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§1.1 Metric spaces.

X, ρ: $X \times X \to \mathbb{R}_+$

Definition. ρ — **metric**

1. $\rho(x,y) \geqslant 0, = 0 \iff x = y$
2. $\rho(x,y) = \rho(y,x)$
3. $\rho(x,y) \leqslant \rho(x,z) + \rho(y,z)$

Definition. (X,ρ) — **Metric space**.

Definition. $x = \lim x_n \iff \rho(x_n, x) \to 0$

$X, \tau = \{G \subset X\}$

1. $\varnothing, X \in \tau$
2. $G_\alpha \in \tau, \alpha \in \mathscr{A} \implies \bigcup\limits_{\alpha} G_\alpha \in \tau$
3. $G_1, \ldots, G_n \in \tau \implies \bigcap\limits_{j=1}^{n} G_j \in \tau$

Definition. (X,τ) — **Topological space**.

$x = \lim x_n \quad \forall G \in \tau : x \in G \quad \exists N : \forall n > N \implies x_n \in G$
G — open in τ
$F = X \setminus G$ — closed

Definition. $B_r(a) = \{\, x \mid \rho(x,a) < r \,\}$ — **open ball**

$\tau = \bigcup B_r(a)$

Statement 1.1. $b \in B_{r_1}(a_1) \cap B_{r_2}(a_2) \implies \exists r_3 > 0 : B_{r_3}(a_3) \subset B_{r_1}(a_1) \cap B_{r_2}(a_2)$

In this sense metric space is just a special case of topological space.

Example 1. $\mathbb{R}, \rho(x,y) = |x-y|$, MS

Example 2. $\bar{x} = (x_1, \dots, x_n) \in \mathbb{R}^n, \rho(\bar{x}, \bar{y}) = \sqrt{\sum_{j=1}^{n} (x_j - y_j)^2}, MS$

Example 3.

$$
\begin{aligned}
&\bar{x} = (x_1, \dots, x_n, \dots) \in \mathbb{R}^\infty \\
&\alpha\bar{x} = (\alpha x_1, \dots, \alpha x_n, \dots) \\
&\bar{x} + \bar{y} = (x_1 + y_1, \dots, x_n + y_n, \dots) \\
&\lim_{m\to\infty} \bar{x}_m ? \\
&\text{in } \mathbb{R}^n \quad \bar{x}_n \to \bar{x} \iff \forall j = 1, \dots, n \qquad x_j^{(m)} \xrightarrow[m\to\infty]{} x_j \\
&\text{in } \mathbb{R}^\infty \quad \bar{x}_m \to \bar{x} \overset{def}{\iff} \forall j = 1, 2, 3, \dots \quad x_j^{(m)} \xrightarrow[m\to\infty]{} x_j
\end{aligned}
$$

Definition. $\rho(\bar{x}, \bar{y}) \overset{\text{def}}{=} \sum_{n=1}^{\infty} \frac{1}{2^n} \underbrace{\frac{|x_n - y_n|}{1 + |x_n - y_N|}}_{\varphi(|x_n - y_n|)}$ — **Urysohn metric**.

$\varphi(t) = \frac{t}{1+t}$
$\varphi(t_1 + t_2) \leqslant \varphi(t_1) + \varphi(t_2)$
$\rho(\bar{x}_m, \bar{x}) \xrightarrow{m\to\infty} 0 \iff x_j^{(m)} \to x_j \ \forall j$
In this way $\mathbb{R}^\infty$ is a metrizable space.

Example 4. $X, \rho(x,y) \overset{\text{def}}{=} \begin{cases} 0, x = y \\ 1, x \neq y \end{cases}$ — **Discrete metric**.

$x_n \to x,\ \varepsilon = \frac{1}{2},\ \exists M :\ m > M \implies \rho(x_m, x) < \frac{1}{2} \implies$
$\rho(x_m, x) = 0 \implies x_m = x$

Definition. $(X, \tau);\ \forall A \subset X;$

$$
\begin{aligned}
&\mathrm{Int}(A) \overset{\text{def}}{=} \bigcup_{G \subset A} G - \text{open}; \\
&\mathrm{Cl}(A) \overset{\text{def}}{=} \bigcap_{A \subset G} G - \text{closed}; \\
&\mathrm{Fr}(A) = \mathrm{Cl}(A) \setminus \mathrm{Int}(A)
\end{aligned}
$$

(X, ρ); Having a metric space one can describe closure of a set.
$\rho(x, A) \overset{\text{def}}{=} \inf_{a \in A} \rho(x, a)$

$\rho(A,B) = \inf\limits_{\substack{a\in A\\ b\in B}} \rho(a,b)$
$\rho(x,A) = f(x), x \in X$

Statement 1.2. *Function f(x) is continuous.*

Proof. $\forall x, y \in X$
$f(x) = \rho(x,A) \underset{\forall \alpha\in A}{\leqslant} \rho(x,\alpha) \leqslant \rho(x,y) + \rho(y,\alpha)$
$\forall \varepsilon > 0\ \exists \alpha_\varepsilon \in A:\ \rho(y,\alpha_\varepsilon) < \rho(y,A) + \varepsilon = f(y) + \varepsilon$
$f(x) \leqslant f(y) + \varepsilon + \rho(x,y),\ \varepsilon \to 0$
$$\begin{cases} f(x) \leqslant f(y) + \rho(x,y) \\ f(y) \leqslant f(x) + \rho(x,y) \end{cases} \implies |f(x) - f(y)| \leqslant \rho(x,y) \quad \blacksquare$$

Statement 1.3. $x \in \mathrm{Cl}(A) \iff \rho(x,A) = 0$

Let's look at the metric spaces in terms of separation of sets from each other by open sets.
x, y
$r = \rho(x,y) > 0$
$B_{\frac{r}{3}}(x),\ B_{\frac{r}{3}}(y)$
In any metric space separability axiom is true.

Theorem 1.1.
Any metric space is a normal space,
i.e. $\forall$ *closed disjoint* $F_1, F_2 \in X,\ \exists$ *open disjoint* $G_1, G_2\colon F_j \in G_j,\ j = 1, 2$

Proof. $g(x) = \dfrac{\rho(x,F_1)}{\rho(x,F_1) + \rho(x,F_2)}$ — continuous on X
$x \in F_1,\ \mathrm{Cl}(F_1) = F_1,\ \rho(x,F_1) = 0,\ g(x) = 0$
$x \in F_2,\ g(x) = 1$
Let's look at $(-\infty; \frac{1}{3}), (\frac{2}{3}, \infty)$ — by continuity their inverse images under g are open.
$G_1 = g^{-1}(-\infty; \frac{1}{3})$
$G_2 = g^{-1}(\frac{2}{3}; \infty) \quad \blacksquare$

Definition. Metric space is **complete** if $\rho(x_n, x_m) \to 0 \implies \exists x = \lim x_n$
$\mathbb{R}^\infty$ – complete (by completeness of the rational numbers).
In complete metric spaces the nested balls principle is true.

Theorem 1.2.
X – complete metric space, $\overline{V}_{r_n}$ – system of closed balls.

1. *$\overline{V}_{r_{n+1}} \subset \overline{V}_{r_n}$ – the system is nested.*
2. *$r_n \to 0$*

Then: $\bigcap\limits_n \overline{V}_{r_n} = \{a\}$

Proof. Let b_n be centers of $\overline{V}_{r_n}$,
$m \geqslant n,\ b_m \in \overline{V}_{r_n},\ \rho(b_m, b_n) \leqslant r_n \to 0\ \forall m \geqslant n$
$\rho(b_m, b_n) \to 0 \xRightarrow{compl.} \exists a = \lim b_n$ Since the balls are closed a $\in$ every ball.
$r_n \to 0 \implies$ there is only one common point ■.

(X, τ) — topological space
$A \subset X,\ \tau_a = \{\, G \cap A, G \in \tau \,\}$ — topology induced on A

Definition. X— metric space, $A \subset X,\ \mathrm{Cl}(A) = X$
Then: A – **dense** in X
If $\mathrm{Int}(\mathrm{Cl}(A)) = \varnothing$ A – **nowhere dense** in X.

Note. It is easy to understand, that in metric spaces nowhere density means the following:
$\forall$ ball $V\ \exists V' \subset V : V'$ contains no points from A.

Definition. X is called **first Baire category set**, if it can be written as at most countable union of x_n each nowhere dense in X.

Theorem 1.3 (Baire category theorem)**.**
Complete metric space is second Baire category set in itself.

Proof. Let X be first Baire category set.
$X = \bigcup\limits_n X_n \quad \forall \overline{V}\ X_1$ is nowhere dense.
$\overline{V}_1 \subset \overline{V}:\ \overline{V}_1 \cap X_1 = \varnothing$
X_2 is nowhere dense $\overline{V}_2 \subset \overline{V}_1 : \overline{V}_2 \cap X_2 = \varnothing$
$r_2 \leqslant \dfrac{r_1}{2}$
$\vdots$
$\{\overline{V}_n\},\ r_n \to 0,\ \bigcap\limits_n \overline{V}_n = \{a\},\ X = \bigcup X_n,\ \exists n_0 : a \in X_{n_0}$
$X_{n_0} \cap \overline{V}_{n_0} = \varnothing \to\leftarrow\ a \in \overline{V}_{n_0}$ ■

Corollary 1.3.1. *Complete metric space without isolated points is uncountable.*

Proof. No isolated points are present $\implies$ every point in the set is nowhere dense in it. Let X be countable: $X = \bigcup\limits_n \{X_n\}$, then it is first Baire category set. $\rightarrow\leftarrow$ ■

Definition. K — **compact** if

1. $K = \mathrm{Cl}(K)$
2. $x_n \in K\ \exists n_1 < n_2 < \dots\ x_{n_j}$ – converges in X.

If only 2 is present, the set is called **precompact**.

Theorem 1.4 (Hausdorff)**.**
Let X — metric space, K — closed in X.
<u>Then:</u> K — compact $\iff$ K — totally bounded,
i.e. $\forall \varepsilon > 0\ \exists a_1, \dots, a_p \in X\colon\ \forall b \in K\ \exists a_j\colon\ \rho(a_j, b) < \varepsilon$
$(a_1, \dots, a_p - finite\ \varepsilon - net)$

Proof.
$\implies$
K— totally bounded, $x_n \in K\ n_1 < n_2 < \dots < n_k < \dots$
x_n – converges in K
$\varepsilon_k \downarrow\to 0\ \varepsilon_1 \quad K \subset \bigcup\limits_{j=1}^{p} \overline{V}_j,\ rad = \varepsilon_1\ (\varepsilon_1 - net)$
n is finite $\implies$ one ball will contain infinetely many x_n elements.
Let's look at $\overline{V}_{j_0} \cap K$ — totally bounded $= K_1,\ \mathrm{diam}(K_1) \leqslant 2\varepsilon_1$
$\varepsilon_2 \quad K_1 \subset \bigcup\limits_{j=1}^{n} \overline{V'}_j,\ rad = \varepsilon_2.$
Then one of $\overline{V'}$ contains infinitely many elements of the sequence contained in K_1
$\overline{V'}_{j_0} \cap K_1 = K_2,\ \mathrm{diam}(K_2) \leqslant 2\varepsilon_2$ and so on.

$$K_n \supset K_{n+1} \supset K_{n+2} \supset \dots,\ \mathrm{diam}(K_N) \leqslant 2\varepsilon_n \xRightarrow{by\ compl.} \underbrace{\bigcap_{n=1}^{\infty} K_n}_{\mathrm{diam}(K_n)\to 0,\ \{x\}} \neq \varnothing$$

Take x_{n_1} from K_1, x_{n_2} from $K_2 \dots$
$\impliedby$
K — compact $\forall \varepsilon\ \exists$ finite ε — net?

By contradiction: $\exists \varepsilon_0 > 0\colon$ finite ε_0-net is impossible to construct.
$\forall x_1 \in K\ \exists x_2 \in K\colon\ \rho(x_1, x_2) > \varepsilon_0$ (or else system of x_1 — finite ε-net)
$\{x_1, x_2\}$ - choose $x_3 \in K\colon \rho(x_3, x_i) > \varepsilon_0,\ i = 1, 2$ and so on.
$x_n \in K\ :\ n \neq m\ \rho(x_n, x_m) > \varepsilon_0$ — contains no converging subsequence $\implies$ set is not a compact. $\rightarrow\leftarrow$ ■

§1.2 Normed spaces

Definition. X — **linear set**, $x + y, \alpha \cdot x$, $\alpha \in \mathbb{R}$
The purpose of norm definition, is to construct a topology on X, so that 2 linear operations are continuous on it.

$\varphi\colon X \to \mathbb{R}\colon$

1. $\varphi(x) \geqslant 0,\ = 0 \iff x = 0$
2. $\varphi(\alpha x) = |\alpha|\varphi(x)$
3. $\varphi(x + y) \leqslant \varphi(x) + \varphi(y)$

Definition. φ — **norm** on X, $\varphi(x) = \|x\|$

$\rho(x, y) \overset{\text{def}}{=} \|x - y\|$ — metric on X.

Definition.
$(X, \|\cdot\|)$ — **normed space** — special case of metrical space.

$x = \lim x_n \overset{def}{\Longrightarrow} \rho(x_n, x) \to 0 \iff \|x_n - x\| \to 0$

Statement 1.4. *In the topology of a normed space linear operations are continuous on X.*

Proof. 1. $x_n \to x,\ y_n \to y;\ \|(x_n + y_n) - (x + y)\| = \|(x_n - x) + (y_n - y)\| \leqslant$

$$\leqslant \underbrace{\|x_n - x\|}_{\downarrow \atop 0} + \underbrace{\|y_n - y\|}_{\downarrow \atop 0}$$

$$\implies x_n + y_n \to x + y$$

2. $\alpha_n \to \alpha,\ x_n \to x;\ \|\alpha_n x_n - \alpha x\| = \|(\alpha_n - \alpha)x_n + \alpha(x_n - x)\| \leqslant$
$$\leqslant \underbrace{|\alpha_n - \alpha|}_{\substack{\downarrow \\ 0}} \cdot \underbrace{\|x_n\|}_{bounded} + \alpha\underbrace{\|x_n - x\|}_{\substack{\downarrow \\ 0}}$$
$x_n \to x \implies \|x_n\|$ — bounded.
$\alpha_x x_n \to \alpha x \quad \blacksquare$

Statement 1.5. *From the triangle inequality* $|\|x\| - \|y\|| \leqslant \|x - y\|$
$x_n \to x \implies \|x_n\| \to \|x\|$
Norm is continious.

Example 5. $\mathbb{R}^n$

1. $\|\bar{x}\| = \sqrt{\sum\limits_{k=1}^{n} x_k^2}$
2. $\|\bar{x}\|_1 \overset{\text{def}}{=} \sum\limits_{k=1}^{n} |x_k|$
3. $\|\bar{x}\|_2 \overset{\text{def}}{=} \max\{|x_1|, \ldots, |x_n|\}$
4. $C[a, b]$ — functions continuous on $[a, b]$; $\|f\| = \max\limits_{x \in [a,b]} |f(x)|$
5. $L_p(E) = \left\{ f - \text{measurable}, \int\limits_E \middle| f|^p < +\infty \right\}$
   $p \geqslant 1,\ \|f\|_p = \left(\int\limits_E |f|^p \right)^{\frac{1}{p}}$

Because the set of points is the same, arises the question about convergence comparison.
$\|\cdot\|_1 \sim \|\cdot\|_2,\ x_n \overset{\|\cdot\|_1}{\to} x \iff x_n \overset{\|\cdot\|_2}{\to} x$

Statement 1.6.
$\|\cdot\|_1 \sim \|\cdot\|_2 \iff \exists a, b > 0 \colon \forall x \in X \implies a\|x_1\|_1 \leqslant \|x\|_2 \leqslant b\|x\|_1$

Theorem 1.5 (Riesz)**.**
$X,\ \dim X < +\infty$ — *linear set.*
<u>*Then:*</u> *Any pair of norms in* X *are equivalent.*

Proof. $l_1, \ldots, l_n$ — linearly independent from X. $\forall x \in X = \sum\limits_{k=1}^{n} \alpha_k l_K$

$\bar{x} \leftrightarrow (l_1, \ldots, l_n) = \bar{l} \in \mathbb{R}^n$

Let $\|\cdot\|$ — some norm in X.

$$\|x\| \underset{\triangle}{\leqslant} \sum_{k=1}^{n} \|l_k\| |\alpha_k| \underset{\text{Cauchy}}{\leqslant} \underbrace{\sqrt{\sum_{k=1}^{n} \|l_k\|^2}}_{const(B), B-basis} \underset{\|\bar{\alpha}\| = \|x\|_1}{\underset{\|}{\sqrt{\sum_{k=1}^{n} |\alpha_k|^2}}} \text{ Let } f(\alpha_1, \ldots, \alpha_n) = \left\| \sum_{k=1}^{n} \alpha_k l_k \right\|$$

$$\|x\|_1 = \sqrt{\sum_{k=1}^{n} \|\alpha_k\|^2}, \; x = \sum \alpha_k l_k$$

$\|x\| \leqslant b\|x\|_1$

$?\exists a > 0$: $a\|x\|_1 \leqslant \|x\| \implies \|\cdot\| \sim \|\cdot\|_1$

$$|f(\bar{\alpha} + \Delta\bar{\alpha}) - f(\bar{\alpha})| = \left| \left\| \sum_{k=1}^{n} \alpha_k l_k + \sum_{k=1}^{n} \Delta\alpha_k l_K \right\| - \left\| \sum_{k=1}^{n} \alpha_k l_k \right\| \right| \leqslant$$
$$\left\| \sum_{k=1}^{n} \Delta\alpha_k l_k \right\| \leqslant \underbrace{\sum \|l_k\| |\Delta\alpha_k|}_{\substack{\downarrow \\ 0, \Delta\alpha_k \to 0}} \implies f\text{— continuous on } \mathbb{R}^n$$

$S_1 = \{\sum\limits_{k=1}^{n} |\alpha_k|^2 = 1\} \subset \mathbb{R}^m$, f — continuous on S_1, S_1 — compact, $\bar{\alpha}^* \in S_1$

By Weierstrass theorem there exists a point $\alpha^* \in S_1$ on a sphere, in which function f achieves its minimum $\implies \forall \alpha \in S_1 \; f(\bar{\alpha}^*) \leqslant f(\bar{\alpha})$

If $f(\bar{\alpha}^*) = 0$, then $\left\| \sum\limits_{k=1}^{n} \alpha_k^* l_k \right\| = 0 \implies \sum\limits_{k=1}^{n} \alpha_k^* l_k = 0, \; \bar{\alpha}^* \in S_1$,

but $l_1 \ldots l_n$ are linearly independent $\rightarrow\leftarrow \implies \min\limits_{S_1} f = m > 0$

$$\|x\| = \left\| \sum_{k=1}^{n} \alpha_k l_k \right\| = f(\bar{\alpha}) = \sqrt{\sum_{k=1}^{n} \alpha_k^2} \cdot \left\| \sum \underbrace{\boxed{\frac{\alpha_k}{\sqrt{\sum_{k=1}^{n} \alpha_k^2}}}}_{\beta_k} \cdot l_k \right\| \geqslant, \; \bar{\beta} = (\beta_1 \ldots \beta_n) \in S_1$$
$$\geqslant m \cdot \|x\|_1, \; a = m \quad \blacksquare$$

Corollary 1.5.1. *X — NS, $Y \subset X, \dim Y < +\infty \implies Y = \mathrm{Cl}(Y)$*

Note. Functional analysis differentiates between linear subset (set of points, closed by addition and scalar multiplication) and linear subspace (closed linear subset).

Proof. $Y = \alpha(l_1, \ldots, l_n) = \left\{ \sum\limits_{i=1}^{n} \alpha_i l_i \;\middle|\; l_1, \ldots, l_n - \text{lin. indep.} \right\}$
$y_m \in Y, y_m \to y$ in $X \implies y \in Y$?
$\|y_m - y\| \to 0 \implies \|y_m - y_p\| \to 0,\ m, p \to \infty$
$\|y\|, y \in Y.$
By Riesz theorem all norm in Y are equivalent.
$y = \sum\limits_{j=1}^{n} \alpha_j l_J, \|y\|_0 = \sqrt{\sum\limits_{j=1}^{n} \alpha_j^2}$ — some norm (by linear independance).
By Riesz theorem $\|y\| \sim \|y\|_0$
$\underbrace{\|y_m - y_p\|}_{\in Y} \to 0 \implies \|y_m - y_p\|_0 \to 0$
$\bar{\alpha} = (\alpha_1, \ldots, l_n) \in \mathbb{R}^n\ y_m = \sum\limits_{i=1}^{n} \alpha_i^{(m)} l_i$
$|\alpha_i^{(m)} - \alpha_i^{(l)}| \to 0\ \forall i = 1, \ldots, n;\ \bar{\alpha} = (\alpha_1^{(m)}, \ldots, \alpha_n^{(m)}) \to \alpha^* = (\alpha_1^*, \ldots, \alpha_n^*)$
$y^* = \sum\limits_{i=1}^{n} \alpha_i^* l_I \in Y,\ \|y_m - y\| \to 0, y = y^* \implies y \in Y$ ■

Definition. If normed space if complete, then it is called **B-space** or **Banach space**.

Example 6. $C[a, b]$ — functions continuous on $[a, b]$.

Example 7. Lebesgue space, $p \geqslant 1, L_p(E) = \left\{ f - \text{measurable on E},\ \int\limits_E |f|^P < +\infty \right\}$.

If X — Banach space,

$$\sum_{n=1}^{\infty} x_n = \lim_{n\to\infty} \sum_{k=1}^{n} x_k,\ \sum_{n=1}^{\infty} \|x_n\| < +\infty$$

$$\|S_n - S_m\| = \left\| \sum_{k=m+1}^{n} x_k \right\| \leqslant \sum_{m+1}^{n} \|x_k\| \xrightarrow[n,m\to\infty]{} 0$$

$$\implies \|S_n - S_m\| \to 0 \implies \exists \lim_{n\to\infty} S_n, \sum_{k=1}^{n} x_k \text{— converges.}$$

In Banach spaces works the theory of absolute convergence of numerical series.

Lemma 1 (Riesz).