1 Describing the model

1. (a) **(2 marks)**

$$\boldsymbol{Y}_3 = \begin{bmatrix} 30 \\ 65 \\ 29 \\ 152 \end{bmatrix}.$$

(b) **(7 marks)**

$$\boldsymbol{X}_3 = \begin{bmatrix} 1 & 2 & -0.72 & 8.2 & -1.44 & 16.4 & -5.904 \\ 1 & 2 & -0.72 & 8.2 & -1.44 & 16.4 & -5.904 \\ 1 & 2 & -0.72 & 8.2 & -1.44 & 16.4 & -5.904 \\ 1 & 2 & -0.72 & 8.2 & -1.44 & 16.4 & -5.904 \end{bmatrix}.$$

(c) **(1 mark)**

$$\boldsymbol{\beta} = \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \\ \beta_5 \\ \beta_6 \end{bmatrix}.$$

(d) **(3 marks)**

$$\boldsymbol{Z}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

(e) **(2 marks)**

$$\boldsymbol{\mu}_3 = \begin{bmatrix} \mu_{03} \end{bmatrix}.$$

(f) **(2 marks)**

$$\boldsymbol{\varepsilon}_3 = \begin{bmatrix} \varepsilon_{13} \\ \varepsilon_{23} \\ \varepsilon_{33} \\ \varepsilon_{43} \end{bmatrix}.$$

2. β_0 is the mean gain in mathematics score for children whose first-grade classroom teacher has 0 years of experience with a score of 0 in mathematical knowledge and where there are 0% of households that are below the poverty level in the neighbourhood of the children's classroom. **(3 marks)**

3. • β_1 is the linear effect of the teacher's years of experience on the mean gain in mathematics score, for children whose first-grade classroom teacher has a score of 0 in mathematical knowledge and where there are 0% of households that are below the poverty level in the neighbourhood of the children's classroom. **(3 marks)**

   • β_2 is the linear effect of the teacher's mathematical knowledge score on the mean gain in mathematics score, for children whose first-grade classroom teacher has 0 years of experience and where there are 0% of households that are below the poverty level in the neighbourhood of the children's classroom.

(3 marks)

- β_3 is the linear effect of HP on the mean gain in mathematics score, for children whose first-grade classroom teacher has 0 years of experience and has a score of 0 in mathematical knowledge. **(3 marks)**

4. - β_4 is the change in the linear effect of the teacher's years of experience on the mean gain in mathematics score, when the mathematical knowledge of the children's first-grade teacher increases by 1 unit, and where the % of households that are below the poverty level in the neighbourhood of the children's classroom is held fixed. or β_4 is the change in the linear effect of the teacher's mathematical knowledge score on the mean gain in mathematics score, when the children's first-grade teacher's years of experience increases by 1 year, and where the % of households that are below the poverty level in the neighbourhood of the children's classroom is held fixed. **(3 marks)**

   - β_5 is the change in the linear effect of the teacher's years of experience on the mean gain in mathematics score, when HP increases by 1%, and where the mathematical knowledge of the teacher is held fixed. or β_5 is the change in the linear effect of HP on the mean gain in mathematics score, when the children's first-grade teacher's years of experience increases by 1 year, and where the mathematical knowledge of the teacher is held fixed. **(3 marks)**

   - β_6 is the change in the linear effect of the teacher's mathematical knowledge score on the mean gain in mathematics score, when HP increases by 1%, and where the years of experience of the teacher is held fixed. or β_6 is the change in the linear effect of HP on the mean gain in mathematics score, when the mathematical knowledge of the children's first-grade teacher increases by 1 unit, and where the years of experience of the teacher is held fixed. **(3 marks)**

5. **(6 marks)**

$$
\begin{aligned}
Var(\mathbf{Y}_3) &= \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} \theta \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} + \begin{bmatrix} \tau & 0 & 0 & 0 \\ 0 & \tau & 0 & 0 \\ 0 & 0 & \tau & 0 \\ 0 & 0 & 0 & \tau \end{bmatrix} \\
&= \begin{bmatrix} \theta \\ \theta \\ \theta \\ \theta \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} + \begin{bmatrix} \tau & 0 & 0 & 0 \\ 0 & \tau & 0 & 0 \\ 0 & 0 & \tau & 0 \\ 0 & 0 & 0 & \tau \end{bmatrix} \\
&= \begin{bmatrix} \theta & \theta & \theta & \theta \\ \theta & \theta & \theta & \theta \\ \theta & \theta & \theta & \theta \\ \theta & \theta & \theta & \theta \end{bmatrix} + \begin{bmatrix} \tau & 0 & 0 & 0 \\ 0 & \tau & 0 & 0 \\ 0 & 0 & \tau & 0 \\ 0 & 0 & 0 & \tau \end{bmatrix} \\
&= \begin{bmatrix} \theta+\tau & \theta & \theta & \theta \\ \theta & \theta+\tau & \theta & \theta \\ \theta & \theta & \theta+\tau & \theta \\ \theta & \theta & \theta & \theta+\tau \end{bmatrix}
\end{aligned}
$$

Compound symmetry **(1 mark)**

2 Testing for the random intercept

6. (a) **(1 mark)**

$$\begin{aligned} MG_{ij} = {} & \beta_0 + \beta_1 YE_j + \beta_2 MK_j + \beta_3 HP_j \\ & + \beta_4 YE_j \times MK_j + \beta_5 YE_j \times HP_j + \beta_6 MK_j \times HP_j \\ & + \mu_{0j} + \varepsilon_{ij}, \end{aligned} \tag{1}$$

   (b) **(1 mark)**

$$\begin{aligned} MG_{ij} = {} & \beta_0 + \beta_1 YE_j + \beta_2 MK_j + \beta_3 HP_j \\ & + \beta_4 YE_j \times MK_j + \beta_5 YE_j \times HP_j + \beta_6 MK_j \times HP_j \\ & + \varepsilon_{ij}, \end{aligned} \tag{2}$$

   (c) p-value $= 6.880054 \times 10^{-10}$ **(1 mark)**

   (d) The reference model since $6.880054 \times 10^{-10} < 0.05$. **(2 marks)**

3 Variance-covariance estimates of the final linear mixed model

7. 205.7458 **(3 marks)**

8. **(3 marks)**

Figure 1: R Matrix

```
> R_hat
         1        2        3        4
1 1014.183    0.000    0.000    0.000
2    0.000 1014.183    0.000    0.000
3    0.000    0.000 1014.183    0.000
4    0.000    0.000    0.000 1014.183
```

9. **(3 marks)**

Figure 2: Variance-covariance matrix of the response vector

```
> Var_Y
```

	1	2	3	4
1	1219.9284	205.7458	205.7458	205.7458
2	205.7458	1219.9284	205.7458	205.7458
3	205.7458	205.7458	1219.9284	205.7458
4	205.7458	205.7458	205.7458	1219.9284

10. **(3 marks)**

$$ICC = \frac{\hat{\theta}}{\hat{\theta} + \hat{\tau}} = \frac{205.7458}{205.7458 + 1014.183} = 0.1687$$

Within a classroom, the correlation between any two measurements of gains in mathematics score is 0.1687. or The proportion of total variation in gain in mathematics score that is due to the variability of the random intercepts of the classrooms is 0.1687. **(2 marks)**

4 Predicted values and residuals of the final linear mixed model

11. **(2 marks)**

Figure 3: Fixed Effects

	Value	Std.Error	DF	t-value	p-value
(Intercept)	54.8655	3.6243	796	15.1382	0.0000
YE	0.2219	0.2328	278	0.9532	0.3413
MK	-1.2255	3.2331	278	-0.3790	0.7049
HP	0.0586	0.1496	278	0.3916	0.6956
YE:MK	0.1379	0.1368	278	1.0081	0.3143
YE:HP	-0.0038	0.0106	278	-0.3613	0.7181
MK:HP	0.0438	0.1100	278	0.3981	0.6909

12. **(6 marks)**

$$\begin{aligned}\widetilde{E(Time)} &= 54.8655 + (0.2219 \times 19) + (-1.2255 \times 1.61) + (0.0586 \times 8.6) + (0.1379 \times 19 \times 1.61) \\ &\quad + (-0.0038 \times 19 \times 8.6) + (0.0438 \times 1.61 \times 8.6) - 8.9031 \\ &= 54.8655 + 4.2161 - 1.973055 + 0.50396 + 4.218361 - 0.62092 + 0.6064548 - 8.9031 \\ &= 52.913301\end{aligned}$$

13. **(6 marks)**

$$
\begin{aligned}
\tilde{\varepsilon} &= 66 - [54.8655 + (0.2219 \times 2) + (-1.2255 \times -0.11) + (0.0586 \times 8.2) \\
&\quad + (0.1379 \times 2 \times -0.11) + (-0.0038 \times 2 \times 8.2) + (0.0438 \times -0.11 \times 8.2) - 4.4535] \\
&= 66 - [54.8655 + 0.4438 + 0.134805 + 0.48052 - 0.030338 - 0.06232 - 0.0395076 - 4.4535] \\
&= 66 - 51.338959 \\
&= 14.661041
\end{aligned}
$$

14. **(6 marks)**

$$
\begin{aligned}
\hat{\varepsilon} &= 28 - [54.8655 + (0.2219 \times 13) + (-1.2255 \times -0.79) + (0.0586 \times 14.8) + (0.1379 \times 13 \times -0.79) \\
&\quad + (-0.0038 \times 13 \times 14.8) + (0.0438 \times -0.79 \times 14.8)] \\
&= 28 - [54.8655 + 2.8847 + 0.968145 + 0.86728 - 1.416233 - 0.73112 - 0.5121096] \\
&= -28.926162
\end{aligned}
$$

15. **(6 marks)**

$$
\begin{aligned}
\widehat{E(Time)} &= 54.8655 + (0.2219 \times 14) + (-1.2255 \times 1.4) + (0.0586 \times 20) + (0.1379 \times 14 \times 1.4) \\
&\quad + (-0.0038 \times 14 \times 20) + (0.0438 \times 1.4 \times 20) \\
&= 54.8655 + 3.1066 - 1.7157 + 1.172 + 2.70284 - 1.064 + 1.1984 \\
&= 60.26564
\end{aligned}
$$