# Functional analysis course by Dodonov N.U.

Sugak A.M.

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# 1    Vector spaces.

## §1.1    Metric spaces.

X, $\rho: X \times X \to \mathbb{R}_+$

**Definition.** $\rho$ — **metric**

    1. $\rho(x, y) \geqslant 0, = 0 \iff x = y$

    2. $\rho(x, y) = \rho(y, x)$

    3. $\rho(x, y) \leqslant \rho(x, z) + \rho(y, z)$

**Definition.** $(X, \rho)$ — **Metric space**.

**Definition.** $x = \lim x_n \iff \rho(x_n, x) \to 0$

    $X, \tau = \{G \subset X\}$

    1. $\varnothing, X \in \tau$

    2. $G_\alpha \in \tau, \alpha \in \mathscr{A} \implies \bigcup_\alpha G_\alpha \in \tau$

    3. $G_1, \ldots, G_n \in \tau \implies \bigcap_{j=1}^{n} G_j \in \tau$

**Definition.** $(X, \tau)$ — **Topological space**.

$x = \lim x_n \quad \forall G \in \tau : x \in G \quad \exists N : \forall n > N \implies x_n \in G$
G — open in $\tau$
$F = X \setminus G$ — closed

**Definition.** $B_r(a) = \{x \mid \rho(x, a) < r\}$ — **open ball**

$\tau = \bigcup B_r(a)$

**Statement 1.1.** $b \in B_{r_1}(a_1) \cap B_{r_2}(a_2) \implies \exists r_3 > 0 : B_{r_3}(a_3) \subset B_{r_1}(a_1) \cap B_{r_2}(a_2)$

In this sense metric space is just a special case of topological space.

**Example 1.** $x \in \mathbb{R}, \rho(x,y) = |x - y|, MS$

**Example 2.** $\bar{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n, \rho(\bar{x}, \bar{y}) = \sqrt{\sum\limits_{j=1}^{n} (x_j - y_j)^2}, MS$

**Example 3.** $\bar{x} = (x_1, \ldots, x_n, \ldots) \in \mathbb{R}^\infty$

$$\alpha\bar{x} = (\alpha x_1, \ldots, \alpha x_n, \ldots)$$
$$\bar{x} + \bar{y} = (x_1 + y_1, \ldots, x_n + y_n, \ldots)$$
$$\lim_{m\to\infty} \bar{x}_m ?$$

in $\mathbb{R}^n$ $\bar{x}_n \to \bar{x} \iff \forall j = 1, \ldots, n \quad x_j^{(m)} \xrightarrow{m\to\infty} x_j$

in $\mathbb{R}^\infty$ $\bar{x}_m \to \bar{x} \overset{def}{\iff} \forall j = 1, 2, 3, \ldots \quad x_j^{(m)} \xrightarrow{m\to\infty} x_j$

$$\rho(\bar{x}, \bar{y}) \overset{def}{=} \sum_{n=1}^{\infty} \frac{1}{2^n} \underbrace{\frac{|x_n - y_n|}{1 + |x_n - y_N|}}_{\varphi(|x_n - y_n|)} ---\textbf{Urysohn metric.}$$

Let $\varphi(t) = \dfrac{t}{1+t}$, then

$$\varphi(t_1 + t_2) \leqslant \varphi(t_1) + \varphi(t_2)$$

**Statement 1.2.** $\rho(\bar{x}_m, \bar{x}) \xrightarrow{m\to\infty} 0 \iff x_j^{(m)} \to x_j \ \forall j$

*Proof.*

- $\Rightarrow$

  $f(|x_k^{(n)} - x_k|) \leqslant 2^k \rho(x^{(n)}, x)$
  Let $\rho(x^{(n)}, x) \leq \frac{\varepsilon}{2^k}$, then $f(|x_k^{(n)} - x_k|) < \varepsilon$
  $|x_k^{(n)} - x_k| = t = \frac{1}{1 - f(t)} - 1$, then $t \to 0$

- $\Leftarrow$

  Lets take such $k_0$ that $\sum\limits_{k=k_0+1}^{\infty} \frac{1}{2^k} < \varepsilon$

  Lets take such $n_0$ that $\forall k \leqslant k_0, n > n_0 : |x_k^{(n)} - x_k| < \varepsilon$.

  Then $\rho(x^{(n)}, x) < \sum\limits_{k=1}^{k_0} \frac{\varepsilon}{2^k} + \varepsilon < 2\varepsilon$

  Letting $x \to 0$, we get what we want $\qquad\qquad \blacksquare$

3

In this way $\mathbb{R}^\infty$ is a metrizable space.

**Example 4.** $x \in X, \rho(x,y) \stackrel{\text{def}}{=} \begin{cases} 0, x = y \\ 1, x \neq y \end{cases}$ — **Discrete metric.**

$x_n \to x, \ \varepsilon = \frac{1}{2}, \ \exists M : \ m > M \implies \rho(x_m, x) < \frac{1}{2} \implies$
$\rho(x_m, x) = 0 \implies x_m = x$

**Definition.** $(X, \tau); \ \forall A \subset X;$

$$\text{Int}(A) \stackrel{\text{def}}{=} \bigcup_{G \subset A} G - \text{open};$$

$$\text{Cl}(A) \stackrel{\text{def}}{=} \bigcap_{A \subset G} G - \text{closed};$$

$$\text{Fr}(A) = \text{Cl}(A) \setminus \text{Int}(A)$$

$(X, \rho);$ Having a metric space one can describe closure of a set.
$\rho(x, A) \stackrel{\text{def}}{=} \inf_{a \in A} \rho(x, a)$
$\rho(A, B) = \inf_{\substack{a \in A \\ b \in B}} \rho(a, b)$
$\rho(x, A) = f(x), x \in X$

**Statement 1.3.** *Function f(x) is continuous.*

*Proof.* $\forall x, y \in X$
$f(x) = \rho(x, A) \underset{\forall \alpha \in A}{\leqslant} \rho(x, \alpha) \leqslant \rho(x, y) + \rho(y, \alpha)$
$\forall \varepsilon > 0 \ \exists \alpha_\varepsilon \in A : \ \rho(y, \alpha_\varepsilon) < \rho(y, A) + \varepsilon = f(y) + \varepsilon$
$f(x) \leqslant f(y) + \varepsilon + \rho(x, y), \ \varepsilon \to 0$
$\begin{cases} f(x) \leqslant f(y) + \rho(x, y) \\ f(y) \leqslant f(x) + \rho(x, y) \end{cases} \implies |f(x) - f(y)| \leqslant \rho(x, y) \quad \blacksquare$

**Statement 1.4.** $x \in \text{Cl}(A) \iff \rho(x, A) = 0$

Let's look at the metric spaces in terms of separation of sets from each other by open sets.
$x, y$
$r = \rho(x, y) > 0$
$B_{\frac{r}{3}}(x), \ B_{\frac{r}{3}}(y)$
In any metric space separability axiom is true.

4

**Theorem 1.1.**
*Any metric space is a normal space,*
*i.e. $\forall$ closed disjoint $F_1, F_2 \in X$, $\exists$ open disjoint $G_1, G_2 \colon F_j \in G_j$, $j = 1, 2$*

*Proof.* $g(x) = \frac{\rho(x, F_1)}{\rho(x, F_1) + \rho(x, F_2)}$ — continuous on X
$x \in F_1$, $\mathrm{Cl}(F_1) = F_1$, $\rho(x, F_1) = 0$, $g(x) = 0$
$x \in F_2$, $g(x) = 1$
Let's look at $(-\infty; \frac{1}{3}), (\frac{2}{3}, \infty)$ — by continuity their inverse images under g are open.
$G_1 = g^{-1}(-\infty; \frac{1}{3})$
$G_2 = g^{-1}(\frac{2}{3}; \infty)$ ∎

**Definition.** Metric space is **complete** if $\rho(x_n, x_m) \to 0 \implies \exists x = \lim x_n$
$\mathbb{R}^\infty$ – complete (by completeness of the rational numbers).
In complete metric spaces the nested balls principle is true.

**Theorem 1.2.**
$X$ – *complete metric space, $\overline{V}_{r_n}$ – system of closed balls.*

1. $\overline{V}_{r_{n+1}} \subset \overline{V}_{r_n}$ – *the system is nested.*

2. $r_n \to 0$

*Then:* $\bigcap_n \overline{V}_{r_n} = \{a\}$

*Proof.* Let $b_n$ be centers of $\overline{V}_{r_n}$,
$m \geqslant n$, $b_m \in \overline{V}_{r_n}$, $\rho(b_m, b_n) \leqslant r_n \to 0 \ \forall m \geqslant n$
$\rho(b_m, b_n) \to 0 \overset{compl.}{\Longrightarrow} \exists a = \lim b_n$ Since the balls are closed a $\in$ every ball.
$r_n \to 0 \implies$ there is only one common point ∎.

$(X, \tau)$ — topological space
$A \subset X$, $\tau_a = \{G \cap A, G \in \tau\}$ — topology induced on $A$

**Definition.** $X$— metric space, $A \subset X$, $\mathrm{Cl}(A) = X$
Then: A – **dense** in X
If $\mathrm{Int}(\mathrm{Cl}(A)) = \varnothing$ A – **nowhere dense** in X.
It is easy to understand, that in metric spaces nowhere density means the following:
$\forall$ *ball* $V$ $\exists V' \subset V \colon V'$ contains no points from A.

**Definition.** X is called **first Baire category set**, if it can be written as at most countable union of $x_n$ each nowhere dense in X.

**Theorem 1.3** (Baire category theorem).
*Complete metric space is second Baire category set in itself.*

*Proof.* Let X be first Baire category set.
$X = \bigcup_n X_n \quad \forall V \; X_1$ is nowhere dense.
$\overline{V}_1 \subset \overline{V} : \; \overline{V}_1 \cap X_1 \; = \varnothing$
$X_2$ is nowhere dense $\overline{V}_2 \subset \overline{V}_1 : \overline{V}_2 \cap X_2 = \varnothing$
$r_2 \leqslant \frac{r_1}{2}$
$\vdots$
$\{\overline{V}_n\}, \; r_n \to 0, \; \bigcap_n \overline{V}_n = \{a\}, \; X = \bigcup X_n, \; \exists n_0 : a \in X_{n_0}$
$X_{n_0} \cap \overline{V}_{n_0} = \varnothing \to\leftarrow \; a \in \overline{V}_{n_0}$ ∎

**Corollary 1.3.1.** *Complete metric space without isolated points is uncountable.*

*Proof.* No isolated points are present $\implies$ every point in the set is nowhere dense in it. Let X be countable: $X = \bigcup_n \{X_n\}$, then it is first Baire category set. $\to\leftarrow$ ∎

**Definition.** K — **compact** if

1. $K = \mathrm{Cl}(K)$

2. $x_n \in K \; \exists n_1 < n_2 < \ldots \; x_{n_j} -$ converges in $X$.

If only 2 is present, the set is called **precompact**.

**Theorem 1.4** (Hausdorff).
*Let $X$ — metric space, $K$ — closed in $X$.*
*Then: $K$ — compact $\iff K$ — totally bounded,*
*i.e. $\forall \varepsilon > 0 \; \exists a_1, \ldots, a_p \in X : \; \forall b \in K \; \exists a_j : \; \rho(a_j, b) < \varepsilon$*
*$(a_1, \ldots, a_p - finite \; \varepsilon - net)$*

*Proof.*
$\implies$
$K$— totally bounded, $x_n \in K \; n_1 < n_2 < \ldots < n_k < \ldots$

6

$x_n$ — converges in K

$\varepsilon_k \downarrow \to 0 \ \varepsilon_1 \quad K \subset \bigcup\limits_{j=1}^{p} \overline{V}_j, \ rad = \varepsilon_1 \ (\varepsilon_1 - net)$

n is finite $\implies$ one ball will contain infinetely many $x_n$ elements.

Let's look at $\overline{V}_{j_0} \cap K$ — totally bounded $= K_1, \ \text{diam}(K_1) \leqslant 2\varepsilon_1$

$\varepsilon_2 \quad K_1 \subset \bigcup\limits_{j=1}^{n} \overline{V'}_j, \ rad = \varepsilon_2.$

Then one of $\overline{V'}$ contains infinitely many elements of the sequence contained in $K_1$

$\overline{V'}_{j_0} \cap K_1 = K_2, \ \text{diam}(K_2) \leqslant 2\varepsilon_2$ and so on.

$K_n \supset K_{n+1} \supset K_{n+2} \supset \ldots, \ \text{diam}(K_N) \leqslant 2\varepsilon_n \xRightarrow{by \ space \ compl.} \underbrace{\bigcap\limits_{n=1}^{\infty} K_n}_{\text{diam}(K_n) \to 0, \ \{x\}} \neq$

$\varnothing$

Take $x_{n_1}$ from $K_1$, $x_{n_2}$ from $K_2 \ldots$

$\impliedby$

K — compact $\forall \varepsilon \ \exists$ finite $\varepsilon$ — net?

By contradiction: $\exists \varepsilon_0 > 0$: finite $\varepsilon_0$-net is impossible to construct.

$\forall x_1 \in K \ \exists x_2 \in K: \ \rho(x_1, x_2) > \varepsilon_0$ (or else system of $x_1$ — finite $\varepsilon$-net)

$\{x_1, x_2\}$ - choose $x_3 \in K: \rho(x_3, x_i) > \varepsilon_0, \ i = 1, 2$ and so on.

$x_n \in K: \ n \neq m \ \rho(x_n, x_m) > \varepsilon_0$ — contains no converging subsequence

$\implies$ set is not a compact. $\to \leftarrow$ $\blacksquare$

# §1.2   Normed spaces

**Definition.** X — **linear set**, x + y, $\alpha \cdot x$, $\alpha \in \mathbb{R}$

The purpose of norm definition, is to construct a topology on X, so that 2 linear operations are continuous on it.

$\varphi : X \to \mathbb{R}$:

1. $\varphi(x) \geqslant 0, \ = 0 \iff x = 0$

2. $\varphi(\alpha x) = |\alpha| \varphi(x)$

3. $\varphi(x + y) \leqslant \varphi(x) + \varphi(y)$

**Definition.** $\varphi$ — **norm** on X, $\varphi(x) = \|x\|$

$\rho(x, y) \overset{\text{def}}{=\joinrel=} \|x - y\|$ — metric on X.

**Definition.**
$(X, \| \cdot \|)$ — **normed space** — special case of metrical space.

$$x = \lim x_n \overset{def}{\Longrightarrow} \rho(x_n, x) \to 0 \iff \|x_n - x\| \to 0$$

**Statement 1.5.** *In the topology of a normed space linear operations are continuous on X.*

*Proof.*   1.  $x_n \to x, \ y_n \to y; \ \|(x_n + y_n) - (x + y)\| = \|(x_n - x) + (y_n - y)\| \leqslant$
$$\leqslant \underbrace{\|x_n - x\|}_{\downarrow \ 0} + \underbrace{\|y_n - y\|}_{\downarrow \ 0}$$
$$\implies x_n + y_n \to x + y$$

2.  $\alpha_n \to \alpha, \ x_n \to x; \ \|\alpha_n x_n - \alpha x\| = \|(\alpha_n - \alpha)x_n + \alpha(x_n - x)\| \leqslant$
$$\leqslant \underbrace{|\alpha_n - \alpha|}_{\downarrow \ 0} \cdot \underbrace{\|x_n\|}_{bounded} + \underbrace{\alpha \|x_n - x\|}_{\downarrow \ 0}$$

$x_n \to x \implies \|x_n\|$ — bounded.
$\alpha_x x_n \to \alpha x$  ■

**Statement 1.6.** *From the triangle inequality $|\|x\| - \|y\|| \leqslant \|x - y\|$*
$x_n \to x \implies \|x_n\| \to \|x\|$
*Norm is continious.*

**Example 5.** $\mathbb{R}^n$

1. $\|\bar{x}\| = \sqrt{\sum_{k=1}^{n} x_k^2}$

2. $\|\bar{x}\|_1 \overset{def}{=} \sum_{k=1}^{n} |x_k|$

3. $\|\bar{x}\|_2 \overset{def}{=} \max\{|x_1| \ldots |x_n|\}$

4. $C[a, b]$ — functions continuous on $[a, b]$; $\|f\| = \max\limits_{x \in [a,b]} |f(x)|$

5. $L_p(E) = \{f - \text{measurable}, \ \int_E |f|^p < +\infty\}$

    $p \geqslant 1, \ \|f\|_p = (\int_E |f|^p)^{\frac{1}{p}}$

Because the set of points is the same, arises the question about convergence comparison.

$\| \cdot \|_1 \sim \| \cdot \|_2, \ x_n \overset{\|\cdot\|_1}{\Rightarrow} x \iff x_n \overset{\|\cdot\|_2}{\Rightarrow}$

**Statement 1.7.**
$\| \cdot \|_1 \sim \| \cdot \|_2 \iff \exists a, b > 0 \colon \forall x \in X \implies a\|x_1\|_1 \leqslant \|x\|_2 \leqslant b\|x\|_1$

**Theorem 1.5** (Riesz).
$X, \ \dim X < +\infty$ — *linear set.*
<u>*Then:*</u> *Any pair of norms in $X$ are equivalent.*

*Proof.* $l_1, \ldots, l_n$ — linearly independent from X. $\forall x \in X = \sum\limits_{k=1}^{n} \alpha_k l_K$

$\bar{x} \leftrightarrow (l_1, \ldots, l_n) = \bar{l} \in \mathbb{R}^n$
Let $\| \cdot \|$ — some norm in X.

$\|x\| \underset{\triangle}{\leqslant} \sum\limits_{k=1}^{n} \|l_k\| |\alpha_k| \underset{Cauchy}{\leqslant} \underbrace{\sqrt{\sum\limits_{1}^{n} \|l_k\|^2}}_{const(B), B-basis} \underset{\|\bar{\alpha}\| = \|x\|_1}{\overset{\|}{\sqrt{\sum\limits_{1}^{n} |\alpha_k|^2}}}$

$\|x\|_1 = \sqrt{\sum\limits_{1}^{n} \|\alpha_k\|^2}, \ x = \sum \alpha_k l_k$
$\|x\| \leqslant b\|x\|_1$
$?\exists a > 0 \colon a\|x\|_1 \leqslant \|x\| \implies \| \cdot \| \sim \| \cdot \|_1$
Let $f(\alpha_1, \ldots, \alpha_n) = \left\| \sum\limits_{k=1}^{n} \alpha_k l_k \right\|$

$|f(\bar{\alpha} + \Delta\bar{\alpha}) - f(\bar{\alpha})| = \left\| \left\| \sum\limits_{1}^{k} \alpha_k l_k + \sum\limits_{1}^{n} \Delta\alpha_k l_K \right\| - \left\| \sum\limits_{1}^{n} \alpha_k l_k \right\| \right\| \leqslant$

$\left\| \sum\limits_{1}^{n} \Delta\alpha_k l_k \right\| \leqslant \underbrace{\sum \|l_k\| |\Delta\alpha_k|}_{\underset{0, \Delta\alpha_k \to 0}{\downarrow}} \implies f$ — continuous on $\mathbb{R}^n$

$S_1 = \{ \sum\limits_{1}^{n} |\alpha_k|^2 = 1 \} \subset \mathbb{R}^m$, f — continuous on $S_1$, $S_1$ — compact, $\bar{\alpha}^* \in S_1$
By Weierstrass theorem there exists a point $\alpha^* \in S_1$ on a sphere, in which function $f$ achieves its minimum $\implies \forall \alpha \in S_1 \ f(\bar{\alpha}^*) \leqslant f(\bar{\alpha})$
If $f(\bar{\alpha}^*) = 0$, then $\left\| \sum\limits_{1}^{n} \alpha_k^* l_k \right\| = 0 \implies \sum\limits_{1}^{n} \alpha_k^* l_k = 0, \ \bar{\alpha}^* \in S_1,$

but $l_1 \ldots l_n$ are linearly independent $\rightarrow\leftarrow \implies \min\limits_{S_1} f = m > 0$

$$\|x\| = \left\| \sum_1^n \alpha_K l_K \right\| = f(\bar{\alpha}) = \sqrt{\sum_1^n \alpha_k^2} \cdot \left\| \sum \boxed{\dfrac{\alpha_k}{\sqrt{\sum_1^n \alpha_k^2}}}_{\beta_k} \cdot l_k \right\| \geqslant, \ \bar{\beta} = (\beta_1 \ldots \beta_n) \in S_1$$

$$\geqslant m \cdot \|x\|_1, \ a = m \quad \blacksquare$$