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# 1 Vector spaces.

## §1.1 Metric spaces.

X, $\rho$: $X \times X \to \mathbb{R}_+$

**Definition.** $\rho$ — **metric**

1. $\rho(x,y) \geqslant 0, = 0 \iff x = y$
2. $\rho(x,y) = \rho(y,x)$
3. $\rho(x,y) \leqslant \rho(x,z) + \rho(y,z)$

**Definition.** $(X,\rho)$ — **Metric space**.

**Definition.** $x = \lim x_n \iff \rho(x_n, x) \to 0$

$X, \tau = \{G \subset X\}$

1. $\varnothing, X \in \tau$
2. $G_\alpha \in \tau, \alpha \in \mathscr{A} \implies \bigcup\limits_\alpha G_\alpha \in \tau$
3. $G_1, \dots, G_n \in \tau \implies \bigcap\limits_{j=1}^n G_j \in \tau$

**Definition.** $(X,\tau)$ — **Topological space**.

$x = \lim x_n \quad \forall G \in \tau : x \in G \quad \exists N : \forall n > N \implies x_n \in G$
G — open in $\tau$
$F = X \setminus G$ — closed

**Definition.** $B_r(a) = \{x \mid \rho(x,a) < r\}$ — **open ball**

$\tau = \bigcup B_r(a)$

**Statement 1.1.** $b \in B_{r_1}(a_1) \cap B_{r_2}(a_2) \implies \exists r_3 > 0 : B_{r_3}(a_3) \subset B_{r_1}(a_1) \cap B_{r_2}(a_2)$

In this sense metric space is just a special case of topological space.

**Example 1.** $\mathbb{R}, \rho(x,y) = |x-y|$, MS

**Example 2.** $\bar{x} = (x_1, \dots, x_n) \in \mathbb{R}^n, \rho(\bar{x}, \bar{y}) = \sqrt{\sum\limits_{j=1}^{n} (x_j - y_j)^2}, MS$

**Example 3.** $\bar{x} = (x_1, \dots, x_n, \dots) \in \mathbb{R}^\infty$

$$\alpha\bar{x} = (\alpha x_1, \dots, \alpha x_n, \dots)$$

$$\bar{x} + \bar{y} = (x_1 + y_1, \dots, x_n + y_n, \dots)$$

$$\lim_{m\to\infty} \bar{x}_m?$$

$$\text{in } \mathbb{R}^n \ \bar{x}_n \to \bar{x} \iff \forall j = 1, \dots, n \quad x_j^{(m)} \xrightarrow{m\to\infty} x_j$$

$$\text{in } \mathbb{R}^\infty \ \bar{x}_m \to \bar{x} \overset{def}{\iff} \forall j = 1, 2, 3, \dots \quad x_j^{(m)} \xrightarrow{m\to\infty} x_j$$

**Definition.** $\rho(\bar{x}, \bar{y}) \overset{\text{def}}{=} \sum\limits_{n=1}^{\infty} \frac{1}{2^n} \underbrace{\frac{|x_n - y_n|}{1 + |x_n - y_N|}}_{\varphi(|x_n - y_n|)}$ — **Urysohn metric**.

$\varphi(t) = \frac{t}{1+t}$
$\varphi(t_1 + t_2) \leqslant \varphi(t_1) + \varphi(t_2)$
$\rho(\bar{x}_m, \bar{x}) \xrightarrow{m\to\infty} 0 \iff x_j^{(m)} \to x_j \ \forall j$
In this way $\mathbb{R}^\infty$ is a metrizable space.

**Example 4.** $X, \rho(x,y) \overset{\text{def}}{=} \begin{cases} 0, x = y \\ 1, x \neq y \end{cases}$ — **Discrete metric**.

$x_n \to x, \ \varepsilon = \frac{1}{2}, \ \exists M : \ m > M \implies \rho(x_m, x) < \frac{1}{2} \implies$
$\rho(x_m, x) = 0 \implies x_m = x$

**Definition.** $(X, \tau); \ \forall A \subset X;$

$$\operatorname{Int}(A) \overset{\text{def}}{=} \bigcup_{G \subset A} G - \text{open};$$

$$\operatorname{Cl}(A) \overset{\text{def}}{=} \bigcap_{A \subset G} G - \text{closed};$$

$$\operatorname{Fr}(A) = \operatorname{Cl}(A) \setminus \operatorname{Int}(A)$$

$(X, \rho)$; Having a metric space one can describe closure of a set.
$\rho(x, A) \overset{\text{def}}{=} \inf\limits_{a \in A} \rho(x, a)$

$\rho(A, B) = \inf\limits_{\substack{a \in A \\ b \in B}} \rho(a, b)$
$\rho(x, A) = f(x), x \in X$

**Statement 1.2.** *Function f(x) is continuous.*

*Proof.* $\forall x, y \in X$
$f(x) = \rho(x, A) \underset{\forall \alpha \in A}{\leqslant} \rho(x, \alpha) \leqslant \rho(x, y) + \rho(y, \alpha)$
$\forall \varepsilon > 0\ \exists \alpha_\varepsilon \in A:\ \rho(y, \alpha_\varepsilon) < \rho(y, A) + \varepsilon = f(y) + \varepsilon$
$f(x) \leqslant f(y) + \varepsilon + \rho(x, y),\ \varepsilon \to 0$
$$\begin{cases} f(x) \leqslant f(y) + \rho(x, y) \\ f(y) \leqslant f(x) + \rho(x, y) \end{cases} \implies |f(x) - f(y)| \leqslant \rho(x, y) \quad \blacksquare$$

**Statement 1.3.** $x \in \mathrm{Cl}(A) \iff \rho(x, A) = 0$

Let's look at the metric spaces in terms of separation of sets from each other by open sets.
$x, y$
$r = \rho(x, y) > 0$
$B_{\frac{r}{3}}(x),\ B_{\frac{r}{3}}(y)$
In any metric space separability axiom is true.

**Theorem 1.1.**
*Any metric space is a normal space,*
*i.e.* $\forall$ *closed disjoint* $F_1, F_2 \in X,\ \exists$ *open disjoint* $G_1, G_2\colon F_j \in G_j,\ j = 1, 2$

*Proof.* $g(x) = \frac{\rho(x, F_1)}{\rho(x, F_1) + \rho(x, F_2)}$ — continuous on X
$x \in F_1,\ \mathrm{Cl}(F_1) = F_1,\ \rho(x, F_1) = 0,\ g(x) = 0$
$x \in F_2,\ g(x) = 1$
Let's look at $(-\infty; \frac{1}{3}), (\frac{2}{3}, \infty)$ — by continuity their inverse images under g are open.
$G_1 = g^{-1}(-\infty; \frac{1}{3})$
$G_2 = g^{-1}(\frac{2}{3}; \infty) \quad \blacksquare$

**Definition.** Metric space is **complete** if $\rho(x_n, x_m) \to 0 \implies \exists x = \lim x_n$
$\mathbb{R}^\infty$ – complete (by completeness of the rational numbers).
In complete metric spaces the nested balls principle is true.

**Theorem 1.2.**
*X – complete metric space,* $\overline{V}_{r_n}$ *– system of closed balls.*

1. $\overline{V}_{r_{n+1}} \subset \overline{V}_{r_n}$ – *the system is nested.*

2. $r_n \to 0$

<u>Then:</u> $\bigcap\limits_n \overline{V}_{r_n} = \{a\}$

*Proof.* Let $b_n$ be centers of $\overline{V}_{r_n}$,
$m \geqslant n,\ b_m \in \overline{V}_{r_n},\ \rho(b_m, b_n) \leqslant r_n \to 0\ \forall m \geqslant n$
$\rho(b_m, b_n) \to 0 \xrightarrow{compl.} \exists a = \lim b_n$ Since the balls are closed a $\in$ every ball.
$r_n \to 0 \implies$ there is only one common point ■.

$(X, \tau)$ — topological space
$A \subset X,\ \tau_a = \{G \cap A, G \in \tau\}$

**Definition.** $X$— metric space, $A \subset X,\ \mathrm{Cl}(A) = X$
<u>Then:</u> A – **dense** in X
<u>If</u> $\mathrm{Int}(\mathrm{Cl}(A)) = \varnothing$ A – **nowhere dense** in X.
It is easy to understand, that in metric spaces nowhere density means the following:
$\forall$ *ball* $V\ \exists V' \subset V : V'$ contains no points from A.

**Definition.** X is called **first Baire category set**, if it can be written as at most countable union of $x_n$ each nowhere dense in X.

**Theorem 1.3** (Baire category theorem)**.**
*Complete metric space is second Baire category set in itself.*

*Proof.* Let X be first Baire category set.
$X = \bigcup\limits_n X_n \quad \forall \overline{V}\ X_1$ is nowhere dense.
$\overline{V}_1 \subset \overline{V}:\ \overline{V}_1 \cap X_1\ = \varnothing$
$X_2$ is nowhere dense $\overline{V}_2 \subset \overline{V}_1 : \overline{V}_2 \cap X_2 = \varnothing$
$r_2 \leqslant \frac{r_1}{2}$
$\vdots$
$\{\overline{V}_n\},\ r_n \to 0,\ \bigcap\limits_n \overline{V}_n = \{a\},\ X = \bigcup X_n,\ \exists n_0 : a \in X_{n_0}$
$X_{n_0} \cap \overline{V}_{n_0} = \varnothing \to\leftarrow\ a \in \overline{V}_{n_0}$ ■

**Corollary 1.3.1.** *Complete metric space without isolated points is uncountable.*

*Proof.* No isolated points are present $\implies$ every point in the set is nowhere dense in it. Let X be countable: $X = \bigcup\limits_{n}\{X_n\}$, then it is first Baire category set. $\rightarrow\leftarrow$ $\blacksquare$

**Definition.** K — **compact** if

1. $K = \mathrm{Cl}(K)$
2. $x_n \in K\ \exists n_1 < n_2 < \ldots\ x_{n_j}$ – converges in $X$.

If only 2 is present, the set is called **precompact**.

**Theorem 1.4** (Hausdorff)**.**
*Let $X$ — metric space, $K$ — closed in $X$.*
*Then:* $K$ *— compact* $\iff$ $K$ *— totally bounded,*
*i.e.* $\forall\varepsilon > 0\ \exists a_1, \ldots, a_p \in X\colon\ \forall b \in K\ \exists a_j\colon\ \rho(a_j, b) < \varepsilon$
$(a_1, \ldots, a_p - finite\ \varepsilon - net)$

*Proof.*
$\implies$
$K$— totally bounded, $x_n \in K\ n_1 < n_2 < \ldots < n_k < \ldots$
$x_n$ – converges in K
$\varepsilon_k \downarrow\to 0\ \varepsilon_1 \quad K \subset \bigcup\limits_{j=1}^{p} \overline{V}_j,\ rad = \varepsilon_1\ (\varepsilon_1 - net)$
n is finite $\implies$ one ball will contain infinetely many $x_n$ elements.
Let's look at $\overline{V}_{j_0} \cap K$ — totally bounded $= K_1,\ \mathrm{diam}(K_1) \leqslant 2\varepsilon_1$
$\varepsilon_2 \quad K_1 \subset \bigcup\limits_{j=1}^{n} \overline{V'}_j,\ rad = \varepsilon_2.$
Then one of $\overline{V'}$ contains infinitely many elements of the sequence contained in $K_1$
$\overline{V'}_{j_0} \cap K_1 = K_2,\ \mathrm{diam}(K_2) \leqslant 2\varepsilon_2$ and so on.

$$K_n \supset K_{n+1} \supset K_{n+2} \supset \ldots,\ \mathrm{diam}(K_N) \leqslant 2\varepsilon_n \xRightarrow{by\ space\ compl.} \underbrace{\bigcap_{n=1}^{\infty} K_n}_{\mathrm{diam}(K_n)\to 0,\ \{x\}} \neq$$

$\varnothing$
Take $x_{n_1}$ from $K_1$, $x_{n_2}$ from $K_2 \ldots$
$\Longleftarrow$
K — compact $\forall\varepsilon\ \exists$ finite $\varepsilon$net?
By contradiction: $\exists\varepsilon_0 > 0\colon$ finite $\varepsilon_0$-net is impossible to construct.
$\forall x_1 \in K\ \exists x_2 \in K\colon\ \rho(x_1, x_2) > \varepsilon_0$ (or else system of $x_1$ — finite $\varepsilon$-net)

$\{x_1, x_2\}$ - choose $x_3 \in K\colon \rho(x_3, x_i) > \varepsilon_0,\ i = 1, 2$ and so on.
$x_n \in K:\ n \neq m\ \rho(x_n, x_m) > \varepsilon_0$ — contains no converging subsequence $\implies$ set is not a compact. $\to\leftarrow$ ■

## §1.2 Normed spaces

**Definition.** X — **linear set**, x + y, $\alpha \cdot x$, $\alpha \in \mathbb{R}$
The purpose of norm definition, is to construct a topology on X, so that 2 linear operations are continuous on it.

$\varphi : X \to \mathbb{R}$:

1. $\varphi(x) \geqslant 0,\ = 0 \iff x = 0$
2. $\varphi(\alpha x) = |\alpha|\varphi(x)$
3. $\varphi(x + y) \leqslant \varphi(x) + \varphi(y)$

**Definition.** $\varphi$ — **norm** on X, $\varphi(x) = \|x\|$

$\rho(x, y) \overset{\text{def}}{=} \|x - y\|$ — metric on X.

**Definition.**
$(X, \|\cdot\|)$ — **normed space** — special case of metrical space.

$x = \lim x_n \overset{def}{\implies} \rho(x_n, x) \to 0 \iff \|x_n - x\| \to 0$

**Statement 1.4.** *In the topology of a normed space linear operations are continuous on X.*

*Proof.* 1. $x_n \to x,\ y_n \to y;\ \|(x_n + y_n) - (x + y)\| = \|(x_n - x) + (y_n - y)\| \leqslant$
$$\leqslant \underbrace{\|x_n - x\|}_{\substack{\downarrow \\ 0}} + \underbrace{\|y_n - y\|}_{\substack{\downarrow \\ 0}}$$
$$\implies x_n + y_n \to x + y$$

2. $\alpha_n \to \alpha,\ x_n \to x;\ \|\alpha_n x_n - \alpha x\| = \|(\alpha_n - \alpha)x_n + \alpha(x_n - x)\| \leqslant$
$$\leqslant \underbrace{|\alpha_n - \alpha|}_{\substack{\downarrow \\ 0}} \cdot \underbrace{\|x_n\|}_{bounded} + \underbrace{\alpha\|x_n - x\|}_{\substack{\downarrow \\ 0}}$$
$x_n \to x \implies \|x_n\|$ — bounded.
$\alpha_x x_n \to \alpha x$ ■

**Statement 1.5.** *From the triangle inequality* $|\|x\|-\|y\||\leqslant\|x-y\|$
$x_n\to x\implies\|x_n\|\to\|x\|$
*Norm is continious.*

**Example 5.** $\mathbb{R}^n$

1. $\|\bar{x}\|=\sqrt{\sum\limits_{k=1}^{n}x_k^2}$

2. $\|\bar{x}\|_1\overset{\text{def}}{=}\sum\limits_{k=1}^{n}|x_k|$

3. $\|\bar{x}\|_2\overset{\text{def}}{=}\max\{|x_1|\dots|x_n|\}$

4. $C[a,b]$ — functions continuous on $[a,b]$; $\|f\|=\max\limits_{x\in[a,b]}|f(x)|$

5. $L_p(E)=\{f-\text{measurable},\ \int\limits_E|f|^p<+\infty\}$
   $p\geqslant 1,\ \|f\|_p=(\int\limits_E|f|^p)^{\frac{1}{p}}$

Because the set of points is the same, arises the question about convergence comparison.
$\|\cdot\|_1\sim\|\cdot\|_2,\ x_n\overset{\|\cdot\|_1}{\to}x\iff x_n\overset{\|\cdot\|_2}{\to}$

**Statement 1.6.**
$\|\cdot\|_1\sim\|\cdot\|_2\iff\exists a,b>0\colon\forall x\in X\implies a\|x_1\|_1\leqslant\|x\|_2\leqslant b\|x\|_1$

**Theorem 1.5** (Riesz)**.**
*$X$, $\dim X<+\infty$ — linear set.*
<u>*Then:*</u> *Any pair of norms in X are equivalent.*

*Proof.* $l_1,\dots,l_n$ — linearly independent from X. $\forall x\in X=\sum\limits_{k=1}^{n}\alpha_k l_K$
$\bar{x}\leftrightarrow(l_1,\dots,l_n)=\bar{l}\in\mathbb{R}^n$
Let $\|\cdot\|$ — some norm in X.

$$\|x\|\underset{\triangle}{\leqslant}\sum_{k=1}^{n}\|l_k\||\alpha_k|\underset{Cauchy}{\leqslant}\underbrace{\sqrt{\sum_{1}^{n}\|l_k\|^2}}_{constB}\underset{\|\bar{\alpha}\|=\|x\|_1}{\underset{\|}{\sqrt{\sum_{1}^{n}|\alpha_k|^2}}}$$

$\|x\|_1 = \sqrt{\sum\limits_1^n \|alpha_k\|^2},\ x = \sum \alpha_k l_k$
$\|x\| \leqslant b\|x\|_1$
$?\exists a > 0\colon a\|x\|_1 \leqslant \|x\| \implies \|\cdot\| \sim \|\cdot\|_1$
Let $f(\alpha_1, \ldots, \alpha_n) = \left\|\sum\limits_{k=1}^n \alpha_k l_k\right\|$

$$|f(\bar\alpha + \Delta\bar\alpha) - f(\bar\alpha)| = \left|\left\|\sum_1^k \alpha_k l_k + \sum_1^n \Delta\alpha_k l_K\right\| - \left\|\sum_1^n \alpha_k l_k\right\|\right| \leqslant$$
$$\left\|\sum_1^n \Delta\alpha_k l_k\right\| \leqslant \underbrace{\sum \|l_k\||\Delta\alpha_k|}_{\substack{\downarrow \\ 0, \Delta\alpha_k \to 0}} \implies f\text{— continuous on } \mathbb{R}^n$$

$S_1 = \{\sum\limits_1^n \alpha_k|^2 = 1\} \subset \mathbb{R}^m$, f — continuous on $S_1,\ \bar\alpha^* \in S_1$
$\forall\alpha \in S_1 \implies f(\bar\alpha^*) \leqslant f(\bar\alpha)$
$f(\bar\alpha^*) = 0$
$\left\|\sum\limits_1^n \alpha_k^* l_k\right\| = 0$
$\sum\limits_1^n \alpha_k^* l_k = 0,\ \bar\alpha^* \in S_1$
$l_1 \ldots l_n$ — linearly independent $\rightarrow\leftarrow$
$\min\limits_{S_1} f = m > 0$

$$\|x\| = \left\|\sum_1^n \alpha_K l_K\right\| = f(\bar\alpha) = \sqrt{\sum_1^n \alpha_k^2} \cdot \left\|\sum \underbrace{\boxed{\frac{\alpha_k}{\sqrt{\sum\limits_1^n \alpha_k^2}}}}_{\beta_k} \cdot l_k\right\| \geqslant,\ \bar\beta = (\beta_1 \ldots \beta_n) \in S_1$$
$$\geqslant m \cdot \|x\|_1,\ a = m \quad \blacksquare$$