

Functional analysis course by Dodonov N.U.

Sugak A.M.

Fall 2015 — Spring 2016

1 Vector spaces

§1.1 Metric spaces.

$X, \rho: X \times X \rightarrow \mathbb{R}_+$

Definition. ρ — **metric**

1. $\rho(x, y) \geq 0, = 0 \iff x = y$
2. $\rho(x, y) = \rho(y, x)$
3. $\rho(x, y) \leq \rho(x, z) + \rho(y, z)$

Definition. (X, ρ) — **Metric space.**

Definition. $x = \lim x_n \iff \rho(x_n, x) \rightarrow 0$

$$X, \tau = \{ G \subset X \}$$

Definition. Let X be arbitrary set. Then system of its subsets τ is called a **topology** if :

1. $\emptyset, X \in \tau$
2. $G_\alpha \in \tau, \alpha \in \mathcal{A} \implies \bigcup_{\alpha} G_\alpha \in \tau$
3. $G_1, \dots, G_n \in \tau \implies \bigcap_{j=1}^n G_j \in \tau$

And any set $G \in \tau$ is called **open**.

Definition. (X, τ) — **Topological space.**

$$x = \lim x_n \quad \forall G \in \tau : x \in G \quad \exists N : \forall n > N \implies x_n \in G$$

G — open in τ

$F = X \setminus G$ — closed

Definition. $B_r(a) = \{ x \mid \rho(x, a) < r \}$ — **open ball**

Statement 1.1. Any metric space gives rise to a topological space in a rather simple way. Let's call the subset $G \subset X$ open if and only if $\forall x \in G$ there is some r such that open ball $B_r(x)$ is contained in G . Then $\tau = \bigcup B_r(a)$

Statement 1.2. $b \in B_{r_1}(a_1) \cap B_{r_2}(a_2) \implies \exists r_3 > 0 : B_{r_3}(a_3) \subset B_{r_1}(a_1) \cap B_{r_2}(a_2)$

Example 1. $\mathbb{R}, \rho(x, y) = |x - y|$, MS

Example 2. $\bar{x} = (x_1, \dots, x_n) \in \mathbb{R}^n, \rho(\bar{x}, \bar{y}) = \sqrt{\sum_{j=1}^n (x_j - y_j)^2}$, MS

Example 3. $\bar{x} = (x_1, \dots, x_n, \dots) \in \mathbb{R}^\infty$

$$\alpha \bar{x} = (\alpha x_1, \dots, \alpha x_n, \dots)$$

$$\bar{x} + \bar{y} = (x_1 + y_1, \dots, x_n + y_n, \dots)$$

Let's define $\lim_{m \rightarrow \infty} \bar{x}_m$

- in \mathbb{R}^n : $\bar{x}_n \rightarrow \bar{x} \iff \forall j = 1, \dots, n \quad x_j^{(m)} \xrightarrow{m \rightarrow \infty} x_j$

- in \mathbb{R}^∞ : $\bar{x}_m \rightarrow \bar{x} \stackrel{\text{def}}{\iff} \forall j = 1, 2, 3, \dots \quad x_j^{(m)} \xrightarrow{m \rightarrow \infty} x_j$

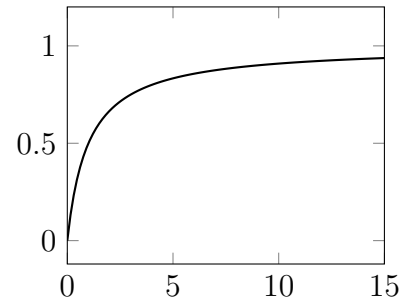
Definition. $\rho(\bar{x}, \bar{y}) \stackrel{\text{def}}{=} \sum_{n=1}^{\infty} \frac{1}{2^n} \underbrace{\frac{|x_n - y_n|}{1 + |x_n - y_n|}}_{\varphi(|x_n - y_n|)}$ — **Urysohn metric.**

$$\varphi(t) = \frac{t}{1+t}$$

$$\varphi(t_1) + \varphi(t_2) \geq \varphi(t_1 + t_2)$$

$$\varphi(t_1) + \varphi(t_2) = \frac{t_1}{1+t_1} + \frac{t_2}{1+t_2} \geq \frac{t_1}{1+t_1+t_2} + \frac{t_2}{1+t_1+t_2}$$

$$\varphi(t_1) + \varphi(t_2) \geq \frac{t_1 + t_2}{1 + t_1 + t_2} = \varphi(t_1 + t_2)$$



Statement 1.3. $\rho(\bar{x}_m, \bar{x}) \xrightarrow{m \rightarrow \infty} 0 \iff x_j^{(m)} \rightarrow x_j \forall j$

Proof.

- \implies

$$f(|x_k^{(n)} - x_k|) \leq 2^k \rho(x^{(n)}, x)$$

$$\text{Let } \rho(x^{(n)}, x) \leq \frac{\varepsilon}{2^k}, \text{ then } f(|x_k^{(n)} - x_k|) < \varepsilon$$

$$|x_k^{(n)} - x_k| = t = \frac{1}{1 - f(t)} - 1, \text{ then } t \rightarrow 0$$

- \Leftarrow

$$\text{Let's choose } k_0 \text{ for which } \sum_{k=k_0+1}^{\infty} \frac{1}{2^k} < \varepsilon$$

$$\text{Let's choose } n_0 \text{ for which } \forall k \leq k_0, n > n_0 : |x_k^{(n)} - x_k| < \varepsilon.$$

$$\text{Then } \rho(x^{(n)}, x) < \sum_{k=1}^{k_0} \frac{\varepsilon}{2^k} + \varepsilon < 2\varepsilon$$

Letting $\varepsilon \rightarrow 0$, we get what we want

■

In this way \mathbb{R}^∞ is a metrizable space.

Example 4. $X, \rho(x, y) \stackrel{\text{def}}{=} \begin{cases} 0, & x = y \\ 1, & x \neq y \end{cases}$ — **Discrete metric.**

$$x_n \rightarrow x, \varepsilon = \frac{1}{2}, \exists M : m > M \implies \rho(x_m, x) < \frac{1}{2} \implies$$

$$\rho(x_m, x) = 0 \implies x_m = x$$

Definition.

$$(X, \tau); \forall A \subset X;$$

$$\text{Int}(A) \stackrel{\text{def}}{=} \bigcup_{G \subset A} G \text{ is open};$$

$$\text{Cl}(A) \stackrel{\text{def}}{=} \bigcap_{A \subset G} G \text{ is closed};$$

$$\text{Fr}(A) \stackrel{\text{def}}{=} \text{Cl}(A) \setminus \text{Int}(A)$$

(X, ρ) ; Having a metric space one can describe closure of a set.

$$\rho(x, A) \stackrel{\text{def}}{=} \inf_{a \in A} \rho(x, a)$$

$$\rho(A, B) \stackrel{\text{def}}{=} \inf_{\substack{a \in A \\ b \in B}} \rho(a, b)$$

$$\rho(x, A) = f(x), x \in X$$

Statement 1.4. Function $f(x)$ is continuous.

Proof. $\forall x, y \in X$

$$f(x) = \rho(x, A) \leq \rho(x, \alpha) \leq \rho(x, y) + \rho(y, \alpha)$$

$$\forall \varepsilon > 0 \exists \alpha_\varepsilon \in A: \rho(y, \alpha_\varepsilon) < \rho(y, A) + \varepsilon = f(y) + \varepsilon$$

$$f(x) \leq f(y) + \varepsilon + \rho(x, y), \varepsilon \rightarrow 0$$

$$\begin{cases} f(x) \leq f(y) + \rho(x, y) \\ f(y) \leq f(x) + \rho(x, y) \end{cases} \implies |f(x) - f(y)| \leq \rho(x, y) \quad \blacksquare$$

Statement 1.5. $x \in \text{Cl}(A) \iff \rho(x, A) = 0$

Let's look at the metric spaces in terms of separation of sets from each other by open sets.

x, y

$$r = \rho(x, y) > 0$$

$$B_r(x), B_r(y)$$

$$\frac{\bar{3}}{3}, \frac{\bar{3}}{3}$$

In any metric space separability axiom is true.

Theorem 1.1.

Any metric space is a normal space,

i.e. \forall closed disjoint $F_1, F_2 \in X$, \exists open disjoint $G_1, G_2: F_j \in G_j, j = 1, 2$

$$\text{Proof. } g(x) = \frac{\rho(x, F_1)}{\rho(x, F_1) + \rho(x, F_2)} \text{ --- continuous on } X$$

$$x \in F_1, \text{Cl}(F_1) = F_1, \rho(x, F_1) = 0, g(x) = 0$$

$$x \in F_2, g(x) = 1$$

Let's look at $(-\infty; \frac{1}{3}), (\frac{2}{3}, \infty)$ — by continuity their inverse images under g are open.

$$G_1 = g^{-1}(-\infty; \frac{1}{3})$$

$$G_2 = g^{-1}(\frac{2}{3}; \infty) \quad \blacksquare$$

Definition. Metric space is **complete** if $\rho(x_n, x_m) \rightarrow 0 \implies \exists x = \lim x_n$

\mathbb{R}^∞ — complete (by completeness of the rational numbers).

In complete metric spaces the nested balls principle is true.

Theorem 1.2.

X — complete metric space, \bar{V}_{r_n} — system of closed balls.

1. $\overline{V}_{r_{n+1}} \subset \overline{V}_{r_n}$ – the system is nested.

2. $r_n \rightarrow 0$

Then: $\bigcap_n \overline{V}_{r_n} = \{a\}$

Proof. Let b_n be centers of \overline{V}_{r_n} ,

$m \geq n$, $b_m \in \overline{V}_{r_n}$, $\rho(b_m, b_n) \leq r_n \rightarrow 0 \forall m \geq n$

$\rho(b_m, b_n) \rightarrow 0 \xrightarrow{\text{compl.}} \exists a = \lim b_n$ Since the balls are closed $a \in$ every ball.

$r_n \rightarrow 0 \implies$ there is only one common point. ■

(X, τ) — topological space

$A \subset X$, $\tau_a = \{G \cap A, G \in \tau\}$ — topology induced on A

Definition. X — metric space, $A \subset X$, $\text{Cl}(A) = X$

Then: A — **dense** in X

If $\text{Int}(\text{Cl}(A)) = \emptyset$ A — **nowhere dense** in X .

Note. It is easy to understand, that in metric spaces nowhere density means the following:
 \forall ball $V \exists V' \subset V$: V' contains no points from A .

Definition. X is called **first Baire category set**, if it can be written as at most countable union of x_n each nowhere dense in X .

Theorem 1.3 (Baire category theorem).

Complete metric space is second Baire category set in itself.

Proof. Let X be first Baire category set.

$X = \bigcup X_n \quad \forall \overline{V} \quad X_1$ is nowhere dense.

$\overline{V}_1 \subset \overline{V}$: $\overline{V}_1 \cap X_1 = \emptyset$

X_2 is nowhere dense $\overline{V}_2 \subset \overline{V}_1$: $\overline{V}_2 \cap X_2 = \emptyset$

$r_2 \leq \frac{r_1}{2}$

\vdots

$\{\overline{V}_n\}$, $r_n \rightarrow 0$, $\bigcap_n \overline{V}_n = \{a\}$, $X = \bigcup X_n$, $\exists n_0$: $a \in X_{n_0}$

$X_{n_0} \cap \overline{V}_{n_0} = \emptyset \rightarrow \leftarrow a \in \overline{V}_{n_0}$ ■

Corollary 1.3.1. *Complete metric space without isolated points is uncountable.*

Proof. No isolated points are present \implies every point in the set is nowhere dense in it. Let X be countable: $X = \bigcup_n \{X_n\}$, then it is first Baire category set. $\rightarrow \leftarrow$ ■

Definition. K — **compact** if

1. $K = \text{Cl}(K)$

2. $x_n \in K \exists n_1 < n_2 < \dots x_{n_j}$ — converges in X .

If only ?? is present, the set is called **precompact**.

Theorem 1.4 (Hausdorff).

Let X — metric space, K — closed in X .

Then: K — compact $\iff K$ — totally bounded,

i.e. $\forall \varepsilon > 0 \exists a_1, \dots, a_p \in X$: $\forall b \in K \exists a_j$: $\rho(a_j, b) < \varepsilon$

(a_1, \dots, a_p) — finite ε — net

Proof.

- Totally bounded \implies compact

K is totally bounded, $x_n \in K$ $n_1 < n_2 < \dots < n_k < \dots$, x_n converges in K

$$\varepsilon_k \downarrow \rightarrow 0 \quad \varepsilon_1 \quad K \subset \bigcup_{j=1}^p \bar{V}_j, \quad \text{rad} = \varepsilon_1 \quad (\varepsilon_1\text{-net})$$

n is finite \implies one ball will contain infinitely many x_n elements.

Let's look at $\bar{V}_{j_0} \cap K$ — totally bounded = K_1 , $\text{diam}(K_1) \leq 2\varepsilon_1$

$$\varepsilon_2 \quad K_1 \subset \bigcup_{j=1}^n \bar{V}'_j, \quad \text{rad} = \varepsilon_2.$$

Then one of \bar{V}' contains infinitely many elements of the sequence contained in K_1 .

$\bar{V}'_{j_0} \cap K_1 = K_2$, $\text{diam}(K_2) \leq 2\varepsilon_2$ and so on.

$$K_n \supset K_{n+1} \supset K_{n+2} \supset \dots, \quad \text{diam}(K_N) \leq 2\varepsilon_n \xrightarrow{\text{by compl.}} \underbrace{\bigcap_{n=1}^{\infty} K_n}_{\text{diam}(K_n) \rightarrow 0, \{x\}} \neq \emptyset$$

Take x_{n_1} from K_1 , x_{n_2} from $K_2 \dots$

- Compact \implies totally bounded

K — compact $\forall \varepsilon \exists$ finite ε -net?

By contradiction: $\exists \varepsilon_0 > 0$: finite ε_0 -net is impossible to construct.

$\forall x_1 \in K \exists x_2 \in K: \rho(x_1, x_2) > \varepsilon_0$ (or else system of x_1 — finite ε -net)

$\{x_1, x_2\}$ - choose $x_3 \in K: \rho(x_3, x_i) > \varepsilon_0, i = 1, 2$ and so on.

$x_n \in K: n \neq m \rho(x_n, x_m) > \varepsilon_0$ — contains no converging subsequence \implies set is not a compact. $\rightarrow \leftarrow$ ■

§1.2 Normed spaces

Definition. X — **linear set**, $x + y, \alpha \cdot x, \alpha \in \mathbb{R}$

The purpose of norm definition, is to construct a topology on X , so that 2 linear operations are continuous on it.

$$\varphi: X \rightarrow \mathbb{R}:$$

$$1. \quad \varphi(x) \geq 0, = 0 \iff x = 0$$

$$2. \quad \varphi(\alpha x) = |\alpha| \varphi(x)$$

$$3. \quad \varphi(x + y) \leq \varphi(x) + \varphi(y)$$

Definition. φ — **norm** on X , $\varphi(x) = \|x\|$

$\rho(x, y) \stackrel{\text{def}}{=} \|x - y\|$ — metric on X .

Definition.

$(X, \|\cdot\|)$ — **normed space** — special case of metrical space.

$$x = \lim x_n \stackrel{\text{def}}{\implies} \rho(x_n, x) \rightarrow 0 \iff \|x_n - x\| \rightarrow 0$$

Statement 1.6. In the topology of a normed space linear operations are continuous on X .

Proof.

$$\begin{aligned}
 1. \quad x_n \rightarrow x, y_n \rightarrow y; \quad & \| (x_n + y_n) - (x + y) \| = \| (x_n - x) + (y_n - y) \| \leq \\
 & \leq \underbrace{\|x_n - x\|}_{\downarrow 0} + \underbrace{\|y_n - y\|}_{\downarrow 0} \\
 & \implies x_n + y_n \rightarrow x + y
 \end{aligned}$$

$$\begin{aligned}
 2. \quad \alpha_n \rightarrow \alpha, x_n \rightarrow x; \quad & \| \alpha_n x_n - \alpha x \| = \| (\alpha_n - \alpha) x_n + \alpha (x_n - x) \| \leq \\
 & \leq \underbrace{\|\alpha_n - \alpha\|}_{\downarrow 0} \cdot \underbrace{\|x_n\|}_{\text{bounded}} + \underbrace{\alpha \|x_n - x\|}_{\downarrow 0}
 \end{aligned}$$

$$\begin{aligned}
 x_n \rightarrow x & \implies \|x_n\| \text{ — bounded.} \\
 \alpha x_n & \rightarrow \alpha x
 \end{aligned}$$

■

Statement 1.7. From the triangle inequality $|\|x\| - \|y\|| \leq \|x - y\|$

$$x_n \rightarrow x \implies \|x_n\| \rightarrow \|x\|$$

Norm is continuous.

Example 5. \mathbb{R}^n

1. $\|\bar{x}\| = \sqrt{\sum_{k=1}^n x_k^2}$
2. $\|\bar{x}\|_1 \stackrel{\text{def}}{=} \sum_{k=1}^n |x_k|$
3. $\|\bar{x}\|_2 \stackrel{\text{def}}{=} \max\{|x_1|, \dots, |x_n|\}$
4. $C[a, b]$ — functions continuous on $[a, b]$; $\|f\| = \max_{x \in [a, b]} |f(x)|$
5. $L_p(E) = \left\{ f \text{ — measurable, } \int_E |f|^p < +\infty \right\}$
 $p \geq 1, \quad \|f\|_p = \left(\int_E |f|^p \right)^{\frac{1}{p}}$

Because the set of points is the same, arises the question about convergence comparison.

$$\|\cdot\|_1 \sim \|\cdot\|_2, \quad x_n \xrightarrow{\|\cdot\|_1} x \iff x_n \xrightarrow{\|\cdot\|_2} x$$

Statement 1.8.

$$\|\cdot\|_1 \sim \|\cdot\|_2 \iff \exists a, b > 0: \forall x \in X \implies a\|x\|_1 \leq \|x\|_2 \leq b\|x\|_1$$

Theorem 1.5 (Riesz).

$X, \dim X < +\infty$ — linear set.

Then: Any pair of norms in X are equivalent.

Proof. l_1, \dots, l_n — linearly independent from X . $\forall x \in X = \sum_{k=1}^n \alpha_k l_k$

$$\bar{x} \leftrightarrow (l_1, \dots, l_n) = \bar{l} \in \mathbb{R}^n$$

Let $\|\cdot\|$ — some norm in X .

$$\|x\| \underset{\triangle}{\leq} \sum_{k=1}^n \|l_k\| |\alpha_k| \underset{\text{Cauchy}}{\leq} \underbrace{\sqrt{\sum_{k=1}^n \|l_k\|^2}}_{\text{const}(B), B\text{-basis}} \underbrace{\sqrt{\sum_{k=1}^n |\alpha_k|^2}}_{\substack{\| \\ \|\bar{\alpha}\|=\|x\|_1}}$$

$$\|x\|_1 = \sqrt{\sum_{k=1}^n \|\alpha_k\|^2}, \quad x = \sum \alpha_k l_k$$

$$\|x\| \leq b \|x\|_1$$

$$?\exists a > 0: a \|x\|_1 \leq \|x\| \implies \|\cdot\| \sim \|\cdot\|_1$$

$$\text{Let } f(\alpha_1, \dots, \alpha_n) = \left\| \sum_{k=1}^n \alpha_k l_k \right\|$$

$$\begin{aligned} |f(\bar{\alpha} + \Delta\bar{\alpha}) - f(\bar{\alpha})| &= \left\| \sum_{k=1}^n \alpha_k l_k + \sum_{k=1}^n \Delta\alpha_k l_k \right\| - \left\| \sum_{k=1}^n \alpha_k l_k \right\| \leq \\ &\left\| \sum_{k=1}^n \Delta\alpha_k l_k \right\| \leq \underbrace{\sum \|l_k\| |\Delta\alpha_k|}_{\substack{\downarrow \\ 0, \Delta\alpha_k \rightarrow 0}} \implies f \text{ is continuous on } \mathbb{R}^n \end{aligned}$$

$$S_1 = \left\{ \sum_{k=1}^n |\alpha_k|^2 = 1 \right\} \subset \mathbb{R}^m, f \text{ — continuous on } S_1, S_1 \text{ — compact, } \bar{\alpha}^* \in S_1$$

By Weierstrass theorem there exists a point $\alpha^* \in S_1$ on a sphere, in which function f achieves its minimum $\implies \forall \alpha \in S_1 \quad f(\bar{\alpha}^*) \leq f(\bar{\alpha})$

$$\text{If } f(\bar{\alpha}^*) = 0, \text{ then } \left\| \sum_{k=1}^n \alpha_k^* l_k \right\| = 0 \implies \sum_{k=1}^n \alpha_k^* l_k = 0, \quad \bar{\alpha}^* \in S_1,$$

but $l_1 \dots l_n$ are linearly independent $\rightarrow \leftarrow \implies \min_{S_1} f = m > 0$

$$\begin{aligned} \|x\| = \left\| \sum_{k=1}^n \alpha_k l_k \right\| = f(\bar{\alpha}) &= \sqrt{\sum_{k=1}^n \alpha_k^2} \cdot \left\| \sum_{\beta_k} \frac{\alpha_k}{\sqrt{\sum_{k=1}^n \alpha_k^2}} \cdot l_k \right\| \geq, \quad \bar{\beta} = (\beta_1 \dots \beta_n) \in S_1 \\ &\geq m \cdot \|x\|_1, \quad a = m \quad \blacksquare \end{aligned}$$

Corollary 1.5.1. $X = NS, Y \subset X, \dim Y < +\infty \implies Y = \text{Cl}(Y)$

Note. Functional analysis differentiates between linear subset (set of points, closed by addition and scalar multiplication) and linear subspace (closed linear subset).

$$\text{Proof. } Y = \mathcal{L}(l_1, \dots, l_n) = \left\{ \sum_{i=1}^n \alpha_i l_i \mid l_1, \dots, l_n - \text{lin. indep.} \right\}$$

$$y_m \in Y, y_m \rightarrow y \text{ in } X \implies y \in Y?$$

$$\|y_m - y\| \rightarrow 0 \implies \|y_m - y_p\| \rightarrow 0, \quad m, p \rightarrow \infty$$

$$\|y\|, y \in Y.$$

By Riesz theorem all norm in Y are equivalent.

$$y = \sum_{j=1}^n \alpha_j l_j, \|y\|_0 = \sqrt{\sum_{j=1}^n \alpha_j^2} \text{ — some norm (by linear independence).}$$

By Riesz theorem $\|y\| \sim \|y\|_0$

$$\underbrace{\|y_m - y_p\|}_{\in Y} \rightarrow 0 \implies \|y_m - y_p\|_0 \rightarrow 0$$

$$\bar{\alpha} = (\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n \quad y_m = \sum_{i=1}^n \alpha_i^{(m)} l_i$$

$$|\alpha_i^{(m)} - \alpha_i^{(l)}| \rightarrow 0 \quad \forall i = 1, \dots, n; \quad \bar{\alpha} = (\alpha_1^{(m)}, \dots, \alpha_n^{(m)}) \rightarrow \alpha^* = (\alpha_1^*, \dots, \alpha_n^*)$$

$$y^* = \sum_{i=1}^n \alpha_i^* l_i \in Y, \quad \|y_m - y\| \rightarrow 0, y = y^* \implies y \in Y \quad \blacksquare$$

Definition. If normed space is complete, then it is called **B-space** or **Banach space**.

Example 6. $C[a, b]$ — functions continuous on $[a, b]$.

Example 7. Lebesgue space, $p \geq 1, L_p(E) = \left\{ f \text{ is measurable on } E, \int_E |f|^p < +\infty \right\}$.

If X — Banach space,

$$\sum_{n=1}^{\infty} x_n = \lim_{n \rightarrow \infty} \sum_{k=1}^n x_k, \quad \sum_{n=1}^{\infty} \|x_n\| < +\infty$$

$$\|S_n - S_m\| = \left\| \sum_{k=m+1}^n x_k \right\| \leq \sum_{m+1}^n \|x_k\| \xrightarrow{n, m \rightarrow \infty} 0$$

$$\implies \|S_n - S_m\| \rightarrow 0 \implies \exists \lim_{n \rightarrow \infty} S_n, \sum_{k=1}^n x_k \text{ — converges.}$$

In Banach spaces works the theory of absolute convergence of numerical series.

Lemma 1 (Riesz's lemma about almost perpendicular). Y — eigen subspace of X — normed space. $\forall \varepsilon \in (0, 1) \exists z_\varepsilon \in X$:

$$1. \quad \|z_\varepsilon\| = 1$$

$$2. \quad \rho(z_\varepsilon, Y) > 1 - \varepsilon$$

Proof. $\exists x \notin Y \quad d = \rho(x, Y), \quad d = 0 \exists y_n \in Y: \|x - y_n\| < \frac{1}{n}, \quad n \rightarrow \infty, \quad y_n \rightarrow x$

$$Y = \text{Cl}(Y) \implies x \in Y \rightarrow \leftarrow x \notin Y, \quad d > 0$$

$$\forall \varepsilon \in (0, 1) \quad \frac{1}{1 - \varepsilon} > 1 \exists y_\varepsilon \in Y: \|x - y_\varepsilon\| < \frac{1}{1 - \varepsilon} d$$

$$z_\varepsilon = \frac{x - y_\varepsilon}{\|x - y_\varepsilon\|}, \quad \|z_\varepsilon\| = 1, \quad \forall y \in Y \quad \|z_\varepsilon - y\| = \left\| \frac{x - y_\varepsilon}{\|x - y_\varepsilon\|} \right\| = \frac{\|x - (\overbrace{y_\varepsilon + \|x - y_\varepsilon\| \cdot y}^{\in Y})\|}{\|x - y_\varepsilon\|} \geq \frac{d}{\frac{1}{1 - \varepsilon} d} > 1 - \varepsilon \quad \blacksquare$$

Corollary 1.5.2. $\dim X = +\infty, S$ — sphere in $X, r_S = 1 \{x \mid \|x\| = 1\} \implies S$ — not a compact.

Proof. $\forall x_1 \in S, Y_1 = \mathcal{L}\{x_1\}$ — finite dimensional linear set. \implies closed in $X \implies Y_1$ — subspace. $\dim X = +\infty \implies Y_1$ — eigen subspace.

Then by the Riesz lemma:

$$\exists x_2 \in S: \|x_2 - x_1\| > \frac{1}{2}$$

$$Y_2 = \mathcal{L}\{x_1, x_2\} \exists x_3 \in S: \|x_3 - x_j\| > \frac{1}{2}, \quad j = 1, 2$$

Continue by induction. Because $\dim X = +\infty$ the process will never stop.

$x_n \in S: \|x_n - x_m\| > \frac{1}{2}, n \neq m$ — obviously we cannot extract converging subsequence.
 $\implies S$ — not a compact. ■

§1.3 Inner product (unitary) spaces

Definition. X — linear space.

$\varphi: X \times X \rightarrow \mathbb{R}$

$$1. \varphi(x, x) \geq 0, \quad \varphi(x, x) = 0 \iff x = 0$$

$$2. \varphi(x, y) = \varphi(y, x)$$

$$3. \varphi(\alpha x + \beta y, z) = \alpha \varphi(x, z) + \beta \varphi(y, z)$$

φ — **inner product**.

$$\varphi(x, y) = \langle x, y \rangle$$

Definition. $(X, \langle \cdot, \cdot \rangle)$ — **inner product space**.

Example 8. $\mathbb{R}^n, \langle \bar{x}, \bar{y} \rangle = \sum_{j=1}^n x_j y_j$

Statement 1.9 (Schwarz). $\forall x, y \in X \quad |\langle x, y \rangle| \leq \sqrt{\langle x, x \rangle} \cdot \sqrt{\langle y, y \rangle}$

Proof. $\lambda \in \mathbb{R}$

$$f(\lambda) = \langle \lambda x + y, \lambda x + y \rangle \geq 0$$

||

$$\lambda^2 \langle x, x \rangle + 2\lambda \langle x, y \rangle + \langle y, y \rangle$$

$$D = 4\langle x, y \rangle^2 - 4\langle x, x \rangle \cdot \langle y, y \rangle \leq 0 \quad \blacksquare$$

Corollary 1.5.3 (Cauchy inequality for sums). Consider $X = \mathbb{R}^n, \|x\| \stackrel{\text{def}}{=} \sqrt{\langle x, x \rangle}$. Then

$$\|x + y\|^2 = \langle x + y, x + y \rangle = \|x\|^2 + 2 \cdot \underbrace{\langle x, y \rangle}_{\leq \|x\| \cdot \|y\|} + \|y\|^2 \leq (\|x\| + \|y\|)^2$$

Any inner product space is a special case of a normed space. The specifics is that we can measure the angles between points:

$$x \perp y \iff \langle x, y \rangle = 0$$

In this case the Pythagorean theorem takes place:

$$\|x + y\|^2 = \|x\|^2 + \|y\|^2$$

In inner product spaces the parallelogram law plays a significant role:

$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2 \quad \forall x, y \in X$$

In an inner product space norm is determined by inner product: $\|x\|^2 = \langle x, x \rangle$

It can be proved that if parallelogram law holds, then the norm must be determined by some inner product. Let X be some normed space, $x \in X$, then $\langle \cdot, \cdot \rangle \mapsto \|x\| = \sqrt{\langle x, x \rangle}$. For any norm satisfying the parallelogram law, the inner product generating the norm is unique.

Example 9. $C_{[a,b]}$, $\|f\| = \max_{x \in [a,b]} |f(x)|$, $\|f\|$ doesn't satisfy the parallelogram law and thus is not determined by any inner product. This fact implies that $C_{[a,b]}$ is not an inner product space.

Definition. Orthonormal set — a set of points $\{l_1, l_2, \dots\}$ (may be finite):

1. $\|l_i\| = 1$
2. $l_i \perp l_j, \quad i \neq j$

Every orthonormal set is linearly independent.

Definition. Let $x \in X, \{l_i\}$ — ONS. Then

$\langle x, l_j \rangle$ — **Fourier coefficient**,
 $\sum_j \langle x, l_j \rangle l_j$ — **Fourier series** of point x .

Fourier series is a special case of orthogonal series.

Definition. $\sum_j x_j$ — orthogonal series $\iff x_i \perp x_j, \quad i \neq j$

Let $\sum_{j=1}^{\infty} x_j, S_m = \sum_{j=1}^m x_j$. Then

$$\|S_m\|^2 = \left\langle \sum_{j=1}^m x_j, \sum_{j=1}^m x_j \right\rangle = \sum_{j=1}^m \|x_j\|^2$$

This fact allows us to effectively build the theory of orthogonal series.

An important problem is concerned with Fourier series. Let X is a normed space, Y is a subspace of X ,

$$\forall x \in X \quad E_Y(x) = \rho(x, Y) = \inf_{y \in Y} \|x - y\|$$

Definition. $E_Y(x)$ — **best approximation** of point x with points of the subspace Y , if $\exists y^* \in Y \quad E_Y(x) = \|x - y^*\|$ — then y^* is an element of best approximation.

Theorem 1.6 (Borel).

$\dim Y < +\infty \implies \forall x \in X \quad \exists y^* \in Y$ — *element of best approximation*.

Proof. $Y = \mathcal{L}(\underbrace{l_1, l_2, \dots, l_n}_{\text{lin. indep.}})$

Consider $f(\alpha_1, \dots, \alpha_n) = \|x - \sum_{k=1}^n \alpha_k l_k\| \rightarrow \min$. By the triangle inequality for norm, $f(\bar{\alpha})$ is continuous on $\mathbb{R}^n, f \geq 0, E_Y(x) = \inf f(\bar{\alpha})$. It is easy to find that there always is a ball $B(0, r) \subset \mathbb{R}^n$, outside of which $f > 2E_Y(x)$. So, $E_Y(x)$ is somewhere inside. But f is continuous, the ball is compact, so, by the Weierstrass theorem, the minimum exists and it is located on the sphere $S(0, r)$. ■

For abstract Fourier series the Borel theorem can be significantly strengthened by specifying the best approximation element.

Theorem 1.7 (extreme quality of Fourier series' partial sums).

$\{e_j\}$ — ONS in X

$H_n = \mathcal{L}(l_1, \dots, l_n)$

$$E_{H_n}(x), S_n(x) = \sum_{j=1}^n \langle x, l_j \rangle l_j \implies E_{H_n}(x) = \|x - S_n(x)\|$$

Proof. $y = \sum_{j=1}^n \alpha_j l_j \in H_n$

$$\begin{aligned} \|x - y\|^2 &= \left\langle x - \sum \alpha_j l_j, x - \sum \alpha_j l_j \right\rangle = \|x\|^2 - 2 \sum \alpha_j \langle x, l_j \rangle + \sum \alpha_j^2 = \\ &= \underbrace{\|x\|^2}_{\text{const}} + \sum (\alpha_j - \langle x, l_j \rangle)^2 - \underbrace{\sum \langle x, l_j \rangle^2}_{\text{const}} \rightarrow \min \end{aligned}$$

So, the sum goes to minimum when the second summand is minimal. Obviously, it's minimal when $\forall (\alpha_j - \langle x, l_j \rangle) = 0$. $E_Y(x)$ — Fourier sum. ■

Corollary 1.7.1 (Bessel's inequality). $\sum_j \langle x, l_j \rangle^2 \leq \|x\|^2$

Proof. $0 \leq \|x - y^*\|^2 = \|x\|^2 - \sum_j \langle x, l_j \rangle^2$ ■

Corollary 1.7.2. *The series of Fourier coefficients' squares always converges*

§1.4 Hilbert space.

Definition. Hilbert space — complete, infinite dimensional, inner product space.

Example 10. $L_2(E)$ — Hilbert space.

$$\langle f, g \rangle = \int_E f \cdot g \, d\mu$$

$$l_2 = \left\{ (x_1, \dots, x_n, \dots) \mid \sum_{n=1}^{\infty} x_n^2 < +\infty \right\}$$

$$\langle \bar{x}, \bar{y} \rangle = \sum_{n=1}^{\infty} x_n y_n, \quad E = \mathbb{N}, \quad \mu\{m\} = 1$$

When we have completeness we can define orthonormal basis.

Definition. $H, \{l_n\}$ — ONS : $\forall x = \sum_{n=1}^{\infty} \alpha_n l_n \implies \langle x, l_m \rangle = \sum_{n=1}^{\infty} \alpha_n \langle l_n, l_m \rangle = \alpha_m$

In this sense basis decomposition is always a Fourier series.

1. Complete ONS: $H = \text{Cl}(\mathcal{L}\{l_1, l_2, \dots\})$

2. Closed ONS: $\forall m \langle x, l_m \rangle = 0 \implies x = 0$.

Statement 1.10. In Hilbert spaces two of the properties outlined above are equivalent.

$\sum_{n=1}^{\infty} y_n$ in H — orthogonal series.

$$\|S_n - S_m\|^2 = \sum_n^m \|y_n\|^2$$

$$\|S_n - S_m\| \rightarrow 0 \iff \sum_n^m \|y_k\|^2 \rightarrow 0$$

$$\text{in } H \sum_1^{\infty} \|y_k\|^2 < +\infty$$

$$\text{ONS } x \mapsto \sum_{k=1}^{\infty} \langle x, l_k \rangle l_k$$