

Functional analysis course by Dodonov N.U.

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# 1 Vector spaces

## §1.1 Metric spaces.

$X, \rho: X \times X \rightarrow \mathbb{R}_+$

**Definition.**  $\rho$  — **metric**

1.  $\rho(x, y) \geq 0, = 0 \iff x = y$
2.  $\rho(x, y) = \rho(y, x)$
3.  $\rho(x, y) \leq \rho(x, z) + \rho(y, z)$

**Definition.**  $(X, \rho)$  — **Metric space.**

**Definition.**  $x = \lim x_n \iff \rho(x_n, x) \rightarrow 0$

$$X, \tau = \{ G \subset X \}$$

**Definition.** Let  $X$  be arbitrary set. Then system of its subsets  $\tau$  is called a **topology** if :

1.  $\emptyset, X \in \tau$
2.  $G_\alpha \in \tau, \alpha \in \mathcal{A} \implies \bigcup_{\alpha} G_\alpha \in \tau$
3.  $G_1, \dots, G_n \in \tau \implies \bigcap_{j=1}^n G_j \in \tau$

And any set  $G \in \tau$  is called **open**.

**Definition.**  $(X, \tau)$  — **Topological space.**

$$x = \lim x_n \quad \forall G \in \tau : x \in G \quad \exists N : \forall n > N \quad x_n \in G$$

$G$  — open in  $\tau$

$F = X \setminus G$  — closed

**Definition.** Given a metric space  $(X, \rho)$  an **open ball** with radius  $r$  around  $a$  is defined as the set  $B_r(a) = \{ x \mid \rho(x, a) < r \}, r \in \mathbb{R}_+$

**Statement 1.1.** Any metric space gives rise to a topological space in a rather simple way. Let's call  $G \subset X$  open if and only if  $\forall x \in G$  there exists some  $r$  such that open ball  $B_r(x)$  is contained in  $G$ .

**Statement 1.2.**  $b \in B_{r_1}(a_1) \cap B_{r_2}(a_2) \implies \exists r_3 > 0 : B_{r_3}(a_3) \subset B_{r_1}(a_1) \cap B_{r_2}(a_2)$

**Example 1.**  $\mathbb{R}, \rho(x, y) = |x - y|$ , MS

**Example 2.**  $\bar{x} = (x_1, \dots, x_n) \in \mathbb{R}^n, \rho(\bar{x}, \bar{y}) = \sqrt{\sum_{j=1}^n (x_j - y_j)^2}$ , MS

**Example 3.**  $\bar{x} = (x_1, \dots, x_n, \dots) \in \mathbb{R}^\infty$

$$\alpha \bar{x} = (\alpha x_1, \dots, \alpha x_n, \dots)$$

$$\bar{x} + \bar{y} = (x_1 + y_1, \dots, x_n + y_n, \dots)$$

Let's define  $\lim_{m \rightarrow \infty} \bar{x}_m$

- in  $\mathbb{R}^n$ :  $\bar{x}_n \rightarrow \bar{x} \iff \forall j = 1, \dots, n \quad x_j^{(m)} \xrightarrow{m \rightarrow \infty} x_j$

- in  $\mathbb{R}^\infty$ :  $\bar{x}_m \rightarrow \bar{x} \stackrel{\text{def}}{\iff} \forall j = 1, 2, 3, \dots \quad x_j^{(m)} \xrightarrow{m \rightarrow \infty} x_j$

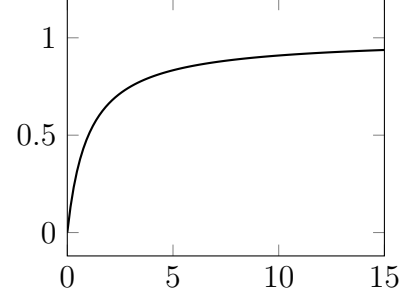
**Definition.**  $\rho(\bar{x}, \bar{y}) \stackrel{\text{def}}{=} \sum_{n=1}^{\infty} \frac{1}{2^n} \underbrace{\frac{|x_n - y_n|}{1 + |x_n - y_n|}}_{\varphi(|x_n - y_n|)}$  — **Urysohn metric.**

$$\varphi(t) = \frac{t}{1+t}$$

$$\varphi(t_1) + \varphi(t_2) \geq \varphi(t_1 + t_2)$$

$$\varphi(t_1) + \varphi(t_2) = \frac{t_1}{1+t_1} + \frac{t_2}{1+t_2} \geq \frac{t_1}{1+t_1+t_2} + \frac{t_2}{1+t_1+t_2}$$

$$\varphi(t_1) + \varphi(t_2) \geq \frac{t_1 + t_2}{1 + t_1 + t_2} = \varphi(t_1 + t_2)$$



**Statement 1.3.**  $\rho(\bar{x}_m, \bar{x}) \xrightarrow{m \rightarrow \infty} 0 \iff x_j^{(m)} \rightarrow x_j \forall j$

*Proof.*

- $\Rightarrow$

$$f(|x_k^{(n)} - x_k|) \leq 2^k \rho(x^{(n)}, x)$$

$$\text{Let } \rho(x^{(n)}, x) \leq \frac{\varepsilon}{2^k}, \text{ then } f(|x_k^{(n)} - x_k|) < \varepsilon$$

$$|x_k^{(n)} - x_k| = t = \frac{1}{1 - f(t)} - 1, \text{ then } t \rightarrow 0$$

- $\Leftarrow$

$$\text{Let's choose } k_0 \text{ for which } \sum_{k=k_0+1}^{\infty} \frac{1}{2^k} < \varepsilon$$

$$\text{Let's choose } n_0 \text{ for which } \forall k \leq k_0, n > n_0 : |x_k^{(n)} - x_k| < \varepsilon.$$

$$\text{Then } \rho(x^{(n)}, x) < \sum_{k=1}^{k_0} \frac{\varepsilon}{2^k} + \varepsilon < 2\varepsilon$$

Letting  $\varepsilon \rightarrow 0$ , we get what we want

■

In this way  $\mathbb{R}^\infty$  is a metrizable space.

**Example 4.**  $X, \rho(x, y) \stackrel{\text{def}}{=} \begin{cases} 0, & x = y \\ 1, & x \neq y \end{cases}$  — **Discrete metric.**

$$x_n \rightarrow x, \varepsilon = \frac{1}{2}, \exists M : m > M \implies \rho(x_m, x) < \frac{1}{2} \implies$$

$$\rho(x_m, x) = 0 \implies x_m = x$$

**Definition.**  $(X, \tau); \forall A \subset X$

$$\text{Int}(A) \stackrel{\text{def}}{=} \bigcup_{G \subset A} G \text{ is open;}$$

$$\text{Cl}(A) \stackrel{\text{def}}{=} \bigcap_{A \subset G} G \text{ is closed;}$$

$$\text{Fr}(A) \stackrel{\text{def}}{=} \text{Cl}(A) \setminus \text{Int}(A)$$

Having a metric space  $(X, \rho)$  one can describe closure of a set.

$$\rho(x, A) \stackrel{\text{def}}{=} \inf_{a \in A} \rho(x, a)$$

$$\rho(A, B) \stackrel{\text{def}}{=} \inf_{\substack{a \in A \\ b \in B}} \rho(a, b)$$

$$\rho(x, A) = f(x), x \in X$$

**Statement 1.4.** Function  $f(x)$  is continuous.

*Proof.*  $\forall x, y \in X$

$$f(x) = \rho(x, A) \leq \rho(x, \alpha) \leq \rho(x, y) + \rho(y, \alpha)$$

$$\forall \varepsilon > 0 \exists \alpha_\varepsilon \in A : \rho(y, \alpha_\varepsilon) < \rho(y, A) + \varepsilon = f(y) + \varepsilon$$

$$f(x) \leq f(y) + \varepsilon + \rho(x, y), \varepsilon \rightarrow 0$$

$$\begin{cases} f(x) \leq f(y) + \rho(x, y) \\ f(y) \leq f(x) + \rho(x, y) \end{cases} \implies |f(x) - f(y)| \leq \rho(x, y) \quad \blacksquare$$

**Statement 1.5.**  $x \in \text{Cl}(A) \iff \rho(x, A) = 0$

Let's look at the metric spaces in terms of separation of sets from each other by open sets.

$x, y$

$$r = \rho(x, y) > 0$$

$$B_{r/3}(x), B_{r/3}(y)$$

In any metric space separability axiom is true.

**Theorem 1.1.**

Any metric space is a normal space,

i.e.  $\forall$  closed disjoint  $F_1, F_2 \in X$ ,  $\exists$  open disjoint  $G_1, G_2$ :  $F_j \in G_j$ ,  $j = 1, 2$

$$\text{Proof. } g(x) = \frac{\rho(x, F_1)}{\rho(x, F_1) + \rho(x, F_2)} \text{ — continuous on } X$$

$$x \in F_1, \text{Cl}(F_1) = F_1, \rho(x, F_1) = 0, g(x) = 0$$

$$x \in F_2, g(x) = 1$$

Let's look at  $(-\infty; \frac{1}{3}), (\frac{2}{3}, \infty)$  — by continuity their inverse images under  $g$  are open.

$$G_1 = g^{-1}(-\infty; \frac{1}{3})$$

$$G_2 = g^{-1}(\frac{2}{3}; \infty) \quad \blacksquare$$

**Definition.** Metric space is **complete** if  $\rho(x_n, x_m) \rightarrow 0 \implies \exists x = \lim x_n$

$\mathbb{R}^\infty$  — complete (by completeness of the rational numbers).

In complete metric spaces the nested balls principle is true.

**Theorem 1.2.**

$X$  — complete metric space,  $\bar{V}_{r_n}$  — system of closed balls.

1.  $\bar{V}_{r_{n+1}} \subset \bar{V}_{r_n}$  — the system is nested.

2.  $r_n \rightarrow 0$

$$\text{Then: } \bigcap_n \bar{V}_{r_n} = \{a\}$$

*Proof.* Let  $b_n$  be centers of  $\bar{V}_{r_n}$ ,

$$m \geq n, b_m \in \bar{V}_{r_n}, \rho(b_m, b_n) \leq r_n \rightarrow 0 \forall m \geq n$$

$$\rho(b_m, b_n) \rightarrow 0 \xrightarrow{\text{compl.}} \exists a = \lim b_n \text{ Since the balls are closed } a \in \text{every ball.}$$

$$r_n \rightarrow 0 \implies \text{there is only one common point.} \quad \blacksquare$$

$(X, \tau)$  — topological space

$A \subset X, \tau_a = \{ G \cap A, G \in \tau \}$  — topology induced on  $A$

**Definition.**  $X$  — metric space,  $A \subset X, \text{Cl}(A) = X$

Then:  $A$  — **dense** in  $X$

If  $\text{Int}(\text{Cl}(A)) = \emptyset$   $A$  — **nowhere dense** in  $X$ .

**Note.** It is easy to understand, that in metric spaces nowhere density means the following:  
 $\forall$  ball  $V \exists V' \subset V: V'$  contains no points from  $A$ .

**Definition.**  $X$  is called **first Baire category set**, if it can be written as at most countable union of  $x_n$  each nowhere dense in  $X$ .

**Theorem 1.3** (Baire category theorem).

*Complete metric space is second Baire category set in itself.*

*Proof.* Let  $X$  be first Baire category set.

$X = \bigcup_n X_n \quad \forall \bar{V} \quad X_1$  is nowhere dense.

$\bar{V}_1 \subset \bar{V}: \bar{V}_1 \cap X_1 = \emptyset$

$X_2$  is nowhere dense  $\bar{V}_2 \subset \bar{V}_1: \bar{V}_2 \cap X_2 = \emptyset$

$r_2 \leq \frac{r_1}{2}$

$\vdots$

$\{\bar{V}_n\}, r_n \rightarrow 0, \bigcap_n \bar{V}_n = \{a\}, X = \bigcup_n X_n, \exists n_0: a \in X_{n_0}$

$X_{n_0} \cap \bar{V}_{n_0} = \emptyset \rightarrow \leftarrow a \in \bar{V}_{n_0}$  ■

**Corollary 1.3.1.** *Complete metric space without isolated points is uncountable.*

*Proof.* No isolated points are present  $\implies$  every point in the set is nowhere dense in it. Let  $X$  be countable:  $X = \bigcup_n \{X_n\}$ , then it is first Baire category set.  $\rightarrow \leftarrow$  ■

**Definition.**  $K$  — **compact** if

1.  $K = \text{Cl}(K)$

2.  $x_n \in K \exists n_1 < n_2 < \dots x_{n_j} -$  converges in  $X$ .

If only 2 is present, the set is called **precompact**.

**Theorem 1.4** (Hausdorff).

*Let  $X$  — metric space,  $K$  — closed in  $X$ .*

Then:  $K$  — compact  $\iff K$  — totally bounded,

*i.e.*  $\forall \varepsilon > 0 \exists a_1, \dots, a_p \in X: \forall b \in K \exists a_j: \rho(a_j, b) < \varepsilon$

$(a_1, \dots, a_p -$  finite  $\varepsilon$ -net)

*Proof.*

• Totally bounded  $\implies$  compact

$K$  is totally bounded,  $x_n \in K \quad n_1 < n_2 < \dots < n_k < \dots, x_n$  converges in  $K$

$$\varepsilon_k \downarrow \rightarrow 0 \quad \varepsilon_1 \quad K \subset \bigcup_{j=1}^p \bar{V}_j, \text{rad} = \varepsilon_1 \quad (\varepsilon_1\text{-net})$$

$n$  is finite  $\implies$  one ball will contain infinitely many  $x_n$  elements.

Let's look at  $\overline{V}_{j_0} \cap K$  — totally bounded =  $K_1$ ,  $\text{diam}(K_1) \leq 2\varepsilon_1$

$$\varepsilon_2 \quad K_1 \subset \bigcup_{j=1}^n \overline{V}'_j, \text{ rad} = \varepsilon_2.$$

Then one of  $\overline{V}'$  contains infinitely many elements of the sequence contained in  $K_1$ .

$$\overline{V}'_{j_0} \cap K_1 = K_2, \text{ diam}(K_2) \leq 2\varepsilon_2 \text{ and so on.}$$

$$K_n \supset K_{n+1} \supset K_{n+2} \supset \dots, \text{ diam}(K_N) \leq 2\varepsilon_n \xrightarrow{\text{by compl.}} \underbrace{\bigcap_{n=1}^{\infty} K_n}_{\text{diam}(K_n) \rightarrow 0, \{x\}} \neq \emptyset$$

Take  $x_{n_1}$  from  $K_1$ ,  $x_{n_2}$  from  $K_2 \dots$

- Compact  $\implies$  totally bounded

$K$  — compact  $\forall \varepsilon \exists$  finite  $\varepsilon$ -net?

By contradiction:  $\exists \varepsilon_0 > 0$ : finite  $\varepsilon_0$ -net is impossible to construct.

$\forall x_1 \in K \exists x_2 \in K: \rho(x_1, x_2) > \varepsilon_0$  (or else system of  $x_1$  — finite  $\varepsilon$ -net)

$\{x_1, x_2\}$  - choose  $x_3 \in K: \rho(x_3, x_i) > \varepsilon_0, i = 1, 2$  and so on.

$x_n \in K: n \neq m \rho(x_n, x_m) > \varepsilon_0$  — contains no converging subsequence  $\implies$  set is not a compact.  $\rightarrow \leftarrow$  ■



## §1.2 Normed spaces

**Definition.**  $X$  — **linear set**,  $x + y, \alpha \cdot x, \alpha \in \mathbb{R}$

The purpose of norm definition, is to construct a topology on  $X$ , so that 2 linear operations are continuous on it.

$$\varphi: X \rightarrow \mathbb{R}:$$

1.  $\varphi(x) \geq 0, = 0 \iff x = 0$
2.  $\varphi(\alpha x) = |\alpha| \varphi(x)$
3.  $\varphi(x + y) \leq \varphi(x) + \varphi(y)$

**Definition.**  $\varphi$  — **norm** on  $X$ ,  $\varphi(x) = \|x\|$

$$\rho(x, y) \stackrel{\text{def}}{=} \|x - y\| \text{ — metric on } X.$$

**Definition.**

$(X, \|\cdot\|)$  — **normed space** — special case of metrical space.

$$x = \lim x_n \stackrel{\text{def}}{\iff} \rho(x_n, x) \rightarrow 0 \iff \|x_n - x\| \rightarrow 0$$

**Statement 1.6.** In the topology of a normed space linear operations are continuous on  $X$ .

*Proof.*

$$\begin{aligned} 1. \quad x_n \rightarrow x, y_n \rightarrow y; \quad \|(x_n + y_n) - (x + y)\| &= \|(x_n - x) + (y_n - y)\| \leq \\ &\leq \underbrace{\|x_n - x\|}_{\downarrow 0} + \underbrace{\|y_n - y\|}_{\downarrow 0} \\ &\implies x_n + y_n \rightarrow x + y \end{aligned}$$

$$\begin{aligned} 2. \quad \alpha_n \rightarrow \alpha, x_n \rightarrow x; \quad \|\alpha_n x_n - \alpha x\| &= \|(\alpha_n - \alpha)x_n + \alpha(x_n - x)\| \leq \\ &\leq \underbrace{|\alpha_n - \alpha|}_{\downarrow 0} \cdot \underbrace{\|x_n\|}_{\text{bounded}} + \underbrace{\alpha\|x_n - x\|}_{\downarrow 0} \end{aligned}$$

$$\begin{aligned} x_n \rightarrow x &\implies \|x_n\| \text{ — bounded.} \\ \alpha x_n &\rightarrow \alpha x \end{aligned}$$

■

**Statement 1.7.** From the triangle inequality  $|\|x\| - \|y\|| \leq \|x - y\|$

$$x_n \rightarrow x \implies \|x_n\| \rightarrow \|x\|$$

Norm is continuous.

**Example 5.**  $\mathbb{R}^n$

$$1. \quad \|\bar{x}\| = \sqrt{\sum_{k=1}^n x_k^2}$$

$$2. \quad \|\bar{x}\|_1 \stackrel{\text{def}}{=} \sum_{k=1}^n |x_k|$$

$$3. \quad \|\bar{x}\|_2 \stackrel{\text{def}}{=} \max\{|x_1|, \dots, |x_n|\}$$

$$4. \quad C[a, b] \text{ — functions continuous on } [a, b]; \quad \|f\| = \max_{x \in [a, b]} |f(x)|$$

$$5. L_p(E) = \left\{ f \text{ — measurable, } \int_E |f|^p < +\infty \right\}$$

$$p \geq 1, \|f\|_p = \left( \int_E |f|^p \right)^{\frac{1}{p}}$$

Because the set of points is the same, arises the question about convergence comparison.  
 $\|\cdot\|_1 \sim \|\cdot\|_2, x_n \xrightarrow{\|\cdot\|_1} x \iff x_n \xrightarrow{\|\cdot\|_2} x$

**Statement 1.8.**

$$\|\cdot\|_1 \sim \|\cdot\|_2 \iff \exists a, b > 0: \forall x \in X \implies a\|x\|_1 \leq \|x\|_2 \leq b\|x\|_1$$

**Theorem 1.5** (Riesz).

$X, \dim X < +\infty$  — linear set.

Then: Any pair of norms in  $X$  are equivalent.

*Proof.*  $l_1, \dots, l_n$  — linearly independent from  $X$ .  $\forall x \in X = \sum_{k=1}^n \alpha_k l_k$

$$\bar{x} \leftrightarrow (l_1, \dots, l_n) = \bar{l} \in \mathbb{R}^n$$

Let  $\|\cdot\|$  — some norm in  $X$ .

$$\|x\| \underset{\triangle}{\leq} \sum_{k=1}^n \|l_k\| |\alpha_k| \underset{\text{Cauchy}}{\leq} \underbrace{\sqrt{\sum_{k=1}^n \|l_k\|^2}}_{\text{const}(B), B\text{-basis}} \underbrace{\sqrt{\sum_{k=1}^n |\alpha_k|^2}}_{\substack{\|\bar{\alpha}\| = \|x\|_1 \\ \|\bar{\alpha}\| = \|x\|_1}}$$

$$\|x\|_1 = \sqrt{\sum_{k=1}^n \|\alpha_k\|^2}, x = \sum \alpha_k l_k$$

$$\|x\| \leq b\|x\|_1$$

$$? \exists a > 0: a\|x\|_1 \leq \|x\| \implies \|\cdot\| \sim \|\cdot\|_1$$

$$\text{Let } f(\alpha_1, \dots, \alpha_n) = \left\| \sum_{k=1}^n \alpha_k l_k \right\|$$

$$|f(\bar{\alpha} + \Delta \bar{\alpha}) - f(\bar{\alpha})| = \left| \left\| \sum_{k=1}^n \alpha_k l_k + \sum_{k=1}^n \Delta \alpha_k l_k \right\| - \left\| \sum_{k=1}^n \alpha_k l_k \right\| \right| \leq$$

$$\left\| \sum_{k=1}^n \Delta \alpha_k l_k \right\| \leq \underbrace{\sum_{k=1}^n \|l_k\| |\Delta \alpha_k|}_{\substack{\downarrow \\ 0, \Delta \alpha_k \rightarrow 0}} \implies f \text{ is continuous on } \mathbb{R}^n$$

$$S_1 = \left\{ \sum_{k=1}^n |\alpha_k|^2 = 1 \right\} \subset \mathbb{R}^m, f \text{ — continuous on } S_1, S_1 \text{ — compact, } \bar{\alpha}^* \in S_1$$

By Weierstrass theorem there exists a point  $\alpha^* \in S_1$  on a sphere, in which function  $f$  achieves its minimum  $\implies \forall \alpha \in S_1 f(\bar{\alpha}^*) \leq f(\bar{\alpha})$

If  $f(\bar{\alpha}^*) = 0$ , then  $\left\| \sum_{k=1}^n \alpha_k^* l_k \right\| = 0 \implies \sum_{k=1}^n \alpha_k^* l_k = 0$ ,  $\bar{\alpha}^* \in S_1$ ,  
but  $l_1 \dots l_n$  are linearly independent  $\rightarrow \leftarrow \implies \min_{S_1} f = m > 0$

$$\|x\| = \left\| \sum_{k=1}^n \alpha_k l_k \right\| = f(\bar{\alpha}) = \sqrt{\sum_{k=1}^n \alpha_k^2} \cdot \left\| \sum_{\beta_k} \frac{\alpha_k}{\sqrt{\sum_{k=1}^n \alpha_k^2}} \cdot l_k \right\| \geq, \bar{\beta} = (\beta_1 \dots \beta_n) \in S_1$$

$$\geq m \cdot \|x\|_1, a = m \quad \blacksquare$$

**Corollary 1.5.1.**  $X = NS, Y \subset X, \dim Y < +\infty \implies Y = \text{Cl}(Y)$

**Note.** Functional analysis differentiates between linear subset (set of points, closed by addition and scalar multiplication) and linear subspace (closed linear subset).

*Proof.*  $Y = \mathcal{L}(l_1, \dots, l_n) = \left\{ \sum_{i=1}^n \alpha_i l_i \mid l_1, \dots, l_n - \text{lin. indep.} \right\}$

$y_m \in Y, y_m \rightarrow y \text{ in } X \implies y \in Y?$

$\|y_m - y\| \rightarrow 0 \implies \|y_m - y_p\| \rightarrow 0, m, p \rightarrow \infty$

$\|y\|, y \in Y$ .

By Riesz theorem all norm in  $Y$  are equivalent.

$y = \sum_{j=1}^n \alpha_j l_j, \|y\|_0 = \sqrt{\sum_{j=1}^n \alpha_j^2}$  — some norm (by linear independence).

By Riesz theorem  $\|y\| \sim \|y\|_0$

$\underbrace{\|y_m - y_p\|}_{\in Y} \rightarrow 0 \implies \|y_m - y_p\|_0 \rightarrow 0$

Notice that convergence by  $\|\cdot\|_0$  is coordinatewise.

$\bar{\alpha} = (\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n, y_m = \sum_{i=1}^n \alpha_i^{(m)} l_i$

$|\alpha_i^{(m)} - \alpha_i^{(l)}| \rightarrow 0 \forall i = 1, \dots, n; \bar{\alpha} = (\alpha_1^{(m)}, \dots, \alpha_n^{(m)}) \rightarrow \alpha^* = (\alpha_1^*, \dots, \alpha_n^*)$

$y^* = \sum_{i=1}^n \alpha_i^* l_i \in Y, \|y_m - y\| \rightarrow 0$

Bu the limit uniqueness  $y = y^* \implies y \in Y$  ■

**Definition.** If normed space is complete, then it is called **B-space** or **Banach space**.

**Example 6.**  $C[a, b]$  — functions continuous on  $[a, b]$ .

**Example 7.** Lebesgue space,  $p \geq 1, L_p(E) = \left\{ f \text{ is measurable on } E, \int_E |f|^p < +\infty \right\}$ .

If  $X$  — Banach space,

$\sum_{n=1}^{\infty} x_n = \lim_{n \rightarrow \infty} \sum_{k=1}^n x_k, \sum_{n=1}^{\infty} \|x_n\| < +\infty$

$\|S_n - S_m\| = \left\| \sum_{k=m+1}^n x_k \right\| \leq \sum_{m+1}^n \|x_k\| \xrightarrow{n, m \rightarrow \infty} 0$

$\implies \|S_n - S_m\| \rightarrow 0 \implies \exists \lim_{n \rightarrow \infty} S_n, \sum_{k=1}^n x_k$  — converges.

In Banach spaces works the theory of absolute convergence of numerical series.

**Lemma 1** (Riesz's lemma about almost perpendicular).  $Y$  — eigen subspace of  $X$  — normed space.  $\forall \varepsilon \in (0, 1) \exists z_\varepsilon \in X$ :

1.  $z_\varepsilon \notin Y$
2.  $\|z_\varepsilon\| = 1$
3.  $\rho(z_\varepsilon, Y) > 1 - \varepsilon$

*Proof.*  $\exists x \in X \setminus Y, d = \rho(x, Y)$

Suppose  $d = 0$  then,  $\exists y_n \in Y: \|x - y_n\| < \frac{1}{n}, n \rightarrow \infty, y_n \rightarrow x$

$Y = \text{Cl}(Y) \implies x \in Y \rightarrow \leftarrow x \notin Y, d > 0$

$\forall \varepsilon \in (0, 1) \frac{1}{1 - \varepsilon} > 1 \exists y_\varepsilon \in Y: \|x - y_\varepsilon\| < \frac{1}{1 - \varepsilon} d$

$z_\varepsilon = \frac{x - y_\varepsilon}{\|x - y_\varepsilon\|}, \|z_\varepsilon\| = 1$

$$\forall y \in Y \|z_\varepsilon - y\| = \left\| \frac{x - y_\varepsilon}{\|x - y_\varepsilon\|} - y \right\| = \frac{\|x - \overbrace{(y_\varepsilon + \|x - y_\varepsilon\| \cdot y)}^{\in Y}\|}{\|x - y_\varepsilon\|} \geq \frac{d}{\frac{1}{1 - \varepsilon} d} > 1 - \varepsilon \quad \blacksquare$$

**Corollary 1.5.2.**  $X$  — normed space,  $\dim X = +\infty, S = \{x \mid \|x\| = 1\}$ , then closed unit ball  $\overline{B}$  is not compact in it.

*Proof.*  $\forall x_1 \in S, Y_1 = \mathcal{L}\{x_1\}$  — finite dimensional linear set.  $\implies$  closed in  $X \implies Y_1$  — subspace.

$\dim X = +\infty > \dim Y_1 \implies Y_1$  — eigen subspace.

Then by the Riesz lemma ( $\varepsilon = \frac{1}{2}$ ):

$\exists x_2 \in X: \|x_2\| = 1, \|x_2 - x_1\| > \frac{1}{2}$  (Notice that  $x_2$  appears to be an element of  $S$ )

$Y_2 = \mathcal{L}\{x_1, x_2\} \exists x_3 \in S: \|x_3 - x_j\| > \frac{1}{2}, j = 1, 2$

Continue by induction. Because  $\dim X = +\infty$  the process will never finish.

$x_n \in S: \|x_n - x_m\| > \frac{1}{2}, n \neq m$  — obviously we cannot extract converging subsequence.

$\implies S$  — not a compact. And that means that  $\overline{B} \supset S$  is not a compact either.  $\blacksquare$

## §1.3 Inner product (unitary) spaces

**Definition.**  $X$  — linear space.

$\varphi: X \times X \rightarrow \mathbb{R}$

1.  $\varphi(x, x) \geq 0, \quad \varphi(x, x) = 0 \iff x = 0$
2.  $\varphi(x, y) = \varphi(y, x)$
3.  $\varphi(\alpha x + \beta y, z) = \alpha \varphi(x, z) + \beta \varphi(y, z)$

$\varphi$  — **inner product**.

$\varphi(x, y) = \langle x, y \rangle$

**Definition.**  $(X, \langle \cdot, \cdot \rangle)$  — **inner product space**.

**Example 8.**  $\mathbb{R}^n, \langle \bar{x}, \bar{y} \rangle = \sum_{j=1}^n x_j y_j$

**Statement 1.9** (Schwarz).  $\forall x, y \in X \quad |\langle x, y \rangle| \leq \sqrt{\langle x, x \rangle} \cdot \sqrt{\langle y, y \rangle}$

*Proof.* Consider  $\lambda \in \mathbb{R}$ .

$$\begin{aligned} f(\lambda) &= \langle \lambda x + y, \lambda x + y \rangle \geq 0 \\ &\quad \parallel \\ &\quad \lambda^2 \langle x, x \rangle + 2\lambda \langle x, y \rangle + \langle y, y \rangle \\ D &= 4\langle x, y \rangle^2 - 4\langle x, x \rangle \cdot \langle y, y \rangle \leq 0 \end{aligned}$$

■

**Corollary 1.5.3** (Cauchy inequality for sums). Consider  $X = \mathbb{R}^n, \|x\| \stackrel{\text{def}}{=} \sqrt{\langle x, x \rangle}$ . Then

$$\|x + y\|^2 = \langle x + y, x + y \rangle = \|x\|^2 + 2 \cdot \underbrace{\langle x, y \rangle}_{\leq \|x\| \cdot \|y\|} + \|y\|^2 \leq (\|x\| + \|y\|)^2$$

Any inner product space is a special case of a normed space. The specifics is that we can measure the angles between points:

$$x \perp y \iff \langle x, y \rangle = 0$$

In this case the Pythagorean theorem takes place:

$$\|x + y\|^2 = \|x\|^2 + \|y\|^2$$

In inner product spaces the parallelogram law plays a significant role:

$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2 \quad \forall x, y \in X$$

In an inner product space norm is determined by inner product:  $\|x\|^2 = \langle x, x \rangle$

It can be proved that if parallelogram law holds, then the norm must be determined by some inner product. Let  $X$  be some normed space,  $x \in X$ , then  $\langle \cdot, \cdot \rangle \mapsto \|x\| = \sqrt{\langle x, x \rangle}$ . For any norm satisfying the parallelogram law, the inner product generating the norm is unique.

**Example 9.**  $C_{[a,b]}, \|f\| = \max_{x \in [a,b]} |f(x)|$ ,  $\|f\|$  doesn't satisfy the parallelogram law and thus is not determined by any inner product. This fact implies that  $C_{[a,b]}$  is not an inner product space.

**Definition. Orthonormal set** — a set of points  $\{e_1, e_2, \dots\}$  (may be finite):

1.  $\|e_i\| = 1$
2.  $e_i \perp e_j, i \neq j$

**Note.** Every orthonormal set is linearly independent.

**Definition.**  $\sum_j x_j$  — **orthogonal series**  $\iff x_i \perp x_j, i \neq j$

**Definition.** Let  $x \in X, \{e_i\}$  — ONS. Then

$\langle x, e_j \rangle$  — **abstract Fourier coefficient**,

$\sum_j \langle x, e_j \rangle e_j$  — **abstract Fourier series** of point  $x$ .

**Note.** Fourier series is a special case of orthogonal series.

Let  $\sum_{j=1}^{\infty} x_j, S_m = \sum_{j=1}^m x_j$ . Then

$$\|S_m\|^2 = \left\langle \sum_{j=1}^m x_j, \sum_{j=1}^m x_j \right\rangle = \sum_{j=1}^m \|x_j\|^2$$

This fact allows us to effectively build the theory of orthogonal series.

An important problem is concerned with Fourier series. Let  $X$  is a normed space,  $Y$  is a subspace of  $X$ ,

$$\forall x \in X \quad E_Y(x) = \rho(x, Y) = \inf_{y \in Y} \|x - y\|$$

**Definition.**  $E_Y(x)$  — **best approximation** of point  $x$  with points of the subspace  $Y$ . If  $\exists y^* \in Y \quad E_Y(x) = \|x - y^*\|$ , then  $y^*$  is called the **element of best approximation**.

**Theorem 1.6** (Borel).

$\dim Y < +\infty \implies \forall x \in X \quad \exists y^* \in Y$  — *element of best approximation*.

*Proof.*  $Y = \mathcal{L}(\underbrace{l_1, l_2, \dots, l_n}_{\text{lin. indep.}})$  Consider  $f(\alpha_1, \dots, \alpha_n) = \|x - \sum_{k=1}^n \alpha_k l_k\| \rightarrow \min$ . By the triangle inequality for norm,  $f(\bar{\alpha})$  is continuous on  $\mathbb{R}^n, f \geq 0, E_Y(x) = \inf_{\bar{\alpha} \in \mathbb{R}^n} f(\bar{\alpha})$ . It is easy to find out that there always is a ball  $B(0, r) \subset \mathbb{R}^n$ , outside of which  $f > 2E_Y(x)$ . So,  $E_Y(x)$  is somewhere inside. But  $f$  is continuous, ball  $B(0, r)$  is compact, so, by the Weierstrass theorem, the minimum exists and is located inside the  $B(0, r)$ . ■

For abstract Fourier series the Borel theorem can be significantly strengthened by specifying the best approximation element.

**Theorem 1.7** (extreme quality of Fourier series' partial sums).

$\{e_j\}$  — ONS in  $X$

$H_n = \mathcal{L}(e_1, \dots, e_n)$

$$E_{H_n}(x), S_n(x) = \sum_{j=1}^n \langle x, e_j \rangle e_j \implies E_{H_n}(x) = \|x - S_n(x)\|$$

*Proof.*  $y = \sum_{j=1}^n \alpha_j e_j \in H_n$

$$\begin{aligned} \|x - y\|^2 &= \left\langle x - \sum \alpha_j e_j, x - \sum \alpha_j e_j \right\rangle = \|x\|^2 - 2 \sum \alpha_j \langle x, e_j \rangle + \sum \alpha_j^2 = \\ &= \underbrace{\|x\|^2}_{\text{const}} + \sum (\alpha_j - \langle x, e_j \rangle)^2 - \underbrace{\sum \langle x, e_j \rangle^2}_{\text{const}} \rightarrow \min \end{aligned}$$

So, the sum goes to minimum when the second summand is minimal. Obviously, it's minimal when  $\forall (\alpha_j - \langle x, e_j \rangle) = 0$ .  $E_Y(x)$  — Fourier sum. ■

**Corollary 1.7.1** (Bessel's inequality).  $\sum_j \langle x, e_j \rangle^2 \leq \|x\|^2$

*Proof.* Bessel's inequality follows from the identity:

$$\begin{aligned} 0 \leq \|x - y^*\|^2 &= \left\| x - \sum_{j=1}^n \langle x, e_j \rangle e_j \right\|^2 = \|x\|^2 - 2 \sum_{j=1}^n |\langle x, e_j \rangle|^2 + \sum_{j=1}^n |\langle x, e_j \rangle|^2 \\ &= \|x\|^2 - \sum_j \langle x, e_j \rangle^2 \end{aligned} \quad \blacksquare$$

**Corollary 1.7.2.** *The series of Fourier coefficients' squares always converges*

## §1.4 Hilbert space.

**Definition. Hilbert space** — complete, infinite dimensional, inner product space.

**Example 10.**  $L_2(E)$  — Hilbert space.

$$\langle f, g \rangle = \int_E f \cdot g \, d\mu$$

$$l_2 = \left\{ (x_1, \dots, x_n, \dots) \mid \sum_{n=1}^{\infty} x_n^2 < +\infty \right\}$$

$$\langle \bar{x}, \bar{y} \rangle = \sum_{n=1}^{\infty} x_n y_n, \quad E = \mathbb{N}, \quad \mu\{m\} = 1$$

When we have completeness we can define orthonormal basis.

**Definition.**  $H, \{e_n\}$  — ONS :  $\forall x = \sum_{n=1}^{\infty} \alpha_n e_n$  Then  $\{e_n\}$  is called **orthonormal basis**.

Let's look at the inner product of arbitrary vector  $x \in H$  and basis vector  $e_m$

$$\langle x, e_m \rangle = \sum_{n=1}^{\infty} \alpha_n \langle e_n, e_m \rangle = \alpha_m$$

In this sense basis decomposition is always a Fourier series.

1. Complete ONS: Let  $L = \mathcal{L}\{e_1, e_2, \dots\}$ , then  $H = \text{Cl}(L)$  ( $L$  is dense in  $H$ )
2. Closed ONS:  $\forall m \langle x, e_m \rangle = 0 \implies x = 0$ .

**Statement 1.10.** In Hilbert spaces two of the properties outlined above are equivalent.

**Statement 1.11.** In Hilbert space Fourier series converges for any point.

*Proof.* Let  $H$  — Hilbert space,  $\sum_{j=1}^{\infty} \langle x, e_j \rangle e_j$  — abstract Fourier series,  $S_n = \sum_{j=1}^n \langle x, e_j \rangle e_j$  — it's partial sum. We need to prove, that  $\exists \lim_{n \rightarrow \infty} S_n$ .  $H$  is a Hilbert space, which means it is also complete. This means we only have to prove that  $\{S_n\}$  is Cauchy sequence, i.e.  $\forall \varepsilon > 0 \exists N \in \mathbb{N} : \forall n, m > N \quad \|S_n - S_m\| < \varepsilon$

$$\text{Consider } \|S_n - S_m\|^2 = \left\| \sum_{j=1}^n \langle x, e_j \rangle e_j - \sum_{j=1}^m \langle x, e_j \rangle e_j \right\|^2 = \left\| \sum_{j=m+1}^n \langle x, e_j \rangle e_j \right\|^2 = \sum_{j=m+1}^n |\langle x, e_j \rangle|^2$$

Because numerical series  $\sum_{j=m+1}^n |\langle x, e_j \rangle|^2$  converges, by the Cauchy criteria we have the following:

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} : \forall n > m > N \quad \left| \sum_{j=m+1}^n |\langle x, e_j \rangle|^2 \right| < \varepsilon^2 \text{ or } \sum_{j=m+1}^n |\langle x, e_j \rangle|^2 < \varepsilon^2, \text{ finally}$$

we get  $\|S_n - S_m\| < \varepsilon$ . ■

*Proof.* 1.10 Complete ONS  $\implies$  Closed ONS

$$\forall x \in H \quad \forall \varepsilon > 0 \quad \exists \sum_{j=1}^p \alpha_{kj} l_{kj} : \underbrace{\left\| x - \sum_{j=1}^p \alpha_{kj} l_{kj} \right\|^2}_{\geq \left\| x - \sum_{j=1}^{k_p} \langle x, l_j \rangle l_j \right\|^2} \leq \varepsilon^2$$

$$S_m(x) \text{ by extremity } \|x - S_{m+p}(x)\|^2 \leq \|x - S_m(x)\|^2$$

Implies partial sums go to  $x$ ,  $x = \sum_{j=1}^{\infty} \langle x, l_j \rangle l_j$  If all fourier coefficients are zero, it means the

ONS is closed ( $x = 0$ ).

Closed ONS  $\implies$  Complete ONS



$$y \in H = \sum_{j=1}^{\infty} \underbrace{\langle x, e_j \rangle}_{=\langle y, e_j \rangle} e_j \implies \langle y, e_j \rangle = \langle x, e_j \rangle \implies \langle y - x, e_j \rangle = 0$$

Because ONS is closed  $y - x = 0$ ,  $y = x$ . Thus we can decompose any point into Fourier series, and this implies that ONS is complete. ■

Considering basis existence.

**Definition.** Topological space is called **separable** if there exists countable dense set in it.  
 $X = \text{Cl}(\{a_1, \dots, a_n, \dots\})$

$H$  — separable,  $a_1, \dots, a_n, \dots$ . We can orthogonalize these dots (Gramm-Shmidt), and we will get complete ONS. This means space separability is equivalent to basis existence.

**Theorem 1.8** (about best approximation in  $H$ ).

$H$  — HS,  $M$  — closed convex subset of  $H$ , then  $\forall x \in H \exists! y \in M: \|x - y\| = \inf_{z \in M} \|x - z\|$ .  
 $M$  has element of best approximation for any  $x$  from  $X$ , and only one.

*Proof.*  $d = \inf_{z \in M} \|x - z\|$  by definition of infimum

$$\forall n \in \mathbb{N} \exists y_n \in M: d \leq \|x - y_n\| < d + \frac{1}{n}$$

$$\exists y = \lim y_n \in M \quad d \leq \|x - y\| \leq d$$

$y_n, y_m \in M$  — convex. Implies

$$\frac{y_n + y_m}{2} \in M \implies d^2 \leq \left\| \frac{y_n + y_m}{2} - x \right\|^2 = \frac{1}{4} \left\| \underbrace{(y_n - x)}_{z_1} + \underbrace{(y_m - x)}_{z_2} \right\|^2$$

Let's use parallelogram law.  $\|z_1 + z_2\|^2 + \|z_1 - z_2\|^2 = 2\|z_1\|^2 + 2\|z_2\|^2$ ,

$$\underbrace{\|(y_n - x) + (y_m - x)\|^2}_{\geq 4d^2} + \|y_n - y_m\|^2 = 2 \overbrace{\|y_n - x\|^2}^{\leq (d + \frac{1}{n})^2} + 2 \overbrace{\|y_m - x\|^2}^{\leq (d + \frac{1}{m})^2}$$

$$\|y_n - y_m\|^2 \leq 2(d + \frac{1}{n})^2 + 2(d + \frac{1}{m})^2 - 4d^2 = 4d\frac{1}{n} + \frac{2}{n^2} + 4d\frac{1}{m} + \frac{2}{m^2} \xrightarrow{n, m \rightarrow 0} 0$$

$$\|y_n - y_m\| \xrightarrow{n, m \rightarrow 0} 0 \implies \exists \lim y_n$$

**Corollary 1.8.1.**  $H$  — HS,  $H_1$  — subspace (closed linear subset).

$H_2 = H_1^\perp = \{y \in H \mid y \perp x, x \in H_1\}$  — **orthogonal addition**.

$\forall x \in H$  can be unambiguously written as  $x = x_1 + x_2$ ,  $x_1 \in H_1$ ,  $x_2 \in H_1^\perp$

**Note.**  $H = H_1 \oplus H_1^\perp$

*Proof.*  $x \in H$ ,  $H_1$ ,  $H_2 = H_1^\perp$

$$\exists x_1 \in H_1: \|x - x_1\| = \inf_{n \in H_1} \|x - n\|$$

$$x_2 = x - x_1 \in H_2?$$

$\forall y \in H_1 y \perp x_2$ ,  $\lambda > 0$ ,  $x_1 + \lambda y \in H_1$  ( $H_1$  — subspace)

By definition of best approximation element  $\|x - (x_1 + \lambda^2)\| \geq \|x - x_1\|^2 \quad \forall \lambda > 0$

Let's expand squared norms via inner product.

$$\underbrace{\langle x - x_1 - \lambda y, x - x_1 - \lambda y \rangle}_{x_2} \geq \langle x - x_1, x - x_1 \rangle$$

$$\langle x_2 - \lambda y, x_2 - \lambda y \rangle \geq \langle x_2, x_2 \rangle$$

$$\langle x_2, x_2 \rangle - 2\lambda \langle y, x_2 \rangle + \lambda^2 \langle y, y \rangle \geq \langle x_2, x_2 \rangle : \lambda > 0$$

$$2\langle y, x_2 \rangle \leq \lambda \langle y, y \rangle, \lambda \rightarrow +0 \implies \langle y, x_2 \rangle \leq 0$$

$$\text{Substitute } y \text{ for } -y: \langle -y, x_2 \rangle \leq 0 \implies \langle y, x_2 \rangle \geq 0 \implies \langle y, x_2 \rangle = 0$$

## §1.5 Countably-normed spaces.

**Definition.**  $X$  — linear set,  $p$  — halfnorm (function satisfying all 3 norm axioms, but the first one is weakened  $p(x) \geq 0$ , but  $p$  can be zero on non-zero  $x$ ).

1.  $p(x) \geq 0$
2.  $p(\lambda x) = |\lambda|p(x)$
3.  $p(x + y)$

**Definition.**  $p_1, p_2, \dots, p_n, \dots$  — halfnorms  $\forall n p_n(x) = 0 \implies x = 0$  ( $X, p_1, p_2, \dots, p_n, \dots$ )  
— **countably normed space**.

$x = \lim x_m \iff \forall n \in \mathbb{N} \lim_{m \rightarrow \infty} (x_m - x) = 0 \parallel \cdot \parallel x_m \rightarrow x' \implies \forall n p_n(x_m - x') \rightarrow 0 x_m \rightarrow x'' \implies \forall n p_n$  If in countably-normed space we assume  $p(x, y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{p_n(x - y)}{1 + p_n(x - y)} \mathbb{R}^{\infty}$

Thus countably-normed space is always metrizable. In countably-normed space two linear operations  $(x + y, \lambda x)$  are continuous, which means any countably normed space is also a topological vector space.

**Example 11.**  $C^{\infty}[a, b] = \{ x(t), t \in [a, b] \mid x(t) \text{— infinitely diff.} \}$   
 $p_n(x) = \max_{[a, b]} |x^{(n)}(t)| \quad n = 0, 1, 2, \dots$

From the next theorem we will see that  $C^{\infty}[a, b]$  is non-normalizable (has no norm convergence by which is equivalent to halfnorm convergence). We will try to deduce the criteria of countably-normed space normalizability.

**Definition.** System of halfnorms is called **monotonous** if  $\forall x \in X, \forall n \in \mathbb{N} p_n(x) \leq p_{n+1}(x)$

**Definition.**  $p_n \sim q_n$  if they have the same convergence (limits in both systems are equal).

**Definition.**  $p_m \text{ in } \{p_n\}$  — **essential** if it can not be majorized by any of the preceding halfnorms.  $p_m$  can be majorized by  $p_n$  if  $\exists C, \forall x \in X p_n(x) \leq C \cdot p_m(x)$ .

**Statement 1.12.** For any halfnorm system there exists equivalent monotonous system.

*Proof.* Let  $q_n(x) = \sum_{k=1}^n p_k(x)$ , it is obvious that every  $q_n$  is halfnorm.  $q_n \sim p_n$ ?  $p_n(x_m - x) \rightarrow 0 \implies \sum_{k=1}^n p_k(x_m - x) \rightarrow 0 \implies q_n(x_m - x) \rightarrow 0$ . Backwards proof is the same. ■

This statement allows us to operate only on monotonous halfnorm systems.

**Statement 1.13.** Two monotonous halfnorm systems are equivalent if and only if they majorize each other, i.e. for any halfnorm from first system there exists majorizing halfnorm from another and vice versa.

## §1.6 Minkowski functional.

$X$  — linear set.  $M \subset X$ ,  $M$  — convex  $\iff x, y \in M \implies \alpha x + \beta y \in M$ ,  $\alpha + \beta = 1, \alpha, \beta \geq 0$

**Definition.**  $M$  — **absorbs**  $A \subset X$  if  $\exists \lambda_0: \forall \lambda: |\lambda| \geq \lambda_0 \implies A \subset \lambda M = \{ \lambda x, x \in M \}$

**Definition.** If  $M$  absorbs any finite number of points, its is called **radial set**.

**Definition.**  $M$  is called **circled**, if  $\forall \lambda: |\lambda| \leq 1 \implies \lambda M \subset M$

**Example 12.**  $X$  — NS,  $\overline{V} = \{ \|x\| \leq 1 \}$  — convex, radial and circled.

Next definition has fundamental meaning for our theory.

**Definition.**  $M$  — radial set.  $\forall x \in X$

$\varphi_M(x) = \inf \{ \lambda \geq 0 \mid x \in \lambda M \}$  — **Minkowski functional**.

**Example 13.**  $\varphi_{\overline{V}}(x) = \|x\|$

It is easy to find out the following:

**Statement 1.14.**  $\varphi_M$  is a halfnorm on  $X$  if and only if  $M$  — radial, convex and circled.

## §1.7 Linear functionals and their dimensionality.

## §1.8 Kolmogorov theorem

## **2 Elements of functional analysis**