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Contents

1 Vector spaces

§1.1 Metric spaces.

X, ρ: $X \times X \to \mathbb{R}_+$

Definition. ρ — **metric**

1. $\rho(x,y) \geqslant 0, = 0 \iff x = y$
2. $\rho(x,y) = \rho(y,x)$
3. $\rho(x,y) \leqslant \rho(x,z) + \rho(y,z)$

Definition. (X, ρ) — **Metric space**.

Definition. $x = \lim x_n \iff \rho(x_n, x) \to 0$

$X, \tau = \{ G \subset X \}$

Definition. Let X be arbitrary set. Then system of its subsets τ is called a **topology** if :

1. $\varnothing, X \in \tau$
2. $G_\alpha \in \tau, \alpha \in \mathscr{A} \implies \bigcup\limits_{\alpha} G_\alpha \in \tau$
3. $G_1, \ldots, G_n \in \tau \implies \bigcap\limits_{j=1}^{n} G_j \in \tau$

And any set $G \in \tau$ is called **open**.

Definition. (X, τ) — **Topological space**.

$x = \lim x_n \quad \forall G \in \tau : x \in G \quad \exists N : \forall n > N \quad x_n \in G$
G — open in τ
$F = X \setminus G$ — closed

Definition. $B_r(a) = \{ x \mid \rho(x,a) < r \}$ — **open ball**

Statement 1.1. Any metric space gives rise to a topological space in a rather simple way. Let's call the subset $G \subset X$ open if and only if $\forall x \in G$ there is some r such that open ball $B_r(x)$ is contained in G. Then $\tau = \bigcup B_r(x)$

Statement 1.2. $b \in B_{r_1}(a_1) \cap B_{r_2}(a_2) \implies \exists r_3 > 0 : B_{r_3}(a_3) \subset B_{r_1}(a_1) \cap B_{r_2}(a_2)$

Example 1. $\mathbb{R}, \rho(x,y) = |x - y|$, MS

Example 2. $\bar{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n, \rho(\bar{x}, \bar{y}) = \sqrt{\sum\limits_{j=1}^{n} (x_j - y_j)^2}, MS$

Example 3. $\bar{x} = (x_1, \ldots, x_n, \ldots) \in \mathbb{R}^\infty$

$$\alpha\bar{x} = (\alpha x_1, \ldots, \alpha x_n, \ldots)$$
$$\bar{x} + \bar{y} = (x_1 + y_1, \ldots, x_n + y_n, \ldots)$$

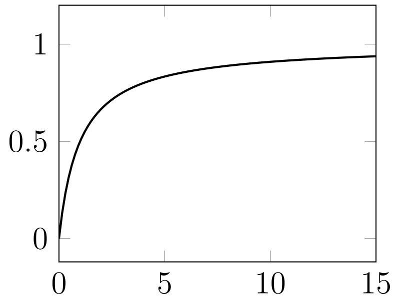
Let's define $\lim\limits_{m\to\infty} \bar{x}_m$

- in $\mathbb{R}^n$: $\bar{x}_n \to \bar{x} \iff \forall j = 1, \ldots, n \qquad x_j^{(m)} \xrightarrow[m\to\infty]{} x_j$

- in $\mathbb{R}^\infty$: $\bar{x}_m \to \bar{x} \overset{def}{\iff} \forall j = 1, 2, 3, \dots \quad x_j^{(m)} \xrightarrow[m\to\infty]{} x_j$

Definition. $\rho(\bar{x}, \bar{y}) \overset{\text{def}}{=} \sum\limits_{n=1}^{\infty} \frac{1}{2^n} \underbrace{\frac{|x_n - y_n|}{1 + |x_n - y_n|}}_{\varphi(|x_n - y_n|)}$ — **Urysohn metric**.

$\varphi(t) = \dfrac{t}{1+t}$



$\varphi(t_1) + \varphi(t_2) \geqslant \varphi(t_1 + t_2)$

$\varphi(t_1) + \varphi(t_2) = \dfrac{t_1}{1+t_1} + \dfrac{t_2}{1+t_2} \geqslant \dfrac{t_1}{1+t_1+t_2} + \dfrac{t_2}{1+t_1+t_2}$

$\varphi(t_1) + \varphi(t_2) \geqslant \dfrac{t_1 + t_2}{1+t_1+t_2} = \varphi(t_1 + t_2)$

Statement 1.3. $\rho(\bar{x}_m, \bar{x}) \xrightarrow[m\to\infty]{} 0 \iff x_j^{(m)} \to x_j \ \forall j$

Proof.

- $\Rightarrow$
  $f(|x_k^{(n)} - x_k|) \leqslant 2^k \rho(x^{(n)}, x)$
  Let $\rho(x^{(n)}, x) \leqslant \dfrac{\varepsilon}{2^k}$, then $f(|x_k^{(n)} - x_k|) < \varepsilon$
  $|x_k^{(n)} - x_k| = t = \dfrac{1}{1 - f(t)} - 1$, then $t \to 0$

- $\Leftarrow$
  Let's choose k_0 for which $\sum\limits_{k=k_0+1}^{\infty} \dfrac{1}{2^k} < \varepsilon$
  Let's choose n_0 for which $\forall k \leqslant k_0, n > n_0 : |x_k^{(n)} - x_k| < \varepsilon$.
  Then $\rho(x^{(n)}, x) < \sum\limits_{k=1}^{k_0} \dfrac{\varepsilon}{2^k} + \varepsilon < 2\varepsilon$
  Letting $\varepsilon \to 0$, we get what we want ■

In this way $\mathbb{R}^\infty$ is a metrizable space.

Example 4. $X, \rho(x, y) \overset{\text{def}}{=} \begin{cases} 0, & x = y \\ 1, & x \neq y \end{cases}$ — **Discrete metric**.

$x_n \to x,\ \varepsilon = \frac{1}{2},\ \exists M :\ m > M \implies \rho(x_m, x) < \frac{1}{2} \implies$
$\rho(x_m, x) = 0 \implies x_m = x$

Definition.

$(X, \tau);\ \forall A \subset X;$

$\text{Int}(A) \overset{\text{def}}{=} \bigcup\limits_{G \subset A} G$ is open;

$\text{Cl}(A) \overset{\text{def}}{=} \bigcap\limits_{A \subset G} G$ is closed;

$\text{Fr}(A) \overset{\text{def}}{=} \text{Cl}(A) \setminus \text{Int}(A)$

(X, ρ); Having a metric space one can describe closure of a set.

$$\rho(x, A) \stackrel{\text{def}}{=} \inf_{a\in A} \rho(x, a)$$
$$\rho(A, B) \stackrel{\text{def}}{=} \inf_{\substack{a\in A \\ b\in B}} \rho(a, b)$$
$$\rho(x, A) = f(x), x \in X$$

Statement 1.4. Function $f(x)$ is continuous.

Proof. $\forall x, y \in X$
$f(x) = \rho(x, A) \underset{\forall \alpha\in A}{\leqslant} \rho(x, \alpha) \leqslant \rho(x, y) + \rho(y, \alpha)$
$\forall \varepsilon > 0\ \exists \alpha_\varepsilon \in A:\ \rho(y, \alpha_\varepsilon) < \rho(y, A) + \varepsilon = f(y) + \varepsilon$
$f(x) \leqslant f(y) + \varepsilon + \rho(x, y),\ \varepsilon \to 0$
$$\begin{cases} f(x) \leqslant f(y) + \rho(x, y) \\ f(y) \leqslant f(x) + \rho(x, y) \end{cases} \implies |f(x) - f(y)| \leqslant \rho(x, y)$$ ■

Statement 1.5. $x \in \mathrm{Cl}(A) \iff \rho(x, A) = 0$

Let's look at the metric spaces in terms of separation of sets from each other by open sets.
x, y
$r = \rho(x, y) > 0$
$B_{r/3}(x),\ B_{r/3}(y)$
In any metric space separability axiom is true.

Theorem 1.1.
Any metric space is a normal space,
i.e. $\forall$ *closed disjoint* $F_1, F_2 \in X,\ \exists$ *open disjoint* G_1, G_2: $F_j \in G_j,\ j = 1, 2$

Proof. $g(x) = \dfrac{\rho(x, F_1)}{\rho(x, F_1) + \rho(x, F_2)}$ — continuous on X
$x \in F_1,\ \mathrm{Cl}(F_1) = F_1,\ \rho(x, F_1) = 0,\ g(x) = 0$
$x \in F_2,\ g(x) = 1$
Let's look at $(-\infty; \frac{1}{3}), (\frac{2}{3}, \infty)$ — by continuity their inverse images under g are open.
$G_1 = g^{-1}(-\infty; \frac{1}{3})$
$G_2 = g^{-1}(\frac{2}{3}; \infty)$ ■

Definition. Metric space is **complete** if $\rho(x_n, x_m) \to 0 \implies \exists x = \lim x_n$
$\mathbb{R}^\infty$ — complete (by completeness of the rational numbers).
In complete metric spaces the nested balls principle is true.

Theorem 1.2.
X — *complete metric space,* $\overline{V}_{r_n}$ — *system of closed balls.*

1. $\overline{V}_{r_{n+1}} \subset \overline{V}_{r_n}$ — *the system is nested.*
2. $r_n \to 0$

Then: $\bigcap_n \overline{V}_{r_n} = \{a\}$

Proof. Let b_n be centers of $\overline{V}_{r_n}$,
$m \geqslant n,\ b_m \in \overline{V}_{r_n},\ \rho(b_m, b_n) \leqslant r_n \to 0\ \forall m \geqslant n$
$\rho(b_m, b_n) \to 0 \xRightarrow{compl.} \exists a = \lim b_n$ Since the balls are closed a $\in$ every ball.
$r_n \to 0 \implies$ there is only one common point. ■

(X, τ) — topological space
$A \subset X,\ \tau_a = \{\, G \cap A, G \in \tau \,\}$ — topology induced on A

Definition. X— metric space, $A \subset X,\ \mathrm{Cl}(A) = X$
Then: A — **dense** in X
If $\mathrm{Int}(\mathrm{Cl}(A)) = \varnothing$ A — **nowhere dense** in X.

Note. It is easy to understand, that in metric spaces nowhere density means the following: $\forall$ ball $V\ \exists V' \subset V \colon V'$ contains no points from A.

Definition. X is called **first Baire category set**, if it can be written as at most countable union of x_n each nowhere dense in X.

Theorem 1.3 (Baire category theorem).
Complete metric space is second Baire category set in itself.

Proof. Let X be first Baire category set.
$X = \bigcup\limits_n X_n \quad \forall \overline{V}\ X_1$ is nowhere dense.
$\overline{V}_1 \subset \overline{V} \colon\ \overline{V}_1 \cap X_1 \ = \varnothing$
X_2 is nowhere dense $\overline{V}_2 \subset \overline{V}_1 \colon \overline{V}_2 \cap X_2 = \varnothing$
$r_2 \leqslant \dfrac{r_1}{2}$
$\vdots$
$\{\overline{V}_n\},\ r_n \to 0,\ \bigcap\limits_n \overline{V}_n = \{a\},\ X = \bigcup X_n,\ \exists n_0 \colon a \in X_{n_0}$
$X_{n_0} \cap \overline{V}_{n_0} = \varnothing \rightarrow\leftarrow\ \ a \in \overline{V}_{n_0}$ ∎

Corollary 1.3.1. *Complete metric space without isolated points is uncountable.*

Proof. No isolated points are present $\implies$ every point in the set is nowhere dense in it. Let X be countable: $X = \bigcup\limits_n \{X_n\}$, then it is first Baire category set. $\rightarrow\leftarrow$ ∎

Definition. K — **compact** if

1. $K = \mathrm{Cl}(K)$
2. $x_n \in K\ \exists n_1 < n_2 < \dots\ x_{n_j}$ – converges in X.

If only 2 is present, the set is called **precompact**.

Theorem 1.4 (Hausdorff).
Let X — metric space, K — closed in X.
Then: K — compact $\iff$ K — totally bounded,
i.e. $\forall \varepsilon > 0\ \exists a_1, \dots, a_p \in X \colon\ \forall b \in K\ \exists a_j \colon\ \rho(a_j, b) < \varepsilon$
($a_1, \dots, a_p$ — finite ε-net)

Proof.

- Totally bounded $\implies$ compact

  K is totally bounded, $x_n \in K\ n_1 < n_2 < \dots < n_k < \cdots$, x_n converges in K

  $$\varepsilon_k \downarrow\to 0\ \varepsilon_1 \quad K \subset \bigcup_{j=1}^{p} \overline{V}_j,\ rad = \varepsilon_1 \qquad (\varepsilon_1\text{-net})$$

  n is finite $\implies$ one ball will contain infinetely many x_n elements.

Let's look at $\overline{V}_{j_0} \cap K$ — totally bounded $= K_1, \operatorname{diam}(K_1) \leqslant 2\varepsilon_1$

$\varepsilon_2 \quad K_1 \subset \bigcup\limits_{j=1}^{n} \overline{V'}_j,\ \text{rad} = \varepsilon_2.$

Then one of $\overline{V'}$ contains infinitely many elements of the sequence contained in K_1.

$\overline{V'}_{j_0} \cap K_1 = K_2,\ \operatorname{diam}(K_2) \leqslant 2\varepsilon_2$ and so on.

$$K_n \supset K_{n+1} \supset K_{n+2} \supset \dots,\ \operatorname{diam}(K_N) \leqslant 2\varepsilon_n \xRightarrow{\text{by compl.}} \underbrace{\bigcap_{n=1}^{\infty} K_n}_{\operatorname{diam}(K_n)\to 0,\ \{x\}} \neq \varnothing$$

Take x_{n_1} from K_1, x_{n_2} from $K_2 \dots$

- Compact $\implies$ totally bounded

  K — compact $\forall \varepsilon\ \exists$ finite ε-net?
  By contradiction: $\exists \varepsilon_0 > 0:$ finite ε_0-net is impossible to construct.
  $\forall x_1 \in K\ \exists x_2 \in K:\ \rho(x_1, x_2) > \varepsilon_0$ (or else system of x_1 — finite ε-net)
  $\{x_1, x_2\}$ - choose $x_3 \in K: \rho(x_3, x_i) > \varepsilon_0,\ i = 1, 2$ and so on.
  $x_n \in K:\ n \neq m\ \rho(x_n, x_m) > \varepsilon_0$ — contains no converging subsequence $\implies$ set is not a compact. $\rightarrow\leftarrow$ ■

§1.2 Normed spaces

Definition. X — **linear set**, $x+y, \alpha\cdot x$, $\alpha\in\mathbb{R}$
The purpose of norm definition, is to construct a topology on X, so that 2 linear operations are continuous on it.

$\varphi\colon X\to\mathbb{R}$:

1. $\varphi(x)\geqslant 0,\ =0 \iff x=0$
2. $\varphi(\alpha x)=|\alpha|\varphi(x)$
3. $\varphi(x+y)\leqslant\varphi(x)+\varphi(y)$

Definition. φ — **norm** on X, $\varphi(x)=\|x\|$

$\rho(x,y)\stackrel{\text{def}}{=}\|x-y\|$ — metric on X.

Definition.
$(X,\|\cdot\|)$ — **normed space** — special case of metrical space.

$x=\lim x_n \stackrel{def}{\Longrightarrow} \rho(x_n,x)\to 0 \iff \|x_n-x\|\to 0$

Statement 1.6. In the topology of a normed space linear operations are continuous on X.

Proof.

1. $x_n\to x,\ y_n\to y;\ \|(x_n+y_n)-(x+y)\|=\|(x_n-x)+(y_n-y)\|\leqslant$
$$\leqslant \underbrace{\|x_n-x\|}_{\substack{\downarrow\\0}}+\underbrace{\|y_n-y\|}_{\substack{\downarrow\\0}}$$
$$\Longrightarrow x_n+y_n\to x+y$$

2. $\alpha_n\to\alpha,\ x_n\to x;\ \|\alpha_n x_n-\alpha x\|=\|(\alpha_n-\alpha)x_n+\alpha(x_n-x)\|\leqslant$
$$\leqslant \underbrace{|\alpha_n-\alpha|}_{\substack{\downarrow\\0}}\cdot\underbrace{\|x_n\|}_{bounded}+\underbrace{\alpha\|x_n-x\|}_{\substack{\downarrow\\0}}$$
$x_n\to x \Longrightarrow \|x_n\|$ — bounded.
$\alpha_x x_n\to\alpha x$ ∎

Statement 1.7. From the triangle inequality $\big|\|x\|-\|y\|\big|\leqslant\|x-y\|$
$x_n\to x \Longrightarrow \|x_n\|\to\|x\|$
Norm is continious.

Example 5. $\mathbb{R}^n$

1. $\|\bar{x}\|=\sqrt{\sum\limits_{k=1}^{n}x_k^2}$
2. $\|\bar{x}\|_1\stackrel{\text{def}}{=}\sum\limits_{k=1}^{n}|x_k|$
3. $\|\bar{x}\|_2\stackrel{\text{def}}{=}\max\{|x_1|,\dots,|x_n|\}$
4. $C[a,b]$ — functions continuous on $[a,b]$; $\|f\|=\max\limits_{x\in[a,b]}|f(x)|$

5. $L_p(E) = \left\{ f \text{ — measurable, } \int\limits_E |f|^p < +\infty \right\}$

$$p \geqslant 1,\ \|f\|_p = \left(\int\limits_E |f|^p \right)^{\dfrac{1}{p}}$$

Because the set of points is the same, arises the question about convergence comparison. $\|\cdot\|_1 \sim \|\cdot\|_2,\ x_n \overset{\|\cdot\|_1}{\to} x \iff x_n \overset{\|\cdot\|_2}{\to} x$

Statement 1.8.
$\|\cdot\|_1 \sim \|\cdot\|_2 \iff \exists a, b > 0 \colon \forall x \in X \implies a\|x_1\|_1 \leqslant \|x\|_2 \leqslant b\|x\|_1$

Theorem 1.5 (Riesz)**.**
$X,\ \dim X < +\infty$ — *linear set.*
Then: *Any pair of norms in* X *are equivalent.*

Proof. $l_1, \ldots, l_n$ — linearly independent from X. $\forall x \in X = \sum\limits_{k=1}^{n} \alpha_k l_K$

$$\bar{x} \leftrightarrow (l_1, \ldots, l_n) = \bar{l} \in \mathbb{R}^n$$

Let $\|\cdot\|$ — some norm in X.

$$\|x\| \underset{\triangle}{\leqslant} \sum_{k=1}^{n} \|l_k\||\alpha_k| \underset{\text{Cauchy}}{\leqslant} \underbrace{\sqrt{\sum_{k=1}^{n} \|l_k\|^2}}_{const(B), B-basis} \underset{\|\bar{\alpha}\| = \|x\|_1}{\underset{\|}{\sqrt{\sum_{k=1}^{n} |\alpha_k|^2}}}$$

$$\|x\|_1 = \sqrt{\sum_{k=1}^{n} \|\alpha_k\|^2},\ x = \sum \alpha_k l_k$$

$$\|x\| \leqslant b\|x\|_1$$

$$?\exists a > 0 \colon a\|x\|_1 \leqslant \|x\| \implies \|\cdot\| \sim \|\cdot\|_1$$

Let $f(\alpha_1, \ldots, \alpha_n) = \left\| \sum\limits_{k=1}^{n} \alpha_k l_k \right\|$

$$|f(\bar{\alpha} + \Delta\bar{\alpha}) - f(\bar{\alpha})| = \left| \left\| \sum_{k=1}^{n} \alpha_k l_k + \sum_{k=1}^{n} \Delta\alpha_k l_K \right\| - \left\| \sum_{k=1}^{n} \alpha_k l_k \right\| \right| \leqslant$$

$$\left\| \sum_{k=1}^{n} \Delta\alpha_k l_k \right\| \leqslant \underbrace{\sum \|l_k\||\Delta\alpha_k|}_{\underset{0, \Delta\alpha_k \to 0}{\downarrow}} \implies f \text{ is continuous on } \mathbb{R}^n$$

$S_1 = \left\{ \sum\limits_{k=1}^{n} |\alpha_k|^2 = 1 \right\} \subset \mathbb{R}^m$, f — continuous on S_1, S_1 — compact, $\bar{\alpha}^* \in S_1$
By Weierstrass theorem there exists a point $\alpha^* \in S_1$ on a sphere, in which function f achieves its minimum $\implies \forall \alpha \in S_1\ f(\bar{\alpha}^*) \leqslant f(\bar{\alpha})$

If $f(\bar{\alpha}^*) = 0$, then $\left\| \sum\limits_{k=1}^{n} \alpha_k^* l_k \right\| = 0 \implies \sum\limits_{k=1}^{n} \alpha_k^* l_k = 0,\ \bar{\alpha}^* \in S_1$,
but $l_1 \dots l_n$ are linearly independent $\to\leftarrow \implies \min\limits_{S_1} f = m > 0$

$$\|x\| = \left\| \sum_{k=1}^{n} \alpha_k l_k \right\| = f(\bar{\alpha}) = \sqrt{\sum_{k=1}^{n} \alpha_k^2} \cdot \left\| \sum \underbrace{\boxed{\frac{\alpha_k}{\sqrt{\sum_{k=1}^{n} \alpha_k^2}}}}_{\beta_k} \cdot l_k \right\| \geqslant,\ \bar{\beta} = (\beta_1 \dots \beta_n) \in S_1$$

$$\geqslant m \cdot \|x\|_1,\ a = m \qquad \blacksquare$$

Corollary 1.5.1. X — *NS*, $Y \subset X, \dim Y < +\infty \implies Y = \mathrm{Cl}(Y)$

Note. Functional analysis differentiates between linear subset (set of points, closed by addition and scalar multiplication) and linear subspace (closed linear subset).

Proof. $Y = \mathscr{L}(l_1, \dots, l_n) = \left\{ \sum\limits_{i=1}^{n} \alpha_i l_i \;\middle|\; l_1, \dots, l_n - \text{lin. indep.} \right\}$
$y_m \in Y, y_m \to y$ in $X \implies y \in Y$?
$\|y_m - y\| \to 0 \implies \|y_m - y_p\| \to 0,\ m, p \to \infty$
$\|y\|, y \in Y$.
By Riesz theorem all norm in Y are equivalent.
$y = \sum\limits_{j=1}^{n} \alpha_j l_J, \|y\|_0 = \sqrt{\sum\limits_{j=1}^{n} \alpha_j^2}$ — some norm (by linear independance).
By Riesz theorem $\|y\| \sim \|y\|_0$
$\underbrace{\|y_m - y_p\|}_{\in Y} \to 0 \implies \|y_m - y_p\|_0 \to 0$
Notice that convergence by $\|\cdot\|_0$ is coordinatewise.
$\bar{\alpha} = (\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n$ $y_m = \sum\limits_{i=1}^{n} \alpha_i^{(m)} l_i$
$|\alpha_i^{(m)} - \alpha_i^{(l)}| \to 0\ \forall i = 1, \dots, n;\ \bar{\alpha} = (\alpha_1^{(m)}, \dots, \alpha_n^{(m)}) \to \alpha^* = (\alpha_1^*, \dots, \alpha_n^*)$
$y^* = \sum\limits_{i=1}^{n} \alpha_i^* l_I \in Y,\ \|y_m - y\| \to 0$
Bu the limit uniqueness $y = y^* \implies y \in Y$ $\blacksquare$

Definition. If normed space if complete, then it is called **B-space** or **Banach space**.

Example 6. $C[a, b]$ — functions continuous on $[a, b]$.

Example 7. Lebesgue space, $p \geqslant 1, L_p(E) = \left\{ f \text{ is measurable on } E,\ \int\limits_E |f|^P < +\infty \right\}$.

If X — Banach space,

$$\sum_{n=1}^{\infty} x_n = \lim_{n\to\infty} \sum_{k=1}^{n} x_k,\ \sum_{n=1}^{\infty} \|x_n\| < +\infty$$

$$\|S_n - S_m\| = \left\| \sum_{k=m+1}^{n} x_k \right\| \leqslant \sum_{m+1}^{n} \|x_k\| \xrightarrow[n,m\to\infty]{} 0$$

$$\implies \|S_n - S_m\| \to 0 \implies \exists \lim_{n\to\infty} S_n, \sum_{k=1}^{n} x_k \text{ converges.}$$

In Banach spaces works the theory of absolute convergence of numerical series.

Lemma 1 (Riesz's lemma about almost perpendicular). *Y — eigen subspace of X — normed space.* $\forall \varepsilon \in (0,1)\ \exists z_\varepsilon \in X$:

1. $z_\varepsilon \notin Y$
2. $\|z_\varepsilon\| = 1$
3. $\rho(z_\varepsilon, Y) > 1 - \varepsilon$

Proof. $\exists x \in X \setminus Y,\ d = \rho(x, Y)$
Suppose $d = 0$ then, $\exists y_n \in Y\colon \|x - y_n\| < \dfrac{1}{n},\ n \to \infty,\ y_n \to x$
$Y = \mathrm{Cl}(Y) \implies x \in Y \to\leftarrow x \notin Y,\ d > 0$
$\forall \varepsilon \in (0,1)\ \dfrac{1}{1-\varepsilon} > 1\ \exists y_\varepsilon \in Y\colon \|x - y_\varepsilon\| < \dfrac{1}{1-\varepsilon} d$
$z_\varepsilon = \dfrac{x - y_\varepsilon}{\|x - y_\varepsilon\|},\ \|z_\varepsilon\| = 1$

$$\forall y \in Y\ \|z_\varepsilon - y\| = \left\| \frac{x - y_\varepsilon}{\|x - y_\varepsilon\|} - y \right\| = \frac{\|x - \overbrace{(y_\varepsilon + \|x - y_\varepsilon\| \cdot y)}^{\in Y}\| \geqslant d}{\|x - y_\varepsilon\| < \frac{1}{1-\varepsilon} d} > 1 - \varepsilon$$ ■

Corollary 1.5.2. *X — normed space,* $\dim X = +\infty,\ S = \{\, x \mid \|x\| = 1 \,\}$, *then closed unit ball $\overline{B}$ is not compact in it.*

Proof. $\forall x_1 \in S,\ Y_1 = \mathscr{L}\{x_1\}$ — finite dimensional linear set. $\implies$ closed in $X \implies Y_1$ — subspace.
$\dim X = +\infty > \dim Y_1 \implies Y_1$ — eigen subspace.
Then by the Riesz lemma ($\varepsilon = \frac{1}{2}$):

$$\exists x_2 \in X\colon \|x_2\| = 1,\ \|x_2 - x_1\| > \frac{1}{2} \text{ (Notice that } x_2 \text{ appears to be an element of } S)$$

$$Y_2 = \mathscr{L}\{x_1, x_2\}\ \exists x_3 \in S\colon \|x_3 - x_j\| > \frac{1}{2},\ j = 1, 2$$

Continue by induction. Because $\dim X = +\infty$ the process willl never finish.
$x_n \in S\colon \|x_n - x_m\| > \dfrac{1}{2},\ n \neq m$ — obviously we cannot extract converging subsequence.
$\implies S$ — not a compact. And that means that $\overline{B} \supset S$ is not a compact either. ■

§1.3 Inner product (unitary) spaces

Definition. X — linear space.
$\varphi\colon X \times X \to \mathbb{R}$

1. $\varphi(x,x) \geqslant 0, \quad \varphi(x,x) = 0 \iff x = 0$
2. $\varphi(x,y) = \varphi(y,x)$
3. $\varphi(\alpha x + \beta y, z) = \alpha\varphi(x,z) + \beta\varphi(y,z)$

φ — **inner product**.
$\varphi(x,y) = \langle x, y\rangle$

Definition. $(X, \langle\cdot,\cdot\rangle)$ — **inner product space**.

Example 8. $\mathbb{R}^n, \langle \bar{x}, \bar{y}\rangle = \sum\limits_{j=1}^{n} x_j y_j$

Statement 1.9 (Schwarz)**.** $\forall x, y \in X \quad |\langle x,y\rangle| \leqslant \sqrt{\langle x,x\rangle}\cdot\sqrt{\langle y,y\rangle}$

Proof. Consider $\lambda \in \mathbb{R}$.

$$f(\lambda) = \underset{\lambda^2\langle x,x\rangle + 2\lambda\langle x,y\rangle + \langle y,y\rangle}{\underset{\|}{\langle \lambda x + y, \lambda x + y\rangle}} \geqslant 0$$

$$D = 4\langle x,y\rangle^2 - 4\langle x,x\rangle\cdot\langle y,y\rangle \leqslant 0$$ ■

Corollary 1.5.3 (Cauchy inequality for sums)**.** *Consider* $X = \mathbb{R}^n$, $\|x\| \overset{def}{=} \sqrt{\langle x,x\rangle}$*. Then*

$$\|x+y\|^2 = \langle x+y, x+y\rangle = \|x\|^2 + 2\cdot\underbrace{\langle x,y\rangle}_{\leqslant\|x\|\cdot\|y\|} + \|y\|^2 \leqslant \big(\|x\| + \|y\|\big)^2$$

Any inner product space is a special case of a normed space. The specifics is that we can measure the angles between points:

$$x \perp y \iff \langle x,y\rangle = 0$$

In this case the Pythagorean theorem takes place:

$$\|x+y\|^2 = \|x\|^2 + \|y\|^2$$

In inner product spaces the parallelogram law plays a significant role:

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2 \quad \forall x, y \in X$$

In an inner product space norm is determined by inner product: $\|x\|^2 = \langle x,x\rangle$
It can be proved that if parallelogram law holds, then the norm must be determined by some inner product. Let X be some normed space, $x \in X$, then $\langle\cdot,\cdot\rangle \mapsto \|x\| = \sqrt{\langle x,x\rangle}$. For any norm satisfying the parallelogram law, the inner product generating the norm is unique.

Example 9. $C_{[a,b]}$, $\|f\| = \max\limits_{x\in[a,b]} |f(x)|$, $\|f\|$ doesn't satisfy the parallelogram law and thus is not determined by any inner product. This fact implies that $C_{[a,b]}$ is not an inner product space.

Definition. Orthonormal set — a set of points $\{ l_1, l_2, \dots \}$ (may be finite):

1. $\|e_i\| = 1$
2. $l_i \perp e_j,\ i \neq j$

Every orthonormal set is linearly independent.

Definition. Let $x \in X, \{ e_i \}$ — ONS. Then
$\langle x, e_j \rangle$ — **abstract Fourier coefficient**,
$\sum\limits_j \langle x, e_j \rangle e_j$ — **abstract Fourier series** of point x.

Fourier series is a special case of orthogonal series.

Definition. $\sum\limits_j x_j$ — **orthogonal series** $\iff x_i \perp x_j, \quad i \neq j$

Let $\sum\limits_{j=1}^{\infty} x_j, S_m = \sum\limits_{j=1}^{m} x_j$. Then

$$\|S_m\|^2 = \Big\langle \sum_{j=1}^{m} x_j, \sum_{j=1}^{m} x_j \Big\rangle = \sum_{j=1}^{m} \|x_j\|^2$$

This fact allows us to effectively build the theory of orthogonal series.

An important problem is concerned with Fourier series. Let X is a normed space, Y is a subspace of X,

$$\forall x \in X \quad E_Y(x) = \rho(x, Y) = \inf_{y \in Y} \|x - y\|$$

Definition. $E_Y(x)$ — **best approximation** of point x with points of the subspace Y. If $\exists y^* \in Y\ E_Y(x) = \|x - y^*\|$, then y^* is called the **element of best approximation**.

Theorem 1.6 (Borel)**.**
$\dim Y < +\infty \implies \forall x \in X\ \exists y^* \in Y$ — *element of best approximation.*

Proof. $Y = \mathscr{L}(\underbrace{l_1, l_2, \dots, l_n}_{\text{lin. indep.}})$ Consider $f(\alpha_1, \dots, \alpha_n) = \Big\|x - \sum\limits_{k=1}^{n} \alpha_k l_k\Big\| \to \min$. By the triangle inequality for norm, $f(\bar{\alpha})$ is continous on $\mathbb{R}^n, f \geqslant 0, E_Y(x) = \inf\limits_{\bar{\alpha} \in \mathbb{R}^n} f(\bar{\alpha})$. It is easy to find out that there always is a ball $B(0, r) \subset \mathbb{R}^n$, outside of which $f > 2E_Y(x)$. So, $E_Y(x)$ is somewhere inside. But f is continuous, ball $B(0, r)$ is compact, so, by the Weierstrass theorem, the minimum exists and is located inside the $B(0, r)$. ■

For abstract Fourier series the Borel theorem can be significantly strengthened by specifying the best approximation element.

Theorem 1.7 (extreme quality of Fourier series' partial sums)**.**
$\{e_j\}$ — *ONS in* X
$H_n = \mathscr{L}(e_1, \dots, e_n)$
$E_{H_n}(x), S_n(x) = \sum\limits_{j=1}^{n} \langle x, e_j \rangle e_j \implies E_{H_n}(x) = \|x - S_n(x)\|$

Proof. $y = \sum\limits_{j=1}^{n} \alpha_j e_j \in H_n$

$$\begin{aligned}\|x-y\|^2 = \left\langle x - \sum \alpha_j e_j, x - \sum \alpha_j e_j \right\rangle = \|x\|^2 - 2\sum \alpha_j \langle x, e_j\rangle + \sum \alpha_j^2 = \\ = \underbrace{\|x\|}_{\text{const}}{}^2 + \sum (\alpha_j - \langle x, e_j\rangle)^2 - \underbrace{\sum \langle x, e_j\rangle^2}_{\text{const}} \to \min\end{aligned}$$

So, the sum goes to minimum when the second summand is minimal. Obviously, it's minimal when $\forall(\alpha_j - \langle x, l_j\rangle) = 0$. $E_Y(x)$ — Fourier sum. ■

Corollary 1.7.1 (Bessel's inequality)**.** $\sum\limits_{j} \langle x, e_j\rangle^2 \leqslant \|x\|^2$

Proof. Bessel's inequality follows from the identity:

$$\begin{aligned}0 \leqslant \|x-y^*\|^2 = \left\| x - \sum_{j=1}^{n} \langle x, e_j\rangle e_j \right\|^2 &= \|x\|^2 - 2\sum_{j=1}^{n} |\langle x, e_j\rangle|^2 + \sum_{j=1}^{n} |\langle x, e_j\rangle|^2 \\ &= \|x\|^2 - \sum_{j} \langle x, e_j\rangle^2\end{aligned}$$

■

Corollary 1.7.2. *The series of Fourier coefficients' squares always converges*

§1.4 Hilbert space.

Definition. Hilbert space — complete, infinite dimensional, inner product space.

Example 10. $L_2(E)$ — Hilbert space.
$\langle f, g\rangle = \int\limits_E f \cdot g \, d\mu$

$l_2 = \left\{ (x_1, \dots, x_n, \dots) \;\middle|\; \sum\limits_{n=1}^{\infty} x_n^2 < +\infty \right\}$

$\langle \bar{x}, \bar{y}\rangle = \sum\limits_{n=1}^{\infty} x_n y_n,\ E = \mathbb{N},\ \mu\{m\} = 1$

When we have completeness we can define orthonormal basis.

Definition. H, $\{e_n\}$ — ONS : $\forall x = \sum\limits_{n=1}^{\infty} \alpha_n e_n$ Then e_n is called **orthonormal basis**.
Let's look at the inner product of arbitrary vector x from H and basis vector e_m
$\langle x, e_m\rangle = \sum\limits_{n=1}^{\infty} \alpha_n \langle e_n, e_m\rangle = \alpha_m$
In this sense basis decomposition is always a Fourier series.

1. Complete ONS: Let $L = \mathscr{L}\{e_1, e_2, \dots\}$, then $H = \mathrm{Cl}(L)$ (L is dense in H)
2. Closed ONS: $\forall m\ \langle x, e_m\rangle = 0 \implies x = 0$.

Statement 1.10. In Hilbert spaces two of the properties outlined above are equivalent.

Statement 1.11. In Hilbert space Fourier series converges for any point.

Proof. Let H — Hilbert space, $\sum\limits_{j=1}^{\infty} \langle x, e_j\rangle e_j$ — abstract Fourier series, $S_n = \sum\limits_{j=1}^{n} \langle x, e_j\rangle e_j$ — it's partial sum. We need to prove, that $\exists \lim\limits_{n\to\infty} S_n$. H is a Hilbert space, which means it is also complete. This means we only have to prove that $\{S_n\}$ is Cauchy sequence, i.e. $\forall \varepsilon > 0\ \exists N \in \mathbb{N}\colon \forall n, m > N\ \|S_n - S_m\| < \varepsilon$
Consider $\|S_n - S_m\|^2 = \left\| \sum\limits_{j=1}^{n} \langle x, e_j\rangle e_j - \sum\limits_{j=1}^{m} \langle x, e_j\rangle e_j \right\|^2 = \left\| \sum\limits_{j=m+1}^{n} \langle x, e_j\rangle e_j \right\|^2 = \sum\limits_{j=m+1}^{n} |\langle x, e_j\rangle|^2$
Because numerical series $\sum\limits_{j=m+1}^{n} |\langle x, e_j\rangle|^2$ converges, by the Cauchy criteria we have the following: $\forall \varepsilon > 0\ \exists N \in \mathbb{N}\colon \forall n > m > N \left| \sum\limits_{j=m+1}^{n} |\langle x, e_j\rangle|^2 \right| < \varepsilon^2$ or $\sum\limits_{j=m+1}^{n} |\langle x, e_j\rangle|^2 < \varepsilon^2$, finally we get $\|S_n - S_m\| < \varepsilon$. ■

Proof. 1.10 Complete ONS $\implies$ Closed ONS
$\forall x \in H\ \forall \varepsilon > 0\ \exists \sum\limits_{j=1}^{p} \alpha_{kj} l_{kj}\colon \underbrace{\|x - \sum\limits_{j=1}^{p} \alpha_{kj} l_{kj}\|^2}_{\geqslant \|x - \sum\limits_{j=1}^{k_p} \langle x, l_j\rangle l_j\|^2} \leqslant \varepsilon^2$

$S_m(x)$ by extremety $\|x - S_{m+p}(x)\|^2 \leqslant \|x - S_m(x)\|^2$
Implies partial sums go to x, $x = \sum\limits_{j=1}^{\infty} \langle x, l_j\rangle l_j$ If all fourier coefficients are zero, it means the ONS is closed ($x = 0$).
Closed ONS $\implies$ Complete ONS

$y \in H = \sum\limits_{j=1}^{\infty} \underbrace{\langle x, e_j \rangle}_{=\langle y, e_j \rangle} e_j \implies \langle y, e_j \rangle = \langle x, e_j \rangle \implies \langle y - x, e_j \rangle = 0$

Because ONS is closed $y - x = 0,\ y = x$. Thus we can decompose any point into Fourier series, and this implies that ONS is complete. ■

Considering basis existance.

Definition. Topological space is called **separable** if there exists countable dense set in it. $X = \mathrm{Cl}(\{a_1, \dots, a_n, \dots\})$

H — separable, $a_1, \dots, a_n, \dots$ We can orthogonalize these dots (Gramm-Shmidt), and we will get complete ONS. This means space separability is equivalent to basis existance.

Theorem 1.8 (about best approximation in H)**.**
H — HS, M — closed convex subset of H, then $\forall x \in H\ \exists! y \in M\colon \|x - y\| = \inf\limits_{z \in M} \|x - z\|$. M has element of best approximation for any x from X, and only one.

Proof. $d = \inf\limits_{z \in M} \|x - z\|$ by definition of infimum
$\forall n \in \mathbb{N}\ \exists y_n \in M\colon d \leqslant \|x - y_n\| < d + \dfrac{1}{n}$
$\exists ? y = \lim y_n \in M\ d \leqslant \|x - y\| \leqslant d$
$y_n, y_m \in M$ — convex. Implies

$$\frac{y_n + y_m}{2} \in M \implies d^2 \leqslant \left\| \frac{y_n + y_m}{2} - x \right\|^2 = \frac{1}{4} \left\| \underbrace{(y_n - x)}_{z_1} + \underbrace{(y_m - x)}_{z_2} \right\|^2$$

Let's use parallelogram law. $\|z_1 + z_2\|^2 + \|z_1 - z_2\|^2 = 2\|z_1\|^2 + 2\|z_2\|^2$,

$\underbrace{\|(y_n - x) + (y_m - x)\|^2}_{\geqslant 4d^2} + \|y_n - y_m\|^2 = 2 \overbrace{\|y_n - x\|^2}^{\leqslant \left(d + \frac{1}{n}\right)^2} + 2 \overbrace{\|y_m - x\|^2}^{\leqslant \left(d + \frac{1}{m}\right)^2}$

$\|y_n - y_m\|^2 \leqslant 2(d + \dfrac{1}{n})^2 + 2(d + \dfrac{1}{m})^2 - 4d^2 = 4d\dfrac{1}{n} + \dfrac{2}{n^2} + 4d\dfrac{1}{m} + \dfrac{2}{m^2} \xrightarrow[n,m \to 0]{} 0$
$\|y_n - y_m\| \xrightarrow[n,m \to 0]{} 0 \implies \exists \lim y_n$ ■

Corollary 1.8.1. *H — HS, H_1 — subspace (closed linear subset).*
*$H_2 = H_1^{\perp} = \{\, y \in H \mid y \perp x, x \in H_1 \,\}$ — **orthogonal addition**.*
$\forall x \in H$ can be unambiguously written as $x = x_1 + x_2,\ x_1 \in H_1,\ x_2 \in H_1^{\perp}$

Note. $H = H_1 \oplus H_1^{\perp}$

Proof. $x \in H,\ H_1,\ H_2 = H_1^{\perp}$
$\exists x_1 \in H_1\colon \|x - x_1\| = \inf\limits_{n \in H_1} \|x - n\|$
$x_2 = x - x_1\ \in H_2?$
$\forall y \in H_1 y \perp x_2,\ \lambda > 0,\ x_1 + \lambda y \in H_1 (H_1$— subspace)
By definition of best approximation element $\|x - (x_1 - \lambda^2)\| \geqslant \|x - x_1\|^2 \quad \forall \lambda > 0$
Let's expand squared norms via inner product.
$\langle \underbrace{x - x_1}_{x_2} - \lambda y, x - x_1 - \lambda y \rangle \geqslant \langle x - x_1, x - x_1 \rangle$
$\langle x_2 - \lambda y, x_2 - \lambda y \rangle \geqslant \langle x_2, x_2 \rangle$
$\langle x_2, x_2 \rangle - 2\lambda \langle y, x_2 \rangle + \lambda^2 \langle y, y \rangle \geqslant \langle x_2, x_2 \rangle \ : \lambda > 0$
$2\langle y, x_2 \rangle \leqslant \lambda \langle y, y \rangle,\ \lambda \to +0 \implies \langle y, x_2 \rangle \leqslant 0$
Substitute y for $-y$: $\langle -y, x_2 \rangle \leqslant 0 \implies \langle y, x_2 \rangle \geqslant 0 \implies \langle y, x_2 \rangle = 0$ ■

§1.5 Countably-normed spaces.

Definition. X — linear set, p — halfnorm (function satisfying all 3 norm axioms, but the first one is weakened $p(x) \geqslant 0$, but p can be zero on non-zero x).

1. $p(x) \geqslant 0$
2. $p(\lambda x) = |\lambda| p(x)$
3. $p(x + y)$

Definition. $p_1, p_2, \ldots, p_n, \ldots$ — halfnorms $\forall n p_n(x) = 0 \implies x = 0$ $(X, p_1, p_2, \ldots, p_n, \ldots)$ — **countably normed space**.

$x = \lim x_m \iff \forall n \in \mathbb{N} \lim\limits_{m \to \infty} (x_m - x) = 0$ $\| \cdot \|$ $x_m \to x'$ $\implies \forall n$ $p_n(x_m - x') \to 0 x_m \to x'' \implies \forall n$ p_n If in countably-normed space we assume $p(x, y) = \sum\limits_{n=1}^{\infty} \frac{1}{2^n} \frac{p_n(x-y)}{1 + p_n(x-y)} \mathbb{R}^{\infty}$ Thus countably-normed space is always metrizable. In countably-normed space two linear operations $(x + y, \lambda x)$ are continuous, which means any countably normed space is also a topological vector space.

Example 11. $C^{\infty}[a, b] = \{\, x(t), t \in [a, b] \mid x(t)$— infinetely diff. $\}$
$p_n(x) = \max\limits_{[a,b]} \left| x^{(n)}(t) \right|$ $n = 0, 1, 2, \ldots$

From the next theorem we will see that $C^{\infty}[a, b]$ is non-normalizable (has no norm convergence by which is equivalent to halfnorm convergence). We will try to deduce the criteria of countably-normed space normalizability.

Definition. System of halfnorms is called **monotonous** if $\forall x \in X$, $\forall n \in \mathbb{N}$ $p_n(x) \leqslant p_{n+1}(X)$

Definition. $p_n \sim q_n$ if they have the same convergence (limits in both systems are equal).

Definition. $p_m in \{p_n\}$ — **essential** if it can not be majorized by any of the preceeding halfnorms. p_m can be majorized by p_n if $\exists C, \forall x \in X p_n(x) \leqslant C \cdot p_m(x)$.

Statement 1.12. For any halfnorm system there exists equivalent monotonous system.

Proof. Let $q_n(x) = \sum\limits_{k=1}^{n} p_k(x)$, it is obvious that every q_n is halfnorm. $q_n \sim p_n$? $p_n(x_m - x) \to 0 \implies \sum\limits_{k=1}^{n} p_k(x_m - x) \to 0 \implies q_n(x_m - x) \to 0$. Backwards proof is the same. ■

This statement allows us to operate only on monotonous halfnorm systems.

Statement 1.13. Two monotonous halfnorm systems are equivalent if and only if they majorize each other, i.e. for any halfnorm from first system there exists majorizing halfnorm from another and vice verca.

Proof. If two systems majorize each other obviously they are equivalent. Let two systems be equivalent. p_n and q_n ■