

Functional analysis course by Dodonov N.U.

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1 Vector spaces

§1.1 Metric spaces.

$X, \rho: X \times X \rightarrow \mathbb{R}_+$

Definition. ρ — **metric**

1. $\rho(x, y) \geq 0, = 0 \iff x = y$
2. $\rho(x, y) = \rho(y, x)$
3. $\rho(x, y) \leq \rho(x, z) + \rho(y, z)$

Definition. (X, ρ) — **Metric space.**

Definition. $x = \lim x_n \iff \rho(x_n, x) \rightarrow 0$

$X, \tau = \{G \subset X\}$

1. $\emptyset, X \in \tau$
2. $G_\alpha \in \tau, \alpha \in \mathcal{A} \implies \bigcup_{\alpha} G_\alpha \in \tau$
3. $G_1, \dots, G_n \in \tau \implies \bigcap_{j=1}^n G_j \in \tau$

Definition. (X, τ) — **Topological space.**

$x = \lim x_n \quad \forall G \in \tau : x \in G \quad \exists N : \forall n > N \implies x_n \in G$

G — open in τ

$F = X \setminus G$ — closed

Definition. $B_r(a) = \{x \mid \rho(x, a) < r\}$ — **open ball**

$\tau = \bigcup B_r(a)$

Statement 1.1. $b \in B_{r_1}(a_1) \cap B_{r_2}(a_2) \implies \exists r_3 > 0 : B_{r_3}(a_3) \subset B_{r_1}(a_1) \cap B_{r_2}(a_2)$

In this sense metric space is just a special case of topological space.

Example 1. $\mathbb{R}, \rho(x, y) = |x - y|$, MS

Example 2. $\bar{x} = (x_1, \dots, x_n) \in \mathbb{R}^n, \rho(\bar{x}, \bar{y}) = \sqrt{\sum_{j=1}^n (x_j - y_j)^2}$, MS

Example 3. $\bar{x} = (x_1, \dots, x_n, \dots) \in \mathbb{R}^\infty$

$\alpha \bar{x} = (\alpha x_1, \dots, \alpha x_n, \dots)$

$\bar{x} + \bar{y} = (x_1 + y_1, \dots, x_n + y_n, \dots)$

Let's define $\lim_{m \rightarrow \infty} \bar{x}_m$

- in \mathbb{R}^n :

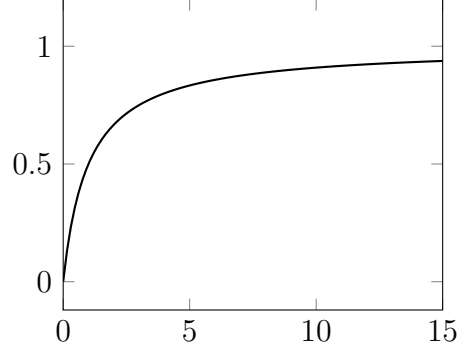
$$\bar{x}_n \rightarrow \bar{x} \iff \forall j = 1, \dots, n \quad x_j^{(m)} \xrightarrow{m \rightarrow \infty} x_j$$

- in \mathbb{R}^∞ :

$$\bar{x}_m \rightarrow \bar{x} \stackrel{\text{def}}{\iff} \forall j = 1, 2, 3, \dots \quad x_j^{(m)} \xrightarrow{m \rightarrow \infty} x_j$$

Definition. $\rho(\bar{x}, \bar{y}) \stackrel{\text{def}}{=} \sum_{n=1}^{\infty} \frac{1}{2^n} \underbrace{\frac{|x_n - y_n|}{1 + |x_n - y_n|}}_{\varphi(|x_n - y_n|)}$ — **Urysohn metric.**

$$\begin{aligned} \varphi(t) &= \frac{t}{1+t} \\ \varphi(t_1 + t_2) &\leq \varphi(t_1) + \varphi(t_2) \end{aligned}$$



$\varphi(t)$ is an upper convex function

Statement 1.2. $\rho(\bar{x}_m, \bar{x}) \xrightarrow{m \rightarrow \infty} 0 \iff x_j^{(m)} \rightarrow x_j \forall j$

Proof.

- \Rightarrow

$$\begin{aligned} f(|x_k^{(n)} - x_k|) &\leq 2^k \rho(x^{(n)}, x) \\ \text{Let } \rho(x^{(n)}, x) &\leq \frac{\varepsilon}{2^k}, \text{ then } f(|x_k^{(n)} - x_k|) < \varepsilon \\ |x_k^{(n)} - x_k| = t &= \frac{1}{1-f(t)} - 1, \text{ then } t \rightarrow 0 \end{aligned}$$

- \Leftarrow

$$\begin{aligned} \text{Lets take such } k_0 &\text{ that } \sum_{k=k_0+1}^{\infty} \frac{1}{2^k} < \varepsilon \\ \text{Lets take such } n_0 &\text{ that } \forall k \leq k_0, n > n_0 : |x_k^{(n)} - x_k| < \varepsilon. \\ \text{Then } \rho(x^{(n)}, x) &< \sum_{k=1}^{k_0} \frac{\varepsilon}{2^k} + \varepsilon < 2\varepsilon \\ \text{Letting } x &\rightarrow 0, \text{ we get what we want} \end{aligned}$$

■

In this way \mathbb{R}^∞ is a metrizable space.

Example 4. $X, \rho(x, y) \stackrel{\text{def}}{=} \begin{cases} 0, & x = y \\ 1, & x \neq y \end{cases}$ — **Discrete metric.**

$$\begin{aligned} x_n \rightarrow x, \varepsilon = \frac{1}{2}, \exists M : m > M &\implies \rho(x_m, x) < \frac{1}{2} \implies \\ \rho(x_m, x) = 0 &\implies x_m = x \end{aligned}$$

Definition. $(X, \tau); \forall A \subset X;$

$$\text{Int}(A) \stackrel{\text{def}}{=} \bigcup_{G \subset A} G - \text{open};$$

$$\text{Cl}(A) \stackrel{\text{def}}{=} \bigcap_{A \subset G} G - \text{closed};$$

$$\text{Fr}(A) = \text{Cl}(A) \setminus \text{Int}(A)$$

(X, ρ) ; Having a metric space one can describe closure of a set.

$$\rho(x, A) \stackrel{\text{def}}{=} \inf_{a \in A} \rho(x, a)$$

$$\rho(A, B) = \inf_{\substack{a \in A \\ b \in B}} \rho(a, b)$$

$$\rho(x, A) = f(x), x \in X$$

Statement 1.3. *Function $f(x)$ is continuous.*

Proof. $\forall x, y \in X$

$$f(x) = \rho(x, A) \leq \rho(x, \alpha) \leq \rho(x, y) + \rho(y, \alpha)$$

$$\forall \varepsilon > 0 \exists \alpha_\varepsilon \in A : \rho(y, \alpha_\varepsilon) < \rho(y, A) + \varepsilon = f(y) + \varepsilon$$

$$f(x) \leq f(y) + \varepsilon + \rho(x, y), \varepsilon \rightarrow 0$$

$$\begin{cases} f(x) \leq f(y) + \rho(x, y) \\ f(y) \leq f(x) + \rho(x, y) \end{cases} \implies |f(x) - f(y)| \leq \rho(x, y) \quad \blacksquare$$

Statement 1.4. $x \in \text{Cl}(A) \iff \rho(x, A) = 0$

Let's look at the metric spaces in terms of separation of sets from each other by open sets.

x, y

$$r = \rho(x, y) > 0$$

$$B_{\frac{r}{3}}(x), B_{\frac{r}{3}}(y)$$

In any metric space separability axiom is true.

Theorem 1.1.

Any metric space is a normal space,

i.e. \forall closed disjoint $F_1, F_2 \in X$, \exists open disjoint G_1, G_2 : $F_j \in G_j$, $j = 1, 2$

$$\text{Proof. } g(x) = \frac{\rho(x, F_1)}{\rho(x, F_1) + \rho(x, F_2)} \text{ — continuous on } X$$

$$x \in F_1, \text{Cl}(F_1) = F_1, \rho(x, F_1) = 0, g(x) = 0$$

$$x \in F_2, g(x) = 1$$

Let's look at $(-\infty; \frac{1}{3}), (\frac{2}{3}, \infty)$ — by continuity their inverse images under g are open.

$$G_1 = g^{-1}(-\infty; \frac{1}{3})$$

$$G_2 = g^{-1}(\frac{2}{3}; \infty) \quad \blacksquare$$

Definition. Metric space is **complete** if $\rho(x_n, x_m) \rightarrow 0 \implies \exists x = \lim x_n$

\mathbb{R}^∞ — complete (by completeness of the rational numbers).

In complete metric spaces the nested balls principle is true.

Theorem 1.2.

X — complete metric space, \bar{V}_{r_n} — system of closed balls.

1. $\bar{V}_{r_{n+1}} \subset \bar{V}_{r_n}$ — the system is nested.

2. $r_n \rightarrow 0$

$$\text{Then: } \bigcap_n \bar{V}_{r_n} = \{a\}$$

Proof. Let b_n be centers of \bar{V}_{r_n} ,

$$m \geq n, b_m \in \bar{V}_{r_n}, \rho(b_m, b_n) \leq r_n \rightarrow 0 \forall m \geq n$$

$$\rho(b_m, b_n) \rightarrow 0 \xrightarrow{\text{compl.}} \exists a = \lim b_n \text{ Since the balls are closed } a \in \text{every ball.}$$

$$r_n \rightarrow 0 \implies \text{there is only one common point.} \quad \blacksquare$$

(X, τ) — topological space

$A \subset X, \tau_a = \{ G \cap A, G \in \tau \}$ — topology induced on A

Definition. X — metric space, $A \subset X, \text{Cl}(A) = X$

Then: A — **dense** in X

If $\text{Int}(\text{Cl}(A)) = \emptyset$ A — **nowhere dense** in X .

Note. It is easy to understand, that in metric spaces nowhere density means the following:
 \forall ball $V \exists V' \subset V : V'$ contains no points from A .

Definition. X is called **first Baire category set**, if it can be written as at most countable union of x_n each nowhere dense in X .

Theorem 1.3 (Baire category theorem).

Complete metric space is second Baire category set in itself.

Proof. Let X be first Baire category set.

$X = \bigcup_n X_n \quad \forall \bar{V} \quad X_1$ is nowhere dense.

$\bar{V}_1 \subset \bar{V} : \bar{V}_1 \cap X_1 = \emptyset$

X_2 is nowhere dense $\bar{V}_2 \subset \bar{V}_1 : \bar{V}_2 \cap X_2 = \emptyset$

$r_2 \leq \frac{r_1}{2}$

\vdots

$\{\bar{V}_n\}, r_n \rightarrow 0, \bigcap_n \bar{V}_n = \{a\}, X = \bigcup_n X_n, \exists n_0 : a \in X_{n_0}$

$X_{n_0} \cap \bar{V}_{n_0} = \emptyset \rightarrow \leftarrow a \in \bar{V}_{n_0}$ ■

Corollary 1.3.1. *Complete metric space without isolated points is uncountable.*

Proof. No isolated points are present \implies every point in the set is nowhere dense in it. Let X be countable: $X = \bigcup_n \{X_n\}$, then it is first Baire category set. $\rightarrow \leftarrow$ ■

Definition. K — **compact** if

1. $K = \text{Cl}(K)$

2. $x_n \in K \exists n_1 < n_2 < \dots x_{n_j}$ — converges in X .

If only 2 is present, the set is called **precompact**.

Theorem 1.4 (Hausdorff).

Let X — metric space, K — closed in X .

Then: K — compact $\iff K$ — totally bounded,

i.e. $\forall \varepsilon > 0 \exists a_1, \dots, a_p \in X : \forall b \in K \exists a_j : \rho(a_j, b) < \varepsilon$

(a_1, \dots, a_p) — finite ε — net

Proof.

• Totally bounded \implies compact

K is totally bounded, $x_n \in K \quad n_1 < n_2 < \dots < n_k < \dots, x_n$ converges in K

$$\varepsilon_k \downarrow \rightarrow 0 \quad \varepsilon_1 \quad K \subset \bigcup_{j=1}^p \bar{V}_j, \text{rad} = \varepsilon_1 \quad (\varepsilon_1\text{-net})$$

n is finite \implies one ball will contain infinitely many x_n elements.

Let's look at $\overline{V}_{j_0} \cap K$ — totally bounded = K_1 , $\text{diam}(K_1) \leq 2\varepsilon_1$

$$\varepsilon_2 \quad K_1 \subset \bigcup_{j=1}^n \overline{V}'_j, \quad \text{rad} = \varepsilon_2.$$

Then one of \overline{V}' contains infinitely many elements of the sequence contained in K_1 .

$$\overline{V}'_{j_0} \cap K_1 = K_2, \quad \text{diam}(K_2) \leq 2\varepsilon_2 \text{ and so on.}$$

$$K_n \supset K_{n+1} \supset K_{n+2} \supset \dots, \quad \text{diam}(K_N) \leq 2\varepsilon_n \xrightarrow{\text{by compl.}} \underbrace{\bigcap_{n=1}^{\infty} K_n}_{\text{diam}(K_n) \rightarrow 0, \{x\}} \neq \emptyset$$

Take x_{n_1} from K_1 , x_{n_2} from $K_2 \dots$

- Compact \implies totally bounded

K — compact $\forall \varepsilon \exists$ finite ε -net?

By contradiction: $\exists \varepsilon_0 > 0$: finite ε_0 -net is impossible to construct.

$\forall x_1 \in K \exists x_2 \in K: \rho(x_1, x_2) > \varepsilon_0$ (or else system of x_1 — finite ε -net)

$\{x_1, x_2\}$ - choose $x_3 \in K: \rho(x_3, x_i) > \varepsilon_0, i = 1, 2$ and so on.

$x_n \in K: n \neq m \rho(x_n, x_m) > \varepsilon_0$ — contains no converging subsequence \implies set is not a compact. $\rightarrow \leftarrow$ ■

§1.2 Normed spaces

Definition. X — **linear set**, $x + y, \alpha \cdot x, \alpha \in \mathbb{R}$

The purpose of norm definition, is to construct a topology on X , so that 2 linear operations are continuous on it.

$$\varphi: X \rightarrow \mathbb{R}:$$

$$1. \quad \varphi(x) \geq 0, \quad = 0 \iff x = 0$$

$$2. \quad \varphi(\alpha x) = |\alpha| \varphi(x)$$

$$3. \quad \varphi(x + y) \leq \varphi(x) + \varphi(y)$$

Definition. φ — **norm** on X , $\varphi(x) = \|x\|$

$$\rho(x, y) \stackrel{\text{def}}{=} \|x - y\| \text{ — metric on } X.$$

Definition.

$(X, \|\cdot\|)$ — **normed space** — special case of metrical space.

$$x = \lim x_n \stackrel{\text{def}}{\implies} \rho(x_n, x) \rightarrow 0 \iff \|x_n - x\| \rightarrow 0$$

Statement 1.5. In the topology of a normed space linear operations are continuous on X .

Proof.

$$\begin{aligned} 1. \quad x_n \rightarrow x, \quad y_n \rightarrow y; \quad \|(x_n + y_n) - (x + y)\| &= \|(x_n - x) + (y_n - y)\| \leq \\ &\leq \underbrace{\|x_n - x\|}_{\downarrow 0} + \underbrace{\|y_n - y\|}_{\downarrow 0} \\ &\implies x_n + y_n \rightarrow x + y \end{aligned}$$

$$\begin{aligned}
2. \quad \alpha_n \rightarrow \alpha, \quad x_n \rightarrow x; \quad \|\alpha_n x_n - \alpha x\| &= \|(\alpha_n - \alpha)x_n + \alpha(x_n - x)\| \leq \\
&\leq \underbrace{|\alpha_n - \alpha|}_{\downarrow 0} \cdot \underbrace{\|x_n\|}_{\text{bounded}} + \underbrace{\alpha\|x_n - x\|}_{\downarrow 0}
\end{aligned}$$

$$x_n \rightarrow x \implies \|x_n\| \text{ --- bounded.}$$

$$\alpha x_n \rightarrow \alpha x$$

■

Statement 1.6. From the triangle inequality $\| \|x\| - \|y\| \| \leq \|x - y\|$

$$x_n \rightarrow x \implies \|x_n\| \rightarrow \|x\|$$

Norm is continuous.

Example 5. \mathbb{R}^n

$$1. \quad \|\bar{x}\| = \sqrt{\sum_{k=1}^n x_k^2}$$

$$2. \quad \|\bar{x}\|_1 \stackrel{\text{def}}{=} \sum_{k=1}^n |x_k|$$

$$3. \quad \|\bar{x}\|_2 \stackrel{\text{def}}{=} \max\{|x_1|, \dots, |x_n|\}$$

$$4. \quad C[a, b] \text{ --- functions continuous on } [a, b]; \quad \|f\| = \max_{x \in [a, b]} |f(x)|$$

$$5. \quad L_p(E) = \left\{ f \text{ --- measurable, } \int_E |f|^p < +\infty \right\}$$

$$p \geq 1, \quad \|f\|_p = \left(\int_E |f|^p \right)^{\frac{1}{p}}$$

Because the set of points is the same, arises the question about convergence comparison.

$$\|\cdot\|_1 \sim \|\cdot\|_2, \quad x_n \xrightarrow{\|\cdot\|_1} x \iff x_n \xrightarrow{\|\cdot\|_2} x$$

Statement 1.7.

$$\|\cdot\|_1 \sim \|\cdot\|_2 \iff \exists a, b > 0: \forall x \in X \implies a\|x\|_1 \leq \|x\|_2 \leq b\|x\|_1$$

Theorem 1.5 (Riesz).

$X, \dim X < +\infty$ --- linear set.

Then: Any pair of norms in X are equivalent.

Proof. l_1, \dots, l_n --- linearly independent from X . $\forall x \in X = \sum_{k=1}^n \alpha_k l_k$

$$\bar{x} \leftrightarrow (l_1, \dots, l_n) = \bar{l} \in \mathbb{R}^n$$

Let $\|\cdot\|$ --- some norm in X .

$$\begin{aligned}
\|x\| &\leq \sum_{k=1}^n \|l_k\| |\alpha_k| \underset{\text{Cauchy}}{\leq} \underbrace{\sqrt{\sum_{k=1}^n \|l_k\|^2}}_{\text{const}(B), B\text{-basis}} \underbrace{\sqrt{\sum_{k=1}^n |\alpha_k|^2}}_{\|\bar{\alpha}\| = \|x\|_1}
\end{aligned}$$

$$\|x\|_1 = \sqrt{\sum_{k=1}^n \|\alpha_k\|^2}, \quad x = \sum \alpha_k l_k$$

$$\|x\| \leq b\|x\|_1$$

$$?\exists a > 0: a\|x\|_1 \leq \|x\| \implies \|\cdot\| \sim \|\cdot\|_1$$

Let $f(\alpha_1, \dots, \alpha_n) = \left\| \sum_{k=1}^n \alpha_k l_k \right\|$

$$|f(\bar{\alpha} + \Delta \bar{\alpha}) - f(\bar{\alpha})| = \left| \left\| \sum_{k=1}^n \alpha_k l_k + \sum_{k=1}^n \Delta \alpha_k l_k \right\| - \left\| \sum_{k=1}^n \alpha_k l_k \right\| \right| \leq \left\| \sum_{k=1}^n \Delta \alpha_k l_k \right\| \leq \underbrace{\sum_{k=1}^n \|l_k\| |\Delta \alpha_k|}_{0, \Delta \alpha_k \rightarrow 0} \implies f \text{ is continuous on } \mathbb{R}^n$$

$$S_1 = \left\{ \sum_{k=1}^n |\alpha_k|^2 = 1 \right\} \subset \mathbb{R}^m, f \text{ — continuous on } S_1, S_1 \text{ — compact, } \bar{\alpha}^* \in S_1$$

By Weierstrass theorem there exists a point $\alpha^* \in S_1$ on a sphere, in which function f achieves its minimum $\implies \forall \alpha \in S_1 f(\bar{\alpha}^*) \leq f(\bar{\alpha})$

$$\text{If } f(\bar{\alpha}^*) = 0, \text{ then } \left\| \sum_{k=1}^n \alpha_k^* l_k \right\| = 0 \implies \sum_{k=1}^n \alpha_k^* l_k = 0, \bar{\alpha}^* \in S_1,$$

but $l_1 \dots l_n$ are linearly independent $\rightarrow \leftarrow \implies \min_{S_1} f = m > 0$

$$\|x\| = \left\| \sum_{k=1}^n \alpha_k l_k \right\| = f(\bar{\alpha}) = \sqrt{\sum_{k=1}^n \alpha_k^2} \cdot \left\| \sum_{\beta_k} \frac{\alpha_k}{\sqrt{\sum_{k=1}^n \alpha_k^2}} \cdot l_k \right\| \geq, \bar{\beta} = (\beta_1 \dots \beta_n) \in S_1$$

$$\geq m \cdot \|x\|_1, a = m \quad \blacksquare$$

Corollary 1.5.1. $X = NS, Y \subset X, \dim Y < +\infty \implies Y = \text{Cl}(Y)$

Note. Functional analysis differentiates between linear subset (set of points, closed by addition and scalar multiplication) and linear subspace (closed linear subset).

$$\text{Proof. } Y = \alpha(l_1, \dots, l_n) = \left\{ \sum_{i=1}^n \alpha_i l_i \mid l_1, \dots, l_n \text{ — lin. indep.} \right\}$$

$$y_m \in Y, y_m \rightarrow y \text{ in } X \implies y \in Y?$$

$$\|y_m - y\| \rightarrow 0 \implies \|y_m - y_p\| \rightarrow 0, m, p \rightarrow \infty$$

$$\|y\|, y \in Y.$$

By Riesz theorem all norm in Y are equivalent.

$$y = \sum_{j=1}^n \alpha_j l_j, \|y\|_0 = \sqrt{\sum_{j=1}^n \alpha_j^2} \text{ — some norm (by linear independance).}$$

$$\text{By Riesz theorem } \|y\| \sim \|y\|_0$$

$$\underbrace{\|y_m - y_p\|}_{\in Y} \rightarrow 0 \implies \|y_m - y_p\|_0 \rightarrow 0$$

$$\bar{\alpha} = (\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n, y_m = \sum_{i=1}^n \alpha_i^{(m)} l_i$$

$$|\alpha_i^{(m)} - \alpha_i^{(l)}| \rightarrow 0 \forall i = 1, \dots, n; \bar{\alpha} = (\alpha_1^{(m)}, \dots, \alpha_n^{(m)}) \rightarrow \alpha^* = (\alpha_1^*, \dots, \alpha_n^*)$$

$$y^* = \sum_{i=1}^n \alpha_i^* l_i \in Y, \|y_m - y\| \rightarrow 0, y = y^* \implies y \in Y \quad \blacksquare$$

Definition. If normed space is complete, then it is called **B-space** or **Banach space**.

Example 6. $C[a, b]$ — functions continuous on $[a, b]$.

Example 7. Lebesgue space, $p \geq 1$, $L_p(E) = \left\{ f \text{ is measurable on } E, \int_E |f|^p < +\infty \right\}$.

If X — Banach space,

$$\sum_{n=1}^{\infty} x_n = \lim_{n \rightarrow \infty} \sum_{k=1}^n x_k, \quad \sum_{n=1}^{\infty} \|x_n\| < +\infty$$

$$\|S_n - S_m\| = \left\| \sum_{k=m+1}^n x_k \right\| \leq \sum_{k=m+1}^n \|x_k\| \xrightarrow{n, m \rightarrow \infty} 0$$

$$\implies \|S_n - S_m\| \rightarrow 0 \implies \exists \lim_{n \rightarrow \infty} S_n, \sum_{k=1}^n x_k \text{ — converges.}$$

In Banach spaces works the theory of absolute convergence of numerical series.

Lemma 1 (Riesz).