# Functional analysis course by Dodonov N.U.

Sugak A.M.

Fall 2015 — Spring 2016

# 1 Vector spaces

## §1.1 Metric spaces.

X, $\rho$: $X \times X \to \mathbb{R}_+$

**Definition.** $\rho$ — **metric**

1. $\rho(x, y) \geqslant 0, = 0 \iff x = y$
2. $\rho(x, y) = \rho(y, x)$
3. $\rho(x, y) \leqslant \rho(x, z) + \rho(y, z)$

**Definition.** $(X, \rho)$ — **Metric space**.

**Definition.** $x = \lim x_n \iff \rho(x_n, x) \to 0$

$X, \tau = \{ G \subset X \}$

1. $\varnothing, X \in \tau$
2. $G_\alpha \in \tau, \alpha \in \mathscr{A} \implies \bigcup\limits_{\alpha} G_\alpha \in \tau$
3. $G_1, \dots, G_n \in \tau \implies \bigcap\limits_{j=1}^{n} G_j \in \tau$

**Definition.** $(X, \tau)$ — **Topological space**.

$x = \lim x_n \quad \forall G \in \tau : x \in G \quad \exists N : \forall n > N \implies x_n \in G$
G — open in $\tau$
$F = X \setminus G$ — closed

**Definition.** $B_r(a) = \{ x \mid \rho(x, a) < r \}$ — **open ball**

$\tau = \bigcup B_r(a)$

**Statement 1.1.** $b \in B_{r_1}(a_1) \cap B_{r_2}(a_2) \implies \exists r_3 > 0 : B_{r_3}(a_3) \subset B_{r_1}(a_1) \cap B_{r_2}(a_2)$

In this sense metric space is just a special case of topological space.

**Example 1.** $\mathbb{R}, \rho(x, y) = |x - y|$, MS

**Example 2.** $\bar{x} = (x_1, \dots, x_n) \in \mathbb{R}^n, \rho(\bar{x}, \bar{y}) = \sqrt{\sum\limits_{j=1}^{n} (x_j - y_j)^2}, MS$

**Example 3.** $\bar{x} = (x_1, \dots, x_n, \dots) \in \mathbb{R}^\infty$

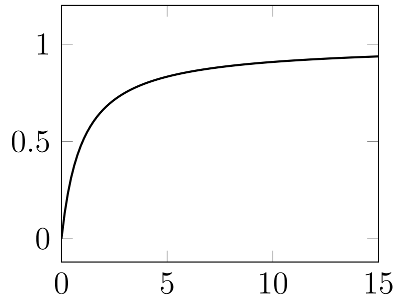
$$\alpha\bar{x} = (\alpha x_1, \dots, \alpha x_n, \dots)$$
$$\bar{x} + \bar{y} = (x_1 + y_1, \dots, x_n + y_n, \dots)$$

Let's define $\lim\limits_{m \to \infty} \bar{x}_m$

- in $\mathbb{R}^n$: $\bar{x}_n \to \bar{x} \iff \forall j = 1, \dots, n \qquad x_j^{(m)} \xrightarrow[m\to\infty]{} x_j$
- in $\mathbb{R}^\infty$: $\bar{x}_m \to \bar{x} \overset{def}{\iff} \forall j = 1, 2, 3, \dots \quad x_j^{(m)} \xrightarrow[m\to\infty]{} x_j$

**Definition.** $\rho(\bar{x}, \bar{y}) \overset{\text{def}}{=} \sum\limits_{n=1}^{\infty} \frac{1}{2^n} \underbrace{\frac{|x_n - y_n|}{1 + |x_n - y_n|}}_{\varphi(|x_n - y_n|)}$ — **Urysohn metric**.

$\varphi(t) = \dfrac{t}{1+t}$




$\varphi(t_1) + \varphi(t_2) \geqslant \varphi(t_1 + t_2)$

$\varphi(t_1) + \varphi(t_2) = \dfrac{t_1}{1+t_1} + \dfrac{t_2}{1+t_2} \geqslant \dfrac{t_1}{1+t_1+t_2} + \dfrac{t_2}{1+t_1+t_2}$

$\varphi(t_1) + \varphi(t_2) \geqslant \dfrac{t_1+t_2}{1+t_1+t_2} = \varphi(t_1+t_2)$

**Statement 1.2.** $\rho(\bar{x}_m, \bar{x}) \xrightarrow[m\to\infty]{} 0 \iff x_j^{(m)} \to x_j \ \forall j$

*Proof.*

- $\Longrightarrow$
  $f(|x_k^{(n)} - x_k|) \leqslant 2^k \rho(x^{(n)}, x)$
  Let $\rho(x^{(n)}, x) \leqslant \dfrac{\varepsilon}{2^k}$, then $f(|x_k^{(n)} - x_k|) < \varepsilon$
  $|x_k^{(n)} - x_k| = t = \dfrac{1}{1 - f(t)} - 1$, then $t \to 0$
- $\Leftarrow$
  Let's choose $k_0$ for which $\sum\limits_{k=k_0+1}^{\infty} \dfrac{1}{2^k} < \varepsilon$
  Let's choose $n_0$ for which $\forall k \leqslant k_0, n > n_0 : |x_k^{(n)} - x_k| < \varepsilon$.
  Then $\rho(x^{(n)}, x) < \sum\limits_{k=1}^{k_0} \dfrac{\varepsilon}{2^k} + \varepsilon < 2\varepsilon$
  Letting $\varepsilon \to 0$, we get what we want ∎

In this way $\mathbb{R}^\infty$ is a metrizable space.

**Example 4.** $X, \rho(x,y) \overset{\text{def}}{=} \begin{cases} 0, & x = y \\ 1, & x \neq y \end{cases}$ — **Discrete metric**.

$x_n \to x,\ \varepsilon = \frac{1}{2},\ \exists M:\ m > M \implies \rho(x_m, x) < \frac{1}{2} \implies \rho(x_m, x) = 0 \implies x_m = x$

**Definition.**

$(X, \tau);\ \forall A \subset X;$

$\operatorname{Int}(A) \overset{\text{def}}{=} \bigcup\limits_{G \subset A} G$ is open;

$\operatorname{Cl}(A) \overset{\text{def}}{=} \bigcap\limits_{A \subset G} G$ is closed;

$\operatorname{Fr}(A) \overset{\text{def}}{=} \operatorname{Cl}(A) \setminus \operatorname{Int}(A)$

$(X, \rho)$; Having a metric space one can describe closure of a set.

$$\rho(x, A) \stackrel{\text{def}}{=} \inf_{a \in A} \rho(x, a)$$

$$\rho(A, B) \stackrel{\text{def}}{=} \inf_{\substack{a \in A \\ b \in B}} \rho(a, b)$$

$$\rho(x, A) = f(x), x \in X$$

**Statement 1.3.** *Function f(x) is continuous.*

*Proof.* $\forall x, y \in X$
$f(x) = \rho(x, A) \underset{\forall \alpha \in A}{\leqslant} \rho(x, \alpha) \leqslant \rho(x, y) + \rho(y, \alpha)$
$\forall \varepsilon > 0\ \exists \alpha_\varepsilon \in A:\ \rho(y, \alpha_\varepsilon) < \rho(y, A) + \varepsilon = f(y) + \varepsilon$
$f(x) \leqslant f(y) + \varepsilon + \rho(x, y),\ \varepsilon \to 0$
$\begin{cases} f(x) \leqslant f(y) + \rho(x, y) \\ f(y) \leqslant f(x) + \rho(x, y) \end{cases} \implies |f(x) - f(y)| \leqslant \rho(x, y)$ ■

**Statement 1.4.** $x \in \mathrm{Cl}(A) \iff \rho(x, A) = 0$

Let's look at the metric spaces in terms of separation of sets from each other by open sets.
$x, y$
$r = \rho(x, y) > 0$
$B_{\frac{r}{3}}(x),\ B_{\frac{r}{3}}(y)$
In any metric space separability axiom is true.

**Theorem 1.1.**
*Any metric space is a normal space,*
*i.e.* $\forall$ *closed disjoint* $F_1, F_2 \in X,\ \exists$ *open disjoint* $G_1, G_2$: $F_j \in G_j,\ j = 1, 2$

*Proof.* $g(x) = \dfrac{\rho(x, F_1)}{\rho(x, F_1) + \rho(x, F_2)}$ — continuous on X
$x \in F_1,\ \mathrm{Cl}(F_1) = F_1,\ \rho(x, F_1) = 0,\ g(x) = 0$
$x \in F_2,\ g(x) = 1$
Let's look at $(-\infty; \frac{1}{3}), (\frac{2}{3}, \infty)$ — by continuity their inverse images under $g$ are open.
$G_1 = g^{-1}(-\infty; \frac{1}{3})$
$G_2 = g^{-1}(\frac{2}{3}; \infty)$ ■

**Definition.** Metric space is **complete** if $\rho(x_n, x_m) \to 0 \implies \exists x = \lim x_n$
$\mathbb{R}^\infty$ – complete (by completeness of the rational numbers).
In complete metric spaces the nested balls principle is true.

**Theorem 1.2.**
*X – complete metric space,* $\overline{V}_{r_n}$ *– system of closed balls.*

1. $\overline{V}_{r_{n+1}} \subset \overline{V}_{r_n}$ *– the system is nested.*
2. $r_n \to 0$

<u>*Then:*</u> $\bigcap\limits_n \overline{V}_{r_n} = \{a\}$

*Proof.* Let $b_n$ be centers of $\overline{V}_{r_n}$,
$m \geqslant n,\ b_m \in \overline{V}_{r_n},\ \rho(b_m, b_n) \leqslant r_n \to 0\ \forall m \geqslant n$
$\rho(b_m, b_n) \to 0 \overset{compl.}{\implies} \exists a = \lim b_n$ Since the balls are closed a $\in$ every ball.
$r_n \to 0 \implies$ there is only one common point. ■

$(X, \tau)$ — topological space
$A \subset X,\ \tau_a = \{\, G \cap A, G \in \tau \,\}$ — topology induced on $A$

**Definition.** $X$— metric space, $A \subset X,\ \mathrm{Cl}(A) = X$
Then: A – **dense** in X
If $\mathrm{Int}(\mathrm{Cl}(A)) = \varnothing$ A – **nowhere dense** in X.

**Note.** It is easy to understand, that in metric spaces nowhere density means the following: $\forall$ ball $V\ \exists V' \subset V\colon V'$ contains no points from A.

**Definition.** $X$ is called **first Baire category set**, if it can be written as at most countable union of $x_n$ each nowhere dense in $X$.

**Theorem 1.3** (Baire category theorem).
*Complete metric space is second Baire category set in itself.*

*Proof.* Let $X$ be first Baire category set.
$X = \bigcup\limits_n X_n \quad \forall \overline{V}\ X_1$ is nowhere dense.
$\overline{V}_1 \subset \overline{V}\colon\ \overline{V}_1 \cap X_1\ = \varnothing$
$X_2$ is nowhere dense $\overline{V}_2 \subset \overline{V}_1\colon \overline{V}_2 \cap X_2 = \varnothing$
$r_2 \leqslant \dfrac{r_1}{2}$
$\vdots$
$\{\overline{V}_n\},\ r_n \to 0,\ \bigcap\limits_n \overline{V}_n = \{a\},\ X = \bigcup X_n,\ \exists n_0\colon a \in X_{n_0}$
$X_{n_0} \cap \overline{V}_{n_0} = \varnothing \rightarrow\leftarrow\ \ a \in \overline{V}_{n_0}$ ∎

**Corollary 1.3.1.** *Complete metric space without isolated points is uncountable.*

*Proof.* No isolated points are present $\implies$ every point in the set is nowhere dense in it. Let $X$ be countable: $X = \bigcup\limits_n \{X_n\}$, then it is first Baire category set. $\rightarrow\leftarrow$ ∎

**Definition.** $K$ — **compact** if

1. $K = \mathrm{Cl}(K)$
2. $x_n \in K\ \exists n_1 < n_2 < \ldots\ x_{n_j}$ – converges in $X$.

If only 2 is present, the set is called **precompact**.

**Theorem 1.4** (Hausdorff).
*Let $X$ — metric space, $K$ — closed in $X$.*
*Then: $K$ — compact $\iff$ $K$ — totally bounded,*
*i.e. $\forall \varepsilon > 0\ \exists a_1, \ldots, a_p \in X\colon\ \forall b \in K\ \exists a_j\colon\ \rho(a_j, b) < \varepsilon$*
*$(a_1, \ldots, a_p - finite\ \varepsilon - net)$*

*Proof.*

- Totally bounded $\implies$ compact

  $K$ is totally bounded, $x_n \in K\ n_1 < n_2 < \cdots < n_k < \cdots$, $x_n$ converges in $K$

  $$\varepsilon_k \downarrow\to 0\ \varepsilon_1 \quad K \subset \bigcup_{j=1}^{p} \overline{V}_j,\ rad = \varepsilon_1 \qquad (\varepsilon_1\text{-net})$$

  $n$ is finite $\implies$ one ball will contain infinetely many $x_n$ elements.

Let's look at $\overline{V}_{j_0} \cap K$ — totally bounded $= K_1, \operatorname{diam}(K_1) \leqslant 2\varepsilon_1$

$\varepsilon_2 \quad K_1 \subset \bigcup\limits_{j=1}^{n} \overline{V'}_j,\ rad = \varepsilon_2.$

Then one of $\overline{V'}$ contains infinitely many elements of the sequence contained in $K_1$.

$\overline{V'}_{j_0} \cap K_1 = K_2,\ \operatorname{diam}(K_2) \leqslant 2\varepsilon_2$ and so on.

$$K_n \supset K_{n+1} \supset K_{n+2} \supset \dots,\ \operatorname{diam}(K_N) \leqslant 2\varepsilon_n \xRightarrow{\text{by compl.}} \underbrace{\bigcap_{n=1}^{\infty} K_n}_{\operatorname{diam}(K_n)\to 0,\ \{x\}} \neq \varnothing$$

Take $x_{n_1}$ from $K_1$, $x_{n_2}$ from $K_2 \dots$

- Compact $\implies$ totally bounded

  $K$ — compact $\forall \varepsilon\ \exists$ finite $\varepsilon$-net?
  By contradiction: $\exists \varepsilon_0 > 0\colon$ finite $\varepsilon_0$-net is impossible to construct.
  $\forall x_1 \in K\ \exists x_2 \in K\colon\ \rho(x_1, x_2) > \varepsilon_0$ (or else system of $x_1$ — finite $\varepsilon$-net)
  $\{x_1, x_2\}$ - choose $x_3 \in K\colon \rho(x_3, x_i) > \varepsilon_0,\ i = 1, 2$ and so on.
  $x_n \in K\ :\ n\ \neq m\ \rho(x_n, x_m) > \varepsilon_0$ — contains no converging subsequence $\implies$ set is not a compact. $\to\leftarrow$ ■

## §1.2 Normed spaces

**Definition.** $X$ — **linear set**, $x + y, \alpha \cdot x,\ \alpha \in \mathbb{R}$
The purpose of norm definition, is to construct a topology on $X$, so that 2 linear operations are continuous on it.

$\varphi\colon X \to \mathbb{R}\colon$

1. $\varphi(x) \geqslant 0,\ = 0 \iff x = 0$
2. $\varphi(\alpha x) = |\alpha|\varphi(x)$
3. $\varphi(x + y) \leqslant \varphi(x) + \varphi(y)$

**Definition.** $\varphi$ — **norm** on $X$, $\varphi(x) = \|x\|$

$\rho(x, y) \overset{\text{def}}{=} \|x - y\|$ — metric on $X$.

**Definition.**
$(X, \|\cdot\|)$ — **normed space** — special case of metrical space.

$x = \lim x_n \overset{def}{\implies} \rho(x_n, x) \to 0 \iff \|x_n - x\| \to 0$

**Statement 1.5.** *In the topology of a normed space linear operations are continuous on $X$.*

*Proof.*

1. $x_n \to x,\ y_n \to y;\ \|(x_n + y_n) - (x + y)\| = \|(x_n - x) + (y_n - y)\| \leqslant$
$$\leqslant \underbrace{\|x_n - x\|}_{\downarrow \atop 0} + \underbrace{\|y_n - y\|}_{\downarrow \atop 0}$$
$$\implies x_n + y_n \to x + y$$

2. $\alpha_n \to \alpha,\ x_n \to x;\ \|\alpha_n x_n - \alpha x\| = \|(\alpha_n - \alpha)x_n + \alpha(x_n - x)\| \leqslant$
$$\leqslant \underbrace{|\alpha_n - \alpha|}_{\substack{\downarrow\\0}} \cdot \underbrace{\|x_n\|}_{bounded} + \underbrace{\alpha\|x_n - x\|}_{\substack{\downarrow\\0}}$$

$x_n \to x \implies \|x_n\|$ — bounded.
$\alpha_x x_n \to \alpha x$ ■

**Statement 1.6.** *From the triangle inequality* $|\|x\| - \|y\|| \leqslant \|x - y\|$
$x_n \to x \implies \|x_n\| \to \|x\|$
*Norm is continious.*

**Example 5.** $\mathbb{R}^n$

1. $\|\bar{x}\| = \sqrt{\sum\limits_{k=1}^{n} x_k^2}$
2. $\|\bar{x}\|_1 \overset{\text{def}}{=} \sum\limits_{k=1}^{n} |x_k|$
3. $\|\bar{x}\|_2 \overset{\text{def}}{=} \max\{|x_1|, \ldots, |x_n|\}$
4. $C[a,b]$ — functions continuous on $[a,b]$; $\|f\| = \max\limits_{x \in [a,b]} |f(x)|$
5. $L_p(E) = \left\{ f \text{ — measurable, } \int\limits_E |f|^p < +\infty \right\}$

   $p \geqslant 1,\ \|f\|_p = \left( \int\limits_E |f|^p \right)^{\frac{1}{p}}$

Because the set of points is the same, arises the question about convergence comparison.
$\|\cdot\|_1 \sim \|\cdot\|_2,\ x_n \overset{\|\cdot\|_1}{\to} x \iff x_n \overset{\|\cdot\|_2}{\to} x$

**Statement 1.7.**
$\|\cdot\|_1 \sim \|\cdot\|_2 \iff \exists a, b > 0\colon \forall x \in X \implies a\|x_1\|_1 \leqslant \|x\|_2 \leqslant b\|x\|_1$

**Theorem 1.5** (Riesz)**.**
$X,\ \dim X < +\infty$ *— linear set.*
*<u>Then:</u> Any pair of norms in $X$ are equivalent.*

*Proof.* $l_1, \ldots, l_n$ — linearly independent from $X$. $\forall x \in X = \sum\limits_{k=1}^{n} \alpha_k l_K$

$$\bar{x} \leftrightarrow (l_1, \ldots, l_n) = \bar{l} \in \mathbb{R}^n$$

Let $\|\cdot\|$ — some norm in $X$.

$$\|x\| \underset{\triangle}{\leqslant} \sum_{k=1}^{n} \|l_k\| |\alpha_k| \underset{\text{Cauchy}}{\leqslant} \underbrace{\sqrt{\sum_{k=1}^{n} \|l_k\|^2}}_{const(B), B-basis} \underset{\|\bar{\alpha}\| = \|x\|_1}{\underset{\|}{\sqrt{\sum_{k=1}^{n} |\alpha_k|^2}}}$$

$$\|x\|_1 = \sqrt{\sum_{k=1}^{n} \|\alpha_k\|^2},\ x = \sum \alpha_k l_k$$

$\|x\| \leqslant b\|x\|_1$
$?\exists a > 0\colon a\|x\|_1 \leqslant \|x\| \implies \|\cdot\| \sim \|\cdot\|_1$

Let $f(\alpha_1, \ldots, \alpha_n) = \left\| \sum\limits_{k=1}^{n} \alpha_k l_k \right\|$

$$|f(\bar{\alpha} + \Delta\bar{\alpha}) - f(\bar{\alpha})| = \left| \left\| \sum_{k=1}^{n} \alpha_k l_k + \sum_{k=1}^{n} \Delta\alpha_k l_K \right\| - \left\| \sum_{k=1}^{n} \alpha_k l_k \right\| \right| \leqslant$$

$$\left\| \sum_{k=1}^{n} \Delta\alpha_k l_k \right\| \leqslant \underbrace{\sum \|l_k\| |\Delta\alpha_k|}_{\substack{\downarrow \\ 0, \Delta\alpha_k \to 0}} \implies f \text{ is continuous on } \mathbb{R}^n$$

$S_1 = \left\{ \sum\limits_{k=1}^{n} |\alpha_k|^2 = 1 \right\} \subset \mathbb{R}^m$, f — continuous on $S_1$, $S_1$ — compact, $\bar{\alpha}^* \in S_1$

By Weierstrass theorem there exists a point $\alpha^* \in S_1$ on a sphere, in which function $f$ achieves its minimum $\implies \forall \alpha \in S_1 \ f(\bar{\alpha}^*) \leqslant f(\bar{\alpha})$

If $f(\bar{\alpha}^*) = 0$, then $\left\| \sum\limits_{k=1}^{n} \alpha_k^* l_k \right\| = 0 \implies \sum\limits_{k=1}^{n} \alpha_k^* l_k = 0, \ \bar{\alpha}^* \in S_1$,
but $l_1 \ldots l_n$ are linearly independent $\rightarrow\leftarrow \implies \min\limits_{S_1} f = m > 0$

$$\|x\| = \left\| \sum_{k=1}^{n} \alpha_k l_k \right\| = f(\bar{\alpha}) = \sqrt{\sum_{k=1}^{n} \alpha_k^2} \cdot \left\| \sum \underbrace{\boxed{\frac{\alpha_k}{\sqrt{\sum_{k=1}^{n} \alpha_k^2}}}}_{\beta_k} \cdot l_k \right\| \geqslant, \ \bar{\beta} = (\beta_1 \ldots \beta_n) \in S_1$$

$$\geqslant m \cdot \|x\|_1, \ a = m \qquad \blacksquare$$

**Corollary 1.5.1.** $X$ — *NS*, $Y \subset X, \dim Y < +\infty \implies Y = \mathrm{Cl}(Y)$

**Note.** Functional analysis differentiates between linear subset (set of points, closed by addition and scalar multiplication) and linear subspace (closed linear subset).

*Proof.* $Y = \mathscr{L}(l_1, \ldots, l_n) = \left\{ \sum\limits_{i=1}^{n} \alpha_i l_i \ \middle| \ l_1, \ldots, l_n - \text{lin. indep.} \right\}$
$y_m \in Y, y_m \to y$ in $X \implies y \in Y$?
$\|y_m - y\| \to 0 \implies \|y_m - y_p\| \to 0, \ m, p \to \infty$
$\|y\|, y \in Y$.
By Riesz theorem all norm in Y are equivalent.
$y = \sum\limits_{j=1}^{n} \alpha_j l_J, \|y\|_0 = \sqrt{\sum\limits_{j=1}^{n} \alpha_j^2}$ — some norm (by linear independance).
By Riesz theorem $\|y\| \sim \|y\|_0$
$\underbrace{\|y_m - y_p\|}_{\in Y} \to 0 \implies \|y_m - y_p\|_0 \to 0$
$\bar{\alpha} = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n \ y_m = \sum\limits_{i=1}^{n} \alpha_i^{(m)} l_i$
$|\alpha_i^{(m)} - \alpha_i^{(l)}| \to 0 \ \forall i = 1, \ldots, n; \ \bar{\alpha} = (\alpha_1^{(m)}, \ldots, \alpha_n^{(m)}) \to \alpha^* = (\alpha_1^*, \ldots, \alpha_n^*)$
$y^* = \sum\limits_{i=1}^{n} \alpha_i^* l_I \in Y, \ \|y_m - y\| \to 0, y = y^* \implies y \in Y$ $\blacksquare$

**Definition.** If normed space if complete, then it is called **B-space** or **Banach space**.

**Example 6.** $C[a, b]$ — functions continuous on $[a, b]$.

**Example 7.** Lebesgue space, $p \geqslant 1, L_p(E) = \left\{ f \text{ is measurable on } E,\ \int\limits_E |f|^P < +\infty \right\}$.

If $X$ — Banach space,

$$\sum_{n=1}^{\infty} x_n = \lim_{n\to\infty} \sum_{k=1}^{n} x_k,\ \sum_{n=1}^{\infty} \|x_n\| < +\infty$$

$$\|S_n - S_m\| = \left\| \sum_{k=m+1}^{n} x_k \right\| \leqslant \sum_{m+1}^{n} \|x_k\| \xrightarrow[n,m\to\infty]{} 0$$

$$\implies \|S_n - S_m\| \to 0 \implies \exists \lim_{n\to\infty} S_n, \sum_{k=1}^{n} x_k \text{— converges.}$$

In Banach spaces works the theory of absolute convergence of numerical series.

**Lemma 1** (Riesz's lemma about almost perpendicular)**.** *$Y$ — eigen subspace of $X$ — normed space.* $\forall \varepsilon \in (0,1)\ \exists z_\varepsilon \in X$:

1. $\|z_\varepsilon\| = 1$
2. $\rho(z_\varepsilon, Y) > 1 - \varepsilon$

*Proof.* $\exists x \notin Y\ d = \rho(x,Y),\ d = 0\ \exists y_n \in Y \colon \|x - y_n\| < \frac{1}{n},\ n \to \infty,\ y_n \to x$
$Y = \mathrm{Cl}(Y) \implies x \in Y \rightarrow\leftarrow x \notin Y,\ d > 0$
$\forall \varepsilon \in (0,1)\ \frac{1}{1-\varepsilon} > 1\ \exists y_\varepsilon \in Y \colon \|x - y_\varepsilon\| < \frac{1}{1-\varepsilon} d$

$$z_\varepsilon = \frac{x - y_\varepsilon}{\|x - y_\varepsilon\|},\ \|z_\varepsilon\| = 1,\ \forall y \in Y\ \|z_\varepsilon - y\| = \left\| \frac{x - y_\varepsilon}{\|x - y_\varepsilon\|} \right\| = \frac{\|x - \overbrace{(y_\varepsilon + \|x - y_\varepsilon\| \cdot y)}^{\in Y}\| \geqslant d}{\|x - y_\varepsilon\| < \frac{1}{1-\varepsilon} d} >$$

$1 - \varepsilon$ ∎

**Corollary 1.5.2.** $\dim X = +\infty$*, $S$ — sphere in $X$, $r_S = 1\,\{\, x \mid \|x\| = 1 \,\} \implies S$ — not a compact.*

*Proof.* $\forall x_1 \in S,\ Y_1 = \mathscr{L}\{x_1\}$ — finite dimensional linear set. $\implies$ closed in $X \implies Y_1$ — subspace. $\dim X = +\infty \implies Y_1$ — eigen subspace.
Then by the Riesz lemma:

$$\exists x_2 \in S \colon \|x_2 - x_1\| > \frac{1}{2}$$

$$Y_2 = \mathscr{L}\{x_1, x_2\}\ \exists x_3 \in S \colon \|x_3 - x_j\| > \frac{1}{2},\ j = 1,2$$

Continue by induction. Because $\dim X = +\infty$ the process willl never stop.
$x_n \in S \colon \|x_n - x_m\| > \frac{1}{2},\ n \neq m$ — obviously we cannot extract converging subsequence.
$\implies S$ — not a compact. ∎

## §1.3 Inner product (unitary) spaces

**Definition.** $X$ — linear space.
$\varphi \colon X \times X \to \mathbb{R}$

1. $\varphi(x,x) \geqslant 0, \quad \varphi(x,x) = 0 \iff x = 0$

2. $\varphi(x, y) = \varphi(y, x)$

3. $\varphi(\alpha x + \beta y, z) = \alpha\varphi(x, z) + \beta\varphi(y, z)$

$\varphi$ — **inner product**.
$\varphi(x, y) = \langle x, y\rangle$

**Definition.** $(X, \langle\cdot,\cdot\rangle)$ — **inner product space**.

**Example 8.** $\mathbb{R}^n, \langle\bar{x}, \bar{y}\rangle = \sum\limits_{j=1}^{n} x_j y_j$

**Statement 1.8** (Schwarz)**.** $\forall x, y \in X \quad |\langle x, y\rangle| \leqslant \sqrt{\langle x, x\rangle} \cdot \sqrt{\langle y, y\rangle}$

*Proof.* $\lambda \in \mathbb{R}$

$$f(\lambda) = \underset{\lambda^2\langle x,x\rangle + 2\lambda\langle x,y\rangle + \langle y,y\rangle}{\overset{\parallel}{\langle \lambda x + y, \lambda x + y\rangle}} \geqslant 0$$

$$D = 4\langle x, y\rangle^2 - 4\langle x, x\rangle \cdot \langle y, y\rangle \leqslant 0 \quad \blacksquare$$

**Corollary 1.5.3** (Cauchy inequality for sums)**.** *Consider* $X = \mathbb{R}^n, \|x\| \overset{def}{=} \sqrt{\langle x, x\rangle}$. *Then*

$$\|x+y\|^2 = \langle x+y, x+y\rangle = \|x\|^2 + 2\cdot\underbrace{\langle x, y\rangle}_{\leqslant \|x\|\cdot\|y\|} + \|y\|^2 \leqslant \left(\|x\| + \|y\|\right)^2$$

Any inner product space is a special case of a normed space. The specifics is that we can measure the angles between points:

$$x \perp y \iff \langle x, y\rangle = 0$$

In this case the Pythagorean theorem takes place:

$$\|x+y\|^2 = \|x\|^2 + \|y\|^2$$

In inner product spaces the parallelogram law plays a significant role:

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2 \quad \forall x, y \in X$$

$\langle x, y\rangle \mapsto \|x\| = \sqrt{\langle x, x\rangle}$
Let $X$ — NS, $\|x\| \overset{?}{=} \sqrt{\langle x, x\rangle}$

It can be proved that it's possible if and only if the norm satisfies the parallelogram law. So, $C_{[a,b]}$ is not an inner product space

**Definition. Orthonormal set** — a set of points $\{\, l_1, l_2, \dots \,\}$ (may be finite):

1. $\|l_i\| = 1$

2. $l_i \perp l_j, \quad i \neq j$

Every orthonormal set is linearly independent.

**Definition.** Let $x \in X, \{\, l_i \,\}$ — ONS. Then
$\langle x, l_j\rangle$ — **Fourier coefficient**,
$\sum\limits_j \langle x, l_j\rangle l_j$ — **Fourier series** of point $x$.

Fourier series is a special case of orthogonal series.

**Definition.** $\sum\limits_j x_j$ — orthogonal series $\iff x_i \perp x_j, \quad i \neq j$

Let $\sum\limits_{j=1}^{\infty} x_j, S_m = \sum\limits_{j=1}^{m} x_j$. Then

$$\|S_m\|^2 = \langle \sum_{j=1}^{m} x_j, \sum_{j=1}^{m} x_j \rangle = \sum_{j=1}^{m} \|x_j\|^2$$

This fact allows us to effectively build the theory of orthogonal series.

An important problem is concerned with Fourier series. Let $X$ is a normed space, $Y$ is a subspace of $X$,

$$\forall x \in X \quad E_Y(x) = \rho(x, Y) = \inf_{y \in Y} \|x - y\|$$

**Definition.** $E_Y(x)$ — **best approximation** of point $x$ with points of the subspace $Y$, if $\exists y^* \in Y \quad E_y(x) = \|x - y^*\|$ — then $y^*$ is an element of best approximation.

**Theorem 1.6** (Borel)**.**
$\dim Y < +\infty \implies \forall x \in X \quad \exists y^* \in Y$ — *element of best approximation.*

*Proof.* $Y = \mathscr{L}(\underbrace{l_1, l_2, \ldots, l_n}_{\text{lin. indep.}})$

Consider $f(\alpha_1, \ldots, \alpha_n) = \left\|x - \sum\limits_{k=1}^{n} \alpha_k l_k\right\| \to \min$. By the triangle inequality for norm, $f(\bar{\alpha})$ is continous on $\mathbb{R}^n, f \geqslant 0, E_Y(x) = \inf f(\bar{\alpha})$. It is easy to find that there always is a ball $B(0, r) \subset \mathbb{R}^n$, outside of which $f > 2E_Y(x)$. So, $E_Y(x)$ is somewhere inside. But $f$ is continuous, the ball is compact, so, by the Weierstrass theorem, the minimum exists and it is located on the sphere $S(0, r)$. ∎

For abstract Fourier series the Borel theorem can be significantly strengthened by specifying the best approximation element.

**Theorem 1.7** (extreme quality of Fourier series' partial sums)**.**
$\{ e_j \}$ — *ONS in X*
$H_n = \mathscr{L}(l1, \ldots, l_n)$
$E_{H_n}(x), S_n(x) = \sum\limits_{j=1}^{n} \langle x, l_j \rangle l_j \implies E_{H_n}(x) = \|x - S_n(x)\|$

*Proof.* $y = \sum\limits_{j=1}^{n} \alpha_j l_j \in H_n$

$$\begin{aligned}\|x - y\|^2 &= \left\langle x - \sum \alpha_j l_j, x - \sum \alpha_j l_j \right\rangle = \|x\|^2 - 2\sum \alpha_j \langle x, l_j \rangle + \sum \alpha_j^2 = \\ &= \underbrace{\|x\|^2}_{\text{const}} + \sum (\alpha_j - \langle x, l_j \rangle)^2 - \underbrace{\sum \langle x, l_j \rangle^2}_{\text{const}} \to \min\end{aligned}$$

So, the sum goes to minimum when the second summand is minimal. Obviously, it's minimal when $\forall (\alpha_j - \langle x, l_j \rangle) = 0$. $E_Y(x)$ — Fourier sum. ∎

**Corollary 1.7.1** (Bessel's inequality)**.** $\sum\limits_j \langle x, l_j \rangle^2 \leqslant \|x\|^2$

*Proof.* $0 \leqslant \|x - y^*\|^2 = \|x\|^2 - \sum\limits_j \langle x, l_j \rangle^2$ ∎

**Corollary 1.7.2.** *The series of Fourier coefficients' squares always converges*