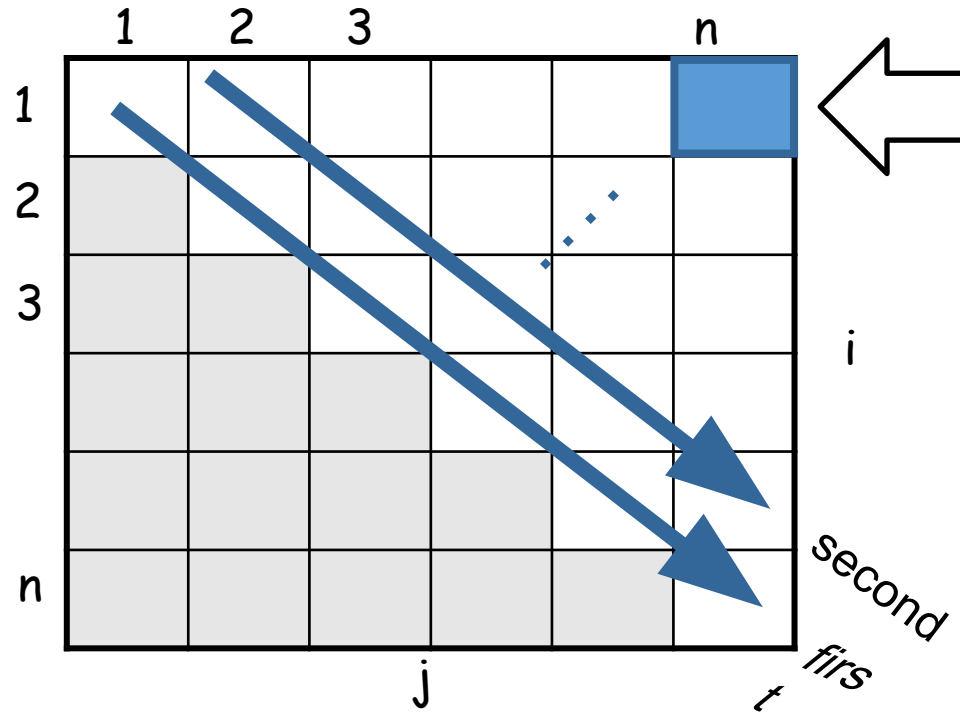
Overlapping Subproblems in RMC Execution



Using Dynamic Programming

$$m[i, j] = \begin{cases} 0 & \text{if } i = j \\ \min_{i \le k < j} \{m[i, k] + m[k+1, j] + p_{i-1} p_k p_j\} & \text{if } i < j \end{cases}$$

- Length = 1: i = j, i = 1, 2, …, n
- Length = 2: j = i + 1, i = 1, 2, …, n-1



m[1, n] gives the optimal solution to the problem

Compute rows from diagonal to top and from left to right
In a similar matrix s, we keep the optimal values of k

Matrix-Chain-Order

Alg.: MATRIX-CHAIN-ORDER(p)

1. $n = p.length-1$
2. let $m[1..n, 1..n]$ and $s[1..n-1, 2..n]$ be new tables
3. **for** $i = 1$ **to** n
4. $\quad m[i, i] = 0$
5. **for** $l = 2$ **to** n
6. $\quad$ **for** $i = 1$ **to** $n - l + 1$
7. $\quad j = i + l - 1$
8. $\quad\quad m[i, j] = \infty$
9. $\quad\quad$ **for** $k = i$ **to** $j - 1$
10. $\quad\quad\quad q = m[i, k] + m[k+1, j] + p_{i-1} p_k p_j$
11. $\quad\quad$ **if** $q < m[i, j]$
12. $\quad\quad\quad m[i, j] = q$
13. $\quad\quad\quad\quad s[i, j] = k$
14. **return** m and s

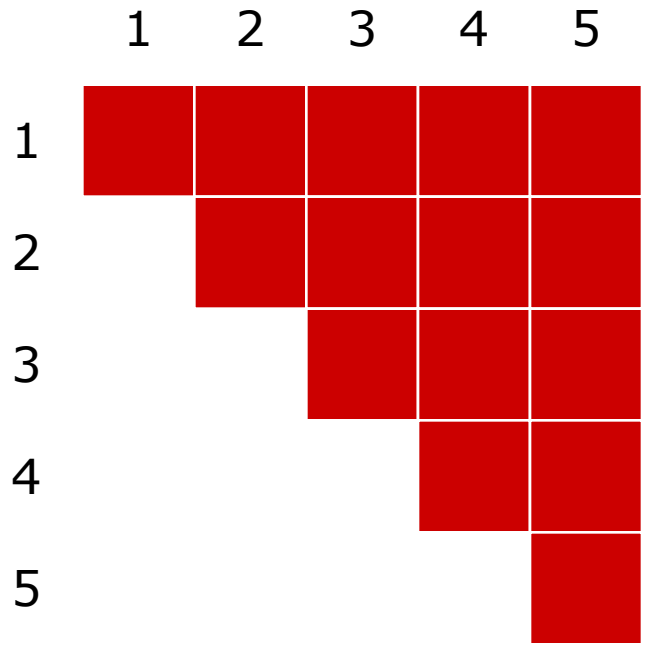
Analysis

- Our algorithm computes the minimum-cost table m and the split table s
- The optimal solution can be constructed from the split table s (shown later)
- Each entry s[i, j]=k shows where to split the product $A_i A_{i+1} \ldots A_j$ for the minimum cost
- There are 3 nested loops and each can iterate at most n times, so the total running time is $\Theta(n^3)$.

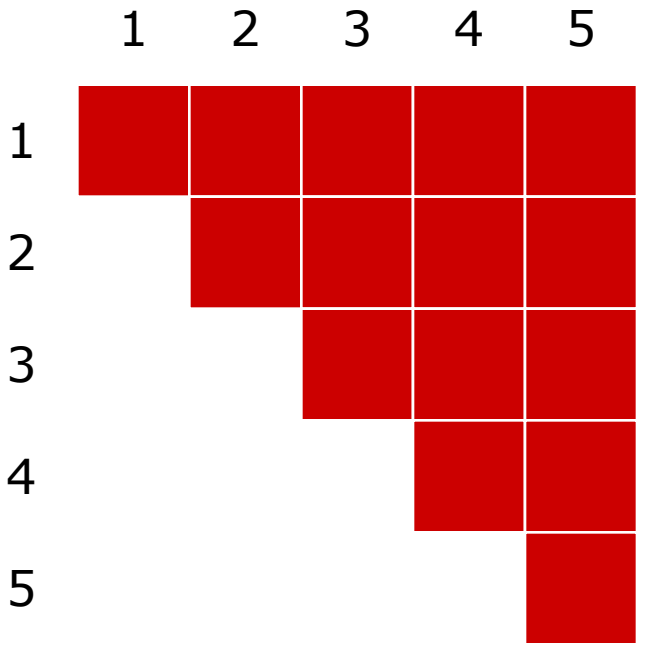
Example:

p = (10, 5, 1, 10, 2, 10)

[10×5]×[5×1]×[1×10]×[10×2]×[2×10]



m(i,j), i ≤ j



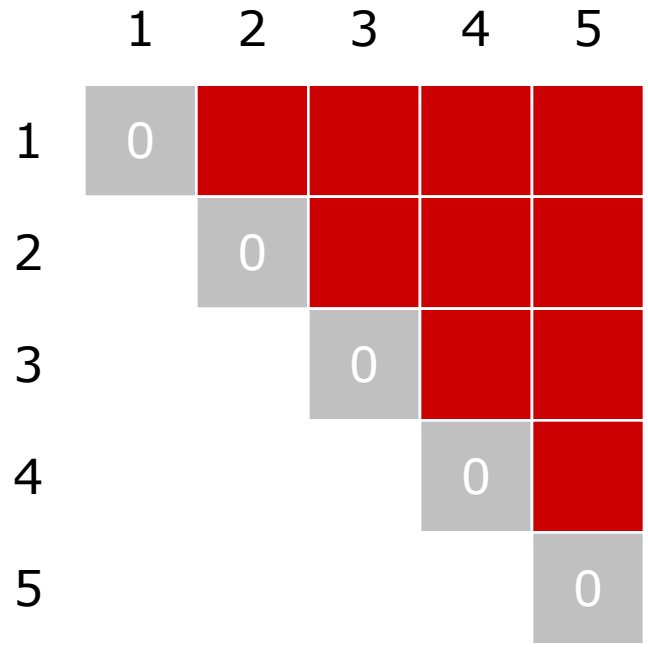
s(i,j), i ≤ j

Example:

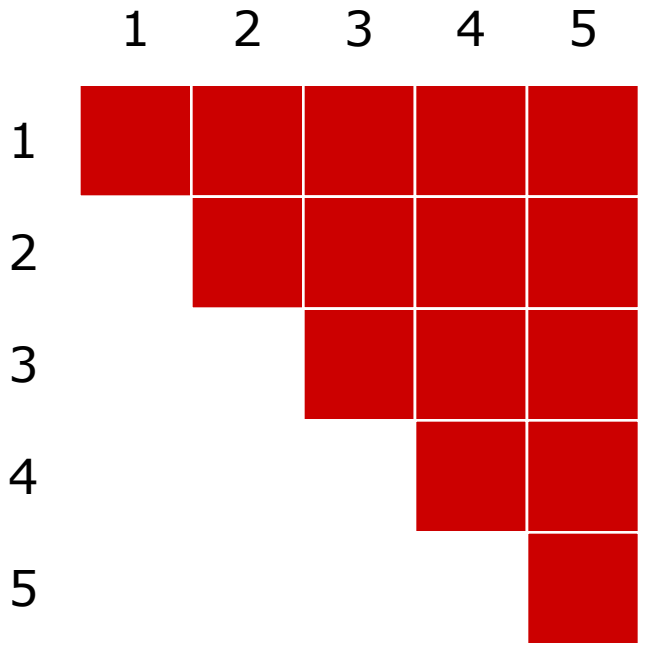
p = (10, 5, 1, 10, 2, 10)

[10×5]×[5×1]×[1×10]×[10×2]×[2×10]

m(i,i) = 0



m(i,j), i ≤ j



s(i,j), i ≤ j

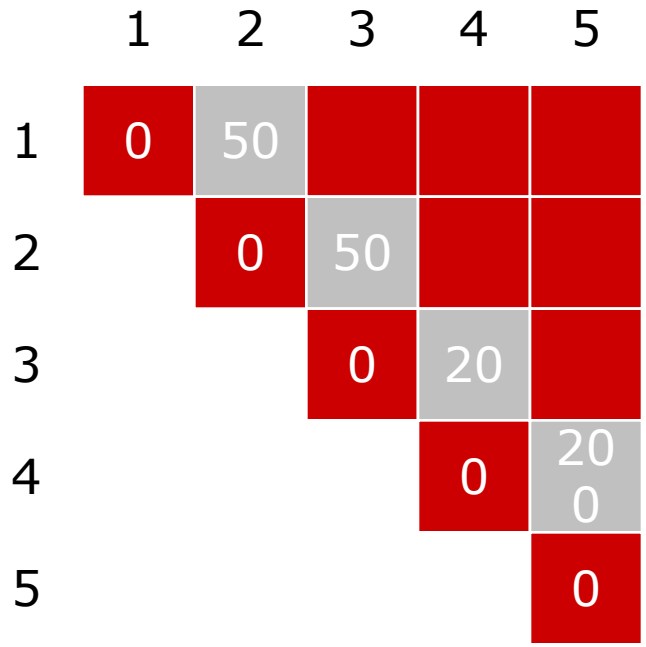
Example:

p = (10, 5, 1, 10, 2, 10)

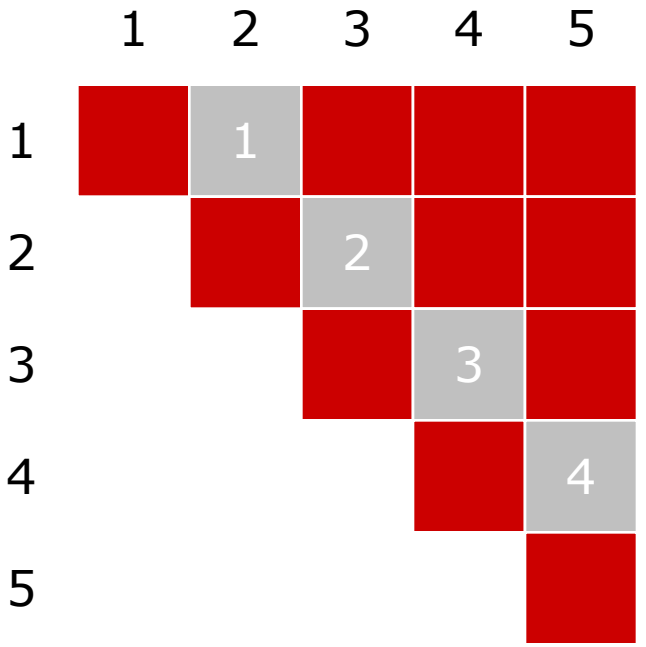
[10×5]×[5×1]×[1×10]×[10×2]×[2×10]

$m(i,i+1) = p_{i-1}p_ip_{i+1}$ e.g., $m(2,3) = p_1p_2p_3 = 5*1*10 = 50$,
$m(1,2)=10*5*1 = 50$, $m(3,4) = 1*10*2=20$, …

s(i,i+1) = i



m(i,j), i ≤ j



s(i,j), i ≤ j

Example:

p = (10, 5, 1, 10, 2, 10)

[10×5]×[5×1]×[1×10]×[10×2]×[2×10]

$m(i,i+2) = \min\{m(i,i) + m(i+1,i+2) + p_{i-1}p_ip_{i+2},$
$m(i,i+1) + m(i+2,i+2) + p_{i-1}p_{i+1}p_{i+2}\}$

	1	2	3	4	5
1	0	50			
2		0	50		
3			0	20	
4				0	200
5					0

m(i,j), i ≤ j

	1	2	3	4	5
1		1			
2			2		
3				3	
4					4
5					

s(i,j), i ≤ j

Example:

p = (10, 5, 1, 10, 2, 10)

[10×5]×[5×1]×[1×10]×[10×2]×[2×10]

$m(2,4) = \min\{m(2,2) + m(3,4) + p_1p_2p_4,\ m(2,3) + m(4,4) + p_1p_3p_4\}$

$= \min\{0+20+5*1*2,\ 50+0+5*10*2\} = 30$

	1	2	3	4	5
1	0	50	150		
2		0	50	30	
3			0	20	40
4				0	200
5					0

m(i,j), i ≤ j

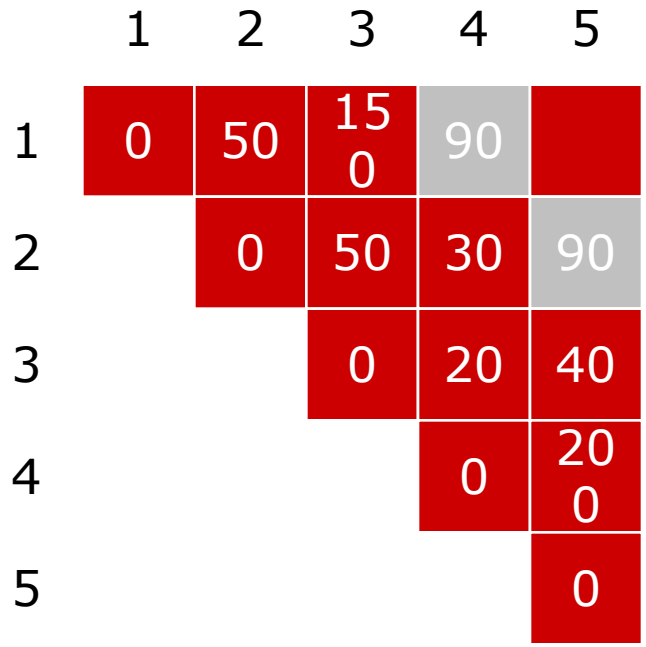
	1	2	3	4	5
1		1	2		
2			2	2	
3				3	3
4					4
5					

s(i,j), i ≤ j

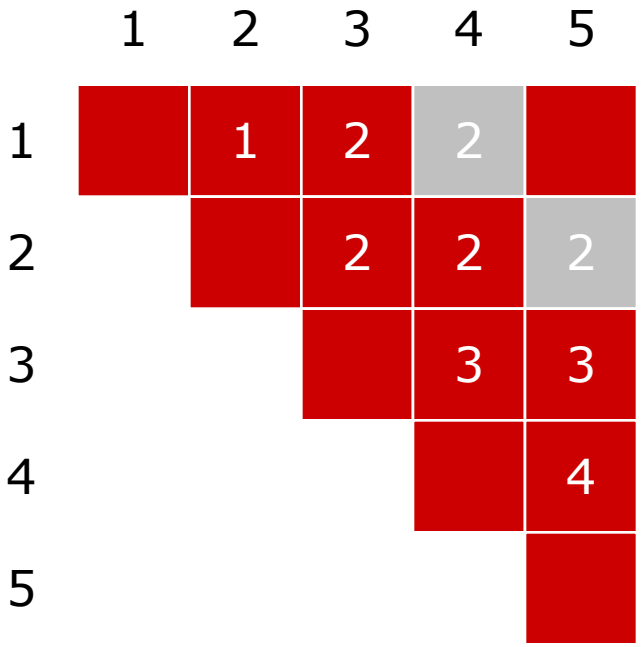
Example:

[10×5]×[5×1]×[1×10]×[10×2]×[2×10]

$m(i,i+3) = \min\{m(i,i) + m(i+1,i+3) + p_{i-1}p_ip_{i+3},$
$m(i,i+1) + m(i+2,i+3) + p_{i-1}p_{i+1}p_{i+3},$
$m(i,i+2) + m(i+3,i+3) + p_{i-1}p_{i+2}p_{i+3}\}$



m(i,j), i ≤ j



s(i,j), i ≤ j

Example:

$[10\times5]\times[5\times1]\times[1\times10]\times[10\times2]\times[2\times10]$

$m(i,i+4) = \min\{m(i,i) + m(i+1,i+4) + p_{i-1}p_ip_{i+4},$
$m(i,i+1) + m(i+2,i+4) + p_{i-1}p_{i+1}p_{i+4},\ m(i,i+2) + m(i+3,i+4) + p_{i-1}p_{i+2}p_{i+4},\ m(i,i+3) + m(i+4,i+4) + p_{i-1}p_{i+3}p_{i+4}\}$

	1	2	3	4	5
1	0	50	150	90	190
2		0	50	30	90
3			0	20	40
4				0	200
5					0

m(i,j), i ≤ j

	1	2	3	4	5
1		1	2	2	2
2			2	2	2
3				3	3
4					4
5					

s(i,j), i ≤ j

Print optimal parenthesis

Alg.: PRINT-OPTIMAL-PARENS(s, i, j)

1. **if** $i == j$
2. print "A_i"
3. **else** print "("
4. PRINT-OPTIMAL-PARENS($s, i, s[i, j]$)
5. PRINT-OPTIMAL-PARENS($s, s[i, j] + 1, j$)
6. print ")"

Example:

Optimal multiplication sequence

s(1,5) = 2

- $A_{15} = A_{12} \times A_{35}$

	1	2	3	4	5
1	0	50	150	90	190
2		0	50	30	90
3			0	20	40
4				0	200
5					0

m(i,j), i ≤ j

	1	2	3	4	5
1		1	2	2	2
2			2	2	2
3				3	3
4					4
5					

s(i,j), i ≤ j

Example:

$A_{15} = A_{12} \times A_{35}$

$s(1,2) = 1$ ▶ $A_{12} = A_{11} \times A_{22}$

$\rightarrow A_{15} = (A_{11} \times A_{22}) \times A_{35}$

	1	2	3	4	5
1	0	50	150	90	190
2		0	50	30	90
3			0	20	40
4				0	200
5					0

m(i,j), i ≤ j

	1	2	3	4	5
1		1	2	2	2
2			2	2	2
3				3	4
4					4
5					

s(i,j), i ≤ j

Example:

$A_{15} = (A_{11} \times A_{22}) \times A_{35}$

$s(3,5) = 4 \blacktriangleright A_{35} = A_{34} \times A_{55}$

$\rightarrow A_{15} = (A_{11} \times A_{22}) \times (A_{34} \times A_{55})$

	1	2	3	4	5
1	0	50	150	90	190
2		0	50	30	90
3			0	20	40
4				0	200
5					0

m(i,j), i ≤ j

	1	2	3	4	5
1		1	2	2	2
2			2	2	2
3				3	4
4					4
5					

s(i,j), i ≤ j

Example:

$A_{15} = (A_{11} \times A_{22}) \times (A_{34} \times A_{55})$

$s(3,4) = 3 \blacktriangleright A_{34} = A_{33} \times A_{44}$

$\rightarrow A_{15} = (A_{11} \times A_{22}) \times ((A_{33} \times A_{44}) \times A_{55})$

	1	2	3	4	5
1	0	50	150	90	190
2		0	50	30	90
3			0	20	40
4				0	200
5					0

m(i,j), i ≤ j

	1	2	3	4	5
1		1	2	2	2
2			2	2	2
3				3	4
4					4
5					

s(i,j), i ≤ j

Memoization

- Offers the efficiency of the usual DP approach while maintaining top-down strategy
- Idea is to memoize the natural, but inefficient, recursive algorithm

Memoization

- Maintains an entry in a table for the soln to each subproblem
- Each table entry contains a special value to indicate that the entry has yet to be filled in
- When the subproblem is first encountered its solution is computed and then stored in the table
- Each subsequent time that the subproblem encountered the value stored in the table is simply looked up and returned

Memoized Matrix-Chain

Alg.: MEMOIZED-MATRIX-CHAIN(p)

```
1.  n ← length[p] – 1
2.  for i ← 1 to n
3.      do for j ← i to n
4.          do m[i, j] ← ∞
5.  return LOOKUP-CHAIN(p, 1, n)
```

Initialize the m table with large values that indicate whether the values of m[i, j] have been computed

← Top-down approach

Memoized Matrix-Chain

Alg.: LOOKUP-CHAIN(p, i, j)

1. **if** $m[i, j] < \infty$
2. **then return** $m[i, j]$
3. **if** $i = j$
4. **then** $m[i, j] \leftarrow 0$
5. **else for** $k \leftarrow i$ **to** $j - 1$
6. **do** $q \leftarrow$ LOOKUP-CHAIN(p, i, k) +
   LOOKUP-CHAIN(p, k+1, j) + $p_{i-1}p_kp_j$
7. **if** $q < m[i, j]$
8. **then** $m[i, j] \leftarrow q$
9. **return** $m[i, j]$