Duel Problem

Dual Problem of an LPP

- Given a LPP (called the primal problem), we shall associate another LPP called the dual problem of the original (primal) problem.

Definition of the dual problem

Given the primal problem (in standard form)

Maximize $$z = c_1x_1 + c_2x_2 + \ldots + c_nx_n$$

subject to

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2$$

$$.$$

$$.$$

$$a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = b_m$$

$$x_1, x_2, \ldots, x_n \geq 0, b_1, b_2, \ldots, b_m \geq 0$$

the dual problem is the LPP

Minimize $w = b_1 y_1 + b_2 y_2 + \ldots + b_m y_m$

subject to

$$a_{11} y_1 + a_{21} y_2 + \ldots + a_{m1} y_m \geq c_1$$

$$a_{12} y_1 + a_{22} y_2 + \ldots + a_{m2} y_m \geq c_2$$

.

.

$$a_{1n} y_1 + a_{2n} y_2 + \ldots + a_{mn} y_m \geq c_n$$

$$y_1, y_2, \ldots, y_n \text{ unrestricted in sign}$$

If the primal problem (in <u>standard</u> form) is

Minimize $z = c_1x_1 + c_2x_2 + \ldots + c_nx_n$

subject to

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2$$

$$.$$

$$.$$

$$a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = b_m$$

$$x_1, x_2, \ldots, x_n \geq 0, b_1, b_2, \ldots, b_m \geq 0$$

Then the dual problem is the LPP

Maximize $w = b_1 y_1 + b_2 y_2 + \ldots + b_m y_m$

subject to

$$a_{11} y_1 + a_{21} y_2 + \ldots + a_{m1} y_m \le c_1$$

$$a_{12} y_1 + a_{22} y_2 + \ldots + a_{m2} y_m \le c_2$$

.

.

$$a_{1n} y_1 + a_{2n} y_2 + \ldots + a_{mn} y_m \le c_n$$

$y_1, y_2, \ldots, y_n$ unrestricted in sign

We thus note the following:

1. In the dual, there are as many (decision) variables as there are constraints in the primal.

   We usually say y_i is the dual variable associated with the ith constraint of the primal.

2. There are as many constraints in the dual as there are variables in the primal.

3. If the primal is maximization then the dual is minimization and all constraints are $\geq$

If the primal is minimization then the dual is maximization and all constraints are $\leq$

4. In the primal, all variables are ≥ 0 while in the **dual** all the variables are **unrestricted in sign**.

5. The objective function coefficients c_j of the primal are the **RHS constants** of the dual constraints.
6. The **RHS constants** b_i of the primal constraints are the **objective function coefficients** of the dual.
7. The **coefficient matrix of the constraints** of the dual is the **transpose of the coefficient matrix of the constraints** of the primal.

Write the dual of the LPP

Maximize $z = -5x_1 + 2x_2$

subject to

$$-x_1 + x_2 \leq -2$$

$$2x_1 + 3x_2 \leq 5$$

$$x_1, x_2 \geq 0$$

Thus the primal in the standard form is:

Maximize $z = -5x_1 + 2x_2 + 0x_3 + 0x_4$

subject to

$$
\begin{aligned}
x_1 - x_2 - x_3 \quad\quad &= 2 \\
2x_1 + 3x_2 \quad\quad + x_4 &= 5 \\
x_1, x_2, x_3, x_4 &\geq 0
\end{aligned}
$$

Hence the dual is:

Minimize $w = 2y_1 + 5y_2$

subject to

$$
\begin{aligned}
y_1 + 2y_2 &\geq -5 \\
-y_1 + 3y_2 &\geq 2 \\
-y_1 &\geq 0 \\
y_2 &\geq 0
\end{aligned}
\quad \Rightarrow \quad y_1 \leq 0,\ y_2 \geq 0
$$

y_1, y_2 unrestricted in sign

Write the dual of the LPP

Minimize $z = 6x_1 + 3x_2$

subject to

$$6x_1 - 3x_2 + x_3 \geq 2$$

$$3x_1 + 4x_2 + x_3 \geq 5$$

$$x_1, x_2, x_3 \geq 0$$

Thus the primal in the standard form is:

Minimize $z = 6x_1 + 3x_2 + 0x_3 + 0x_4 + 0x_5$

subject to

$$
\begin{aligned}
6x_1 - 3x_2 + x_3 - x_4 \quad &= 2 \\
3x_1 + 4x_2 + x_3 \quad - x_5 &= 5 \\
x_1, x_2, x_3, x_4, x_5 &\geq 0
\end{aligned}
$$

Hence the dual is:

Maximize $w = 2y_1 + 5y_2$

subject to

$$
\begin{aligned}
6y_1 + 3y_2 &\le 6 \\
-3y_1 + 4y_2 &\le 3 \\
y_1 + y_2 &\le 0 \\
-y_1 &\le 0 \\
-y_2 &\le 0
\end{aligned}
\quad \Rightarrow \quad y_1, y_2 \ge 0
$$

~~y_1, y_2 unrestricted in sign~~

Write the dual of the LPP

Maximize $z = x_1 + x_2$

subject to

$$2x_1 + x_2 = 5$$

$$3x_1 - x_2 = 6$$

$$x_1, x_2 \text{ unrestricted in sign}$$

Thus the primal in the standard form is:

Maximize $z = x_1^+ - x_1^- + x_2^+ - x_2^-$

subject to

$$2x_1^+ - 2x_1^- + x_2^+ - x_2^- = 5$$

$$3x_1^+ - 3x_1^- - x_2^+ + x_2^- = 6$$

$$x_1^+, x_1^-, x_2^+, x_2^- \geq 0$$

Hence the dual is:

Minimize $w = 5y_1 + 6y_2$

subject to

$$\left.\begin{aligned} 2y_1 + 3y_2 &\geq 1 \\ -2y_1 - 3y_2 &\geq -1 \end{aligned}\right\} \Rightarrow 2y_1 + 3y_2 = 1$$

$$\left.\begin{aligned} y_1 - y_2 &\geq 1 \\ -y_1 + y_2 &\geq -1 \end{aligned}\right\} \Rightarrow y_1 - y_2 = 1$$

y_1, y_2 **unrestricted in sign**

From the above examples we get the following **SOB** rules for writing the dual:

Label	Maximization	Minimization
	Constraints	**Variables**
Sensible	$\leq$ form	≥ 0
Odd	$=$ form	unrestricted
Bizarre	$\geq$ form	≤ 0
	Variables	**Constraints**
Sensible	≥ 0	$\geq$ form
Odd	unrestricted	$=$ form
Bizarre	≤ 0	$\leq$ form