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1 Vector spaces

§1.1 Metric spaces.

X, ρ: $X \times X \to \mathbb{R}_+$

Definition. ρ — **metric**

1. $\rho(x, y) \geqslant 0, = 0 \iff x = y$
2. $\rho(x, y) = \rho(y, x)$
3. $\rho(x, y) \leqslant \rho(x, z) + \rho(y, z)$

Definition. (X, ρ) — **Metric space**.

Definition. $x = \lim x_n \iff \rho(x_n, x) \to 0$

$X, \tau = \{ G \subset X \}$

1. $\varnothing, X \in \tau$
2. $G_\alpha \in \tau, \alpha \in \mathscr{A} \implies \bigcup\limits_\alpha G_\alpha \in \tau$
3. $G_1, \dots, G_n \in \tau \implies \bigcap\limits_{j=1}^{n} G_j \in \tau$

Definition. (X, τ) — **Topological space**.

$x = \lim x_n \quad \forall G \in \tau : x \in G \quad \exists N : \forall n > N \implies x_n \in G$
G — open in τ
$F = X \setminus G$ — closed

Definition. $B_r(a) = \{ x \mid \rho(x, a) < r \}$ — **open ball**

$\tau = \bigcup B_r(a)$

Statement 1.1. $b \in B_{r_1}(a_1) \cap B_{r_2}(a_2) \implies \exists r_3 > 0 : B_{r_3}(a_3) \subset B_{r_1}(a_1) \cap B_{r_2}(a_2)$

In this sense metric space is just a special case of topological space.

Example 1. $\mathbb{R}, \rho(x, y) = |x - y|$, MS

Example 2. $\bar{x} = (x_1, \dots, x_n) \in \mathbb{R}^n, \rho(\bar{x}, \bar{y}) = \sqrt{\sum\limits_{j=1}^{n} (x_j - y_j)^2}, MS$

Example 3. $\bar{x} = (x_1, \dots, x_n, \dots) \in \mathbb{R}^\infty$

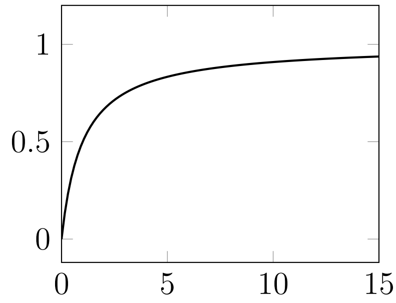
$$\alpha\bar{x} = (\alpha x_1, \dots, \alpha x_n, \dots)$$
$$\bar{x} + \bar{y} = (x_1 + y_1, \dots, x_n + y_n, \dots)$$

Let's define $\lim\limits_{m\to\infty} \bar{x}_m$

- in $\mathbb{R}^n$: $\bar{x}_n \to \bar{x} \iff \forall j = 1, \dots, n \qquad x_j^{(m)} \xrightarrow[m\to\infty]{} x_j$
- in $\mathbb{R}^\infty$: $\bar{x}_m \to \bar{x} \overset{def}{\iff} \forall j = 1, 2, 3, \dots \quad x_j^{(m)} \xrightarrow[m\to\infty]{} x_j$

Definition. $\rho(\bar{x}, \bar{y}) \overset{\text{def}}{=} \sum\limits_{n=1}^{\infty} \frac{1}{2^n} \underbrace{\frac{|x_n - y_n|}{1 + |x_n - y_n|}}_{\varphi(|x_n - y_n|)}$ — **Urysohn metric**.

$\varphi(t) = \dfrac{t}{1+t}$



$\varphi(t_1) + \varphi(t_2) \geqslant \varphi(t_1 + t_2)$

$\varphi(t_1) + \varphi(t_2) = \dfrac{t_1}{1+t_1} + \dfrac{t_2}{1+t_2} \geqslant \dfrac{t_1}{1+t_1+t_2} + \dfrac{t_2}{1+t_1+t_2}$

$\varphi(t_1) + \varphi(t_2) \geqslant \dfrac{t_1 + t_2}{1+t_1+t_2} = \varphi(t_1 + t_2)$

Statement 1.2. $\rho(\bar{x}_m, \bar{x}) \xrightarrow[m\to\infty]{} 0 \iff x_j^{(m)} \to x_j \ \forall j$

Proof.

- $\implies$
  $f(|x_k^{(n)} - x_k|) \leqslant 2^k \rho(x^{(n)}, x)$
  Let $\rho(x^{(n)}, x) \leqslant \dfrac{\varepsilon}{2^k}$, then $f(|x_k^{(n)} - x_k|) < \varepsilon$
  $|x_k^{(n)} - x_k| = t = \dfrac{1}{1 - f(t)} - 1$, then $t \to 0$
- $\Leftarrow$
  Let's choose k_0 for which $\sum\limits_{k=k_0+1}^{\infty} \dfrac{1}{2^k} < \varepsilon$
  Let's choose n_0 for which $\forall k \leqslant k_0, n > n_0 : |x_k^{(n)} - x_k| < \varepsilon$.
  Then $\rho(x^{(n)}, x) < \sum\limits_{k=1}^{k_0} \dfrac{\varepsilon}{2^k} + \varepsilon < 2\varepsilon$
  Letting $\varepsilon \to 0$, we get what we want ■

In this way $\mathbb{R}^\infty$ is a metrizable space.

Example 4. $X, \rho(x, y) \overset{\text{def}}{=} \begin{cases} 0, & x = y \\ 1, & x \neq y \end{cases}$ — **Discrete metric**.

$x_n \to x,\ \varepsilon = \frac{1}{2},\ \exists M :\ m > M \implies \rho(x_m, x) < \frac{1}{2} \implies$
$\rho(x_m, x) = 0 \implies x_m = x$

Definition.

$(X, \tau);\ \forall A \subset X;$

$$\mathrm{Int}(A) \overset{\text{def}}{=} \bigcup_{G \subset A} G \text{ is open;}$$

$$\mathrm{Cl}(A) \overset{\text{def}}{=} \bigcap_{A \subset G} G \text{ is closed;}$$

$$\mathrm{Fr}(A) \overset{\text{def}}{=} \mathrm{Cl}(A) \setminus \mathrm{Int}(A)$$

(X, ρ); Having a metric space one can describe closure of a set.

$$\rho(x, A) \stackrel{\text{def}}{=} \inf_{a \in A} \rho(x, a)$$

$$\rho(A, B) \stackrel{\text{def}}{=} \inf_{\substack{a \in A \\ b \in B}} \rho(a, b)$$

$$\rho(x, A) = f(x), x \in X$$

Statement 1.3. *Function f(x) is continuous.*

Proof. $\forall x, y \in X$
$f(x) = \rho(x, A) \underset{\forall \alpha \in A}{\leqslant} \rho(x, \alpha) \leqslant \rho(x, y) + \rho(y, \alpha)$
$\forall \varepsilon > 0\ \exists \alpha_\varepsilon \in A: \ \rho(y, \alpha_\varepsilon) < \rho(y, A) + \varepsilon = f(y) + \varepsilon$
$f(x) \leqslant f(y) + \varepsilon + \rho(x, y), \ \varepsilon \to 0$
$$\begin{cases} f(x) \leqslant f(y) + \rho(x, y) \\ f(y) \leqslant f(x) + \rho(x, y) \end{cases} \implies |f(x) - f(y)| \leqslant \rho(x, y)$$ ∎

Statement 1.4. $x \in \mathrm{Cl}(A) \iff \rho(x, A) = 0$

Let's look at the metric spaces in terms of separation of sets from each other by open sets.
x, y
$r = \rho(x, y) > 0$
$B_{\frac{r}{3}}(x), \ B_{\frac{r}{3}}(y)$
In any metric space separability axiom is true.

Theorem 1.1.
Any metric space is a normal space,
i.e. $\forall$ *closed disjoint* $F_1, F_2 \in X, \ \exists$ *open disjoint* G_1, G_2: $F_j \in G_j, \ j = 1, 2$

Proof. $g(x) = \dfrac{\rho(x, F_1)}{\rho(x, F_1) + \rho(x, F_2)}$ — continuous on X
$x \in F_1, \ \mathrm{Cl}(F_1) = F_1, \ \rho(x, F_1) = 0, \ g(x) = 0$
$x \in F_2, \ g(x) = 1$
Let's look at $(-\infty; \frac{1}{3}), (\frac{2}{3}, \infty)$ — by continuity their inverse images under g are open.
$G_1 = g^{-1}(-\infty; \frac{1}{3})$
$G_2 = g^{-1}(\frac{2}{3}; \infty)$ ∎

Definition. Metric space is **complete** if $\rho(x_n, x_m) \to 0 \implies \exists x = \lim x_n$
$\mathbb{R}^\infty$ – complete (by completeness of the rational numbers).
In complete metric spaces the nested balls principle is true.

Theorem 1.2.
X – complete metric space, $\overline{V}_{r_n}$ *– system of closed balls.*

1. $\overline{V}_{r_{n+1}} \subset \overline{V}_{r_n}$ *– the system is nested.*
2. $r_n \to 0$

<u>*Then:*</u> $\bigcap\limits_n \overline{V}_{r_n} = \{a\}$

Proof. Let b_n be centers of $\overline{V}_{r_n}$,
$m \geqslant n, \ b_m \in \overline{V}_{r_n}, \ \rho(b_m, b_n) \leqslant r_n \to 0\ \forall m \geqslant n$
$\rho(b_m, b_n) \to 0 \xrightarrow{\text{compl.}} \exists a = \lim b_n$ Since the balls are closed a $\in$ every ball.
$r_n \to 0 \implies$ there is only one common point. ∎

(X, τ) — topological space
$A \subset X,\ \tau_a = \{\, G \cap A, G \in \tau \,\}$ — topology induced on A

Definition. X— metric space, $A \subset X,\ \mathrm{Cl}(A) = X$
<u>Then:</u> A – **dense** in X
<u>If</u> $\mathrm{Int}(\mathrm{Cl}(A)) = \varnothing$ A – **nowhere dense** in X.

Note. It is easy to understand, that in metric spaces nowhere density means the following: $\forall$ ball $V\ \exists V' \subset V \colon V'$ contains no points from A.

Definition. X is called **first Baire category set**, if it can be written as at most countable union of x_n each nowhere dense in X.

Theorem 1.3 (Baire category theorem).
Complete metric space is second Baire category set in itself.

Proof. Let X be first Baire category set.
$X = \bigcup\limits_n X_n \quad \forall \overline{V}\ X_1$ is nowhere dense.
$\overline{V}_1 \subset \overline{V} \colon\ \overline{V}_1 \cap X_1\ = \varnothing$
X_2 is nowhere dense $\overline{V}_2 \subset \overline{V}_1 \colon \overline{V}_2 \cap X_2 = \varnothing$
$r_2 \leqslant \dfrac{r_1}{2}$
$\vdots$
$\{\overline{V}_n\},\ r_n \to 0,\ \bigcap\limits_n \overline{V}_n = \{a\},\ X = \bigcup X_n,\ \exists n_0 \colon a \in X_{n_0}$
$X_{n_0} \cap \overline{V}_{n_0} = \varnothing \to\leftarrow\ a \in \overline{V}_{n_0}$ ■

Corollary 1.3.1. *Complete metric space without isolated points is uncountable.*

Proof. No isolated points are present $\implies$ every point in the set is nowhere dense in it. Let X be countable: $X = \bigcup\limits_n \{X_n\}$, then it is first Baire category set. $\to\leftarrow$ ■

Definition. K — **compact** if

1. $K = \mathrm{Cl}(K)$
2. $x_n \in K\ \exists n_1 < n_2 < \ldots\ x_{n_j}$ – converges in X.

If only 2 is present, the set is called **precompact**.

Theorem 1.4 (Hausdorff).
Let X — metric space, K — closed in X.
<u>Then:</u> K — compact $\iff$ K — totally bounded,
i.e. $\forall \varepsilon > 0\ \exists a_1, \ldots, a_p \in X \colon\ \forall b \in K\ \exists a_j \colon\ \rho(a_j, b) < \varepsilon$
$(a_1, \ldots, a_p - finite\ \varepsilon - net)$

Proof.

- Totally bounded $\implies$ compact

  K is totally bounded, $x_n \in K\ n_1 < n_2 < \cdots < n_k < \cdots$, x_n converges in K

$$\varepsilon_k \downarrow\to 0\ \varepsilon_1 \quad K \subset \bigcup_{j=1}^{p} \overline{V}_j,\ rad = \varepsilon_1 \qquad (\varepsilon_1\text{-net})$$

  n is finite $\implies$ one ball will contain infinetely many x_n elements.

Let's look at $\overline{V}_{j_0} \cap K$ — totally bounded $= K_1, \operatorname{diam}(K_1) \leqslant 2\varepsilon_1$

$\varepsilon_2 \quad K_1 \subset \bigcup\limits_{j=1}^{n} \overline{V'}_j,\ rad = \varepsilon_2.$

Then one of $\overline{V'}$ contains infinitely many elements of the sequence contained in K_1.

$\overline{V'}_{j_0} \cap K_1 = K_2,\ \operatorname{diam}(K_2) \leqslant 2\varepsilon_2$ and so on.

$$K_n \supset K_{n+1} \supset K_{n+2} \supset \ldots,\ \operatorname{diam}(K_N) \leqslant 2\varepsilon_n \xRightarrow{\text{by compl.}} \underbrace{\bigcap_{n=1}^{\infty} K_n}_{\operatorname{diam}(K_n)\to 0,\ \{x\}} \neq \varnothing$$

Take x_{n_1} from K_1, x_{n_2} from $K_2 \ldots$

- Compact $\implies$ totally bounded

  K — compact $\forall\varepsilon\ \exists$ finite ε-net?
  By contradiction: $\exists\varepsilon_0 > 0:$ finite ε_0-net is impossible to construct.
  $\forall x_1 \in K\ \exists x_2 \in K:\ \rho(x_1, x_2) > \varepsilon_0$ (or else system of x_1 — finite ε-net)
  $\{x_1, x_2\}$ - choose $x_3 \in K: \rho(x_3, x_i) > \varepsilon_0,\ i = 1, 2$ and so on.
  $x_n \in K:\ n \neq m\ \rho(x_n, x_m) > \varepsilon_0$ — contains no converging subsequence $\implies$ set is not a compact. $\to\leftarrow$ ∎

§1.2 Normed spaces

Definition. X — **linear set**, $x + y, \alpha \cdot x,\ \alpha \in \mathbb{R}$
The purpose of norm definition, is to construct a topology on X, so that 2 linear operations are continuous on it.

$\varphi\colon X \to \mathbb{R}:$

1. $\varphi(x) \geqslant 0,\ = 0 \iff x = 0$
2. $\varphi(\alpha x) = |\alpha|\varphi(x)$
3. $\varphi(x + y) \leqslant \varphi(x) + \varphi(y)$

Definition. φ — **norm** on X, $\varphi(x) = \|x\|$

$\rho(x, y) \overset{\text{def}}{=} \|x - y\|$ — metric on X.

Definition.
$(X, \|\cdot\|)$ — **normed space** — special case of metrical space.

$$x = \lim x_n \overset{def}{\implies} \rho(x_n, x) \to 0 \iff \|x_n - x\| \to 0$$

Statement 1.5. *In the topology of a normed space linear operations are continuous on X.*

Proof.

1. $x_n \to x,\ y_n \to y;\ \|(x_n + y_n) - (x + y)\| = \|(x_n - x) + (y_n - y)\| \leqslant$
$$\leqslant \underbrace{\|x_n - x\|}_{\downarrow\, 0} + \underbrace{\|y_n - y\|}_{\downarrow\, 0}$$
$$\implies x_n + y_n \to x + y$$

2. $\alpha_n \to \alpha,\ x_n \to x;\ \|\alpha_n x_n - \alpha x\| = \|(\alpha_n - \alpha)x_n + \alpha(x_n - x)\| \leqslant$
$$\leqslant \underbrace{|\alpha_n - \alpha|}_{\substack{\downarrow \\ 0}} \cdot \underbrace{\|x_n\|}_{bounded} + \underbrace{\alpha\|x_n - x\|}_{\substack{\downarrow \\ 0}}$$

$x_n \to x \implies \|x_n\|$ — bounded.

$\alpha_x x_n \to \alpha x$ ■

Statement 1.6. *From the triangle inequality* $|\|x\| - \|y\|| \leqslant \|x - y\|$
$x_n \to x \implies \|x_n\| \to \|x\|$
Norm is continious.

Example 5. $\mathbb{R}^n$

1. $\|\bar{x}\| = \sqrt{\sum\limits_{k=1}^{n} x_k^2}$
2. $\|\bar{x}\|_1 \overset{\text{def}}{=} \sum\limits_{k=1}^{n} |x_k|$
3. $\|\bar{x}\|_2 \overset{\text{def}}{=} \max\{|x_1|, \ldots, |x_n|\}$
4. $C[a,b]$ — functions continuous on $[a,b]$; $\|f\| = \max\limits_{x \in [a,b]} |f(x)|$
5. $L_p(E) = \left\{ f \text{ — measurable, } \int\limits_E |f|^p < +\infty \right\}$

   $p \geqslant 1,\ \|f\|_p = \left(\int\limits_E |f|^p \right)^{\frac{1}{p}}$

Because the set of points is the same, arises the question about convergence comparison.
$\|\cdot\|_1 \sim \|\cdot\|_2,\ x_n \overset{\|\cdot\|_1}{\to} x \iff x_n \overset{\|\cdot\|_2}{\to} x$

Statement 1.7.
$\|\cdot\|_1 \sim \|\cdot\|_2 \iff \exists a, b > 0 \colon \forall x \in X \implies a\|x_1\|_1 \leqslant \|x\|_2 \leqslant b\|x\|_1$

Theorem 1.5 (Riesz)**.**
$X,\ \dim X < +\infty$ — *linear set.*
Then: *Any pair of norms in* X *are equivalent.*

Proof. $l_1, \ldots, l_n$ — linearly independent from X. $\forall x \in X = \sum\limits_{k=1}^{n} \alpha_k l_K$

$$\bar{x} \leftrightarrow (l_1, \ldots, l_n) = \bar{l} \in \mathbb{R}^n$$

Let $\|\cdot\|$ — some norm in X.

$$\|x\| \underset{\triangle}{\leqslant} \sum_{k=1}^{n} \|l_k\||\alpha_k| \underset{\text{Cauchy}}{\leqslant} \underbrace{\sqrt{\sum_{k=1}^{n} \|l_k\|^2}}_{const(B), B-basis} \underset{\|\bar{\alpha}\| = \|x\|_1}{\underset{\|}{\sqrt{\sum_{k=1}^{n} |\alpha_k|^2}}}$$

$$\|x\|_1 = \sqrt{\sum_{k=1}^{n} \|\alpha_k\|^2},\ x = \sum \alpha_k l_k$$

$\|x\| \leqslant b\|x\|_1$

$?\exists a > 0 \colon a\|x\|_1 \leqslant \|x\| \implies \|\cdot\| \sim \|\cdot\|_1$

Let $f(\alpha_1,\dots,\alpha_n)=\left\|\sum\limits_{k=1}^n \alpha_k l_k\right\|$

$$|f(\bar\alpha+\Delta\bar\alpha)-f(\bar\alpha)| = \left|\left\|\sum_{k=1}^n \alpha_k l_k+\sum_{k=1}^n \Delta\alpha_k l_K\right\|-\left\|\sum_{k=1}^n \alpha_k l_k\right\|\right| \leqslant$$
$$\left\|\sum_{k=1}^n \Delta\alpha_k l_k\right\| \leqslant \underbrace{\sum \|l_k\||\Delta\alpha_k|}_{\substack{\downarrow\\ 0,\Delta\alpha_k\to 0}} \implies f \text{ is continuous on } \mathbb{R}^n$$

$S_1=\left\{\sum\limits_{k=1}^n |\alpha_k|^2=1\right\}\subset\mathbb{R}^m$, f — continuous on S_1, S_1 — compact, $\bar\alpha^*\in S_1$

By Weierstrass theorem there exists a point $\alpha^*\in S_1$ on a sphere, in which function f achieves its minimum $\implies \forall\alpha\in S_1\ f(\bar\alpha^*)\leqslant f(\bar\alpha)$

If $f(\bar\alpha^*)=0$, then $\left\|\sum\limits_{k=1}^n \alpha_k^* l_k\right\|=0 \implies \sum\limits_{k=1}^n \alpha_k^* l_k=0,\ \bar\alpha^*\in S_1$,
but $l_1\dots l_n$ are linearly independent $\to\leftarrow \implies \min\limits_{S_1} f=m>0$

$$\|x\|=\left\|\sum_{k=1}^n \alpha_k l_k\right\|=f(\bar\alpha)=\sqrt{\sum_{k=1}^n \alpha_k^2}\cdot\left\|\sum \underbrace{\boxed{\frac{\alpha_k}{\sqrt{\sum_{k=1}^n \alpha_k^2}}}}_{\beta_k}\cdot l_k\right\|\geqslant,\ \bar\beta=(\beta_1\dots\beta_n)\in S_1$$
$$\geqslant m\cdot\|x\|_1,\ a=m \qquad \blacksquare$$

Corollary 1.5.1. X — *NS*, $Y\subset X,\dim Y<+\infty \implies Y=\mathrm{Cl}(Y)$

Note. Functional analysis differentiates between linear subset (set of points, closed by addition and scalar multiplication) and linear subspace (closed linear subset).

Proof. $Y=\alpha(l_1,\dots,l_n)=\left\{\sum\limits_{i=1}^n \alpha_i l_i \,\middle|\, l_1,\dots,l_n-\text{lin. indep.}\right\}$
$y_m\in Y, y_m\to y$ in $X \implies y\in Y$?
$\|y_m-y\|\to 0 \implies \|y_m-y_p\|\to 0,\ m,p\to\infty$
$\|y\|, y\in Y$.
By Riesz theorem all norm in Y are equivalent.
$y=\sum\limits_{j=1}^n \alpha_j l_J, \|y\|_0=\sqrt{\sum\limits_{j=1}^n \alpha_j^2}$ — some norm (by linear independance).
By Riesz theorem $\|y\|\sim\|y\|_0$
$\underbrace{\|y_m-y_p\|}_{\in Y}\to 0 \implies \|y_m-y_p\|_0\to 0$
$\bar\alpha=(\alpha_1,\dots,\alpha_n)\in\mathbb{R}^n\ y_m=\sum\limits_{i=1}^n \alpha_i^{(m)} l_i$
$|\alpha_i^{(m)}-\alpha_i^{(l)}|\to 0\ \forall i=1,\dots,n;\ \bar\alpha=(\alpha_1^{(m)},\dots,\alpha_n^{(m)})\to\alpha^*=(\alpha_1^*,\dots,\alpha_n^*)$
$y^*=\sum\limits_{i=1}^n \alpha_i^* l_I\in Y,\ \|y_m-y\|\to 0, y=y^* \implies y\in Y$ $\blacksquare$

Definition. If normed space if complete, then it is called **B-space** or **Banach space**.

Example 6. $C[a,b]$ — functions continuous on $[a,b]$.

Example 7. Lebesgue space, $p \geqslant 1, L_p(E) = \left\{ f \text{ is measurable on } E,\ \int\limits_E |f|^P < +\infty \right\}$.

If X — Banach space,

$$\sum_{n=1}^{\infty} x_n = \lim_{n\to\infty} \sum_{k=1}^{n} x_k,\ \sum_{n=1}^{\infty} \|x_n\| < +\infty$$

$$\|S_n - S_m\| = \left\| \sum_{k=m+1}^{n} x_k \right\| \leqslant \sum_{m+1}^{n} \|x_k\| \xrightarrow[n,m\to\infty]{} 0$$

$$\implies \|S_n - S_m\| \to 0 \implies \exists \lim_{n\to\infty} S_n, \sum_{k=1}^{n} x_k \text{— converges.}$$

In Banach spaces works the theory of absolute convergence of numerical series.

Lemma 1 (Riesz's lemma about almost perpendicular)**.** *Y — eigen subspace of X — normed space.* $\forall \varepsilon \in (0,1)\ \exists z_\varepsilon \in X$:

1. $\|z_\varepsilon\| = 1$
2. $\rho(z_\varepsilon, Y) > 1 - \varepsilon$

Proof. $\exists x \notin Y\ d = \rho(x, Y),\ d = 0\ \exists y_n \in Y\colon \|x - y_n\| < \dfrac{1}{n},\ n \to \infty,\ y_n \to x$
$Y = \mathrm{Cl}(Y) \implies x \in Y \to\leftarrow x \notin Y,\ d > 0$
$\forall \varepsilon \in (0,1)\ \dfrac{1}{1-\varepsilon} > 1\ \exists y_\varepsilon \in Y\colon \|x - y_\varepsilon\| < \dfrac{1}{1-\varepsilon} d$

$$z_\varepsilon = \frac{x - y_\varepsilon}{\|x - y_\varepsilon\|},\ \|z_\varepsilon\| = 1,\ \forall y \in Y\ \|z_\varepsilon - y\| = \left\| \frac{x - y_\varepsilon}{\|x - y_\varepsilon\|} \right\| = \frac{\|x - \overbrace{(y_\varepsilon + \|x - y_\varepsilon\| \cdot y)}^{\in Y}\| \geqslant d}{\|x - y_\varepsilon\| < \dfrac{1}{1-\varepsilon} d} >$$

$1 - \varepsilon$ ■

Corollary 1.5.2. $\dim X = +\infty$, *S — sphere in X, $r_S = 1\,\{\, x \mid \|x\| = 1 \,\} \implies S$ — not a compact.*

Proof. $\forall x_1 \in S,\ Y_1 = \alpha\{x_1\}$ — finite dimensional linear set. $\implies$ closed in $X \implies Y_1$ — subspace. $\dim X = +\infty \implies Y_1$ — eigen subspace.
Then by the Riesz lemma:

$$\exists x_2 \in S\colon \|x_2 - x_1\| > \frac{1}{2}$$

$$Y_2 = \alpha\{x_1, x_2\}\ \exists x_3 \in S\colon \|x_3 - x_j\| > \frac{1}{2},\ j = 1, 2$$

Continue by induction. Because $\dim X = +\infty$ the process willl never stop.
$x_n \in S\colon \|x_n - x_m\| > \dfrac{1}{2},\ n \neq m$ — obviously we cannot extract converging subsequence.
$\implies S$ — not a compact. ■

§1.3 Inner product (unitary) spaces

Definition. X — linear space.
$\varphi\colon X \times X \to \mathbb{R}$

1. $\varphi(x, x) \geqslant 0, \quad \varphi(x, x) = 0 \iff x = 0$

2. $\varphi(x,y)=\varphi(y,x)$

3. $\varphi(\alpha x+\beta y,z)=\alpha\varphi(x,z)+\beta\varphi(y,z)$

φ — **inner product**.
$\varphi(x,y)=\langle x,y\rangle$

Definition. $(X,\langle\cdot,\cdot\rangle)$ — **inner product space**.

Example 8. $\mathbb{R}^n, \langle\bar{x},\bar{y}\rangle=\sum\limits_{j=1}^{n}x_jy_j$

Statement 1.8 (Schwarz)**.** $\forall x,y\in X \quad |\langle x,y\rangle|\leqslant\sqrt{\langle x,x\rangle}\cdot\sqrt{\langle y,y\rangle}$

Proof. $\lambda\in\mathbb{R}$

$$f(\lambda)=\underset{\lambda^2\langle x,x\rangle+2\lambda\langle x,y\rangle+\langle y,y\rangle}{\underset{\|}{\langle\lambda x+y,\lambda x+y\rangle}}\geqslant 0$$

$$D=4\langle x,y\rangle^2-4\langle x,x\rangle\cdot\langle y,y\rangle\leqslant 0 \quad \blacksquare$$

Corollary 1.5.3 (Cauchy inequality for sums)**.** *Consider* $X=\mathbb{R}^n, \|x\|\overset{def}{=}\sqrt{\langle x,x\rangle}$. *Then*

$$\|x+y\|^2=\langle x+y,x+y\rangle=\|x\|^2+2\cdot\underbrace{\langle x,y\rangle}_{\leqslant\|x\|\cdot\|y\|}+\|y\|^2\leqslant\big(\|x\|+\|y\|\big)^2$$

Any inner product space is a special case of a normed space. The specifics is that we can measure the angles between points:

$$x\perp y \iff \langle x,y\rangle=0$$

In this case the Pythagorean theorem takes place:

$$\|x+y\|^2=\|x\|^2+\|y\|^2$$

In inner product spaces the parallelogram law plays a significant role:

$$\|x+y\|^2+\|x-y\|^2=2\|x\|^2+2\|y\|^2 \quad \forall x,y\in X$$

$\langle x,y\rangle\mapsto\|x\|=\sqrt{\langle x,x\rangle}$
Let X — NS, $\|x\|\overset{?}{=}\sqrt{\langle x,x\rangle}$

It can be proved that it's possible if and only if the norm satisfies the parallelogram law. So, $C_{[a,b]}$ is not an inner product space

Definition. **Orthonormal set** — a set of points $\{\, l_1,l_2,\dots\,\}$ (may be finite):

1. $\|l_i\|=1$

2. $l_i\perp l_j, \quad i\neq j$

Every orthonormal set is linearly independent.

Definition. Let $x\in X, \{\, l_i\,\}$ — ONS. Then
$\langle x,l_j\rangle$ — **Fourier coefficient**,
$\sum\limits_j\langle x,l_j\rangle l_j$ — **Fourier series** of point x.

Fourier series is a special case of orthogonal series.

Definition. $\sum\limits_j x_j$ — orthogonal series $\iff x_i \perp x_j, \quad i \neq j$

Let $\sum\limits_{j=1}^{\infty} x_j, S_m = \sum\limits_{j=1}^{m} x_j$. Then

$$\|S_m\|^2 = \langle \sum_{j=1}^{m} x_j, \sum_{j=1}^{m} x_j \rangle = \sum_{j=1}^{m} \|x_j\|^2$$

This fact allows us to effectively build the theory of orthogonal series.

An important problem is concerned with Fourier series. Let X is a normed space, Y is a subspace of X,

$$\forall x \in X \quad E_Y(x) = \rho(x, Y) = \inf_{y \in Y} \|x - y\|$$

Definition. $E_Y(x)$ — **best approximation** of point x with points of the subspace Y, if $\exists y^* \in Y \quad E_y(x) = \|x - y^*\|$ — then y^* is an element of best approximation.

Theorem 1.6 (Borel)**.**
$\dim Y < +\infty \implies \forall x \in X \quad \exists y^* \in Y$ — *element of best approximation.*

Proof. $Y = \mathscr{L}(\underbrace{l_1, l_2, \ldots, l_n}_{\text{lin. indep.}})$

Consider $f(\alpha_1, \ldots, \alpha_n) = \left\|x - \sum\limits_{k=1}^{n} \alpha_k l_k\right\| \to$ min. By the triangle inequality for norm, $f(\bar{\alpha})$ is continous on $\mathbb{R}^n, f \geqslant 0, E_Y(x) = \inf f(\bar{\alpha})$. It is easy to find that there always is a ball $B(0, r) \subset \mathbb{R}^n$, outside of which $f > 2E_Y(x)$. So, $E_Y(x)$ is somewhere inside. But f is continuous, the ball is compact, so, by the Weierstrass theorem, the minimum exists and it is located on the sphere $S(0, r)$. ■

For abstract Fourier series the Borel theorem can be significantly strengthened by specifying the best approximation element.

Theorem 1.7 (extreme quality of Fourier series' partial sums)**.**
$\{ e_j \}$ — ONS in X
$H_n = \mathscr{L}(l1, \ldots, l_n)$
$E_{H_n}(x), S_n(x) = \sum\limits_{j=1}^{n} \langle x, l_j \rangle l_j \implies E_{H_n}(x) = \|x - S_n(x)\|$

Proof. $y = \sum\limits_{j=1}^{n} \alpha_j l_j \in H_n$

$$\begin{aligned}\|x - y\|^2 &= \left\langle x - \sum \alpha_j l_j, x - \sum \alpha_j l_j \right\rangle = \|x\|^2 - 2\sum \alpha_j \langle x, l_j \rangle + \sum \alpha_j^2 = \\ &= \underbrace{\|x\|^2}_{\text{const}} + \sum (\alpha_j - \langle x, l_j \rangle)^2 - \underbrace{\sum \langle x, l_j \rangle^2}_{\text{const}} \to \min\end{aligned}$$

So, the sum goes to minimum when the second summand is minimal. Obviously, it's minimal when $\forall (\alpha_j - \langle x, l_j \rangle) = 0$. $E_Y(x)$ — Fourier sum. ■

Corollary 1.7.1 (Bessel's inequality)**.** $\sum\limits_j \langle x, l_j \rangle^2 \leqslant \|x\|^2$

Proof. $0 \leqslant \|x - y^*\|^2 = \|x\|^2 - \sum\limits_j \langle x, l_j \rangle^2$ ■

Corollary 1.7.2. *The series of Fourier coefficients' squares always converges*