Functional analysis course by Dodonov N.U.

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1 Vector spaces

§1.1 Metric spaces.

X, ρ: $X \times X \to \mathbb{R}_+$

Definition. ρ — **metric**

1. $\rho(x,y) \geqslant 0, = 0 \iff x = y$
2. $\rho(x,y) = \rho(y,x)$
3. $\rho(x,y) \leqslant \rho(x,z) + \rho(y,z)$

Definition. (X,ρ) — **Metric space**.

Definition. $x = \lim x_n \iff \rho(x_n, x) \to 0$

$X, \tau = \{ G \subset X \}$

Definition. Let X be arbitrary set. Then system of its subsets τ is called a **topology** if :

1. $\varnothing, X \in \tau$
2. $G_\alpha \in \tau, \alpha \in \mathscr{A} \implies \bigcup\limits_\alpha G_\alpha \in \tau$
3. $G_1, \dots, G_n \in \tau \implies \bigcap\limits_{j=1}^n G_j \in \tau$

And any set $G \in \tau$ is called **open**.

Definition. (X,τ) — **Topological space**.

$x = \lim x_n \quad \forall G \in \tau : x \in G \quad \exists N : \forall n > N \quad x_n \in G$
G — open in τ
$F = X \setminus G$ — closed

Definition. Given a metric space (X,ρ) an **open ball** with radius r around a is defined as the set $B_r(a) = \{ x \mid \rho(x,a) < r \},\ r \in \mathbb{R}_+$

Statement 1.1.1. Any metric space gives rise to a topological space in a rather simple way. Let's call $G \subset X$ open if and only if $\forall x \in G$ there exists some r such that open ball $B_r(x)$ is contained in G.

Statement 1.1.2. $b \in B_{r_1}(a_1) \cap B_{r_2}(a_2) \implies \exists r_3 > 0 : B_{r_3}(a_3) \subset B_{r_1}(a_1) \cap B_{r_2}(a_2)$

Example 1.1.1. $\mathbb{R}, \rho(x,y) = |x-y|$, MS

Example 1.1.2. $\bar{x} = (x_1, \dots, x_n) \in \mathbb{R}^n, \rho(\bar{x},\bar{y}) = \sqrt{\sum\limits_{j=1}^n (x_j - y_j)^2}, MS$

Example 1.1.3. $\bar{x} = (x_1, \dots, x_n, \dots) \in \mathbb{R}^\infty$

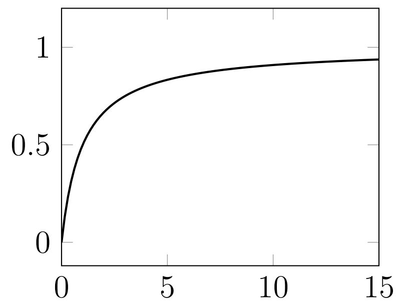
$$\alpha\bar{x} = (\alpha x_1, \dots, \alpha x_n, \dots)$$
$$\bar{x} + \bar{y} = (x_1 + y_1, \dots, x_n + y_n, \dots)$$

Let's define $\lim\limits_{m\to\infty} \bar{x}_m$

- in $\mathbb{R}^n$: $\bar{x}_m \to \bar{x} \iff \forall j = 1, \dots, n \qquad x_j^{(m)} \xrightarrow[m\to\infty]{} x_j$

- in $\mathbb{R}^\infty$: $\bar{x}_m \to \bar{x} \overset{def}{\iff} \forall j = 1, 2, 3, \ldots \quad x_j^{(m)} \xrightarrow[m\to\infty]{} x_j$

Definition. $\rho(\bar{x}, \bar{y}) \overset{\text{def}}{=} \sum\limits_{n=1}^{\infty} \frac{1}{2^n} \underbrace{\frac{|x_n - y_n|}{1 + |x_n - y_n|}}_{\varphi(|x_n - y_n|)}$ — **Urysohn metric**.



$\varphi(t) = \frac{t}{1+t}$

$\varphi(t_1) + \varphi(t_2) \geqslant \varphi(t_1 + t_2)$

$\varphi(t_1) + \varphi(t_2) = \frac{t_1}{1+t_1} + \frac{t_2}{1+t_2} \geqslant \frac{t_1}{1+t_1+t_2} + \frac{t_2}{1+t_1+t_2}$

$\varphi(t_1) + \varphi(t_2) \geqslant \frac{t_1+t_2}{1+t_1+t_2} = \varphi(t_1 + t_2)$

Statement 1.1.3. $\rho(\bar{x}_m, \bar{x}) \xrightarrow[m\to\infty]{} 0 \iff x_j^{(m)} \to x_j \ \forall j$

Proof.

- $\Rightarrow$
  $\varphi(|x_k^{(m)} - x_k|) \leqslant 2^k \rho(x^{(m)}, x)$
  Let $\rho(x^{(m)}, x) \leqslant \frac{\varepsilon}{2^k}$, then $\varphi(|x_k^{(m)} - x_k|) < \varepsilon$
  $|x_k^{(m)} - x_k| = t = \frac{1}{1-\varphi(t)} - 1$, then $t \to 0$
- $\Leftarrow$
  Let's choose k_0 for which $\sum\limits_{k=k_0+1}^{\infty} \frac{1}{2^k} < \varepsilon$
  Let's choose m_0 for which $\forall k \leqslant k_0, m > m_0 : |x_k^{(m)} - x_k| < \varepsilon$.
  Then $\rho(x^{(m)}, x) < \sum\limits_{k=1}^{k_0} \frac{\varepsilon}{2^k} + \varepsilon < 2\varepsilon$
  Letting $\varepsilon \to 0$, we get what we want ∎

In this way $\mathbb{R}^\infty$ is a metrizable space.

Example 1.1.4. $X, \rho(x, y) \overset{\text{def}}{=} \begin{cases} 0, & x = y \\ 1, & x \neq y \end{cases}$ — **Discrete metric**.

$x_n \to x,\ \varepsilon = \frac{1}{2},\ \exists M :\ m > M \implies \rho(x_m, x) < \frac{1}{2} \implies$
$\rho(x_m, x) = 0 \implies x_m = x$

Definition. $(X, \tau);\ \forall A \subset X$

$$\operatorname{Int}(A) \overset{\text{def}}{=} \bigcup_{G \subset A} G \text{ is open;}$$

$$\operatorname{Cl}(A) \overset{\text{def}}{=} \bigcap_{A \subset G} G \text{ is closed;}$$

$$\operatorname{Fr}(A) \overset{\text{def}}{=} \operatorname{Cl}(A) \setminus \operatorname{Int}(A)$$

Having a metric space (X, ρ) one can describe closure of a set.

$$\rho(x,A) \overset{\text{def}}{=} \inf_{a\in A} \rho(x,a)$$

$$\rho(A,B) \overset{\text{def}}{=} \inf_{\substack{a\in A\\ b\in B}} \rho(a,b)$$

$$\rho(x,A) = f(x), x\in X$$

Statement 1.1.4. Function $f(x)$ is continuous.

Proof. $\forall x,y\in X$
$f(x) = \rho(x,A) \underset{\forall\alpha\in A}{\leqslant} \rho(x,\alpha) \leqslant \rho(x,y)+\rho(y,\alpha)$
$\forall\varepsilon>0\ \exists\alpha_\varepsilon\in A:\ \rho(y,\alpha_\varepsilon) < \rho(y,A)+\varepsilon = f(y)+\varepsilon$
$f(x)\leqslant f(y)+\varepsilon+\rho(x,y),\ \varepsilon\to 0$

$$\begin{cases} f(x)\leqslant f(y)+\rho(x,y) \\ f(y)\leqslant f(x)+\rho(x,y)\end{cases} \implies |f(x)-f(y)|\leqslant\rho(x,y)$$ ■

Statement 1.1.5. $x\in \mathrm{Cl}(A) \iff \rho(x,A)=0$

Let's look at the metric spaces in terms of separation of sets from each other by open sets.
x,y
$r=\rho(x,y)>0$
$B_{r/3}(x),\ B_{r/3}(y)$
In any metric space separability axiom is true.

Theorem 1.1.
Any metric space is a normal space,
i.e. $\forall$ *closed disjoint* $F_1,F_2\in X,\ \exists$ *open disjoint* G_1,G_2: $F_j\in G_j,\ j=1,2$

Proof. $g(x)=\dfrac{\rho(x,F_1)}{\rho(x,F_1)+\rho(x,F_2)}$ — continuous on X
$x\in F_1,\ \mathrm{Cl}(F_1)=F_1,\ \rho(x,F_1)=0,\ g(x)=0$
$x\in F_2,\ g(x)=1$
Let's look at $(-\infty;\frac{1}{3}),(\frac{2}{3},\infty)$ — by continuity their inverse images under g are open.
$G_1=g^{-1}(-\infty;\frac{1}{3})$
$G_2=g^{-1}(\frac{2}{3};\infty)$ ■

Definition. Metric space is **complete** if $\rho(x_n,x_m)\to 0 \implies \exists x=\lim x_n$
$\mathbb{R}^\infty$ — complete (by completeness of rational numbers).
In complete metric spaces the nested balls principle is true.

Theorem 1.2.
X — *complete metric space,* $\overline{V}_{r_n}$ — *system of closed balls.*

1. $\overline{V}_{r_{n+1}}\subset\overline{V}_{r_n}$ — *the system is nested.*
2. $r_n\to 0$

<u>*Then:*</u> $\exists! a\in\bigcap_n \overline{V}_{r_n}$

Proof. Let b_n be centers of $\overline{V}_{r_n}$,
$m\geqslant n,\ b_m\in\overline{V}_{r_n},\ \rho(b_m,b_n)\leqslant r_n\to 0\ \forall m\geqslant n$
$\rho(b_m,b_n)\to 0 \overset{compl.}{\implies} \exists a=\lim b_n$ Since the balls are closed a $\in$ every ball.
$r_n\to 0 \implies$ there is only one common point. ■

(X, τ) — topological space
$A \subset X,\ \tau_a = \{\, G \cap A, G \in \tau \,\}$ — topology induced on A

Definition. X— metric space, $A \subset X,\ \mathrm{Cl}(A) = X$
Then: A — **dense** in X
If $\mathrm{Int}(\mathrm{Cl}(A)) = \varnothing$ A — **nowhere dense** in X.

Note. It is easy to understand, that in metric spaces nowhere density means the following: $\forall$ ball $V\ \exists V' \subset V\colon V'$ contains no points from A.

Definition. X is called **first Baire category set**, if it can be written as at most countable union of X_n each nowhere dense in X.

Theorem 1.3 (Baire category theorem)**.**
Complete metric space is second Baire category set in itself.

Proof. Let X be first Baire category set.
$X = \bigcup\limits_n X_n \quad \forall \overline{V}\ X_1$ is nowhere dense.
$\overline{V}_1 \subset \overline{V}\colon\ \overline{V}_1 \cap X_1\ = \varnothing$
X_2 is nowhere dense $\overline{V}_2 \subset \overline{V}_1\colon \overline{V}_2 \cap X_2 = \varnothing$
$r_2 \leqslant \dfrac{r_1}{2}$
$\vdots$
$\{\overline{V}_n\},\ r_n \to 0,\ \bigcap\limits_n \overline{V}_n = \{a\},\ X = \bigcup X_n,\ \exists n_0\colon a \in X_{n_0}$
$X_{n_0} \cap \overline{V}_{n_0} = \varnothing \to\leftarrow\ a \in \overline{V}_{n_0}$ ■

Corollary 1.3.1. *Complete metric space without isolated points is uncountable.*

Proof. No isolated points are present $\implies$ every point in the set is nowhere dense in it. Let X be countable: $X = \bigcup\limits_n \{X_n\}$, then it is first Baire category set. $\to\leftarrow$ ■

Definition. K — **compact** if

1. $K = \mathrm{Cl}(K)$
2. $x_n \in K\ \exists n_1 < n_2 < \ldots\ x_{n_j}$ – converges in X.

If only 2 is present, the set is called **precompact**.

Theorem 1.4 (Hausdorff)**.**
Let X — metric space, K — closed in X.
Then: K — compact $\iff$ K — totally bounded,
i.e. $\forall \varepsilon > 0\ \exists a_1, \ldots, a_p \in X\colon\ \forall b \in K\ \exists a_j\colon\ \rho(a_j, b) < \varepsilon$
($a_1, \ldots, a_p$ — finite ε-net)

Proof.

- Totally bounded $\implies$ compact

  K is totally bounded, $x_n \in K\ n_1 < n_2 < \cdots < n_k < \cdots$, x_{n_k} converges in K?

$$\varepsilon_k \downarrow\to 0 \quad \varepsilon_1 \quad K \subset \bigcup_{j=1}^{p} \overline{V}_j,\ rad = \varepsilon_1 \qquad (\varepsilon_1\text{-net})$$

p is finite $\implies$ one ball will contain infinetely many x_n elements. Let's name that ball $\overline{V}_{j_0}$

Let's look at $\overline{V}_{j_0} \cap K$ — totally bounded $= K_1, \operatorname{diam}(K_1) \leqslant 2\varepsilon_1$

$\varepsilon_2 \quad K_1 \subset \bigcup\limits_{j=1}^{n} \overline{V'}_j,$ rad $= \varepsilon_2$.

Then one of $\overline{V'}$ contains infinitely many elements of the sequence contained in K_1.

$\overline{V'}_{j_0} \cap K_1 = K_2,$ $\operatorname{diam}(K_2) \leqslant 2\varepsilon_2$ and so on.

$$K_n \supset K_{n+1} \supset K_{n+2} \supset \dots, \ \operatorname{diam}(K_N) \leqslant 2\varepsilon_n \xRightarrow{\text{by compl.}} \qquad \underbrace{\bigcap_{n=1}^{\infty} K_n}_{\operatorname{diam}(K_n)\to 0,\ \{x\}} \neq \varnothing$$

Take x_{n_1} from K_1, x_{n_2} from $K_2 \dots$

- Compact $\implies$ totally bounded

  K — compact $\forall \varepsilon\ \exists$ finite ε-net?
  By contradiction: $\exists \varepsilon_0 > 0\colon$ finite ε_0-net is impossible to construct.
  $\forall x_1 \in K\ \exists x_2 \in K\colon\ \rho(x_1, x_2) > \varepsilon_0$ (or else system of x_1 — finite ε-net)
  $\{x_1, x_2\}$ - choose $x_3 \in K\colon \rho(x_3, x_i) > \varepsilon_0,\ i = 1, 2$ and so on.
  $x_n \in K\ :\ n \neq m\ \rho(x_n, x_m) > \varepsilon_0$ — contains no converging subsequence $\implies$ set is not a compact. $\to\leftarrow$ ■

§1.2 Normed spaces

Definition. X — **linear set**, $x+y, \alpha \cdot x$, $\alpha \in \mathbb{R}$
The purpose of norm definition, is to construct a topology on X, so that 2 linear operations are continuous on it.

$\varphi\colon X \to \mathbb{R}$:

1. $\varphi(x) \geqslant 0,\ = 0 \iff x = 0$
2. $\varphi(\alpha x) = |\alpha| \varphi(x)$
3. $\varphi(x+y) \leqslant \varphi(x) + \varphi(y)$

Definition. φ — **norm** on X, $\varphi(x) = \|x\|$

$\rho(x,y) \overset{\text{def}}{=} \|x-y\|$ — metric on X.

Definition.
$(X, \|\cdot\|)$ — **normed space** — special case of metrical space.

$x = \lim x_n \overset{def}{\Longrightarrow} \rho(x_n, x) \to 0 \iff \|x_n - x\| \to 0$

Statement 1.2.1. In the topology of a normed space linear operations are continuous on X.

Proof.

1. $x_n \to x,\ y_n \to y;\ \|(x_n+y_n) - (x+y)\| = \|(x_n - x) + (y_n - y)\| \leqslant$
$$\leqslant \underbrace{\|x_n - x\|}_{\substack{\downarrow \\ 0}} + \underbrace{\|y_n - y\|}_{\substack{\downarrow \\ 0}}$$
$$\Longrightarrow x_n + y_n \to x + y$$
2. $\alpha_n \to \alpha,\ x_n \to x;\ \|\alpha_n x_n - \alpha x\| = \|(\alpha_n - \alpha)x_n + \alpha(x_n - x)\| \leqslant$
$$\leqslant \underbrace{|\alpha_n - \alpha|}_{\substack{\downarrow \\ 0}} \cdot \underbrace{\|x_n\|}_{\textit{bounded}} + \underbrace{\alpha\|x_n - x\|}_{\substack{\downarrow \\ 0}}$$
$x_n \to x \implies \|x_n\|$ — bounded.
$\alpha_x x_n \to \alpha x$

■

Statement 1.2.2. From the triangle inequality $\big|\|x\| - \|y\|\big| \leqslant \|x - y\|$
$x_n \to x \implies \|x_n\| \to \|x\|$
Norm is continious.

Example 1.2.1. $\mathbb{R}^n$

1. $\|\bar{x}\| = \sqrt{\sum\limits_{k=1}^{n} x_k^2}$
2. $\|\bar{x}\|_1 \overset{\text{def}}{=} \sum\limits_{k=1}^{n} |x_k|$
3. $\|\bar{x}\|_2 \overset{\text{def}}{=} \max\{|x_1|, \dots, |x_n|\}$

4. $C[a,b]$ — functions continuous on $[a,b]$; $\|f\| = \max\limits_{x\in[a,b]} |f(x)|$

5. $L_p(E) = \left\{ f \text{ — measurable, } \int\limits_E |f|^p < +\infty \right\}$

$$p \geqslant 1,\ \|f\|_p = \left(\int\limits_E |f|^p\right)^{\dfrac{1}{p}}$$

Because the set of points is the same, arises the question about convergence comparison.
$\|\cdot\|_1 \sim \|\cdot\|_2,\ x_n \overset{\|\cdot\|_1}{\to} x \iff x_n \overset{\|\cdot\|_2}{\to} x$

Statement 1.2.3.
$\|\cdot\|_1 \sim \|\cdot\|_2 \iff \exists a,b>0\colon \forall x\in X \implies a\|x_1\|_1 \leqslant \|x\|_2 \leqslant b\|x\|_1$

Theorem 1.5 (Riesz)**.**
$X,\ \dim X < +\infty$ — linear set.
<u>Then:</u> Any pair of norms in X are equivalent.

Proof. $l_1,\dots,l_n$ — linearly independent from X. $\forall x \in X = \sum\limits_{k=1}^{n} \alpha_k l_k$

$$\bar{x} \leftrightarrow (l_1,\dots,l_n) = \bar{l} \in \mathbb{R}^n$$

Let $\|\cdot\|$ — some norm in X.

$$\|x\| \underset{\triangle}{\leqslant} \sum_{k=1}^{n} \|l_k\||\alpha_k| \underset{\text{Cauchy}}{\leqslant} \underbrace{\sqrt{\sum_{k=1}^{n}\|l_k\|^2}}_{const(B), B-basis} \underset{\|\bar{\alpha}\|=\|x\|_1}{\underset{\|}{\sqrt{\sum_{k=1}^{n}|\alpha_k|^2}}}$$

$$\|x\|_1 = \sqrt{\sum_{k=1}^{n}\|\alpha_k\|^2},\ x = \sum \alpha_k l_k$$

$$\|x\| \leqslant b\|x\|_1$$

$$?\exists a>0\colon a\|x\|_1 \leqslant \|x\| \implies \|\cdot\| \sim \|\cdot\|_1$$

Let $f(\alpha_1,\dots,\alpha_n) = \left\|\sum\limits_{k=1}^{n}\alpha_k l_k\right\|$

$$|f(\bar{\alpha}+\Delta\bar{\alpha}) - f(\bar{\alpha})| = \left|\left\|\sum_{k=1}^{n}\alpha_k l_k + \sum_{k=1}^{n}\Delta\alpha_k l_K\right\| - \left\|\sum_{k=1}^{n}\alpha_k l_k\right\|\right| \leqslant$$

$$\left\|\sum_{k=1}^{n}\Delta\alpha_k l_k\right\| \leqslant \underbrace{\sum \|l_k\||\Delta\alpha_k|}_{\substack{\downarrow \\ 0,\Delta\alpha_k\to 0}} \implies f \text{ is continuous on } \mathbb{R}^n$$

$S_1 = \left\{ \sum\limits_{k=1}^{n}|\alpha_k|^2 = 1 \right\} \subset \mathbb{R}^m$, f — continuous on S_1, S_1 — compact, $\bar{\alpha}^* \in S_1$
By Weierstrass theorem there exists a point $\alpha^* \in S_1$ on a sphere, in which function f achieves its minimum $\implies \forall\alpha\in S_1\ f(\bar{\alpha}^*) \leqslant f(\bar{\alpha})$

If $f(\bar{\alpha}^*) = 0$, then $\left\|\sum\limits_{k=1}^{n} \alpha_k^* l_k\right\| = 0 \implies \sum\limits_{k=1}^{n} \alpha_k^* l_k = 0,\ \bar{\alpha}^* \in S_1$,
but $l_1 \dots l_n$ are linearly independent $\to\leftarrow \implies \min\limits_{S_1} f = m > 0$

$$\|x\| = \left\|\sum_{k=1}^{n} \alpha_k l_k\right\| = f(\bar{\alpha}) = \sqrt{\sum_{k=1}^{n} \alpha_k^2} \cdot \left\| \sum \underbrace{\boxed{\frac{\alpha_k}{\sqrt{\sum_{k=1}^n \alpha_k^2}}}}_{\beta_k} \cdot l_k \right\| \geqslant,\ \bar{\beta} = (\beta_1 \dots \beta_n) \in S_1$$

$$\geqslant m \cdot \|x\|_1,\ a = m \qquad \blacksquare$$

Corollary 1.5.1. X — *NS*, $Y \subset X, \dim Y < +\infty \implies Y = \mathrm{Cl}(Y)$

Note. Functional analysis differentiates between linear subset (set of points, closed by addition and scalar multiplication) and linear subspace (closed linear subset).

Proof. $Y = \mathscr{L}(l_1, \dots, l_n) = \left\{ \sum\limits_{i=1}^{n} \alpha_i l_i \;\middle|\; l_1, \dots, l_n - \text{lin. indep.} \right\}$
$y_m \in Y, y_m \to y$ in $X \implies y \in Y$?
$\|y_m - y\| \to 0 \implies \|y_m - y_p\| \to 0,\ m, p \to \infty$
$\|y\|, y \in Y$.
By Riesz theorem all norm in Y are equivalent.
$y = \sum\limits_{j=1}^{n} \alpha_j l_J, \|y\|_0 = \sqrt{\sum\limits_{j=1}^{n} \alpha_j^2}$ — some norm (by linear independance).
By Riesz theorem $\|y\| \sim \|y\|_0$
$\underbrace{\|y_m - y_p\|}_{\in Y} \to 0 \implies \|y_m - y_p\|_0 \to 0$
Notice that convergence by $\|\cdot\|_0$ is coordinatewise.
$\bar{\alpha} = (\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n$ $y_m = \sum\limits_{i=1}^{n} \alpha_i^{(m)} l_i$
$|\alpha_i^{(m)} - \alpha_i^{(l)}| \to 0\ \forall i = 1, \dots, n;\ \bar{\alpha} = (\alpha_1^{(m)}, \dots, \alpha_n^{(m)}) \to \alpha^* = (\alpha_1^*, \dots, \alpha_n^*)$
$y^* = \sum\limits_{i=1}^{n} \alpha_i^* l_i \in Y,\ \|y_m - y^*\| \to 0$
Bu the limit uniqueness $y = y^* \implies y \in Y$ $\blacksquare$

Definition. If normed space if complete, then it is called **B-space** or **Banach space**.

Example 1.2.2. $C[a, b]$ — functions continuous on $[a, b]$, $\|f\| = \max\limits_{t \in (a,b)} |f(t)|$

Example 1.2.3. Lebesgue space, $p \geqslant 1, L_p(E) = \left\{ f \text{ is measurable on } E,\ \int\limits_E |f|^P < +\infty \right\}$,
$\|f\| = \sqrt[p]{\int\limits_E |f|^p}$.

If X — Banach space,

$$\sum_{n=1}^{\infty} x_n = \lim_{n\to\infty} \sum_{k=1}^{n} x_k,\ \sum_{n=1}^{\infty} \|x_n\| < +\infty$$

$$\|S_n - S_m\| = \left\|\sum_{k=m+1}^{n} x_k\right\| \leqslant \sum_{m+1}^{n} \|x_k\| \xrightarrow[n,m\to\infty]{} 0$$

$$\implies \|S_n - S_m\| \to 0 \implies \exists \lim_{n\to\infty} S_n, \sum_{k=1}^{n} x_k \text{— converges.}$$

In Banach spaces works the theory of absolute convergence of numerical series.

Lemma 1 (Riesz's lemma about almost perpendicular)**.** *Y — eigen subspace of X — normed space.* $\forall \varepsilon \in (0,1)\ \exists z_\varepsilon \in X$:

1. $z_\varepsilon \notin Y$
2. $\|z_\varepsilon\| = 1$
3. $\rho(z_\varepsilon, Y) > 1 - \varepsilon$

Proof. $\exists x \in X \setminus Y,\ d = \rho(x, Y)$
Suppose $d = 0$ then, $\exists y_n \in Y \colon \|x - y_n\| < \dfrac{1}{n},\ n \to \infty,\ y_n \to x$
$Y = \mathrm{Cl}(Y) \implies x \in Y \to\leftarrow x \notin Y,\ d > 0$
$\forall \varepsilon \in (0,1)\ \dfrac{1}{1-\varepsilon} > 1\ \exists y_\varepsilon \in Y \colon \|x - y_\varepsilon\| < \dfrac{1}{1-\varepsilon} d$
$z_\varepsilon = \dfrac{x - y_\varepsilon}{\|x - y_\varepsilon\|},\ \|z_\varepsilon\| = 1$

$$\forall y \in Y\ \|z_\varepsilon - y\| = \left\| \frac{x - y_\varepsilon}{\|x - y_\varepsilon\|} - y \right\| = \frac{\|x - \overbrace{(y_\varepsilon + \|x - y_\varepsilon\| \cdot y)}^{\in Y}\| \geqslant d}{\|x - y_\varepsilon\| < \dfrac{1}{1-\varepsilon} d} > 1 - \varepsilon \qquad \blacksquare$$

Corollary 1.5.2. *X — normed space,* $\dim X = +\infty,\ S = \{\, x \mid \|x\| = 1 \,\}$, *then closed unit ball $\overline{B}$ is not compact in it.*

Proof. $\forall x_1 \in S,\ Y_1 = \mathscr{L}\{x_1\}$ — finite dimensional linear set. $\implies$ closed in X $\implies$ Y_1 — subspace.
$\dim X = +\infty > \dim Y_1 \implies Y_1$ — eigen subspace.
Then by the Riesz lemma ($\varepsilon = \dfrac{1}{2}$):

$\exists x_2 \in X \colon \|x_2\| = 1,\ \|x_2 - x_1\| > \dfrac{1}{2}$ (Notice that x_2 appears to be an element of S)

$Y_2 = \mathscr{L}\{x_1, x_2\}\ \exists x_3 \in S \colon \|x_3 - x_j\| > \dfrac{1}{2},\ j = 1, 2$

Continue by induction. Because $\dim X = +\infty$ the process willl never finish.
$x_n \in S \colon \|x_n - x_m\| > \dfrac{1}{2},\ n \neq m$ — obviously we cannot extract converging subsequence.
$\implies S$ — not a compact. And that means that $\overline{B} \supset S$ is not a compact either. $\blacksquare$

§1.3 Inner product (unitary) spaces

Definition. X — linear space.
$\varphi\colon X \times X \to \mathbb{R}$

1. $\varphi(x,x) \geqslant 0, \quad \varphi(x,x) = 0 \iff x = 0$
2. $\varphi(x,y) = \varphi(y,x)$
3. $\varphi(\alpha x + \beta y, z) = \alpha\varphi(x,z) + \beta\varphi(y,z)$

φ — **inner product**.
$\varphi(x,y) = \langle x, y\rangle$

Definition. $(X, \langle\cdot,\cdot\rangle)$ — **inner product space**.

Example 1.3.1. $\mathbb{R}^n$, $\langle \bar{x}, \bar{y}\rangle = \sum\limits_{j=1}^{n} x_j y_j$

Statement 1.3.1 (Schwarz)**.** $\forall x, y \in X \quad |\langle x,y\rangle| \leqslant \sqrt{\langle x,x\rangle} \cdot \sqrt{\langle y,y\rangle}$

Proof. Consider $\lambda \in \mathbb{R}$.

$$f(\lambda) = \underbrace{\langle \lambda x + y, \lambda x + y\rangle}_{\lambda^2\langle x,x\rangle + 2\lambda\langle x,y\rangle + \langle y,y\rangle} \geqslant 0 \qquad \text{by the first axiom of inner product}$$

$$D = 4\langle x,y\rangle^2 - 4\langle x,x\rangle \cdot \langle y,y\rangle \leqslant 0$$

$$\langle x,y\rangle^2 \leqslant \langle x,x\rangle \cdot \langle y,y\rangle$$

∎

Corollary 1.5.3 (Cauchy inequality for sums)**.** *Consider* $X = \mathbb{R}^n$, $\|x\| \overset{def}{=} \sqrt{\langle x,x\rangle}$. *Then*

$$\|x+y\|^2 = \langle x+y, x+y\rangle = \|x\|^2 + 2\cdot\underbrace{\langle x,y\rangle}_{\leqslant \|x\|\cdot\|y\|} + \|y\|^2 \leqslant \big(\|x\| + \|y\|\big)^2$$

Any inner product space is a special case of a normed space. The specifics is that we can measure the angles between points:

$$x \perp y \iff \langle x,y\rangle = 0$$

In this case the Pythagorean theorem takes place:

$$\|x+y\|^2 = \|x\|^2 + \|y\|^2$$

In inner product spaces the parallelogram law plays a significant role:

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2 \quad \forall x, y \in X$$

In an inner product space norm is determined by inner product: $\|x\|^2 = \langle x,x\rangle$
It can be proved that if parallelogram law holds, then the norm must be determined by some inner product. Let X be some normed space, $x \in X$, then $\langle\cdot,\cdot\rangle \mapsto \|x\| = \sqrt{\langle x,x\rangle}$. For any norm satisfying the parallelogram law, the inner product generating the norm is unique.

Example 1.3.2. $C_{[a,b]}$, $\|f\| = \max\limits_{x\in[a,b]} |f(x)|$, $\|f\|$ doesn't satisfy the parallelogram law and thus is not determined by any inner product. This fact implies that $C_{[a,b]}$ is not an inner product space.

Definition. Orthonormal set — a set of points $\{e_1, e_2, \dots\}$ (may be finite):

1. $\|e_i\| = 1$
2. $e_i \perp e_j,\ i \neq j$

Note. Every orthonormal set is linearly independent.

Definition. $\sum\limits_j x_j$ — **orthogonal series** $\iff x_i \perp x_j, \quad i \neq j$

Definition. Let $x \in X, \{\, e_i \,\}$ — ONS. Then
$\langle x, e_j\rangle$ — **abstract Fourier coefficient**,
$\sum\limits_j \langle x, e_j\rangle e_j$ — **abstract Fourier series** of point x.

Note. Fourier series is a special case of orthogonal series.

Let $\sum\limits_{j=1}^{\infty} x_j, S_m = \sum\limits_{j=1}^{m} x_j$. Then

$$\|S_m\|^2 = \Big\langle \sum_{j=1}^{m} x_j, \sum_{j=1}^{m} x_j \Big\rangle = \sum_{j=1}^{m} \|x_j\|^2$$

This fact allows us to effectively build the theory of orthogonal series.

An important problem is concerned with Fourier series. Let X is a normed space, Y is a subspace of X,

$$\forall x \in X \quad E_Y(x) = \rho(x, Y) = \inf_{y \in Y} \|x - y\|$$

Definition. $E_Y(x)$ — **best approximation** of point x with points of the subspace Y. If $\exists y^* \in Y\ E_Y(x) = \|x - y^*\|$, then y^* is called the **element of best approximation**.

Theorem 1.6 (Borel)**.**
$\dim Y < +\infty \implies \forall x \in X\ \exists y^* \in Y$ — *element of best approximation.*

Proof. $Y = \mathscr{L}(\underbrace{e_1, e_2, \dots, e_n}_{\text{lin. indep.}})$ Consider $f(\alpha_1, \dots, \alpha_n) = \Big\|x - \sum\limits_{k=1}^{n} \alpha_k e_k\Big\| \to \min$. By the triangle inequality for norm, $f(\bar\alpha)$ is continous on $\mathbb{R}^n, f \geqslant 0, E_Y(x) = \inf\limits_{\bar\alpha \in \mathbb{R}^n} f(\bar\alpha)$. It is easy to find out that there always is a ball $B(0, r) \subset \mathbb{R}^n$, outside of which $f > 2E_Y(x)$. So, $E_Y(x)$ is somewhere inside. But f is continuous, ball $B(0, r)$ is compact, so, by the Weierstrass theorem, the minimum exists and is located inside the $B(0, r)$. ■

For abstract Fourier series the Borel theorem can be significantly strengthened by specifying the best approximation element.

Theorem 1.7 (extremal propertirs of Fourier series' partial sums)**.**
$\{e_j\}$ — *ONS in* X
$H_n = \mathscr{L}(e_1, \dots, e_n)$
$E_{H_n}(x), S_n(x) = \sum\limits_{j=1}^{n} \langle x, e_j\rangle e_j \implies E_{H_n}(x) = \|x - S_n(x)\|$

Proof. $y = \sum\limits_{j=1}^{n} \alpha_j e_j \in H_n$

$$\begin{aligned}\|x-y\|^2 &= \left\langle x - \sum \alpha_j e_j, x - \sum \alpha_j e_j \right\rangle = \|x\|^2 - 2\sum \alpha_j \langle x, e_j\rangle + \sum \alpha_j^2 = \\ &= \underbrace{\|x\|}_{\text{const}}{}^2 + \sum (\alpha_j - \langle x, e_j\rangle)^2 - \underbrace{\sum \langle x, e_j\rangle^2}_{\text{const}} \to \min\end{aligned}$$

So, the sum goes to minimum when the second summand is minimal. Obviously, it's minimal when $\forall(\alpha_j - \langle x, l_j\rangle) = 0$. $E_Y(x)$ — Fourier sum. ∎

Corollary 1.7.1 (Bessel's inequality). $\sum\limits_j \langle x, e_j\rangle^2 \leqslant \|x\|^2$

Proof. Bessel's inequality follows from the identity:

$$\begin{aligned}0 \leqslant \|x - y^*\|^2 = \left\| x - \sum_{j=1}^{n} \langle x, e_j\rangle e_j \right\|^2 &= \|x\|^2 - 2\sum_{j=1}^{n} |\langle x, e_j\rangle|^2 + \sum_{j=1}^{n} |\langle x, e_j\rangle|^2 \\ &= \|x\|^2 - \sum_j \langle x, e_j\rangle^2\end{aligned}$$ ∎

Corollary 1.7.2. *The series of Fourier coefficients' squares always converges*

§1.4 Hilbert spaces

Definition. Hilbert space — complete, infinite dimensional, inner product space.

Example 1.4.1. $L_2(E)$ — Hilbert space.
$\langle f, g\rangle = \int\limits_E f \cdot g \, d\mu$
$l_2 = \left\{ (x_1, \dots, x_n, \dots) \;\middle|\; \sum\limits_{n=1}^{\infty} x_n^2 < +\infty \right\}$
$\langle \bar{x}, \bar{y}\rangle = \sum\limits_{n=1}^{\infty} x_n y_n,\ E = \mathbb{N},\ \mu\{m\} = 1$

When we have completeness we can define orthonormal basis.

Definition. $H,\ \{e_n\}$ — ONS : $\forall x = \sum\limits_{n=1}^{\infty} \alpha_n e_n$ Then $\{e_n\}$ is called **orthonormal basis**.
Let's look at the inner product of arbitrary vector $x \in H$ and basis vector e_m
$\langle x, e_m\rangle = \sum\limits_{n=1}^{\infty} \alpha_n \langle e_n, e_m\rangle = \alpha_m$
In this sense basis decomposition is always a Fourier series.

1. Complete ONS: Let $L = \mathscr{L}\{e_1, e_2, \dots\}$, then $H = \mathrm{Cl}(L)$ (L is dense in H)
2. Closed ONS: $\forall m\ \langle x, e_m\rangle = 0 \implies x = 0$.

Statement 1.4.1. In Hilbert spaces two of the properties outlined above are equivalent.

Statement 1.4.2. In Hilbert space Fourier series converges for any point.

Proof. Let H — Hilbert space, $\sum\limits_{j=1}^{\infty} \langle x, e_j\rangle e_j$ — abstract Fourier series, $S_n = \sum\limits_{j=1}^{n} \langle x, e_j\rangle e_j$ — it's partial sum. We need to prove the existance of $\lim\limits_{n\to\infty} S_n$. H is a Hilbert space, which means it is also complete. This means it is sufficient to prove that $\{S_n\}$ is a Cauchy sequence, i.e.

$$\forall \varepsilon > 0\ \exists N \in \mathbb{N} \colon \forall n, m > N\ \|S_n - S_m\| < \varepsilon$$

Consider

$$\|S_n - S_m\|^2 = \left\| \sum_{j=1}^{n} \langle x, e_j\rangle e_j - \sum_{j=1}^{m} \langle x, e_j\rangle e_j \right\|^2 = \left\| \sum_{j=m+1}^{n} \langle x, e_j\rangle e_j \right\|^2 = \sum_{j=m+1}^{n} |\langle x, e_j\rangle|^2$$

Because numerical series $\sum\limits_{j=m+1}^{n} |\langle x, e_j\rangle|^2$ converges, by the Cauchy criteria we have the following:

$$\forall \varepsilon > 0\ \exists N \in \mathbb{N} \colon \forall n > m > N \left| \sum_{j=m+1}^{n} |\langle x, e_j\rangle|^2 \right| < \varepsilon^2$$

or

$$\sum_{j=m+1}^{n} |\langle x, e_j\rangle|^2 < \varepsilon^2$$

Which is essentially the same as:

$$\|S_n - S_m\| < \varepsilon.$$

■

Proof. 1.4.1 Complete ONS $\implies$ Closed ONS
$\forall x \in H\ \forall \varepsilon > 0\ \exists \sum\limits_{j=1}^{p} \alpha_{kj} e_{kj} \colon \underbrace{\|x - \sum\limits_{j=1}^{p} \alpha_{kj} e_{kj}\|^2}_{\geqslant \|x - \sum\limits_{j=1}^{k_p} \langle x, e_j \rangle e_j\|^2} \leqslant \varepsilon^2$
$S_m(x)$ by extremality $\|x - S_{m+p}(x)\|^2 \leqslant \|x - S_m(x)\|^2$
Implies partial sums go to x, $x = \sum\limits_{j=1}^{\infty} \langle x, e_j \rangle e_j$ If all fourier coefficients are zero, it means the ONS is closed ($x = 0$).
Closed ONS $\implies$ Complete ONS
Let $x \in H$, $y \coloneqq \sum\limits_{j=1}^{\infty} \langle x, e_j \rangle e_j$

$$\begin{aligned}\forall n\ \langle x - y, e_n \rangle = \left\langle x - \sum_{j=1}^{\infty} \langle x, e_j \rangle e_j, e_n \right\rangle = \langle x, e_n \rangle - \left\langle \sum_{j=1}^{\infty} \langle x, e_j \rangle e_j, e_n \right\rangle = \\ = \langle x, e_n \rangle - \sum_{j=1}^{\infty} \langle x, e_j \rangle \langle e_j, e_n \rangle = \langle x, e_n \rangle - \sum_{j=1}^{\infty} \langle x, e_j \rangle \delta_{j,n} = 0.\end{aligned}$$

Because ONS is closed $\langle x - y, e_n \rangle = 0 \implies x - y = 0 \implies x = y$. Thus we can decompose any point into Fourier series, and this implies that ONS is complete. ∎

Definition. Topological space is called **separable** if there exists countable dense set in it. $X = \mathrm{Cl}(a_1, \dots, a_n, \dots)$

Note. If H is separable, $\{a_1, \dots, a_n, \dots\}$ — countable dense set in H. We can orthogonalize these dots using the Gramm-Shmidt process, and we will get complete orthonormal system. This means space separability is equivalent to basis existance.

Theorem 1.8 (about best approximation in Hilbert space)**.**
Let H be Hilbert space, M — closed convex subset of H, then $\forall x \in H\ \exists! y \in M \colon \|x - y\| = \inf\limits_{z \in M} \|x - z\|$. In other words for any $x \in H$, the best approximation element exists, is contained in M and is unique.

Proof. Let $d = \inf\limits_{z \in M} \|x - z\|$
Then by definition of infimum:

$$\begin{aligned}&\forall n \in \mathbb{N}\ \exists y_n \in M \colon d \leqslant \|x - y_n\| < d + \frac{1}{n} \\ &\exists ? y = \lim y_n \in M\ d \leqslant \|x - y\| \leqslant d\end{aligned}$$

$y_n, y_m \in M$. M is convex, which means

$$\frac{y_n + y_m}{2} \in M \implies d^2 \leqslant \left\| \frac{y_n + y_m}{2} - x \right\|^2 = \frac{1}{4} \left\| \underbrace{(y_n - x)}_{z_1} + \underbrace{(y_m - x)}_{z_2} \right\|^2$$

Applying the parallelogram law, we get

$$\|z_1 + z_2\|^2 + \|z_1 - z_2\|^2 = 2\|z_1\|^2 + 2\|z_2\|^2$$

$$\underbrace{\|(y_n - x) + (y_m - x)\|^2}_{\geqslant 4d^2} + \|y_n - y_m\|^2 = 2\overbrace{\|y_n - x\|^2}^{\leqslant\left(d+\frac{1}{n}\right)^2} + 2\overbrace{\|y_m - x\|^2}^{\leqslant\left(d+\frac{1}{m}\right)^2}$$

$$\|y_n - y_m\|^2 \leqslant 2(d + \frac{1}{n})^2 + 2(d + \frac{1}{m})^2 - 4d^2 = 4d\frac{1}{n} + \frac{2}{n^2} + 4d\frac{1}{m} + \frac{2}{m^2} \xrightarrow[n,m\to 0]{} 0$$

$$\|y_n - y_m\| \xrightarrow[n,m\to 0]{} 0 \implies \exists \lim y_n$$ ■

Corollary 1.8.1. *Let H be Hilbert space, H_1 — subspace of H (closed linear subset). $H_2 = H_1^{\perp} = \{ y \in H \mid y \perp x, x \in H_1 \}$ is called* ***orthogonal addition****.*
$\forall x \in H$ can be unambiguously written as $x = x_1 + x_2,\ x_1 \in H_1,\ x_2 \in H_1^{\perp}$

Note. $H = H_1 \oplus H_1^{\perp}$

Proof. $x \in H,\ H_1,\ H_2 = H_1^{\perp}$

$$\exists x_1 \in H_1 \colon \|x - x_1\| = \inf_{n \in H_1} \|x - n\|$$

$$x_2 = x - x_1 \ \in H_2?$$

$$\forall y \in H_1 y \perp x_2,\ \lambda > 0,\ x_1 + \lambda y \in H_1 (H_1\text{— subspace})$$

By definition of best approximation element

$$\|x - (x_1 - \lambda^2)\| \geqslant \|x - x_1\|^2 \ \forall \lambda > 0$$

Let's expand squared norms via inner product.

$$\langle \underbrace{x - x_1}_{x_2} - \lambda y, x - x_1 - \lambda y \rangle \geqslant \langle x - x_1, x - x_1 \rangle$$

$$\langle x_2 - \lambda y, x_2 - \lambda y \rangle \geqslant \langle x_2, x_2 \rangle$$

$$\langle x_2, x_2 \rangle - 2\lambda \langle y, x_2 \rangle + \lambda^2 \langle y, y \rangle \geqslant \langle x_2, x_2 \rangle \ : \lambda > 0$$

$$2\langle y, x_2 \rangle \leqslant \lambda \langle y, y \rangle,\ \lambda \to +0 \implies \langle y, x_2 \rangle \leqslant 0$$

Then substitute y for $-y$:

$$\langle -y, x_2 \rangle \leqslant 0 \implies \langle y, x_2 \rangle \geqslant 0 \implies \langle y, x_2 \rangle = 0$$ ■

§1.5 Countably-normed spaces.

Definition. X — linear set, p — **seminorm**:

1. $p(x) \geqslant 0$
2. $p(\lambda x) = |\lambda| p(x)$
3. $p(x+y) \leqslant p(x) + p(y)$

Definition. $p_1, p_2, \dots, p_n, \dots$ — seminorms
$\forall n \; p_n(x) = 0 \implies x = 0$ $(X, p_1, p_2, \dots, p_n, \dots)$ — **countably normed space**.

$x = \lim x_m \iff \forall n \in \mathbb{N} \lim\limits_{m \to \infty} (x_m - x) = 0$ with respect to corresponding seminorn p_n

If in countably-normed space we assume $\rho(x, y) = \sum\limits_{n=1}^{\infty} \frac{1}{2^n} \frac{p_n(x-y)}{1 + p_n(x-y)}$, we will get a valid metric. Thus countably-normed space is always metrizable. In countably-normed space two linear operations $(x+y, \lambda x)$ are continuous, which means any countably normed space is also a topological vector space.

Example 1.5.1. $C^\infty[a,b] = \{\, x(t), t \in [a,b] \mid x(t)$— infinetely diff. $\}$
$p_n(x) = \max\limits_{[a,b]} \left| x^{(n)}(t) \right| \; n = 0, 1, 2, \dots$

From the next theorem we will see that $C^\infty[a,b]$ is non-normalizable (has no norm convergence by which is equivalent to seminorm convergence). We will also try to deduce the criterion of countably-normed space normalizability.

Definition. System of seminorms is called **monotone** if $\forall x \in X, \; \forall n \in \mathbb{N} \; p_n(x) \leqslant p_{n+1}(x)$

Definition. $\{p_n\} \sim \{q_n\}$ if they have the same convergence (limits in both systems are equal).

Definition. $p_m \in \{p_n\}$ is called **essential** if it can not be majorized by any of the preceeding seminorms. p_m is majorized by p_n if $\exists C : \forall x \in X \; p_m(x) \leqslant C \cdot p_n(x)$.

Statement 1.5.1. For system of seminorms there exists equivalent monotone system.

Proof. Let $q_n(x) = \sum\limits_{k=1}^{n} p_k(x)$, it is obvious that every q_n is seminorm. $\{q_n\} \sim \{p_n\}$?
$p_n(x_m - x) \to 0 \implies \sum\limits_{k=1}^{n} p_k(x_m - x) \to 0 \implies q_n(x_m - x) \to 0$. Backwards proof is the same. ∎

This statement allows us to operate only on monotone seminorm systems.

Statement 1.5.2. Two monotone seminorm systems are equivalent if and only if they majorize each other, i.e. for any seminorm p_n from $\{p_n\}$ there exists majorizing seminorm q_m from $\{q_m\}$ and vice verca.

Proof. If two systems majorize each other obviously they are equivalent. Let two systems be equivalent. $\{p_n\} \sim \{q_n\}$
$\{p_n\}, \; \{q_n\} \; \forall p_n \; \exists q_m : \; \exists C \; \forall x \in X \; p_n(x) \leqslant C \cdot q'_m(x)$
Proof by contradiction. Let there be some p_{n_0}:

$$\forall q_m \exists x_m \in X : p_{n_0}(x_m) \geqslant m \cdot q_m(x_m)$$

$$q_m\left(\frac{x_m}{p_{n_0}(x_m)}\right) \leqslant \frac{1}{m}, \; y_m = \frac{x_m}{p_{n_0}(x_m)}, \; q_m(y_m) \leqslant \frac{1}{m}, \; p_{n_0}(y_m) = 1$$

Let's fix m_0, because q_m is monotone:

$$m \geqslant m_0,\ q_{m_0}(y_m) \leqslant q_m(y_m) \leqslant \frac{1}{m}$$
$$q_{m_0}(y_m) \leqslant \frac{1}{m},\ m \geqslant m_0,\ m \to +\infty$$
$$q_{m_0}(y_m) \to 0 \quad \forall m_0\ q_m(y_m) \to 0$$

But $p_{n_0}(y_m) = 1 \not\to 0,\ m \to \infty\ \ \to\leftarrow$ ∎

Theorem 1.9 (Normalizability criterion)**.**
Countably-normed space with monotone seminorm system is normalizable if and only if this system has finite number of essential seminorms.

Proof. $\Leftarrow$
Let system has finite amount of essential seminorms. $\{p_{n_1}, \dots p_{n_m}\}$
$\|x\| = \sum\limits_{k=1}^{n_m} p_k(x)$ — norm. Given system and this majorize each other.
$\Rightarrow$
Easily proovable using statement 1.5.2 ∎

Example 1.5.2. $\mathbb{R}^\infty$ $\bar{x} = (x_1, \dots, x_n, \dots)$ $p_n(\bar{x}) = |x_n|$. All seminorms are essential, thus $\mathbb{R}^\infty$ is not normalizable. The same applies to $C^\infty[a, b]$.

§1.6 Minkowski functional.

X — linear set. $M \subset X$, M — convex $\iff$ $x, y \in M \implies \alpha x + \beta y \in M,\ \alpha, \beta \geqslant 0, \alpha + \beta = 1$

Definition. M — **absorbs** $A \subset X$ if $\exists \lambda_0 \colon \forall \lambda \colon |\lambda| \geqslant \lambda_0 \implies A \subset \lambda M = \{\, \lambda x, x \in M \,\}$

Definition. If M absorbs any finite number of points, its is called **radial set**.

Definition. M is called **circled**, if $\forall \lambda \colon |\lambda| \leqslant 1\ \lambda M \subset M$

Example 1.6.1. X — normed space, $\overline{V} = \{\, x \mid \|x\| \leqslant 1 \,\}$ — convex, radial and circled.

Next definition has fundamental meaning for our theory.

Definition. M — radial set. $\forall x \in X$
$\varphi_M(x) = \inf \{\, \lambda \geqslant 0 \mid x \in \lambda M \,\}$ — **Minkowski functional**.

Example 1.6.2. $\varphi_{\overline{V}}(x) = \|x\|$

It is easy to find out the following:

Statement 1.6.1. φ_M is a seminorm on X if and only if M — radial, convex and circled.

§1.7 Linear functionals and their dimensionality.

Definition. X, Y — linear subset of X, $\forall x, y \in X,\ x \sim y \iff x - y \in Y$
$[x] = \{\, y \mid y \sim x \,\}$
$X/Y = \{\, [x] \,\}$ — **factor set**.
$[x] + [y] = [x + y],\ \alpha[x] = [\alpha x] \implies X/Y$ — linear set.

Definition. $\operatorname{codim} Y \overset{\text{def}}{=} \dim X/Y$

Statement 1.7.1. Let $\dim X/Y = p$, then $\exists e_1, \dots, e_p \in X$ — lin. indep.
$\forall x \in X\ x = \sum\limits_{k=1}^{p} \alpha_k e_k + y,\ y \in Y$

Definition. If $\operatorname{codim} Y = 1$, Y is called **hyperplane** in X.

Analytical definition of hyperplane is given via so called linear functionals.

Definition. $f \colon X \to \mathbb{R}$
$f(\alpha x_1 + \beta x_2) = \alpha f(x_1) + \beta f(x_2),\ f(0) = 0$
f — **linear functional**.

Definition. $\ker f \overset{\text{def}}{=} \{\, x \mid f(x) = 0 \,\}$

Statement 1.7.2. Any hyperplane can be written as a kernel of some linear functional.

Proof. 1. f — nontrivial linear functional. $Y = \ker f$, $\operatorname{codim} Y = 1$?
By nontriviality of f, $\exists x_0 \in X \colon f(x_0) \neq 0,\ e := \dfrac{x_0}{f(x_0)}$

$$
\begin{aligned}
& f(e) = \frac{f(x_0)}{f(x_0)} = 1.\ \forall x \in X\ \exists ? t \in \mathbb{R} \colon x - te \in \ker f \\
& x - te = y,\ x = te + y \\
& f(x - te) = 0 \\
& f(x) - t f(e) = f(x) - t = 0 \implies t = f(x)
\end{aligned}
$$

2. Let $\operatorname{codim} Y = 1, \forall x \in X, x = te + y,\ y \in Y$
Then $f(x) = t$, by uniqueness of x. $\ker f = Y$.

∎

§1.8 Kolmogorov theorem

Definition. X — linear set, τ — topology on X, αx, $x+y$ are continuous on τ.
$E(x) \subset X$ is called a neighbourhood of $x \in X$ if there exists open set $U \in \tau : x \in U \subset E(x)$.
(X, τ) is called a **topological vector space**.
αx continuity means:

$$\forall E(\alpha_0 x_0)\ \exists \delta > 0,\ \exists E(x_0) \colon |\alpha - \alpha_0| < \delta,\ x \in E(x_0) \implies \alpha x \in E(\alpha_0 x_0),$$

or

$$\alpha_0 x_0 = \lim_{\substack{\alpha \to \alpha_0 \\ x \to x_0}} \alpha x.$$

$x + y$ continuity means:

$$\forall E(x_0 + y_0)\ \exists E(x_0), E(y_0) \colon x \in E(x_0), y \in E(y_0) \implies x + y \in E(x_0 + y_0),$$

or

$$x_0 + y_0 = \lim_{\substack{x \to x_0 \\ y \to y_0}} (x + y).$$

It is clear that normed and countably-normed spaces are just special cases of topological vector space.
Let's fix x_0.
It is clear that $f(x) = x + x_0$ is a bijection $X \to X$
$f^{-1}(y) = y - x_0$

Definition. A function $f \colon X \to Y$ between two topological spaces (X, τ_X) and (Y, τ_Y) is called a **homeomorphism** if it has the following properties:

1. f is a bijection
2. f is continuous
3. the inverse function f^{-1} is continuous

If G is open in X, $G \in \tau$, then $x_0 + G = \{\, x_0 + x, x \in G \,\}$ is open too.
This means any topology is invariant relative to shift.

Definition. A **neighbourhood basis** or **local basis** for a point x is a

$$\mathcal{B}(x) \subset \mathcal{V}(x)$$

such that

$$\forall V \in \mathcal{V}(x) \quad \exists B \in \mathcal{B}(x) : B \subset V.$$

Where $\mathcal{V}(x)$ is the collection of all neighbourhoods for the point x. We will use Σ to denote neighbourhood basis for point 0.

$x \to 0$ in τ $x + x \to 0 + 0 = 0$
$\forall V \in \sigma\ \exists U \in \sigma\colon U + U \subset V,\ 2 \cdot U \subset U + U \subset V$
$\lambda x \to 0,\ \lambda \to 0,\ x \to 0\ \forall V\sigma\ \exists \varepsilon > 0,\ U \in \sigma\ |\lambda| \leqslant \varepsilon \implies \lambda U \subset V$
$\bigcup\limits_{|\lambda|\leqslant\varepsilon} \lambda U$ — radial, also zero neighbourhood. That means if X is a topological vector space then system of open sets is shift-invariant and such base can be created that: σ

1. $\forall V \in \sigma\ \exists U \in \sigma : U + U \subset V$

2. All elements of σ are radial, circled sets.

Definition. Topological space is called Hausdorff space, if any pair of points can be divided by their neighbourhoods, i.e. there exist disjoint neighbourhoods of such points.

Theorem 1.10 (Kolmogorov)**.**
Hausdorff topological vector space is normalizable if and only if zero has at least one bounded, convex neighbourhood (the set is bounded if it is absorbed by any zero's neighbourhood).

Proof. Minkowski functional is seminorm if and only if it is produced by radial, circled set. If space is normalizable then unit ball is bounded, convex zero's neighbourhood. By the former characterization of vector space topology, having bounded, convex zero's neighbourhood we can assume we have radial, circled neighbourhood. Produce Minkowski functional using it (it will be seminorm). We can assume the set is bounded. V — bounded, convex, circled. $\left\{ \dfrac{1}{n}V, b \in \mathbb{N} \right\}$ $\forall$ bounded. W $|\lambda| \geqslant \lambda_0,\ V \subset \lambda W\ \exists n_0 \in \mathbb{N} > \lambda_0\ V \subset n_0 W\ \dfrac{1}{n_0}V \subset W$
Now we only have to check that functional is norm. $p_V(x) = 0 \implies x = 0?$
$x \in \bigcap\limits_{n=1}^{\infty} \left(\dfrac{1}{n}V\right) = \{0\}$ ■

Example 1.8.1. $\mathbb{R}^\infty = \{\, (x_1, \dots, x_n, \dots)\,\}$
$\{\, \bar{x}\colon x_{i_1} \in (-\delta_1, \delta_1),\ x_{i_2} \in (-\delta_2, \delta_2), \dots, x_{i_p} \in (-\delta_p, \delta_p),\ \delta_j > 0 \,\}$ — this system will produce zero's base and insure coordinatewise convergence. $P_n(\bar{x}) = |x_n|$ all of the seminorms are essential, which means the space is not normalizable.

2 Elements of functional analysis

§2.1 Continuous functionals. Hahn-Banach theorem

X — linear set, $f\colon X \to \mathbb{R}$ — linear functional if $f(\alpha x_1 + \beta x_2) = \alpha f(x_1) + \beta f(x_2)$
Let X be arbitrary normed space. We call functional f continuous on X if $x_n \to x \implies f(x_n) \to f(x)$. Because in normed space addition and scalar multiplication are continuous, by functional lineriarity $f(x_n) - f(x) = f(x_n - x),\ x_n \to x \iff x_n - x \to 0,\ f(0) = 0$

Statement 2.1.1. f is continuous on X if and only if f is continuous in 0.

Definition. $\sup\limits_{\|x\|\leqslant 1} |f(x)| = \|f\|$, f — bounded $\iff \|f\| < +\infty$

Theorem 2.1.
Linear functional is continuous if and only if it is bounded.

Proof. Let f be bounded linear functional:

$$\|f\| < +\infty,\ |f(x)| \leqslant \|f\| \cdot \|x\|,\ x_n \to 0 \implies \|x_n\| \to 0$$
$$|f(x_n)| \leqslant \underbrace{\|f\| \cdot \|x_n\|}_{\substack{\downarrow \\ 0}} \implies f(x_n) \to 0$$

Let f be continuous functional:

$$\|f\| = +\infty = \sup_{\|x\|\leqslant 1} |f(x)|,\ \forall b \in \mathbb{N} : \exists x_n : \|x_n\| \leqslant 1,\ |f(x_n)| > b$$
$$\left|f(\frac{x_n}{n})\right| > 1,\ \left\|\frac{x_n}{n}\right\| = \frac{\|x_n\|}{n} \leqslant \frac{1}{n}\ \frac{x_n}{n} \to 0 \implies f(\frac{x_n}{n}) \to 0 \to\leftarrow$$ ■

Note. By functional lineriarity it's kernel is linear set.

Theorem 2.2.
Functional f is continuous if and only if $\ker f$ *is closed in X.*

Proof. Let f be continuous, then

$$x_n \in \ker f,\ x_n \to x \implies \underbrace{f(x_n)}_{\substack{\downarrow \\ 0}} \to f(x) \implies f(x) = 0 \implies x \in \ker f$$

Let $\ker f$ be closed, $\operatorname{codim}(\ker f) = 1$

$$\exists e \in X : \forall x = y + t \cdot e,\ y \in \ker f,\ t \in \mathbb{R}$$
$$f(x) = f(y) + t \cdot f(e) = t \cdot f(e)$$
$$x_n = y_n + t_n \cdot e \to x = y + t \cdot e,\ f(x_n) = t_n \cdot f(e)$$
$$?t_n \to t \implies t_n \cdot f(e) \to t \cdot f(e) = f(x) \implies f(x_n) \to f(x).$$
$$x_n = y_n + t_n \cdot e,\ x = y + t \cdot e,\ x_n \to x,\ y_n, y \in \ker f.$$

If we can prove that all converging subsequences $t_{n_k} \to t^*$ then t_n will converge too.

$$y_{n_k} = x_{n_k} - t_{n_k} \cdot e$$
$$y_{n_k} \in \ker f \to x - t^* \cdot e \implies z = x - t^* \cdot e \in \ker f$$
$$x = z + t^* \cdot e,\ x = y + t \cdot e \implies t^* = t.$$ ■

X — normed space, Y — linear set in X, Y dense in X. $C[0,1]$
$\forall \varepsilon > 0\ \exists P_n(x) : |f(x) - P_n(x)| < \varepsilon\ \forall x \in [0,1]$
S.N. Bernstein constructively showed these polynoms.

$$B_n(f,x) = \sum_{k=0}^{n} C_n^k f(\frac{k}{n}) x^k (1-x)^{n-k}$$

From functional analysis point of view Weierstrass theorem means that:

$$Y = \{\text{all } P_n(x)\} \text{ — linear set in } C[0,1]$$

That means Y is dense in $C[0,1]$.

Note. Let f_0 be continuous, linear functional on Y. $\exists?$ continuous, linear functional f on $X : f|_Y = f_0$

Theorem 2.3.
Given task is always has (unique) solution, also $\|f\|_X = \|f_0\|_Y$

Proof. f — continuous on X, g — continuous on X, $f|_Y = g|_Y$

$$\forall x \in X,\ \mathrm{Cl}(Y) = X\ \exists y_n \in Y : y_n \to x$$
$$f(y_n) \to f(x),\ g(y_n) \to g(x) \implies f(x) = g(x)$$
$$\mathrm{Cl}(Y) = X\ \forall x \in X\ \exists y_n \in Y : y_n \to x$$
$$f_0 \text{ — continuous on } Y \implies \|f_0\|_Y < +\infty$$
$$|f_0(y_m) - f_0(y_n)| = |f_0(y_n - y_m)| \leqslant \|f_0\|_Y \cdot \underbrace{\|y_n - y_m\|}_{\downarrow\ 0} \implies \{f_0(y_n)\} \text{ — Cauchy seq.}$$
$$f(x) \coloneqq \lim f_0(y_n),\ y_n \to x,\ y_n' \to x \implies \lim f_0(y_n) = \lim f_0(y_n')$$
$$|f_0(y_n) - f_0(y_n')| \leqslant \|f_0\|_Y \cdot \underbrace{\|y_n - y_n'\|}_{\downarrow\ 0}$$
$$\underbrace{f_0(y_n)}_{\downarrow\ a} - \underbrace{f_0(y_n')}_{\downarrow\ b} \to 0 \implies a = b$$
$$f(x) = \lim f_0(y_n),\ y_n \to x$$

$f(x)$ — linear functional.
$\|f\|_X = \|f_0\|_Y$

$$\begin{cases} \|f\|_X = \sup\limits_{\|x\| \leqslant 1} |f(x)| \\ \|f_0\|_Y = \sup\limits_{\|y\| \leqslant 1} |f_0(y)| \end{cases} \implies \|f_0\|_Y \leqslant \|f\|_X$$

By the norm definition it is clear that supremum is achievable on the sphere.

$$\sup_{\|x\| \leqslant 1} |f(x)| \leqslant \sup_{\|x'\| = 1} |f(x')|$$
$$\|x\| = 1,\ \exists y_n \in Y : y_n \to x,\ \|y_n\| \to \|x\| = 1$$
$$y_n' = \frac{y_n}{\|y_n\|} \to \frac{x}{1} = 1$$
$$\|y_n'\| = 1,\ y_n' \in Y$$
$$\|x\| = 1\ \exists y_n \in Y : \|y_n\| = 1 \implies y_n \to x$$
$$f_0(y_n) \to f(x)$$
$$|f_0(y_n)| \leqslant \|f_0\|_Y,\ \|y_n\| = 1$$
$$|f(x)| \leqslant \|f_0\|_Y \implies \|f\|_x \leqslant \|f_0\|_Y \qquad \blacksquare$$

X, $p(x)$ — seminorm, Y — linear set in X.
f_0 — linear functional on Y, satisfying the so called seminorm submission property, i.e.

$$\forall y \in Y \ |f_0(y)| \leqslant p(y)$$

Theorem 2.4 (Hahn-banach)**.**
Any linear functional on linear set in X satisfying some seminorm submission property on Y, can be continued to X saving this property.
$\exists f\colon X \to \mathbb{R}$ — *linear functional*

1. $f|_Y = f_0$
2. $\forall x \in X \implies |f(x)| \leqslant p(x)$

Lemma 2 (Banach)**.** *X — linear set, $p(x)$ — seminorm, Y — linear eigen subset X. $f_0\colon Y \to \mathbb{R}$ — linear functional. $|f_0(y)| \leqslant p(y)$*
$e \notin Y$. $Y_1 = \mathscr{L}(Y, e) = \{\, \alpha y + \beta e,\ y \in Y, \alpha, \beta \in \mathbb{R} \,\}$
$\implies \exists f\colon Y_1 \to R$ — linear.

1. $f|_Y = f_0$
2. *on* Y_1 $|f(y)| \leqslant p(y)$

Proof. $\alpha y + \beta e \in Y_1$, f — linear $f(\alpha y + \beta e) = \alpha f(y) + \beta f(e)$, $f|_Y = f_0$
$f(\alpha y + \beta e) = \alpha f_0(y) + \beta f(e)$
With any value of $f(e)$ we will get f_0 on Y. $c \coloneqq f(e)$.

$$|f(\alpha y + \beta e)| \leqslant p(\alpha y + \beta e)$$
$$\alpha y \in Y,\ \alpha y, y \in Y$$
$$f(y \in Y + t \cdot e) = f_0(y) + t \cdot c$$
$$|f_0(y) - t \cdot c| \leqslant p(y - t \cdot e)\ \forall y \in Y,\ \forall t \in \mathbb{R}$$
$|f_0(y)| \leqslant p(y)$— given.
$f_0(y) - p(y - t \cdot e) \leqslant t \cdot c \leqslant f_0(y) + p(y - t \cdot e),\ t > 0$ ($t < 0$)is the same.
$$f_0(\frac{y}{t}) - p(\frac{y}{t} - e) \leqslant c \leqslant f_0(\frac{y}{t}) + p(\frac{y}{t} - e),\ \frac{y}{t} \in Y$$
$$f_0(y) - p(y - e) \leqslant c \leqslant f_0(y) + p(y - e)$$
$$\forall y_1, y_2 \in Y\ f_0(y_1) - p(y_1 - e) \leqslant c \leqslant f_0(y_2) + p(y_2 - e)$$

$$\begin{aligned} f_0(y_1) - f_0(y_2) \leqslant p(y_1 - e) + p(y_2 - e) f_0(y_1 - y_2) \leqslant \\ \leqslant p(y_1 - y_2) = p((y_1 - e) - (y_2 - e)) \leqslant p(y_1 - e) + p(y_2 - e) \end{aligned}$$

$$\sup_{y \in Y}(f_0(y) - p(y - e)) \leqslant f_0(y_2) + p(y_2 - e)$$
$$\underbrace{\sup_Y(f_0(y) - p(y - e))}_{a} \leqslant \underbrace{\inf_Y(f_0(y) + p(y - e))}_{b} \in \mathbb{R}$$
$$c \in [a, b] \neq \varnothing$$ ■

Note. If $a < b$ there exists infinitely many continuations.

To fully prove Hahn-Banach theorem, we need to enter the theory of semi-ordered sets and discuss Zorn's lemma. We will prove only the separable normed space case with $p(x) = a \cdot \|x\|,\ a > 0$

Theorem 2.5 (Hahn-Banach (separable normed space case))**.**
Let X be separable normed space, i.e. there exists countable, dense set in it. Let Y be linear set (not subspace) in X, f_0 — continuous linear functional on Y, then there exists continuous linear functional f on X:

1. $f|_Y = f_0$
2. $\|f\|_X = \|f_0\|_Y$

Proof. If X is separable, $A = \{a_1, \dots, a_n, \dots\} : \mathrm{Cl}(A) = X$. We will assume that all a_n are distinct. It is clear that separability gives us a way to define $L_0 = Y,\ L_n = \mathscr{L}\{L_{n-1}, a_n\} \subset L_{n+1},\ L = \bigcup\limits_{n=1}^{\infty} L_n$ — linear set and $\mathrm{Cl}(L) = X$. By the lemma we can continue f_0 from L_0 to L_1, from L_1 to L_2 and so on. And then by the first theorem we can continue it on the X. ■

Corollary 2.5.1. *Let X be normed space then $\forall x_0 \neq 0\ \exists f$ — linear functional:*

1. $\|f\| = 1$
2. $f(x_0) = \|x_0\|$

Proof. On $Y = \{\, tx_0,\ t \in \mathbb{R} \,\}$ we build linear functional $f_0 : \|f_0\|_Y = 1,\ f_0(x) = \|x_0\|$

$$\begin{aligned}
&f_0(tx_0) = t \cdot f_0(x_0) \\
&f_0(x_0) = \|x_0\|,\ t = 1 \\
&f_0(tx_0) = t \cdot \|x_0\|
\end{aligned}$$

■

Corollary 2.5.2. $\forall x_1 \neq x_2\ \exists$ *linear bounded functional* $: f(x_1) \neq f(x_2)$

Theorem 2.6 (Riesz)**.**
H — Hilbert space, f — linear, bounded functional on H, then $\exists y \in H$:

1. $\forall x \in H\ f(x) = \langle x, y\rangle$
2. $\|f\| = \|y\|$

Proof. First, we will prove that $\|f\| = \|y\|$. Let $f(x) = \langle x, y\rangle$, then by Shwarz inequality:

$$\begin{aligned}
&|f(x)| = |\langle x.y\rangle| \leqslant \|y\| \cdot \|x\| \implies \|f\| \leqslant \|y\| \\
&x_0 = \frac{y}{\|y\|},\ \|x_0\| = 1,\ |f(x_0)| = \frac{|f(y)|}{\|f(y)\|} = \frac{|\langle y, y\rangle|}{\|y\|} = \frac{\|y\|^2}{\|y\|} = \|y\| \\
&\|f\| = \sup_{\|x\| \leqslant 1} |f(x)|,\ \|x_0\| = 1, \|f\| \leqslant \|y\| \implies \|f\| = \|y\|
\end{aligned}$$

Now we prove that $\forall x \in H\ f(x) = \langle x, y\rangle$
$H_1 = \ker f$ — closed, H_1 — subspace of H. Let $H_2 = H_1^{\perp},\ H = H_1 \oplus H_2$
H_1 — hyperplane $\implies$ $\operatorname{codim} H_1 = 1, \dim H_1 = 1\ H_2 = \{\, te,\ t \in \mathbb{R},\ e \in H_2 \,\}$
$\forall x \in H\ x \overset{!}{=} x_1 \in H_1 + t \cdot e$
$f(x) = f(x_1) + t \cdot f(e) = t \cdot f(e)$. Let $y = \alpha \cdot e,\ \alpha \in \mathbb{R}\ \langle x, \alpha e\rangle = \langle x_1 + te, \alpha e\rangle = \alpha t \|e\|^2$.
Choose such α, so that $f(x) = \langle x, \alpha e\rangle\ \forall x \in H,\ t \cdot f(e) = \alpha t \|e\|^2,\ \alpha = \dfrac{f(e)}{\|e\|^2},\ y = \dfrac{f(e)}{\|e\|^2} \cdot e$ ■

Example 2.1.1. $H = L_2[0,1]$
$\forall$ linear bounded functional $L_2[0,1]$
$\exists g \in L_2[0,1]$
$f(\hat{g}) = \int\limits_0^1 f(t)g(t)dt$

Example 2.1.2. $C[0,1]$ f — limear bounded functional on $C[0,1]$
$\exists g$ — bounded on [0, 1] $f(\hat{g}) = \int\limits_0^1 \hat{g}(t)dg(t)$ — Riemann–Stieltjes integral.

§2.2 Linear bounded operators

Definition. X, Y
$A\colon X \xrightarrow{linear} Y$
$A(\alpha_1 x_1 + \alpha_2 x_2) = \alpha_1 A(x_1) + \alpha_2 A(x_2)$
Then A is called **linear operator**. In case $Y = \mathbb{R}$ we get linear functional.

Definition. A — continuous if $x_n \to x \implies Ax_n \to Ax \iff x_n \to 0 \implies Ax_n \to 0 = A0$

Definition. A — bounded, if $\exists const\ M : \|Ax\| \leqslant M\|x\|\ \forall x \in X \iff \sup\limits_{\|x\|\leqslant 1} \|Ax\| < +\infty$

Definition. $\|A\| = \sup\limits_{\|x\|\leqslant 1} \|Ax\|$
$\|Ax\| \leqslant \|A\| \cdot \|x\|\ \forall x \in X$

Theorem 2.7.
A — continuous $\iff$ A — bounded.

Proof. Is identical to proof for functionals. ■

Note. $L(X, Y)$ — linear space of linear, bounded operators from X to Y, which is normed by the operator norm.

Theorem 2.8.
Y — Banach space $\implies$ $L(X, Y)$ is also a Banach space.

Proof. Let A_n be sequence of linear operators $A_n \in L(X, Y) : \|A_n - A_n\| \to 0$

$$?\exists A \in L(X, Y) : \|A_n - A\| \to 0$$
$$\|A_n - A\| \to 0 \iff A_n \overset{\overline{V}}{\to} A, \overline{V} = \{\ \|\ x\| \leqslant 1\ \}$$
$$\forall \varepsilon > 0\ \exists N \forall n > N, \forall x \in \overline{V} \implies \|A_n x - Ax\| < \varepsilon$$
$$\forall x \in X\ \|A_n x - A_m x\| = \|(A_n - A_m)x\| \leqslant \underbrace{\|A_n - A_m\|}_{\downarrow\atop 0} \|x\|$$
$$\implies A_n x - A_m x \to 0 \text{ in } Y \text{ — Banach space.} \implies \exists \lim_{n\to\infty} A_n x = Ax$$

It is obvious that A is linear operator. We only have to prove that A is bounded and $A = \lim A_n$ in $L(X, Y)$

$$x \in \overline{V}\ \|Ax\| \leqslant \|A_n x - Ax\| + \|A_n x\|$$
$$\|A_n x - A_m x\| \leqslant \varepsilon,\ \forall n, m > N,\ \forall x \in \overline{V},\ m \to \infty$$
$$\|A_n x - Ax\| \leqslant \varepsilon,\ \forall n > N,\ \forall x \in \overline{V}$$
$$\|(A_n - A)x\| \leqslant \varepsilon \implies \|A_n - A\| \leqslant \varepsilon, \forall n > N$$
$$A_n - A \in L(X, Y),\ A = \underbrace{A_n}_{\text{bounded}} - \underbrace{(A_n - A)}_{\text{bounded}} \implies A \text{ — bounded.}$$
$$A = \lim A_n \text{ in } L(X, Y).$$
■

Note. X — normed space, $\mathbb{R}$ — Banach space, then $L(X, R)$ is Banach space.
$X^* \overset{\text{def}}{=} L(X, R)$ — space dual to X.

Theorem 2.9 (Banach-Steinhaus).
X — Banach space, Y — normed space, $A_n \in L(X,Y)$
$\forall x \in X\ \sup\limits_n \|A_n x\| < +\infty$. (Operator sequence is pointwise bounded). Then $\sup\limits_n \|A_n\| < +\infty$ (Uniform bounded.)

Proof. $\forall x \in \overline{V}, n = 1, 2, \dots\ \|A_n x\| \leqslant M?$
Assume $\exists \overline{V}_r(a) : \forall x \in \overline{V}_r(a),\ n = 1, 2 \dots \implies \|A_n x\| \leqslant P$, then
$x = ry + a,\ y\overline{V}, \overline{x} \in \overline{V}_r(a)\ y = \dfrac{x-a}{r}$
$\|A_n y\| = \dfrac{1}{r}\|A_n x - A_n a\| \leqslant \dfrac{1}{2}(\overbrace{\|A_n x\|}^{\in \overline{V}_r(a)} + \overbrace{\|A_n a\|}^{\in \overline{V}_r(a)}) \leqslant \dfrac{2P}{r} = M$ Proof by contradiction, let there be no such $\overline{V}_r(a)$. $\forall \overline{V}_{r_1}(a_1)$ it does not satisfy our requirements $\implies \exists n_1, \exists x_1 \in \overline{V}_{r_1}(a_1) : \|A_{n_1} x_1\| > 1$. But A_{n_1} is continuous, then by continuity $\exists \overline{V}_{r_2}(x_1) : \forall x \in \overline{V}_{r_2}(x_1) \implies \|A_{n_1}(x)\| > 1$
$\exists n_2 > n_1\ \exists x_2 \in \overline{V}_{r_2}(x_1) : \|A_{n_2}(x_2)\| > 2\ \overline{V}_{r_3}(x_2)$. We can assume that each consecutive radius is at least two times smaller than the previous one. Continuing by induction, we get the sequence of nested balls with radiuses going to 0. Then they will have commmon point.
$\|A_{n_k}(b)\| > k$
$\|A_{n_k}(b)\| \to +\infty$
$\sup\limits_n \|A_n(b)\| < +\infty \to\leftarrow$ ■

Corollary 2.9.1. *X, Y – banach spaces, $A_n \in L(X,Y)\ \forall x \in X\ A_n x - A_m x \to 0$, then $\exists A \in L(X,Y) : \forall x \in X\ A_n x \to Ax$*

§2.3 Continuously reversible operators

Definition. $A \in L(X, Y)$
$A\colon X \xrightarrow{bijection} Y\ \exists A^{-1}\colon Y \xrightarrow{linear} X$
$A^{-1} \in L(Y, X)$ A — **continuously reversible**.

Theorem 2.10 (Banach)**.**
X — Banach space, $C \in L(X)$, $\|C$
$< 1 \implies I - C,\ Ix = x$ — continuously reversible.

Proof. $\sum\limits_{n=0}^{\infty} C^n,\ S_n = \sum\limits_{k=0}^{n} C^k$
$S_n(I - C) = S_n - S_n \cdot CS_n - \sum\limits_{k=1}^{n+1} C^k = I - C^{n+1}$
$S_n(I - C) = I - C^{n+1}$
$(I - C)S_n = I - C^{n+1}$
$\|C^n\| \leqslant \|C\|^n$
$\sum\limits_{n=0}^{\infty} \|C^n\| \leqslant \sum\limits_{n=0}^{\infty} \|C\|^n < +\infty,\ \|C\| < 1$
$\exists S = \sum\limits_{n=0}^{\infty},\ S \in L(X),\ C^n \to 0$
$S_n(I - C) = I - C^{n+1},\ n \to \infty$
$S(I - C) = I$
$(I - C)S = I$
$S = (I - C)^{-1} \in L(X)$ $I - C$ — continuously reversible. ■

$A\colon X \to Y$
$R(A) = \{\, x \mid x \in X \,\}$ — lienar set in Y. Closed $R(A)$. $y \in R(A)$ $Ax = y$

Definition. If $\exists const \alpha > 0 : \exists \hat{x} Ax = y,\ \forall y \in R(A) : \|\hat{x} \leqslant \alpha \|y\|$, then $Ax = y$ has for its solutions.

Theorem 2.11.
If operator equation $Ax = y$ has for its solutions, then $R(A) = \mathrm{Cl}(R(A))$

§2.4 Steinhaus theorem