Properties of Regular Languages

Chapter-4

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Topics

1) How to prove whether a given language is regular or not?

2) Closure properties of regular languages

3) Minimization of DFAs

Some languages are *not* regular

When a language is regular?

if we are able to construct one of the following: DFA *or* NFA *or* ε -NFA *or* regular expression

When is it not?

If we can show that no FA can be built for a language

How to prove languages are ***not*** regular?

What if we cannot come up with any FA?

A) Can it be language that is not regular?

B) Or is it that we tried wrong approaches?

How do we *decisively* prove that a language is not regular?

> *"The hardest thing of all is to find a black cat in a dark room, especially if there is no cat!" -Confucius*

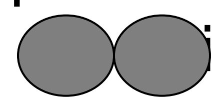
Example of a non-regular language

Let L = {w | w is of the form 0^n1^n , for all n≥0}

- *Hypothesis:* *L is not regular*
- Intuitive rationale: How do you keep track of a running count in an FA?
- A more formal rationale:
  - By contradition, if L is regular then there should exist a DFA for L.
  - Let k = number of states in that DFA.
  - Consider the special word w= 0^k1^k => w ∈ L
  - DFA is in some state p_i, after consuming the first i symbols in w

Uses Pigeon Hole Principle

Rationale…

- Let $\{p_0, p_1, \ldots p_k\}$ be the sequence of states that the DFA should have visited after consuming the first k symbols in w which is 0^k
- But there are only k states in the DFA!
- ==> at least one state should repeat somewhere along the path (by  + inciple)
- ==> Let the repeating state be $p_i = p_J$ for $i < j$
- ==> We can fool the DFA by inputing $0^{(k-(j-i))}1^k$ and still get it to accept (note: k-(j-i) is at most k-1).
- ==> DFA accepts strings w/ unequal number of 0s and 1s, implying that the DFA is wrong!

The Pumping Lemma for Regular Languages

What it is?

The Pumping Lemma is a property of all regular languages.

How is it used?

A technique that is used to show that a given language is not regular

Pumping Lemma for Regular Languages

Let L be a regular language

Then *there exists* some constant ***N*** such that *for every* string $w \in L$ s.t. $|w| \geq N$, *there exists* a way to break w into three parts, $w = xyz$, such that:

1. $y \neq \varepsilon$
2. $|xy| \leq N$
3. For all $k \geq 0$, all strings of the form $xy^kz \in L$

This property should hold for all regular languages.

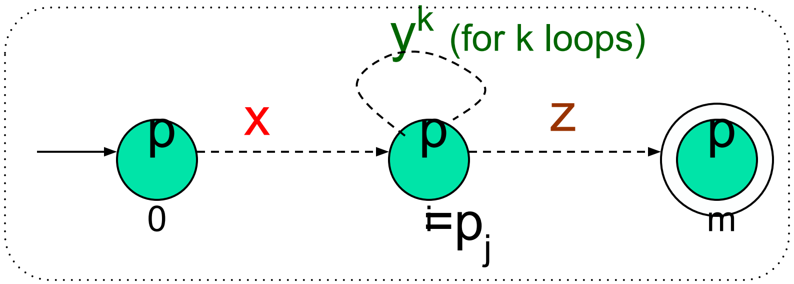
Definition: *N* is called the "Pumping Lemma Constant"

Pumping Lemma: Proof

- L is regular => it should have a DFA.
  - Set N := number of states in the DFA
- Any string w∈L, s.t. $|w| \geq N$, should have the form: $w=a_1a_2\ldots a_m$, where $m \geq N$
- Let the states traversed after reading the first N symbols be: $\{p_0, p_1, \ldots p_N\}$
  - ==> There are N+1 p-states, while there are only N DFA states
  - ==> at least one state has to repeat i.e, $p_i = p_J$ where $0 \leq i < j \leq N$ (by PHP)

Pumping Lemma: Proof…

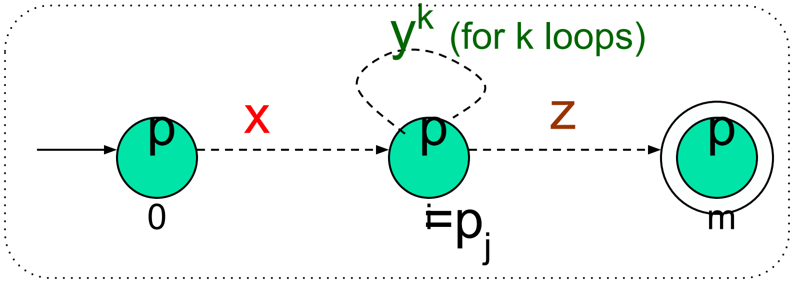
- => We should be able to break w=xyz as follows:
  - $x=a_1a_2..a_i$; $y=a_{i+1}a_{i+2}..a_J$; $z=a_{J+1}a_{J+2}..a_m$
  - x's path will be $p_0..p_i$
  - y's path will be $p_i\, p_{i+1}..p_J$ (but $p_i=p_J$ implying a loop)
  - z's path will be $p_Jp_{J+1}..p_m$
- Now consider another string $w_k=xy^kz$, where k≥0
- Case k=0
  - DFA will reach the accept state p_m
- Case k>0
  - DFA will loop for y^k, and finally reach the accept state p_m for z
- In either case, $w_k \in L$



This proves part (3) of the lemma

Pumping Lemma: Proof…

- For part (1):
  - Since i<j, y ≠ ε



- For part (2):
  - By PHP, the repetition of states has to occur within the first N symbols in w
  - ==> |xy|≤N

□

The Purpose of the Pumping Lemma for RL

- To prove that some languages *cannot be* regular.

How to use the pumping lemma?

Think of playing a 2 person game

- <u>Role 1:</u> ***We*** claim that the language cannot be regular
- <u>Role 2:</u> An ***adversary*** who claims the language is regular
- We show that the adversary's statement will lead to a contradiction that implyies pumping lemma *cannot* hold for the language.
- We win!!

How to use the pumping lemma? (The Steps)

1. (we) L is not regular.
2. (adv.) Claims that L is regular and gives you a value for N as its P/L constant
3. (we) Using N, choose a string $w \in L$ s.t.,
    1. $|w| \geq N$,
    2. Using w as the template, construct other words w_k of the form xy^kz and show that at least one such $w_k \notin L$

        => this implies we have successfully broken the pumping lemma for the language, and hence that the adversary is wrong.

(Note: In this process, we may have to try many values of k, starting with k=0, and then 2, 3, .. so on, until $w_k \notin L$)

Note: We don't have any control over N, except that it is positive.
We also don't have any control over how to split w=xyz,
but xyz should respect the P/L conditions (1) and (2).

Using the Pumping Lemma

- What WE do?

  3. Using *N*, we construct our template string *w*
  4. Demonstrate to the adversary, either through pumping up or down on *w*, that some string $w_k \notin L$
     (this should happen regardless of w=xyz)

- <u>What the Adversary does?</u>
  1. Claims L is regular
  2. Provides *N*

Note: This N can be anything (need not necessarily be the #states in the DFA. It's the adversary's choice.)

Example of using the Pumping Lemma to prove that a language is not regular

Let L_{eq} = {w | w is a binary string with equal number of 1s and 0s}

- Your Claim: L_{eq} is not regular
- Proof:
  - By contradiction, let L_{eq} be regular — adv.
  - P/L constant should exist — adv.
    - Let N = that P/L constant
  - Consider input w = $0^N 1^N$ — you
    (your choice for the template string)
  - By pumping lemma, we should be able to break w=xyz, such that: — you
    1) $y \neq \varepsilon$
    2) $|xy| \leq N$
    3) For all k≥0, the string xy^kz is also in L

Template string $w = 0^N1^N = 00 \ldots 011 \ldots 1$, with the run of 0s of length N and the run of 1s of length N

Proof…

- Because $|xy| \leq N$, xy should contain only 0s
  - (This and because $y \neq \varepsilon$, implies $y=0^+$)
- Therefore x can contain *at most* N-1 0s
- Also, all the N 1s must be inside z
- By (3), any string of the form $xy^kz \in L_{eq}$ for all $k \geq 0$

Case k=0: xz has at most N-1 0s but has N 1s

Therefore, $xy^0z \notin L_{eq}$

- This violates the P/L (a contradiction)

□ you

Setting k=0 is referred to as **"pumping down"**

Setting k>1 is referred to as **"pumping up"**

Another way of proving this will be to show that if the #0s is arbitrarily pumped up (e.g., k=2), then the #0s will become exceed the #1s

Exercise 2

Prove $L = \{0^n10^n \mid n \geq 1\}$ is not regular

<u>Note:</u> This n is not to be confused with the pumping lemma constant N. That *can* be different.

In other words, the above question is same as proving:

- $L = \{0^m10^m \mid m \geq 1\}$ is not regular

Example 3: Pumping Lemma

Claim: $L = \{ 0^i \mid i$ is a perfect square$\}$ is not regular

- Proof:
  - By contradiction, let L be regular.
  - P/L should apply
  - Let N = P/L constant
  - Choose $w=0^{N2}$
  - By pumping lemma, $w=xyz$ satisfying all three rules
  - By rules (1) & (2), y has between 1 and N 0s
  - By rule (3), any string of the form xy^kz is also in L for all $k\geq0$
  - Case k=0:
    - #zeros (xy^0z) = #zeros (xyz) - #zeros (y)
    - $N^2 - N \leq$ #zeros $(xy^0z) \leq N^2 - 1$
    - $\mathbf{(N-1)^2} < N^2 - N \leq$ #zeros $(xy^0z) \leq N^2 - 1 < \mathbf{N^2}$
    - $xy^0z \notin L$
    - But the above will complete the proof ONLY IF N>1.
    - … (proof contd.. Next slide)

Example 3: Pumping Lemma

- (proof contd…)
  - If the adversary pick N=1, then $\mathbf{(N-1)^2} \leq N^2 - N$, and therefore the #zeros(xy^0z) could end up being a perfect square!
  - This means that pumping down (i.e., setting k=0) is not giving us the proof!
  - So lets try pumping up next…
- Case k=2:
  - #zeros (xy^2z) = #zeros (xyz) + #zeros (y)
  - $N^2 + 1 \leq$ #zeros (xy^2z) $\leq N^2 + N$
  - $\mathbf{N^2} < N^2 + 1 \leq$ #zeros (xy^2z) $\leq N^2 + N < \mathbf{(N+1)^2}$
  - $xy^2z \notin L$ ↯
  - (Notice that the above should hold for all possible N values of N>0. Therefore, this completes the proof.)

Closure properties of Regular Languages

Closure properties for Regular Languages (RL)

This is different from Kleene closure

- *Closure property:*
  - If a set of regular languages are combined using an operator, then the resulting language is also regular
- Regular languages are *closed* under:
  - Union, intersection, complement, difference
  - Reversal
  - Kleene closure
  - Concatenation
  - Homomorphism
  - Inverse homomorphism

Now, lets prove all of this!

RLs are closed under union

- IF L and M are two RLs THEN:
  - they both have two corresponding regular expressions, R and S respectively
  - (L U M) can be represented using the regular expression R+S
  - Therefore, (L U M) is also regular

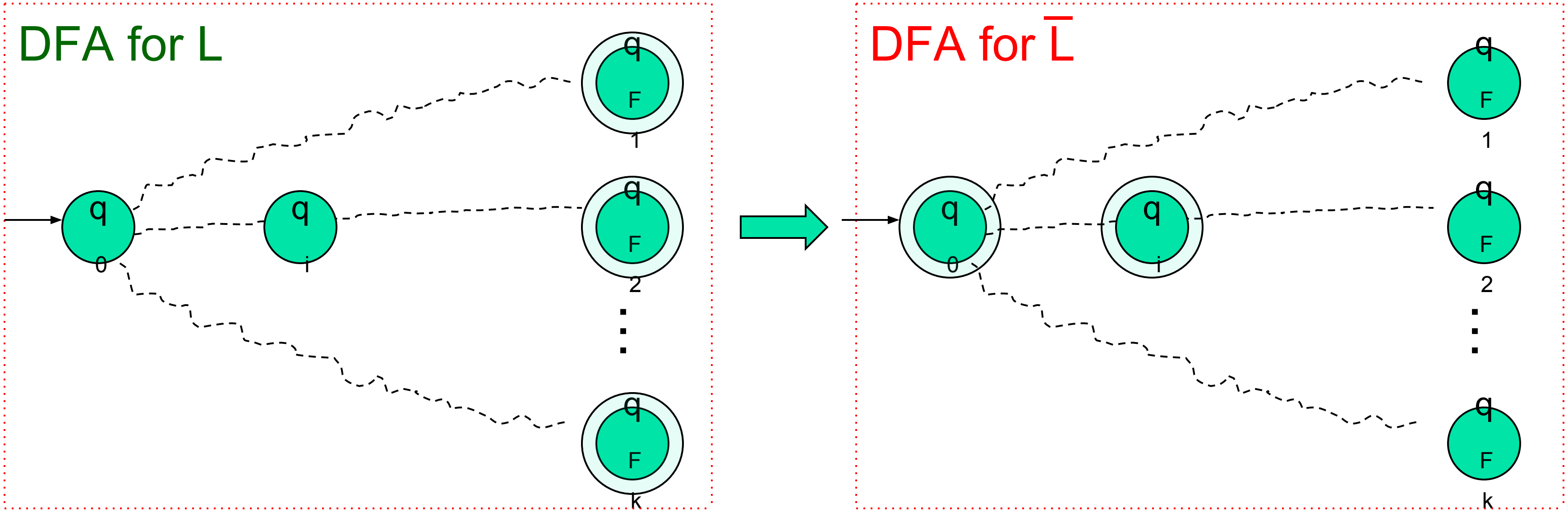
□

How can this be proved using FAs?

RLs are closed under complementation

- If L is an RL over ∑, then $\overline{L}$=∑*-L
- To show $\overline{L}$ is also regular, make the following construction

Convert every final state into non-final, and every non-final state into a final state



Assumes q0 is a non-final state. If not, do the opposite.

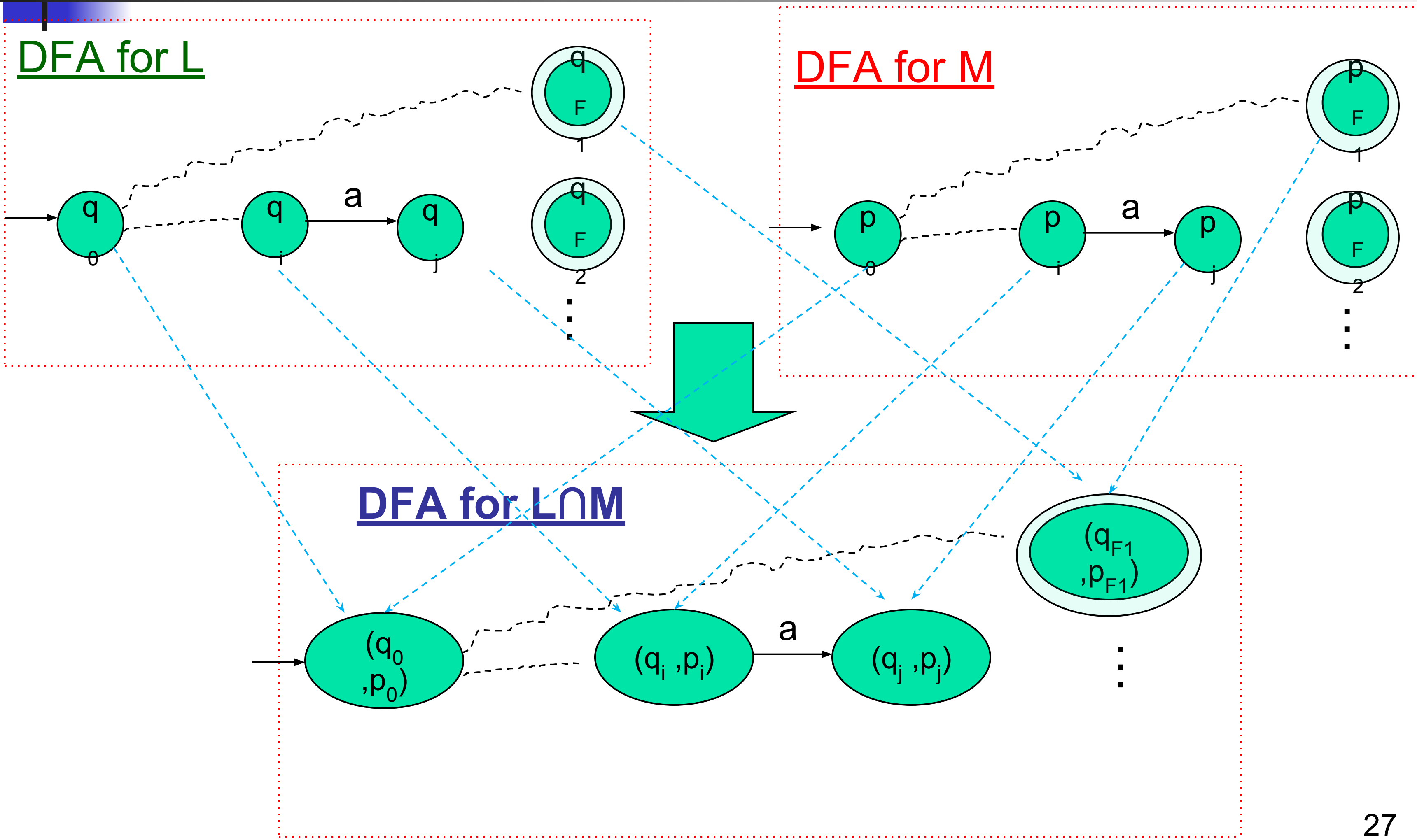
RLs are closed under intersection

- A quick, indirect way to prove:
  - By DeMorgan's law:
  - $L \cap M = (\overline{\overline{L} \cup \overline{M}})$
  - Since we know RLs are closed under union and complementation, they are also closed under intersection
- A more direct way would be construct a finite automaton for $L \cap M$

DFA construction for L ∩ M

- A_L = DFA for L = $\{Q_L, \Sigma, q_L, F_L, \delta_L\}$
- A_M = DFA for M = $\{Q_M, \Sigma, q_M, F_M, \delta_M\}$
- Build $A_{L \cap M} = \{Q_L \times Q_M, \Sigma, (q_L, q_M), F_L \times F_M, \delta\}$ such that:
  - $\delta((p,q),a) = (\delta_L(p,a), \delta_M(q,a))$, where p in Q_L, and q in Q_M
- This construction ensures that a string w will be accepted if and only if w reaches an accepting state in <u>both</u> input DFAs.

DFA construction for L ∩ M



RLs are closed under set difference

- We observe:
  - $L - M = L \cap \overline{M}$

Closed under intersection

Closed under complementation

- Therefore, L - M is also regular

RLs are closed under reversal

Reversal of a string w is denoted by w^R

- E.g., w=00111, w^R=11100

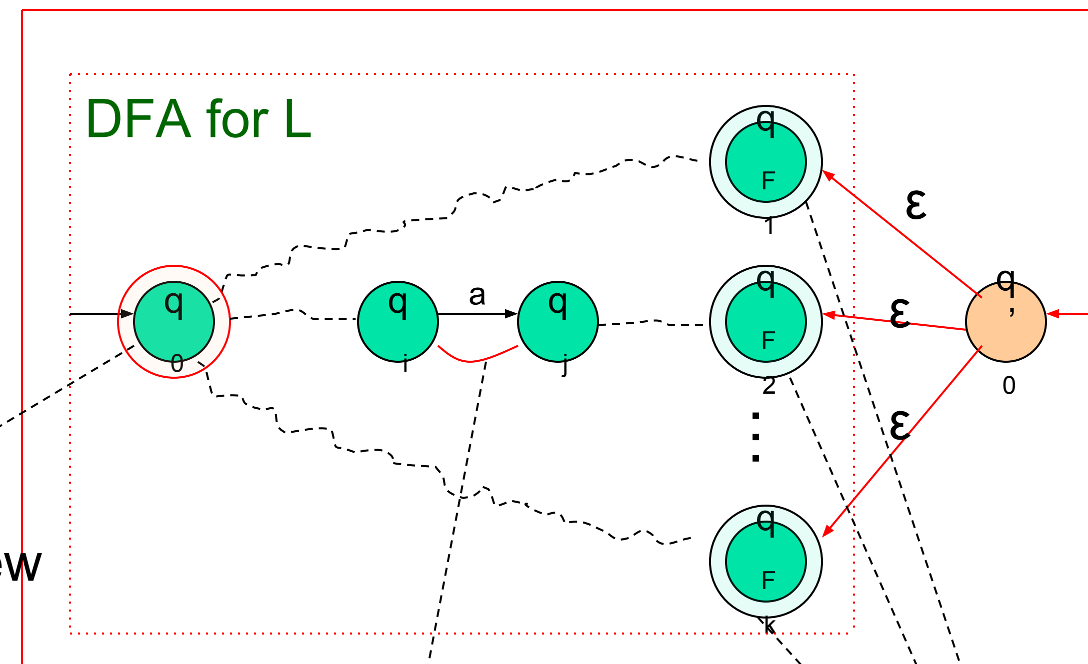
<u>Reversal of a language:</u>

- L^R = The language generated by reversing <u>all</u> strings in L

<u>Theorem:</u> If L is regular then L^R is also regular

ε -NFA Construction for L^R

New ε-NFA for L^R



New start state

Make the old start state as the only new final state

Reverse all transitions

Convert the old set of final states into <u>non-final</u> states

What to do if q_0 was one of the final states in the input DFA?

If L is regular, L^R is regular (proof using regular expressions)

- Let E be a regular expression for L
- Given E, how to build E^R?
- <u>Basis:</u> If E= ε, Ø, or a, then E^R=E
- <u>Induction:</u> Every part of E (refer to the part as “F”) can be in only *one* of the three following forms:

1. $F = F_1+F_2$
   - $F^R = F_1^R+F_2^R$
2. $F = F_1F_2$
   - $F^R = F_2^RF_1^R$
3. $F = (F_1)^*$
   - $(F^R)^* = (F_1^R)^*$

Homomorphisms

- Substitute each <u>symbol</u> in ∑ (main alphabet) by a corresponding <u>string</u> in T (another alphabet)
  - h: ∑--->T*
- <u>Example</u>:
  - Let ∑={0,1} and T={a,b}
  - Let a homomorphic function h on ∑ be:
    - h(0)=ab, h(1)=ε
  - If w=10110, then h(w) = εabεεab = abab
- In general,
  - $h(w) = h(a_1)\ h(a_2)\ldots\ h(a_n)$

RLs are closed under homomorphisms

- Theorem: If L is regular, then so is h(L)
- Proof: If E is a RE for L, then show L(h(E)) = h(L(E))
- Basis: If E= ε, Ø, or a, then the claim holds.
- Induction: There are three forms of E:

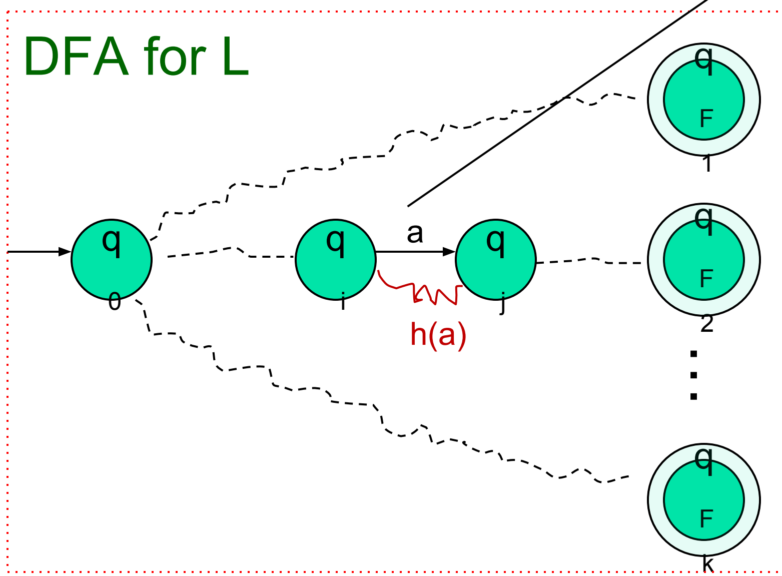
1. $E = E_1+E_2$
   - $L(h(E)) = L(h(E_1) + h(E_2)) = L(h(E_1)) \cup L(h(E_2))$ ----- (1)
   - $h(L(E)) = h(L(E_1) + L(E_2)) = h(L(E_1)) \cup h(L(E_2))$ ----- (2)
   - By inductive hypothesis, $L(h(E_1))= h(L(E_1))$ and $L(h(E_2))= h(L(E_2))$
   - Therefore, L(h(E)= h(L(E)
2. $E = E_1E_2$
3. $E = (E_1)^*$

Similar argument

Think of a DFA based construction

Given a DFA for L, how to convert it into an FA for h(L)?

FA Construction for h(L)



Replace every <u>edge</u> "a" by a <u>path</u> labeled h(a) in the new DFA

- Build a new FA that simulates h(a) for every symbol a transition in the above DFA
- The resulting FA may or may not be a DFA, but will be a FA for h(L)

Given a DFA for M, how to convert it into an FA for $h^{-1}(M)$?

The set of strings in $\sum^*$ whose homomorphic translation results in the strings of M

Inverse homomorphism

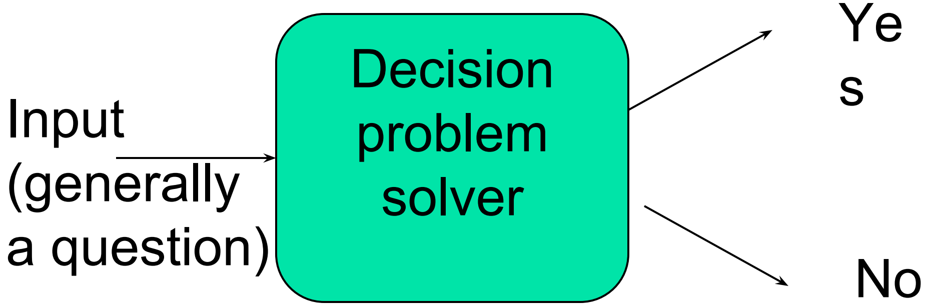
- Let h: $\sum$--->T*
- Let M be a language over alphabet T
- $h^{-1}(M) = \{w \mid w \in \sum^* \text{ s.t., } h(w) \in M\}$

Claim: If M is regular, then so is $h^{-1}(M)$

- Proof:
  - Let A be a DFA for M
  - Construct another DFA A' which encodes $h^{-1}(M)$
  - A' is an exact replica of A, except that its transition functions are s.t. for any input symbol *a* in $\sum$, A' will simulate *h(a)* in A.
    - $\delta(p,a) = \hat{\delta}(p,h(a))$

Decision properties of regular languages

<u>Any “decision problem” looks like this:</u>



Membership question

- <u>Decision Problem:</u> Given L, is w in L?
- <u>Possible answers:</u> Yes or No
- <u>Approach:</u>
  1. Build a DFA for L
  2. Input w to the DFA
  3. If the DFA ends in an accepting state, then yes; otherwise no.

Emptiness test

- Decision Problem: Is L=Ø ?
- Approach:

  On a DFA for L:

  1. From the start state, run a *reachability* test, which returns:
     1. success: if there is at least one final state that is reachable from the start state
     2. failure: otherwise
  2. L=Ø if and only if the reachability test fails

How to implement the reachability test?

Finiteness

- Decision Problem: Is L finite or infinite?
- Approach:

  On a DFA for L:

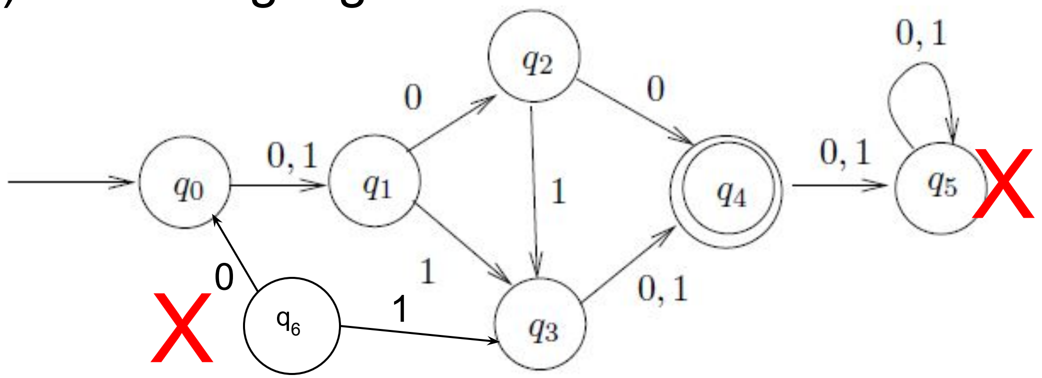
  1. Remove all states unreachable from the start state
  2. Remove all states that cannot lead to any accepting state.
  3. After removal, check for cycles in the resulting FA
  4. L is finite if there are no cycles; otherwise it is infinite

- Another approach
  - Build a regular expression and look for Kleene closure

How to implement steps 2 and 3?

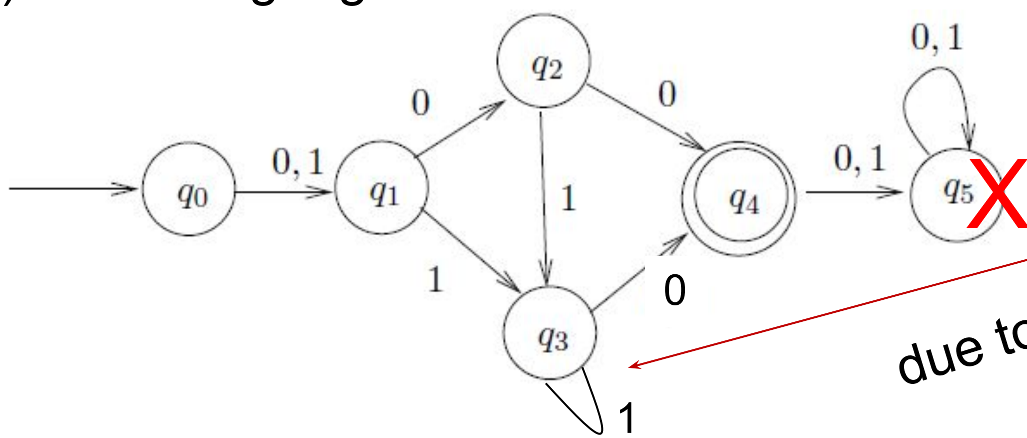
Finiteness test - examples

Ex 1) Is the language of this DFA finite or infinite?



FINITE

Ex 2) Is the language of this DFA finite or infinite?



INFINITE

due to this

Equivalence & Minimization of DFAs

Applications of interest

- Comparing two DFAs:
  - $L(DFA_1) == L(DFA_2)$?

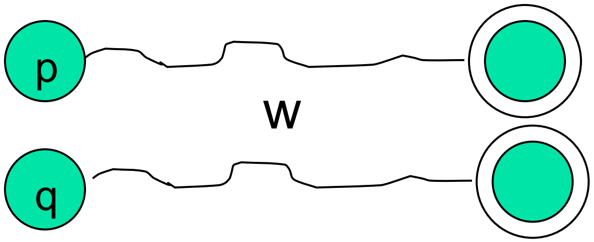
- How to minimize a DFA?
  1. Remove unreachable states
  2. Identify & condense equivalent states into one

When to call two states in a DFA "equivalent"?

Past doesn't matter - only future does!

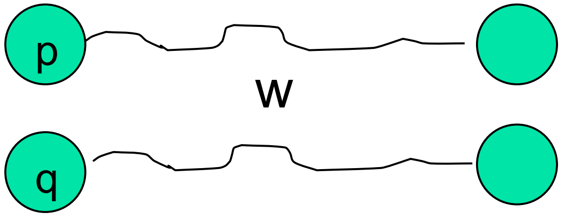
Two states p and q are said to be *equivalent* iff:

i) Any string w accepted by starting at p is also accepted by starting at q;



AND

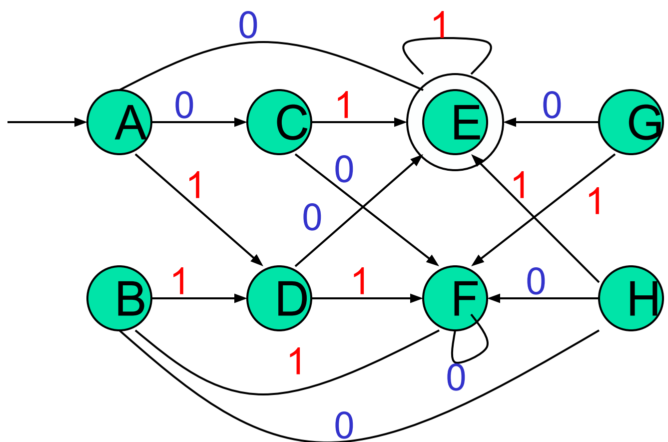
ii) Any string w rejected by starting at p is also rejected by starting at q.



□ $p \equiv q$

Computing equivalent states in a DFA

Table Filling Algorithm



A	=							
B	=	=						
C	x	x	=					
D	x	x	x	=				
E	x	x	x	x	=			
F	x	x	x	x	x	=		
G	x	x	x	=	x	x	=	
H	x	x	=	x	x	x	x	=
	A	B	C	D	E	F	G	H

Pass #0

1. Mark accepting states ≠ non-accepting states

Pass #1

2. Compare every pair of states
3. Distinguish by one symbol transition
4. Mark = or ≠ or blank(tbd)

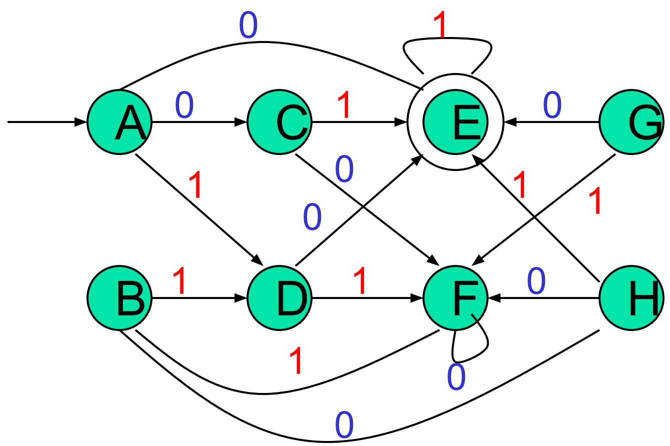
Pass #2

5. Compare every pair of states
6. Distinguish by up to two symbol transitions (until different or same or tbd)

….

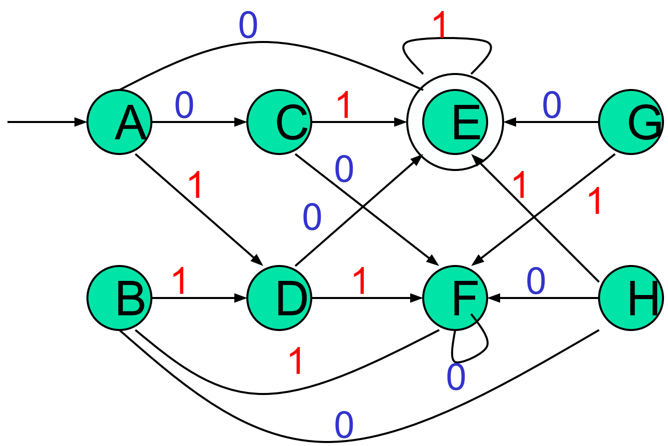
(keep repeating until table complete)

Table Filling Algorithm - step by step



A	=							
B		=						
C			=					
D				=				
E					=			
F						=		
G							=	
H								=
	A	B	C	D	E	F	G	H

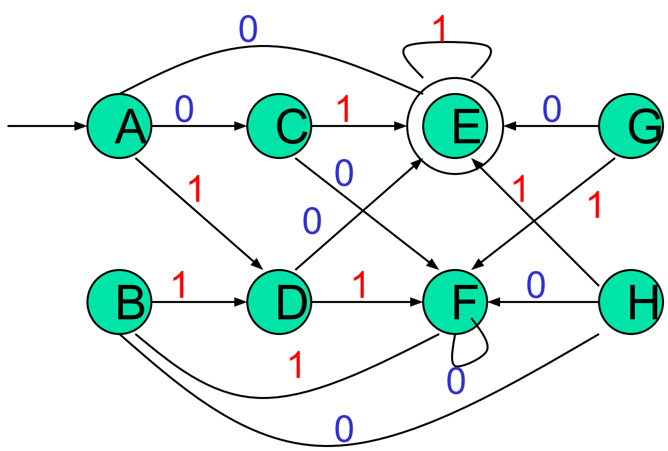
Table Filling Algorithm - step by step



	A	B	C	D	E	F	G	H
A	=							
B		=						
C			=					
D				=				
E	X	X	X	X	=			
F					X	=		
G					X		=	
H					X			=

1. Mark X between accepting vs. non-accepting state

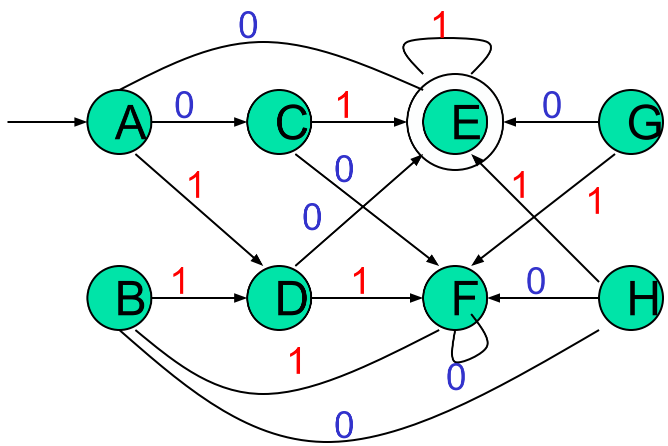
Table Filling Algorithm - step by step



	A	B	C	D	E	F	G	H
A	=							
B		=						
C	X		=					
D	X			=				
E	X	X	X	X	=			
F					X	=		
G	X				X		=	
H	X				X			=

1. Mark X between accepting vs. non-accepting state
2. Look 1- hop away for distinguishing states or strings

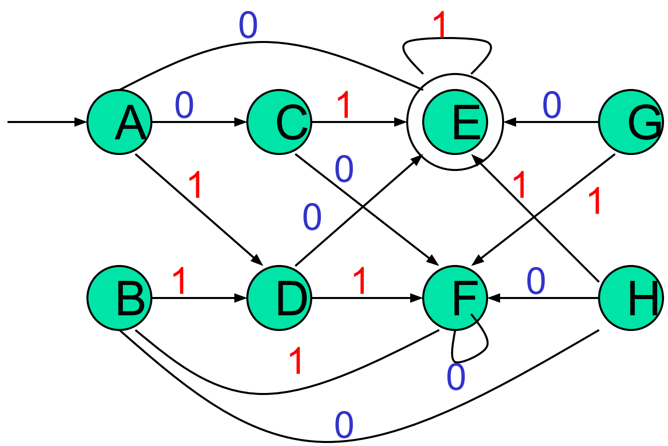
Table Filling Algorithm - step by step



	A	B	C	D	E	F	G	H
A	=							
B		=						
C	X	X	=					
D	X	X		=				
E	X	X	X	X	=			
F					X	=		
G	X	X			X		=	
H	X	X			X			=

1. Mark X between accepting vs. non-accepting state
2. Look 1- hop away for distinguishing states or strings

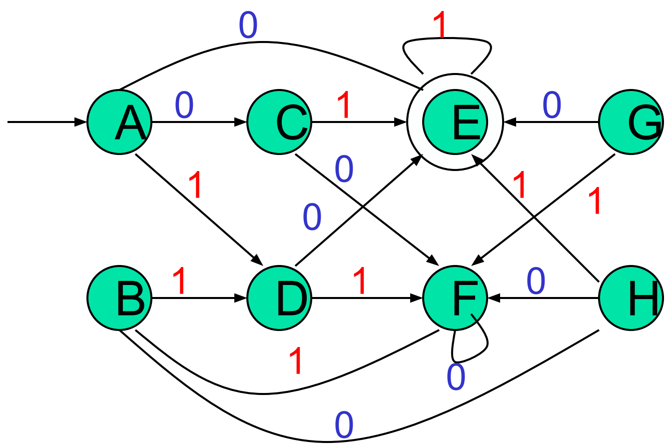
Table Filling Algorithm - step by step



A	=							
B		=						
C	X	X	=					
D	X	X	X	=				
E	X	X	X	X	=			
F			X		X	=		
G	X	X	X		X		=	
H	X	X	**=**		X			=
	A	B	C	D	E	F	G	H

1. Mark X between accepting vs. non-accepting state
2. Look 1- hop away for distinguishing states or strings

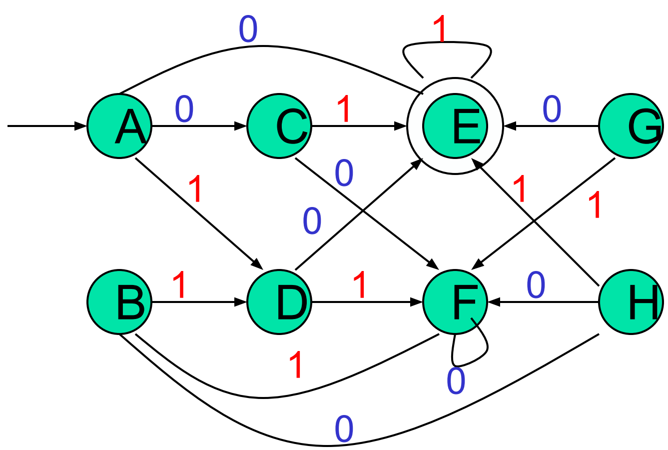
Table Filling Algorithm - step by step



	A	B	C	D	E	F	G	H
A	=							
B		=						
C	X	X	=					
D	X	X	X	=				
E	X	X	X	X	=			
F			X	X	X	=		
G	X	X	X	=	X		=	
H	X	X	=	X	X			=

1. Mark X between accepting vs. non-accepting state
2. Look 1- hop away for distinguishing states or strings

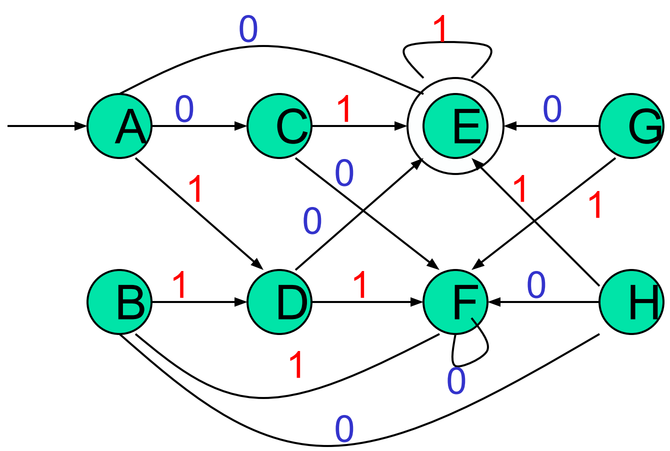
Table Filling Algorithm - step by step



	A	B	C	D	E	F	G	H
A	=							
B		=						
C	X	X	=					
D	X	X	X	=				
E	X	X	X	X	=			
F			X	X	X	=		
G	X	X	X	=	X	X	=	
H	X	X	=	X	X	X		=

1. Mark X between accepting vs. non-accepting state
2. Look 1- hop away for distinguishing states or strings

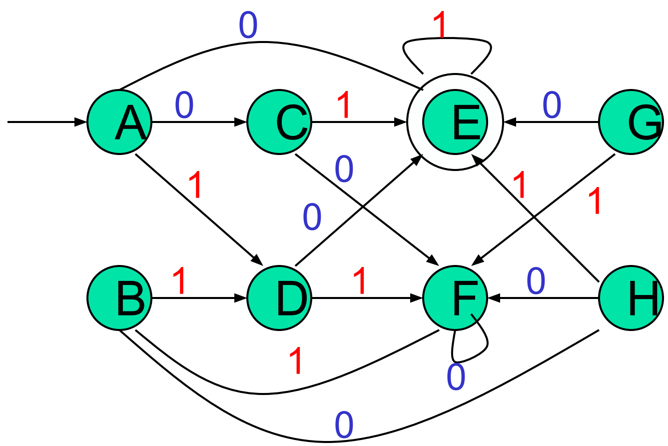
Table Filling Algorithm - step by step



A	=							
B		=						
C	X	X	=					
D	X	X	X	=				
E	X	X	X	X	=			
F			X	X	X	=		
G	X	X	X	=	X	X	=	
H	X	X	=	X	X	X	X	=
	A	B	C	D	E	F	G	H

1. Mark X between accepting vs. non-accepting state
2. Look 1- hop away for distinguishing states or strings

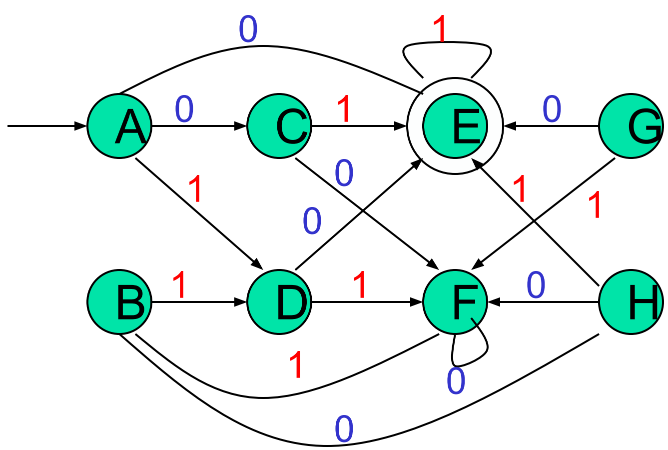
Table Filling Algorithm - step by step



A	=							
B	=	=						
C	X	X	=					
D	X	X	X	=				
E	X	X	X	X	=			
F	X	X	X	X	X	=		
G	X	X	X	=	X	X	=	
H	X	X	=	X	X	X	X	=
	A	B	C	D	E	F	G	H

1. Mark X between accepting vs. non-accepting state
2. Pass 1:
   Look 1- hop away for distinguishing states or strings
3. Pass 2:
   Look 1-hop away again for distinguishing states or strings
   continue….

Table Filling Algorithm - step by step

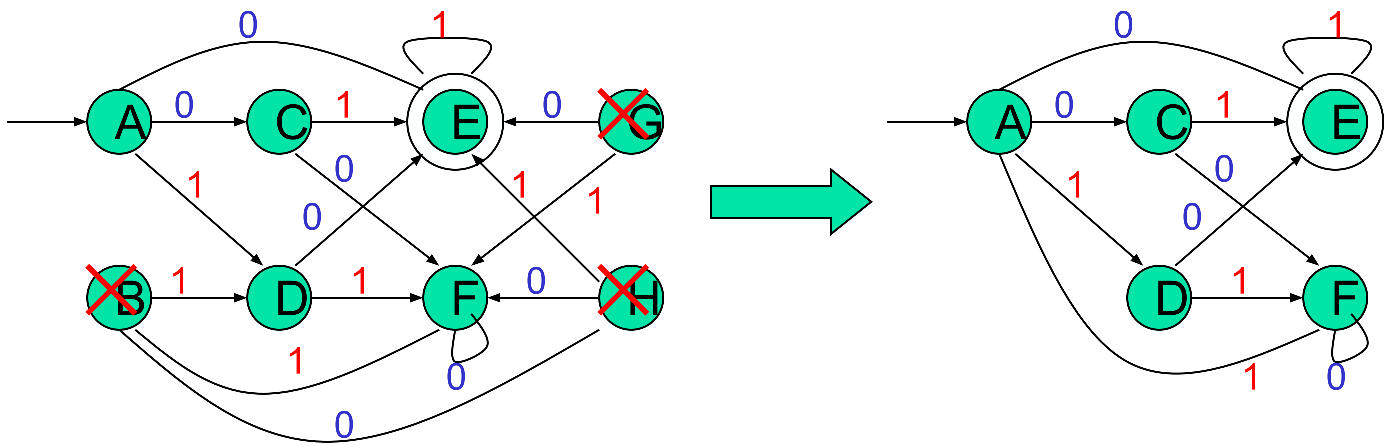


A	=							
B	=	=						
C	X	X	=					
D	X	X	X	=				
E	X	X	X	X	=			
F	X	X	X	X	X	=		
G	X	X	X	=	X	X	=	
H	X	X	=	X	X	X	X	=
	A	B	C	D	E	F	G	H

1. Mark X between accepting vs. non-accepting state
2. Pass 1:
   Look 1- hop away for distinguishing states or strings
3. Pass 2:
   Look 1-hop away again for distinguishing states or strings
   continue….

Equivalences:
- A=B
- C=H
- D=G

Table Filling Algorithm - step by step

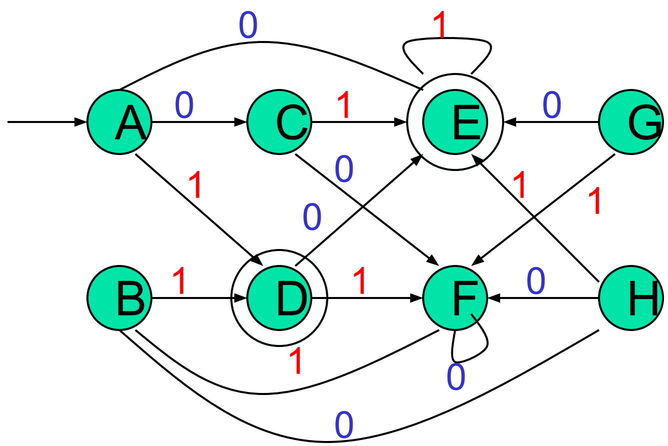


Retrain only one copy for
each equivalence set of states

Equivalences:

- A=B
- C=H
- D=G

Table Filling Algorithm – special case



A	=							
B		=						
C			=					
D				=				
E				?	=			
F						=		
G							=	
H								=
	A	B	C	D	E	F	G	H

Q) What happens if the input DFA has more than one final state?
Can all final states initially be treated as equivalent to one another?

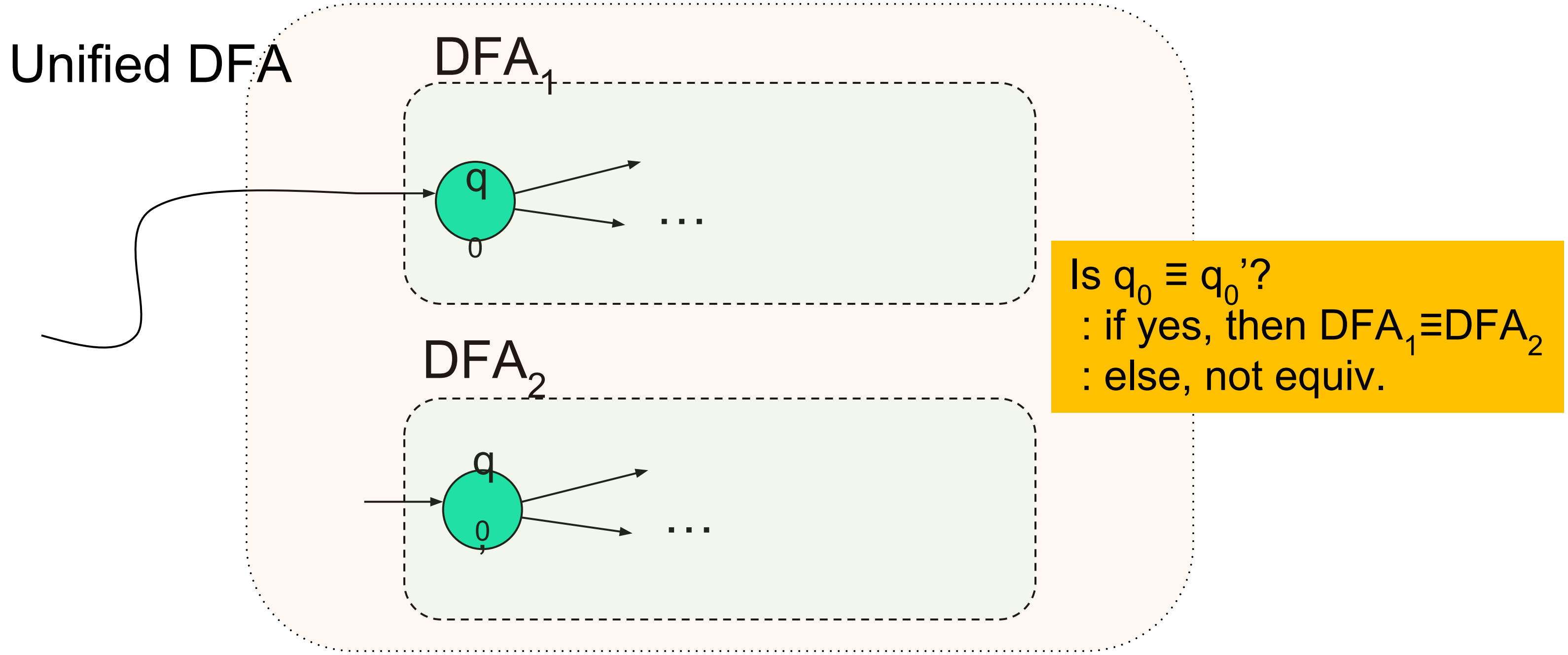
How to minimize a DFA?

- Goal: Minimize the number of states in a DFA
- Algorithm:
  1. Eliminate states unreachable from the start state
  2. Identify and remove equivalent states
  3. Output the resultant DFA

Depth-first traversal from the start state

Table filling algorithm

Are Two DFAs Equivalent?



1. Make a new dummy DFA by just putting together both DFAs
2. Run table-filling algorithm on the unified DFA
3. *IF* the start states of both DFAs are found to be equivalent,

   THEN: $DFA_1 \equiv DFA_2$

   ELSE: different

Summary

- How to prove languages are not regular?
  - Pumping lemma & its applications

- Closure properties of regular languages

- Simplification of DFAs
  - How to remove unreachable states?
  - How to identify and collapse equivalent states?
  - How to minimize a DFA?
  - How to tell whether two DFAs are equivalent?