

# CSE 301

# Combinatorial Optimization

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## Greedy Algorithms

# Greedy Algorithm

- Solves an optimization problem
  - For many optimization problems, greedy algorithm can be used. (not always)
- Greedy algorithm for optimization problems typically go through a sequence of steps, with a set of choices at each step.
- Current choice does not depend on evaluating potential future choices or pre-solving repeatedly occurring subproblems (a.k.a., *overlapping subproblems*).
- With each step, the original problem is reduced to a smaller problem.
- Greedy algorithm always makes the choice that looks best at the moment.
- It makes a locally optimal choice in the hope that this choice will lead to a globally optimal solution.

# When can we use Greedy algorithms?

We can use a greedy algorithm when the following are true:

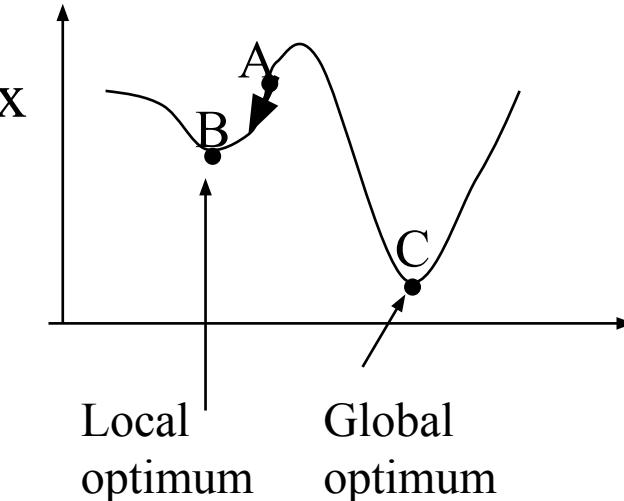
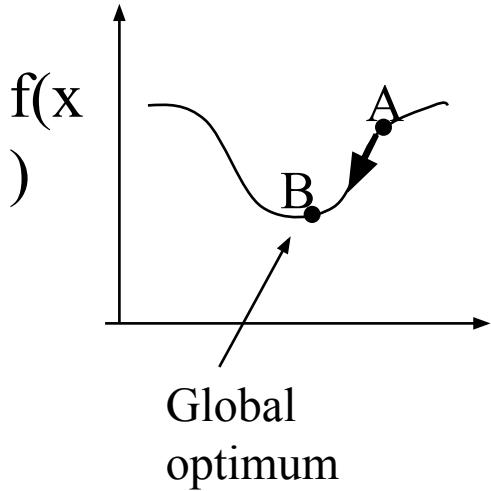
- 1) **The greedy choice property:** A globally optimal solution can be arrived at by making a locally optimal (greedy) choice.
- 2) **The optimal substructure property:** The optimal solution contains within its optimal solutions to subproblems.

# Designing Greedy Algorithms

1. Cast the optimization problem as one for which:
  - we make a choice and are left with only one subproblem to solve
2. Prove the **GREEDY CHOICE**
  - that there is always an optimal solution to the original problem that makes the greedy choice
3. Prove the **OPTIMAL SUBSTRUCTURE**:
  - the greedy choice + an optimal solution to the resulting subproblem leads to an optimal solution

# Finding the global minima of a function

- start from an arbitrary value of  $x$
- If  $f(x+1) < f(x)$  then set  $x = x+1$ 
  - Otherwise if  $f(x-1) < f(x)$  then set  $x = x-1$
- Continue until changing  $x$  doesn't decrease  $f(x)$



If we start at A and move in the direction of descent, we will end up at the local optimum, B.

On the left graph, B is also at the global optimum.

On the right graph, the global optimum is elsewhere, at C.

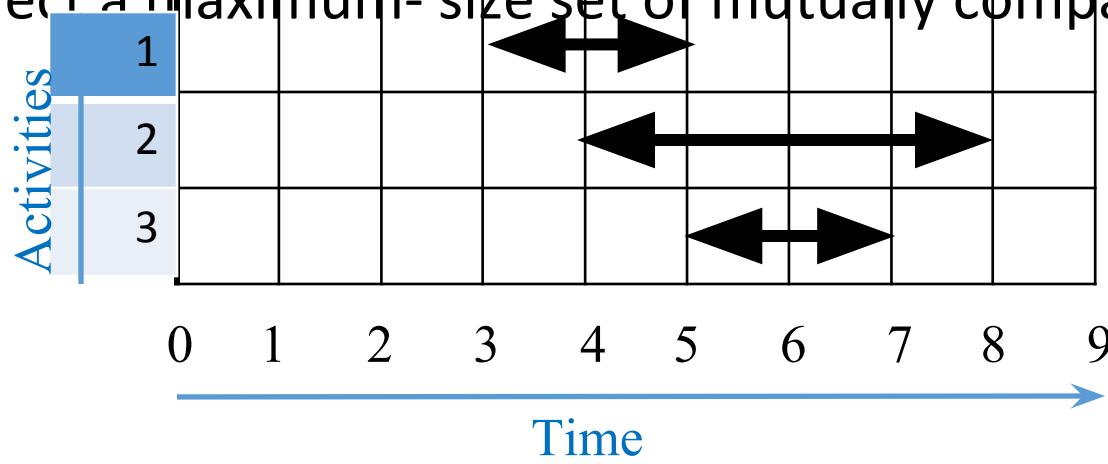
# Activity Selection aka. Interval Scheduling Problem

**Definition:** Scheduling a resource among several competing activities.

**Elaboration:** Suppose we have a set  $S = \{1, 2, \dots, n\}$  of  $n$  proposed activities that wish to allocate a resource, such as a lecture hall, which can be used by only one activity at a time. Each activity  $i$  has a **start time**  $s_i$  and **finish time**  $f_i$  where  $s_i \leq f_i$ .

**Compatibility:** Activities  $i$  and  $j$  are compatible if the interval  $[s_i, f_i]$  and  $[s_j, f_j]$  do not overlap (i.e.  $s_i \geq f_j$  or  $s_j \geq f_i$ ). E.g. in the example below, activities 1 & 2 (as well as activities 2 & 3) are incompatible, and activities 1 & 3 are compatible.

**Goal:** To select a maximum- size set of mutually compatible activities.



# Activity Selection a.k.a. Interval Scheduling Problem

Here are a set of start and finish times

i	1	2	3	4	5	6	7	8	9	10	11
$S_j$	1	3	0	5	3	5	6	8	8	2	12
$f_j$	4	5	6	7	9	9	10	11	12	14	16

What is the maximum number of activities that can be completed?

- {3,9,11} can be completed (not an optimal solution)
- But so can {1,4,8,11} which is a larger set (*an optimal solution*)
- Solution is not unique, consider {2,4,9,11} (another optimal solution)

# The Activity Selection Problem

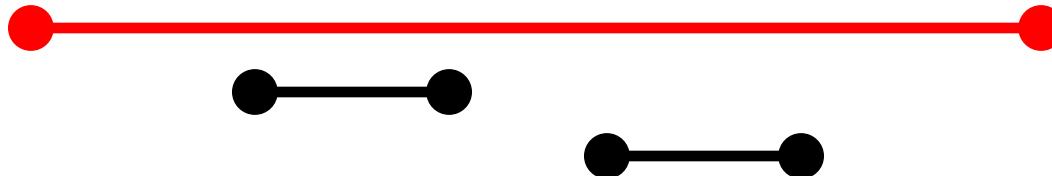
Algorithm 1:

1. sort the activities by the starting time
2. pick the first activity  $a$
3. remove all activities conflicting with  $a$
4. repeat

# The Activity Selection Problem

Algorithm 1:

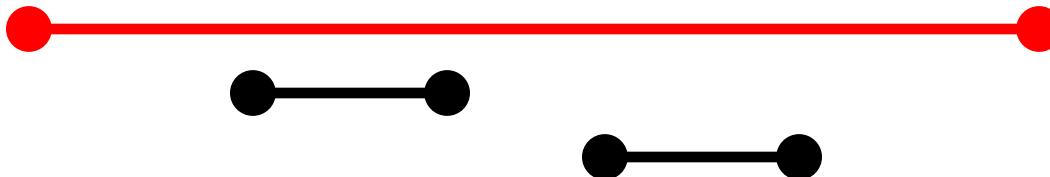
1. sort the activities by the starting time
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4. repeat



# The Activity Selection Problem

Algorithm 1:

1. sort the activities by the starting time
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# The Activity Selection Problem

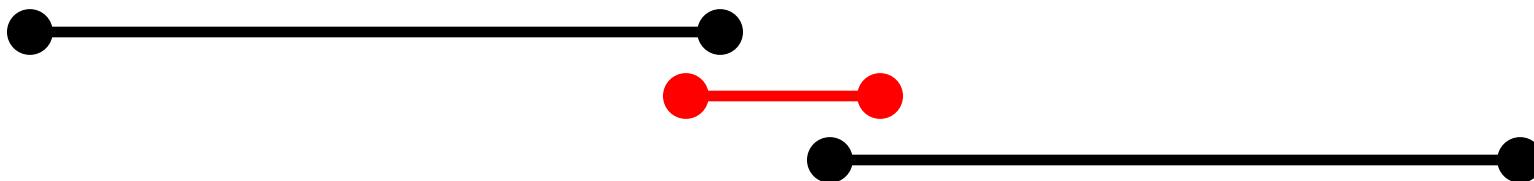
Algorithm 2:

1. sort the activities by length
2. pick the shortest activity  $a$
3. remove all activities conflicting with  $a$
4. repeat

# The Activity Selection Problem

Algorithm 2:

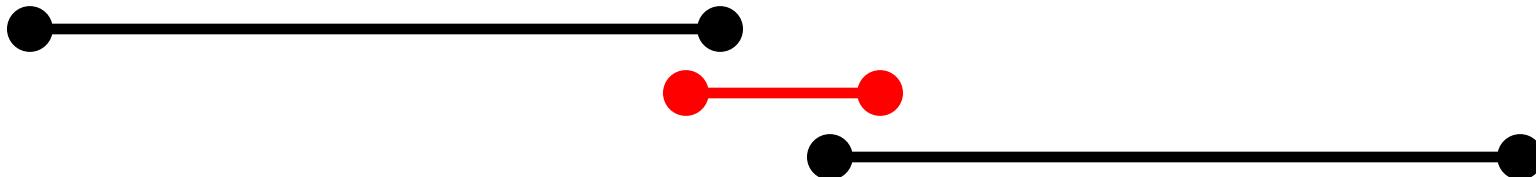
1. sort the activities by length
2. pick the shortest activity  $a$
3. remove all activities conflicting with  $a$
4. repeat



# The Activity Selection Problem

Algorithm 2:

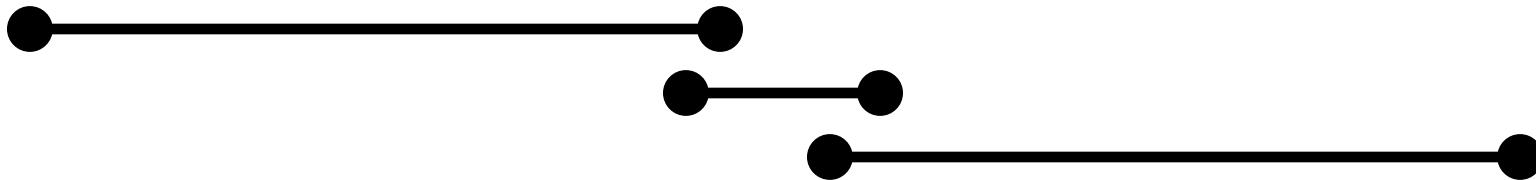
1. sort the activities by length
2. pick the shortest activity  $a$
3. remove all activities conflicting with  $a$
4. repeat



# The Activity Selection Problem

Algorithm 3:

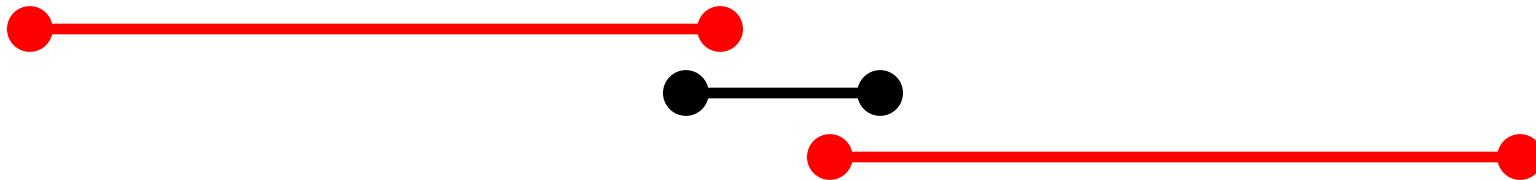
1. sort the activities by **ending time**
2. **pick** the activity  $a$  which **ends first**
3. remove all activities conflicting with  $a$
4. repeat



# The Activity Selection Problem

Algorithm 3:

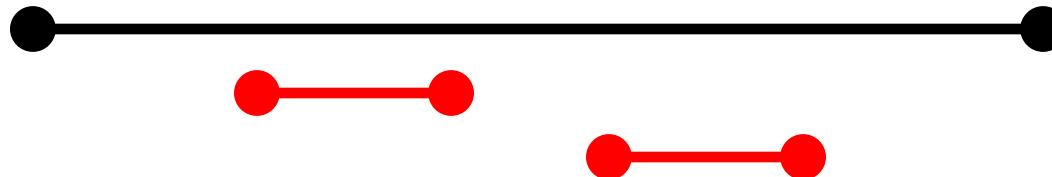
1. sort the activities by **ending time**
2. pick the activity  $a$  which **ends first**
3. remove all activities conflicting with  $a$
4. repeat



# The Activity Selection Problem

Algorithm 3:

1. sort the activities by **ending time**
2. pick the activity  $a$  which **ends first**
3. remove all activities conflicting with  $a$
4. repeat



# Iterative Greedy Algorithm

**GREEDY-ACTIVITY-SELECTOR( $s, f$ )**

// $s$ : array containing start times of input activities

// $f$ : array containing finishing times of input activities

1.  $n = s.length$
2.  $A = \{a_1\}$
3.  $k = 1$  // $k$ : previously chosen activity
4. **for**  $m = 2$  **to**  $n$
5.     **if**  $s[m] \geq f[k]$  //find earliest-finishing activity  $m$  which is compatible with  $k$
6.          $A = A \cup \{a_m\}$
7.          $k = m$  // $f[k] = \max\{f[k]: k \in A\}$ , since activities are sorted
8. **return**  $A$

Running time is  $\Theta(n)$

GREEDY-ACTIVITY-SELECTOR schedules a set of  $n$  activities in  $\Theta(n)$  time, assuming that the activities were already sorted initially by their finish times.

# Recursive Greedy Algorithm

**Initial call:** RECURSIVE-ACTIVITY-SELECTOR( $s, f, 0$ )

**RECURSIVE-ACTIVITY-SELECTOR( $s, f, i$ )**

//Solves the subproblem  $S_{i+1,n} = \{a_{i+1}, a_{i+2}, \dots, a_n\}$  of the problem  $S_{i,n} = \{a_i, a_{i+1}, \dots, a_n\}$ , i.e.,  
//computes mutually compatible activities in the set  $S_{i+1,n}$  given that  $a_i \in A_{i,n}$ : solution set of  $S_{i,n}$

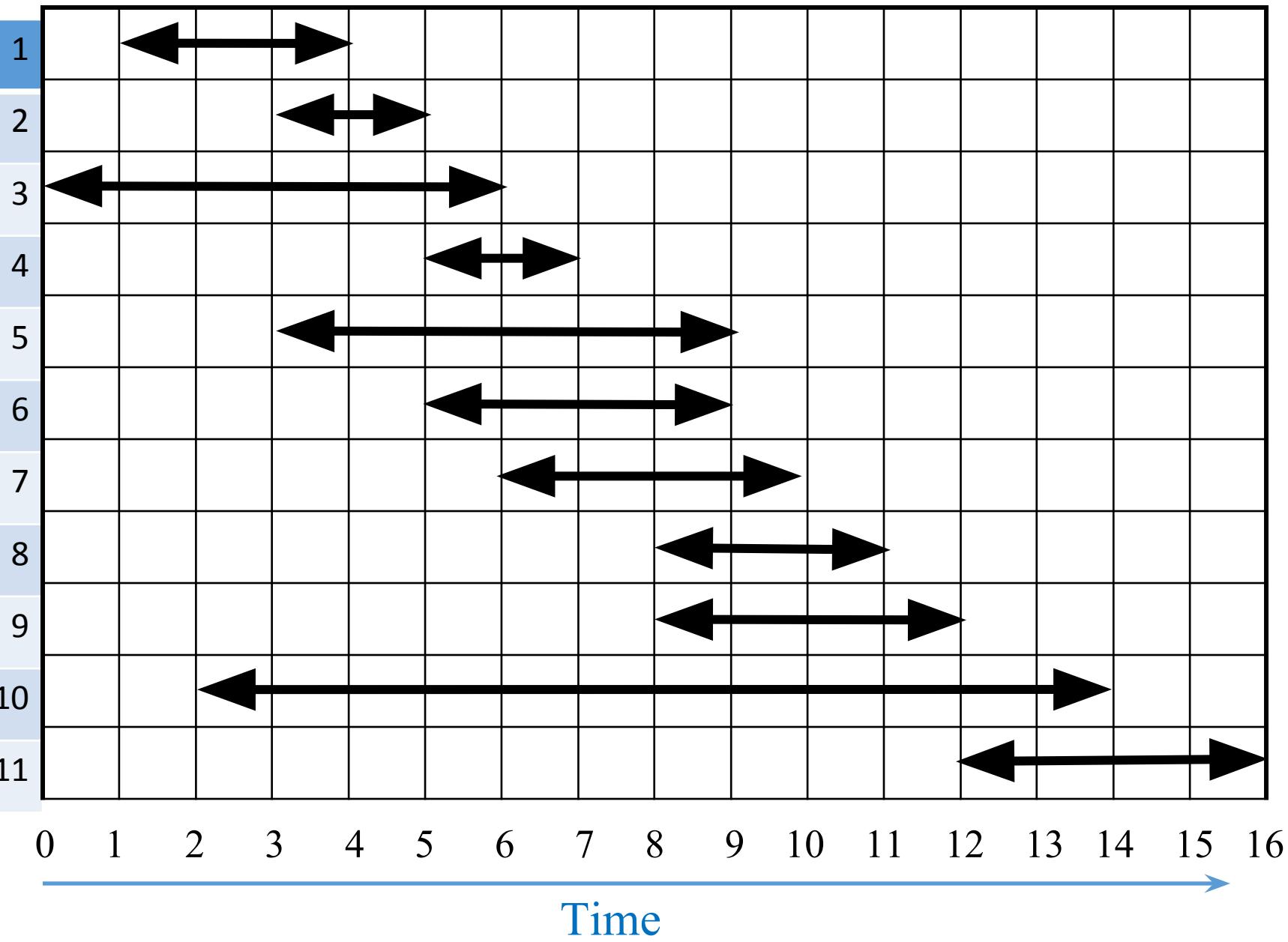
1.  $n = s.length$
2.  $m = i+1$
3. **while**  $m \leq n$  **and**  $s[m] < f[i]$       //skip all activities which are incompatible with  $i$
4.      $m = m + 1$
5. **if**  $m \leq n$     //if an activity is found in  $S_{i+1,n}$  which is compatible with  $i$
6.     return  $\{a_m\} \cup \text{RECURSIVE-ACTIVITY-SELECTOR}(s, f, m)$  //add  $a_m$  to solution set
6. **else** return  $\emptyset$

Running time is  $\Theta(n)$

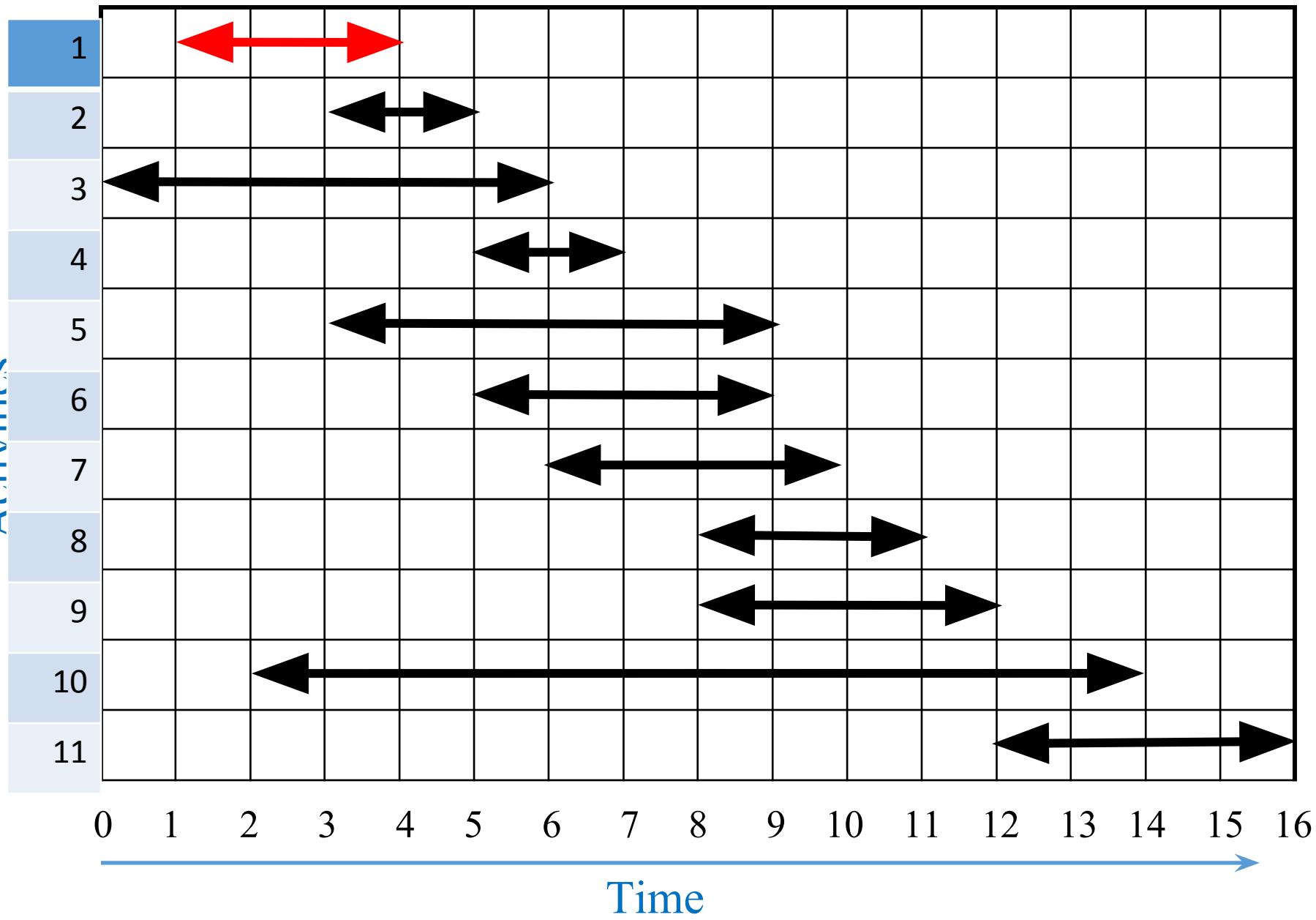
Assuming that the activities have already been sorted by finish times,  
the running time of the call RECURSIVE-ACTIVITY-SELECTOR is  $\Theta(n)$

Because, over all recursive calls, each activity is examined exactly once in the  
while loop test of line 3.

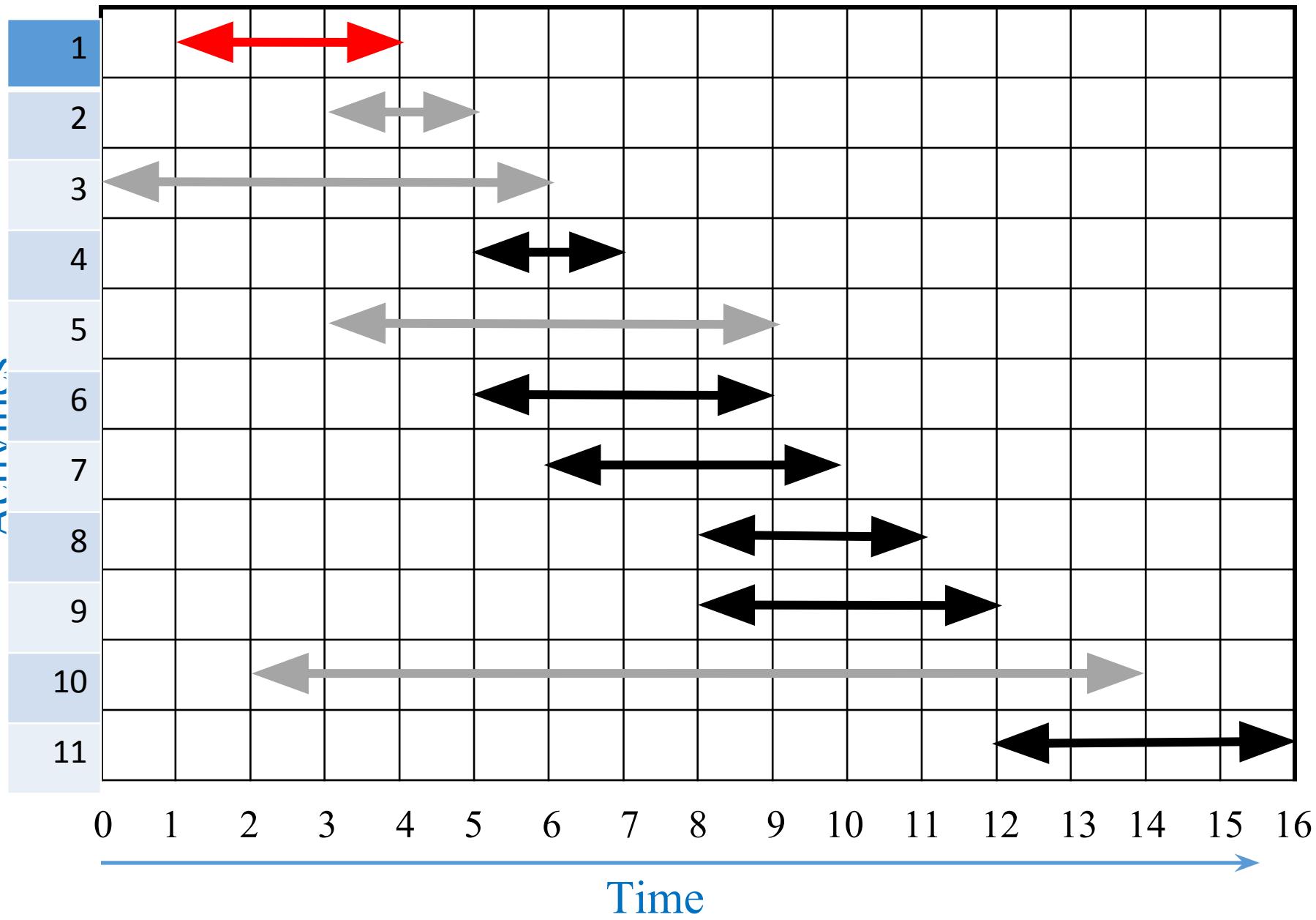
Activities



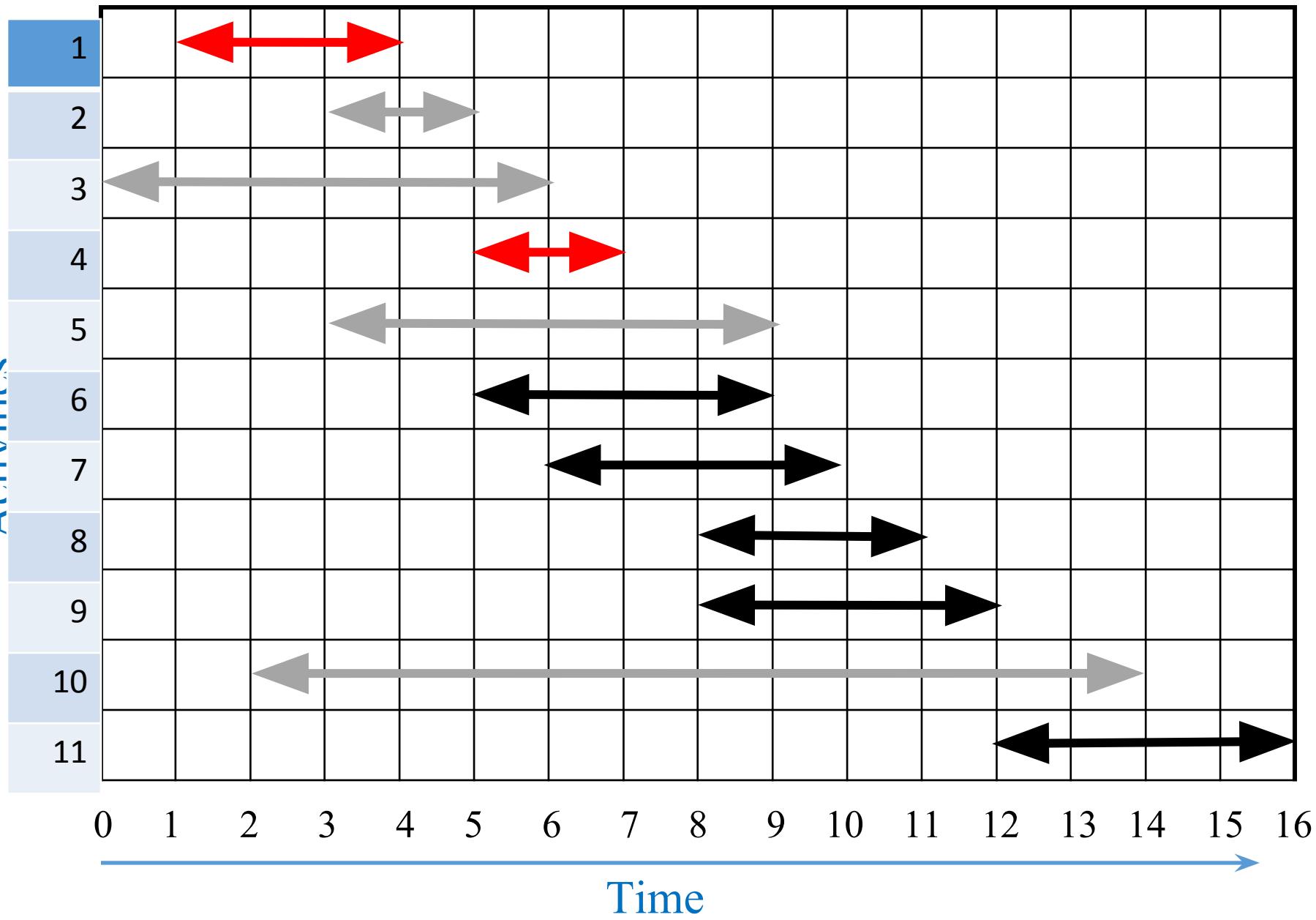
Activities



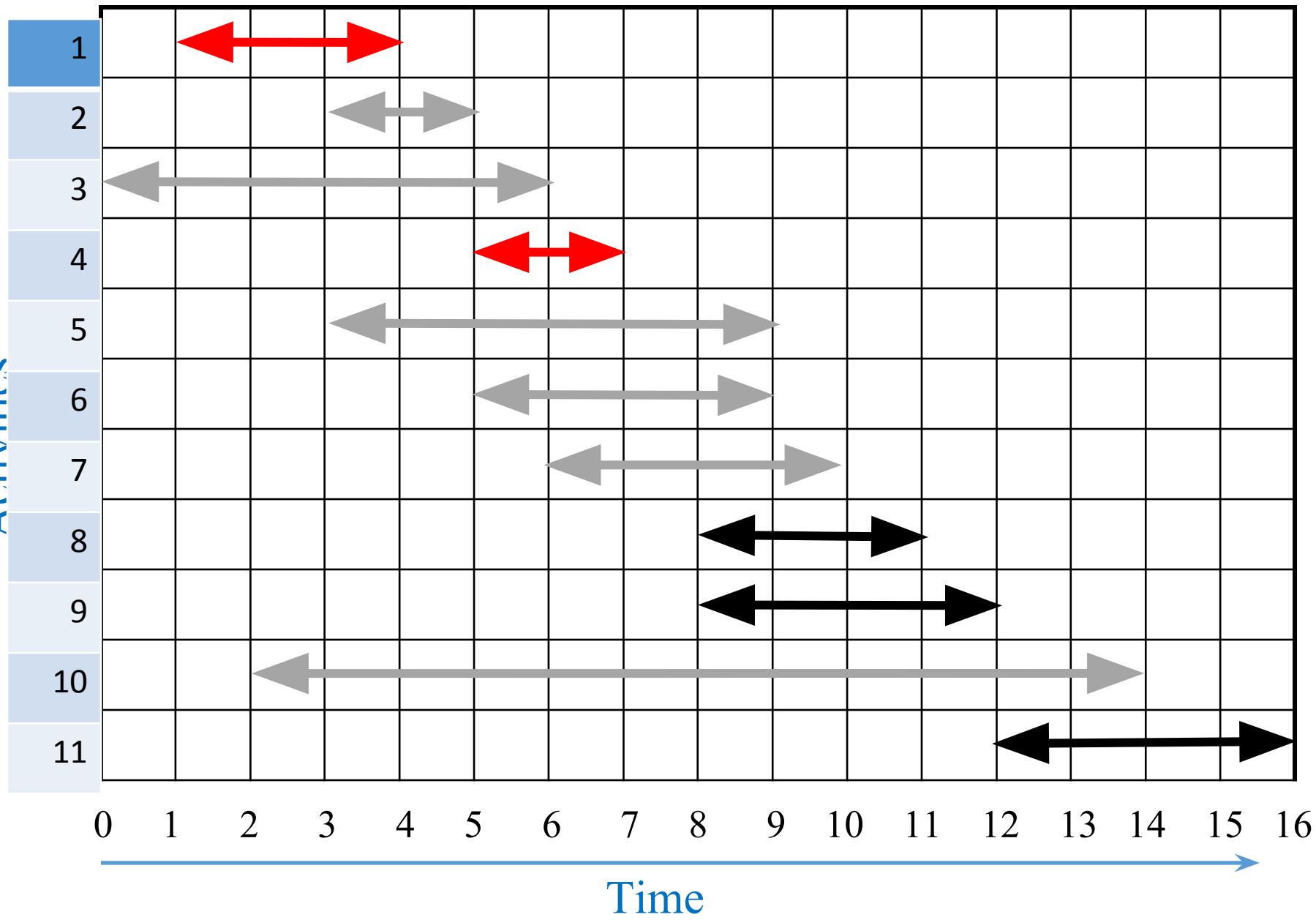
Activities

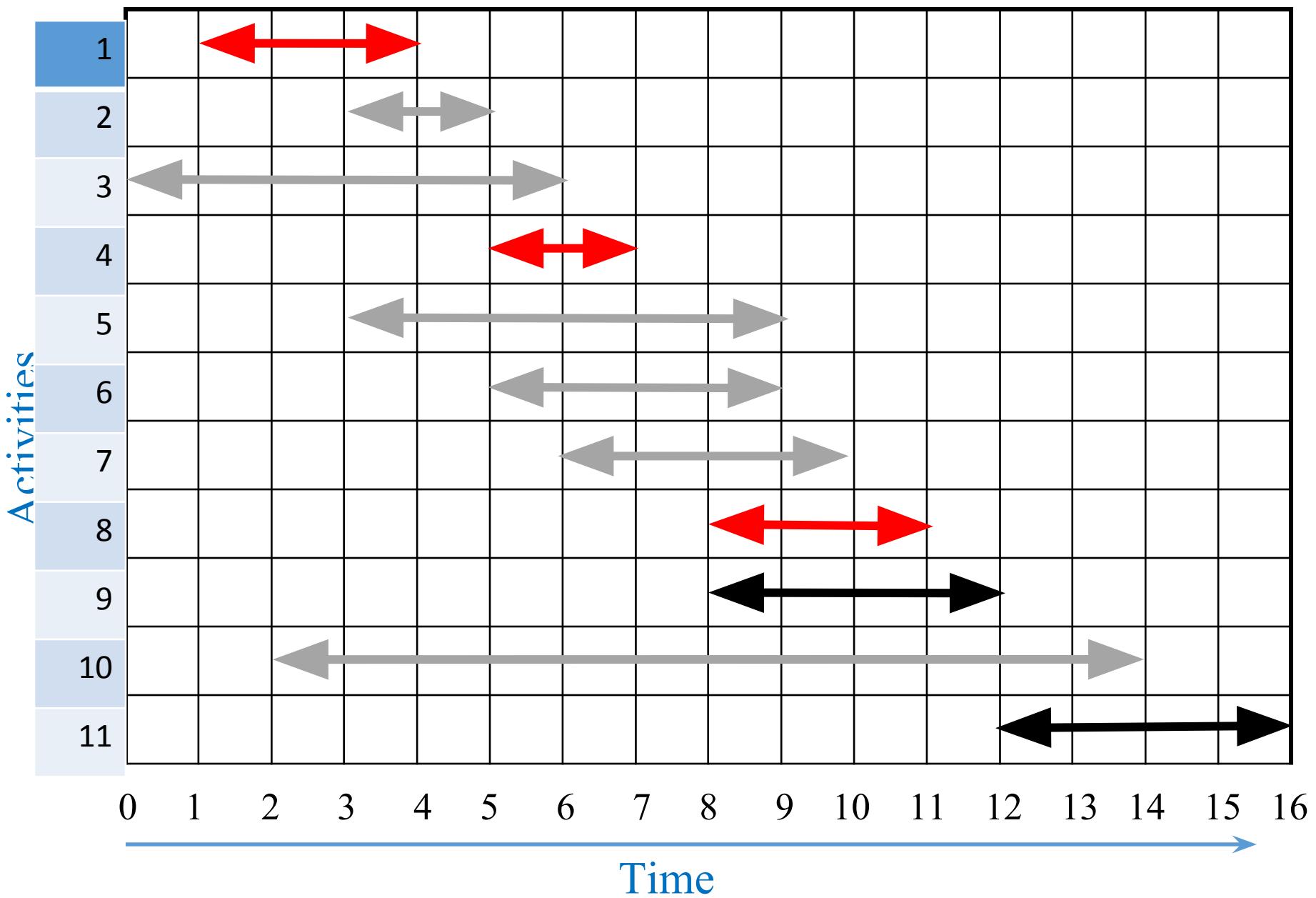


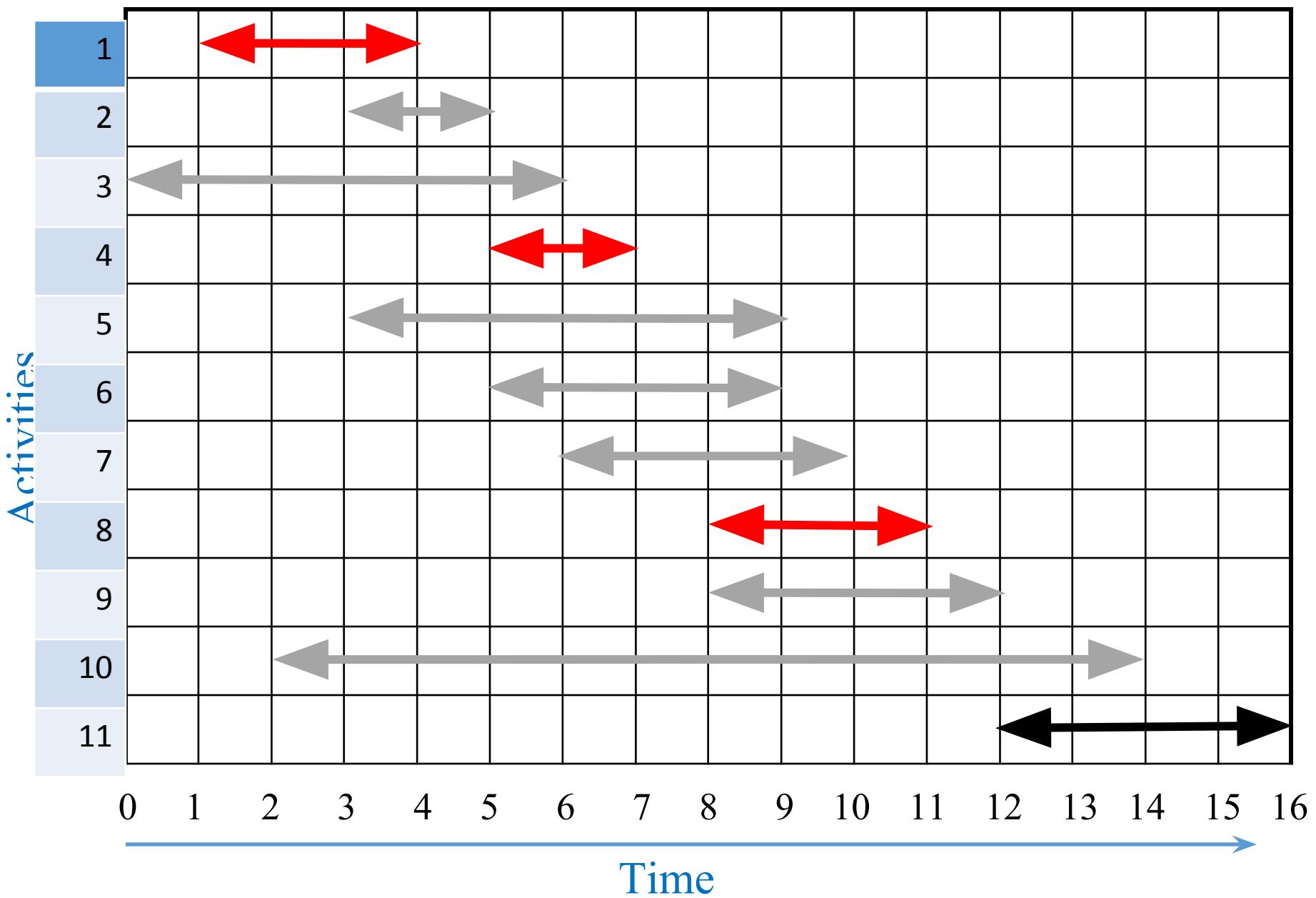
Activities



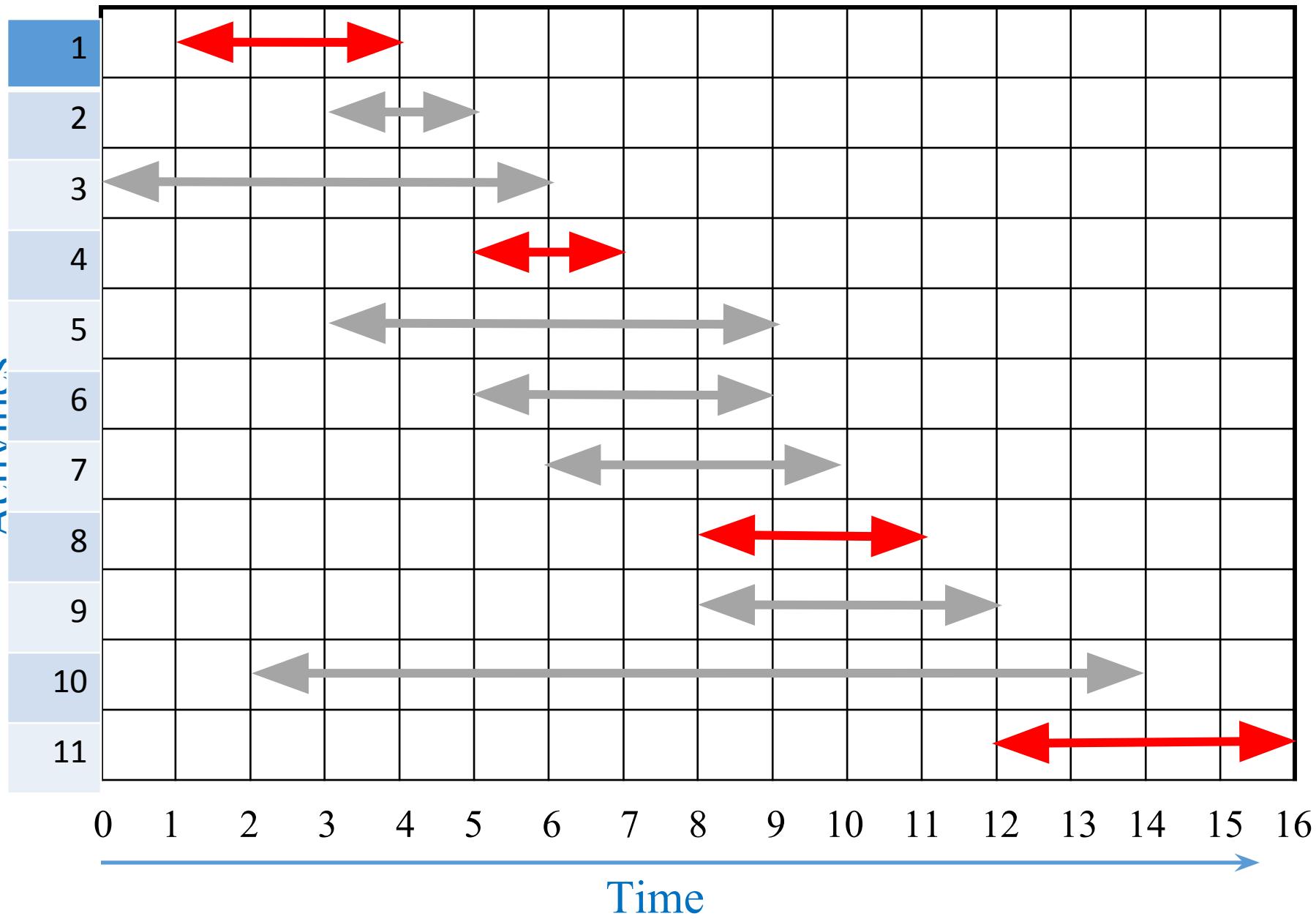
Activities







Activities



# Properties of Greedy Problems

How can one tell if a greedy algorithm will be able to solve an optimization problem i.e., whether a problem is a “Greedy problem”?

There is no way in general. But there are 2 ingredients exhibited by most Greedy Problems (problems solvable via a greedy algorithm):

**1. Greedy Choice Property:** there is an optimal solution that contains the first greedy choice

- It implies that a globally optimal solution can be arrived at by making a locally optimal (Greedy) choice.
- So we can make whatever choice seems best at this moment (i.e., greedy choice) and then solve the subproblem that results after the choice is made.
- Thus, a greedy strategy usually progresses in a **top-down** fashion, making one greedy choice after another, iteratively reducing each given problem instance to a smaller one.

**2. Optimal Sub Structure Property:** Optimal solution of a problem contains within it optimal solution of its subproblem(s)

- It implies that the optimal solution of subproblem(s) can be combined together to obtain the optimal solution of the problem itself

# Greedy Choice Property Illustration

i	1	2	3	4	5	6	7	8	9	10	11
$s_j$	1	3	0	5	3	5	6	8	8	2	12
$f_j$	4	5	6	7	9	9	10	11	12	14	16

- An optimal solution which does not begin with activity 1 is:  $A = \{2, 4, 9, 11\}$
- Let  $B = A - \{2\} \cup \{1\} = \{1, 4, 9, 11\}$ ; B is an optimal solution, too. **Why?**
  - Activity 2 doesn't clash with 4, 9, or 11. Activity 1 finishes before 2 finishes and as such 1 cannot clash with 4, 9, or 11, either. Also, activities 4, 9, and 11 do not clash with each other (otherwise they wouldn't be members of A).
- Hence  $B = \{1, 4, 9, 11\}$  is a valid solution. Since  $|B| = |A|$ , B is an optimal solution.

# Optimal Substructure Property Illustration

i	1	2	3	4	5	6	7	8	9	10	11
$s_j$	1	3	0	5	3	5	6	8	8	2	12
$f_j$	4	5	6	7	9	9	10	11	12	14	16

- Take an optimal solution of  $S = \{1,2,3,4,5,6,7,8,9,10,11\}$  which contains activity 1:  $A = \{1,4,9,11\}$
- According to the optimal substructure property:
  - $A' = A - \{1\} = \{4,9,11\}$  is an optimal solution of  $S' = \{4,6,7,8,9,11\}$  (a.k.a. a subproblem of the original problem S)

# Implications of greedy choice and optimal substructure properties

- After each greedy choice is made, we are left with an smaller optimization problem (called a subproblem) of the same form as the original problem
- By induction, making the greedy choice for every subproblem produces an optimal solution of the whole problem

i	1	2	3	4	5	6	7	8	9	10	11
$s_j$	1	3	0	5	3	5	6	8	8	2	12
$f_j$	4	5	6	7	9	9	10	11	12	14	16

Current Problem	Greedy Choice for Current Problem	Solution to current problem
$S = \{1,2,3,4,5,6,7,8,9,10,11\}$	1	$A = \{1, \dots\}$
$S' = \{4,6,7,8,9,11\}$	4	$A' = \{4, \dots\}$
$S'' = \{6,7,8,9,11\}$	6	$A'' = \{6, \dots\}$
$S''' = ?$	?	?

# Practice Problem on Activity Selection

i	1	2	3	4	5	6	7	8	9	10	11
$s_j$	1	4	5	6	9	11	10	6	8	13	15
$f_j$	5	6	9	10	12	15	18	8	10	14	20

Current Problem	Greedy Choice for Current Problem	Solution to current problem

# Greedy-Choice Property of Activity Selection Problem

Claim: There is an optimal solution that contains the first greedy choice (*i.e.*, activity 1, which has the earliest finish time)

Proof:

Let  $S = \{1, 2, \dots, n\}$  be the set of input activities sorted by their finishing times, *i.e.*,  $f_1 \leq f_2 \leq \dots \leq f_n$

An optimal solution will be a subset of  $S$

Suppose  $A \subseteq S$  is an optimal solution

Order the activities in  $A$  by finish time. Let  $k$  be the first activity in  $A$  in this order.

If  $k = 1$ , the solution  $A$  contains activity 1 and the proof completes.

If  $k \neq 1$ , we have to show that there is an optimal solution  $B \subseteq S$  which contains activity 1

# Greedy-Choice Property of Activity Selection Problem

Let  $B = A - \{k\} \cup \{1\}$ .

- activity 1 has the earliest finish time in  $S \Rightarrow f_1 \leq f_k \dots \dots (1)$  and
- activity  $k$  finishes before other activities in  $A$  starts (otherwise, activity  $k$  would clash with other activities in set  $A$ )  $\Rightarrow f_k \leq s_j \dots \dots (2)$  for all  $j \in A - \{k\}$

(1)&(2)  $\Rightarrow f_1 \leq s_j$  for all  $j \in A - \{k\} \Rightarrow$  activity 1 doesn't clash with other activities in  $B$

Also, activities in  $A - \{k\}$  do not clash with each other (Since,  $A$  is a valid solution)

Hence activities in  $B$  are mutually compatible. Also,  $|B| = |A|$

Thus,  $B$  is an optimal solution which contains activity 1 (the first greedy choice)

# Optimal Substructure Property of Activity Selection Problem

Once the greedy choice of activity 1 is made, the problem reduces to finding an optimal solution for the activity-selection problem over those activities in S that are compatible with activity 1

## Optimal Substructure property

If A is an optimal solution of input S which contains the first greedy choice (activity 1), then  $A' = A - \{1\}$  is an optimal solution of input  $S' = \{i \in S : s_i \geq f_1\}$

### Proof:

If we could find an optimal solution, B' of S' where  $|B'| > |A'|$ , then  $B = B' \cup \{1\}$  would be an optimal solution of S which contains more activities than A

(since  $|B| = |B'| + 1 > |A'| + 1 = |A|$ ),

But this contradicts our premise that A is an optimal solution.

# The Fractional Knapsack Problem

**Given:** A set  $S$  of  $n$  items, with each item  $i$  having

$b_i$  - a positive benefit

$w_i$  - a positive weight

**Goal:** Choose items with maximum total benefit but with weight at most  $W$ .

Items:	1	2	3	4	5
Weight:	4 ml	8 ml	2 ml	6 ml	1 ml
Benefit:	\$12	\$32	\$40	\$30	\$50
Value: (\$ per ml)	3	4	20	5	50



**Solution:** P

- 1 ml of 5 50\$
- 2 ml of 3 40\$
- 6 ml of 4 30\$
- 1 ml of 2 4\$

• Total Profit: 124\$

# The Fractional Knapsack Problem

**Given:** A set  $S$  of  $n$  items, with each item  $i$  having

$b_i$  - a positive benefit

$w_i$  - a positive weight

**Goal:** Choose items with maximum total benefit but with weight at most  $W$ .

If we are allowed to take fractional amounts, then this is the **fractional knapsack problem**.

In this case, we let  $x_i$  denote the amount we take of item  $i$

Objective: maximize

Constraint:

$$\sum_{i \in S} b_i(x_i / w_i)$$

$$\sum_{i \in S} x_i \leq W, 0 \leq x_i \leq w_i$$

# The Fractional Knapsack Algorithm

Greedy choice: Keep taking item with highest **value** (benefit to weight ratio)

Since

$$\sum_{i \in S} b_i(x_i / w_i) = \sum_{i \in S} (b_i / w_i)x_i$$

**Algorithm** *fractionalKnapsack(S, W)*

**Input:** set  $S$  of items w/ benefit  $b_i$  and weight  $w_i$ ; max. weight  $W$

**Output:** amount  $x_i$  of each item  $i$  to maximize benefit w/ weight at most  $W$

**for** *each item i in S*

$x_i \leftarrow 0$

$v_i \leftarrow b_i / w_i$       {value}

$w \leftarrow 0$                           {total weight}

**while**  $w < W$

*remove item i with highest  $v_i$*

$x_i \leftarrow \min\{w_i, W - w\}$

$w \leftarrow w + x_i$

# The Fractional Knapsack Algorithm

## Running time:

Given a collection  $S$  of  $n$  items, such that each item  $i$  has a benefit  $b_i$  and weight  $w_i$ , we can construct a maximum-benefit subset of  $S$ , allowing for fractional amounts, that has a total weight  $W$  in  $O(n \log n)$  time.  
(how?)

Use heap-based priority queue to store  $S$

Removing the item with the highest value takes  $O(\log n)$  time

In the worst case, need to remove all items

# Practice Problem on Fractional Knapsack

Item	1	2	3	4	5	6	7
Weight(Kg)	2	3	4	2	5	4	1
Price	7	6	9	6	12	10	3

*Total capacity of Knapsack = 15Kg*

Remaining capacity of knapsack	Item chosen	Per Kg Price of chosen item	Weight of chosen item	Total benefits/prices of the item chosen

# Huffman Codes

Widely used technique for data compression

Assume the data to be a sequence of characters

Looking for an effective way of storing the data

***Binary character code***

Uniquely represents a character by a binary string

# Fixed-Length Codes

*E.g.:* Data file containing 100,000 characters

	a	b	c	d	e	f
Frequency (thousands)	45	13	12	16	9	5

3 bits needed

a = 000, b = 001, c = 010, d = 011, e = 100, f = 101

Requires:  $100,000 \cdot 3 = 300,000$  bits

Compression ratio =  $300000/800000 = 0.375$  using fixed length encoding.

# Huffman Codes

Idea:

Use the frequencies of occurrence of characters to build a optimal way of representing each character

	a	b	c	d	e	f
Frequency (thousands)	45	13	12	16	9	5

# Variable-Length Codes

*E.g.:* Data file containing 100,000 characters

	a	b	c	d	e	f
Frequency (thousands)	45	13	12	16	9	5

Assign short codewords to frequent characters and long codewords to infrequent characters

$$a = 0, b = 101, c = 100, d = 111, e = 1101, f = 1100$$

File size after encoding using these codewords =

$$(45 \cdot 1 + 13 \cdot 3 + 12 \cdot 3 + 16 \cdot 3 + 9 \cdot 4 + 5 \cdot 4) \cdot 1,000 \\ = 224,000 \text{ bits}$$

Compression ratio using this encoding =  
 $224000/800000 = 0.28$

# Prefix Codes

Prefix codes:

Codes for which no codeword is also a prefix of some other codeword

Better name would be “prefix-free codes”

We can achieve optimal data compression using prefix codes

We will restrict our attention to prefix codes

# Encoding with Binary Character Codes

## Encoding

Concatenate the codewords representing each character in the file

*E.g.:*

$a = 0, b = 101, c = 100, d = 111, e = 1101, f = 1100$

$abc = 0 \cdot 101 \cdot 100 = 0101100$

# Decoding with Binary Character Codes

Prefix codes simplify decoding

No codeword is a prefix of another  $\Rightarrow$  the codeword that begins an encoded file is unambiguous

Approach

Identify the initial codeword

Translate it back to the original character

Repeat the process on the remainder of the file

*E.g.:*

a = 0, b = 101, c = 100, d = 111, e = 1101, f = 1100

001011101 =

0 . . . . = aabe  
0 101 1101

# Constructing a Huffman Code

A greedy algorithm that constructs an optimal prefix code called a **Huffman code**

Assume that:

$C$  is a set of  $n$  characters

Each character has a frequency  $f(c)$

The tree  $T$  is built in a bottom up manner

Idea:

Start with a set of  $|C|$  leaves

f: 5	e: 9	c: 12	b: 13	d: 16	a: 45
------	------	-------	-------	-------	-------

At each step, merge the two least frequent objects: the frequency of the new node = sum of two frequencies

Use a min-priority queue  $Q$ , keyed on  $f$  to identify the two least frequent objects

# Constructing a Huffman tree

Alg.: HUFFMAN( $C$ )

1.  $n \leftarrow |C|$

Running time:  $O(nlgn)$

2.  $Q \leftarrow C$

3. **for**  $i \leftarrow 1$  **to**  $n - 1$   $O(n)$

4.     **do** allocate a new node  $z$

5.          $\text{left}[z] \leftarrow x \leftarrow \text{EXTRACT-MIN}(Q)$

6.          $\text{right}[z] \leftarrow y \leftarrow \text{EXTRACT-MIN}(Q)$

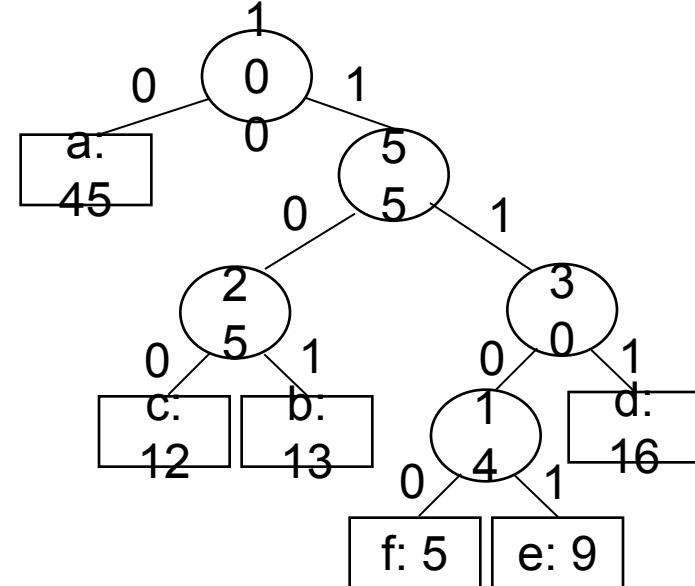
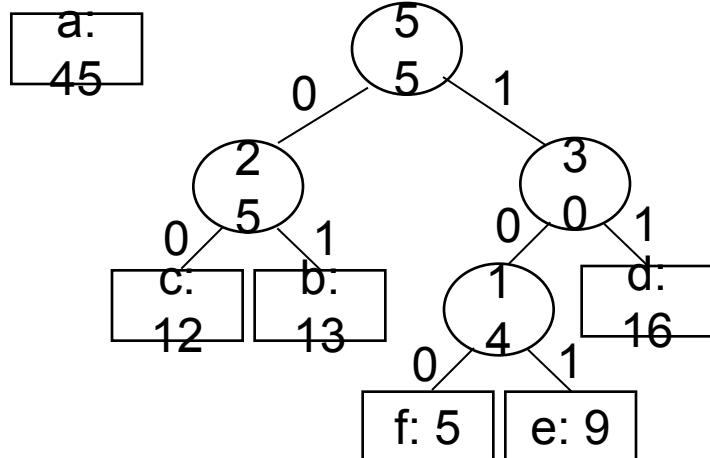
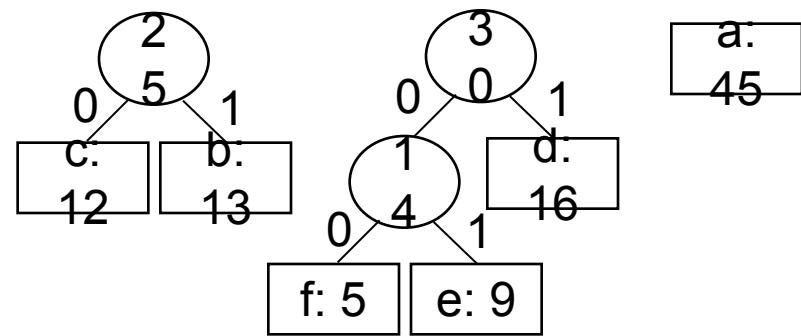
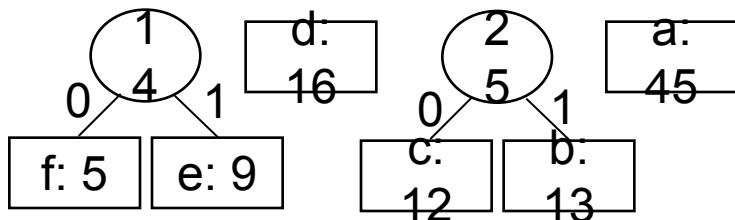
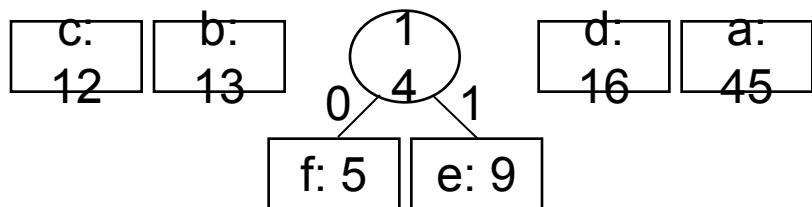
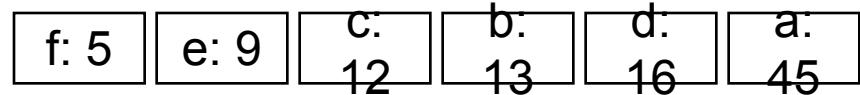
7.          $f[z] \leftarrow f[x] + f[y]$

8.          $\text{INSERT}(Q, z)$

9. **return**  $\text{EXTRACT-MIN}(Q)$

$O(lgn)$

# Example



# Huffman encoding/decoding using Huffman tree

## **Encoding:**

- Construct Huffman tree using the previous algorithm
- Traverse the Huffman tree to assign a code to each leaf (representing an input character )
- Use these codes to encode the file

## **Decoding:**

- Construct Huffman tree using the previous algorithm
- Traverse the Huffman tree starting from root node according to the bits you encounter until you reach a leaf node at which point you output the character represented by that leaf node & then go back to the root node.
- Continue in this fashion until all the bits in the file are read.

# Huffman Coding Practice

A file has the following characters along with their frequencies:

a: 20, b: 30, c: 78, d: 12, e: 27, f: 67 g: 5, h: 80.

Draw Huffman coding tree and compute the optimal Huffman code of each character mentioned above. Also compute the compression ratio achieved via this encoding.