Functional Analysis Assignment 1

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Question 1

Theorem 1 *Suppose that B_k is the kth Bernoulli polynomial, and $e_n(t) = \exp(2\pi int)$. Then for $n \neq 0$, we have*

$$\langle B_k, e_n \rangle = -\frac{k!}{(2\pi in)^k}$$

and

$$\langle B_k, e_0 \rangle = 0.$$

Proof Put $n \neq 0$.
Consider first $k = 1$. By definition, $B_1(t) = t - \frac{1}{2}$. Then,

$$\begin{aligned}\langle B_1, e_n \rangle &= \int_0^1 (x - \frac{1}{2})e^{-2\pi inx}dx \\ &= \int_0^1 xe^{-2\pi inx}dx\end{aligned}$$

since $\int_0^1 e^{2\pi inx}dx = 0$. Then using integration by parts,

$$\int_0^1 xe^{-2\pi inx}dx = -\frac{1}{2\pi in}.$$

Hence the result holds for $k = 1$. Suppose now that $k > 1$.
The kth Bernoulli polynomial can be found by the fomula $kB_{k-1}(t) = B_k'(t)$ when $k > 1$. Since $B_k(0) = B_k(1)$, and $e_n(0) = e_n(1)$, we have the relation

$$\langle B_k', e_n \rangle = -\int_0^1 B_k(t)\frac{d}{dt}[e^{-2\pi int}]dt$$

This gives us

$$2\pi in\langle B_k, e_n \rangle = k\langle B_{k-1}, e_n \rangle.$$

Thus we have a recurrence relation:

$$\langle B_k, e_n \rangle = \frac{k}{2\pi in}\langle B_{k-1}, e_n \rangle$$

valid for $k > 1$. Hence by induction,

$$\langle B_k, e_n \rangle = \frac{k!}{(2\pi in)^{k-1}}\langle B_1, e_n \rangle.$$

So finally,

$$\langle B_k, e_n \rangle = -\frac{k!}{(2\pi in)^k}$$

for $n \neq 0$.
For $n = 0$, we have

$$\begin{aligned}\langle B_k, e_0 \rangle &= \int_0^1 B_k(x)dx \\ &= \int_0^1 \frac{1}{k+1} B'_{k+1}(x)dx \\ &= \frac{1}{k+1}(B_{k+1}(1) - B_{k+1}(0)) \\ &= 0\end{aligned}$$

as required. $\square$

Hence we have the expansion,

$$B_k(t) = \sum_{n \in \mathbb{Z} \setminus \{0\}} -\frac{k!}{(2\pi i n)^k} e^{2\pi i n t}.$$

Question 2

For this question, t is a variable defined on $[-1, 1]$, and we consider functions in $L^2(-1, 1)$.

Definition The nth Legendre polynomials L_n is

$$L_n(t) := \frac{1}{2^n n!} \sqrt{\frac{2n+1}{2}} \frac{d^n}{dx^n} [(t^2 - 1)^n]$$

and $L_0(t) := 1$. Define $v_n(t) = t^n$.

Lemma 2

$$\langle L_n, v_m \rangle = \begin{cases} 0 & \textit{if } n > m \\ 0 & \textit{if } m - n \textit{ is odd and positive.} \\ \frac{1}{2^n}\sqrt{\frac{2n+1}{2}}\binom{m}{n} B(\frac{m-n+1}{2}, n+1) & \textit{if } m - n \textit{ is even and positive.} \end{cases}$$

where B is the beta function.

Proof Since the function $f(t) = t^2 - 1$ has $f(0) = f(1) = 0$, we can use the integration by parts rule,

$$\begin{aligned}\langle L_n, v_m \rangle &= \frac{1}{2^n n!} \sqrt{\frac{2n+1}{2}} \int_{-1}^1 \frac{d^n}{dx^n} [(x^2 - 1)^n] x^m dx \\ &= \frac{(-1)^n}{2^n n!} \sqrt{\frac{2n+1}{2}} \int_{-1}^1 (x^2 - 1)^n \frac{d^n}{dx^n} (x^m) dx.\end{aligned}$$

If $n > m$, $\frac{d^n}{dx^n} x^m$ vanishes, and we get $\langle L_n, v_m \rangle = 0$.
In the case $n \leq m$, we have $\frac{d^n}{dx^n}(x^m) = \frac{m!}{(m-n)!} x^{m-n}$. Hence,

$$\langle L_n, v_m \rangle = \frac{1}{2^n} \sqrt{\frac{2n+1}{2}} \binom{m}{n} \int_{-1}^1 (1 - x^2)^n x^{m-n} dx.$$

Now if $n - m$ is odd, this is an integral of an odd function over a symmetric domain. So $\langle L_n, v_m \rangle = 0$ when $n - m$ is odd.

Now if $n-m$ is even, use the substitution $u = x^2$ so that

$$\langle L_n, v_m \rangle = \frac{1}{2^n} \binom{m}{n} \sqrt{\frac{2n+1}{2}} \int_0^1 (1-u)^n u^{\frac{m-n}{2} - \frac{1}{2}} du.$$

Hence we have by the definition of the beta function,

$$\langle L_n, v_m \rangle = \frac{1}{2^n} \binom{m}{n} \sqrt{\frac{2n+1}{2}} B(\frac{m-n+1}{2}, n+1)$$

as required. □

Theorem 3 *The set $\{L_n\}_{n=1}^{\infty}$, when normalised, is the result of Gram-Schmidt orthonormalisation applied to the set $\{v_n\}_{n=0}^{\infty}$.*

Proof Define the set $\{B_n\}_{n=0}^{\infty}$ as the result of Gram-Schmidt orthogonalisation applied to $\{v_n\}_{n=0}^{\infty}$. That is,

$$\begin{aligned} B_0 &= v_0 \\ B_n &= v_n - \sum_{k=0}^{n-1} \frac{\langle B_k, v_n \rangle}{\langle B_k, B_k \rangle} B_k. \end{aligned}$$

Write $P_n = \text{span}\{v_k\}_{k=0}^{n}$ as the set of polynomials of degree at most n. Note that by construction $\text{span}\{B_k\}_{k=0}^{n} = P_n$.
We show that B_n is parallel L_n for $n \geq 0$. For $n = 0$, we have $B_0(t) = 1 = L_0(t)$.
By construction, B_n is in the orthogonal complement of P_{n-1} in P_n. Since P_{n-1} is n dimensional, and P_n is $n+1$ dimensional, the orthogonal complement of P_{n-1} in P_n is one dimensional.
Hence to show that L_n is parallel to B_n it will be enough to show that L_n lies in the orthogonal complement to P_{n-1} in P_n, since then both B_n and L_n lie in the same one dimensional subspace of P_n.
Clearly since L_n is a polynomial of degree n, we have $L_n \in P_n$.
By lemma 2, $\rangle L_n, v_k \langle = 0$ for $k < n$. Hence L_n is orthogonal to every element P_{n-1}.
Hence, L_n is parallel to B_n. So if we normalise each L_n, we obtain the result of Gram-Schmidt orthonormalisation applied to the set $\{v_n\}_{n=0}^{\infty}$. □

Question 3

Lemma 2 gives us the decomposition of v_n in terms of L_n. So,

$$\begin{aligned} v_m &= \sum_{n=0}^{\infty} \langle L_n, v_m \rangle L_n \\ &= \sum_{n=0}^{m} \langle L_n, v_m \rangle L_n \end{aligned}$$

since $\langle L_n, v_m \rangle$ vanishes for $n > m$. So if m is even, we have

$$v_m = \sum_{k=0}^{m/2} \frac{1}{2^{2k}} \binom{m}{2k} \sqrt{\frac{4k+1}{2}} B(\frac{m-2k+1}{2}, n+1) L_{2k}.$$

If m is odd, we have

$$v_m = \sum_{k=0}^{(m-1)/2} \frac{1}{2^{2k+1}} \binom{m}{2k+1} \sqrt{\frac{4k+2}{2}} B(\frac{m-2k-1}{2}, n+1) L_{2k+1}.$$