

# Functional analysis course by Dodonov N.U.

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# 1 Vector spaces

## §1.1 Metric spaces.

$X, \rho: X \times X \rightarrow \mathbb{R}_+$

**Definition.**  $\rho$  — **metric**

1.  $\rho(x, y) \geq 0, = 0 \iff x = y$
2.  $\rho(x, y) = \rho(y, x)$
3.  $\rho(x, y) \leq \rho(x, z) + \rho(y, z)$

**Definition.**  $(X, \rho)$  — **Metric space.**

**Definition.**  $x = \lim x_n \iff \rho(x_n, x) \rightarrow 0$

$X, \tau = \{G \subset X\}$

1.  $\emptyset, X \in \tau$
2.  $G_\alpha \in \tau, \alpha \in \mathcal{A} \implies \bigcup_{\alpha} G_\alpha \in \tau$
3.  $G_1, \dots, G_n \in \tau \implies \bigcap_{j=1}^n G_j \in \tau$

**Definition.**  $(X, \tau)$  — **Topological space.**

$x = \lim x_n \quad \forall G \in \tau : x \in G \quad \exists N : \forall n > N \implies x_n \in G$

$G$  — open in  $\tau$

$F = X \setminus G$  — closed

**Definition.**  $B_r(a) = \{x \mid \rho(x, a) < r\}$  — **open ball**

$\tau = \bigcup B_r(a)$

**Statement 1.1.**  $b \in B_{r_1}(a_1) \cap B_{r_2}(a_2) \implies \exists r_3 > 0 : B_{r_3}(a_3) \subset B_{r_1}(a_1) \cap B_{r_2}(a_2)$

In this sense metric space is just a special case of topological space.

**Example 1.**  $\mathbb{R}, \rho(x, y) = |x - y|$ , MS

**Example 2.**  $\bar{x} = (x_1, \dots, x_n) \in \mathbb{R}^n, \rho(\bar{x}, \bar{y}) = \sqrt{\sum_{j=1}^n (x_j - y_j)^2}$ , MS

**Example 3.**  $\bar{x} = (x_1, \dots, x_n, \dots) \in \mathbb{R}^\infty$

$\alpha \bar{x} = (\alpha x_1, \dots, \alpha x_n, \dots)$

$\bar{x} + \bar{y} = (x_1 + y_1, \dots, x_n + y_n, \dots)$

Let's define  $\lim_{m \rightarrow \infty} \bar{x}_m$

- in  $\mathbb{R}^n$ :  $\bar{x}_n \rightarrow \bar{x} \iff \forall j = 1, \dots, n \quad x_j^{(m)} \xrightarrow{m \rightarrow \infty} x_j$

- in  $\mathbb{R}^\infty$ :  $\bar{x}_m \rightarrow \bar{x} \stackrel{\text{def}}{\iff} \forall j = 1, 2, 3, \dots \quad x_j^{(m)} \xrightarrow{m \rightarrow \infty} x_j$

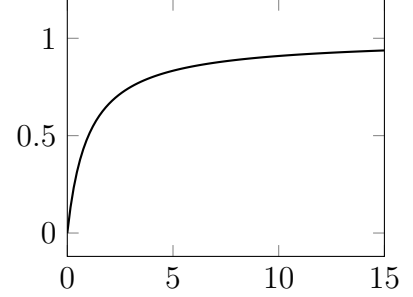
**Definition.**  $\rho(\bar{x}, \bar{y}) \stackrel{\text{def}}{=} \sum_{n=1}^{\infty} \frac{1}{2^n} \underbrace{\frac{|x_n - y_n|}{1 + |x_n - y_n|}}_{\varphi(|x_n - y_n|)}$  — **Urysohn metric.**

$$\varphi(t) = \frac{t}{1+t}$$

$$\varphi(t_1) + \varphi(t_2) \geq \varphi(t_1 + t_2)$$

$$\varphi(t_1) + \varphi(t_2) = \frac{t_1}{1+t_1} + \frac{t_2}{1+t_2} \geq \frac{t_1}{1+t_1+t_2} + \frac{t_2}{1+t_1+t_2}$$

$$\varphi(t_1) + \varphi(t_2) \geq \frac{t_1 + t_2}{1 + t_1 + t_2} = \varphi(t_1 + t_2)$$



**Statement 1.2.**  $\rho(\bar{x}_m, \bar{x}) \xrightarrow{m \rightarrow \infty} 0 \iff x_j^{(m)} \rightarrow x_j \forall j$

*Proof.*

- $\rightarrow$

$$f(|x_k^{(n)} - x_k|) \leq 2^k \rho(x^{(n)}, x)$$

$$\text{Let } \rho(x^{(n)}, x) \leq \frac{\varepsilon}{2^k}, \text{ then } f(|x_k^{(n)} - x_k|) < \varepsilon$$

$$|x_k^{(n)} - x_k| = t = \frac{1}{1 - f(t)} - 1, \text{ then } t \rightarrow 0$$

- $\Leftarrow$

$$\text{Let's choose } k_0 \text{ for which } \sum_{k=k_0+1}^{\infty} \frac{1}{2^k} < \varepsilon$$

$$\text{Let's choose } n_0 \text{ for which } \forall k \leq k_0, n > n_0 : |x_k^{(n)} - x_k| < \varepsilon.$$

$$\text{Then } \rho(x^{(n)}, x) < \sum_{k=1}^{k_0} \frac{\varepsilon}{2^k} + \varepsilon < 2\varepsilon$$

Letting  $\varepsilon \rightarrow 0$ , we get what we want ■

In this way  $\mathbb{R}^\infty$  is a metrizable space.

**Example 4.**  $X, \rho(x, y) \stackrel{\text{def}}{=} \begin{cases} 0, & x = y \\ 1, & x \neq y \end{cases}$  — **Discrete metric.**

$$x_n \rightarrow x, \varepsilon = \frac{1}{2}, \exists M : m > M \implies \rho(x_m, x) < \frac{1}{2} \implies \rho(x_m, x) = 0 \implies x_m = x$$

**Definition.**  $(X, \tau); \forall A \subset X;$

$$\text{Int}(A) \stackrel{\text{def}}{=} \bigcup_{G \subset A} G - \text{open};$$

$$\text{Cl}(A) \stackrel{\text{def}}{=} \bigcap_{A \subset G} G - \text{closed};$$

$$\text{Fr}(A) = \text{Cl}(A) \setminus \text{Int}(A)$$

$(X, \rho);$  Having a metric space one can describe closure of a set.

$$\rho(x, A) \stackrel{\text{def}}{=} \inf_{a \in A} \rho(x, a)$$

$$\rho(A, B) = \inf_{\substack{a \in A \\ b \in B}} \rho(a, b)$$

$$\rho(x, A) = f(x), x \in X$$

**Statement 1.3.** *Function  $f(x)$  is continuous.*

*Proof.*  $\forall x, y \in X$

$$f(x) = \rho(x, A) \leq \rho(x, \alpha) \leq \rho(x, y) + \rho(y, \alpha)$$

$$\forall \varepsilon > 0 \exists \alpha_\varepsilon \in A: \rho(y, \alpha_\varepsilon) < \rho(y, A) + \varepsilon = f(y) + \varepsilon$$

$$f(x) \leq f(y) + \varepsilon + \rho(x, y), \varepsilon \rightarrow 0$$

$$\begin{cases} f(x) \leq f(y) + \rho(x, y) \\ f(y) \leq f(x) + \rho(x, y) \end{cases} \implies |f(x) - f(y)| \leq \rho(x, y)$$

■

**Statement 1.4.**  $x \in \text{Cl}(A) \iff \rho(x, A) = 0$

Let's look at the metric spaces in terms of separation of sets from each other by open sets.

$x, y$

$$r = \rho(x, y) > 0$$

$$B_{\frac{r}{3}}(x), B_{\frac{r}{3}}(y)$$

In any metric space separability axiom is true.

**Theorem 1.1.**

*Any metric space is a normal space,*

*i.e.  $\forall$  closed disjoint  $F_1, F_2 \in X$ ,  $\exists$  open disjoint  $G_1, G_2: F_j \in G_j, j = 1, 2$*

$$\text{Proof. } g(x) = \frac{\rho(x, F_1)}{\rho(x, F_1) + \rho(x, F_2)} \text{ — continuous on } X$$

$$x \in F_1, \text{Cl}(F_1) = F_1, \rho(x, F_1) = 0, g(x) = 0$$

$$x \in F_2, g(x) = 1$$

Let's look at  $(-\infty; \frac{1}{3}), (\frac{2}{3}, \infty)$  — by continuity their inverse images under  $g$  are open.

$$G_1 = g^{-1}(-\infty; \frac{1}{3})$$

$$G_2 = g^{-1}(\frac{2}{3}; \infty)$$

■

**Definition.** Metric space is **complete** if  $\rho(x_n, x_m) \rightarrow 0 \implies \exists x = \lim x_n$

$\mathbb{R}^\infty$  – complete (by completeness of the rational numbers).

In complete metric spaces the nested balls principle is true.

**Theorem 1.2.**

$X$  – complete metric space,  $\overline{V}_{r_n}$  – system of closed balls.

1.  $\overline{V}_{r_{n+1}} \subset \overline{V}_{r_n}$  – the system is nested.

2.  $r_n \rightarrow 0$

$$\text{Then: } \bigcap_n \overline{V}_{r_n} = \{a\}$$

*Proof.* Let  $b_n$  be centers of  $\overline{V}_{r_n}$ ,

$$m \geq n, b_m \in \overline{V}_{r_n}, \rho(b_m, b_n) \leq r_n \rightarrow 0 \forall m \geq n$$

$$\rho(b_m, b_n) \rightarrow 0 \xrightarrow{\text{compl.}} \exists a = \lim b_n \text{ Since the balls are closed } a \in \text{every ball.}$$

$$r_n \rightarrow 0 \implies \text{there is only one common point.}$$

■

$(X, \tau)$  — topological space

$A \subset X, \tau_a = \{G \cap A, G \in \tau\}$  — topology induced on  $A$

**Definition.**  $X$  — metric space,  $A \subset X, \text{Cl}(A) = X$

Then:  $A$  – **dense** in  $X$

If  $\text{Int}(\text{Cl}(A)) = \emptyset$   $A$  – **nowhere dense** in  $X$ .

**Note.** It is easy to understand, that in metric spaces nowhere density means the following:  
 $\forall \text{ ball } V \exists V' \subset V : V' \text{ contains no points from } A.$

**Definition.**  $X$  is called **first Baire category set**, if it can be written as at most countable union of  $x_n$  each nowhere dense in  $X$ .

**Theorem 1.3** (Baire category theorem).

*Complete metric space is second Baire category set in itself.*

*Proof.* Let  $X$  be first Baire category set.

$X = \bigcup_n X_n \quad \forall \bar{V} \quad X_1 \text{ is nowhere dense.}$

$\bar{V}_1 \subset \bar{V} : \bar{V}_1 \cap X_1 = \emptyset$

$X_2 \text{ is nowhere dense } \bar{V}_2 \subset \bar{V}_1 : \bar{V}_2 \cap X_2 = \emptyset$

$r_2 \leq \frac{r_1}{2}$

$\vdots$

$\{\bar{V}_n\}, r_n \rightarrow 0, \bigcap_n \bar{V}_n = \{a\}, X = \bigcup_n X_n, \exists n_0 : a \in X_{n_0}$

$X_{n_0} \cap \bar{V}_{n_0} = \emptyset \rightarrow \leftarrow a \in \bar{V}_{n_0}$  ■

**Corollary 1.3.1.** *Complete metric space without isolated points is uncountable.*

*Proof.* No isolated points are present  $\implies$  every point in the set is nowhere dense in it. Let  $X$  be countable:  $X = \bigcup_n \{X_n\}$ , then it is first Baire category set.  $\rightarrow \leftarrow$  ■

**Definition.**  $K$  — **compact** if

1.  $K = \text{Cl}(K)$
2.  $x_n \in K \exists n_1 < n_2 < \dots x_{n_j} - \text{converges in } X.$

If only 2 is present, the set is called **precompact**.

**Theorem 1.4** (Hausdorff).

*Let  $X$  — metric space,  $K$  — closed in  $X$ .*

Then:  $K$  — compact  $\iff K$  — totally bounded,

*i.e.  $\forall \varepsilon > 0 \exists a_1, \dots, a_p \in X : \forall b \in K \exists a_j : \rho(a_j, b) < \varepsilon$   
 $(a_1, \dots, a_p - \text{finite } \varepsilon - \text{net})$*

*Proof.*

- Totally bounded  $\implies$  compact

$K$  is totally bounded,  $x_n \in K \quad n_1 < n_2 < \dots < n_k < \dots, x_n$  converges in  $K$

$$\varepsilon_k \downarrow \rightarrow 0 \quad \varepsilon_1 \quad K \subset \bigcup_{j=1}^p \bar{V}_j, \text{ rad} = \varepsilon_1 \quad (\varepsilon_1\text{-net})$$

$n$  is finite  $\implies$  one ball will contain infinitely many  $x_n$  elements.

Let's look at  $\bar{V}_{j_0} \cap K$  — totally bounded =  $K_1, \text{diam}(K_1) \leq 2\varepsilon_1$

$$\varepsilon_2 \quad K_1 \subset \bigcup_{j=1}^n \bar{V}'_j, \text{ rad} = \varepsilon_2.$$

Then one of  $\bar{V}'$  contains infinitely many elements of the sequence contained in  $K_1$ .

$\bar{V}'_{j_0} \cap K_1 = K_2, \text{diam}(K_2) \leq 2\varepsilon_2$  and so on.

$$K_n \supset K_{n+1} \supset K_{n+2} \supset \dots, \text{diam}(K_N) \leq 2\varepsilon_n \xrightarrow{\text{by compl.}} \underbrace{\bigcap_{n=1}^{\infty} K_n}_{\text{diam}(K_n) \rightarrow 0, \{x\}} \neq \emptyset$$

Take  $x_{n_1}$  from  $K_1$ ,  $x_{n_2}$  from  $K_2 \dots$

- Compact  $\implies$  totally bounded

$K$  — compact  $\forall \varepsilon \exists$  finite  $\varepsilon$ -net?

By contradiction:  $\exists \varepsilon_0 > 0$ : finite  $\varepsilon_0$ -net is impossible to construct.

$\forall x_1 \in K \exists x_2 \in K: \rho(x_1, x_2) > \varepsilon_0$  (or else system of  $x_1$  — finite  $\varepsilon$ -net)

$\{x_1, x_2\}$  - choose  $x_3 \in K: \rho(x_3, x_i) > \varepsilon_0, i = 1, 2$  and so on.

$x_n \in K: n \neq m \rho(x_n, x_m) > \varepsilon_0$  — contains no converging subsequence  $\implies$  set is not a compact.  $\rightarrow \leftarrow$  ■

## §1.2 Normed spaces

**Definition.**  $X$  — **linear set**,  $x + y, \alpha \cdot x, \alpha \in \mathbb{R}$

The purpose of norm definition, is to construct a topology on  $X$ , so that 2 linear operations are continuous on it.

$\varphi: X \rightarrow \mathbb{R}$ :

1.  $\varphi(x) \geq 0, = 0 \iff x = 0$
2.  $\varphi(\alpha x) = |\alpha| \varphi(x)$
3.  $\varphi(x + y) \leq \varphi(x) + \varphi(y)$

**Definition.**  $\varphi$  — **norm** on  $X$ ,  $\varphi(x) = \|x\|$

$\rho(x, y) \stackrel{\text{def}}{=} \|x - y\|$  — metric on  $X$ .

**Definition.**

$(X, \|\cdot\|)$  — **normed space** — special case of metrical space.

$$x = \lim x_n \xrightarrow{\text{def}} \rho(x_n, x) \rightarrow 0 \iff \|x_n - x\| \rightarrow 0$$

**Statement 1.5.** In the topology of a normed space linear operations are continuous on  $X$ .

*Proof.*

$$\begin{aligned} 1. \quad x_n \rightarrow x, y_n \rightarrow y; \quad \|(x_n + y_n) - (x + y)\| &= \|(x_n - x) + (y_n - y)\| \leq \\ &\leq \underbrace{\|x_n - x\|}_{\downarrow 0} + \underbrace{\|y_n - y\|}_{\downarrow 0} \\ &\implies x_n + y_n \rightarrow x + y \end{aligned}$$

$$\begin{aligned} 2. \quad \alpha_n \rightarrow \alpha, x_n \rightarrow x; \quad \|\alpha_n x_n - \alpha x\| &= \|(\alpha_n - \alpha)x_n + \alpha(x_n - x)\| \leq \\ &\leq \underbrace{|\alpha_n - \alpha|}_{\downarrow 0} \cdot \underbrace{\|x_n\|}_{\text{bounded}} + \underbrace{\alpha\|x_n - x\|}_{\downarrow 0} \end{aligned}$$

$x_n \rightarrow x \implies \|x_n\|$  — bounded.

$\alpha x_n \rightarrow \alpha x$  ■

**Statement 1.6.** From the triangle inequality  $|\|x\| - \|y\|| \leq \|x - y\|$   
 $x_n \rightarrow x \implies \|x_n\| \rightarrow \|x\|$   
 Norm is continuous.

**Example 5.**  $\mathbb{R}^n$

1.  $\|\bar{x}\| = \sqrt{\sum_{k=1}^n x_k^2}$
2.  $\|\bar{x}\|_1 \stackrel{\text{def}}{=} \sum_{k=1}^n |x_k|$
3.  $\|\bar{x}\|_2 \stackrel{\text{def}}{=} \max\{|x_1|, \dots, |x_n|\}$
4.  $C[a, b]$  — functions continuous on  $[a, b]$ ;  $\|f\| = \max_{x \in [a, b]} |f(x)|$
5.  $L_p(E) = \left\{ f \text{ — measurable, } \int_E |f|^p < +\infty \right\}$   
 $p \geq 1, \|f\|_p = \left( \int_E |f|^p \right)^{\frac{1}{p}}$

Because the set of points is the same, arises the question about convergence comparison.  
 $\|\cdot\|_1 \sim \|\cdot\|_2, x_n \xrightarrow{\|\cdot\|_1} x \iff x_n \xrightarrow{\|\cdot\|_2} x$

**Statement 1.7.**

$$\|\cdot\|_1 \sim \|\cdot\|_2 \iff \exists a, b > 0: \forall x \in X \implies a\|x\|_1 \leq \|x\|_2 \leq b\|x\|_1$$

**Theorem 1.5** (Riesz).

$X, \dim X < +\infty$  — linear set.

Then: Any pair of norms in  $X$  are equivalent.

*Proof.*  $l_1, \dots, l_n$  — linearly independent from  $X$ .  $\forall x \in X = \sum_{k=1}^n \alpha_k l_k$

$$\bar{x} \leftrightarrow (l_1, \dots, l_n) = \bar{l} \in \mathbb{R}^n$$

Let  $\|\cdot\|$  — some norm in  $X$ .

$$\|x\| \underset{\triangle}{\leq} \sum_{k=1}^n \|l_k\| |\alpha_k| \underset{\text{Cauchy}}{\leq} \underbrace{\sqrt{\sum_{k=1}^n \|l_k\|^2}}_{\text{const}(B), B\text{-basis}} \sqrt{\sum_{k=1}^n |\alpha_k|^2} \underset{\substack{\| \\ \|\bar{\alpha}\| = \|x\|_1}}{\leq}$$

$$\|x\|_1 = \sqrt{\sum_{k=1}^n \|\alpha_k\|^2}, \quad x = \sum \alpha_k l_k$$

$$\|x\| \leq b\|x\|_1$$

$$?\exists a > 0: a\|x\|_1 \leq \|x\| \implies \|\cdot\| \sim \|\cdot\|_1$$

$$\text{Let } f(\alpha_1, \dots, \alpha_n) = \left\| \sum_{k=1}^n \alpha_k l_k \right\|$$

$$\begin{aligned} |f(\bar{\alpha} + \Delta \bar{\alpha}) - f(\bar{\alpha})| &= \left| \left\| \sum_{k=1}^n \alpha_k l_k + \sum_{k=1}^n \Delta \alpha_k l_k \right\| - \left\| \sum_{k=1}^n \alpha_k l_k \right\| \right| \leq \\ &= \left\| \sum_{k=1}^n \Delta \alpha_k l_k \right\| \leq \underbrace{\sum_{k=1}^n \|l_k\| |\Delta \alpha_k|}_{0, \Delta \alpha_k \rightarrow 0} \implies f \text{ is continuous on } \mathbb{R}^n \end{aligned}$$

$$S_1 = \left\{ \sum_{k=1}^n |\alpha_k|^2 = 1 \right\} \subset \mathbb{R}^m, f \text{ — continuous on } S_1, S_1 \text{ — compact, } \bar{\alpha}^* \in S_1$$

By Weierstrass theorem there exists a point  $\alpha^* \in S_1$  on a sphere, in which function  $f$  achieves its minimum  $\implies \forall \alpha \in S_1 \ f(\bar{\alpha}^*) \leq f(\bar{\alpha})$

$$\text{If } f(\bar{\alpha}^*) = 0, \text{ then } \left\| \sum_{k=1}^n \alpha_k^* l_k \right\| = 0 \implies \sum_{k=1}^n \alpha_k^* l_k = 0, \bar{\alpha}^* \in S_1,$$

but  $l_1 \dots l_n$  are linearly independent  $\rightarrow \leftarrow \implies \min_{S_1} f = m > 0$

$$\begin{aligned} \|x\| &= \left\| \sum_{k=1}^n \alpha_k l_k \right\| = f(\bar{\alpha}) = \sqrt{\sum_{k=1}^n \alpha_k^2} \cdot \left\| \sum_{\beta_k} \frac{\alpha_k}{\sqrt{\sum_{k=1}^n \alpha_k^2}} \cdot l_k \right\| \geq, \bar{\beta} = (\beta_1 \dots \beta_n) \in S_1 \\ &\geq m \cdot \|x\|_1, a = m \quad \blacksquare \end{aligned}$$

**Corollary 1.5.1.**  $X = NS, Y \subset X, \dim Y < +\infty \implies Y = \text{Cl}(Y)$

**Note.** Functional analysis differentiates between linear subset (set of points, closed by addition and scalar multiplication) and linear subspace (closed linear subset).

$$\text{Proof. } Y = \alpha(l_1, \dots, l_n) = \left\{ \sum_{i=1}^n \alpha_i l_i \mid l_1, \dots, l_n \text{ — lin. indep.} \right\}$$

$$y_m \in Y, y_m \rightarrow y \text{ in } X \implies y \in Y?$$

$$\|y_m - y\| \rightarrow 0 \implies \|y_m - y_p\| \rightarrow 0, m, p \rightarrow \infty$$

$$\|y\|, y \in Y.$$

By Riesz theorem all norm in  $Y$  are equivalent.

$$y = \sum_{j=1}^n \alpha_j l_j, \|y\|_0 = \sqrt{\sum_{j=1}^n \alpha_j^2} \text{ — some norm (by linear independance).}$$

$$\text{By Riesz theorem } \|y\| \sim \|y\|_0$$

$$\underbrace{\|y_m - y_p\|}_{\in Y} \rightarrow 0 \implies \|y_m - y_p\|_0 \rightarrow 0$$

$$\bar{\alpha} = (\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n \ y_m = \sum_{i=1}^n \alpha_i^{(m)} l_i$$

$$|\alpha_i^{(m)} - \alpha_i^{(l)}| \rightarrow 0 \ \forall i = 1, \dots, n; \ \bar{\alpha} = (\alpha_1^{(m)}, \dots, \alpha_n^{(m)}) \rightarrow \alpha^* = (\alpha_1^*, \dots, \alpha_n^*)$$

$$y^* = \sum_{i=1}^n \alpha_i^* l_i \in Y, \|y_m - y\| \rightarrow 0, y = y^* \implies y \in Y \quad \blacksquare$$

**Definition.** If normed space is complete, then it is called **B-space** or **Banach space**.

**Example 6.**  $C[a, b]$  — functions continuous on  $[a, b]$ .



**Example 7.** Lebesgue space,  $p \geq 1$ ,  $L_p(E) = \left\{ f \text{ is measurable on } E, \int_E |f|^p < +\infty \right\}$ .

If  $X$  — Banach space,

$$\sum_{n=1}^{\infty} x_n = \lim_{n \rightarrow \infty} \sum_{k=1}^n x_k, \quad \sum_{n=1}^{\infty} \|x_n\| < +\infty$$

$$\|S_n - S_m\| = \left\| \sum_{k=m+1}^n x_k \right\| \leq \sum_{k=m+1}^n \|x_k\| \xrightarrow{n, m \rightarrow \infty} 0$$

$$\implies \|S_n - S_m\| \rightarrow 0 \implies \exists \lim_{n \rightarrow \infty} S_n, \sum_{k=1}^n x_k \text{ — converges.}$$

In Banach spaces works the theory of absolute convergence of numerical series.

**Lemma 1** (Riesz).