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Abstract: *New concise notations identify the 226 fully supported (non-reentrant) stellations of a familiar thirty-faced icosahedral solid, the rhombic triacontahedron (RTC). Systematic methods of labeling are described beginning with elements of the stellation diagram and ending with names of RTC stellations and their exposed facets. Two-dimensional properties of the stellation diagram are correlated with the three-dimensional properties of stellations. Besides identifying the names and facial patterns of some interesting RTC stellations for the model builder, this paper introduces a general approach to practical nomenclature which can be readily applied to the largely unexplored stellations of other convex isohedra, for example, the duals of the Archimedean polyhedra.*

This article originally appeared in the last issue of *Structural Topology* (1995), 21, 25-46, University of Quebec, Montreal, before that unique journal ceased publication. This updated (with very few changes and corrections) version is followed by a new supplement which contains new images, a complete catalog of the fully supported stellations of RTC, and a modified method for naming chiral stellations which is better suited for computerizing the stellation process.

THE PROCESS OF STELLATION AND ENUMERATION

The *stellation process* is a transformation that extends the faces of a polyhedral core until they meet and thus additional volumes (*cells*) are enclosed which preserve the rotational symmetries of the core. The process continues outwardly from the core until facial planes no longer intersect. Sufficiently far from the core, therefore, cells are infinite and the corresponding unbounded stellations are usually not considered. Embedded in a given facial plane of the core is the *stellation diagram*, a collection of lines which represent the intersections with the other facial planes. If the core is an *isohedron*, a polyhedron having congruent faces, then one stellation diagram is sufficient to describe the entire stellation process.

Stellation theory and its systematic application to various uniform polyhedra and their isohedral duals are discussed in the works listed in the references. The simplest isohedra produce several well-known stellated forms. The regular octahedron, for example, generates a single stellation known a the stella octangula or a uniform compound of two tetrahedra. The regular dodecahedron generates sequentially three non-convex regular polyhedra: the small stellated dodecahedron, the great dodecahedron, and the great stellated dodecahedron. The regular tetrahedron and cube are unable to form finite stellations.

The icosahedron and more complex cores can generate stellations that are so numerous that investigators have devised restrictive rules that conveniently reduce the stellations to more manageable subsets. Least restrictive are the rules of J. C. P. Miller (Coxeter et al. 1938) which still allow, among other features, so-called "reentrant conditions", i.e., solids that expose negative regions of facial planes. An insect crawling on the surface of such a stellation can find itself on the opposite side or same side of the plane as the solid's centre. The former surface is "positive" while the latter surface is "negative".

Accepting Miller's rules, Coxeter at al. (1938) described the complete set of 59 icosahedra, i.e., the icosahedron core and its 58 stellations. The core itself is not considered to be a stellation in this article. The analogous set of triacontahedra using the same rules is a formidable collection of 358,833,098 members which includes the RTC core. This computer-assisted enumeration was first attempted by John A. Gingrich (Toronto, Canada, 1989), corroborated by Paul A. Gingrich, but later altered slightly by Robert Webb (Thornbury, Australia, 2002) who fully realized the subtleties of Miller's rules. Robert Webb used his general stellating program *Great Stella* which is available at http://www.software3d.com. Preferring further restrictive rules for model building purposes however, J. A. Gingrich requires cells to interconnect by their faces into a single continuous solid. Accordingly, the Gingrich team counted 155,014,690 of such rigid triacontahedra, almost all of which are still reentrant.

If one intends to describe in detail each stellation of a defined set that is not too large, it is convenient, at least for the RTC, to consider only non-reentrant cases. Because cells consist of top (outer) facets which are positive and bottom (inner) faces which are negative, a simple rule for non-reentry is: no cell shall have any of its bottom surface uncovered. This implies that each component cell of a stellation must be *fully supported* by facet-to-facet connections with certain other component cells that belong to lower layers. The analogy is much like a stack of balls in which the top ball depends on the support of all balls below it. Rather than the term "non-reentrant", the more positive designation *fully supported* is preferred for these maximally connected forms.

As it turns out, there is indeed a manageable set of 226 fully supported stellations of the RTC of which 114 have the full icosahedral symmetry of the RTC core and each of the remaining 112 is a member of an enantiomorphic (chiral) pair of stellations. Although right-handed and left-handed members of a chiral pair have all the rotational symmetries, they are not "reflexible", i.e., they lack a symmetry plane ("mirror"). Pawley (1975) first catalogued the fully supported, fully symmetric (reflexible) RTC stellations - but without accompanying facial descriptions. Included in this list were the number of chiral pairs that can be derived from a fully symmetric "parent" stellation. It is often possible that a reflexible parent can be reduced to a chiral state by simply removing one or more kinds of chiral cells while preserving both rotational symmetry and full support. Pawley's combined total of exactly 226 fully supported stellations has been verified independently by the author who used inspection methods and by P. A. Gingrich who used a computer approach.

A notable subset of fully supported cases are the *main-line stellations*. Ede (1958) described all 12 of them for the RTC which are also identified in Table 3. The term *main-line* refers to the outward sequence of stellations obtained by adding the next complete set of outer cells.

Each such set of cells constitutes a different *layer* of which only the top surface is exposed. In the case of the RTC, each of the last three layers has some break in continuity. Such a noncontinuous layer does not completely encase the previous layer beneath it. Layers and the corresponding main-line stellations are numbered consecutively outwardly. The outermost layer combined with any exposed portions of lower layers, is commonly named the *final stellation*. For the RTC, the 12th layer determines this final form (Figure 3). Other interesting RTC main-line stellations are the 11th (penultimate form) and the 4th (uniform compound of 5 cubes) shown in Figure 4 and 5, respectively.

A	144°
8*d*	3.804
7*c*	1.176
3*b*	1.902
1*a*	1.176
0	1.902
3*c*	1.176
5*c*	1.902
7*d*	1.902
9*e*	3.078

B	B	108°
10*e*	10*f*	1.902
9*d*	9*f*	3.078
7*b*	7*e*	0.727
6*b*	6*d*	1.176
5*b*	5*d*	1.176
3*a*	3*c*	0.727
2*a*	2*b*	1.176

C	H	90°
12*c*	12*a*	8.472
10*g*	10*b*	3.236
9*f*	9*b*	2.000
7*d*	7*a*	2.000
6*c*	6*b*	1.236
4*b*	4*c*	2.000
4*a*	4*d*	1.236

D	120°
10*b*	2.803
8*a*	4.535
5*a*	1.732
3*a*	1.070
1*a*	1.732
2*b*	1.070
3*b*	1.732
7*b*	1.732
9*c*	2.803
11*c*	7.337

E	72°
11*d*	4.980
9*c*	1.902
7*a*	3.078
6*a*	1.902
5*b*	1.176
5*c*	1.902
7*e*	1.176
9*g*	1.902
10*g*	4.980

F	F	36°
12*b*	12*c*	4.980
11*c*	11*f*	4.980
11*b*	11*e*	1.902
10*a*	10*f*	1.176
9*a*	9*e*	1.902
9*b*	9*d*	1.176
10*c*	10*d*	1.902

G	G	60°
12*a*	12*b*	7.337
11*a*	11*d*	2.803
10*a*	10*e*	1.732
8*a*	8*d*	2.803
8*b*	8*e*	1.732
8*c*	8*f*	1.070
7*c*	7*f*	0.662
6*c*	6*d*	1.070

Table 1: Table of line connections - constant in-radius k from origin to center of isohedral face is distance $\tau^2 \approx 2.618$.

Each such set of cells constitutes a different *layer* of which only the top surface is exposed. In the case of the RTC, each of the last three layers has some break in continuity. Such a noncontinuous layer does not completely encase the previous layer beneath it. Layers and the corresponding main-line stellations are numbered consecutively outwardly. The outermost layer combined with any exposed portions of lower layers, is commonly named the *final stellation*. For the RTC, the 12th layer determines this final form (Figure 3). Other interesting RTC main-line stellations are the 11th (penultimate form) and the 4th (uniform compound of 5 cubes) shown in Figure 4 and 5, respectively.

ELEMENTS OF THE STELLATION DIAGRAM

Messer and Wenninger (1989) introduced tentative methods for labeling elements of the stellations diagram having one line of bilateral symmetry. In that discussion the term *stellation pattern* was used, but as Hudson and Kingston (1988) prefer, that designation is best reserved for the collection of exposed facets, embedded in the stellation diagram, which identify a particular stellation.

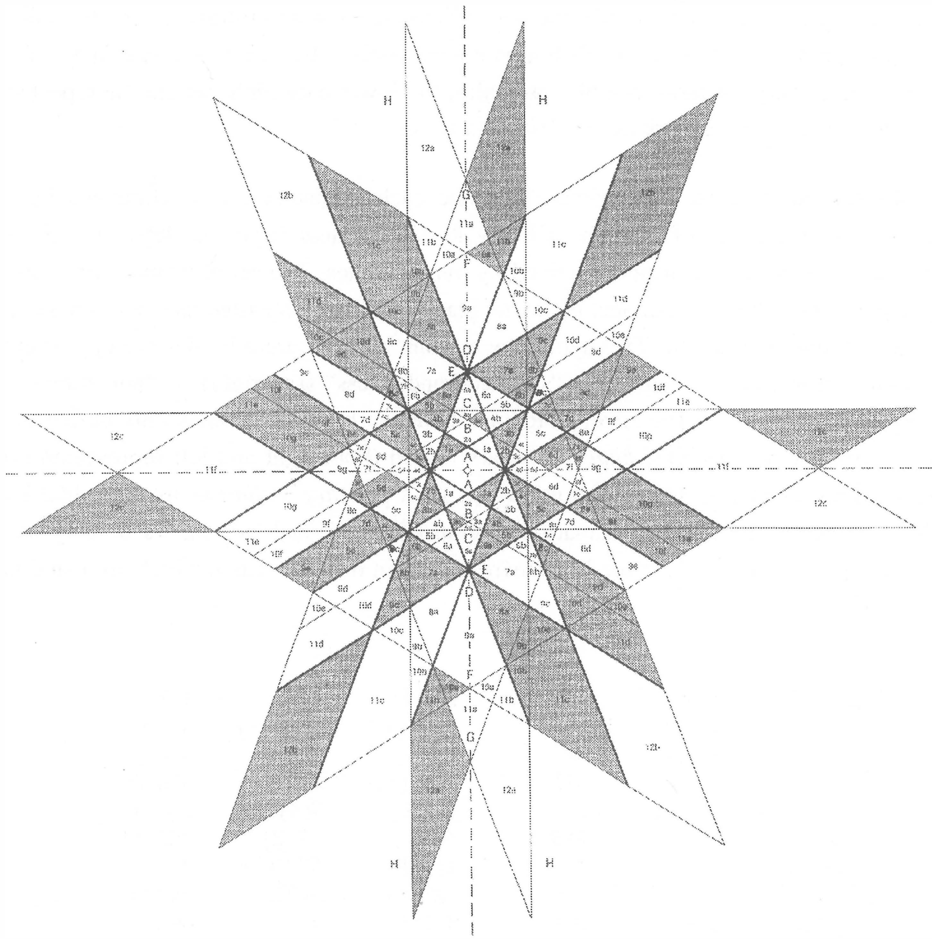

Figure 1: Stellation diagram for the rhombic triacontahedron

Details of the stellation diagram for the RTC are shown in Figure 1. Using capital letters, lines are labelled alphabetically along a vertical symmetry line in an outward and bilateral direction from the core region. The bilateral arrangement of lines, both vertically and horizontally, is clearly consistent with the two-fold rotational symmetry of the RTC face. Note that there are three kinds of lines: *primary line* (heavy), *secondary line* (light), and *symmetry line* (broken). Only the last line does not represent an actual intersection of facial planes and, therefore, it does not contribute to the dissection of the plane into finite *elementary regions*. Each elementary region corresponds to a top facet of one cell kind and a bottom facet of another cell kind (except of course if the region is already outermost). Elementary regions are labeled according to a layer number prefix and a lower case letter which follows a circular order in the diagram. This pattern of labeling is conveniently reflected by the two symmetry lines. If necessary for more complex isohedra, the lower case alphabet can be repeated by using a single prime mark as in: *a'*, *b'*, and so on.

Unlike secondary lines, each primary line and each symmetry line is contained by a different mirror plane of the core. Clearly the handedness (right vs. left) of regions switches only at line segments contained by mirror planes. The set of primary lines and symmetry lines, therefore, dissect the facial plane into *symmetry regions* - both bounded (triangles) and unbounded. Two symmetry regions which share a border are oppositely handed - shown in the diagram as "stippled-handedness" (say right) or "non-stippled-handedness" (say left). The smallest regions bounded by primary lines only define the set of *primary regions*. One *kind* of primary region (both right and left representation) corresponds exactly to the stellation pattern of a *primary stellation* as described by Messer (1989). Therefore, a primary stellation (Figure 7 and 8) consists of only one kind of surface facet and for the RTC there are seven different cases which are listed in Table 3.

A	*GN*	*LQ*(1)	*PQ*(4)	*RW*	*UY*
B	*GQ*(1)	*LU*	*PQR*(4)	*RWX*(1)	*V*(5)
C	*GU*	*M*	*PR*(1)	*RX*(3)	*VW*(1)
CD	*H*	*MN*	*PRU*(1)	*R2A*	*VWX*(4)
D	*HI*	*MO*	*PU*(1)	*S*(4)	*VX*(6)
DE	*I*	*MR*	*Q*(1)	*ST*(2)	*V2A*(1)
DG	*IJ*	*N*	*QR*(1)	*STU*(1)	*W*
E	*J*	*NP*(1)	*QRT*(1)	*SU*(2)	*WX*(1)
EF(1)	*JK*(1)	*NT*	*QT*(1)	*T*	*WY*
EI	*JM*	*O*	*R*	*TU*	*X*(3)
F(1)	*JMN*	*OP*(1)	*RS*(4)	*TUV*(2)	*XZ*(7)
FG(1)	*JN*	*OPQ*(4)	*RST*(2)	*TV*(3)	*Y*
G	*JQ*(1)	*OPU*(1)	*RSTU*(1)	*TVW*(1)	*YZ*(1)
GH	*JU*	*OQ*(1)	*RSU*(2)	*TW*	*YZ2A*(1)
GHI	*K*(1)	*OQT*(1)	*RT*	*TZ*(2)	*Y2A*
GI	*L*	*OT*	*RTU*	*U*	*Z*(2)
GK(1)	*LM*	*OTU*	*RTW*	*UV*(3)	*Z2A*(2)
GM	*LMN*	*OU*	*RU*	*UVX*(5)	*2A*
GMN	*LN*	*P*(1)	*RUX*(2)	*UX*(2)	*2B*(2)

Table 2: Catalog of the 114 fully supported, fully symmetric stellations of the rhombic triacontahedron. The number of fully supported chiral pairs derived from the corresponding reflexible parent stellations are shown in brackets.

Main-line (1st to 12th):	*A, B, CD, EF, GHI, JK, LMN, OPQ, RSTU, VWX, YZ2A, 2B*
Primary:	*A, D, G, I, J, U, T*
Small stellated triacontahedron (narrow rhombic faces whose obtuse vertices are hidden):	*G*
Great stellated triacontahedron (wide rhombic faces):	*JU*
Uniform compound of 5 cubes:	*EF*
Compound of 30 rhombic disphenoids (tetrahedra with scalene faces, all alike):	*VX*
Compound of 15 rhombic disphenoids (chiral form derived from one half of *VX*):	$Q^RS^RV^RX^R$, new name: V_RX_R
Isohedral-isogonal forms: equal facial figures and equal vertex figures are each single continuous polygonal paths of which all the peripheral portions are exposed on the solid's surface: here vertices are 3-gons and faces are self-intersecting irregular 12-gons):	*K, 2B*

Table 3: Notable fully supported stellations of the RTC. Models of the well-known stellations *G*, *JU*, and *EF* are picture in Cundy and Rollett (1961), Coxeter (1973) and Bruckner (1900). Bruckner (1900) also shows models of isohedral-isogonal forms *K* and *2B*. Photographs of three miscellaneous chiral stellations appear in Pawley (1975) in the following order (revised names): $Q^RS^RX^R$, V^RZ^R and YZ^R2A. Other stellations of the RTC are practically unknown in the literature. The special cases are summarized below and photographs of several examples are shown in Figures 3-14.

	Isohedral Core Noyau isoèdre	Layers Couches	Cell Types Types de cellules	Stellations Étoilements
(8)	regular octahedron	1	1 (1,0)	1 (1,0)
(12)	regular dodecahedron	3	3 (3,0)	3 (3,0)
(12)	rhombic dodecahedron	3	3 (3,0)	3 (3,0)
(20)	regular icosahedron	7	10 (9,1)	17 (15,2)
(24)	triakis tetrahedron	5	8 (6,2)	20 (16,4)
(24)	tetrakis hexahedron	9	30 (17,13)	1761 (371,1390)
(24)	triakis octahedron	9	31 (18,13)	3082 (564,2518)
(24)	trapezoidal icositetrahedron	9	31 (18,13)	1200 (385,815)
(24)	pentagonal icositetrahedron	11	68 (0,0)	72620 (0,72620)
(30)	rhombic triacontahedron	12	28 (19,9)	226 (114,112)
(48)	hexakis octahedron	21		
(60)	trapezoidal hexecontahedron	28	225 (82,143)	
(60)	pentagonal hexecontahedron	31	– (0,0)	– (0,–)
(60)	pentakis dodecahedron			
(60)	triakis icosahedron			
(120)	hexakis icosahedron			

Table 4: This chart summarizes the author's systematic enumerations for the fully supported stellations of convex isohedra, including the RTC for comparison. A blank entry means that data is not readily accessible without computer assistance. In the appropriate column, brackets enclose the following totals: core faces; reflexible cell types followed by pairs of chiral cells; reflexible stellations followed by pairs of chiral stellations. If clarification of technical names is required, the reader should consult Cundy et al. (1961). Several more of the blank entries have been updated and filled in by Robert Webb using his general stellating program *Great Stella*. The current results are at http://home.connexus.net.au/~robandfi/Enumerate.html.

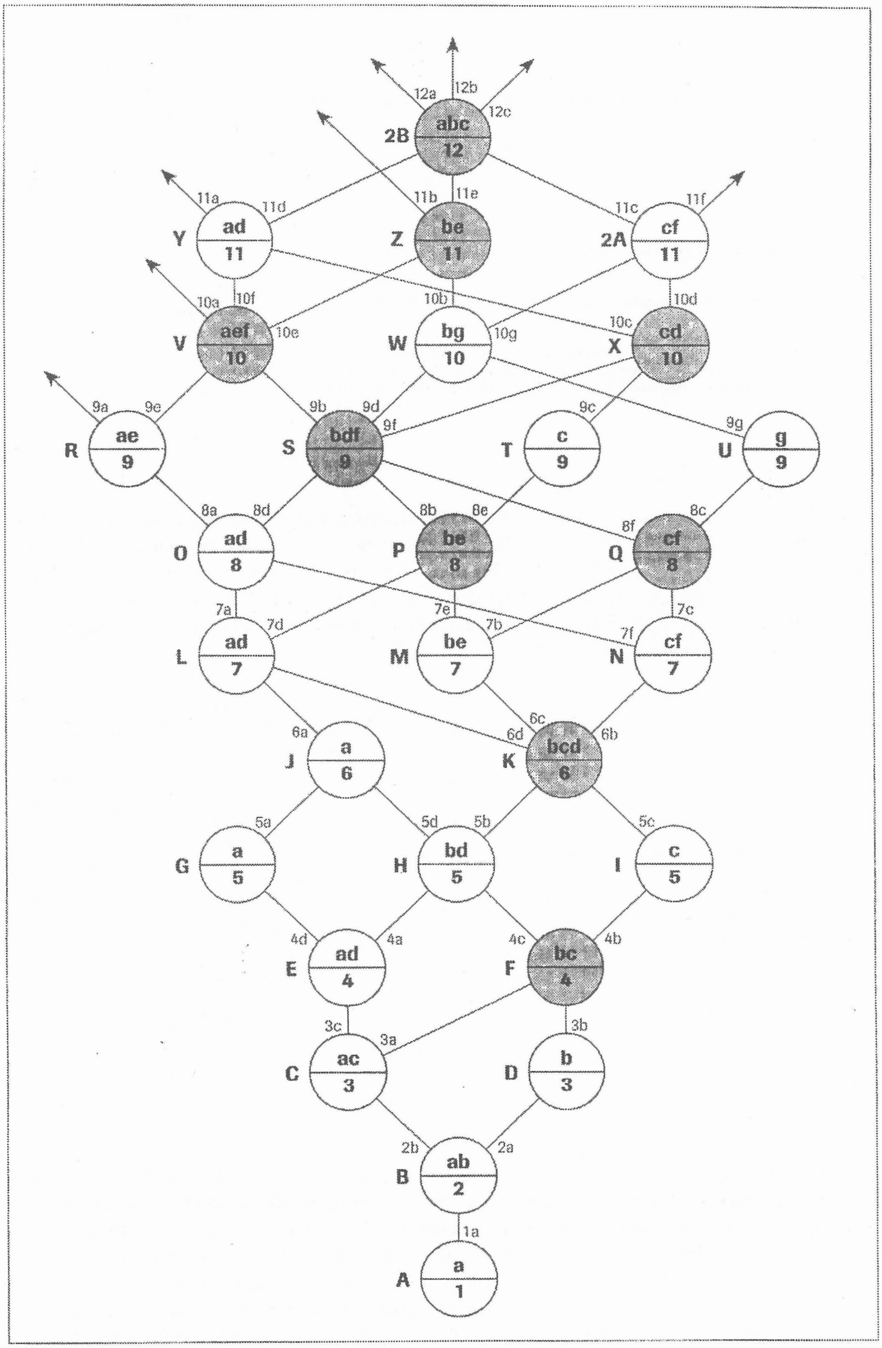

Figure 2: Graph of cell connectivity for the fully stellated rhombic triacontahedron

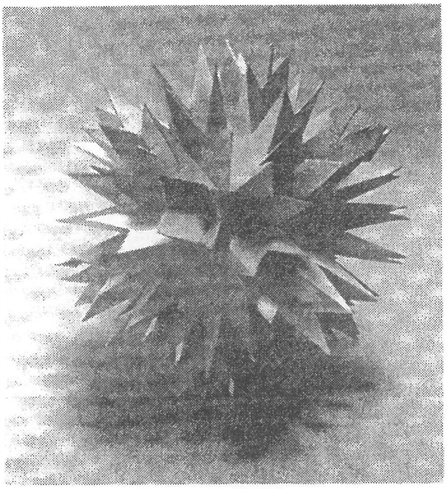

Figure 3: $2B = 9a$, $10a$, $11abf$, $12abc$ (final stellation)

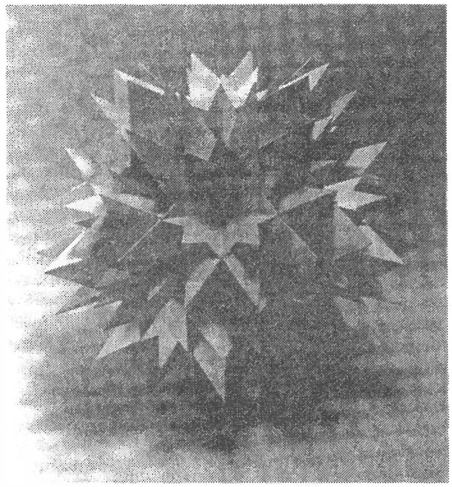

Figure 4: $YZ2A = 9a$, $10a$, $11abcdef$

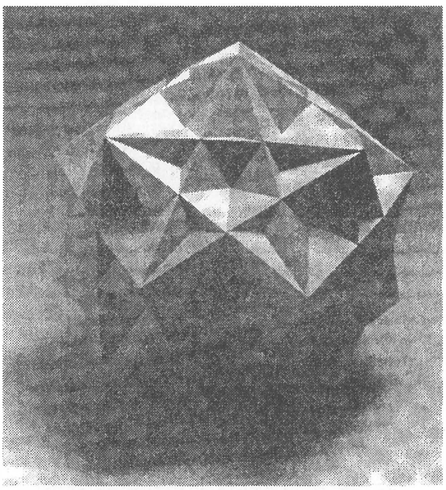

Figure 5: $EF = 4abcd$ (uniform compound of 5 cubes)

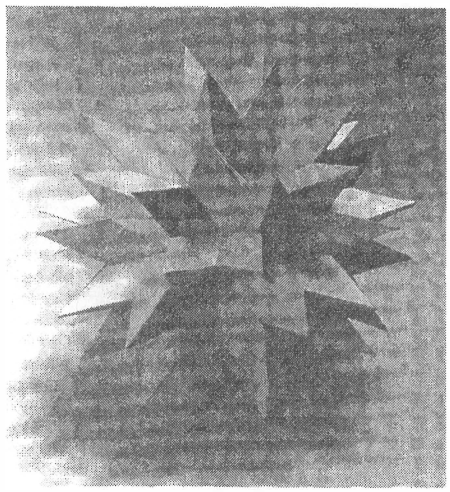

Figure 6: $Y = 8c$, $9ad$, $10ade$, $11ad$

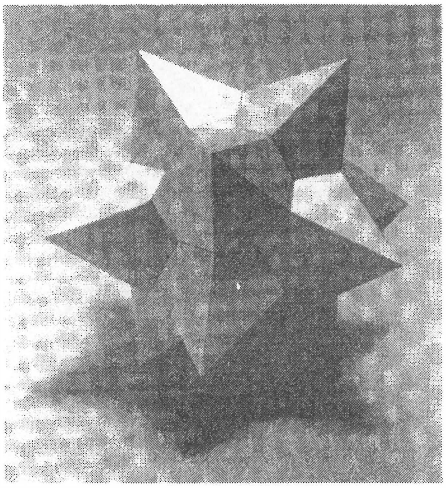

Figure 7: $G = 2a, 3a, 4a, 5a$
(small stellated triacontahedron)

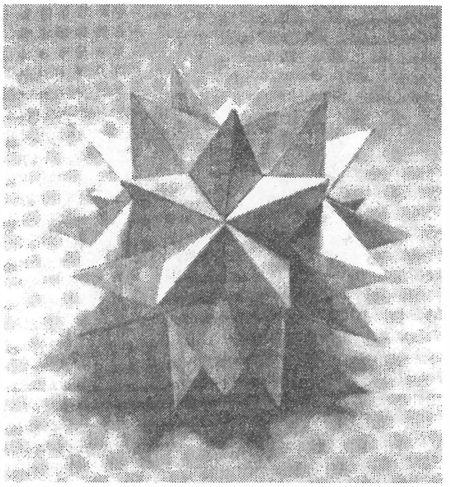

Figure 8: $T = 6b, 7ab, 8b, 9c$

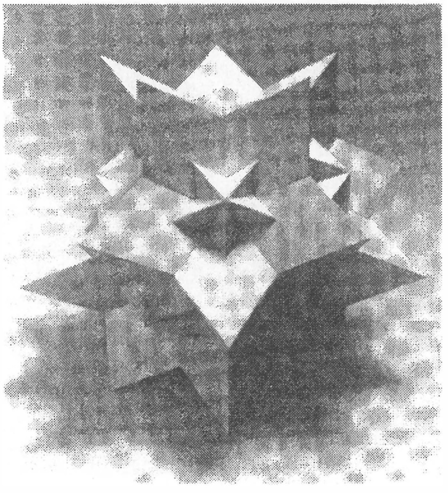

Figure 9: $JU = 6ad, 7ef, 8f, 9g$
(great stellated triacontahedron)

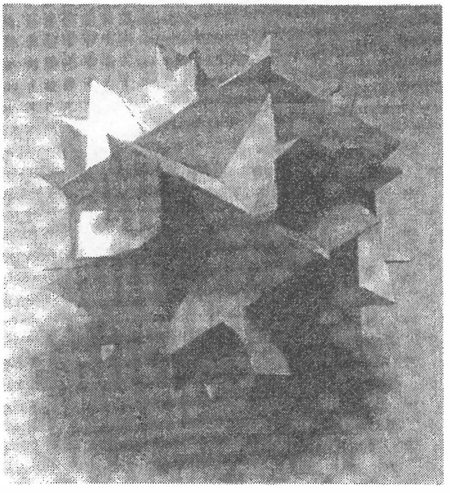

Figure 10: $Q^R S^R V^R X^R$ (new name: $VrXr$) =
= $(8c, 9d, 10acdef)^R$, $(7bc, 8bd, 9ce)^L$, $(9a)$

Those elementary regions that appear to be cut in half by symmetry lines naturally have portions of either handedness. So that borders of regions are not confused, such *ambidextrous regions* appear entirely unstippled in the diagram.

It is useful to identify points in the stellation diagram that corresponds to points on the n-fold rotational axes. The latter points occur at various distances from the core's solid centre, also known as the *origin*. The icosahedral symmetry of the RTC limits n to 2, 3 or 5. Such information helps the model builder visualise how surface parts radiate from a common central point. The underlying principle is that an n-fold rotational axis is a line in space common to n intersecting mirror planes. Therefore, an n-fold axis point corresponds to a site in the diagram where a combination of n lines of the primary and/or symmetry kind intersect. This simple relationship is summarised by

$$n = p + s \tag{1}$$

where p primary lines and s symmetry lines share the given point.

It is evident that each bounded symmetry region is a triangle of which each vertex corresponds to a different rotational symmetry (Messer and Wenninger 1989). For the RTC, the regions are variously shaped "2-3-5" triangles. If two borders of an unbounded symmetry region are parallel, a useful interpretation is that the third vertex exists at infinity where the parallel lines meet. Such n-fold axis points are discussed in later sections.

INSTRUCTIONS FOR DRAWING A STELLATION DIAGRAM WITH WORKABLE PROPORTIONS

Readers will gain much from actually drawing their own stellation diagram for the RTC, at a scale significantly larger than the one shown in Figure 1. The enlargement, which complies with the numeric data in Table 1, makes handling of surface parts easier during model building. The recommended unit of measurement is the centimetre.

First to be drawn are the lines that connect the 12 peripheral points (vertices) of the diagram in a single continuous polygonal path as evident from Figure 1. This self-intersecting figure, which is defined by lines C, F, G and H, is in fact an irregular star 12-gon. The vertices may be accurately plotted on a large sheet of paper, at least 40 x 40 cm, using a rectangular coordinate system (horizontal X-axis, vertical Y-axis). The desired proportions are obtained when the 12 vertices are ($\pm a$, $\pm b$); ($\pm b$, $\pm a$); and ($\pm c$, $\pm d$) where all combination of signs are permitted; and

$$a = \tau + 1 \approx 2.618; \qquad b = 9\tau + 5 \approx 19.56;$$
$$c = 5\tau + 3 \approx 11.09; \qquad d = 7\tau + 5 \approx 16.33;$$

where the golden ratio $\tau = (1 + \sqrt{5}) / 2 \approx 1.618$.

Continuing to use Figure 1 as a guide, all remaining lines can be drawn from previously defined intersection points. Lines drawn in the order of *E, D, B*, and *A* works well. When finished, the four *A* lines define a core figure that is a rhombus (RTC face) which has a short diagonal equal to 2 and a long diagonal equal to $2\tau \approx 3.236$.

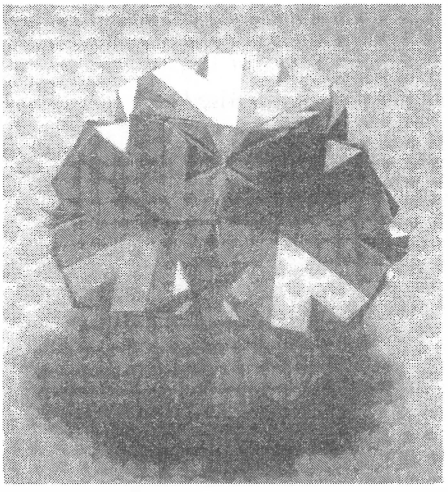

Figure 11: $K = 4d, 5d, 6bcd$

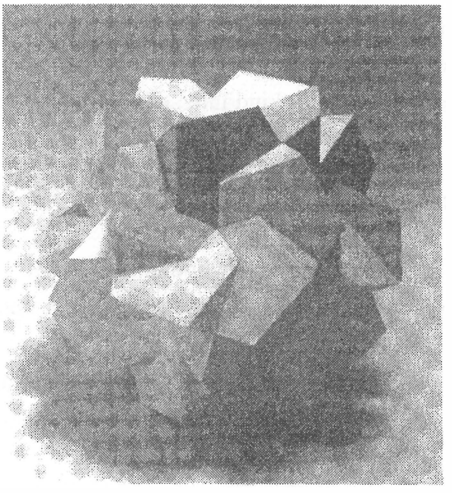

Figure 12: $K^R = (6bcd)^R, (5bc)^L, (4d, 5d)$

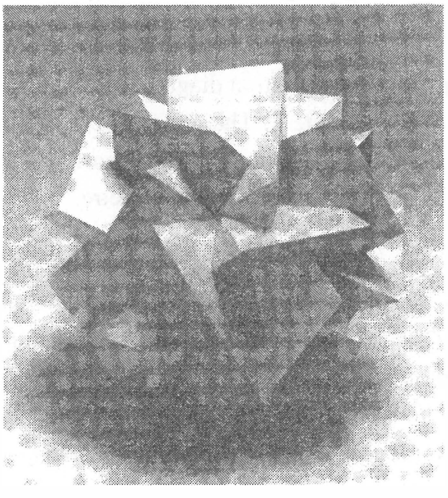

Figure 13: $Q^R S^R X^R$ (new name: X_r) =
$= (8c, 9bd, 10cd)^R, (7bc, 8bd, 9c)^L, (8a)$

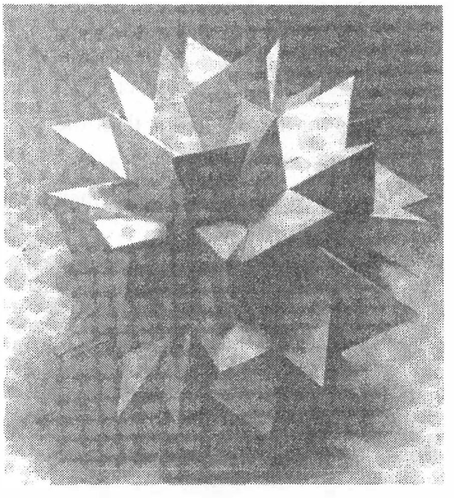

Figure 14: $V^R Z^R$ (new name: Z_r) =
$= (10af, 11be)^R, (9be, 10b)^L, (8e, 9af, 10g)$

CONNECTIVITY OF LINES AND REGIONS

A labeled stellation diagram and a table of line connections provide the necessary information for determining the way in which various lines and elementary regions connect (*match*) in three-dimensions. Matching two lines according to their consecutively numbered elementary segments, as described previously (Messer and Wenninger 1989), is both tedious and prone to error when one actually counts outward to the desired segments. The new method of presenting proper connections is a table (Table 1 for the RTC) that immediately matches two elementary regions along their corresponding connecting line borders. One can find desired lines and regions in the stellation diagram by following the diagram's methodical system of labeling.

The details of Table 1 require some explanation by way of specific examples. Line A is a primary line which matches a different, opposite handed line A in the diagram (*pair-matching*). Here one column suffices for understanding that a right-handed region 8d matches a left-handed region 8d along their corresponding line A borders which have relative lengths of 3.804. It is important to note that for the entire table, matched regions have been selected to lie on the same side of the corresponding lines as the centre of the stellation diagram. That means matching borders consistently produce convex (not concave) edges on the surface of the solid. It is also understood that two matching regions can always make the proper connections by sliding them together in the plane of the stellation diagram - never flipping over one region. Bringing two facial parts together in this way begins the process of net assembly used in model building.

Also noted in Table 1 is an internal dihedral angle of 144 degrees which occurs at the edge of the corresponding line A. Dihedral angles between intersecting facial planes become more apparent as the stellation takes shape during model construction. Two lines of a matching pair are equidistant from the centre of the stellation diagram and their associated dihedral angles decrease as their distances from the centre increase. An exact relationship for isohedra is

$$d \tan(\delta / 2) = k \qquad (2)$$

where δ is the internal dihedral angle, d is the distance of the line from the diagram's centre, and distance k is the constant in-radius of origin to core face.

Because pair-matching lines C and H are secondary lines, for example, they make connections of the *same* handedness. More specifically, border C of right region $10g$ matches border H of right region $10b$ to produce a convex edge of a right handed cell. The connecting edge has length 3.236 and it is associated with an internal dihedral angle of 90 degrees. The heavy dot shown in that double column indicates the relative position of a point which is symmetric to the consecutively matched segments along lines C and H.

Line B is an example of a *self-matching* secondary line, i.e., border B of right region $3a$ matches border B of right region $3c$ from the same line. The point of symmetry for such a line is understood to be at the bottom of the double column. Whether pair-matching or self- matching, the point of symmetry for *primary* lines always coincides with an n-fold axis point on a symmetry line; for matching *secondary* lines, however, that point does not lie on a symmetry line.

A given line may actually contain more than one point that is symmetric with respect to that line and other distinct lines in the stellation diagram. Not too surprisingly, the number of such symmetry points that are assigned to a line is equal to the total number of symmetry lines occurring in the diagram. However, when no symmetry lines exist, as in duals of snub solids, one symmetry point is still assigned to every line. In the case of the RTC the number assignment for each line is clearly two. For example, one primary line E intersects symmetry lines at two such points. Along the same secondary line C there exists two points that are each symmetric with respect to that line C and a different line H. Likewise, self-matching line B contains one such point at its midpoint. The second symmetry point occurs at infinity in either direction along line B. In this more abstract case there is indeed a pair-matching of two parallel secondary B lines.

SOME MATHEMATICAL PROPERTIES OF THE STELLATION DIAGRAM

Besides equation (1) there are other ways to determine the number n of a particular n-fold axis point located in the stellation diagram. An interesting observation is

$$n = 2m + i + 1 \tag{3}$$

where m pairs of matching lines are symmetric to the axis point (m = 0, 1, or 2) and i self-matching lines are symmetric to the axis point (i = 0 or 1). The four sets of combined possibilities for the icosahedral and octahedral symmetry groups are:

n	m	i
2	0	1
3	1	0
4	1	1
5	2	0

Lacking mirror planes, a dual of a snub Archimedean solid has only secondary lines in its stellation diagram, in which case equation (3) is especially useful.

As an exercise in contriving new relationships, consider

$$i = (1 + (-1)^n)/2 \tag{4}$$

which fits the fact that i is 0 or 1 depending on weather n is odd or even. Combining equation (3) and (4) leads to

$$m = (2n - 3 - (-1)^n)/4 \tag{5}$$

Other relationships involve n and various angles and distances associated with the stellation diagram. For example, let α be the measure of the angle formed by the sides of a matching pair of lines. The vertex of this angle is both an n-fold axis point and the point symmetric with respect to the given pair of lines. Clearly, the angle bisector of α, contains the diagram's centre and when $\alpha = \pi$ radians or 180 degrees, the two lines reduce to a single self-matching line. In three-dimensions the exposed vertex corresponds to a point on an n-fold rotation axis surrounded by n equal face angles, α, and n equal dihedral angles, δ. A relationship for such a configuration of parts is

$$\cos(\pi/n) = \cos(\alpha/2)\sin(\delta/2) \tag{6}$$

where n = 2, 3, 4, 5, or 5/2. The last n is in fractional form which represents another instance of 5-fold symmetry (numerator) but with a density of two (denominator) (Coxeter 1973).

For a given pair of matching lines, if r is the distance of the symmetry point from the diagram's centre, then from equation (2)

$$d = k / \tan(\delta / 2) = r \sin(\alpha / 2) \tag{7}$$

Combining equation (6) and (7) leads to

$$\sin^2(\pi / n) = (1/k^2 + 1/r^2) / (1/k^2 + 1/d^2) \tag{8}$$

and so n may be determined from distance measurements r and d when constant k is known.

As further analysis, when $r = d$ the pair of matching lines degenerates to a single self-matching line ($n = 2$) whose symmetry point is the foot of the perpendicular from the diagram's centre. When considering a 5-fold axis point, for which m is always 2, the first pair of matching lines produce $n = 5$ using either equation (6) or (8); the second pair of matching lines produce $n = 5/2$. This distance is consistent with the two sets of different values observed for d and α. Note that if a symmetry point is shared by both a self-matching line and a pair of matching lines, the latter case takes precedence in solving for n ($n = 4$) when using either equation (6) or (8).

The previous equations are applicable to the abstract case of pair-matching of the parallel lines wherein the conditions $1/r = 0$ and $\alpha = 0$ occur. In the RTC diagram, for example, a matching par of parallel B lines is parallel to a matching pair of parallel F lines. Each pair is symmetric to the same n-fold axis point at infinity in which case $m = 2$ and thus $n = 5$. As further confirmation for such a point, equation (6) and (8) each yield $n = 5$ for the B lines and $n = 5/2$ for F lines. Furthermore, equation (6) for pair-matching of two parallel lines can be reduced to

$$n = 2\pi / (\pi - \delta) \tag{9}$$

Therefore, possible sets of (n, δ) are limited to $(2, 0)$, $(3, \pi/3)$, $(4, \pi/2)$, $(5, 3\pi/5)$, and $(5/2, \pi/5)$. Clearly, only the first set does not occur. In practice, however, the n value of an infinite axis point can be found by simple inspection of the stellation diagram, i.e., parallel to the matching pair of parallel lines are primary lines which define certain unbounded symmetry regions, all of which share the same point of infinity. The common point is in fact the third vertex (n-fold axis point) for each of the imaginary symmetry triangles.

There is a curious relationship between the distance properties of two similar figures embedded in the stellation diagram. The related figures arise only for isohedra that are centrally symmetric, therefore the triakis tetrahedron and snub duals are excluded. The first figure is the familiar core region (*core figure*). The proportionately larger second figure, named the *projected figure*, surrounds the first and is rotated a half-turn with respect to the first. In the case of the RTC, A lines define the core figure while F lines define the projected figure. Here the half-turn is not obvious because of the inherent 2-fold rotational symmetry of the RTC face. In general, borders of the core figure are equidistant from the diagram's centre, say distance d_1, and associated with a constant internal dihedral angle say D. Dihedral angles for isohedra are well known (Cundy and Rollett 1961) and for the RTC, $D = 4\pi/5$ radians or 144 degrees. Accordingly, the borders of the projected figure has an internal dihedral angle equal to $(\pi - D)$ radians. If the constant distance of projected border to diagram centre is d_2, then the following occurs:

$$k = d_1 \tan (D / 2) = d_2 \tan (\pi - D) / 2 = d_2 / \tan (D / 2) \qquad (10)$$

Then from

$$d_2 / d_1 = \tan^2 (D / 2) \qquad (11)$$

the projected figure is determined to be larger by a factor of $\tan^2(D/2)$ compared to the core figure.

If published dihedral angles, D, for convex isohedra (duals of the convex uniform polyhedra, Table 4) are not readily available, the reader may calculate them using the following approach. Basically, there are three general kinds of isohedral faces (core figures): a triangle; a quadrilateral having at least two opposite angles that are equal; a pentagon having at least four equal angles. Accordingly, an isohedral solid consists of at most three kinds of vertices at which n_1, n_2, or n_3 equal face angles meet. Each vertex is assigned such a characteristic *vertex number* which is either equal to or double the *n-fold number* of the symmetry axis containing the vertex. If a particular vertex of an isolated core figure lies symmetric with respect to the figure's other vertices, then the two numbers are equal, but without this symmetry the vertex number is doubled. The exceptions are pentagon figures of snub solids for which vertex numbers are consistently equal to the corresponding n-fold numbers. Furthermore, there exists a pair of snub vertices not incident to rotation axes but they have a vertex numbers of 3 like that of another similar pair of snub vertices.

Let $a = \cos(\pi / n_1)$, $b = \cos(\pi / n_2)$, and $c = \cos(\pi / n_3)$. These measurements represent distance and correspond to the sides of yet another important figure - the *vertex figure* (Coxeter 1973) of the reciprocal uniform solid which has edge = 1. The sides of the vertex figure follow the same cylindrical order as the corresponding vertex numbers of the isohedral core figure. The diameter of the circle that simultaneously circumscribes the vertex figure and inscribes the core figure is equal to $\sin(D/2)$. It is of further interest that

$$\sin(D/2) = \cos\phi \tag{12}$$

where ϕ is the orthoschematic central angle subtended by the half-edge of the reciprocal uniform solid.

Simplification occurs if semi-perimeters of the vertex figures are defined: $s = (a+b+c)/2$ for the triangle case and $s' = (a + 2b + c) / 2$ for the quadrilateral case. Let the isohedral core figure be identified according to its vertex numbers n_1, n_2, n_3 with appropriate coefficients of multiplicity. Then the following general formulae solve the diameter, $\sin(D / 2)$, of the common circle of duality described above:

For triangle (n_1, n_2, n_3),

$$\sin(D/2) = \frac{abc}{2\sqrt{s(s-a)(s-b)(s-c)}} \tag{13}$$

For quadrilateral $(n_1, 2n_2, n_3)$,

$$\sin(D/2) = b\sqrt{\frac{b^2+ac}{(s'-a)(s'-b)}} \tag{14}$$

For pentagon $(n_1, 4n_2)$,

$$\sin(D/2) = \frac{2b}{\sqrt{4-x^2}} \tag{15}$$

where x is a positive root of

$$x^3 - 2x - \frac{a}{b} = 0 \tag{16}$$

The RTC quadrilateral core figure, for example, has $n_1 = n_3 = 3$ and two $n_2 = 5$.

Both equations (13) and (14) are derived from the well-know general case of the cylindrical quadrilateral (sides *a* , *b, c, d*) for which the circumdiameter is

$$\frac{1}{2}\sqrt{\frac{(ac+bd)(ad+bc)(ab+cd)}{(s-a)(s-b)(s-c)(s-d)}}$$

where semi-perimeter is $s = (a + b + c + d)/2$. Clearly, $d = 0$ in the reduced case of a triangle.

If isohedral face angles A, B, and C correspond to vertex numbers n_1, n_2, and n_3, respectively, then the face angles can be solved using equation (6), that is

$$\cos(A/2) = \frac{a}{\sin(D/2)}\ ; \quad \cos(B/2) = \frac{b}{\sin(D/2)}\ ;$$

$$\cos(C/2) = \frac{c}{\sin(D/2)} \tag{17}$$

CELL IDENTITIES AND THEIR INTERCONNECTIONS

Introductory methods have been described (Messer and Wenninger 1989) for solving elementary regions that identify the top and bottom facets of various cell members in each layer. One further comment on this matter is that a particular cell kind is reflexible if it is found to have either an ambidextrous top facet or top facets connected along a primary line border. However, the cell is chiral (either left or right) if only secondary line connections involve top facets. Once all the top and bottom facets are known and if the numbers are not too overwhelming, then it is useful to construct a *graph of cell connectivity* also referred to as simply *cell graph*. For the complete stellation of the RTC, such a graph (Figure 2) schematically depicts the 12 layers of the cells as horizontally grouped circles. It is understood that a stippled circle represents a pair of chiral cells of which each handed member has independent connections with other cells. Not counting the RTC core, there are a total of 28 cell kinds of which 19 are reflexible and 9 exist as chiral pairs.

Emanating from the circles are lines that represent cell-to-cell connections. More specifically, each line joins a top facet of a lower cell to a bottom facet of an upper cell. The identity of the common facet is an elementary region which is indicated in the graph among other top facets of the same cell. Choice of handedness for the involved cells and their top facets is important only when considering chiral stellations later. Also depicted in the cell graph are arrows which point upwardly from those top facets that are never covered - either because they are always exposed by upper noncontinuous layers or because they are already outermost atop the 12^{th} layer.

A particular cell may be identified by its unique set of top facets. However, for purpose of naming stellations later, it is more convenient to assign a single character to each cell kind. The systematic nomenclature presented here differs from Pawley's cell designation (Pawley 1975). Beginning with the innermost layer, cells are assigned capital letters which follow the alphabetical order of the sets of lower case letters associated with the top of each layer. When letters run out, the alphabet repeats by using number prefix 2 before each capital letter. In this way, for the RTC, 2*A* follows *Z*. For isohedra more complex than the RTC, 3*A* follows 2*Z* and so on. As discussed later, stellations are named according to fully top-exposed cells so a hypothetical name such as *Z2A2B2D* is preferably reduced to the form *Z2ABD*. In this way a number prefix is in effect until it is replaced by the next one when reading cell labels arranged alphabetically from the left.

There is a partial solution to the problem of how many different cell *kinds* radiate in space from a common point on an n-fold rotational axis given only the stellation diagram to inspect. The stellation diagram of the RTC, and of many other isohedra, contains only "simple" n-fold axis points. However, a few isohedral cases are complicated by the existence of "split" pairs of n-fold axis points. Each member of such split pair usually occupies a symmetrically identical site on the corresponding rotation axis in space, but in marked contrast to simple points, each split member occupies a symmetrically different site within the plane of the stellation diagram. Of course a pair of split points must lie equidistant from the diagram's centre. In the majority of cases then, one observes $p + 2q + 2$ kinds of cells, both bounded and unbounded, that share a given simple n-fold axis point defined by the intersection of p primary lines and q secondary lines in the stellation diagram.

STELLATION NAMES AND THEIR ASSOCIATED STELLATION PATTERNS

Knowledge of cell identities and cell interconnections are prerequisites for systematically naming individual stellations that are fully supported - both reflexible and chiral. In contrast with Pawley's names which express every exposed cell, the new stellation names presented here denote only those participating cells that are fully top-exposed (*cell peaks*). The one exception discussed later is that the name must identify all participating chiral cells which have missing counterparts. It is understood that the cell peaks, expressed in a given stellation name, must all be fully supported. Besides the tops of designated cell peaks, additional surface parts are contributed by supporting cells which are partially top-exposed. This information is readily determined by inspecting the cell graph.

The new nomenclature for stellation names offers conciseness without sacrificing uniqueness. For example, Pawley's notation for the final main-line stellation is *A(bcdgk)* which translate to the new and simpler name 2*B*. Here cell 2*B* is identified as the only peak and thus regions 12*a*, 12*b*, and 12*c* (abbreviated as set 12*abc*) are exposed. From the graph of cell connectivity one observes that parts of supporting cells, *R, V, Y, Z*, and 2*A* are also exposed, namely regions 9*a*, 10*a* and 11*abf*. It is useful to equate a stellation name to the names of the exposed regions which in turn determine the stellation pattern. The previous example is written as 2*B* = 9*a*, 10*a*, 11*abf*, 12*abc*. For the sake of brevity, however, stellation patterns are omitted from the alphabetical catalog of the fully supported, fully symmetric stellations of the RTC (Table 2). Some examples of stellation names equated to patterns are shown in Figures 3-14. When searching the stellation diagram for exposed facial parts of a reflexible stellation, one must remember to include both right and left regions, i.e., stippled and unstippled regions.

As discussed earlier, various chiral stellations may be derived from a single reflexible parent form by deleting either right or left members from one or more components chiral cell pairs. Accordingly, it suffices to name one handed member from each of the 112 chiral pairs of fully supported stellations. The number of chiral pairs obtainable from a reflexible parent stellation are shown in brackets in Table 2. Superscripts (*R* for right, *L* for left) are used for identifying both the handedness of chiral cells and the handedness of sets of exposed regions. Examples are shown in Figures 3-14. No superscript for a cell or set of regions implies that there exists both right and left representation. Clearly, if an oppositely handed stellation is desired, one needs to switch all of the participating superscripts.

The name given to a chiral stellation requires special attention. All component chiral cells, whose opposites are missing, must be identified in the name even if a chiral cell is not fully top-exposed. Methods for finding the set of all permissible chiral derivatives from a reflexible parent was introduced by Pawley (1975). Further development of a systematic approach by the author has proved successful for the more complex isohedral cores and is a subject for a future report.

Having established the subset of chiral cell interconnections for a particular stellation, all exposed top facets can then be found from a careful study of the cell graph. In order to ensure both fully supported conditions and accurate selection of handed facets from the graph, the following fundamental facts should be remembered:

1) A reflexible cell has both right and left top facets (and sometimes also an ambidextrous top facet).

2) A pair of chiral cells, shown as a single stippled circle, consist of a right cell and a left cell.

3) A right cell has only right top facets; left top facets for the opposite cell.

4) A given right cell rests on the right top facet of either a reflexible cell or another right cell.

5) A right cell and left cell together must support a given higher reflexible cell or a given higher chiral cell pair.

The reader who is interested in solving stellation patterns will find the following practical steps useful when studying the cell graph. First, place small transparent markers (say yellow) over each cell (circle) identified in the stellation name. However, place different coloured markers (right vs. left) over those chiral cell participants whose opposites are missing. Next, using more yellow markers and moving toward inner layers, cover all lower cells that must contribute to the support of the cell peaks. Finally read the exposed top facets from each participating layer by noting where graph lines remain uninvolved in cell connections. When the handedness of a facet is an issue, refer to the fundamental facts listed above.

FINAL REMARKS

The author has successfully applied the general approach presented in this paper to several other convex isohedra (Table 4). While emphasis has been on predicting the fully supported stellations from a system of nomenclature, there is the realization that with some further modification, reentrant cases can be treated similarly. Cells with exposed bottoms, for example, could be identified and grouped together within brackets. In this way, literally millions of pleasing shapes can be assigned names; stellations which otherwise might not have been predicted.

ACKNOWLEDGEMENT

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SUPPLEMENT

Some time after the publication of the original article, the author recommended naming chiral stellations in exactly the same way that reflexible stellations were named in the section "Stellation names and their associated stellation patterns". That means only the fully top-exposed chiral cells should appear in the stellation name, not also the partially exposed chiral cells as originally required. This consistency in nomenclature simplifies the task of enumerating and identifying the complete set of fully supported stellations using general computer algorithms written by the author and others in recent years. Such automated approaches easily supersede the tedious manual counting methods that were used to compile Table 4. The new "computerized" names for the chiral stellations are inserted in both the reprinted article and this supplement. Such improved names are recognized by the symbols *r* and *s* rather than the *R* and *L* used in the original names to denote right and left handedness, respectively.

New versions of Figures 3-14 are shown here as 2, 3, 5-fold axis views generated by the author's program in Mathematica (Wolfram Research, Inc.).

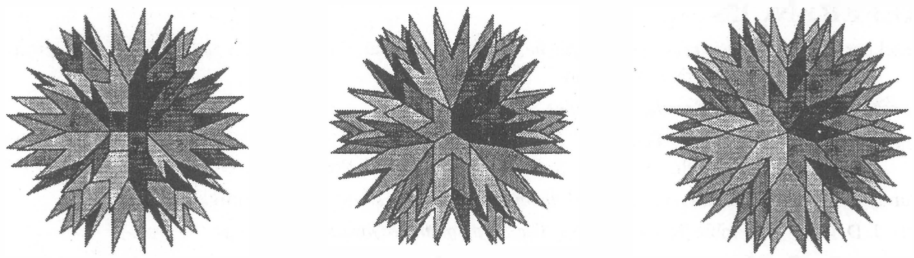


Figure 3: *2B* = 9*a*, 10*a*, 11*abf*, 12*abc*

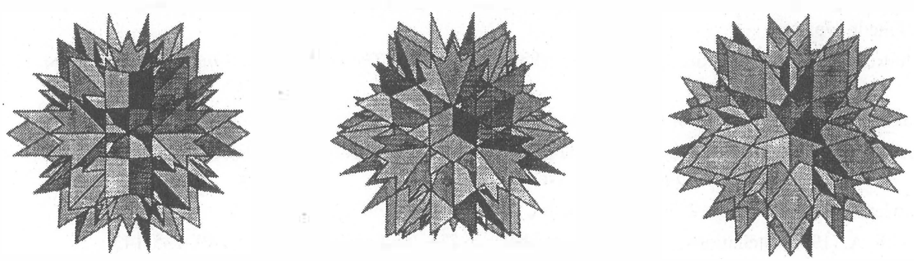


Figure 4: *YZ2A* = 9*a*, 10*a*, 11*abcdef*

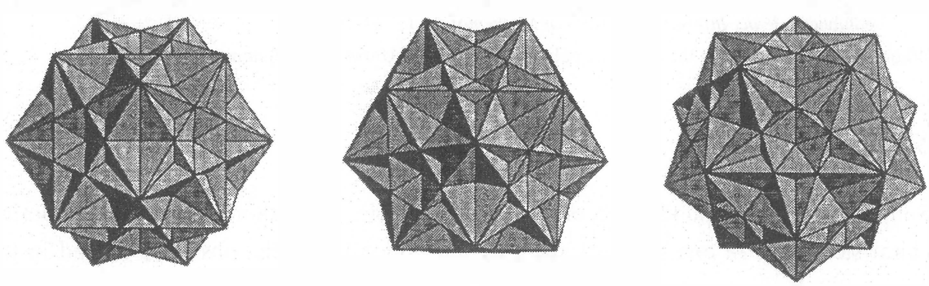


Figure 5: *EF* = 4*abcd*

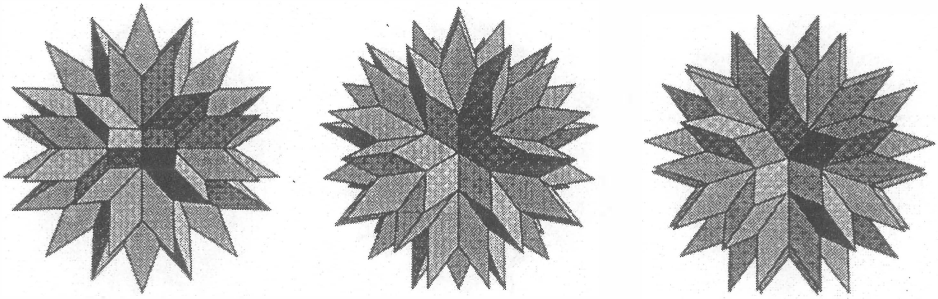


Figure 6: $Y = 8c$, $9ad$, $10ade$, $11ad$

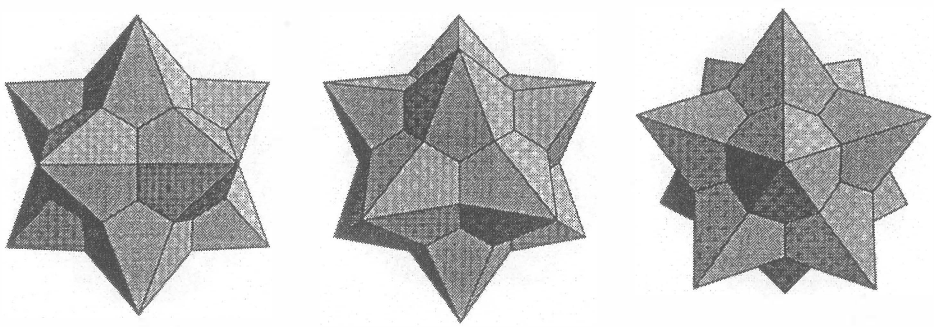


Figure 7: $G = 2a$, $3a$, $4a$, $5a$

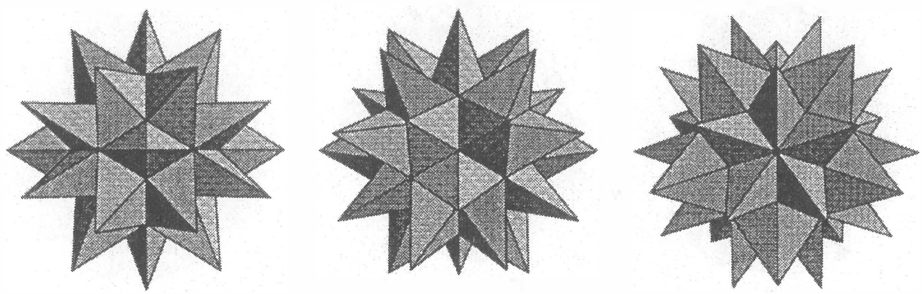


Figure 8: $T = 6b$, $7ab$, $8b$, $9c$

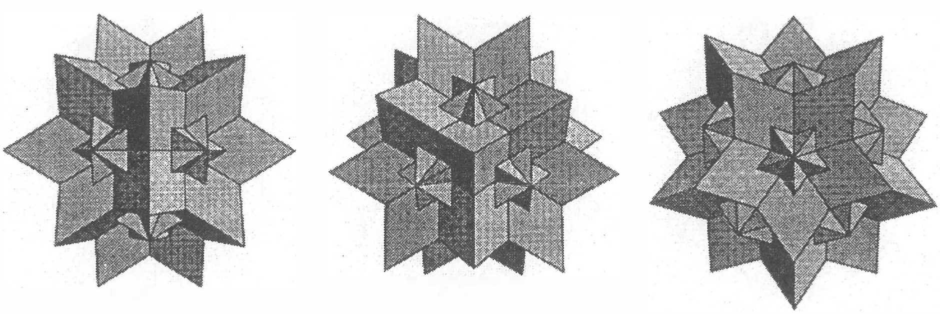


Figure 9: $JU = 6ad$, $7ef$, $8f$, $9g$

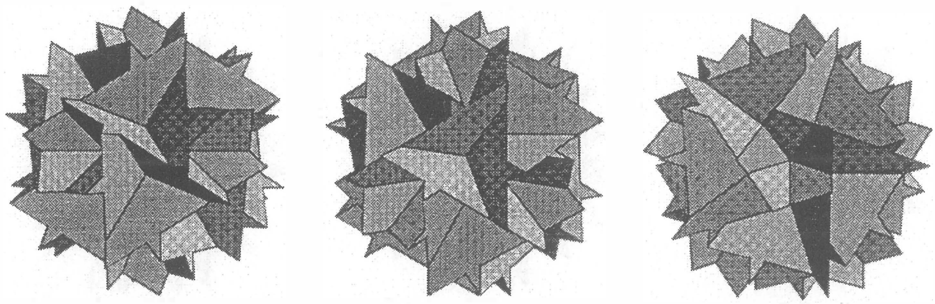


Figure 10: *QrSrVrXr* (improved name: *VrXr*) = (8*c*, 9*d*, 10*acdef*)*R*, (7*bc*, 8*bd*, 9*ce*)*L*, (9*a*)

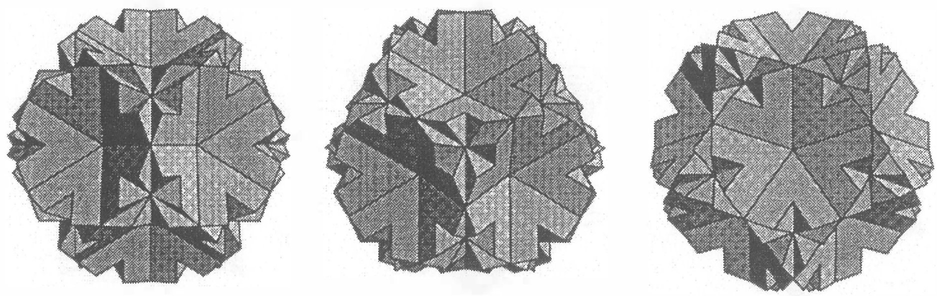


Figure 11: *K* = 4*d*, 5*d*, 6*bcd*

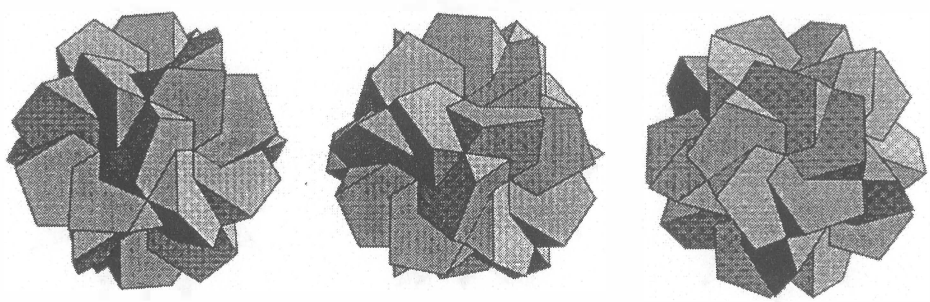


Figure 12: *Kr* = (6*bcd*)*R*, (5*bc*)*L*, (4*d*, 5*d*)

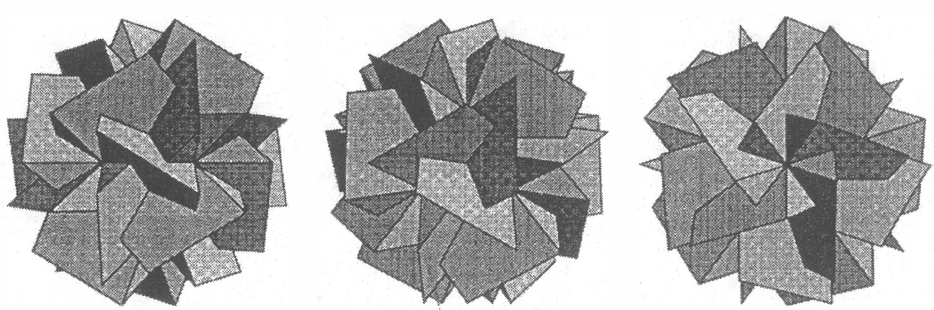


Figure 13: *QrSrXr* (improved name: *Xr*) = (8*c*, 9*bd*, 10*cd*)*R*, (7*bc*, 8*bd*, 9*c*)*L*, (8*a*)

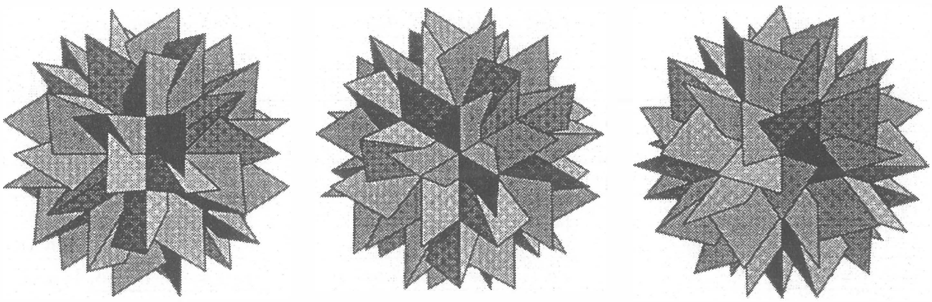


Figure 14: *VrZr* (improved name: *Zr*) = (10*af*, 11*be*)*R*, (9*be*, 10*b*)*L*, (8*e*, 9*af*, 10*g*)

SUPPLEMENT TO TABLE 2

Catalog of the 114 fully supported, fully symmetric stellations of the rhombic triacontahedron. Each stellation name is equated to the exposed facets of the stellation pattern.

1	A = 1a	39	LQ = 7adef, 8cf	77	RW = 8e, 9abef, 10bg
2	B = 2ab	40	LU = 7adef, 8f, 9g	78	RWX = 9abe, 10bcdg
3	C = 2a, 3ac	41	M = 4d, 5d, 6bd, 7be	79	RX = 8c, 9abde, 10cd
4	CD = 3abc	42	MN = 4d, 5d, 6d, 7bcef	80	R_2A = 9abe, 10bc, 11cf
5	D = 2b, 3b	43	MO = 7bcde, 8ad	81	S = 8ace, 9bdf
6	DE = 3ab, 4ad	44	MR = 7bcde, 8d, 9ae	82	ST = 8ac, 9bcdf
7	DG = 3ab, 4a, 5a	45	N = 4d, 5d, 6cd, 7cf	83	STU = 8a, 9bcdfg
8	E = 2a, 3a, 4ad	46	NP = 7abcf, 8be	84	SU = 8ae, 9bdfg
9	EF = 4abcd	47	NT = 7abcf, 8b, 9c	85	T = 6b, 7ab, 8b, 9c
10	EI = 4acd, 5c	48	O = 6c, 7cd, 8ad	86	TU = 7af, 8bf, 9cg
11	F = 3c, 4bc	49	OP = 7bc, 8abde	87	TUV = 9acdfg, 10aef
12	FG = 4abc, 5a	50	OPQ = 8abcdef	88	TV = 8c, 9acdf, 10aef
13	G = 2a, 3a, 4a, 5a	51	OPU = 8abdef, 9g	89	TVW = 9acf, 10abefg
14	GH = 4b, 5abd	52	OQ = 7de, 8acdf	90	TW = 8a, 9bcf, 10bg
15	GHI = 5abcd	53	OQT = 8abcdf, 9c	91	TZ = 9acf, 10afg, 11be
16	GI = 4ac, 5ac	54	OT = 7bc, 8abd, 9c	92	U = 4d, 5d, 6d, 7ef, 8f, 9g
17	GK = 5ad, 6bcd	55	OTU = 8abdf, 9cg	93	UV = 8e, 9adfg, 10aef
18	GM = 5ad, 6bd, 7be	56	OU = 7de, 8adf, 9g	94	UVX = 9adg, 10acdef
19	GMN = 5ad, 6d, 7bcef	57	P = 6b, 7ab, 8be	95	UX = 8a, 9bdg, 10cd
20	GN = 5ad, 6cd, 7cf	58	PQ = 7af, 8bcef	96	UY = 9adg, 10ade, 11ad
21	GQ = 5ad, 6d, 7ef, 8cf	59	PQR = 8bcdef, 9ae	97	V = 8ce, 9adf, 10aef
22	GU = 5ad, 6d, 7ef, 8f, 9g	60	PR = 7bc, 8bde, 9ae	98	VW = 8e, 9af, 10abefg
23	H = 4bd, 5bd	61	PRU = 8bdef, 9aeg	99	VWX = 9a, 10abcdefg
24	HI = 4d, 5bcd	62	PU = 7af, 8bef, 9g	100	VX = 8c, 9ad, 10acdef
25	I = 3c, 4c, 5c	63	Q = 4d, 5d, 6d, 7ef, 8cf	101	V_2A = 9a, 10abcef, 11cf

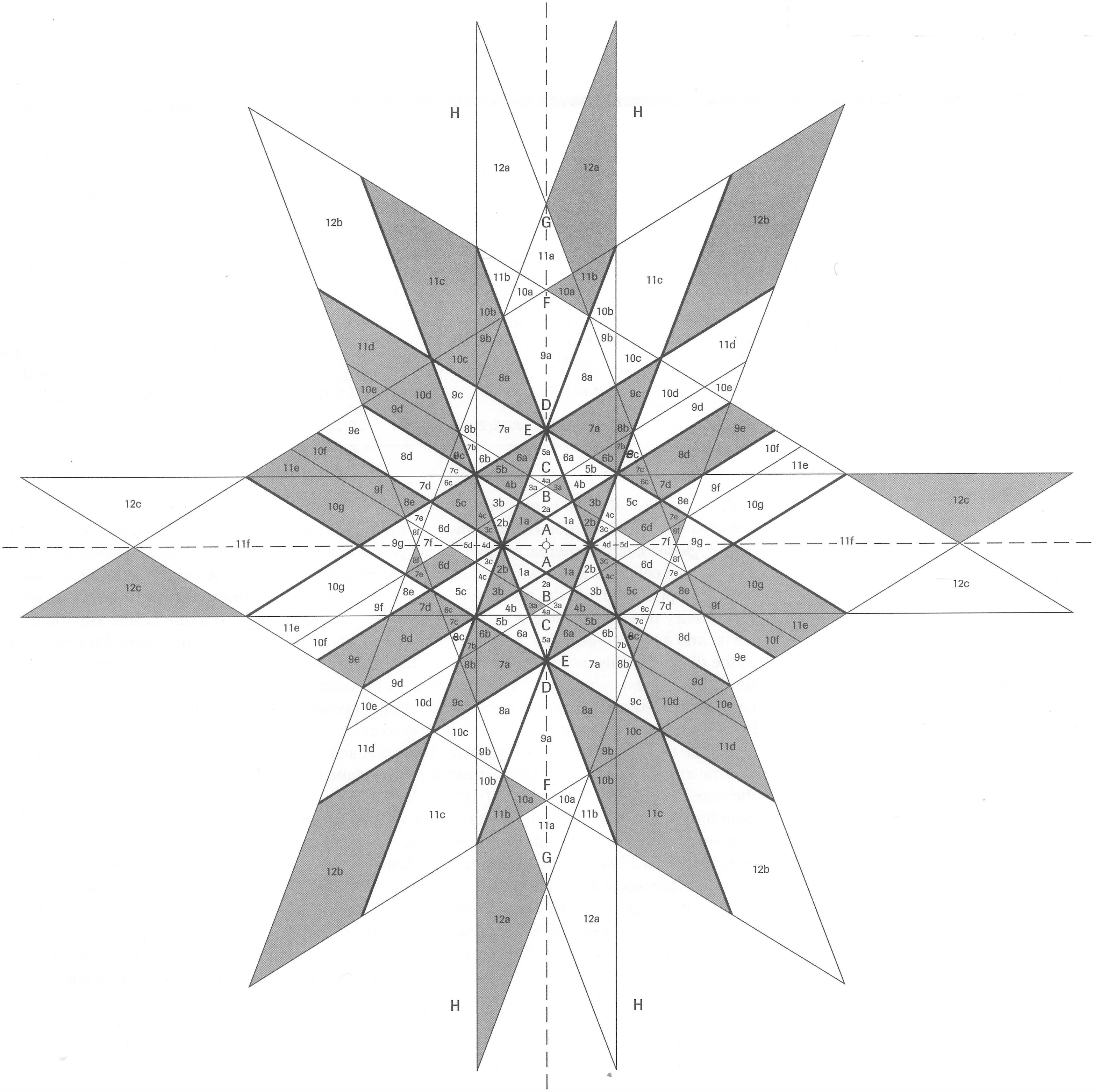
26	IJ = 5bc, 6a	64	QR = 7de, 8cdf, 9ae	102	W = 8ae, 9bf, 10bg
27	J = 4b, 5b, 6a	65	QRT = 8bcdf, 9ace	103	WX = 8a, 9b, 10bcdg
28	JK = 6abcd	66	QT = 7af, 8bcf, 9c	104	WY = 9a, 10abdeg, 11ad
29	JM = 6abd, 7be	67	R = 6c, 7cd, 8d, 9ae	105	X = 8ac, 9bd, 10cd
30	JMN = 6ad, 7bcef	68	RS = 8ce, 9abdef	106	XZ = 9a, 10acdfg, 11be
31	JN = 6acd, 7cf	69	RST = 8c, 9abcdef	107	Y = 8c, 9ad, 10ade, 11ad
32	JQ = 6ad, 7ef, 8cf	70	RSTU = 9abcdefg	108	YZ = 9a, 10adg, 11abde
33	JU = 6ad, 7ef, 8f, 9g	71	RSU = 8e, 9abdefg	109	VZ_2A = 9a, 10a, 11abcdef
34	K = 4d, 5d, 6bcd	72	RT = 7bc, 8bd, 9ace	110	Y_2A = 9a, 10abe, 11acdf
35	L = 6bc, 7ad	73	RTU = 8bdf, 9aceg	111	Z = 8e, 9af, 10afg, 11be
36	LM = 6b, 7abde	74	RTW = 9abcef, 10bg	112	Z_2A = 9a, 10acf, 11bcef
37	LMN = 7abcdef	75	RU = 7de, 8df, 9aeg	113	$_2A$ = 8a, 9b, 10bc, 11cf
38	LN = 6c, 7acdf	76	RUX = 9abdeg, 10cd	114	$_2B$ = 9a, 10a, 11abf, 12abc

Catalog of the 112 fully supported chiral stellations of the rhombic triacontahedron. The second column shows the number of chiral names obtainable from the reflexible parent stellation.

	Reflexible parent	Chiral form derived from its reflexible parent	Exposed facets define the stellation pattern	New computerized names show only the peak cells: r = R, s = L
1	EF(1)	EF^R	= $(4bc)^R$, $(3ab)^L$, (4ad)	= Efr
2	F(1)	F^R	= $(4bc)^R$, $(3ab)^L$, (3c)	= Fr
3	FG (1)	F^RG	= $(4bc)^R$, $(3ab)^L$, (4a, 5a)	= FrG
4	GK (1)	GK^R	= $(6bcd)^R$, $(5bc)^L$, (5ad)	= GKr
5	GQ (1)	GQ^R	= $(8cf)^R$, $(7bc)^L$, (5ad, 6d, 7ef)	= GQr
6	JK (1)	JK^R	= $(6bcd)^R$, $(5bc)^L$, (6a)	= JKr
7	JQ(1)	JQ^R	= $(8cf)^R$, $(7bc)^L$, (6ad, 7ef)	= JQr
8	K(1)	K^R	= $(6bcd)^R$, $(5bc)^L$, (4d, 5d)	= Kr
9	LQ(1)	LQ^R	= $(8cf)^R$, $(7bc)^L$, (7adef)	= LQr
10	NP(1)	NP^R	= $(8be)^R$, $(7de)^L$, (7abcf)	= NPr
11	OP(1)	OP^R	= $(8be)^R$, $(7de)^L$, (7bc, 8ad)	= OPr
12	OPQ(4)	OP^RQ^R	= $(8bcef)^R$, $(7bcde)^L$,(8ad)	= OPrQr
13		OP^RQ^L	= $(7bc, 8be)^R$, $(7de, 8cf)^L$, (8ad)	= OPrQs
14		OP^RQ	= $(8be)^R$, $(7de)^L$, (8acdf)	= OPrQ
15		OPQ^R	= $(8cf)^R$, $(7bc)^L$, (8abde)	= OPQr
16	OPU(1)	OP^RU	= $(8be)^R$, $(7de)^L$, (8adf, 9g)	= OPrU
17	OQ (1)	OQ^R	= $(8cf)^R$, $(7bc)^L$, (7de, 8ad)	= OQr
18	OQT(1)	OQ^RT	= $(8cf)^R$, $(7bc)^L$, (8abd, 9c)	= OQrT
19	P(1)	P^R	= $(8be)^R$, $(7de)^L$, (6b, 7ab)	= Pr
20	PQ(4)	P^RQ^R	= $(8bcef)^R$, $(7bcde)^L$, (7af)	= PrQr
21		P^RQ^L	= $(7bc, 8be)^R$, $(7de, 8cf)^L$, (7af)	= PrQs

22		P^RQ	= $(8be)^R$, $(7de)^L$, (7af, 8cf)	= PrQ
23		PQ^R	= $(8cf)^R$, $(7bc)^L$, (7af, 8be)	= PQr
24	PQR(4)	P^RQ^RR	= $(8bcef)^R$, $(7bcde)^L$, (8d, 9ae)	= PrQrR
25		P^RQ^LR	= $(7bc, 8be)^R$, $(7de, 8cf)^L$, (8d, 9ae)	= PrQsR
26		P^RQR	= $(8be)^R$, $(7de)^L$, (8cdf, 9ae)	= PrQR
27		PQ^RR	= $(8cf)^R$, $(7bc)^L$, (8bde, 9ae)	= PQrR
28	PR(1)	P^RR	= $(8be)^R$, $(7de)^L$, (7bc, 8d, 9ae)	= PrR
29	PRU(1)	P^RRU	= $(8be)^R$, $(7de)^L$, (8df, 9aeg)	= PrRU
30	PU(1)	P^RU	= $(8be)^R$, $(7de)^L$, (7af, 8f, 9g)	= PrU
31	Q(1)	Q^R	= $(8cf)^R$, $(7bc)^L$, (4d, 5d, 6d, 7ef)	= Qr
32	QR(1)	Q^RR	= $(8cf)^R$, $(7bc)^L$, (7de, 8d, 9ae)	= QrR
33	QRT(1)	Q^RRT	= $(8cf)^R$, $(7bc)^L$, (8bd, 9ace)	= QrRT
34	QT(1)	Q^RT	= $(8cf)^R$, $(7bc)^L$, (7af, 8b, 9c)	= QrT
35	RS(4)	$P^RQ^RRS^R$	= $(8ce, 9bdf)^R$, $(7bcde, 8d)^L$, (9ae)	= RSr
36		P^RRS^R	= $(8e, 9bdf)^R$, $(7de, 8df)^L$, (8c, 9ae)	= QsRSr
37		Q^RRS^R	= $(8c, 9dbf)^R$, $(7bc, 8bd)^L$, (8e, 9ae)	= PsRSr
38		RS^R	= $(9bdf)^R$, $(8bdf)^L$, (8ce, 9ae)	= PsQsRSr
39	RST(2)	Q^RRS^RT	= $(8c, 9bdf)^R$, $(7bc, 8bd)^L$, (9ace)	= RSrT
40		RS^RT	= $(9bdf)^R$, $(8bdf)^L$, (8c, 9ace)	= QsRSrT
41	RSTU(1)	RS^RTU	= $(9bdf)^R$, $(8bdf)^L$, (9aceg)	= RSrTU
42	RSU(2)	P^RRS^RU	= $(8e, 9bdf)^R$, $(7de, 8df)^L$, (9aeg)	= RSrU
43		RS^RU	= $(9bdf)^R$, $(8bdf)^L$, (8e, 9aeg)	= PsRSrU
44	RUX(2)	RS^RUX^R	= $(9bd, 10cd)^R$, $(8bdf, 9c)^L$, (9aeg)	= RUXr
45		RUX^R	= $(10cd)^R$, $(9cf)^L$, (9abdeg)	= RSsUXr
46	RWX(1)	RWX^R	= $(10cd)^R$, $(9cf)^L$, (9abe, 10bg)	= RWXr
47	RX(3)	$Q^RRS^RX^R$	= $(8c, 9bd, 10cd)^R$, $(7bc, 8bd, 9c)^L$, (9ae)	= RXr
48		RS^RX^R	= $(9bd, 10cd)^R$, $(8bdf, 9c)^L$, (8c, 9ae)	= QsRXr
49		RX^R	= $(10cd)^R$, $(9cf)^L$, (8c, 9abde)	= RSsXr
50	S(4)	$P^RQ^RS^R$	= $(8ce, 9bdf)^R$, $(7bcde, 8d)^L$, (8a)	= Sr
51		P^RS^R	= $(8e, 9bdf)^R$, $(7de, 8df)^L$, (8ac)	= QsSr
52		Q^RS^R	= $(8c, 9bdf)^R$, $(7bc, 8bd)^L$, (8ae)	= PsSr
53		S^R	= $(9bdf)^R$, $(8bdf)^L$, (8ace)	= PsQsSr
54	ST(2)	Q^RS^RT	= $(8c, 9bdf)^R$, $(7bc, 8bd)^L$, (8a, 9c)	= SrT
55		S^RT	= $(9bdf)^R$, $(8bdf)^L$, (8ac, 9c)	= QsSrT
56	STU(1)	S^RTU	= $(9bdf)^R$, $(8bdf)^L$, (8a, 9cg)	= SrTU
57	SU(2)	P^RS^RU	= $(8e, 9bdf)^R$, $(7de, 8df)^L$, (8a, 9g)	= SrU
58		S^RU	= $(9bdf)^R$, $(8bdf)^L$, (8ae, 9g)	= PsSrU
59	TUV(2)	S^RTUV^R	= $(9df, 10aef)^R$, $(8bdf, 9e)^L$, (9acg)	= TUVr
60		TUV^R	= $(10aef)^R$, $(9be)^L$, (9acdfg)	= SsTUVr
61	TV(3)	$Q^RS^RTV^R$	= $(8c, 9df, 10aef)^R$, $(7bc, 8bd, 9e)^L$, (9ac)	= TVr
62		S^RTV^R	= $(9df, 10aef)^R$, $(8bdf, 9e)^L$, (8c, 9ac)	= QsTVr
63		TV^R	= $(10aef)^R$, $(9be)^L$, (8c, 9acdf)	= SsTVr
64	TVW(1)	TV^RW	= $(10aef)^R$, $(9be)^L$, (9acf, 10bg)	= TVrW
65	TZ (2)	TV^RZ^R	= $(10af, 11be)^R$, $(9be, 10b)^L$, (9acf, 10g)	= TZr
66		TZ^R	= $(11be)^R$, $(10be)^L$, (9acf, 10afg)	= TVsZr
67	UV(3)	$P^RS^RUV^R$	= $(8e, 9df, 10aef)^R$, $(7de, 8df, 9e)^L$, (9ag)	= UVr
68		S^RUV^R	= $(9df, 10aef)^R$, $(8bdf, 9e)^L$, (8e, 9ag)	= PsUVr
69		UV^R	= $(10aef)^R$, $(9be)^L$, (8e, 9adfg)	= SsUVr

70	UVX(5)	$S^RUV^RX^R$	= (9d, 10acdef)^R, (8bdf, 9ce)^L, (9ag)	= UVrXr
71		UV^RX^R	= (10acdef)^R, (9bcef)^L, (9dg)	= SsUVrXr
72		UV^LX^R	= (9be, 10cd)^R, (9cf, 10aef)^L, (9adg)	= UVsXr
73		UVX^R	= (10cd)^R, (9cf)^L, (9adg, 10aef)	= UVXr
74		UV^RX	= (10aef)^R, (9be)^L, (9adg, 10cd)	= UVrX
75	UX(2)	S^RUX^R	= (9bd, 10cd)^R, (8bdf, 9c)^L, (8a, 9g)	= UXr
76		UX^R	= (10cd)^R, (9cf)^L, (8a, 9bdg)	= SsUXr
77	V(5)	$P^RQ^RS^RV^R$	= (8ce, 9df, 10aef)^R, (7bcde, 8d, 9e)^L, (9a)	= Vr
78		$P^RS^RV^R$	= (8e, 9df, 10aef)^R, (7de, 8df, 9e)^L, (8c, 9a)	= QsVr
79		$Q^RS^RV^R$	= (8c, 9df, 10aef)^R, (7bc, 8bd, 9e)^L, (8e, 9a)	= PsVr
80		S^RV^R	= (9df, 10aef)^R, (8bdf, 9e)^L, (8ce, 9a)	= PsQsVr
81		V^R	= (10aef)^R, (9be)^L, (8ce, 9adf)	= SsVr
82	VW(1)	V^RW	= (10aef)^R, (9be)^L, (8e, 9af, 10bg)	= VrW
83	VWX(4)	V^RWX^R	= (10acdef)^R, (9bcef)^L, (9a, 10bg)	= VrWXr
84		V^LWX^R	= (9be, 10cd)^R, (9cf, 10aef)^L, (9a, 10bg)	= VsWXr
85		VWX^R	= (10cd)^R, (9cf)L, (9a, 10abefg)	= VWXr
86		V^RWX	= (10aef)^R, (9be)L, (9a, 10bcdg)	= VrWX
87	VX(6)	$Q^RS^RV^RX^R$	= (8c, 9d, 10acdef)^R, (7bc, 8bd, 9ce)^L, (9a)	= VrXr
88		$S^RV^RX^R$	= (9d, 10acdef)^R, (8bdf, 9ce)^L, (8c, 9a)	= QsVrXr
89		V^RX^R	= (10acdef)^R, (9bcef)^L, (8c, 9ad)	= SsVrXr
90		V^LX^R	= (9be, 10cd)^R, (9cf, 10aef)^L, (8c, 9ad)	= VsXr
91		VX^R	= (10cd)^R, (9cf)^L, (8c, 9ad, 10aef)	= VXr
92		V^RX	= (10aef)^R, (9be)^L, (8c, 9ad, 10cd)	= VrX
93	$V_2A(1)$	$V^R{}_2A$	= (10aef)^R, (9be)^L, (9a, 10bc, 11cf)	= Vr2A
94	WX(1)	WX^R	= (10cd)^R, (9cf)^L, (8a, 9b, 10bg)	= WXr
95	X(3)	$Q^RS^RX^R$	= (8c, 9bd, 10cd)^R, (7bc, 8bd, 9c)^L, (8a)	= Xr
96		S^RX^R	= (9bd, 10cd)^R, (8bdf, 9c)^L, (8ac)	= QsXr
97		X^R	= (10cd)^R, (9cf)^L, (8ac, 9bd)	= SsXr
98	XZ(7)	$V^RX^RZ^R$	= (10acdf, 11be)^R, (9bcef, 10b)^L, (9a, 10g)	= XrZr
99		$V^RX^LZ^R$	= (9cf, 10af, 11be)^R, (9be, 10bcd)^L, (9a, 10g)	= XsZr
100		V^RXZ^R	= (10af, 11be)^R, (9be, 10b)^L, (9a, 10cdg)	= XZr
101		X^RZ^R	= (10cd, 11be)^R, (9cf, 10be)^L, (9a, 10afg)	= VsXrZr
102		X^LZ^R	= (9cf, 11be)^R, (10bcde)^L, (9a, 10afg)	= VsXsZr
103		XZ^R	= (11be)^R, (10be)^L, (9a, 10acdfg)	= VsXZr
104		X^RZ	= (10cd) ^{R}, (9cf)^L, (9a, 10afg, 11be)	= XrZ
105	YZ(1)	YZ^R	= (11be)^R, (10be)^L, (9a, 10adg, 11ad)	= YZr
106	$YZ_2A(1)$	$YZ^R{}_2A$	= (11be)^R, (10be)^L, (9a, 10a, 11acdf)	= YZr2A
107	Z(2)	V^RZ^R	= (10af, 11be)^R, (9be, 10b)^L, (8e, 9af, 10g)	= Zr
108		Z^R	= (11be)^R, (10be)^L, (8e, 9af, 10afg)	= VsZr
109	$Z_2A(2)$	$V^RZ^R{}_2A$	= (10af, 11be)^R, (9be, 10b)^L, (9a, 10c, 11cf)	= Zr2A
110		$Z^R{}_2A$	= (11be)^R, (10be)^L, (9a, 10acf, 11cf)	= VsZr2A
111	$_2B(2)$	$Z^R{}_2B^R$	= (11b, 12abc)^R, (10be, 11cd)^L, (9a, 10a, 11af)	= 2Br
112		$_2B^R$	= (12abc) ^{R}, (11cde)^L, (9a, 10a, 11abf)	= Zs2Br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