ASEN 5044 Spring 2020 - Project Progress Report 1

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Our group has decided to solve the state estimation problem for system A - Cooperative Air-Ground Robot Localization

Please note that for a and b, we double checked our work.

a)
First, let us set up the CT system

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$$x = \begin{bmatrix} \varepsilon_g \\ \eta_g \\ \theta_g \\ \varepsilon_a \\ \eta_a \\ \theta_a \end{bmatrix} \qquad \dot{x} = \begin{bmatrix} u_1 \cos(x_3) + \tilde{\omega}_1 \\ u_1 \sin(x_3) + \tilde{\omega}_2 \\ \frac{u_1}{L} \tan(u_2) + \tilde{\omega}_3 \\ u_3 \cos(x_6) + \tilde{\omega}_4 \\ u_3 \sin(x_6) + \tilde{\omega}_5 \\ u_4 + \tilde{\omega}_6 \end{bmatrix}$$

$$u = \begin{bmatrix} v_g \\ \phi_g \\ v_a \\ \omega_a \end{bmatrix} \qquad \tilde{\omega} = \begin{bmatrix} \omega_{xg} \\ \omega_{yg} \\ \omega_{zg} \\ \omega_{xa} \\ \omega_{ya} \\ \omega_{za} \end{bmatrix}$$

$$y = \begin{bmatrix} \tan^{-1}\left(\frac{x_5 - x_2}{x_4 - x_1}\right) - x_3 \\ \left[(x_1 - x_4)^2 + (x_2 - x_5)^2\right]^{\frac{1}{2}} \\ \tan^{-1}\left(\frac{x_2 - x_5}{x_1 - x_4}\right) - x_6 \\ x_4 \\ x_5 \end{bmatrix} + \tilde{v}(t)$$

Now let's compute the Jacobians by taking the partial derivative with respect to vectors x and u

$$\tilde{A} = \frac{\partial \dot{x}}{\partial x} = \begin{bmatrix} 0 & 0 & -u_1 \sin(x_3) & 0 & 0 & 0 \\ 0 & 0 & u_1 \cos(x_3) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -u_3 \sin(x_6) \\ 0 & 0 & 0 & 0 & 0 & u_3 \cos(x_6) \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\tilde{B} = \frac{\partial \dot{x}}{\partial u} = \begin{bmatrix} \cos(x_3) & 0 & 0 & 0 \\ \sin(x_3) & 0 & 0 & 0 \\ \frac{\tan(u_2)}{L} & \frac{u_1}{L} \cdot \frac{1}{\cos^2(u_2)} & 0 & 0 \\ 0 & 0 & \cos(x_6) & 0 \\ 0 & 0 & \sin(x_6) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\frac{\partial \dot{x}}{\partial \tilde{\omega}} = I_6 \qquad \frac{\partial y}{\partial u} = 0 \qquad \frac{\partial y}{\partial \tilde{v}} = I_5$$

for $\tilde{A}$, we use $\frac{d}{dx} \sin(x) = \cos(x)$

and $\frac{d}{dx} \cos(x) = -\sin(x)$.

We use the following definitions to compute C~

To compute $\frac{\partial y}{\partial x}$ we use

$$\frac{d}{dx}\tan^{-1}(x) = \frac{1}{x^2+1}$$

and
$$\frac{d}{dx}\frac{1}{a-x} = (a-x)^{-2}$$

$$\frac{d}{dx}\frac{1}{x+a} = -(x+a)^{-2}$$

$$\Rightarrow \frac{d}{dx_1}\tan^{-1}\left(\frac{x_5-x_2}{x_4-x_1}\right) - \cancel{x_3}^{\,0}$$

$$= \frac{1}{\left[\frac{x_5-x_2}{x_4-x_1}\right]^2+1}\cdot(x_5-x_2)\cdot(x_4-x_1)^{-2}$$

$$= \frac{\frac{x_5-x_2}{(x_4-x_1)^2}}{\left[\frac{x_5-x_2}{x_4-x_1}\right]^2+1}$$

Similarly
$$\frac{d}{dx_5}\tan^{-1}\left[\frac{x_5-x_2}{x_4-x_1}\right] - \cancel{x_3}^{\,0}$$

$$= \frac{1}{\left[\frac{x_5-x_2}{x_4-x_1}\right]^2+1}\cdot\frac{1}{(x_4-x_1)}$$

using the fact that $\frac{x_5 - x_2}{x_4 - x_1} \equiv \frac{x_2 - x_5}{x_1 - x_4}$ (4)

we can apply the same diff process to row 3 of $y(t)$ to find $\frac{\partial y}{\partial x}$.

Next we note:

$$((x_1 - x_4)^2 + (x_2 - x_5)^2)^{\frac{1}{2}}$$

can be differentiated with the chain rule for any of x_1 x_4 x_2 or x_5

the derivative of the outsid fn is THE SAME for any of these particals. w/ resp to x_1, x_2, x_4, x_5

The derivicte is

$$\frac{1}{2}((x_1 - x_4)^2 + (x_2 - x_5)^2)^{-\frac{1}{2}} \cdot \frac{\partial(\text{inside})}{\partial x_i}$$

let $P = \frac{1}{2}((x_1 - x_4)^2 + (x_2 - x_5)^2)^{-\frac{1}{2}}$ let $K = \frac{x_5 - x_2}{x_4 - x_1}$ (5)

$$\frac{\partial y}{\partial x} = \begin{bmatrix} \frac{K}{(K^2+1)(x_4-x_1)} & \frac{-1}{(K^2+1)(x_4-x_1)} & -1 & \frac{-K}{(K^2+1)(x_4-x_1)} & \frac{1}{(K^2+1)(x_4-x_1)} & 0 \\ 2P(x_1-x_4) & 2P(x_2-x_5) & 0 & -2P(x_1-x_4) & -2P(x_2-x_5) & 0 \\ \frac{K}{(K^2+1)(x_4-x_1)} & \frac{-1}{(K^2+1)(x_4-x_1)} & 0 & \frac{-K}{(K^2+1)(x_4-x_1)} & \frac{1}{(K^2+1)(x_4-x_1)} & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

We will simply get identity matrices (for jacobians) when we take partial derivatives with respect to noise vectors.

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a) Given that:

$$x(t) = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} \xi_g \\ \eta_g \\ \theta_g \\ \xi_a \\ \eta_a \\ \theta_a \end{bmatrix}_{6\times1} ; \quad u(t) = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} v_g \\ \phi_g \\ v_a \\ \omega_a \end{bmatrix}_{4\times1}$$

$$\tilde{\omega}(t) = \begin{bmatrix} \tilde{\omega}_1 \\ \tilde{\omega}_2 \\ \tilde{\omega}_3 \\ \tilde{\omega}_4 \\ \tilde{\omega}_5 \\ \tilde{\omega}_6 \end{bmatrix} = \begin{bmatrix} \tilde{\omega}_{x,g} \\ \tilde{\omega}_{y,g} \\ \tilde{\omega}_{\omega,g} \\ \tilde{\omega}_{x,a} \\ \tilde{\omega}_{y,a} \\ \tilde{\omega}_{\omega,a} \end{bmatrix}_{6\times1}$$

$$\dot{x}(t) = f\left[x(t), u(t), \tilde{\omega}(t)\right]$$

$$= \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_5 \\ \dot{x}_6 \end{bmatrix} = \begin{bmatrix} \dot{\xi}_g \\ \dot{\eta}_g \\ \dot{\theta}_g \\ \dot{\xi}_a \\ \dot{\eta}_a \\ \dot{\theta}_a \end{bmatrix} = \begin{bmatrix} v_g \cos\theta_g + \tilde{\omega}_{x,g} \\ v_g \sin\theta_g + \tilde{\omega}_{y,g} \\ (v_g \tan\phi_g)/L + \tilde{\omega}_{\omega,g} \\ v_a \cos\theta_a + \tilde{\omega}_{x,a} \\ v_a \sin\theta_a + \tilde{\omega}_{y,a} \\ \omega_a + \tilde{\omega}_{\omega,a} \end{bmatrix}_{6\times1}$$

$$\therefore \dot{x}(t) = \mathcal{F}\left[x(t), u(t), \tilde{\omega}(t)\right]$$

$$= \begin{bmatrix} u_1 \cos(x_3) + \tilde{\omega}_1 \\ u_1 \sin(x_3) + \tilde{\omega}_2 \\ \frac{u_1}{L} \tan(\cancel{u_2}) + \tilde{\omega}_3 \\ u_3 \cos(x_6) + \tilde{\omega}_4 \\ u_3 \sin(x_6) + \tilde{\omega}_5 \\ u_4 + \tilde{\omega}_6 \end{bmatrix}_{6\times 1} = \begin{bmatrix} \mathcal{F}_1 \\ \mathcal{F}_2 \\ \mathcal{F}_3 \\ \mathcal{F}_4 \\ \mathcal{F}_5 \\ \mathcal{F}_6 \end{bmatrix}$$

Also,

$$y(t) = \mathcal{H}\left[x(t), \tilde{v}(t)\right]$$

$$= \begin{bmatrix} \tan^{-1}\left(\frac{\eta_a - \eta_g}{\xi_a - \xi_g}\right) - \theta_g \\ \sqrt{(\xi_g - \xi_a)^2 + (\eta_g - \eta_a)^2} \\ \tan^{-1}\left(\frac{\eta_g - \eta_a}{\xi_g - \xi_a}\right) - \theta_a \\ \xi_a \\ \eta_a \end{bmatrix}_{5\times 1} + \tilde{v}(t) = \begin{bmatrix} \mathcal{H}_1 \\ \mathcal{H}_2 \\ \mathcal{H}_3 \\ \mathcal{H}_4 \\ \mathcal{H}_5 \end{bmatrix}$$

$\tilde{v}(t) \rightarrow$ AWGN $\in \mathbb{R}^5$

$$\therefore y(t) = \mathcal{H}\left[x(t), \tilde{v}(t)\right]$$

$$= \begin{bmatrix} \tan^{-1}\left(\dfrac{x_5 - x_2}{x_4 - x_1}\right) - x_3 \\ \sqrt{(x_1 - x_4)^2 + (x_2 - x_5)^2} \\ \tan^{-1}\left(\dfrac{x_2 - x_5}{x_1 - x_4}\right) - x_6 \\ x_4 \\ x_5 \end{bmatrix} + \tilde{v}(t) = \begin{bmatrix} \mathcal{H}_1 \\ \mathcal{H}_2 \\ \mathcal{H}_3 \\ \mathcal{H}_4 \\ \mathcal{H}_5 \end{bmatrix}$$

We have been asked to calculate the CT Jacobians $\tilde{A}$, $\tilde{B}$

① $\dfrac{\partial \mathcal{F}[x(t), u(t), \tilde{w}(t)]}{\partial x(t)}$

$$= \begin{bmatrix} \dfrac{\partial \mathcal{F}_1}{\partial x_1} & \dfrac{\partial \mathcal{F}_1}{\partial x_2} & \cdots & \dfrac{\partial \mathcal{F}_1}{\partial x_6} \\ \vdots & & & \\ \dfrac{\partial \mathcal{F}_6}{\partial x_1} & \dfrac{\partial \mathcal{F}_6}{\partial x_2} & \cdots & \dfrac{\partial \mathcal{F}_6}{\partial x_6} \end{bmatrix}$$

$$\mathcal{F}_1 = u_1 \cos(x_3) + \tilde{w}_1$$

$$\therefore \frac{\partial \mathcal{F}_1}{\partial x_1} = \frac{\partial \mathcal{F}_1}{\partial x_2} = \frac{\partial \mathcal{F}_1}{\partial x_4} = \frac{\partial \mathcal{F}_1}{\partial x_5} = \frac{\partial \mathcal{F}_1}{\partial x_6} = 0$$

$$\frac{\partial \mathcal{F}_1}{\partial x_3} = -u_1 \sin(x_3)$$

$F_2 = u_1 \sin(x_3) + \tilde{\omega}_2$

$$\therefore \frac{\partial F_2}{\partial x_1} = \frac{\partial F_2}{\partial x_2} = \frac{\partial F_2}{\partial x_4} = \frac{\partial F_2}{\partial x_5} = \frac{\partial F_2}{\partial x_6} = 0$$

$$\frac{\partial F_2}{\partial x_3} = u_1 \cos(x_3)$$

$F_3 = \frac{u_1}{L} \tan(u_2) + \tilde{\omega}_3$

$$\therefore \frac{\partial F_3}{\partial x_1} = \frac{\partial F_3}{\partial x_2} = \frac{\partial F_3}{\partial x_3} = \frac{\partial F_3}{\partial x_4} = \frac{\partial F_3}{\partial x_5} = \frac{\partial F_3}{\partial x_6} = 0$$

$F_4 = u_3 \cos(x_6) + \tilde{\omega}_4$

$$\therefore \frac{\partial F_4}{\partial x_1} = \frac{\partial F_4}{\partial x_2} = \frac{\partial F_4}{\partial x_3} = \frac{\partial F_4}{\partial x_4} = \frac{\partial F_4}{\partial x_5} = 0$$

$$\frac{\partial F_4}{\partial x_6} = -u_3 \sin(x_6)$$

$F_5 = u_3 \sin(x_6) + \tilde{\omega}_5$

$$\therefore \frac{\partial F_5}{\partial x_1} = \frac{\partial F_5}{\partial x_2} = \frac{\partial F_5}{\partial x_3} = \frac{\partial F_5}{\partial x_4} = \frac{\partial F_5}{\partial x_5} = 0$$

$$\frac{\partial F_5}{\partial x_6} = u_3 \cos(x_6)$$

$F_6 = u_4 + \tilde{\omega}_6$

$$\therefore \frac{\partial F_6}{\partial x_1} = \frac{\partial F_6}{\partial x_2} = \frac{\partial F_6}{\partial x_3} = \frac{\partial F_6}{\partial x_4} = \frac{\partial F_6}{\partial x_5} = \frac{\partial F_6}{\partial x_6} = 0$$

$$\therefore \frac{\partial F[x(t), u(t), \tilde{\omega}(t)]}{\partial x(t)} = \begin{bmatrix} 0 & 0 & -u_1 \sin(x_3) & 0 & 0 & 0 \\ 0 & 0 & u_1 \cos(x_3) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -u_3 \sin(x_6) \\ 0 & 0 & 0 & 0 & 0 & u_3 \cos(x_6) \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}_{6 \times 6}$$

It's dimensions are 6×6

(2) $$\frac{\partial F[x(t), u(t), \tilde{\omega}(t)]}{\partial u(t)} = \begin{bmatrix} \frac{\partial F_1}{\partial u_1} & \frac{\partial F_1}{\partial u_2} & \cdots & \frac{\partial F_1}{\partial u_4} \\ \vdots & & & \\ \frac{\partial F_6}{\partial u_1} & \frac{\partial F_6}{\partial u_2} & \cdots & \frac{\partial F_6}{\partial u_4} \end{bmatrix}$$

$F_1 = u_1 \cos(x_3) + \tilde{\omega}_1$

$\therefore \frac{\partial F_1}{\partial u_1} = \cos x_3 \qquad \frac{\partial F_1}{\partial u_2} = \frac{\partial F_1}{\partial u_3} = \frac{\partial F_1}{\partial u_4} = 0$

$F_2 = u_1 \sin(x_3) + \tilde{\omega}_2$

$\therefore \frac{\partial F_2}{\partial u_1} = \sin x_3 \qquad \frac{\partial F_2}{\partial u_2} = \frac{\partial F_2}{\partial u_3} = \frac{\partial F_2}{\partial u_4} = 0$

$F_3 = \frac{u_1}{L} \tan(u_2) + \tilde{\omega}_3$

$\therefore \frac{\partial F_3}{\partial u_1} = \frac{\tan(u_2)}{L} \qquad \frac{\partial F_3}{\partial u_2} = \frac{u_1}{L} \sec^2(u_2) \qquad \frac{\partial F_3}{\partial u_3} = \frac{\partial F_3}{\partial u_4} = 0$

$$F_4 = u_3 \cos(x_6) + \tilde{w}_4$$

$$\therefore \frac{\partial F_4}{\partial u_1} = \frac{\partial F_4}{\partial u_2} = \frac{\partial F_4}{\partial u_4} = 0 \qquad \frac{\partial F_4}{\partial u_3} = \cos(x_6)$$

$$F_5 = u_3 \sin(x_6) + \tilde{w}_5$$

$$\therefore \frac{\partial F_5}{\partial u_1} = \frac{\partial F_5}{\partial u_2} = \frac{\partial F_5}{\partial u_4} = 0 \qquad \frac{\partial F_5}{\partial u_3} = \sin(x_6)$$

$$F_6 = u_4 + \tilde{w}_6$$

$$\therefore \frac{\partial F_6}{\partial u_1} = \frac{\partial F_6}{\partial u_2} = \frac{\partial F_6}{\partial u_3} = 0 \qquad \frac{\partial F_6}{\partial u_4} = 1$$

$$\therefore \frac{\partial F[x(t), u(t), \tilde{w}(t)]}{\partial u(t)} = \begin{bmatrix} \cos(x_3) & 0 & 0 & 0 \\ \sin(x_3) & 0 & 0 & 0 \\ \frac{\tan(u_2)}{L} & \frac{u_1 \sec^2(u_2)}{L} & 0 & 0 \\ 0 & 0 & \cos(x_6) & 0 \\ 0 & 0 & \sin(x_6) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

6×4

<u>Its dimensions are</u> 6×4

3

$$\frac{\partial F[x(t),u(t),\tilde{\omega}(t)]}{\partial \tilde{\omega}(t)} = \begin{bmatrix} \frac{\partial F_1}{\partial \tilde{\omega}_1} & \frac{\partial F_1}{\partial \tilde{\omega}_2} & \cdots & \frac{\partial F_1}{\partial \tilde{\omega}_6} \\ \vdots & & & \\ \frac{\partial F_6}{\partial \tilde{\omega}_1} & \frac{\partial F_6}{\partial \tilde{\omega}_2} & \cdots & \frac{\partial F_6}{\partial \tilde{\omega}_6} \end{bmatrix}$$

$F_1 = u_1 \cos(x_3) + \tilde{\omega}_1$

$\therefore \frac{\partial F_1}{\partial \tilde{\omega}_1} = 1 \qquad \frac{\partial F_1}{\partial \tilde{\omega}_2} = \frac{\partial F_1}{\partial \tilde{\omega}_3} = \frac{\partial F_1}{\partial \tilde{\omega}_4} = \frac{\partial F_1}{\partial \tilde{\omega}_5} = \frac{\partial F_1}{\partial \tilde{\omega}_6} = 0$

$F_2 = u_1 \sin(x_3) + \tilde{\omega}_2$

$\therefore \frac{\partial F_2}{\partial \tilde{\omega}_2} = 1 \qquad \frac{\partial F_2}{\partial \tilde{\omega}_1} = \frac{\partial F_2}{\partial \tilde{\omega}_3} = \frac{\partial F_2}{\partial \tilde{\omega}_4} = \frac{\partial F_2}{\partial \tilde{\omega}_5} = \frac{\partial F_2}{\partial \tilde{\omega}_6} = 0$

$F_3 = \frac{u_1}{L} \tan(u_2) + \tilde{\omega}_3$

$\therefore \frac{\partial F_3}{\partial \tilde{\omega}_3} = 1 \qquad \frac{\partial F_3}{\partial \tilde{\omega}_1} = \frac{\partial F_3}{\partial \tilde{\omega}_2} = \frac{\partial F_3}{\partial \tilde{\omega}_4} = \frac{\partial F_3}{\partial \tilde{\omega}_5} = \frac{\partial F_3}{\partial \tilde{\omega}_6} = 0$

$F_4 = u_3 \cos(x_6) + \tilde{\omega}_4$

$\therefore \frac{\partial F_4}{\partial \tilde{\omega}_4} = 1 \qquad \frac{\partial F_4}{\partial \tilde{\omega}_1} = \frac{\partial F_4}{\partial \tilde{\omega}_2} = \frac{\partial F_4}{\partial \tilde{\omega}_3} = \frac{\partial F_4}{\partial \tilde{\omega}_5} = \frac{\partial F_4}{\partial \tilde{\omega}_6} = 0$

$F_5 = u_3 \sin(x_6) + \tilde{\omega}_5$

$\therefore \frac{\partial F_5}{\partial \tilde{\omega}_5} = 1 \qquad \frac{\partial F_5}{\partial \tilde{\omega}_1} = \frac{\partial F_5}{\partial \tilde{\omega}_2} = \frac{\partial F_5}{\partial \tilde{\omega}_3} = \frac{\partial F_5}{\partial \tilde{\omega}_4} = \frac{\partial F_5}{\partial \tilde{\omega}_6} = 0$

$F_6 = u_4 + \tilde{\omega}_6$

$\therefore \frac{\partial F_6}{\partial \tilde{\omega}_6} = 1 \qquad \frac{\partial F_6}{\partial \tilde{\omega}_1} = \frac{\partial F_6}{\partial \tilde{\omega}_2} = \frac{\partial F_6}{\partial \tilde{\omega}_3} = \frac{\partial F_6}{\partial \tilde{\omega}_4} = \frac{\partial F_6}{\partial \tilde{\omega}_5} = 0$

$$\therefore \frac{\partial F[x(t),u(t),\tilde{\omega}(t)]}{\partial \tilde{\omega}(t)} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}_{6\times 6} = I_6$$

Its dimensions are 6×6

(4) $$\frac{\partial \mathcal{H}[x(t),\tilde{v}(t)]}{\partial u(t)} = 0$$ (since there are no $u(t)$ components in $\mathcal{H}[x(t), v(t)]$)

(5) We know $\mathcal{H}[x(t),\tilde{v}(t)]$ = Some complicated term independent of $\tilde{v}(t)$ + $\tilde{v}(t)$

$$\therefore \frac{\partial \mathcal{H}[x(t),\tilde{v}(t)]}{\partial \tilde{v}(t)} = \frac{\partial}{\partial \tilde{v}(t)}\left[\text{Some complicated term independent of } \tilde{v}(t)\right] + \frac{\partial \tilde{v}(t)}{\partial \tilde{v}(t)}$$

$$= 0 + \frac{\partial \tilde{v}(t)}{\partial \tilde{v}(t)}$$

Since $\tilde{v}(t) \in \mathbb{R}^5$, $\frac{\partial \tilde{v}(t)}{\partial \tilde{v}(t)} = I_5$

$$\therefore \frac{\partial \mathcal{H}[x(t),\tilde{v}(t)]}{\partial \tilde{v}(t)} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}_{5\times5} = I_5$$

Its dimensions are 5×5

(6) $$\frac{\partial \mathcal{H}[x(t), \tilde{v}(t)]}{\partial x(t)} = \begin{bmatrix} \frac{\partial \mathcal{H}_1}{\partial x_1} & \frac{\partial \mathcal{H}_1}{\partial x_2} & \cdots & \frac{\partial \mathcal{H}_1}{\partial x_6} \\ \vdots & & & \\ \frac{\partial \mathcal{H}_5}{\partial x_1} & \frac{\partial \mathcal{H}_5}{\partial x_2} & \cdots & \frac{\partial \mathcal{H}_5}{\partial x_6} \end{bmatrix}$$

$$\mathcal{H}_1 = \tan^{-1}\left(\frac{x_5 - x_2}{x_4 - x_1}\right) - x_3$$

(Note: $\frac{\partial \tan^{-1}(x)}{\partial x} = \frac{1}{1+x^2}$)

$$\therefore \frac{\partial \mathcal{H}_1}{\partial x_1} = \frac{-1 \times -1 \times \left(\frac{x_5 - x_2}{(x_4 - x_1)^2}\right)}{1 + \left(\frac{x_5 - x_2}{x_4 - x_1}\right)^2} = \frac{\frac{x_5 - x_2}{(x_4 - x_1)^2}}{1 + \left(\frac{x_5 - x_2}{x_4 - x_1}\right)^2}$$

$$\therefore \frac{\partial \mathcal{H}_1}{\partial x_2} = \frac{-1 \times \frac{1}{x_4 - x_1}}{1 + \left(\frac{x_5 - x_2}{x_4 - x_1}\right)^2} = \frac{\frac{-1}{x_4 - x_1}}{1 + \left(\frac{x_5 - x_2}{x_4 - x_1}\right)^2}$$

$$\frac{\partial \mathcal{H}_1}{\partial x_3} = -1$$

$$\frac{\partial \mathcal{H}_1}{\partial x_4} = \frac{-1 \times \left(\frac{x_5 - x_2}{(x_4 - x_1)^2}\right)}{1 + \left(\frac{x_5 - x_2}{x_4 - x_1}\right)^2} = \frac{\frac{-(x_5 - x_2)}{(x_4 - x_1)^2}}{1 + \left(\frac{x_5 - x_2}{x_4 - x_1}\right)^2}$$

$$\frac{\partial H_1}{\partial x_5} = \frac{1 \times \frac{1}{x_4 - x_1}}{1 + \left(\frac{x_5 - x_2}{x_4 - x_1}\right)^2} = \frac{\frac{1}{(x_4 - x_1)}}{1 + \left(\frac{x_5 - x_2}{x_4 - x_1}\right)^2}$$

$$\frac{\partial H_1}{\partial x_6} = 0$$

$$H_2 = \sqrt{(x_1 - x_4)^2 + (x_2 - x_5)^2}$$

$$\therefore \frac{\partial H_2}{\partial x_1} = \frac{1}{2} \times \frac{2(x_1 - x_4)}{\sqrt{(x_1 - x_4)^2 + (x_2 - x_5)^2}} = \frac{x_1 - x_4}{\sqrt{(x_1 - x_4)^2 + (x_2 - x_5)^2}}$$

$$\frac{\partial H_2}{\partial x_2} = \frac{1}{2} \times \frac{2(x_2 - x_5)}{\sqrt{(x_1 - x_4)^2 + (x_2 - x_5)^2}} = \frac{x_2 - x_5}{\sqrt{(x_1 - x_4)^2 + (x_2 - x_5)^2}}$$

$$\frac{\partial H_2}{\partial x_3} = 0$$

$$\frac{\partial H_2}{\partial x_4} = \frac{1}{2} \times \frac{-2(x_1 - x_4)}{\sqrt{(x_1 - x_4)^2 + (x_2 - x_5)^2}} = \frac{-(x_1 - x_4)}{\sqrt{(x_1 - x_4)^2 + (x_2 - x_5)^2}}$$

$$\frac{\partial H_2}{\partial x_5} = \frac{1}{2} \times \frac{-2(x_2 - x_5)}{\sqrt{(x_1 - x_4)^2 + (x_2 - x_5)^2}} = \frac{-(x_2 - x_5)}{\sqrt{(x_1 - x_4)^2 + (x_2 - x_5)^2}}$$

$$\frac{\partial H_2}{\partial x_6} = 0$$

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$$\therefore \frac{\partial H[x(t), \tilde{v}(t)]}{\partial x(t)} =$$

$$\begin{bmatrix}
\dfrac{\frac{x_5-x_2}{(x_4-x_1)^2}}{1+\left(\frac{x_5-x_2}{x_4-x_1}\right)^2} & \dfrac{\frac{-1}{x_4-x_1}}{1+\left(\frac{x_5-x_2}{x_4-x_1}\right)^2} & -1 & \dfrac{-\frac{(x_5-x_2)}{(x_4-x_1)^2}}{1+\left(\frac{x_5-x_2}{x_4-x_1}\right)^2} & \dfrac{\frac{1}{x_4-x_1}}{1+\left(\frac{x_5-x_2}{x_4-x_1}\right)^2} & 0 \\
\dfrac{x_1-x_4}{\sqrt{(x_1-x_4)^2+(x_2-x_5)^2}} & \dfrac{x_2-x_5}{\sqrt{(x_1-x_4)^2+(x_2-x_5)^2}} & 0 & \dfrac{-(x_1-x_4)}{\sqrt{(x_1-x_4)^2+(x_2-x_5)^2}} & \dfrac{-(x_2-x_5)}{\sqrt{(x_1-x_4)^2+(x_2-x_5)^2}} & 0 \\
\dfrac{\frac{x_5-x_2}{(x_4-x_1)^2}}{1+\left(\frac{x_5-x_2}{x_4-x_1}\right)^2} & \dfrac{\frac{-1}{x_4-x_1}}{1+\left(\frac{x_5-x_2}{x_4-x_1}\right)^2} & 0 & \dfrac{-\frac{(x_5-x_2)}{(x_4-x_1)^2}}{1+\left(\frac{x_5-x_2}{x_4-x_1}\right)^2} & \dfrac{\frac{1}{(x_4-x_1)}}{1+\left(\frac{x_5-x_2}{x_4-x_1}\right)^2} & -1 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0
\end{bmatrix}_{5\times 6}$$

Its dimensions are 5×6

This is the huge H matrix. Kindly zoom into the image to see the actual values.

b)

b) From discussions in the class, we know that DT jacobians can be obtained using Euler approximation.

The formulas/equations for DT Jacobians from CT Jacobians are

$$\tilde{F}_k = \frac{\partial f}{\partial x_k}\bigg|_{nom[k]} \approx I + \Delta T \cdot \tilde{A}\Big|_{(x^*, u^*, t=t_k)}$$

$$\tilde{G}_k = \frac{\partial f}{\partial u_k}\bigg|_{nom[k]} \approx \Delta T \cdot \tilde{B}\Big|_{(x^*, u^*, t=t_k)}$$

$$\tilde{\Omega}_k = \frac{\partial f}{\partial w_k}\bigg|_{nom[k]} \approx \Delta T \cdot \Gamma(t)\Big|_{t=t_k}$$

$$\tilde{H}_{k+1}\Big|_{nom[k+1]} = \frac{\partial h}{\partial x_k}\bigg|_{nom[k]} = \tilde{C}\Big|_{(x^*, u^*, t=t_{k+1})}$$

(where x^*, u^* is the nominal trajectory)

Next, let us go from Jacobians to and LTI system.
To do this, we compute the Jacboians at nominal. Sub in x_nom values to A~, B~, C~, and D~ to find:

$\xi_g = 10 \quad \eta_g = 10 \quad \theta_g = \frac{\pi}{2} \quad v_g = 2 \quad \phi_g = -\frac{\pi}{18}$

$\xi_a = -60 \quad \eta_a = 0 \quad \theta_a = -\frac{\pi}{2}$

6

$$X_{nom} = \begin{bmatrix} 10 \\ 0 \\ \frac{\pi}{2} \\ -60 \\ 0 \\ -\frac{\pi}{2} \end{bmatrix} \qquad U_{nom} = \begin{bmatrix} 2 \\ -\frac{\pi}{18} \\ 12 \\ \frac{\pi}{25} \end{bmatrix}$$

$$\tilde{C}_{nom} = \begin{bmatrix} 0 & \frac{1}{70} & -1 & 0 & \frac{-1}{70} & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & \frac{1}{70} & 0 & 0 & \frac{-1}{70} & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

(7)

$$\tilde{A}_{nom} = \begin{bmatrix} 0 & 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1.2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\tilde{B}_{nom} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ -0.0882 & 4.1244 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\tilde{D}_{nom} = 0$$

Next, we use our A~, B~, C~ and D~ to compute F, G, H and M.
C~ = H and D~ = M.

We discretize to compute F and G. Performing this numerically gives:
F = I + dt*A~ =

1.0000	0	-0.2000	0	0	0

```
     0    1.0000         0         0         0         0
     0         0    1.0000         0         0         0
     0         0         0    1.0000         0    1.2000
     0         0         0         0    1.0000         0
     0         0         0         0         0    1.0000
```

```
G =
array([[ 0.        ,  0.        ,  0.        ,  0.        ],
       [ 0.1       ,  0.        ,  0.        ,  0.        ],
       [-0.0352654 ,  0.41243648,  0.        ,  0.        ],
       [ 0.        ,  0.        ,  0.        ,  0.        ],
       [ 0.        ,  0.        , -0.1       ,  0.        ],
       [ 0.        ,  0.        ,  0.        ,  0.1       ]])
```

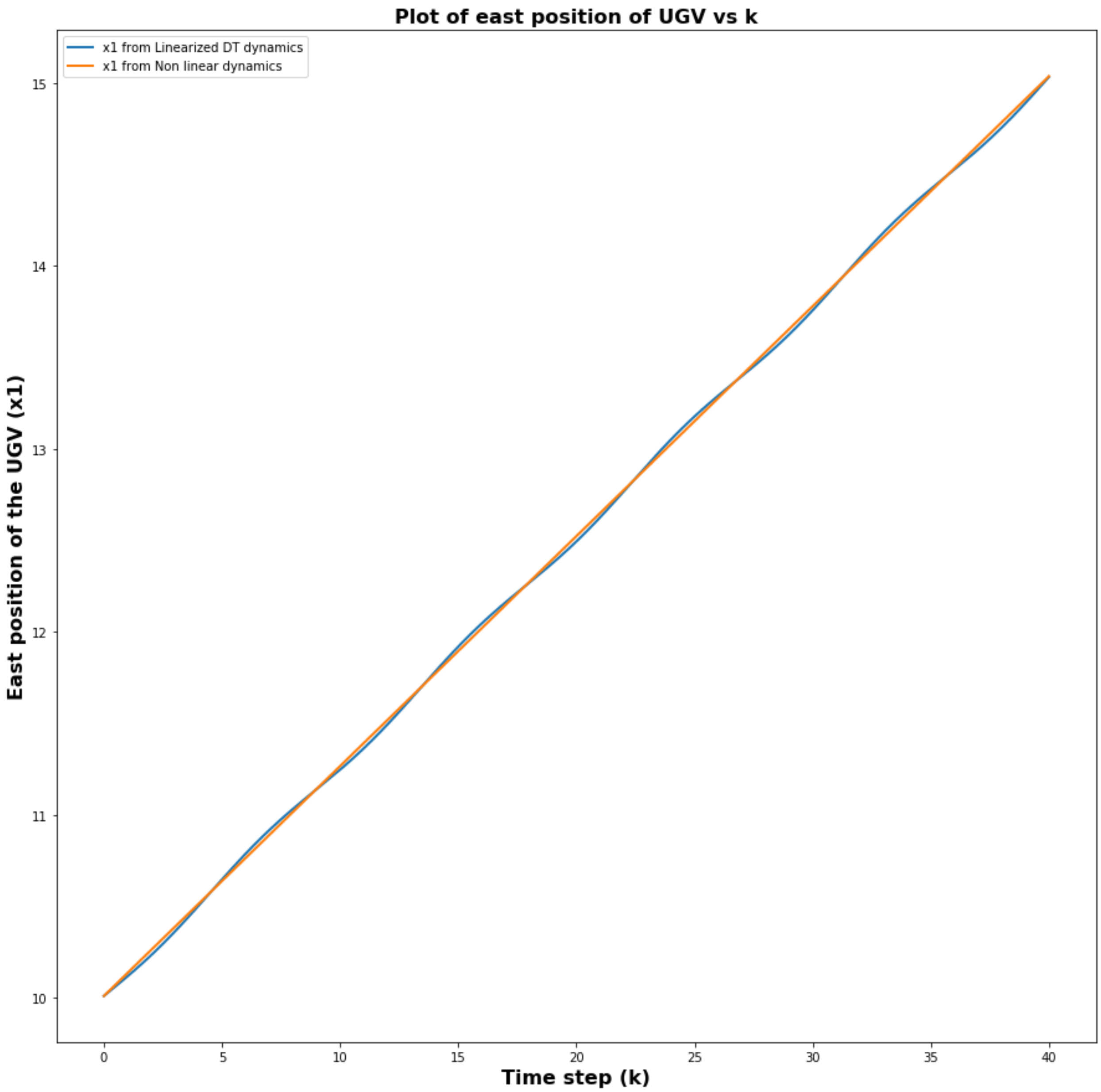
```
H =

         0    0.0143   -1.0000         0   -0.0143         0
    1.0000         0         0   -1.0000         0         0
         0    0.0143         0         0   -0.0143   -1.0000
         0         0         0    1.0000         0         0
         0         0         0         0    1.0000         0
```

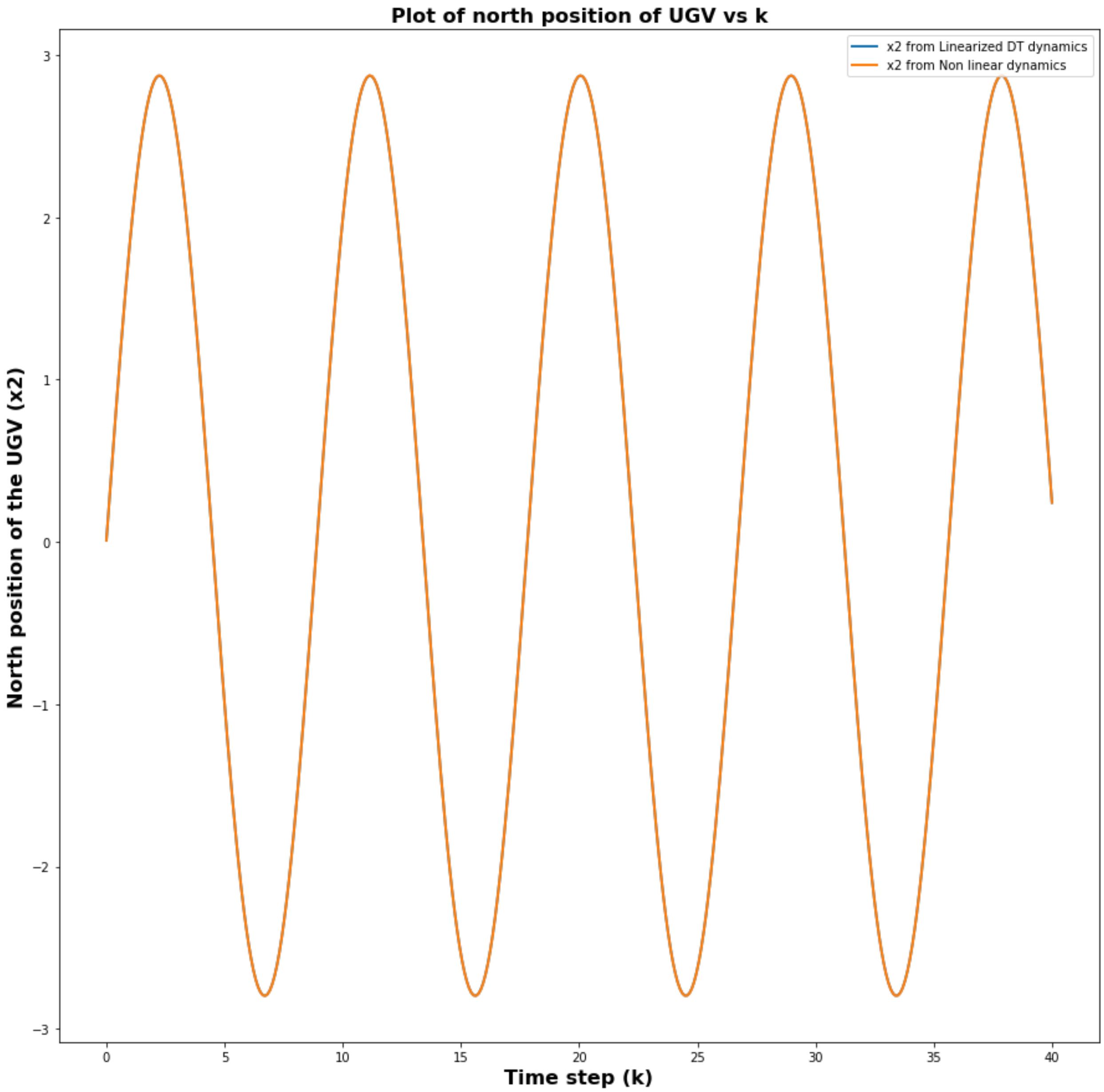
M = 0

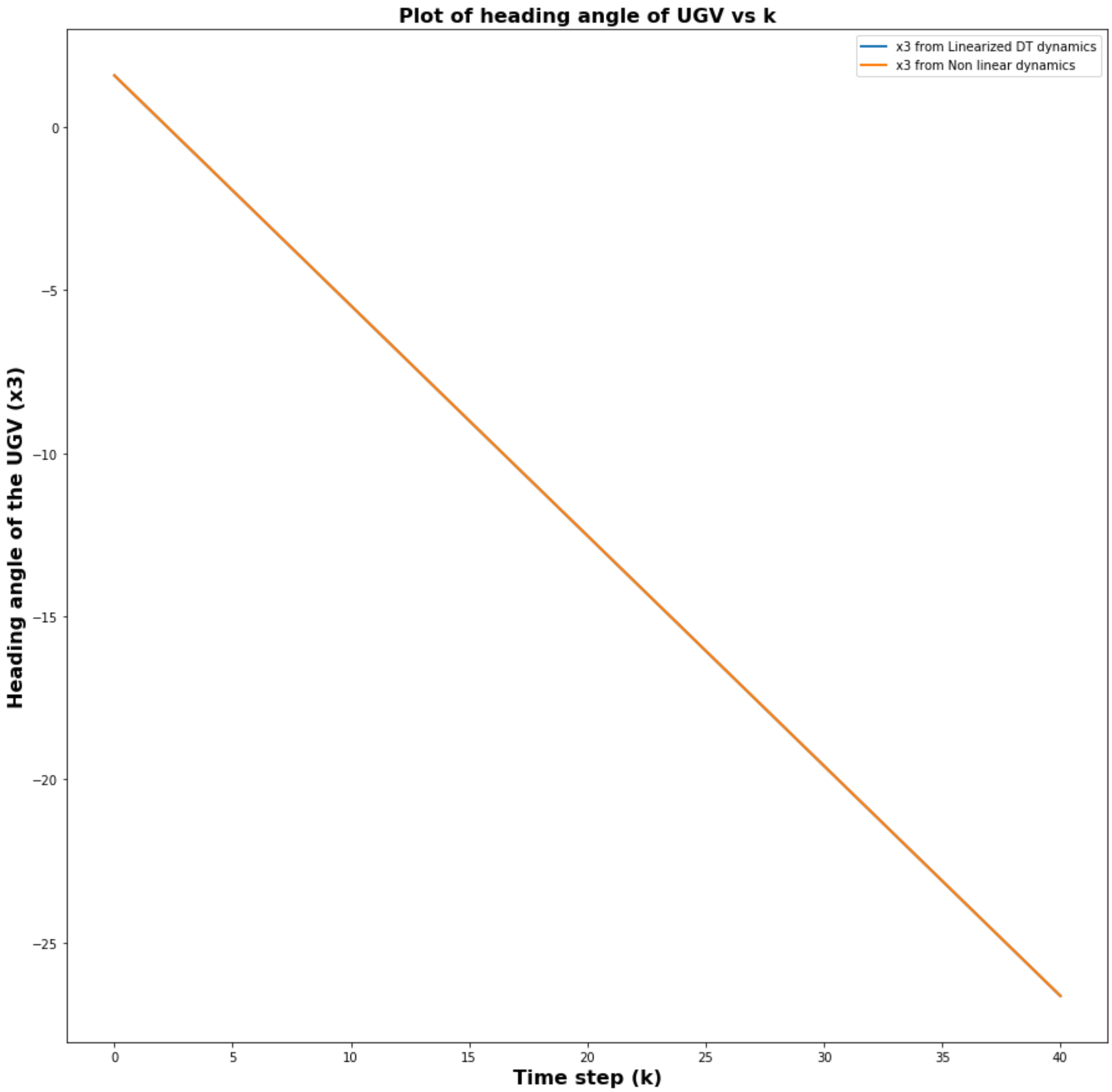
The DT jacobians that we obtained are state (and thus time) dependent. Thus, we will have to check for observability and controllability at every time step. Since the result is time varying, we won't go into that analysis as mentioned in the question.

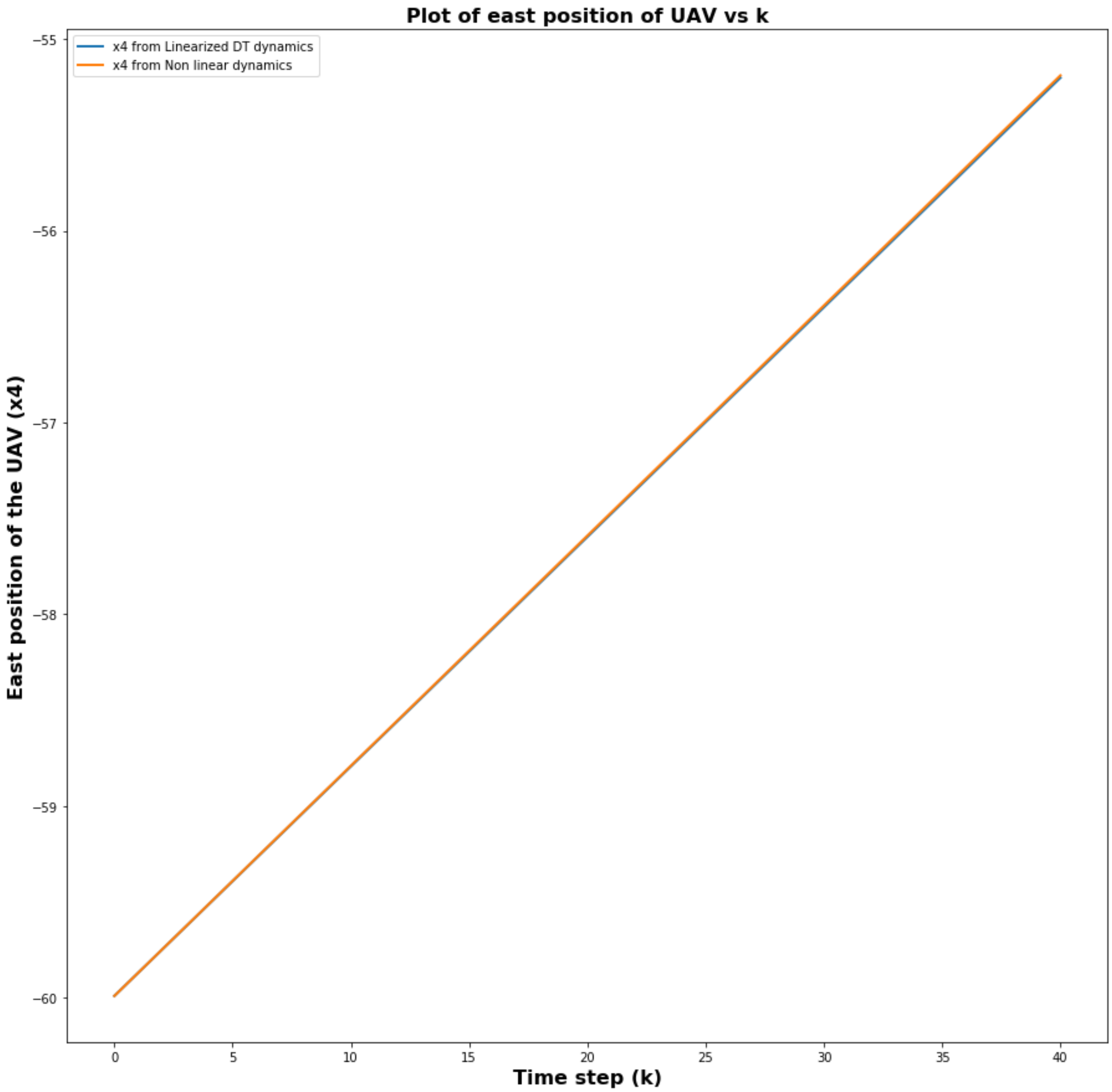
c)

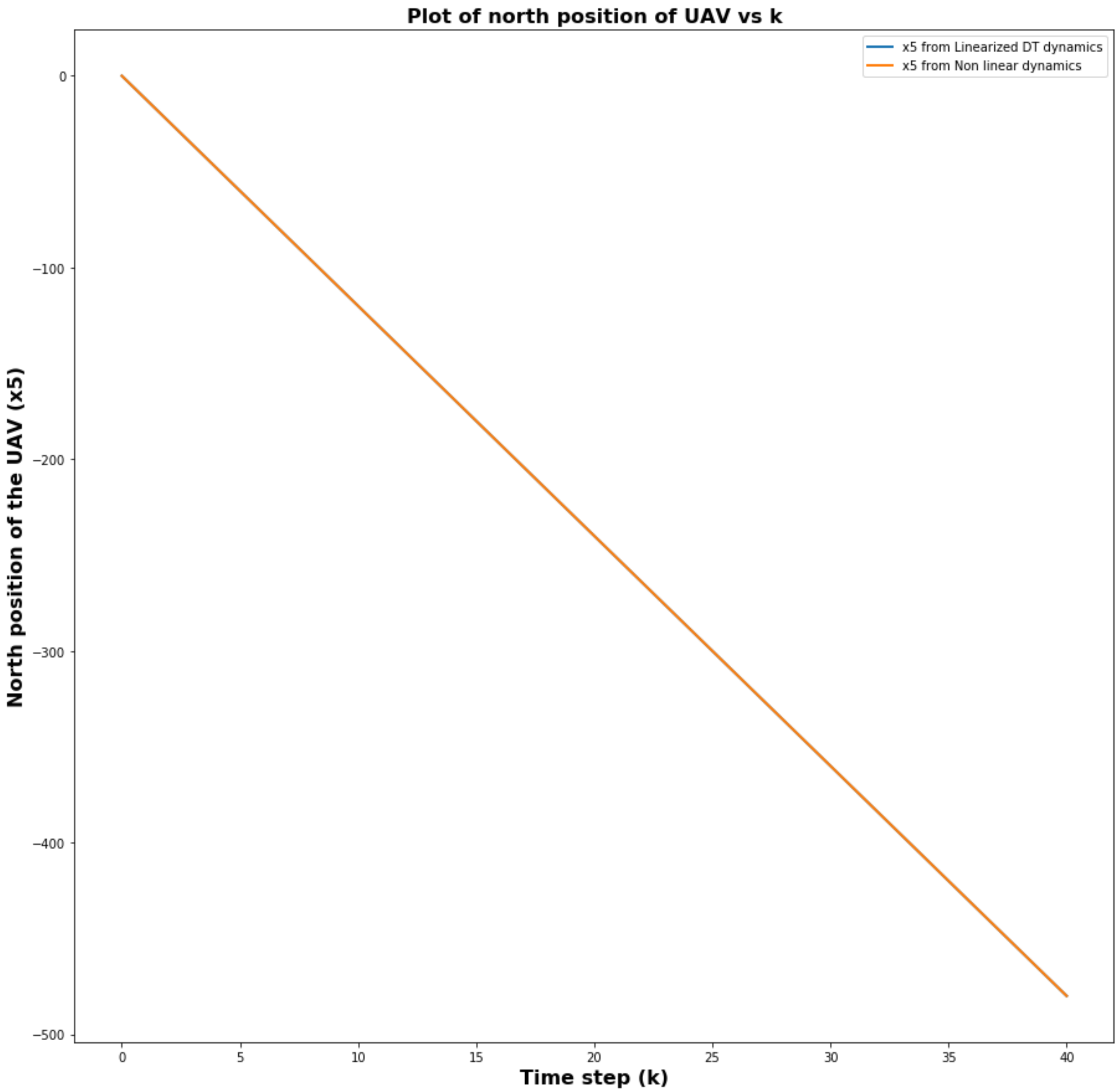
Various different plots are as follows :

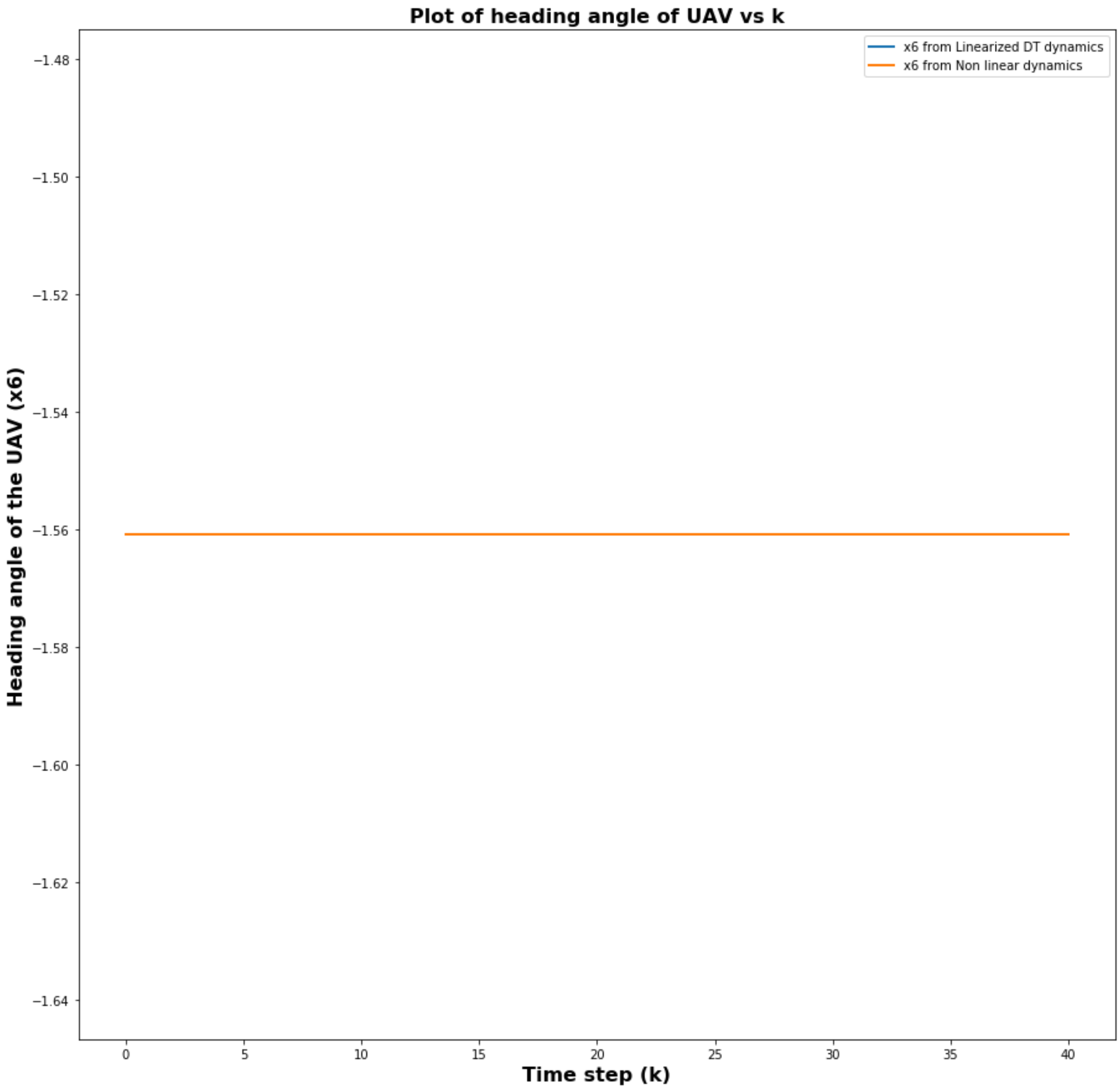



Plot of north position of UGV vs k
x2 from Linearized DT dynamics
x2 from Non linear dynamics
North position of the UGV (x2)
Time step (k)
3
2
1
0
−1
−2
−3
0
5
10
15
20
25
30
35
40

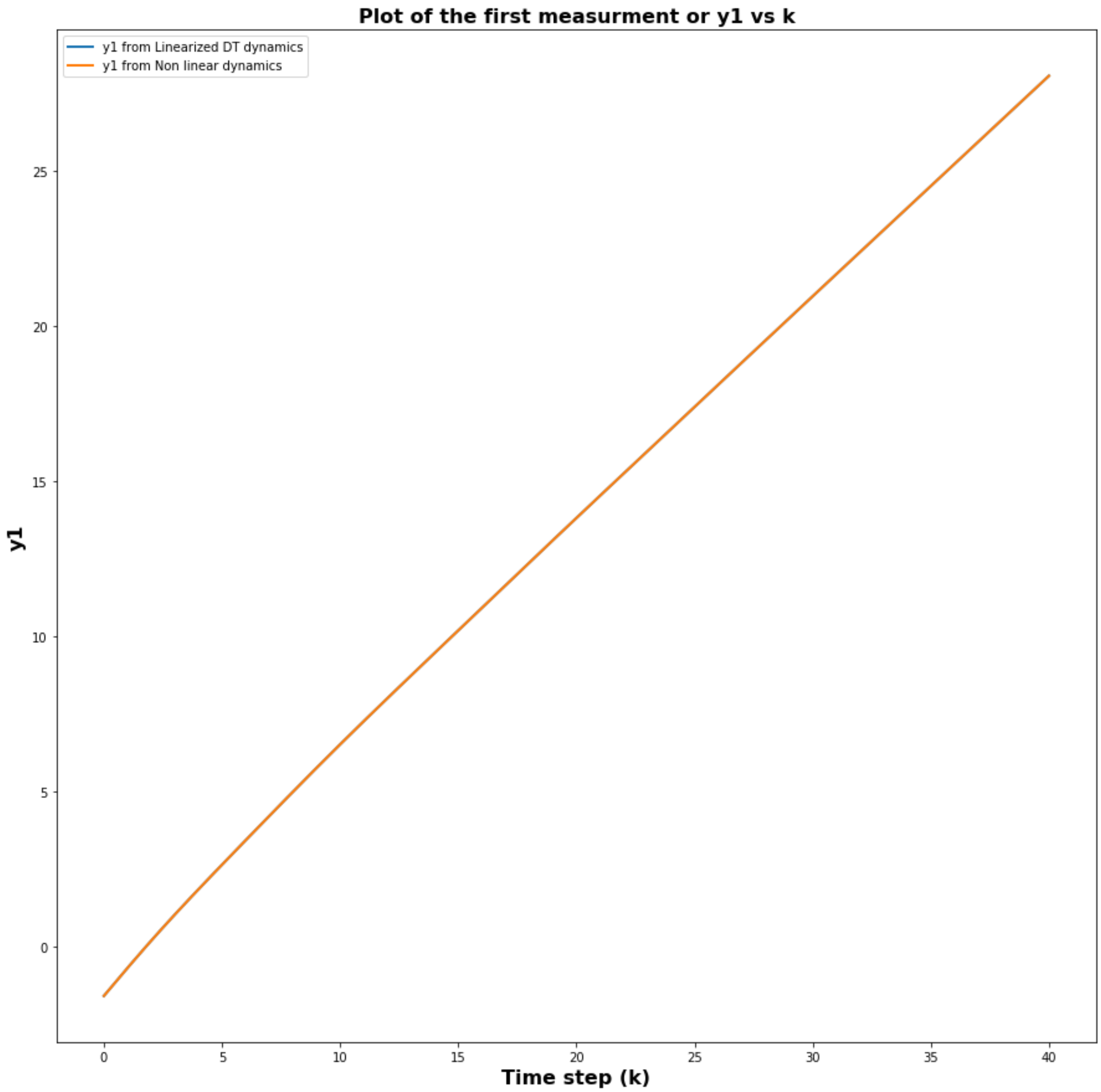

Plot of heading angle of UGV vs k
x3 from Linearized DT dynamics
x3 from Non linear dynamics
Heading angle of the UGV (x3)
0
−5
−10
−15
−20
−25
Time step (k)
0
5
10
15
20
25
30
35
40

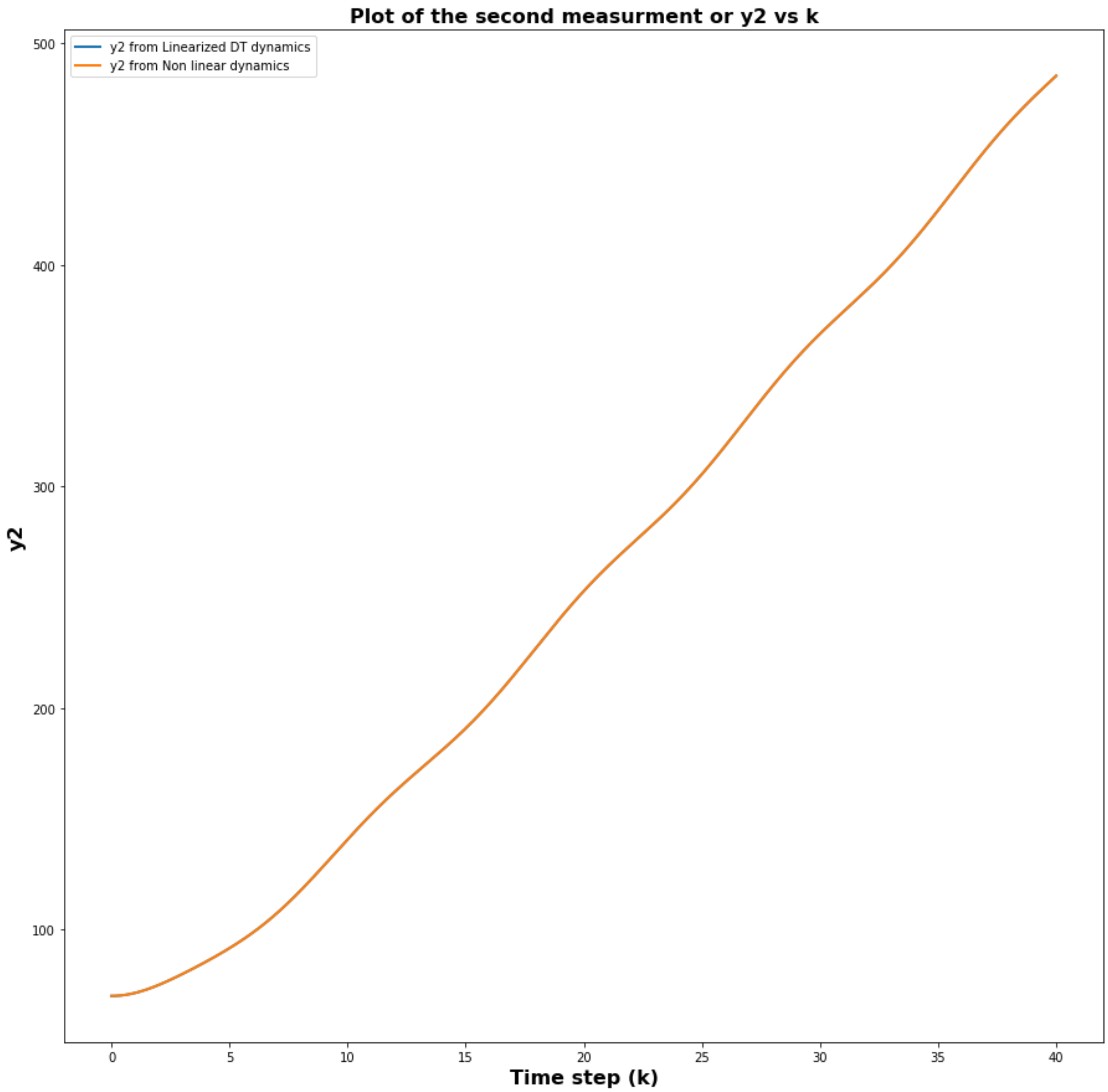

Plot of east position of UAV vs k
x4 from Linearized DT dynamics
x4 from Non linear dynamics
East position of the UAV (x4)
Time step (k)
−55
−56
−57
−58
−59
−60
0
5
10
15
20
25
30
35
40


Plot of north position of UAV vs k
x5 from Linearized DT dynamics
x5 from Non linear dynamics
North position of the UAV (x5)
0
−100
−200
−300
−400
−500
0
5
10
15
20
25
30
35
40
Time step (k)

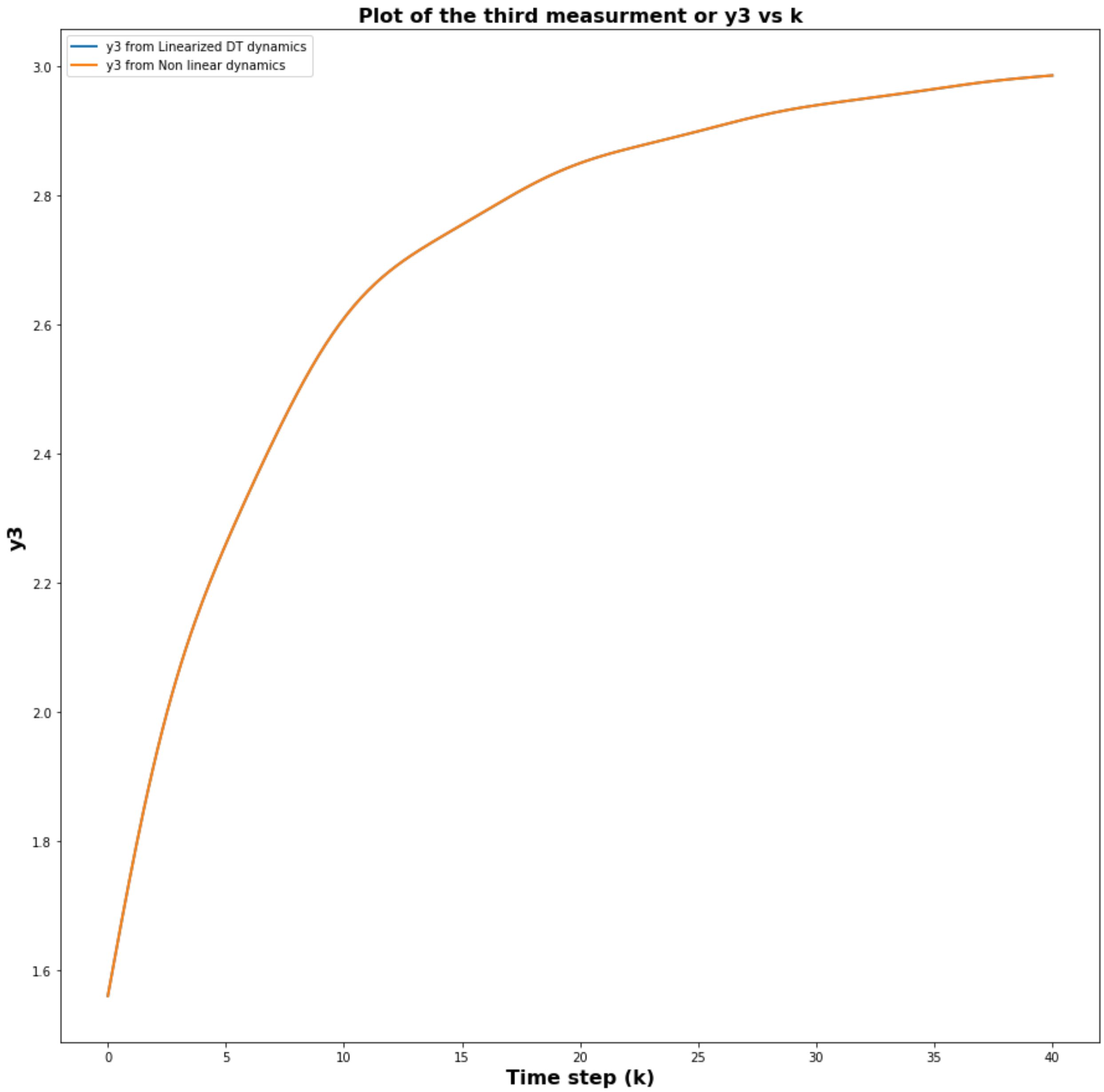


Plot of heading angle of UAV vs k
x6 from Linearized DT dynamics
x6 from Non linear dynamics
−1.48
−1.50
−1.52
−1.54
−1.56
−1.58
−1.60
−1.62
−1.64
Heading angle of the UAV (x6)
0
5
10
15
20
25
30
35
40
Time step (k)

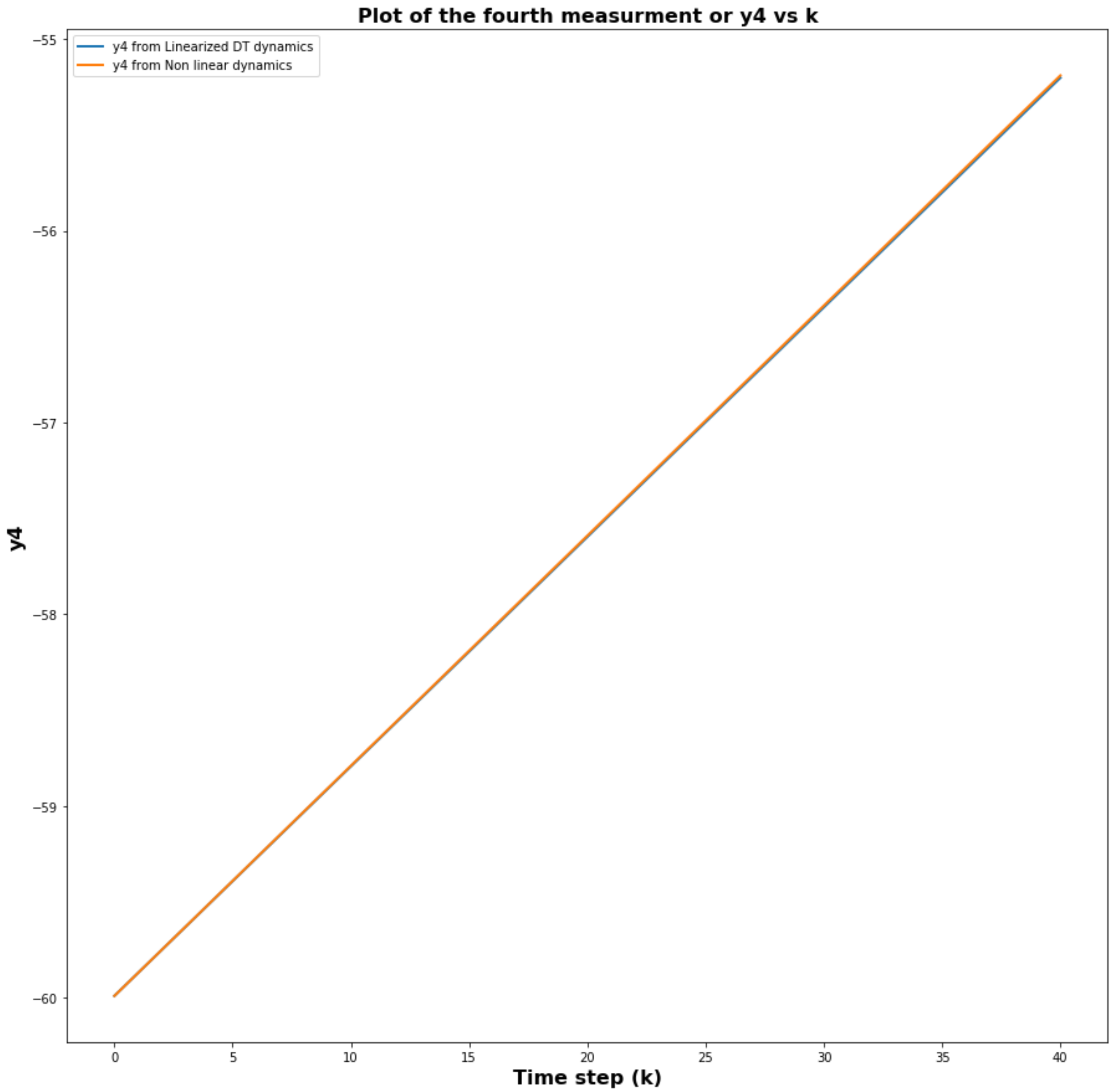

Plot of the first measurment or y1 vs k
y1 from Linearized DT dynamics
y1 from Non linear dynamics
y1
Time step (k)
0
5
10
15
20
25
30
35
40
0
5
10
15
20
25

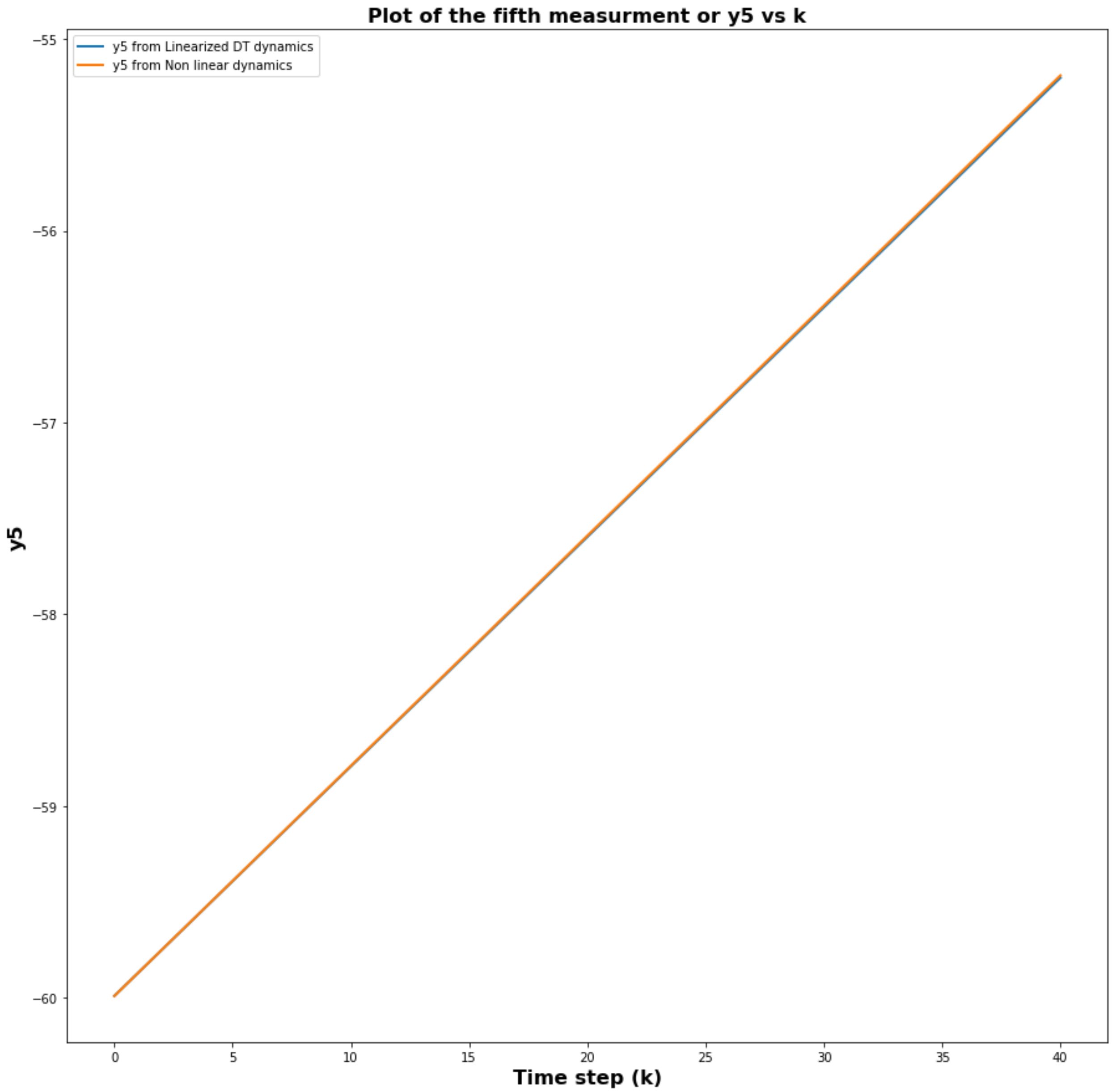


Plot of the second measurment or y2 vs k
y2 from Linearized DT dynamics
y2 from Non linear dynamics
500
400
300
200
100
y2
0
5
10
15
20
25
30
35
40
Time step (k)



Plot of the third measurment or y3 vs k
y3 from Linearized DT dynamics
y3 from Non linear dynamics
3.0
2.8
2.6
2.4
2.2
2.0
1.8
1.6
y3
0
5
10
15
20
25
30
35
40
Time step (k)


Plot of the fourth measurment or y4 vs k
y4 from Linearized DT dynamics
y4 from Non linear dynamics
y4
Time step (k)
−55
−56
−57
−58
−59
−60
0
5
10
15
20
25
30
35
40


Plot of the fifth measurment or y5 vs k
y5 from Linearized DT dynamics
y5 from Non linear dynamics
y5
Time step (k)
−55
−56
−57
−58
−59
−60
0
5
10
15
20
25
30
35
40