AFRICA BUSINESS SCHOOL – UM6P

Program FEOR – Module 6 – Optimization II

Rachid Ellaia & Agnès Gorge

How considering uncertainties in an optimization problem ?

Reminder : what is a mathematical program ?

$$\begin{cases} \text{Min} \quad f_0(x) \\ \text{S.t.} \quad f_1(x) \leq 0 \\ \quad \vdots \\ \quad f_m(x) = 0 \\ \boldsymbol{x} \in \mathbb{R}^{\boldsymbol{n}} \end{cases}$$

Consider uncertainties in data

$$\begin{cases} \text{Min} \quad f_0(x, \xi) \\ \text{S.t.} \quad f_1(x, \xi) \leq 0 \\ \quad \vdots \\ \quad f_m(x, \xi) = 0 \\ \boldsymbol{x} \in \mathbb{R}^{\boldsymbol{n}}, \xi \in \mathbb{R}^{\boldsymbol{p}} \end{cases}$$

Ill-defined

Operators $h_0, \ldots, h_m$

3 criteria for the choice of h_j:

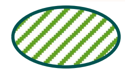
- Reflect decision-maker’s risk profile
- Make the best use of our knowledge of ξ
- Lead to a tractable deterministic counterpart

Determine its deterministic counterpart :

$$\begin{cases} \text{Min} \quad g_0(x) = h_0(f_0(x, \xi)) \\ \text{S.t.} \quad g_1(x) = h_1(f_1(x, \xi)) \leq 0 \\ \quad \vdots \\ \quad g_m(x) = h_m(f_m(x, \xi)) = 0 \\ \boldsymbol{x} \in \mathbb{R}^{\boldsymbol{n}}, \xi \in \mathbb{R}^{\boldsymbol{p}} \end{cases}$$

Well-defined

Two main paradigms co-exist for choosing operators h_i

	 **Stochastic optimization**	 **Robust optimization**
Uncertain data representation	Probability distribution *Approximation through a bunch of scenario*	Support = all possible values
Objective function	h_0 = Expected value	h_0 = Min/max
Constraint	h_i = Quantile	h_i = Min/max
Philosophy	**Average approach** *Requires a good knowledge of the uncertainties*	**Worst-case approach** *Very conservative*
Typical application field	 Finance	 Engineering

TP5 : The newsvendor problem

Problem presentation

Decision variable

- **x :** purchase

Uncertain data : demand to satisfy **D** of distribution F

Certain datas :

- SalesPrice p
- LiqPrice l (<p)
- UnitCost c

$$\text{Max} \quad E[p \min(\mathbf{x}, D) + l \max(0, \mathbf{x} - D)] - c\,\mathbf{x}$$

Resolution if F is Gaussian

Solution : $x^* = F^{-1}\left(\frac{p-c}{p}\right)$

→ **One of the rare example where a closed formula can be found !!**

TP5 : The newsvendor problem through LP with a discrete approximation of the uncertain demand (scenarios)

Problem presentation

Decision variable
- Purchase
- Sales[scenarios] : how much to sell
- Liq[scenarios] : how much to liquidate
- Revenue[scenarios] : the revenue of sales and liquidation

Uncertain data : demand following a discrete distribution (scenarios)
Certain datas :
- SalesPrice
- LiqPrice
- UnitCost

Resolution through LP

1. **Formulate the problem as a linear program, for maximization of expected profit**
2. **What's the gain compared to a deterministic optimization based on the expected demand**
3. **What about considering the minimization of the variance in the objective function ?**
4. **Consider chance-constraint : Proba[sold out] <= 1%**
5. **Bonus** : What is the optimal solution if the demand follows a Gaussian distribution (with same expected value and standard deviation) ?

Chance-constraint (or probability constraint)

Define a binary variable per scenario to « flag » whether or not the sold-out happen in this scenario

Let X[s] be this variable : we want that « sold-out on scenario s implies X[s] = 1 »

Then we limit the number of X[s] = 1

Reminder : BigM tips

- X a continuous variable
- B a binary variable
- **We want : X > 0 implies B = 1 :** for this we add a « bigM » constraint, i.e.,
  - **X <= M. B with M sufficiently big**