

Functional analysis course by Dodonov N.U.

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Fall 2015 — Spring 2016

# 1 Vector spaces

## §1.1 Metric spaces.

$X, \rho: X \times X \rightarrow \mathbb{R}_+$

**Definition.**  $\rho$  — **metric**

1.  $\rho(x, y) \geq 0, = 0 \iff x = y$
2.  $\rho(x, y) = \rho(y, x)$
3.  $\rho(x, y) \leq \rho(x, z) + \rho(y, z)$

**Definition.**  $(X, \rho)$  — **Metric space.**

**Definition.**  $x = \lim x_n \iff \rho(x_n, x) \rightarrow 0$

$X, \tau = \{ G \subset X \}$

1.  $\emptyset, X \in \tau$
2.  $G_\alpha \in \tau, \alpha \in \mathcal{A} \implies \bigcup_{\alpha} G_\alpha \in \tau$
3.  $G_1, \dots, G_n \in \tau \implies \bigcap_{j=1}^n G_j \in \tau$

**Definition.**  $(X, \tau)$  — **Topological space.**

$x = \lim x_n \quad \forall G \in \tau : x \in G \quad \exists N : \forall n > N \implies x_n \in G$

$G$  — open in  $\tau$

$F = X \setminus G$  — closed

**Definition.**  $B_r(a) = \{ x \mid \rho(x, a) < r \}$  — **open ball**

$\tau = \bigcup B_r(a)$

**Statement 1.1.**  $b \in B_{r_1}(a_1) \cap B_{r_2}(a_2) \implies \exists r_3 > 0 : B_{r_3}(a_3) \subset B_{r_1}(a_1) \cap B_{r_2}(a_2)$

In this sense metric space is just a special case of topological space.

**Example 1.**  $\mathbb{R}, \rho(x, y) = |x - y|$ , MS

**Example 2.**  $\bar{x} = (x_1, \dots, x_n) \in \mathbb{R}^n, \rho(\bar{x}, \bar{y}) = \sqrt{\sum_{j=1}^n (x_j - y_j)^2}$ , MS

**Example 3.**  $\bar{x} = (x_1, \dots, x_n, \dots) \in \mathbb{R}^\infty$

$\alpha \bar{x} = (\alpha x_1, \dots, \alpha x_n, \dots)$

$\bar{x} + \bar{y} = (x_1 + y_1, \dots, x_n + y_n, \dots)$

Let's define  $\lim_{m \rightarrow \infty} \bar{x}_m$

- in  $\mathbb{R}^n$ :  $\bar{x}_n \rightarrow \bar{x} \iff \forall j = 1, \dots, n \quad x_j^{(m)} \xrightarrow{m \rightarrow \infty} x_j$
- in  $\mathbb{R}^\infty$ :  $\bar{x}_m \rightarrow \bar{x} \stackrel{\text{def}}{\iff} \forall j = 1, 2, 3, \dots \quad x_j^{(m)} \xrightarrow{m \rightarrow \infty} x_j$

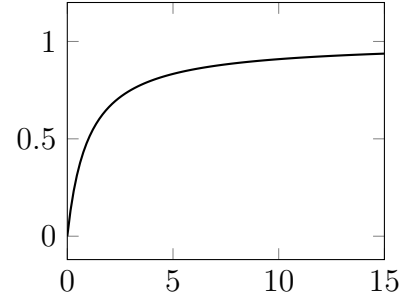
**Definition.**  $\rho(\bar{x}, \bar{y}) \stackrel{\text{def}}{=} \sum_{n=1}^{\infty} \frac{1}{2^n} \underbrace{\frac{|x_n - y_n|}{1 + |x_n - y_n|}}_{\varphi(|x_n - y_n|)}$  — **Urysohn metric.**

$$\varphi(t) = \frac{t}{1+t}$$

$$\varphi(t_1) + \varphi(t_2) \geq \varphi(t_1 + t_2)$$

$$\varphi(t_1) + \varphi(t_2) = \frac{t_1}{1+t_1} + \frac{t_2}{1+t_2} \geq \frac{t_1}{1+t_1+t_2} + \frac{t_2}{1+t_1+t_2}$$

$$\varphi(t_1) + \varphi(t_2) \geq \frac{t_1 + t_2}{1 + t_1 + t_2} = \varphi(t_1 + t_2)$$



**Statement 1.2.**  $\rho(\bar{x}_m, \bar{x}) \xrightarrow{m \rightarrow \infty} 0 \iff x_j^{(m)} \rightarrow x_j \quad \forall j$

*Proof.*

•  $\implies$

$$f(|x_k^{(n)} - x_k|) \leq 2^k \rho(x^{(n)}, x)$$

$$\text{Let } \rho(x^{(n)}, x) \leq \frac{\varepsilon}{2^k}, \text{ then } f(|x_k^{(n)} - x_k|) < \varepsilon$$

$$|x_k^{(n)} - x_k| = t = \frac{1}{1 - f(t)} - 1, \text{ then } t \rightarrow 0$$

•  $\Leftarrow$

$$\text{Let's choose } k_0 \text{ for which } \sum_{k=k_0+1}^{\infty} \frac{1}{2^k} < \varepsilon$$

$$\text{Let's choose } n_0 \text{ for which } \forall k \leq k_0, n > n_0 : |x_k^{(n)} - x_k| < \varepsilon.$$

$$\text{Then } \rho(x^{(n)}, x) < \sum_{k=1}^{k_0} \frac{\varepsilon}{2^k} + \varepsilon < 2\varepsilon$$

Letting  $\varepsilon \rightarrow 0$ , we get what we want

■

In this way  $\mathbb{R}^\infty$  is a metrizable space.

**Example 4.**  $X, \rho(x, y) \stackrel{\text{def}}{=} \begin{cases} 0, & x = y \\ 1, & x \neq y \end{cases}$  — **Discrete metric.**

$$x_n \rightarrow x, \varepsilon = \frac{1}{2}, \exists M : m > M \implies \rho(x_m, x) < \frac{1}{2} \implies \rho(x_m, x) = 0 \implies x_m = x$$

**Definition.**

$$(X, \tau); \quad \forall A \subset X;$$

$$\text{Int}(A) \stackrel{\text{def}}{=} \bigcup_{G \subset A} G \text{ is open;}$$

$$\text{Cl}(A) \stackrel{\text{def}}{=} \bigcap_{A \subset G} G \text{ is closed;}$$

$$\text{Fr}(A) \stackrel{\text{def}}{=} \text{Cl}(A) \setminus \text{Int}(A)$$

$(X, \rho)$ ; Having a metric space one can describe closure of a set.

$$\rho(x, A) \stackrel{\text{def}}{=} \inf_{a \in A} \rho(x, a)$$

$$\rho(A, B) \stackrel{\text{def}}{=} \inf_{\substack{a \in A \\ b \in B}} \rho(a, b)$$

$$\rho(x, A) = f(x), x \in X$$

**Statement 1.3.** *Function  $f(x)$  is continuous.*

*Proof.*  $\forall x, y \in X$

$$f(x) = \rho(x, A) \leq \rho(x, \alpha) \leq \rho(x, y) + \rho(y, \alpha)$$

$$\forall \varepsilon > 0 \exists \alpha_\varepsilon \in A: \rho(y, \alpha_\varepsilon) < \rho(y, A) + \varepsilon = f(y) + \varepsilon$$

$$f(x) \leq f(y) + \varepsilon + \rho(x, y), \varepsilon \rightarrow 0$$

$$\begin{cases} f(x) \leq f(y) + \rho(x, y) \\ f(y) \leq f(x) + \rho(x, y) \end{cases} \implies |f(x) - f(y)| \leq \rho(x, y)$$

■

**Statement 1.4.**  $x \in \text{Cl}(A) \iff \rho(x, A) = 0$

Let's look at the metric spaces in terms of separation of sets from each other by open sets.

$x, y$

$$r = \rho(x, y) > 0$$

$$B_{\frac{r}{3}}(x), B_{\frac{r}{3}}(y)$$

In any metric space separability axiom is true.

**Theorem 1.1.**

*Any metric space is a normal space,*

*i.e.  $\forall$  closed disjoint  $F_1, F_2 \in X$ ,  $\exists$  open disjoint  $G_1, G_2: F_j \in G_j, j = 1, 2$*

$$\text{Proof. } g(x) = \frac{\rho(x, F_1)}{\rho(x, F_1) + \rho(x, F_2)} \text{ — continuous on } X$$

$$x \in F_1, \text{Cl}(F_1) = F_1, \rho(x, F_1) = 0, g(x) = 0$$

$$x \in F_2, g(x) = 1$$

Let's look at  $(-\infty; \frac{1}{3}), (\frac{2}{3}, \infty)$  — by continuity their inverse images under  $g$  are open.

$$G_1 = g^{-1}(-\infty; \frac{1}{3})$$

$$G_2 = g^{-1}(\frac{2}{3}; \infty)$$

■

**Definition.** Metric space is **complete** if  $\rho(x_n, x_m) \rightarrow 0 \implies \exists x = \lim x_n$

$\mathbb{R}^\infty$  — complete (by completeness of the rational numbers).

In complete metric spaces the nested balls principle is true.

**Theorem 1.2.**

$X$  — complete metric space,  $\overline{V}_{r_n}$  — system of closed balls.

1.  $\overline{V}_{r_{n+1}} \subset \overline{V}_{r_n}$  — the system is nested.

2.  $r_n \rightarrow 0$

Then:  $\bigcap_n \overline{V}_{r_n} = \{a\}$

*Proof.* Let  $b_n$  be centers of  $\overline{V}_{r_n}$ ,

$$m \geq n, b_m \in \overline{V}_{r_n}, \rho(b_m, b_n) \leq r_n \rightarrow 0 \forall m \geq n$$

$$\rho(b_m, b_n) \rightarrow 0 \xrightarrow{\text{compl.}} \exists a = \lim b_n \text{ Since the balls are closed } a \in \text{every ball.}$$

$$r_n \rightarrow 0 \implies \text{there is only one common point.}$$

■

$(X, \tau)$  — topological space

$A \subset X, \tau_a = \{ G \cap A, G \in \tau \}$  — topology induced on  $A$

**Definition.**  $X$  — metric space,  $A \subset X, \text{Cl}(A) = X$

Then:  $A$  — **dense** in  $X$

If  $\text{Int}(\text{Cl}(A)) = \emptyset$   $A$  — **nowhere dense** in  $X$ .

**Note.** It is easy to understand, that in metric spaces nowhere density means the following:  
 $\forall$  ball  $V \exists V' \subset V : V'$  contains no points from  $A$ .

**Definition.**  $X$  is called **first Baire category set**, if it can be written as at most countable union of  $x_n$  each nowhere dense in  $X$ .

**Theorem 1.3** (Baire category theorem).

*Complete metric space is second Baire category set in itself.*

*Proof.* Let  $X$  be first Baire category set.

$X = \bigcup_n X_n \quad \forall \bar{V} \quad X_1$  is nowhere dense.

$\bar{V}_1 \subset \bar{V} : \bar{V}_1 \cap X_1 = \emptyset$

$X_2$  is nowhere dense  $\bar{V}_2 \subset \bar{V}_1 : \bar{V}_2 \cap X_2 = \emptyset$

$r_2 \leq \frac{r_1}{2}$

$\vdots$

$\{\bar{V}_n\}, r_n \rightarrow 0, \bigcap_n \bar{V}_n = \{a\}, X = \bigcup_n X_n, \exists n_0 : a \in X_{n_0}$

$X_{n_0} \cap \bar{V}_{n_0} = \emptyset \rightarrow \leftarrow a \in \bar{V}_{n_0}$  ■

**Corollary 1.3.1.** *Complete metric space without isolated points is uncountable.*

*Proof.* No isolated points are present  $\implies$  every point in the set is nowhere dense in it. Let  $X$  be countable:  $X = \bigcup_n \{X_n\}$ , then it is first Baire category set.  $\rightarrow \leftarrow$  ■

**Definition.**  $K$  — **compact** if

1.  $K = \text{Cl}(K)$

2.  $x_n \in K \exists n_1 < n_2 < \dots x_{n_j}$  — converges in  $X$ .

If only 2 is present, the set is called **precompact**.

**Theorem 1.4** (Hausdorff).

*Let  $X$  — metric space,  $K$  — closed in  $X$ .*

Then:  $K$  — compact  $\iff K$  — totally bounded,

*i.e.*  $\forall \varepsilon > 0 \exists a_1, \dots, a_p \in X : \forall b \in K \exists a_j : \rho(a_j, b) < \varepsilon$

$(a_1, \dots, a_p)$  — finite  $\varepsilon$  — net

*Proof.*

• Totally bounded  $\implies$  compact

$K$  is totally bounded,  $x_n \in K \quad n_1 < n_2 < \dots < n_k < \dots, x_n$  converges in  $K$

$$\varepsilon_k \downarrow \rightarrow 0 \quad \varepsilon_1 \quad K \subset \bigcup_{j=1}^p \bar{V}_j, \text{rad} = \varepsilon_1 \quad (\varepsilon_1\text{-net})$$

$n$  is finite  $\implies$  one ball will contain infinitely many  $x_n$  elements.

Let's look at  $\overline{V}_{j_0} \cap K$  — totally bounded =  $K_1$ ,  $\text{diam}(K_1) \leq 2\varepsilon_1$

$$\varepsilon_2 \quad K_1 \subset \bigcup_{j=1}^n \overline{V}'_j, \quad \text{rad} = \varepsilon_2.$$

Then one of  $\overline{V}'$  contains infinitely many elements of the sequence contained in  $K_1$ .

$$\overline{V}'_{j_0} \cap K_1 = K_2, \quad \text{diam}(K_2) \leq 2\varepsilon_2 \text{ and so on.}$$

$$K_n \supset K_{n+1} \supset K_{n+2} \supset \dots, \quad \text{diam}(K_N) \leq 2\varepsilon_n \xrightarrow{\text{by compl.}} \underbrace{\bigcap_{n=1}^{\infty} K_n}_{\text{diam}(K_n) \rightarrow 0, \{x\}} \neq \emptyset$$

Take  $x_{n_1}$  from  $K_1$ ,  $x_{n_2}$  from  $K_2 \dots$

- Compact  $\implies$  totally bounded

$K$  — compact  $\forall \varepsilon \exists$  finite  $\varepsilon$ -net?

By contradiction:  $\exists \varepsilon_0 > 0$ : finite  $\varepsilon_0$ -net is impossible to construct.

$\forall x_1 \in K \exists x_2 \in K: \rho(x_1, x_2) > \varepsilon_0$  (or else system of  $x_1$  — finite  $\varepsilon$ -net)

$\{x_1, x_2\}$  - choose  $x_3 \in K: \rho(x_3, x_i) > \varepsilon_0, i = 1, 2$  and so on.

$x_n \in K: n \neq m \rho(x_n, x_m) > \varepsilon_0$  — contains no converging subsequence  $\implies$  set is not a compact.  $\rightarrow \leftarrow$  ■

## §1.2 Normed spaces

**Definition.**  $X$  — **linear set**,  $x + y, \alpha \cdot x, \alpha \in \mathbb{R}$

The purpose of norm definition, is to construct a topology on  $X$ , so that 2 linear operations are continuous on it.

$$\varphi: X \rightarrow \mathbb{R}:$$

$$1. \varphi(x) \geq 0, = 0 \iff x = 0$$

$$2. \varphi(\alpha x) = |\alpha| \varphi(x)$$

$$3. \varphi(x + y) \leq \varphi(x) + \varphi(y)$$

**Definition.**  $\varphi$  — **norm** on  $X$ ,  $\varphi(x) = \|x\|$

$\rho(x, y) \stackrel{\text{def}}{=} \|x - y\|$  — metric on  $X$ .

**Definition.**

$(X, \|\cdot\|)$  — **normed space** — special case of metrical space.

$$x = \lim x_n \xrightarrow{\text{def}} \rho(x_n, x) \rightarrow 0 \iff \|x_n - x\| \rightarrow 0$$

**Statement 1.5.** In the topology of a normed space linear operations are continuous on  $X$ .

*Proof.*

$$\begin{aligned} 1. \quad x_n \rightarrow x, y_n \rightarrow y; \quad \|(x_n + y_n) - (x + y)\| &= \|(x_n - x) + (y_n - y)\| \leq \\ &\leq \underbrace{\|x_n - x\|}_{\downarrow 0} + \underbrace{\|y_n - y\|}_{\downarrow 0} \\ &\implies x_n + y_n \rightarrow x + y \end{aligned}$$

$$\begin{aligned}
2. \quad \alpha_n \rightarrow \alpha, \quad x_n \rightarrow x; \quad \|\alpha_n x_n - \alpha x\| &= \|(\alpha_n - \alpha)x_n + \alpha(x_n - x)\| \leq \\
&\leq \underbrace{|\alpha_n - \alpha|}_{\downarrow 0} \cdot \underbrace{\|x_n\|}_{\text{bounded}} + \underbrace{\alpha\|x_n - x\|}_{\downarrow 0}
\end{aligned}$$

$$x_n \rightarrow x \implies \|x_n\| \text{ --- bounded.}$$

$$\alpha x_n \rightarrow \alpha x$$

■

**Statement 1.6.** From the triangle inequality  $\| \|x\| - \|y\| \| \leq \|x - y\|$

$$x_n \rightarrow x \implies \|x_n\| \rightarrow \|x\|$$

Norm is continuous.

**Example 5.**  $\mathbb{R}^n$

$$1. \quad \|\bar{x}\| = \sqrt{\sum_{k=1}^n x_k^2}$$

$$2. \quad \|\bar{x}\|_1 \stackrel{\text{def}}{=} \sum_{k=1}^n |x_k|$$

$$3. \quad \|\bar{x}\|_2 \stackrel{\text{def}}{=} \max\{|x_1|, \dots, |x_n|\}$$

$$4. \quad C[a, b] \text{ --- functions continuous on } [a, b]; \quad \|f\| = \max_{x \in [a, b]} |f(x)|$$

$$5. \quad L_p(E) = \left\{ f \text{ --- measurable, } \int_E |f|^p < +\infty \right\}$$

$$p \geq 1, \quad \|f\|_p = \left( \int_E |f|^p \right)^{\frac{1}{p}}$$

Because the set of points is the same, arises the question about convergence comparison.

$$\|\cdot\|_1 \sim \|\cdot\|_2, \quad x_n \xrightarrow{\|\cdot\|_1} x \iff x_n \xrightarrow{\|\cdot\|_2} x$$

**Statement 1.7.**

$$\|\cdot\|_1 \sim \|\cdot\|_2 \iff \exists a, b > 0: \forall x \in X \implies a\|x\|_1 \leq \|x\|_2 \leq b\|x\|_1$$

**Theorem 1.5** (Riesz).

$X, \dim X < +\infty$  --- linear set.

Then: Any pair of norms in  $X$  are equivalent.

*Proof.*  $l_1, \dots, l_n$  --- linearly independent from  $X$ .  $\forall x \in X = \sum_{k=1}^n \alpha_k l_k$

$$\bar{x} \leftrightarrow (l_1, \dots, l_n) = \bar{l} \in \mathbb{R}^n$$

Let  $\|\cdot\|$  --- some norm in  $X$ .

$$\begin{aligned}
\|x\| &\leq \sum_{k=1}^n \|l_k\| |\alpha_k| \stackrel{\text{Cauchy}}{\leq} \underbrace{\sqrt{\sum_{k=1}^n \|l_k\|^2}}_{\text{const}(B), B\text{-basis}} \underbrace{\sqrt{\sum_{k=1}^n |\alpha_k|^2}}_{\|\bar{\alpha}\| = \|x\|_1} \\
\|x\|_1 &= \sqrt{\sum_{k=1}^n \|\alpha_k\|^2}, \quad x = \sum \alpha_k l_k
\end{aligned}$$

$$\|x\| \leq b\|x\|_1$$

$$? \exists a > 0: a\|x\|_1 \leq \|x\| \implies \|\cdot\| \sim \|\cdot\|_1$$

$$\text{Let } f(\alpha_1, \dots, \alpha_n) = \left\| \sum_{k=1}^n \alpha_k l_k \right\|$$

$$\begin{aligned} |f(\bar{\alpha} + \Delta\bar{\alpha}) - f(\bar{\alpha})| &= \left| \left\| \sum_{k=1}^n \alpha_k l_k + \sum_{k=1}^n \Delta\alpha_k l_k \right\| - \left\| \sum_{k=1}^n \alpha_k l_k \right\| \right| \leq \\ &= \left\| \sum_{k=1}^n \Delta\alpha_k l_k \right\| \leq \underbrace{\sum_{k=1}^n \|\alpha_k\| \|\Delta\alpha_k\|}_{0, \Delta\alpha_k \rightarrow 0} \implies f \text{ is continuous on } \mathbb{R}^n \end{aligned}$$

$$S_1 = \left\{ \sum_{k=1}^n |\alpha_k|^2 = 1 \right\} \subset \mathbb{R}^n, f \text{ — continuous on } S_1, S_1 \text{ — compact, } \bar{\alpha}^* \in S_1$$

By Weierstrass theorem there exists a point  $\alpha^* \in S_1$  on a sphere, in which function  $f$  achieves its minimum  $\implies \forall \alpha \in S_1 \ f(\bar{\alpha}^*) \leq f(\bar{\alpha})$

$$\text{If } f(\bar{\alpha}^*) = 0, \text{ then } \left\| \sum_{k=1}^n \alpha_k^* l_k \right\| = 0 \implies \sum_{k=1}^n \alpha_k^* l_k = 0, \bar{\alpha}^* \in S_1,$$

but  $l_1 \dots l_n$  are linearly independent  $\rightarrow \leftarrow \implies \min_{S_1} f = m > 0$

$$\begin{aligned} \|x\| &= \left\| \sum_{k=1}^n \alpha_k l_k \right\| = f(\bar{\alpha}) = \sqrt{\sum_{k=1}^n \alpha_k^2} \cdot \left\| \sum_{k=1}^n \frac{\alpha_k}{\sqrt{\sum_{k=1}^n \alpha_k^2}} \cdot l_k \right\| \geq, \bar{\beta} = (\beta_1 \dots \beta_n) \in S_1 \\ &\geq m \cdot \|x\|_1, a = m \quad \blacksquare \end{aligned}$$

**Corollary 1.5.1.**  $X = NS, Y \subset X, \dim Y < +\infty \implies Y = \text{Cl}(Y)$

**Note.** Functional analysis differentiates between linear subset (set of points, closed by addition and scalar multiplication) and linear subspace (closed linear subset).

$$\text{Proof. } Y = \alpha(l_1, \dots, l_n) = \left\{ \sum_{i=1}^n \alpha_i l_i \mid l_1, \dots, l_n \text{ — lin. indep.} \right\}$$

$$y_m \in Y, y_m \rightarrow y \text{ in } X \implies y \in Y?$$

$$\|y_m - y\| \rightarrow 0 \implies \|y_m - y_p\| \rightarrow 0, m, p \rightarrow \infty$$

$$\|y\|, y \in Y.$$

By Riesz theorem all norm in  $Y$  are equivalent.

$$y = \sum_{j=1}^n \alpha_j l_j, \|y\|_0 = \sqrt{\sum_{j=1}^n \alpha_j^2} \text{ — some norm (by linear independance).}$$

$$\text{By Riesz theorem } \|y\| \sim \|y\|_0$$

$$\underbrace{\|y_m - y_p\|}_{\in Y} \rightarrow 0 \implies \|y_m - y_p\|_0 \rightarrow 0$$

$$\bar{\alpha} = (\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n \ y_m = \sum_{i=1}^n \alpha_i^{(m)} l_i$$

$$|\alpha_i^{(m)} - \alpha_i^{(l)}| \rightarrow 0 \ \forall i = 1, \dots, n; \ \bar{\alpha} = (\alpha_1^{(m)}, \dots, \alpha_n^{(m)}) \rightarrow \alpha^* = (\alpha_1^*, \dots, \alpha_n^*)$$

$$y^* = \sum_{i=1}^n \alpha_i^* l_i \in Y, \|y_m - y\| \rightarrow 0, y = y^* \implies y \in Y \quad \blacksquare$$

**Definition.** If normed space is complete, then it is called **B-space** or **Banach space**.

**Example 6.**  $C[a, b]$  — functions continuous on  $[a, b]$ .



**Example 7.** Lebesgue space,  $p \geq 1$ ,  $L_p(E) = \left\{ f \text{ is measurable on } E, \int_E |f|^p < +\infty \right\}$ .

If  $X$  — Banach space,

$$\sum_{n=1}^{\infty} x_n = \lim_{n \rightarrow \infty} \sum_{k=1}^n x_k, \quad \sum_{n=1}^{\infty} \|x_n\| < +\infty$$

$$\|S_n - S_m\| = \left\| \sum_{k=m+1}^n x_k \right\| \leq \sum_{k=m+1}^n \|x_k\| \xrightarrow{n, m \rightarrow \infty} 0$$

$$\implies \|S_n - S_m\| \rightarrow 0 \implies \exists \lim_{n \rightarrow \infty} S_n, \sum_{k=1}^n x_k \text{ — converges.}$$

In Banach spaces works the theory of absolute convergence of numerical series.

**Lemma 1** (Riesz).  $Y$  — eigen subspace of  $X$  — normed space.  $\forall \varepsilon \in (0, 1) \exists z_\varepsilon \in X$ :

1.  $\|z_\varepsilon\| = 1$
2.  $\rho(z_\varepsilon, Y) > 1 - \varepsilon$

*Proof.*  $\exists x \notin Y$   $d = \rho(x, Y)$ ,  $d = 0 \exists y_n \in Y: \|x - y_n\| < \frac{1}{n}$ ,  $n \rightarrow \infty$ ,  $y_n \rightarrow x$

$Y = \text{Cl}(Y) \implies x \in Y \rightarrow \leftarrow x \notin Y, d > 0$

$\forall \varepsilon \in (0, 1) \frac{1}{1 - \varepsilon} > 1 \exists y_\varepsilon \in Y: \|x - y_\varepsilon\| < \frac{1}{1 - \varepsilon} d$

$$z_\varepsilon = \frac{x - y_\varepsilon}{\|x - y_\varepsilon\|}, \quad \|z_\varepsilon\| = 1, \quad \forall y \in Y \quad \|z_\varepsilon - y\| = \left\| \frac{x - y_\varepsilon}{\|x - y_\varepsilon\|} \right\| = \frac{\overbrace{\|x - (y_\varepsilon + \|x - y_\varepsilon\| \cdot y)\|}^{\in Y}}{\|x - y_\varepsilon\|} \geq d > 1 - \varepsilon$$

■

**Corollary 1.5.2.**  $\dim X = +\infty$ ,  $S$  — sphere in  $X$ ,  $r_S = 1 \{x \mid \|x\| = 1\} \implies S$  — not a compact.

*Proof.*  $\forall x_1 \in S$ ,  $Y_1 = \alpha\{x_1\}$  — finite dimensional linear set.  $\implies$  closed in  $X \implies Y_1$  — subspace.  $\dim X = +\infty \implies Y_1$  — eigen subspace.

Then by the Riesz lemma:

$$\exists x_2 \in S: \|x_2 - x_1\| > \frac{1}{2}$$

$$Y_2 = \alpha\{x_1, x_2\} \exists x_3 \in S: \|x_3 - x_j\| > \frac{1}{2}, \quad j = 1, 2$$

Continue by induction. Because  $\dim X = +\infty$  the process will never stop.

$x_n \in S: \|x_n - x_m\| > \frac{1}{2}$ ,  $n \neq m$  — obviously we cannot extract converging subsequence.  $\implies S$  — not a compact. ■

### §1.3 Inner product (unitary) spaces

**Definition.**  $X$  — linear space.

$\varphi: X \times X \rightarrow \mathbb{R}$

1.  $\varphi(x, x) \geq 0$ ,  $\varphi(x, x) = 0 \iff x = 0$
2.  $\varphi(x, y) = \varphi(y, x)$

$$3. \varphi(\alpha x + \beta y, z) = \alpha \varphi(x, z) + \beta \varphi(y, z)$$

$\varphi$  — **inner product**.

$$\varphi(x, y) = \langle x, y \rangle$$

**Definition.**  $(X, \langle \cdot, \cdot \rangle)$  — **inner product space**.

**Example 8.**  $\mathbb{R}^n, \langle \bar{x}, \bar{y} \rangle = \sum_{j=1}^n x_j y_j$

**Statement 1.8** (Schwarz).  $\forall x, y \in X \quad |\langle x, y \rangle| \leq \sqrt{\langle x, x \rangle} \cdot \sqrt{\langle y, y \rangle}$

*Proof.*  $\lambda \in \mathbb{R}$

$$f(\lambda) = \langle \lambda x + y, \lambda x + y \rangle \geq 0$$

||

$$\lambda^2 \langle x, x \rangle + 2\lambda \langle x, y \rangle + \langle y, y \rangle$$

$$D = 4\langle x, y \rangle^2 - 4\langle x, x \rangle \cdot \langle y, y \rangle \leq 0 \quad \blacksquare$$

**Corollary 1.5.3** (Cauchy inequality for sums). Consider  $X = \mathbb{R}^n, \|x\| \stackrel{\text{def}}{=} \sqrt{\langle x, x \rangle}$ . Then

$$\|x + y\|^2 = \langle x + y, x + y \rangle = \|x\|^2 + 2 \cdot \underbrace{\langle x, y \rangle}_{\leq \|x\| \cdot \|y\|} + \|y\|^2 \leq (\|x\| + \|y\|)^2$$

Any inner product space is a special case of a normed space. The specifics is that we can measure the angles between points:

$$x \perp y \iff \langle x, y \rangle = 0$$

In this case the Pythagoras theorem takes place:

$$\|x + y\|^2 = \|x\|^2 + \|y\|^2$$

In inner product spaces the parallelogram equality plays a significant part:

$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2 \quad \forall x, y \in X$$

$$\langle x, y \rangle \mapsto \|x\| = \sqrt{\langle x, x \rangle}$$

Let  $X \text{ — NP}, \|x\| \stackrel{?}{=} \sqrt{\langle x, x \rangle}$

It can be proved that it's possible iff the norm complies to the parallelogram equality. So,  $C_{[a,b]}$  is not an inner product space

**Definition. Orthonormal set** is a set of points  $\{l_1, l_2, \dots\}$  (may be finite):

$$1. \|l_i\| = 1$$

$$2. l_i \perp l_j, \quad i \neq j$$

Every orthonormal set is linearly independent.

**Definition.** Let  $x \in X, \{l_i\}$  — ONS. Then

$\langle x, l_j \rangle$  — **Fourier coefficient**,

$\sum_j \langle x, l_j \rangle l_j$  — **Fourier series** of point  $x$ .

Fourier series is a special case of orthogonal series.

**Definition.**  $\sum_j x_j$  — orthogonal series  $\iff x_i \perp x_j, \quad i \neq j$

Let  $\sum_{j=1}^{\infty} x_j, S_m = \sum_{j=1}^m x_j$ . Then

$$\|S_m\|^2 = \left\langle \sum_{j=1}^m x_j, \sum_{j=1}^m x_j \right\rangle = \sum_{j=1}^m \|x_j\|^2$$

This fact lets us build the orthogonal series theory effectively.

An important problem is concerned with Fourier series. Let  $X$  is a normed space,  $Y$  is a subspace of  $X$ ,

$$\forall x \in X \quad E_Y(x) = \rho(x, Y) = \inf_{y \in Y} \|x - y\|$$

**Definition.**  $E_Y(x)$  is **best approximation** of point  $x$  with points of the subspace  $Y$ , if  $\exists y^* \in Y \quad E_Y(x) = \|x - y^*\|$  — then  $y^*$  is an element of best approximation.

**Theorem 1.6** (Borel).

$\dim Y < +\infty \implies \forall x \in X \quad \exists y^* \in Y$  — *element of best approximation*.

*Proof.*  $Y = \mathcal{L}(\underbrace{l_1, l_2, \dots, l_n}_{\text{lin. indep.}})$

Consider  $f(\alpha_1, \dots, \alpha_n) = \|x - \sum_{k=1}^n \alpha_k l_k\| \rightarrow \min$ . By the triangle inequality for norm,  $f(\bar{\alpha})$  is continuous on  $\mathbb{R}^n, f \geq 0, E_Y(x) = \inf f(\bar{\alpha})$ . Easy to find that there always is a ball  $B(0, r) \subset \mathbb{R}^n$ , outside which  $f > 2E_Y(x)$ . So,  $E_Y(x)$  is somewhere inside. But  $f$  is continuous, the ball is compact, so, by the Weierstrass theorem, the minimum exists and is on the sphere. ■

For abstract Fourier series the Borel theorem can be significantly reinforced by specifying the best approximation element.

**Theorem 1.7** (extremal quality of Fourier series' partial sums).

$\{e_j\}$  — ONS in  $X$

$H_n = \mathcal{L}(l_1, \dots, l_n)$

$$E_{H_n}(x), S_n(x) = \sum_{j=1}^n \langle x, l_j \rangle l_j \implies E_{H_n}(x) = \|x - S_n(x)\|$$

*Proof.*  $y = \sum_{j=1}^n \alpha_j l_j \in H_n$

$$\begin{aligned} \|x - y\|^2 &= \left\langle x - \sum \alpha_j l_j, x - \sum \alpha_j l_j \right\rangle = \|x\|^2 - 2 \sum \alpha_j \langle x, l_j \rangle + \sum \alpha_j^2 = \\ &= \underbrace{\|x\|^2}_{\text{const}} + \sum (\alpha_j - \langle x, l_j \rangle)^2 - \underbrace{\sum \langle x, l_j \rangle^2}_{\text{const}} \rightarrow \min \end{aligned}$$

So, the sum goes to minimum when the second summand is minimal. Obviously, it's minimal when  $\forall (\alpha_j - \langle x, l_j \rangle) = 0$ .  $E_Y(x)$  — Fourier sum. ■

**Corollary 1.7.1** (Bessel's inequality).  $\sum_j \langle x, l_j \rangle^2 \leq \|x\|^2$

*Proof.*  $0 \leq \|x - y^*\|^2 = \|x\|^2 - \sum_j \langle x, l_j \rangle^2$  ■

**Corollary 1.7.2.** *The series of Fourier coefficients' squares always converges*