University of New South Wales

School of Mathematics and Statistics

Assignment 3

Functional Analysis

Author:
Edward McDonald

Student Number:
3375335

Question 1

For this question, X is a normed space and X_0 is a closed subspace.

Proposition 1. *There is an isometric embedding,* $X/X_0 \hookrightarrow (X_0^\perp)^*$.

Proof. Define $\rho : X/X_0 \to (X_0^\perp)^*$ as $\rho(x+X_0)(f) = f(x)$, for $x \in X$ and $f \in X_0^\perp$.
This is a well defined linear map, since if $x' + X_0 = x + X_0$ then $f(x'-x) = 0$ as $f \in X_0^\perp$.
See that ρ is linear, since $\rho(\alpha x + y + X_0) = f(\alpha x + y) = \alpha f(x) + f(y)$ for $x, y \in X$ and $\alpha \in \mathbb{C}$.
We need to show that ρ is an isometry. That is,

$$\|\rho(x+X_0)\|_{(X_0^\perp)^*} = \|x+X_0\|_{X/X_0}.$$

for any $x \in X$. This will also prove that for each $x \in X$, $\rho(x+X_0)$ is bounded, so ρ is indeed well defined.
Expanding this out into definitions, we must prove that

$$\sup_{\|f\|_{X^*}\leq 1, f \in X_0^\perp} |f(x)| = \inf_{y \in X_0} \|x-y\|. \tag{1}$$

for any $x \in X$.
Suppose that $f \in X_0^\perp$ with $\|f\|_{X^*} \leq 1$, then for any $y \in X_0$ and $x \in X$.

$$|f(x)| = |f(x-y)| \leq \|x-y\|.$$

Hence,

$$|f(x)| \leq \inf_{y \in X_0} \|x-y\|.$$

So

$$\sup_{\|f\|_{X^*}\leq 1, f \in X_0^\perp} |f(x)| \leq \inf_{y \in X_0} \|x-y\|$$

follows. Now we must prove the opposite inequality.
Fix $x \in X$. Then on the subspace $V := \mathbb{C}x$, define the functional

$$\omega(\lambda x) = \lambda \|x+X_0\|_{X/X_0}.$$

ω is linear, and for any $y \in V$, $|\omega(y)| = \|y+X_0\| \leq \|y\|$. So $\|\omega\| \leq 1$.
So ω is a linear functional on a subspace V of X bounded by the seminorm $\|\cdot\|_{X/X_0}$.
So by the Hahn-Banach theorem (as in Rudin 1991, Theorem 3.2 or Wikipedia *Hahn-Banach Theorem* 14/5/2014), there is a functional $f \in X^*$, with $f(y) = \omega(y)$ for $y \in V$ and $|f(z)| \leq \|z+X_0\|$ for any $z \in X$. Hence for $z \in X_0$, $f(z) = 0$, so $f \in X_0^\perp$ and $|f(z)| \leq \|z\|$, so $\|f\|_{X^*} \leq 1$.
Therefore, $|f(x)| = \|x+X_0\|_{X/X_0}$. Hence,

$$\sup_{\|f\|_{X^*}\leq 1, f \in X_0^\perp} |f(x)| \geq \|x+X_0\|_{X/X_0} = \inf_{y \in X_0} \|x-y\|.$$

So the equality 1 holds. Hence ρ is an isometric embedding. □

Question 2

Theorem 1. *Suppose $T : X \to Y$ is a linear mapping between normed spaces X and Y. Then T is bounded if and only if T has the property if that if U is open in Y then $T^{-1}(U)$ is open in X.*

Proof. Suppose that T is bounded, and let $U \subset Y$ be an open set with $T^{-1}(U) \neq \emptyset$.
Then let $x \in T^{-1}(U)$, and let ε be small enough such that $B_Y(Tx, \varepsilon) \subset U$.
Then choose $\varepsilon' = \varepsilon/\|T\|$. Then if $y \in B_X(x, \varepsilon')$,

$$\|Ty - Tx\| \leq \|T\|\|y - x\| < \varepsilon.$$

Hence $Ty \in B_Y(Tx, \varepsilon)$, so $Ty \in U$. Therefore $B_X(x, \varepsilon') \subset T^{-1}(U)$, and so $T^{-1}(U)$ is open.
Conversely, suppose that T has the property that $T^{-1}(U)$ is open in X whenever U is open in Y.
Let $U = B_Y(0, 1) \subset Y$. Then since $T^{-1}(U)$ is open, there is some $\varepsilon > 0$ such that $B_X(0, \varepsilon) \subset T^{-1}(U)$.
So $TB_X(0, \varepsilon) \subseteq B_Y(0, 1)$. Hence, by linearity, $TB_X(0, 1) \subseteq B_Y(0, \frac{1}{\varepsilon})$.
Hence, $\|T\| \leq 1/\varepsilon$. So T is bounded.

□

Question 3

Let $X = \ell^1(\mathbb{N})$, and X_0 is the subspace defined by

$$X_0 = \{(\xi_k)_{k\geq 0} \in X \ : \ \sum_{k\geq 0} \xi_k = 0\}.$$

Theorem 2. *There is an isometric isomorphism,*

$$X/X_0 \cong \mathbb{C}.$$

Proof. Define the function $S : X \to \mathbb{C}$ by

$$S((\xi_k)_{k\geq 0}) = \sum_{k\geq 0} \xi_k.$$

S is clearly linear and S is surjective since $S(\lambda, 0, 0, 0, \ldots) = \lambda$ for any $\lambda \in \mathbb{C}$.
So by the first isomorphism theorem, there is a vector space isomorphism,

$$X/\ker S = X/X_0 \cong \mathbb{C}.$$

Let Ψ be this isomorphism, that is, $\Psi(x + X_0) = S(x)$ for $x \in X$.
Now we must show that this is an isometry.

Let $e_0 = (1, 0, 0, \ldots) \in X$. Then if $x = (\xi_k)_{k\geq 0} \in X_0$,

$$\begin{aligned}
\|e_0 + x\| &= |1 + \xi_0| + \sum_{k\geq 1} |\xi_k| \\
&\geq |1 + \xi_0| + |\sum_{k\geq 1} \xi_k| \\
&= |1 + \xi_0| + |\xi_0| \\
&\geq 1.
\end{aligned}$$

Hence, $\|e_0 + X_0\|_{X/X_0} = 1$.
Since we have shown X/X_0 is one dimensional, $X/X_0 = \mathbb{C}(e_0 + X_0)$. Hence, for all $y + X_0 \in X/X_0$, $y = \lambda(e_0 + X_0)$ for some $\lambda \in \mathbb{C}$. Then,

$$|\Psi(y)| = |S(\lambda e_0)| = |\lambda| = \|y\|_{X/X_0}.$$

So Ψ is an isometry. $\square$

Theorem 3. *$X_0^\perp$ is the one dimensional subspace of ℓ^∞, $\mathbb{C}(1, 1, 1, \ldots)$, where $(1, 1, 1, \ldots)$ is a constant sequence with value 1.*

Proof. We identify X^* with ℓ^∞. For $(\eta_k)_{k\geq 0} \in X_0^\perp$, we require for all $(\xi_k)_{k\geq 0} \in X_0$,

$$\sum_{k\geq 0} \overline{\xi_k}\eta_k = 0.$$

Choose the sequence $(\xi_k)_{k\geq 0} \in X_0$ as $\xi_0 = 1$ and $\xi_p = -1$ for some $p > 0$, and $\xi_k = 0$ otherwise. Then,

$$\eta_0 - \eta_p = 0.$$

Hence $(\eta_k)_{k\geq 0}$ is a constant sequence. So $X_0^\perp$ consists of constant sequences. Clearly any constant sequence is in $X_0^\perp$, so $X_0^\perp = \mathbb{C}(1, 1, 1, \ldots)$. $\square$