TOI-OHOMAI Institute of Technology	COMP.5202 Fundamentals of Programming and Problem Solving	Mathematics: Matrices

Definitions

Matrix

A matrix is an ordered set of numbers listed in a rectangular form.

Example:

We shall call the matrix shown below as A:

$$A = \begin{bmatrix} 2 & 5 \\ 7 & 8 \end{bmatrix}$$ elements

This matrix A has two rows and two columns. We say it is a 2 x 2 matrix.

We can also denote (or reference) the elements in the matrix as following:

- A [1,2] = 5
- $a_{1,2} = 5$

[1,2] is a reference to the element in row 1 and column 2 that has the element 5 in it.

Referencing Rows and Columns

To be able to refer to specific elements in a matrix, we use a "double" subscript notation like this:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & \boxed{a_{23}} \end{bmatrix} \leftarrow \text{Element: Row 2, Column}$$

Square Matrix

The number of columns and rows in a matrix do not have to be the same but a matrix that has the same number of rows and columns such as in example 1 above, is called a square matrix.

Example:

$$\begin{pmatrix} 3 & -1 \\ 1 & 2 \\ 6 & 1 \end{pmatrix}$$ **is <u>NOT</u> a square matrix**

Identity Matrix (I)

A square matrix that has a 1 for each element on the main diagonal and 0 for all other elements.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Row Matrix

A matrix with one row is called a row matrix.

[2 5 -1 5]

Column Matrix

A matrix with one column is called a column matrix.

$$\begin{bmatrix} 2.9 \\ -4.6 \\ 0.0 \end{bmatrix}$$

Each number of the column matrix is called an <u>element</u>. The numbers are *real numbers*. The number of elements in a vector is called its <u>dimension</u>.

Zero Matrix

If all the elements in the matrix are zero, it is called a “Zero matrix”.

Equality of Matrices

Two matrices are said to be equal if and only if they are **IDENTICAL**.

This means, they must have the same number of columns, the same number of rows and the elements must respectively be equal.

Therefore if

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} 1 & -5 \\ 4 & 6 \end{bmatrix}$$

Then the following must be true:

$a_{11} = 1$

$a_{12} = -5$

$a_{21} = 4$

$a_{22} = 6$

Matrix Addition and Matrix Subtraction

How to Add Matrices

Two matrices may be added **only** if they have the same dimension; that is, they must have the same number of rows and columns.

Addition is accomplished by adding corresponding elements.

Example: Consider matrix **A** and matrix **B**.

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 7 & 8 & 9 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 5 & 6 & 7 \\ 3 & 4 & 5 \end{bmatrix}$$

Both matrices have the same number of rows and columns (2 rows and 3 columns), so they can be added. Thus:

$$\mathbf{A} + \mathbf{B} = \begin{bmatrix} 1+5 & 2+6 & 3+7 \\ 7+3 & 8+4 & 9+5 \end{bmatrix} = \begin{bmatrix} \mathbf{6} & \mathbf{8} & \mathbf{10} \\ \mathbf{10} & \mathbf{12} & \mathbf{14} \end{bmatrix}$$

And finally, note that the order in which matrices are added is not important.
Thus **A** + **B** = **B** + **A**.

How to Subtract Matrices

Subtraction is accomplished by subtracting corresponding elements.

Example: Consider matrix **A** and matrix **B**.

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 7 & 8 & 9 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 5 & 6 & 7 \\ 3 & 4 & 5 \end{bmatrix}$$

Both matrices have the same number of rows and columns (2 rows and 3 columns), so they can be subtracted. Thus:

$$\mathbf{A} - \mathbf{B} = \begin{bmatrix} 1-5 & 2-6 & 3-7 \\ 7-3 & 8-4 & 9-5 \end{bmatrix} = \begin{bmatrix} \mathbf{-4} & \mathbf{-4} & \mathbf{-4} \\ \mathbf{4} & \mathbf{4} & \mathbf{4} \end{bmatrix}$$

Exercise:

Consider the matrices shown below: **A**, **B**, **C**, and **D**

A = $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ **B** = $\begin{bmatrix} 3 & 5 \\ 4 & 6 \end{bmatrix}$ **C** = $\begin{bmatrix} 4 & 5 \\ 6 & 6 \end{bmatrix}$ **D** = $\begin{bmatrix} -1 & 0 \\ -2 & 0 \end{bmatrix}$

1. A + B
2. C - D
3. B + D
4. B – D
5. B + C
6. Add the following matrices

   a.

   $$\begin{bmatrix} 2 & 3 \\ -5 & 4 \end{bmatrix} + \begin{bmatrix} -1 & 7 \\ 5 & -2 \end{bmatrix}$$

   b.

   $$\begin{bmatrix} 1 & 0 & 9 \\ 3 & -5 & -2 \end{bmatrix} + \begin{bmatrix} 4 & -1 & 7 \\ 2 & 0 & -3 \end{bmatrix}$$

7. Determine the values of the letters in the following matrices:

a.

$$\begin{bmatrix} a & b \\ x & y \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ 4 & 7 \end{bmatrix}$$

b.

$$\begin{bmatrix} x & 2y & z \\ b/4 & -a & -5k \end{bmatrix} = \begin{bmatrix} -2 & 10 & -9 \\ 12 & -4 & 5 \end{bmatrix}$$

c.

$$\begin{bmatrix} 2x - 3 \\ x + 4 \end{bmatrix} = \begin{bmatrix} 13 \\ 1 \end{bmatrix}$$

Matrix Multiplication

In matrix algebra, there are two kinds of matrix multiplication:

- Multiplication of a matrix by a number
- Multiplication of a matrix by another matrix.

How to Multiply a Matrix by a Number

When you multiply a matrix by a number, you multiply every element in the matrix by the same number. This operation produces a new matrix, which is called a **scalar multiple**.

Example:

If the matrix **A** is: $\begin{bmatrix} 100 & 200 \\ 300 & 400 \end{bmatrix}$

Then **5A** would look like this

$$A= \begin{bmatrix} 100 & 200 \\ 300 & 400 \end{bmatrix} \quad 5A= \begin{bmatrix} 5 * 100 & 5 * 200 \\ 5 * 300 & 5 * 400 \end{bmatrix} = \begin{bmatrix} 500 & 1000 \\ 1500 & 2000 \end{bmatrix}$$

In the example above, every element of **A** is multiplied by 5 to produce the scalar multiple, which we can call **B**.

$$B = \begin{bmatrix} 500 & 1000 \\ 1500 & 2000 \end{bmatrix}$$

How to Multiply a Matrix by a Matrix

- The matrix product **AB** is defined only when the number of columns in **A** is equal to the number of rows in **B**.
- Similarly, the matrix product **BA** is defined only when the number of columns in **B** is equal to the number of rows in **A**.

Thus the matrix product **AB** results in a matrix **C**.

<u>Example 2</u>: Suppose we want to compute **AB**, given the matrices below.

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 2 \\ 3 & 4 & 5 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 6 & 7 \\ 8 & 9 \\ 10 & 11 \end{bmatrix}$$

To give us C **(AB** = **C**)

Because **A** has 2 rows, we know that **C** will have two rows; and because **B** has 2 columns, we know that **C** will have 2 columns.

To compute the value of every element in the 2 x 2 matrix **C**, we use the sum shown below.

All you have to do is multiply row elements in Matrix **A** by corresponding column elements in Matrix **B**.

- ***C* = 0*6 + 1*8 +2*10 = 0 + 8 + 20 = 28** *(row 1 of A * Col 1 of B)*
- ***C* = 0*7 + 1*9 +2*11 = 0 + 9 + 22 = 31** *(row 1 of A * Col 2 of B)*
- ***C* = 3*6 + 4*8 +5*10 = = 18 + 32 + 50 = 100** *(row 2 of A * Col 1 of B)*
- ***C* = 3*7 + 4*9 +5*11 = 21 + 36 +55 = 112** *(row 2 of A * Col 2 of B)*

Based on the above calculations, we can say

$$\mathbf{AB} = \mathbf{C} = \begin{bmatrix} 28 & 31 \\ 100 & 112 \end{bmatrix}$$

How to Multiply 2 × 2 Matrices

The process is the same for any size matrix. We multiply **across** rows of the first matrix and **down** columns of the second matrix, element by element.

We then add the products:

Example 1:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{pmatrix}$$

In this case, we multiply a 2 × 2 matrix by a 2 × 2 matrix and we get a 2 × 2 matrix as the result.

Example 2:

Multiply:

$$\begin{pmatrix} 8 & 9 \\ 5 & -1 \end{pmatrix}\begin{pmatrix} -2 & 3 \\ 4 & 0 \end{pmatrix}$$

Answer:

$$\begin{aligned}\begin{pmatrix} 8 & 9 \\ 5 & -1 \end{pmatrix}\begin{pmatrix} -2 & 3 \\ 4 & 0 \end{pmatrix} &= \begin{pmatrix} 8\times -2 \; + \; 9\times 4 & 8\times 3 \; + \; 9\times 0 \\ 5\times -2 \; + \; -1\times 4 & 5\times 3 \; + \; -1\times 0 \end{pmatrix} \\ &= \begin{pmatrix} -16+36 & 24+0 \\ -10+-4 & 15+0 \end{pmatrix} \\ &= \begin{pmatrix} 20 & 24 \\ -14 & 15 \end{pmatrix}\end{aligned}$$

Multiplying by the Identity Matrix (*I*)

For the matrix *A*, find *AI (ie. A x I)*

$$A = \begin{pmatrix} -3 & 1 & 6 \\ 3 & -1 & 0 \\ 4 & 2 & 5 \end{pmatrix}$$

$$\begin{aligned} AI &= \begin{pmatrix} -3 & 1 & 6 \\ 3 & -1 & 0 \\ 4 & 2 & 5 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} -3+0+0 & 0+1+0 & 0+0+6 \\ 3+0+0 & 0+-1+0 & 0+0+0 \\ 4+0+0 & 0+2+0 & 0+0+5 \end{pmatrix} \\ &= \begin{pmatrix} -3 & 1 & 6 \\ 3 & -1 & 0 \\ 4 & 2 & 5 \end{pmatrix} \\ &= A \end{aligned}$$

The Identity Matrix I, has all 1's in the major diagonal.

We see that multiplying by the identity matrix does not change the value of the original matrix.

Therefore, ***AI = A***

Evaluating Matrix Multiplications

Does *AB* = *BA*?

Example:

$$A = \begin{pmatrix} 0 & -1 & 2 \\ 4 & 11 & 2 \end{pmatrix} \qquad B = \begin{pmatrix} 3 & -1 \\ 1 & 2 \\ 6 & 1 \end{pmatrix}$$

Therefore:

$$AB = \begin{pmatrix} 11 & 0 \\ 35 & 20 \end{pmatrix}$$

And *BA* is (3 × 2)(2 × 3) which will give a 3 × 3 matrix:

$$\begin{aligned} BA &= \begin{pmatrix} 3 & -1 \\ 1 & 2 \\ 6 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 & 2 \\ 4 & 11 & 2 \end{pmatrix} \\ &= \begin{pmatrix} 0-4 & -3-11 & 6-2 \\ 0+8 & -1+22 & 2+4 \\ 0+4 & -6+11 & 12+2 \end{pmatrix} \\ &= \begin{pmatrix} -4 & -14 & 4 \\ 8 & 21 & 6 \\ 4 & 5 & 14 \end{pmatrix} \end{aligned}$$

So in this case, *AB* does NOT equal *BA.*

In fact, for most matrices, you cannot reverse the order of multiplication and get the same result.

Exercises:

Consider the matrices shown below: **A**, **B**, and **C**

A = $\begin{bmatrix} 8 & 5 \\ 2 & 3 \end{bmatrix}$ **B** = $\begin{bmatrix} 5 & 3 & 1 \\ 4 & 7 & 2 \end{bmatrix}$ **C** = $\begin{bmatrix} 6 & 5 \\ 1 & 8 \end{bmatrix}$ **D** = $\begin{bmatrix} 4 & 2 \\ 3 & 7 \end{bmatrix}$

Solve the following:

a) 3A

b) 2C

c) A * C

d) B * C

e) C * D

f) A * D

g) A – 2D

h) 3A – C

i) -4A

j) 2A + C

k) 2C + A

Online Matrix Calculators:

There is a wide range of matrix calculators:

https://matrixcalc.org/en/

https://www.mathportal.org/calculators/matrices-calculators/matrix-operations-calculator.php

https://www.desmos.com/matrix