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1 Vector spaces

§1.1 Metric spaces.

X, ρ: $X \times X \to \mathbb{R}_+$

Definition. ρ — **metric**

1. $\rho(x, y) \geqslant 0, = 0 \iff x = y$
2. $\rho(x, y) = \rho(y, x)$
3. $\rho(x, y) \leqslant \rho(x, z) + \rho(y, z)$

Definition. (X, ρ) — **Metric space**.

Definition. $x = \lim x_n \iff \rho(x_n, x) \to 0$

$X, \tau = \{G \subset X\}$

1. $\varnothing, X \in \tau$
2. $G_\alpha \in \tau, \alpha \in \mathscr{A} \implies \bigcup\limits_\alpha G_\alpha \in \tau$
3. $G_1, \ldots, G_n \in \tau \implies \bigcap\limits_{j=1}^{n} G_j \in \tau$

Definition. (X, τ) — **Topological space**.

$x = \lim x_n \quad \forall G \in \tau : x \in G \quad \exists N : \forall n > N \implies x_n \in G$
G — open in τ
$F = X \setminus G$ — closed

Definition. $B_r(a) = \{\, x \mid \rho(x, a) < r \,\}$ — **open ball**

$\tau = \bigcup B_r(a)$

Statement 1.1. $b \in B_{r_1}(a_1) \cap B_{r_2}(a_2) \implies \exists r_3 > 0 : B_{r_3}(a_3) \subset B_{r_1}(a_1) \cap B_{r_2}(a_2)$

In this sense metric space is just a special case of topological space.

Example 1. $\mathbb{R}, \rho(x, y) = |x - y|$, MS

Example 2. $\bar{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n, \rho(\bar{x}, \bar{y}) = \sqrt{\sum\limits_{j=1}^{n} (x_j - y_j)^2}, MS$

Example 3. $\bar{x} = (x_1, \ldots, x_n, \ldots) \in \mathbb{R}^\infty$

$$\alpha\bar{x} = (\alpha x_1, \ldots, \alpha x_n, \ldots)$$

$$\bar{x} + \bar{y} = (x_1 + y_1, \ldots, x_n + y_n, \ldots)$$

Let's define $\lim\limits_{m\to\infty} \bar{x}_m$

- in $\mathbb{R}^n$:

$$\bar{x}_n \to \bar{x} \iff \forall j = 1, \ldots, n \qquad x_j^{(m)} \xrightarrow[m\to\infty]{} x_j$$

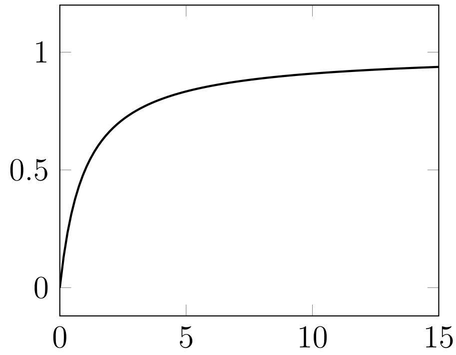
- in $\mathbb{R}^\infty$:

$$\bar{x}_m \to \bar{x} \overset{def}{\iff} \forall j = 1, 2, 3, \dots \quad x_j^{(m)} \xrightarrow[m\to\infty]{} x_j$$

Definition. $\rho(\bar{x}, \bar{y}) \overset{\text{def}}{=} \sum\limits_{n=1}^{\infty} \frac{1}{2^n} \underbrace{\dfrac{|x_n - y_n|}{1 + |x_n - y_n|}}_{\varphi(|x_n - y_n|)}$ — **Urysohn metric**.

$$\varphi(t) = \frac{t}{1+t}$$
$$\varphi(t_1 + t_2) \leqslant \varphi(t_1) + \varphi(t_2)$$
$$\rho(\bar{x}_m, \bar{x}) \xrightarrow[m\to\infty]{} 0 \iff x_j^{(m)} \to x_j \ \forall j$$

In this way $\mathbb{R}^\infty$ is a metrizable space.



$\varphi(t)$ is an upper convex function

Example 4. $X, \rho(x, y) \overset{\text{def}}{=} \begin{cases} 0, & x = y \\ 1, & x \neq y \end{cases}$ — **Discrete metric**.
$x_n \to x,\ \varepsilon = \frac{1}{2},\ \exists M:\ m > M \implies \rho(x_m, x) < \frac{1}{2} \implies$
$\rho(x_m, x) = 0 \implies x_m = x$

Definition. $(X, \tau);\ \forall A \subset X;$

$$\operatorname{Int}(A) \overset{\text{def}}{=} \bigcup_{G \subset A} G - \text{open};$$
$$\operatorname{Cl}(A) \overset{\text{def}}{=} \bigcap_{A \subset G} G - \text{closed};$$
$$\operatorname{Fr}(A) = \operatorname{Cl}(A) \setminus \operatorname{Int}(A)$$

(X, ρ); Having a metric space one can describe closure of a set.
$\rho(x, A) \overset{\text{def}}{=} \inf\limits_{a \in A} \rho(x, a)$
$\rho(A, B) = \inf\limits_{\substack{a \in A \\ b \in B}} \rho(a, b)$
$\rho(x, A) = f(x), x \in X$

Statement 1.2. *Function f(x) is continuous.*

Proof. $\forall x, y \in X$
$f(x) = \rho(x, A) \underset{\forall \alpha \in A}{\leqslant} \rho(x, \alpha) \leqslant \rho(x, y) + \rho(y, \alpha)$
$\forall \varepsilon > 0\ \exists \alpha_\varepsilon \in A:\ \rho(y, \alpha_\varepsilon) < \rho(y, A) + \varepsilon = f(y) + \varepsilon$
$f(x) \leqslant f(y) + \varepsilon + \rho(x, y),\ \varepsilon \to 0$
$\begin{cases} f(x) \leqslant f(y) + \rho(x, y) \\ f(y) \leqslant f(x) + \rho(x, y) \end{cases} \implies |f(x) - f(y)| \leqslant \rho(x, y)$ ■

Statement 1.3. $x \in \operatorname{Cl}(A) \iff \rho(x, A) = 0$

Let's look at the metric spaces in terms of separation of sets from each other by open sets.
x, y
$r = \rho(x, y) > 0$
$B_{\frac{r}{3}}(x),\ B_{\frac{r}{3}}(y)$
In any metric space separability axiom is true.

Theorem 1.1.
Any metric space is a normal space,
i.e. $\forall$ *closed disjoint* $F_1, F_2 \in X,\ \exists$ *open disjoint* $G_1, G_2\colon F_j \in G_j,\ j = 1,2$

Proof. $g(x) = \dfrac{\rho(x,F_1)}{\rho(x,F_1)+\rho(x,F_2)}$ — continuous on X
$x \in F_1,\ \mathrm{Cl}(F_1) = F_1,\ \rho(x,F_1) = 0,\ g(x) = 0$
$x \in F_2,\ g(x) = 1$
Let's look at $(-\infty; \frac{1}{3}), (\frac{2}{3}, \infty)$ — by continuity their inverse images under g are open.
$G_1 = g^{-1}(-\infty; \frac{1}{3})$
$G_2 = g^{-1}(\frac{2}{3}; \infty)$ ∎

Definition. Metric space is **complete** if $\rho(x_n, x_m) \to 0 \implies \exists x = \lim x_n$
$\mathbb{R}^\infty$ – complete (by completeness of the rational numbers).
In complete metric spaces the nested balls principle is true.

Theorem 1.2.
X – complete metric space, $\overline{V}_{r_n}$ – system of closed balls.

1. *$\overline{V}_{r_{n+1}} \subset \overline{V}_{r_n}$ – the system is nested.*
2. *$r_n \to 0$*

<u>*Then:*</u> $\bigcap\limits_n \overline{V}_{r_n} = \{a\}$

Proof. Let b_n be centers of $\overline{V}_{r_n}$,
$m \geqslant n,\ b_m \in \overline{V}_{r_n},\ \rho(b_m, b_n) \leqslant r_n \to 0\ \forall m \geqslant n$
$\rho(b_m, b_n) \to 0 \xRightarrow{compl.} \exists a = \lim b_n$ Since the balls are closed a $\in$ every ball.
$r_n \to 0 \implies$ there is only one common point. ∎

(X, τ) — topological space
$A \subset X,\ \tau_a = \{\, G \cap A, G \in \tau \,\}$ — topology induced on A

Definition. X— metric space, $A \subset X,\ \mathrm{Cl}(A) = X$
<u>Then:</u> A – **dense** in X
<u>If</u> $\mathrm{Int}(\mathrm{Cl}(A)) = \varnothing$ A – **nowhere dense** in X.

Note. It is easy to understand, that in metric spaces nowhere density means the following: $\forall$ ball $V\ \exists V' \subset V\colon V'$ contains no points from A.

Definition. X is called **first Baire category set**, if it can be written as at most countable union of x_n each nowhere dense in X.

Theorem 1.3 (Baire category theorem)**.**
Complete metric space is second Baire category set in itself.

Proof. Let X be first Baire category set.
$X = \bigcup\limits_n X_n \quad \forall \overline{V}\ X_1$ is nowhere dense.
$\overline{V}_1 \subset \overline{V}\colon\ \overline{V}_1 \cap X_1\ = \varnothing$
X_2 is nowhere dense $\overline{V}_2 \subset \overline{V}_1\colon \overline{V}_2 \cap X_2 = \varnothing$
$r_2 \leqslant \dfrac{r_1}{2}$
$\vdots$
$\{\overline{V}_n\},\ r_n \to 0,\ \bigcap\limits_n \overline{V}_n = \{a\},\ X = \bigcup X_n,\ \exists n_0\colon a \in X_{n_0}$
$X_{n_0} \cap \overline{V}_{n_0} = \varnothing \to\leftarrow\ a \in \overline{V}_{n_0}$ ∎

Corollary 1.3.1. *Complete metric space without isolated points is uncountable.*

Proof. No isolated points are present $\implies$ every point in the set is nowhere dense in it. Let X be countable: $X = \bigcup_n \{X_n\}$, then it is first Baire category set. $\rightarrow\leftarrow$ ∎

Definition. K — **compact** if

1. $K = \mathrm{Cl}(K)$
2. $x_n \in K\ \exists n_1 < n_2 < \dots\ x_{n_j}$ – converges in X.

If only 2 is present, the set is called **precompact**.

Theorem 1.4 (Hausdorff)**.**
Let X — metric space, K — closed in X.
<u>Then:</u> K — compact $\iff$ K — totally bounded,
i.e. $\forall \varepsilon > 0\ \exists a_1, \dots, a_p \in X\colon\ \forall b \in K\ \exists a_j\colon\ \rho(a_j, b) < \varepsilon$
$(a_1, \dots, a_p - finite\ \varepsilon - net)$

Proof.

- Totally bounded $\implies$ compact

  K is totally bounded, $x_n \in K\ n_1 < n_2 < \dots < n_k < \cdots$, x_n converges in K

  $$\varepsilon_k \downarrow\to 0\ \varepsilon_1 \quad K \subset \bigcup_{j=1}^{p} \overline{V}_j,\ rad = \varepsilon_1 \qquad (\varepsilon_1\text{-net})$$

  n is finite $\implies$ one ball will contain infinetely many x_n elements.
  Let's look at $\overline{V}_{j_0} \cap K$ — totally bounded $= K_1, \mathrm{diam}(K_1) \leqslant 2\varepsilon_1$
  $\varepsilon_2 \quad K_1 \subset \bigcup_{j=1}^{n} \overline{V'}_j,\ rad = \varepsilon_2.$

  Then one of $\overline{V'}$ contains infinitely many elements of the sequence contained in K_1.
  $\overline{V'}_{j_0} \cap K_1 = K_2,\ \mathrm{diam}(K_2) \leqslant 2\varepsilon_2$ and so on.

  $$K_n \supset K_{n+1} \supset K_{n+2} \supset \dots,\ \mathrm{diam}(K_N) \leqslant 2\varepsilon_n \xrightarrow{\text{by compl.}} \underbrace{\bigcap_{n=1}^{\infty} K_n}_{\mathrm{diam}(K_n)\to 0,\ \{x\}} \neq \varnothing$$

  Take x_{n_1} from K_1, x_{n_2} from $K_2 \dots$

- Compact $\implies$ totally bounded

  K — compact $\forall \varepsilon\ \exists$ finite ε-net?
  By contradiction: $\exists \varepsilon_0 > 0\colon$ finite ε_0-net is impossible to construct.
  $\forall x_1 \in K\ \exists x_2 \in K\colon\ \rho(x_1, x_2) > \varepsilon_0$ (or else system of x_1 — finite ε-net)
  $\{x_1, x_2\}$ - choose $x_3 \in K\colon \rho(x_3, x_i) > \varepsilon_0,\ i = 1, 2$ and so on.
  $x_n \in K\ :\ n \neq m\ \rho(x_n, x_m) > \varepsilon_0$ — contains no converging subsequence $\implies$ set is not a compact. $\rightarrow\leftarrow$ ∎

§1.2 Normed spaces

Definition. X — **linear set**, $x+y, \alpha\cdot x$, $\alpha\in\mathbb{R}$
The purpose of norm definition, is to construct a topology on X, so that 2 linear operations are continuous on it.

$\varphi\colon X\to\mathbb{R}$:

1. $\varphi(x)\geqslant 0,\ =0 \iff x=0$
2. $\varphi(\alpha x)=|\alpha|\varphi(x)$
3. $\varphi(x+y)\leqslant\varphi(x)+\varphi(y)$

Definition. φ — **norm** on X, $\varphi(x)=\|x\|$

$\rho(x,y)\overset{\text{def}}{=}\|x-y\|$ — metric on X.

Definition.
$(X,\|\cdot\|)$ — **normed space** — special case of metrical space.

$x=\lim x_n \overset{def}{\Longrightarrow} \rho(x_n,x)\to 0 \iff \|x_n-x\|\to 0$

Statement 1.4. *In the topology of a normed space linear operations are continuous on X.*

Proof.

1. $$x_n\to x,\ y_n\to y;\ \|(x_n+y_n)-(x+y)\|=\|(x_n-x)+(y_n-y)\|\leqslant$$
$$\leqslant\underbrace{\|x_n-x\|}_{\downarrow\atop 0}+\underbrace{\|y_n-y\|}_{\downarrow\atop 0}$$
$$\Longrightarrow x_n+y_n\to x+y$$

2. $$\alpha_n\to\alpha,\ x_n\to x;\ \|\alpha_n x_n-\alpha x\|=\|(\alpha_n-\alpha)x_n+\alpha(x_n-x)\|\leqslant$$
$$\leqslant\underbrace{|\alpha_n-\alpha|}_{\downarrow\atop 0}\cdot\underbrace{\|x_n\|}_{bounded}+\underbrace{\alpha\|x_n-x\|}_{\downarrow\atop 0}$$

   $x_n\to x \Longrightarrow \|x_n\|$ — bounded.
   $\alpha_x x_n\to\alpha x$ ■

Statement 1.5. *From the triangle inequality* $|\|x\|-\|y\||\leqslant\|x-y\|$
$x_n\to x \implies \|x_n\|\to\|x\|$
Norm is continious.

Example 5. $\mathbb{R}^n$

1. $\|\bar{x}\|=\sqrt{\sum\limits_{k=1}^{n}x_k^2}$
2. $\|\bar{x}\|_1\overset{\text{def}}{=}\sum\limits_{k=1}^{n}|x_k|$
3. $\|\bar{x}\|_2\overset{\text{def}}{=}\max\{|x_1|,\dots,|x_n|\}$
4. $C[a,b]$ — functions continuous on $[a,b]$; $\|f\|=\max\limits_{x\in[a,b]}|f(x)|$

5. $L_p(E) = \left\{ f \text{ — measurable}, \int\limits_E |f|^p < +\infty \right\}$

$$p \geqslant 1,\ \|f\|_p = \left(\int\limits_E |f|^p \right)^{\frac{1}{p}}$$

Because the set of points is the same, arises the question about convergence comparison. $\|\cdot\|_1 \sim \|\cdot\|_2,\ x_n \overset{\|\cdot\|_1}{\to} x \iff x_n \overset{\|\cdot\|_2}{\to} x$

Statement 1.6.
$\|\cdot\|_1 \sim \|\cdot\|_2 \iff \exists a, b > 0 \colon \forall x \in X \implies a\|x_1\|_1 \leqslant \|x\|_2 \leqslant b\|x\|_1$

Theorem 1.5 (Riesz)**.**
X, $\dim X < +\infty$ — linear set.
<u>Then:</u> Any pair of norms in X are equivalent.

Proof. $l_1, \ldots, l_n$ — linearly independent from X. $\forall x \in X = \sum\limits_{k=1}^{n} \alpha_k l_K$

$$\bar{x} \leftrightarrow (l_1, \ldots, l_n) = \bar{l} \in \mathbb{R}^n$$

Let $\|\cdot\|$ — some norm in X.

$$\|x\| \underset{\triangle}{\leqslant} \sum_{k=1}^{n} \|l_k\||\alpha_k| \underset{\text{Cauchy}}{\leqslant} \underbrace{\sqrt{\sum_{k=1}^{n} \|l_k\|^2}}_{const(B), B-basis} \underset{\|\bar{\alpha}\| = \|x\|_1}{\underset{\|}{\sqrt{\sum_{k=1}^{n} |\alpha_k|^2}}}$$

$$\|x\|_1 = \sqrt{\sum_{k=1}^{n} \|\alpha_k\|^2},\ x = \sum \alpha_k l_k$$

$$\|x\| \leqslant b\|x\|_1$$

$$?\exists a > 0 \colon a\|x\|_1 \leqslant \|x\| \implies \|\cdot\| \sim \|\cdot\|_1$$

Let $f(\alpha_1, \ldots, \alpha_n) = \left\| \sum\limits_{k=1}^{n} \alpha_k l_k \right\|$

$$|f(\bar{\alpha} + \Delta\bar{\alpha}) - f(\bar{\alpha})| = \left| \left\| \sum_{k=1}^{n} \alpha_k l_k + \sum_{k=1}^{n} \Delta\alpha_k l_K \right\| - \left\| \sum_{k=1}^{n} \alpha_k l_k \right\| \right| \leqslant$$

$$\left\| \sum_{k=1}^{n} \Delta\alpha_k l_k \right\| \leqslant \underbrace{\sum \|l_k\||\Delta\alpha_k|}_{\underset{0, \Delta\alpha_k \to 0}{\downarrow}} \implies f \text{ is continuous on } \mathbb{R}^n$$

$S_1 = \left\{ \sum\limits_{k=1}^{n} |\alpha_k|^2 = 1 \right\} \subset \mathbb{R}^m$, f — continuous on S_1, S_1 — compact, $\bar{\alpha}^* \in S_1$
By Weierstrass theorem there exists a point $\alpha^* \in S_1$ on a sphere, in which function f achieves its minimum $\implies \forall \alpha \in S_1\ f(\bar{\alpha}^*) \leqslant f(\bar{\alpha})$

If $f(\bar{\alpha}^*) = 0$, then $\left\| \sum\limits_{k=1}^{n} \alpha_k^* l_k \right\| = 0 \implies \sum\limits_{k=1}^{n} \alpha_k^* l_k = 0,\ \bar{\alpha}^* \in S_1$,
but $l_1 \dots l_n$ are linearly independent $\rightarrow\leftarrow \implies \min\limits_{S_1} f = m > 0$

$$\|x\| = \left\| \sum_{k=1}^{n} \alpha_k l_k \right\| = f(\bar{\alpha}) = \sqrt{\sum_{k=1}^{n} \alpha_k^2} \cdot \left\| \sum \underbrace{\boxed{\frac{\alpha_k}{\sqrt{\sum_{k=1}^{n} \alpha_k^2}}}}_{\beta_k} \cdot l_k \right\| \geqslant,\ \bar{\beta} = (\beta_1 \dots \beta_n) \in S_1$$

$$\geqslant m \cdot \|x\|_1,\ a = m \qquad \blacksquare$$

Corollary 1.5.1. *X — NS, $Y \subset X, \dim Y < +\infty \implies Y = \mathrm{Cl}(Y)$*

Note. Functional analysis differentiates between linear subset (set of points, closed by addition and scalar multiplication) and linear subspace (closed linear subset).

Proof. $Y = \alpha(l_1, \dots, l_n) = \left\{ \sum\limits_{i=1}^{n} \alpha_i l_i \;\middle|\; l_1, \dots, l_n - \text{lin. indep.} \right\}$
$y_m \in Y, y_m \to y$ in $X \implies y \in Y$?
$\|y_m - y\| \to 0 \implies \|y_m - y_p\| \to 0,\ m, p \to \infty$
$\|y\|, y \in Y$.
By Riesz theorem all norm in Y are equivalent.
$y = \sum\limits_{j=1}^{n} \alpha_j l_J, \|y\|_0 = \sqrt{\sum\limits_{j=1}^{n} \alpha_j^2}$ — some norm (by linear independance).
By Riesz theorem $\|y\| \sim \|y\|_0$
$\underbrace{\|y_m - y_p\|}_{\in Y} \to 0 \implies \|y_m - y_p\|_0 \to 0$
$\bar{\alpha} = (\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n$ $y_m = \sum\limits_{i=1}^{n} \alpha_i^{(m)} l_i$
$|\alpha_i^{(m)} - \alpha_i^{(l)}| \to 0\ \forall i = 1, \dots, n;\ \bar{\alpha} = (\alpha_1^{(m)}, \dots, \alpha_n^{(m)}) \to \alpha^* = (\alpha_1^*, \dots, \alpha_n^*)$
$y^* = \sum\limits_{i=1}^{n} \alpha_i^* l_I \in Y,\ \|y_m - y\| \to 0, y = y^* \implies y \in Y$ $\blacksquare$

Definition. If normed space if complete, then it is called **B-space** or **Banach space**.

Example 6. $C[a, b]$ — functions continuous on $[a, b]$.

Example 7. Lebesgue space, $p \geqslant 1, L_p(E) = \left\{ f \text{ is measurable on } E,\ \int\limits_E |f|^P < +\infty \right\}$.

If X — Banach space,
$\sum\limits_{n=1}^{\infty} x_n = \lim\limits_{n\to\infty} \sum\limits_{k=1}^{n} x_k,\ \sum\limits_{n=1}^{\infty} \|x_n\| < +\infty$

$$\|S_n - S_m\| = \left\| \sum_{k=m+1}^{n} x_k \right\| \leqslant \sum_{m+1}^{n} \|x_k\| \xrightarrow[n,m\to\infty]{} 0$$

$$\implies \|S_n - S_m\| \to 0 \implies \exists \lim_{n\to\infty} S_n, \sum_{k=1}^{n} x_k \text{— converges.}$$

In Banach spaces works the theory of absolute convergence of numerical series.

Lemma 1 (Riesz)**.**