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1 Bartor spaces.

§1.1 Metric spaces.

X, ρ: $X \times X \to \mathbb{R}_+$

Definition. ρ — **metric**

1. $\rho(x,y) \geq 0, = 0 \iff x = y$
2. $\rho(x,y) = \rho(y,x)$
3. $\rho(x,y) \leq \rho(x,z) + \rho(y,z)$

Definition. (X,ρ) — **Metric space**.

Definition. $x = \lim x_n \iff \rho(x_n, x) \to 0$

$X, \tau = \{G \subset X\}$

1. $\varnothing, X \in \tau$
2. $G_\alpha \in \tau, \alpha \in \mathcal{A} \implies \bigcup\limits_{\alpha} G_\alpha \in \tau$
3. $G_1, \ldots, G_n \in \tau \implies \bigcap\limits_{j=1}^{n} G_j \in \tau$

Definition. (X,τ) — **Topological space**.

$x = \lim x_n \quad \forall G \in \tau : x \in G \quad \exists N : \forall n > N \implies x_n \in G$
G — open in τ
$F = X \setminus G$ — closed

Definition. $B_r(a) = \{x : \rho(x,a) < r\}$ — **open ball**

$\tau = \bigcup B_r(a)$

Statement 1.1. $b \in B_{r_1}(a_1) \cap B_{r_2}(a_2) \implies \exists r_3 > 0 : B_{r_3}(a_3) \subset B_{r_1}(a_1) \cap B_{r_2}(a_2)$

In this sense metric space is just a special case of topological space.

Example 1. $\mathbb{R}, \rho(x,y) = |x-y|$, MS

Example 2. $\bar{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n, \rho(\bar{x}, \bar{y}) = \sqrt{\sum_{j=1}^{n} (x_j - y_j)^2}, MS$

Example 3. $\bar{x} = (x_1, \ldots, x_n, \ldots) \in \mathbb{R}^\infty$
$\alpha\bar{x} = (\alpha x_1, \ldots, \alpha x_n, \ldots)$
$\bar{x} + \bar{y} = (x_1 + y_1, \ldots, x_n + y_n, \ldots)$
$\lim\limits_{m\to\infty} \bar{x}_m?$
in $\mathbb{R}^n$ $\bar{x}_n \to \bar{x} \iff \forall j = 1 \ldots n \quad x_j^{(m)} \underset{m\to\infty}{\to} x_j$
in $\mathbb{R}^\infty$ $\bar{x}_m \to \bar{x} \overset{def}{\iff} \forall j = 1, 2, 3, \ldots \quad x_j^{(m)} \underset{m\to\infty}{\to} x_j$

Definition. $\rho(\bar{x}, \bar{y}) \overset{def}{=} \sum\limits_{n=1}^{\infty} \frac{1}{2^n} \underbrace{\frac{|x_n - y_n|}{1 + |x_n - y_N|}}_{\phi(|x_n - y_n|)}$ — **Urysohn metric**.

$\phi(t) = \frac{t}{1+t}$
$\phi(t_1 + t_2) \leq \phi(t_1) + \phi(t_2)$
$\rho(\bar{x_m}, \bar{x}) \underset{m\to\infty}{\to} 0 \iff x_j^{(m)} \to x_j \ \forall j$
In this way $\mathbb{R}^\infty$ is a metrizable space.

Example 4. $X, \rho(x,y) \overset{def}{=} \begin{cases} 0, x = y \\ 1, x \neq y \end{cases}$ — **Discrete metric**.
$x_n \to x \ \mathcal{E} = \frac{1}{2} \ \exists M: \ m > M \implies \rho(x_m, x) < \frac{1}{2} \implies$
$\rho(x_m, x) = 0 \implies x_m = x$

Definition. (X, τ); $\forall A \subset X$;
$\mathrm{Int}(A) \overset{def}{=} \bigcup\limits_{G \subset A} G -$ open;
$\mathrm{Cl}(A) \overset{def}{=} \bigcap\limits_{A \subset G} G -$ closed;
$\mathrm{Fr}(A) = \mathrm{Cl}(A) \setminus \mathrm{Int}(A)$

(X, ρ); Having a metric space one can describe closure of a set.
$\rho(x, A) \overset{def}{=} \inf\limits_{a \in A} \rho(x, a)$

$\rho(A,B) = \inf\limits_{\substack{a\in A\\ b\in B}} \rho(a,b)$
$\rho(x,A) = f(x), x \in X$

Statement 1.2. *Function f(x) is continuous.*

Proof. $\forall x, y \in X$
$f(x) = \rho(x,A) \underset{\forall \alpha \in A}{\leq} \rho(x,\alpha) \leq \rho(x,y) + \rho(y,\alpha)$
$\forall \mathcal{E} > 0\ \exists \alpha_\epsilon \in A:\ \rho(y,\alpha_\epsilon) < \rho(y,A) + \mathcal{E} = f(y) + \mathcal{E}$
$f(x) \leq f(y) + \mathcal{E} + \rho(x,y),\ \mathcal{E} \to 0$
$\begin{cases} f(x) \leq f(y) + \rho(x,y) \\ f(y) \leq f(x) + \rho(x,y) \end{cases} \implies |f(x) - f(y)| \leq \rho(x,y)$ ■

Statement 1.3. $x \in \mathrm{Cl}(A) \iff \rho(x,A) = 0$

Let's look at the metric spaces in terms of separation of sets from each other by open sets.
x, y
$r = \rho(x,y) > 0$
$B_{\frac{r}{3}}(x),\ B_{\frac{r}{3}}(y)$
In any metric space separability axiom is true.

Theorem 1.1.
Any metric space is a normal space,
i.e. $\forall$ *closed disjoint* $F_1, F_2 \in X,\ \exists$ *open disjoint* G_1, G_2: $F_j \in G_j,\ j = 1, 2$

Proof. $g(x) = \frac{\rho(x,F_1)}{\rho(x,F_1)+\rho(x,F_2)}$ - continuous on X
$x \in F_1,\ \mathrm{Cl}(F_1) = F_1,\ \rho(x,F_1) = 0,\ g(x) = 0$
$x \in F_2,\ g(x) = 1$
Let's look at $(-\infty; \frac{1}{3}), (\frac{2}{3}, \infty)$ — by continuity their inverse images under g are open.
$G_1 = g^{-1}(-\infty; \frac{1}{3})$
$G_2 = g^{-1}(\frac{2}{3}; \infty)$ ■

Definition. Metric space is **complete** if $\rho(x_n, x_m) \to 0 \implies \exists x = \lim x_n$
$\mathbb{R}^\infty$ – complete (by completeness of the rational numbers).
In complete metric spaces the nested balls principle is true.

Theorem 1.2.
X – complete metric space, $\overline{V}_{r_n}$ *– system of closed balls.*

1. $\overline{V}_{r_{n+1}} \subset \overline{V}_{r_n}$ – *the system is nested.*

2. $r_n \to 0$

<u>*Then:*</u> $\bigcap\limits_n \overline{V}_{r_n} = \{a\}$

Proof. Let b_n be centers of $\overline{V}_{r_n}$,
$m \geq n,\ b_m \in \overline{V}_{r_n},\ \rho(b_m, b_n) \leq r_n \to 0\ \forall m \geq n$
$\rho(b_m, b_n) \to 0 \overset{comp.}{\Longrightarrow} \exists a = \lim b_n$ Since the balls are closed a $\in$ every ball.
$r_n \to 0 \implies$ there is only one common point $\blacksquare$.

(X, τ) — topological space
$A \subset X,\ \tau_a = \{G \cap A, G \in \tau\}$

Definition. X— metric space, $A \subset X,\ \mathrm{Cl}(A) = X$
<u>Then:</u> A – **dense** in X
<u>If</u> $\mathrm{Int}(\mathrm{Cl}(A)) = \varnothing$ A – **nowhere dense** in X.
It is easy to understand, that in metric spaces nowhere density means the following: $\forall$ *ball* $V\ \exists V' \subset V : V'$ contains no points from A.

X is called **first Baire category set**, if it can be written as at most countable union of x_n each nowhere dense in X.

Theorem 1.3 (Baire category theorem)**.**
Complete metric space is second Baire category set in itself.

Proof. Let X be first Baire category set.
$X = \bigcup\limits_n X_n \quad \forall \overline{V}\ X_1$ is nowhere dense.
$\overline{V}_1 \subset \overline{V}:\ \overline{V}_1 \cap X_1\ = \varnothing$
X_2 is nowhere dense $\overline{V}_2 \subset \overline{V}_1 \colon \overline{V}_2 \cap X_2 = \varnothing$
$r_2 \leq \frac{r_1}{2}$
$\vdots$
$\{\overline{V}_n\},\ r_n \to 0,\ \bigcap\limits_n \overline{V}_n = \{a\},\ X = \bigcup X_n,\ \exists n_0 \colon a \in X_{n_0}$
$X_{n_0} \cap \overline{V}_{n_0} = \varnothing \to\leftarrow\ a \in \overline{V}_{n_0}\quad \blacksquare$

Corollary 1.3.1. *Complete metric space without isolated points is uncountable.*

Proof. No isolated points are present $\implies$ every point in the set is nowhere dense in it. Let X be countable: $X = \bigcup_n \{X_n\}$, then it is first Baire category set. $\rightarrow\leftarrow$ ■

Definition. K — **compact** if

1. $K = \mathrm{Cl}(K)$
2. $x_n \in K\ \exists n_1 < n_2 < \dots\ x_{n_j}$ – converges in X.

If only **??** is present, the set is called **precompact**.

Theorem 1.4 (Hausdorff)**.**
Let X — metric space, K — closed in X.
<u>Then:</u> K — compact $\iff$ K — totally bounded,
i.e. $\forall \mathcal{E} > 0\ \exists a_1, \dots, a_p \in X\colon\ \forall b \in K\ \exists a_j\colon\ \rho(a_j, b) < \mathcal{E}$
$(a_1, \dots, a_p - finite\ \mathcal{E} - net)$

Proof.
$\Longrightarrow$
K— totally bounded, $x_n \in K\ n_1 < n_2 < \dots < n_k < \dots$
x_n – converges in K
$\mathcal{E}_k \downarrow\rightarrow 0\ \mathcal{E}_1 \quad K \subset \bigcup\limits_{j=1}^{p} \overline{V}_j,\ rad = \mathcal{E}_1\ (\mathcal{E}_1 - net)$
n is finite $\implies$ one ball will contain infinetely many x_n elements.
Let's look at $\overline{V}_{j_0} \cap K$ — totally bounded $= K_1,\ \mathrm{diam}(K_1) \leq 2\mathcal{E}_1$
$\mathcal{E}_2 \quad K_1 \subset \bigcup\limits_{j=1}^{n} \overline{V'}_j,\ rad = \mathcal{E}_2,$ then one of $\overline{V'}$
contains infinitely many elements of the sequence contained in K_1
$\overline{V'}_{j_0} \cap K_1 = K_2,\ \mathrm{diam}(K_2) \leq 2\mathcal{E}_2$ and so on.

$$K_n \supset K_{n+1} \supset K_{n+2} \supset \dots,\ \mathrm{diam}(K_N) \leq 2\mathcal{E}_n \overset{by\ space\ comp.}{\Longrightarrow} \underbrace{\bigcap_{n=1}^{\infty} K_n}_{\mathrm{diam}(K_n)\to 0,\ \{x\}} \neq \varnothing$$

Take x_{n_1} from K_1, x_{n_2} from $K_2 \dots$
$\Longleftarrow$
K — compact $\forall \mathcal{E}\ \exists$ finite $\mathcal{E}$net?
By contradiction: $\exists \mathcal{E}_0 > 0\colon$ finite $\mathcal{E}_0$-net is impossible to construct.
$\forall x_1 \in K\ \exists x_2 \in K\colon\ \rho(x_1, x_2) > \mathcal{E}_0$ (or else system of x_1 — finite $\mathcal{E}$-net)
$\{x_1, x_2\}$ - choose $x_3 \in K\colon \rho(x_3, x_i) > \mathcal{E}_0,\ i = 1, 2$ and so on.
$x_n \in K\ :\ n \neq m\ \rho(x_n, x_m) > \mathcal{E}_0$ — contains no converging subsequence $\implies$ set is not a compact. $\rightarrow\leftarrow$ ■

§1.2 Normed spaces

Definition. X — **linear set**, x + y, $\alpha \cdot x$, $\alpha \in \mathbb{R}$
The purpose of norm definition, is to construct a topology on X, so that 2 linear operations are continuous on it.

$\phi : X \to \mathbb{R}$:

1. $\phi(x) \geq 0, \ = 0 \iff x = 0$
2. $\phi(\alpha x) = |\alpha| \phi(x)$
3. $\phi(x + y) \leq \phi(x) + \phi(y)$

Definition. ϕ — **norm** on X, $\phi(x) = \|x\|$

$\rho(x, y) \overset{def}{=} \|x - y\|$ — metric on X.

Definition.
$(X, \|\cdot\|)$ — **normed space** — special case of metrical space.

$$x = \lim x_n \overset{def}{\Longrightarrow} \rho(x_n, x) \to 0 \iff \|x_n - x\| \to 0$$

Statement 1.4. *In the topology of a normed space linear operations are continuous on X.*

Proof. 1. $x_n \to x, \ y_n \to y; \ \|(x_n + y_n) - (x + y)\| = \|(x_n - x) + (y_n - y)\| \leq \underbrace{\|x_n - x\|}_{\substack{\downarrow \\ 0}} + \underbrace{\|y_n - y\|}_{\substack{\downarrow \\ 0}} \implies x_n + y_n \to x + y$

2. $\alpha_n \to \alpha, \ x_n \to x; \ \|\alpha_n x_n - \alpha x\| = \|(\alpha_n - \alpha) x_n + \alpha(x_n - x)\| \leq \underbrace{|\alpha_n - \alpha|}_{\substack{\downarrow \\ 0}} \cdot \underbrace{\|x_n\|}_{bounded} + \underbrace{\alpha \|x_n - x\|}_{\substack{\downarrow \\ 0}}$

   $x_n \to x \implies \|x_n\|$ — bounded.
   $\alpha_x x_n \to \alpha x$ ■

Statement 1.5. *From the triangle inequality* $|\|x\| - \|y\|| \leq \|x - y\|$
$x_n \to x \implies \|x_n\| \to \|x\|$
Norm is continious.

Example 5. $\mathbb{R}^n$

1. $\|\bar{x}\| = \sqrt{\sum\limits_{k=1}^{n} x_k^2}$

2. $\|\bar{x}\|_1 \overset{def}{=} \sum\limits_{k=1}^{n} |x_k|$

3. $\|\bar{x}\|_2 \overset{def}{=} \max\{|x_1| \dots |x_n|\}$

4. $C[a,b]$ — functions continuous on $[a,b]$; $\|f\| = \max\limits_{x\in[a,b]} |f(x)|$

5. $L_p(E) = \{f -$ measurable, $\int\limits_E |f|^p < +\infty\}$

   $p \geq 1,\ \|f\|_p = (\int\limits_E |f|^p)^{\frac{1}{p}}$

Because the set of points is the same, arises the question about convergence comparison.
$\|\cdot\|_1 \sim \|\cdot\|_2,\ x_n \overset{\|\cdot\|_1}{\to} x \iff x_n \overset{\|\cdot\|_2}{\to}$

Statement 1.6.
$\|\cdot\|_1 \sim \|\cdot\|_2 \iff \exists a,b > 0 \colon \forall x \in X \implies a\|x_1\|_1 \leq \|x\|_2 \leq b\|x\|_1$

Theorem 1.5 (Riesz)**.**
$X,\ \dim X < +\infty$ — *linear set.*
Then: *Any pair of norms in X are equivalent.*

Proof. $l_1, \dots, l_n$ — linearly independent from X. $\forall x \in X = \sum\limits_{k=1}^{n} \alpha_k l_K$
$\bar{x} \leftrightarrow (l_1, \dots, l_n) = \bar{l} \in \mathbb{R}^n$
Let $\|\cdot\|$ — some norm in X.

$$\|x\| \underset{\triangle}{\leq} \sum_{k=1}^{n} \|l_k\||\alpha_k| \underset{Cauchy}{\leq} \underbrace{\sqrt{\sum_{1}^{n} \|l_k\|^2}}_{const B} \underset{\|\bar{\alpha}\|=\|x\|_1}{\underset{\|}{\sqrt{\sum_{1}^{n} |\alpha_k|^2}}}$$

$\|x\|_1 = \sqrt{\sum\limits_{1}^{n} \|alpha_k\|^2},\ x = \sum \alpha_k l_k$
$\|x\| \leq b\|x\|_1$
$?\exists a > 0 \colon a\|x\|_1 \leq \|x\| \implies \|\cdot\| \sim \|\cdot\|_1$

Let $f(\alpha_1, \dots, \alpha_n) = \left\| \sum\limits_{k=1}^{n} \alpha_k l_k \right\|$

$$|f(\bar{\alpha} + \Delta\bar{\alpha}) - f(\bar{\alpha}) = \left| \left\| \sum_{1}^{k} \alpha_k l_k + \sum_{1}^{n} \Delta\alpha_k l_K \right\| - \left\| \sum_{1}^{n} \alpha_k l_k \right\| \right| \leq$$

$$\left\| \sum_{1}^{n} \Delta\alpha_k l_k \right\| \leq \underbrace{\sum \|l_k\| |\Delta\alpha_k|}_{\substack{\downarrow \\ 0, \Delta\alpha_k \to 0}} \implies f\text{— continuous on } \mathbb{R}^n$$

$S_1 = \{\sum\limits_{1}^{n} \alpha_k|^2 = 1\} \subset \mathbb{R}^m$, f — continuous on S_1, $\bar{\alpha}^* \in S_1$

$\forall \alpha \in S_1 \implies f(\bar{\alpha}^*) \leq f(\bar{\alpha})$

$f(\bar{\alpha}^*) = 0$

$\left\| \sum\limits_{1}^{n} \alpha_k^* l_k \right\| = 0$

$\sum\limits_{1}^{n} \alpha_k^* l_k = 0, \ \bar{\alpha}^* \in S_1$

$l_1 \dots l_n$ — linearly independent $\to\leftarrow$

$\min\limits_{S_1} f = m > 0$

$$\|x\| = \left\| \sum_{1}^{n} \alpha_K l_K \right\| = f(\bar{\alpha}) = \sqrt{\sum_{1}^{n} \alpha_k^2} \cdot \left\| \sum \underbrace{\frac{\alpha_k}{\sqrt{\sum\limits_{1}^{n} \alpha_k^2}}}_{\beta_k} l_k \right\|, \ \bar{\beta} = (\beta_1 \dots \beta_n) \in S_1$$

$$\geq m \cdot \|x\|_1, \ a = m \quad \blacksquare$$