MATH 2116: Linear Algebra

ClassNote: 02

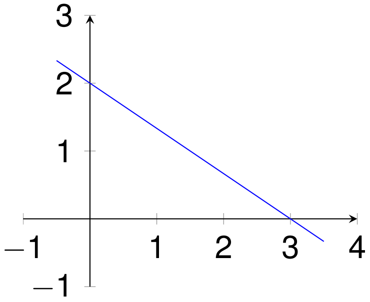
Insitute of Information Technology (IIT), DU

Mohd. Zulfiquar Hafiz

2023/2024

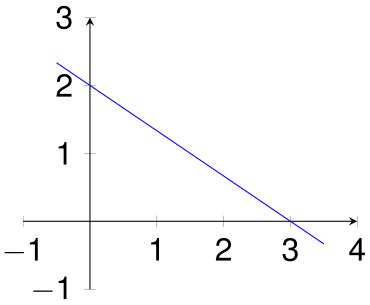
Linear Equation

The equation $2x + 3y = 6$ is called linear because its solution set is a straight line in $\mathbb{R}^2$. A solution of the equation is a pair of numbers $(\alpha, \beta) \in \mathbb{R}^2$ such that $2\alpha + 3\beta = 6$.



$2x + 3y = 6$

Linear Equation



For example, $(3, 0)$ and $(0, 2)$ are solutions. Alternatively, we can write the first solution as $x = 3$, $y = 0$.

General equation of a line: $ax + by = c$, where x, y are variables and a, b, c are constants (except for the case $a = b = 0$).

Linear Equation

Definition: A linear equation in variables $x_1, x_2, \cdots, x_n$ is an equation of the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b \tag{1}$$

where $a_1, \cdots, a_n$, and b are constants.
A solution of the equation 1 is an array of numbers $(\gamma_1, \gamma_2, \cdots, \gamma_n) \in \mathbb{R}^n$ such that

$$a_1\gamma_1 + a_2\gamma_2 + \cdots + a_n\gamma_n = b$$

The set of all such solutions of a linear equation is called the **solution set**.

Linear Equation

Homogeneous/Non-homogeneous linear equation

If $b = 0$, then the equation 1 is called a **homogeneous linear equation** and if $b \neq 0$, then the equation 1 is called a **non-homogeneous linear equation**.

Linear Equation

Non-degenerate/degenerate linear equation

The general linear equation $a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$ is also called **Non-degenerate linear equation**. A linear equation is said to be **degenerate linear equation** if it has the form $0x_1 + 0x_2 + \cdots + 0x_n = b$. That is, if every coefficient of the variable is equal to zero. The solution of such a degenerate linear equation is as follows:

1. If the constant $b \neq 0$, then the above equation has no solution.
2. If the constant $b = 0$, then every vector $u = (u_1, u_2, \cdots, u_n)$ is a solution of the above equation.

System of Linear Equations

A general form of a **System of Linear Equations** (or a set of m **Simultaneous Linear Equations** or **Linear System**) is as following

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \cdots\cdots\cdots\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}\right\} \tag{2}$$

Here $x_1, x_2, \cdots, x_n$ are variables and a_{ij}, b_i are constants. A *solution* of the system is a common solution of all equations in the system. A system of linear equations can have **one solution**, **infinitely many solutions**, or **no solution** at all.

System of Linear Equations

Consistent/Inconsistent System

A system of linear equations is called **consistent** if it has at least one solution and **inconsistent** if it has no solution.

Determinate/Indeterminate System

A consistent system is called **determinate** if it has a unique solution and **indeterminate** if it has more than one solution. An indeterminate system of linear equations always has an infinite number of solutions.

System of Linear Equations

Homogeneous/Non-homogeneous System

If all the b_i of the system 2 are zero, then the system 2 is called a **homogeneous system of linear equations** and if at least one b_i of the system 2 is not zero, then the system 2 is called a **non-homogeneous system of linear equations**.

Equivalent System

Two systems of linear equations are called **equivalent** if every solution of the first system is a solution of the second and conversely (vice versa).

Solution of homogeneous SLE

A system of linear equations of the following form is called **homogeneous system of linear equations**

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= 0 \\ \cdots\cdots\cdots\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= 0 \end{aligned}\right\} \tag{3}$$

Every homogeneous system of linear equations is consistent, since $x_1 = 0, x_2 = 0, \cdots, x_n = 0$ is always a solution of the system. This solution is called **trivial (or zero) solution**. If the other solutions exist, they a called the **non-trivial (or non-zero) solutions**.

Solution of homogeneous SLE

Thus the above homogeneous system can always be reduced to an equivalent homogeneous system in **echelon form**:

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1r}x_r + \cdots + a_{1n}x_n &= 0 \\ c_{22}x_2 + c_{23}x_3 + \cdots + c_{2r}x_r + \cdots + c_{2n}x_n &= 0 \\ d_{33}x_3 + \cdots + d_{3r}x_r + \cdots + d_{3n}x_n &= 0 \\ \cdots\cdots\cdots & \\ k_{rr}x_r + \cdots + k_{rn}x_n &= 0 \end{aligned}\right\}$$

Solution of homogeneous SLE

Hence we have the following two possibilities:

1. If $r = n$ i.e. the number of equations is equal to the number of unknowns then the system has only the **trivial (or zero) solution**.
2. If $r < n$ i.e. the number of equations is less than the number of unknowns then the system also has an infinite number of **non-trivial (or non-zero) solutions**.

Solution of non-homogeneous SLE

A system of linear equations of the following form is called **non-homogeneous system of linear equations** if at least one b_i of the system 4 is not equal to zero.

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \cdots\cdots\cdots\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}\right\} \tag{4}$$

Solution of non-homogeneous SLE

If we reduced the system 4 to a simpler equivalent system in **echelon form**:

$$\left.\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1r}x_r + \cdots + a_{1n}x_n &= b_1 \\
c_{22}x_2 + c_{23}x_3 + \cdots + c_{2r}x_r + \cdots + c_{2n}x_n &= \bar{b}_2 \\
d_{33}x_3 + \cdots + d_{3r}x_r + \cdots + d_{3n}x_n &= \bar{b}_3 \\
\cdots\cdots\cdots\cdots & \\
k_{rr}x_r + \cdots + k_{rn}x_n &= \bar{b}_r \\
0 &= \bar{b}_{r+1} \\
\cdots & \\
0 &= \bar{b}_m
\end{aligned}\right\} \tag{5}$$

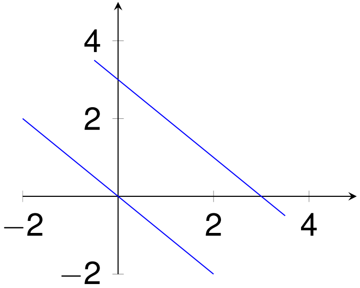
Solution of non-homogeneous SLE

where $r \leq n$. We see that there are three possible cases:

1. **Inconsistent system or no solution**, if $r < n$ and one of the numbers $\bar{b}_{r+1}, \cdots, \bar{b}_m$ is not zero.
2. **Determinate system or precisely one solution**, If $r = n$ and $\bar{b}_{r+1}, \cdots, \bar{b}_m$ if present are zero.
3. **Indeterminate system or infinitely many solutions**, If $r < n$ and $\bar{b}_{r+1}, \cdots, \bar{b}_m$ if present, are zero. Then any of these solutions is obtained by choosing values for the unknowns $x_{r+1}, \cdots, x_n$. The unknowns $x_{r+1}, \cdots, x_n$ are also called **free variables**.

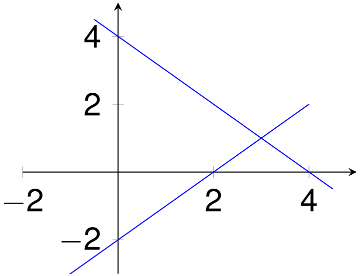
Solution of non-homogeneous SLE

Example 1: The linear system $\left.\begin{array}{c} x+y=3 \\ x+y=0 \end{array}\right\}$ has no solution.



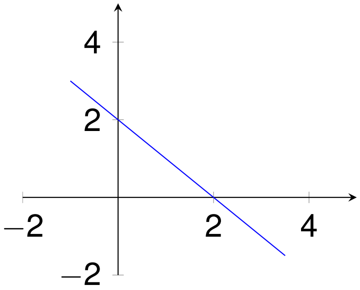
Solution of non-homogeneous SLE

Example 2: The linear system $\left.\begin{array}{l} x+y=4 \\ x-y=2 \end{array}\right\}$ has only one solution.



Solution of non-homogeneous SLE

Example 3: The linear system $\left.\begin{aligned} x+y&=2 \\ 3x+3y&=6 \end{aligned}\right\}$ has infinitely many solutions.



Solving systems of linear equations

Cross Multiplication Method

Cross-multiplication is a technique to determine the solution of linear equations in two variables. It proves to be the simplest, easiest and fastest method to solve a pair of linear equations. For a given pair of linear equations in two variables:

$$a_1x + b_1y + c_1 = 0$$
$$a_2x + b_2y + c_2 = 0$$

Using following **cross multiplication formula** we can solve this linear system:

$$\frac{x}{b_1c_2 - b_2c_1} = \frac{-y}{a_1c_2 - a_2c_1} = \frac{1}{a_1b_2 - a_2b_1}$$

Solving systems of linear equations

Substitution Method

The substitution method involves substituting the value of any one of the variables from one equation into the other equations. The steps to apply are given below:

- Step-1: Solve any one variable from one of the equations.
- Step-2: Substitute that value in the other equation.
- Step-3: Now, simplify the new equation to obtain the value of one variable.
- Step-4: Now, substitute the value obtained in Step 3 in any of the given equations to solve for the other variable.

Solving systems of linear equations

Elimination method

The steps to use the elimination method are:

- Step-1: Multiply each equation with a non-zero number to get a common coefficient of any one of the variables in both equations.
- Step-2: Add or subtract both equations to eliminate the same terms.
- Step-3: Simplify the result to get a value for the left out variable.
- Step-4: Substitute this value in any of the given equations to find the value of the other variable.

Solving systems of linear equations

Gaussian elimination method

Gaussian elimination method involves a successive "Whiting away" of the variables in order to isolate their values. This method is based on the following **three elementary operations** which alter the form of the equations, but not the solutions:

1. Interchange a pair of equations.
2. Multiplying an equation by non-zero number.
3. Adding a multiple of one equation to another eq.

Theorem (i) Applying elementary operations to a SLE does not change the solution set of the system. **(ii)** Any elementary operation can be undone by another elementary operation