Vectors - A more in-depth understanding

Note: This short write-up assumes a basic knowledge of vectors and some set theory. It is intended to introduce some of the more abstract maths underpinning vectors.

Section 1: What is a vector, really?

Formally, a vector is any element (member) of a **vector space** V. A **vector space** is just a collection (set) of objects that follow a specific set of rules. The vectors that we are most familiar with are the vectors of the vector spaces $\mathbb{R}^n$, whose elements are n-tuples over the real numbers $\mathbb{R}$.

$$(a_1, a_2, ..., a_n) \text{ where } a_1, a_2, ..., a_n \in \mathbb{R}$$

For example, the 2-tuple (ordered pair) $(2, 2)$ is an element of the vector space $\mathbb{R}^2$.

It is important to note that all sorts of things form vector spaces. For example, the set of all $n \times n$ matrices form their own vector space. So, in the context of that vector space, matrices are also vectors as well.

Vector spaces are defined *over a field* F. A field is a set of numbers that follow their own specific set of rules. The most common field used with vectors is that of the real numbers $\mathbb{R}$.

Vector spaces define a **binary operation** and a **binary function**. A binary operation is an operation that takes two elements from a set and returns another element from that set. A binary function is just a function that takes in two inputs and returns a single output.

The binary operation for vector spaces is *vector addition*, which assigns any two vectors $\mathbf{v}$ and $\mathbf{w}$ to a third vector in V written as $\mathbf{v} + \mathbf{w}$. For example, consider the ordered pairs $(2, 2), (3, 4) \in \mathbb{R}^2$. The binary operation of vector addition in $\mathbb{R}^2$ assigns $(2, 2)$ and $(3, 4)$ to the vector $(5, 6)$ (another vector in $\mathbb{R}^2$).

The binary function of a vector space is called **scalar multiplication**. It takes some scalar in the field the vector space is defined over $k \in F$ and an element of the vector space $\mathbf{v} \in V$ and returns another vector in V written as $k\mathbf{v}$. For example, consider the vector $(2, 2) \in \mathbb{R}^2$ and the scalar $3 \in \mathbb{R}$. Then, the binary function of scalar multiplication takes in the inputs of $(2, 2)$ and 3 and returns the vector $(6, 6) \in \mathbb{R}^2$.

It is important to note that our understanding of vectors as "representing a direction and magnitude" is only an **interpretation**. Vectors do **NOT** inherently encode this information. Instead, we have chosen to interpret $(3, 2) \in \mathbb{R}^2$ as some point on the Cartesian plane, and we have chosen for it to represent some displacement from an origin O. In other contexts, they can mean other things.

One example you are probably familiar with is RGB values. An RGB value like $(255, 0, 0)$ is a 3-tuple, and hence would technically be an element of the vector space $\mathbb{R}^3$. However, we don't think of RGB values in the same way we normally think about vectors representing a magnitude and direction. **Context matters**.