A Textbook of

Mathematics

For Grade XII

Test Edition

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"Stop Corruption"
"Save the Nation."

A Textbook of

Mathematics

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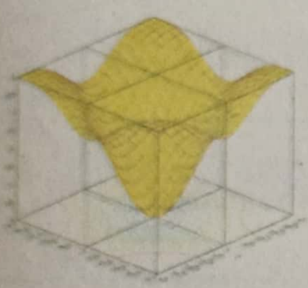

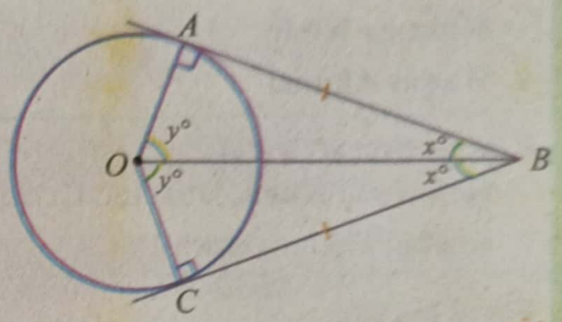

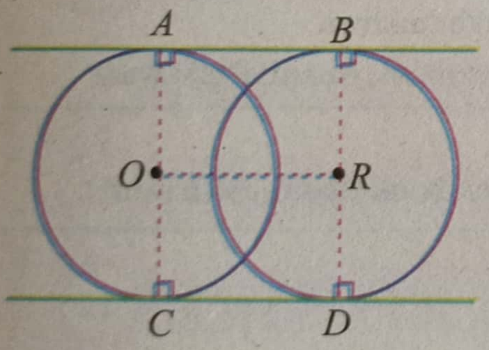

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Table of Contents

Unit 1

INTRODUCTION TO SYMBOLIC PACKAGE: MAPLE

By the end of this unit, the students will be able to:

1.1 Introduction

i. Recognize MAPLE environment.
ii. Recognize basic MAPLE commands.
iii. Use MAPLE as a calculator.
iv. Use online MAPLE help.

1.2 Polynomials

i. Use MAPLE commands for.
- factoring a polynomial,
- expanding an expression,
- simplifying an expression,
- simplifying a rational expression,
- substituting into an expression.

1.3 Graphics

i. Plot a two-dimensional graph.
ii. Demonstrate domain and range of a plot.
iii. Sketch parametric equations.
iv. Know plotting options.

1.4 Matrices

i. Recognize matrix and vector entry arrangement.
ii. Apply matrix operations.
iii. Compute inverse and transpose of a matrix.

1.1 Introduction

In the modern age of science and technology, the technical computation has become the heart of problem solving in engineering and mathematics. To help us, MAPLE offers a vast repository of mathematical algorithms converting a wide range of applications. It is a symbolic and numeric computing tools as well as a multi-paradigm programming language maple was conceived at the university of waterloo in 1980. From the first day it continues to be the benchmark software for mathematical and symbolic computation. Maple user interface allows us to harness all the computational power by using context sensitive means, interactive assistant and task templates. In this unit we will learn how to use the basic commands that will allow and lead us into the creative, dynamic and captivating world of MAPLE explorations.

1.1.1 Recognition of MAPLE Environment

Maple software consists of three different parts.

i. User interface

It handles the input of mathematical expressions and different commands. User interface also handles the display of output and the control of the MAPLE worksheet environment.

ii. Kernel

It is a small collection of compiled C code. The entire kernel is loaded when a MAPLE session is started. It contains those essential facilities that required to run maple and perform basic mathematical operations. The components of kernel include the maple programming language interpreter, arithmetics and memory management facilities and fundamental functions. The small size of kernel ensures that the maple system is portable, compact and efficient.

iii. Library

It contains most of the maple routines including functions related to linear algebra, statistics, calculus, graphics and other topics. This library also consists of individual routines and different packages of routines. All the library routines which are implemented in the high level maple programming language that can be viewed and modified. Hence, it is useful to learn the maple programming language so, we can modify the existing code to produce the required routines.

A. Getting started with Maple

The maple software runs on different systems and platforms. It depends on the platform and system. It is convenient to use if you have windows based operating system (installed maple software package 14, 18 or any latest package).

When a maple session is started, the maple prompt command (>) is displayed like [>.

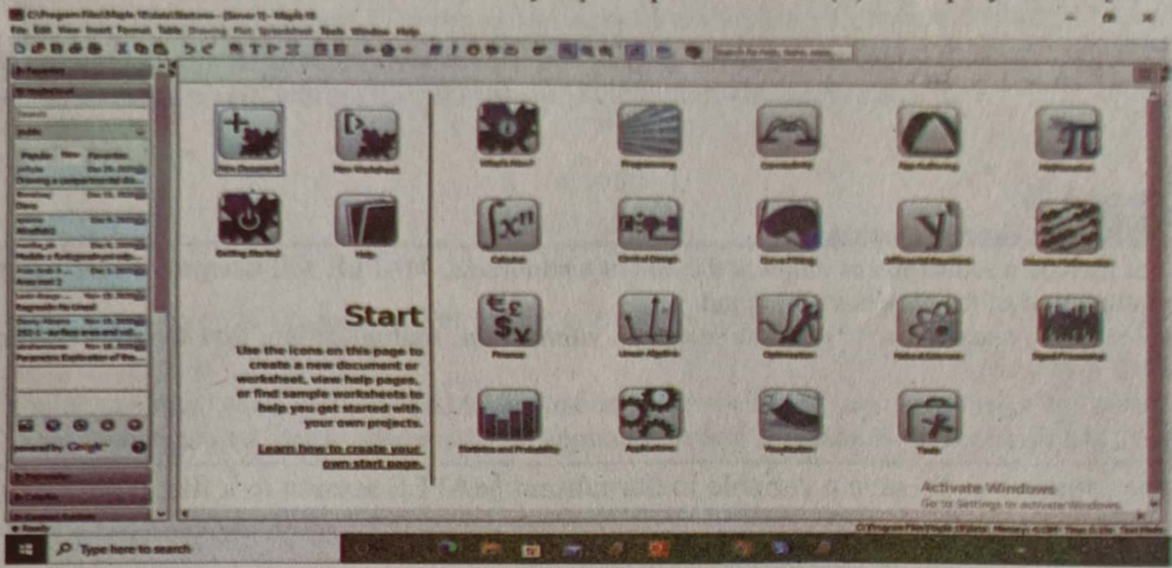


Figure 1.1 Maple start menu

This prompt character will show at the upper left of the worksheet indicates that maple is waiting to receive input in the form of maple statement.

When you are finished with the maple session, you will leave the program by selecting "Exit" under the file menu (upper left of the maple tools bar) as shown in Figure 1.2.

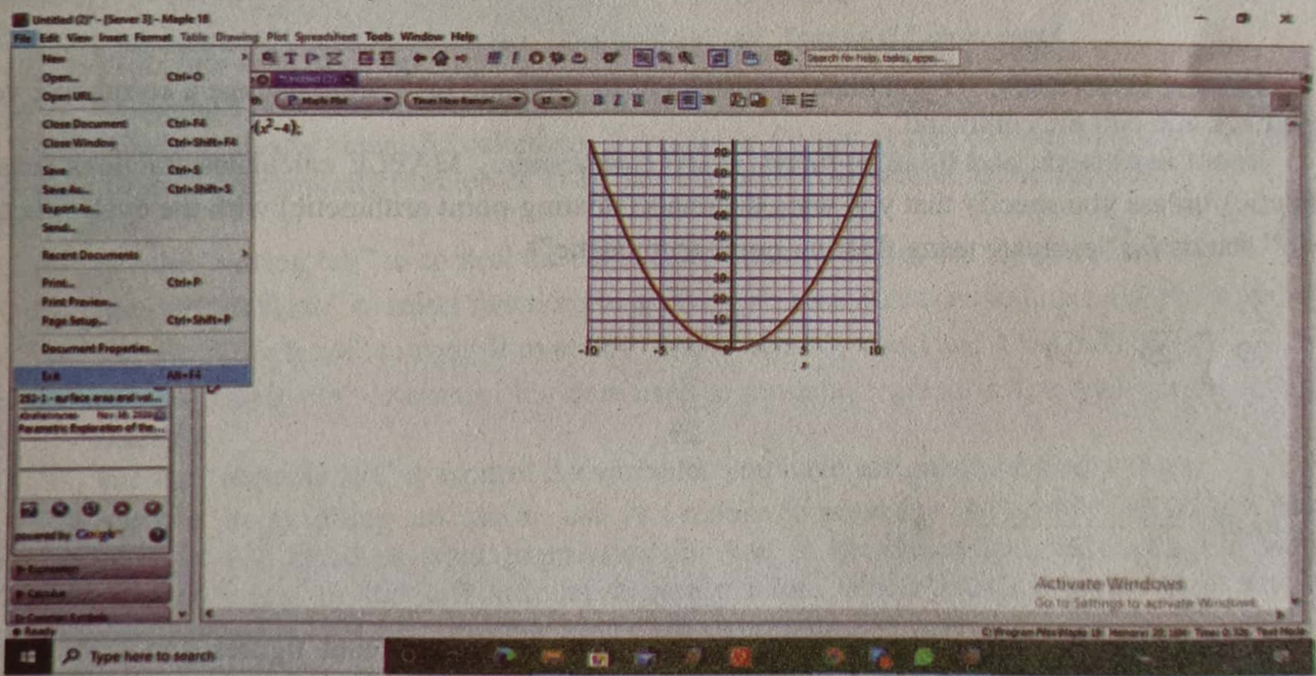


Figure 1.2

1.1.2 Recognition of basic MAPLE commands

A MAPLE command is a statement of calculation followed by a semicolon or a colon. Following are some commands followed by the displayed results.

Enter the commands on your worksheet and verify the given results. When you get to "Save the file", select "Save" under the "File" menu or use CTRL-S to save your working. For example,

> $7+17;$

$$24 \quad (1)$$

> $12^2+15\cdot 2-3;$

$$171 \quad (2)$$

> $12+4*(6+1-10);$

$$0 \quad (3)$$

>

Remember

If you do not include a semicolon or colon at the end of a command, MAPLE will interpret the next command line as a continuation of the previous command.
The symbol +, –, *, / and ^ (or **) denote addition, subtraction, multiplication, division and exponential ($4\text{^}2 = 4**2 = 4^2 = 16$).
When a string of operations are specified in a command, MAPLE first does exponentiations then multiplication and divisions, then additions and subtractions. To change the order, we use parentheses.

i. **Save command:** To save a variable in our current MAPLE session to a file, type the following at the command prompt.

Save variablename, "Filename.m";

Replace the variable name with your variable name and replace the file name with your file name, but keep **.m** extension.

The Save command saves the variable as a MAPLE assignment statement. If the value of your variable depends on other variables, you must save them as well.

You can save more than one variable by giving all variable names to the save command.

Save variablename1, variablename2, variablename3, "filename.m";

ii. **Editing commands:** If you make a mistake in a command or want to change a command, you can go back and edit the command.

iii. **Exact arithmetic and floating-point (evalf) commands:** MAPLE calculates fractions (exact arithmetic) unless you specify that you want decimals (floating-point arithmetic) with the evalf function ("evalf" stands for "evaluate using floating-point arithmaetic").

> $\frac{27}{29}+\frac{14}{29}-\frac{4}{29};$

$$\frac{37}{29} \quad (4)$$

> $evalf\left(\frac{27}{29}+\frac{14}{29}-\frac{4}{29},4\right);$

$$1.276 \quad (5)$$

The argument 4 in the evalf command specifies the number of significant figures you want in the result. If you omit this command, you will get ten significant figures:

> $evalf\left(\frac{27}{29}+\frac{14}{29}-\frac{4}{29}\right);$

$$1.275862069 \quad (6)$$

>

iv. **Maple internal memory clearing command:** To clear the internal memory during a maple session. We use 'restart' command or click the restart icon on the tool bar of the worksheet. e.g

```
> restart
```

when you enter this command, the maple session returns to its startup state. All the values reset to their initial values.

v. **Enlistment of variables:** Use the colon-equal symbol (:=) to define variables that is, to assign values to them. Once you have defined a variable, simply typing its name will show its value, and using the name in a formula will cause the value to be substituted. For example,

> $A := 25;\ B := 145;$

$$A := 25$$
$$B := 145 \qquad (7)$$

>

If you want to string commands together on the same line, then:

> $A := 5; B := 10; C := 12;$

$$A := 5$$
$$B := 10$$
$$C := 12 \qquad (8)$$

> $2 \cdot A + 4 \cdot B + 5 \cdot C;$

$$110 \qquad (9)$$

> $\frac{4 \cdot A}{B};$

$$2 \qquad (10)$$

>

1.1.3 Use of MAPLE as a calculator

You click-start, then select-program <Maple 18 < click-"Maple Calculator" to obtain;

(a) **Maplesoft (TM) Graphing Calculator Overview**

This graphically scientific calculator is available for use as part of your Maple(TM) installation or via a Web Server running MapleNet(TM). The calculator use Maple for calculations.

On toolbar,

- use the "setting tab" to control the basic computation settings for the calculator.
- use the "Math tab" to select functions to apply, from basic functions to linear algebra to statistics.
- use the "Graph tab" to control over how graphs are displayed and what they display.
- use the "Data tab" to control the data used to produce a graph or the data you have tabulated directly.
- use the "variable tab" to control the variables you have assigned and their values.

To involve the graphing calculator, use you mouse to press the "Math tab" and select functions to apply. This will build up your expression for you in the input area, which is just below the session history area on the left side of the calculator. When you are ready to evaluate your expression, press ENTER key on your keyboard. Alternatively, you can press the "Graph button", to graph the expression, or the "data button", to tabulate values for the expression.

Example 1 Differentiate $f(x)=x^2+4x+4$ with respect to x at a point $x=2$. The steps required for obtaining the graphing-calculator result are:

Solution Click-Math tab < Click-Calculus < Click-Differentiate to obtain:

Diff(1)

Cursor "1" requires A: the expression x^2+4x+4, X: the differentiation of $f(x)$ with respect to x and P: the differentiation of $f(x)$ at a point $x=2$:

Diff(A,X,P)

Diff(x^ 2 + 4x + 4, x,2)

Click-ENTER

8

Example 2 Integrate x^2+4x+4, with respect to x over the interval [0,1]. The graphing – calculator result is as under:

Solution

Int(1)

Cursor "1" requires A: the function x^2+4x+4, X: the integration of $f(x)$ with respect to x, P: the lower limit $x=0$ and Q: the upper limit $x=1$ of the integral.

Int(A,X,P,Q)

Int(x ^ 2 + 4x + 4,x,0,1)

6.333333

(b) Unit calculator

To use maple as a calculator simply type in your worksheet command prompt e.g.

|>? Assistant calculator

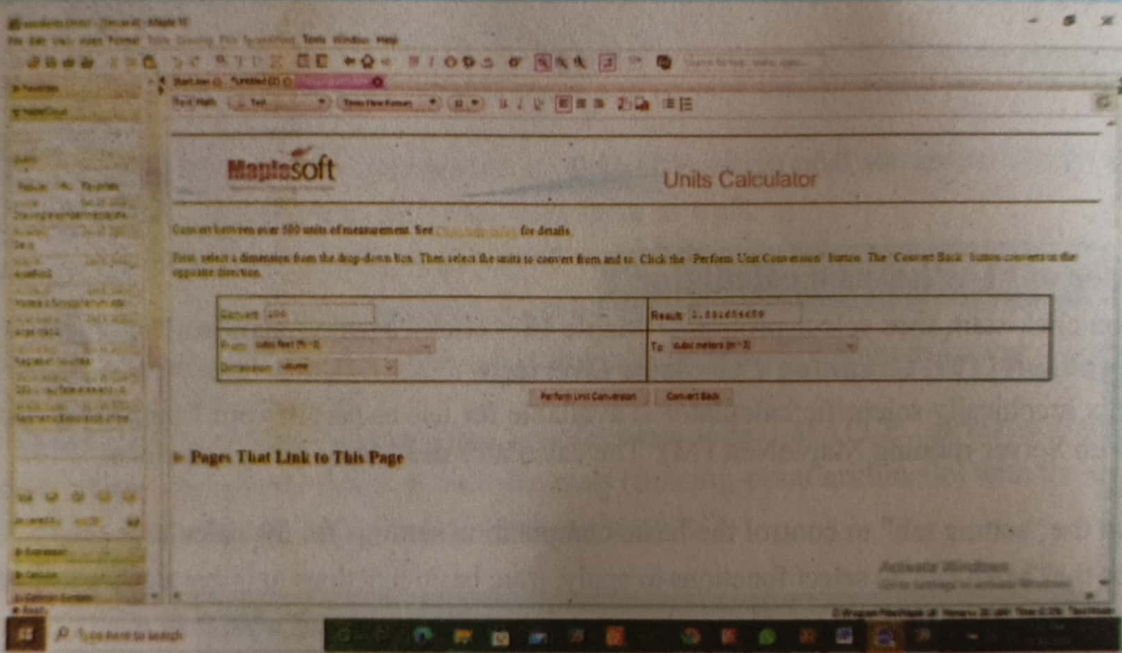


Figure 1.3

Then you will see the unit calculator as shown in the Figure 1.3.

1.1.4 Online MAPLE help

You can get help with MAPLE syntax by using the HELP menu, as described previously. If you have a question about a particular command, you can quickly get help by typing a question mark followed by the name (no semicolon). For example,

|> ? Addition

will open a window containing information about what the "addition" does and how to use it as shown in Figure 1.4. Click on the little "Cross" box at the upper left of the window to close down the little window.

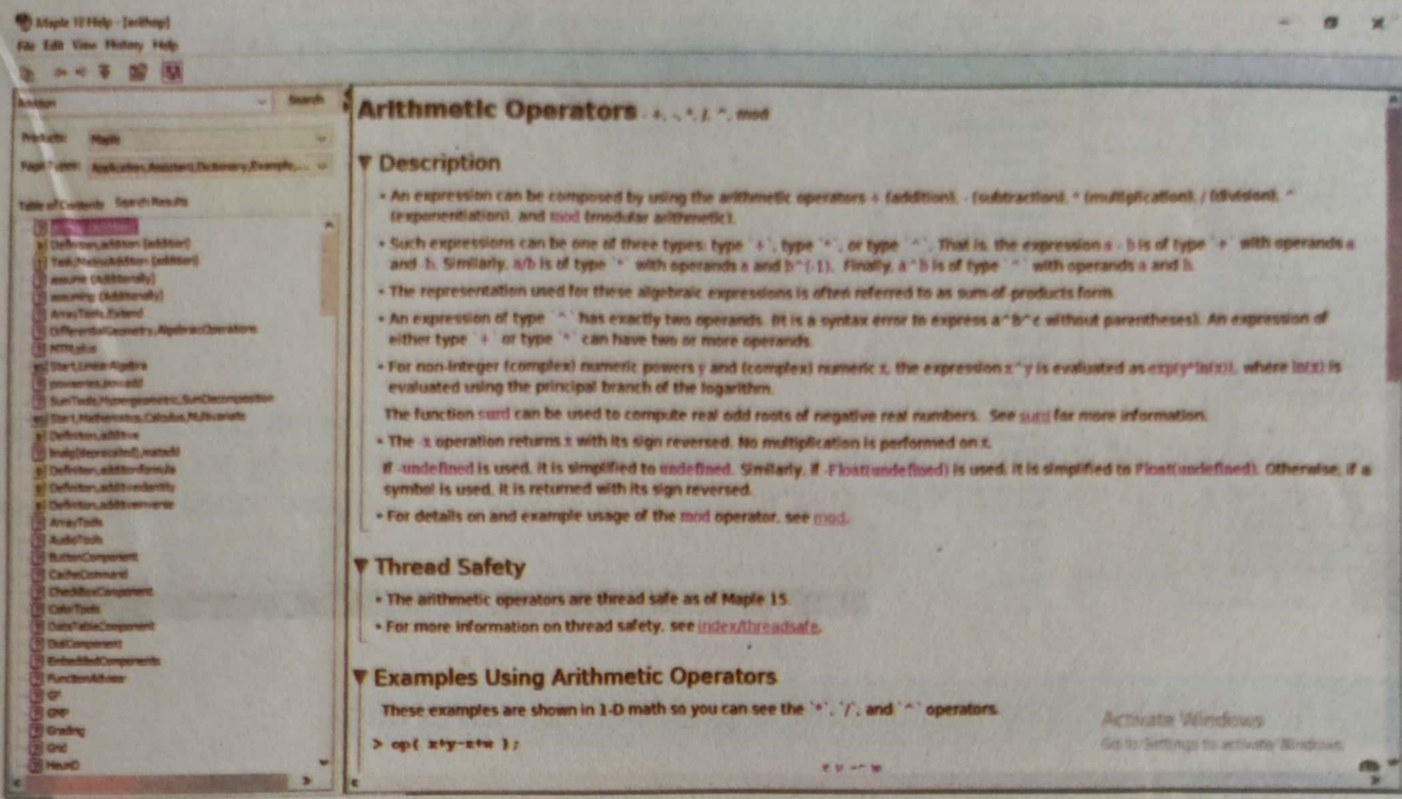


Figure 1.4

1.2 Polynomials

We are familiar about polynomials since our previous grades. The factorization of a polynomial, expansion of an expression can be solve through direct MAPLE commands and context menus.

1.2.1 Use of MAPLE commands for factoring a polynomial

(a) Commands

> $factor(x^2 + 7\cdot x + 12);$

$$(x+4)(x+3) \quad (1)$$

>

(b) Context menu result

You can use Maple's context menus to perform a wide variety of mathematical and other operations. Enter the polynomial and place your cursor on the last end of the polynomial or expressions and right-click. The command will show you full information about factorization on line by typing. Then choose the factor option from the open window as shown in Figure 1.5. The context menu offers several operation to choose form according to the expression that you are using. The above result through context menu is as under:

> $x^2 + 7\cdot x + 12$

> factor (x^2+7*x+12);

$$(x+4)(x+3) \quad (2)$$

>

The result is obtained through right-click on the last end of the expression by selecting "Factor" on the context menus. As shown in Figure 1.5.

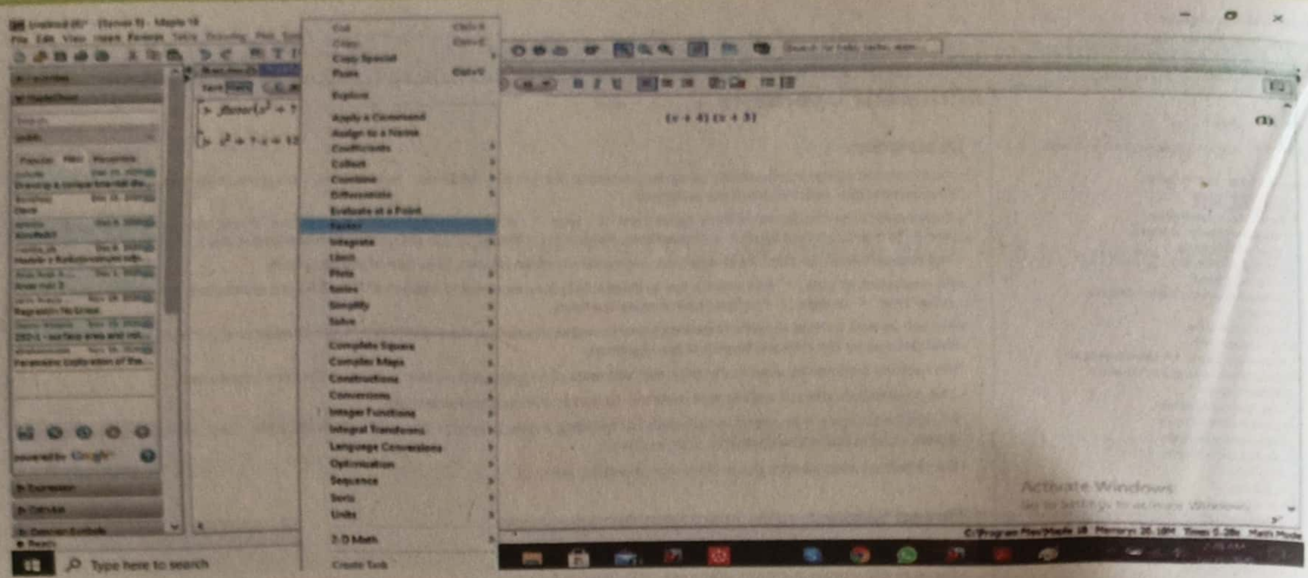


Figure 1.5

1.2.2 Use of MAPLE commands for expanding an expression

(a) Command

Use MAPLE command **“expand”** before parenthesis to expand the given expression.

> $expand((x^2+2)\cdot(x^3+4\cdot x-3));$

$$x^5+6x^3-3x^2+8x-6 \quad (3)$$

>

(b) Context menu result

Entre the given expression and place your cursor on the last end of the expression and right click. Then choose expand option from the opened window as shown in the Figure 1.6.

> $(x^2+2)\cdot(x^3+4\cdot x-3)$

> expand ((x^2+2) * (x^3+4*x–3)) ;

$$x^5+6x^3-3x^2+8x-6 \quad (4)$$

>

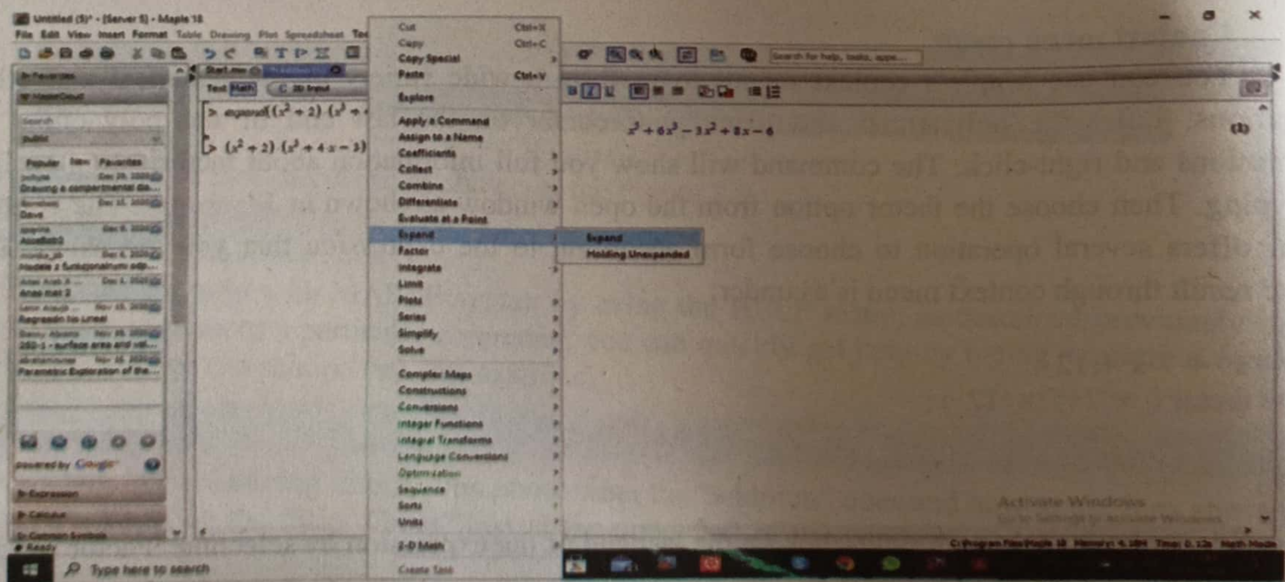


Figure 1.6

1.2.3 Use of MAPLE commands for simplifying an expression

(a) Command

$> simplify\left((25)^{\frac{1}{2}}+\frac{9}{6}-\frac{2}{3}\right);$

$$\frac{35}{6} \tag{5}$$

$>$

(b) Context menu result

Enter the given expression and place your cursor on the last end of the expression and right click. Then choose "simplify" from the opened window as shown in the Figure 1.7.

$> (25)^{\frac{1}{2}}+\frac{9}{6}-\frac{2}{3}$

> simplify (25^ (1/2) +9* (1/6) –2/3) ;

$$\frac{35}{6} \tag{6}$$

$>$

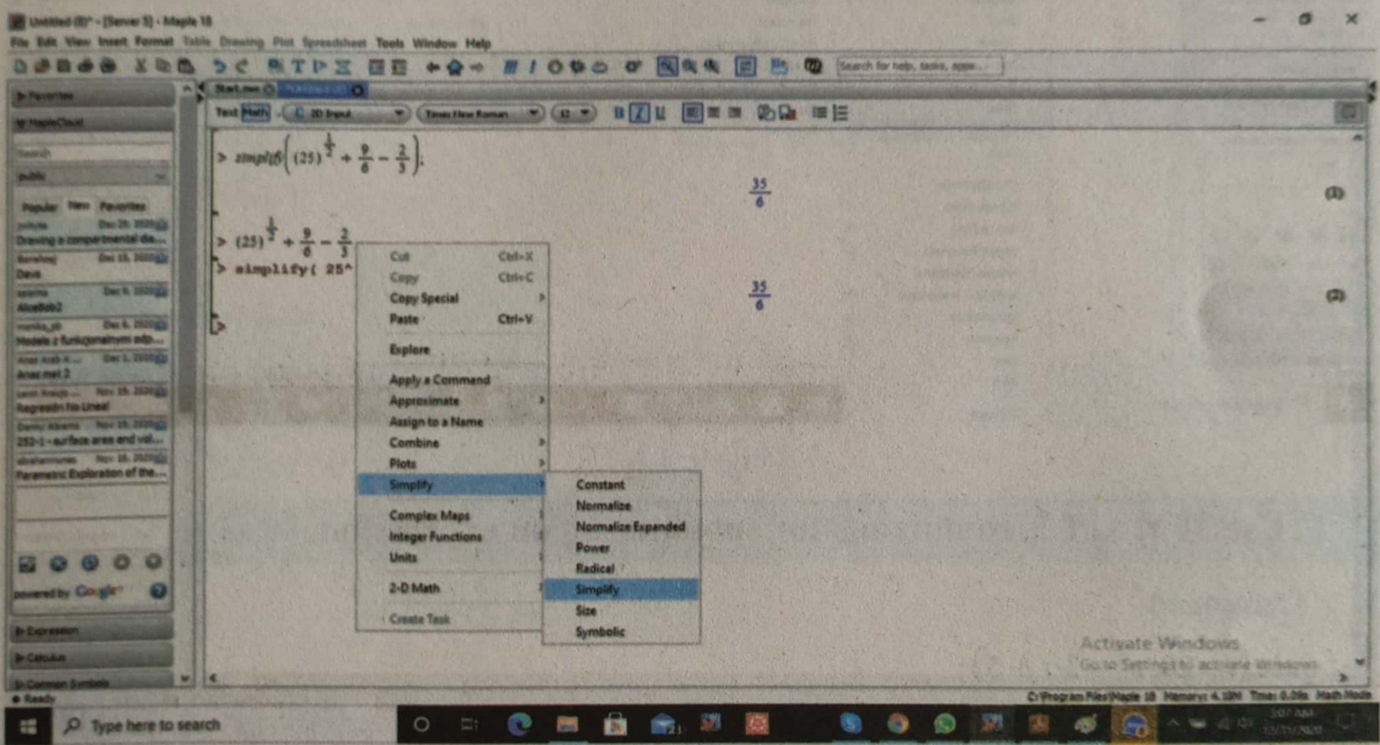


Figure 1.7

1.2.4 Use of MAPLE commands for simplifying a rational expression

(a) Command

$> simplify\left(\frac{t^2-9}{t^2+7\cdot t+12}\right);$

$$\frac{t-3}{t+4} \tag{7}$$

$>$

(b) Context menu result

Enter the given expression and place your cursor on the last end of the expression and right click. Then select "simplify" option as shown in the Figure 1.8.

> $\dfrac{t^2-9}{t^2+7\cdot t+12}$

> simplify ((t^2-9) / (t^2+7* t+12));

$$\frac{t-3}{t+4} \qquad (8)$$

>

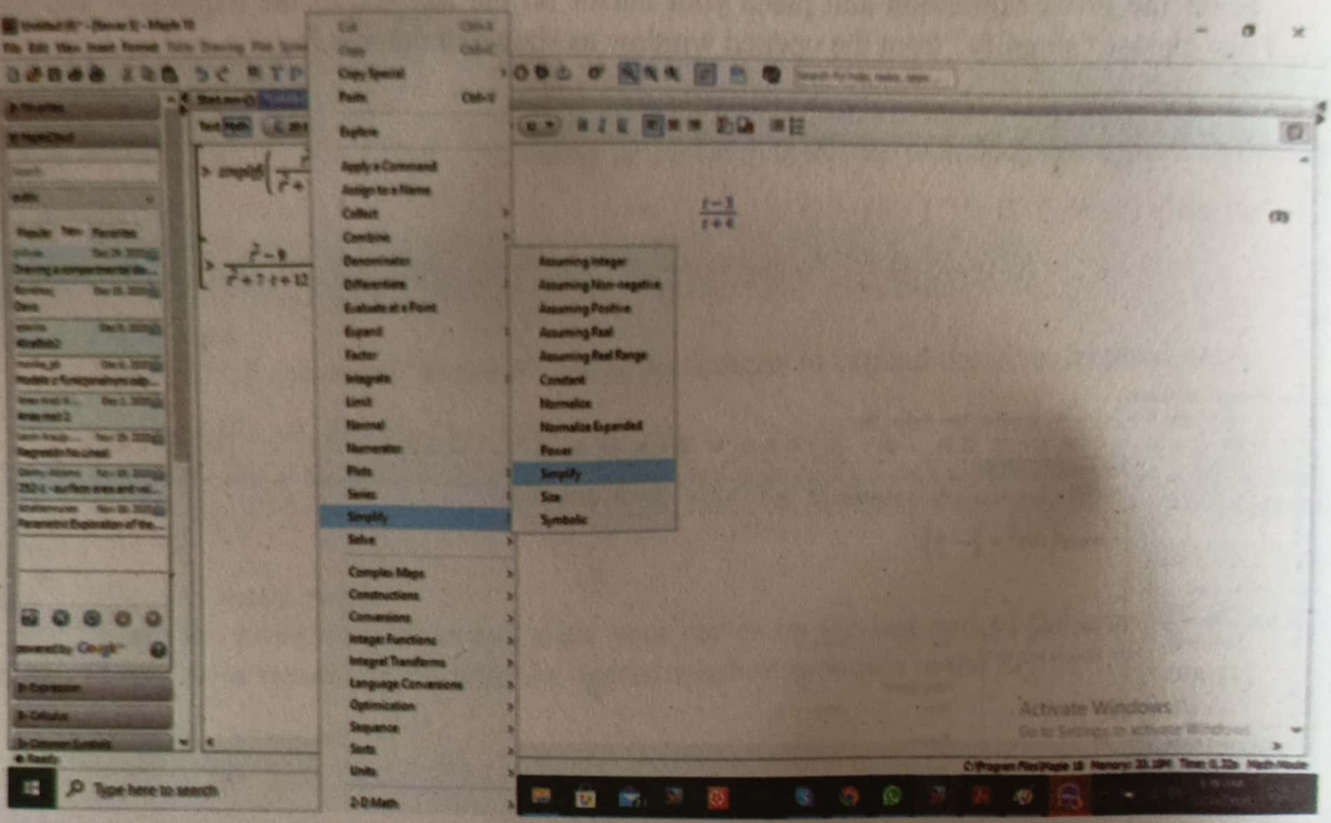


Figure 1.8

1.2.5 Use of MAPLE commands for substituting on expression

(a) Command

> $subs(t=20, t^2-2\cdot t+5);$

$$365 \qquad (9)$$

>

(b) Context menu result

Enter the given expression and place your cursor on last end of the expression and right click. Then select "evaluate at a point" option as shown in the Figure 1.9 and Figure 1.10 respectively.

> $t^2-2\cdot t+5$

> eval (t^2-2*t+5, t = 20);

$$365 \qquad (10)$$

>

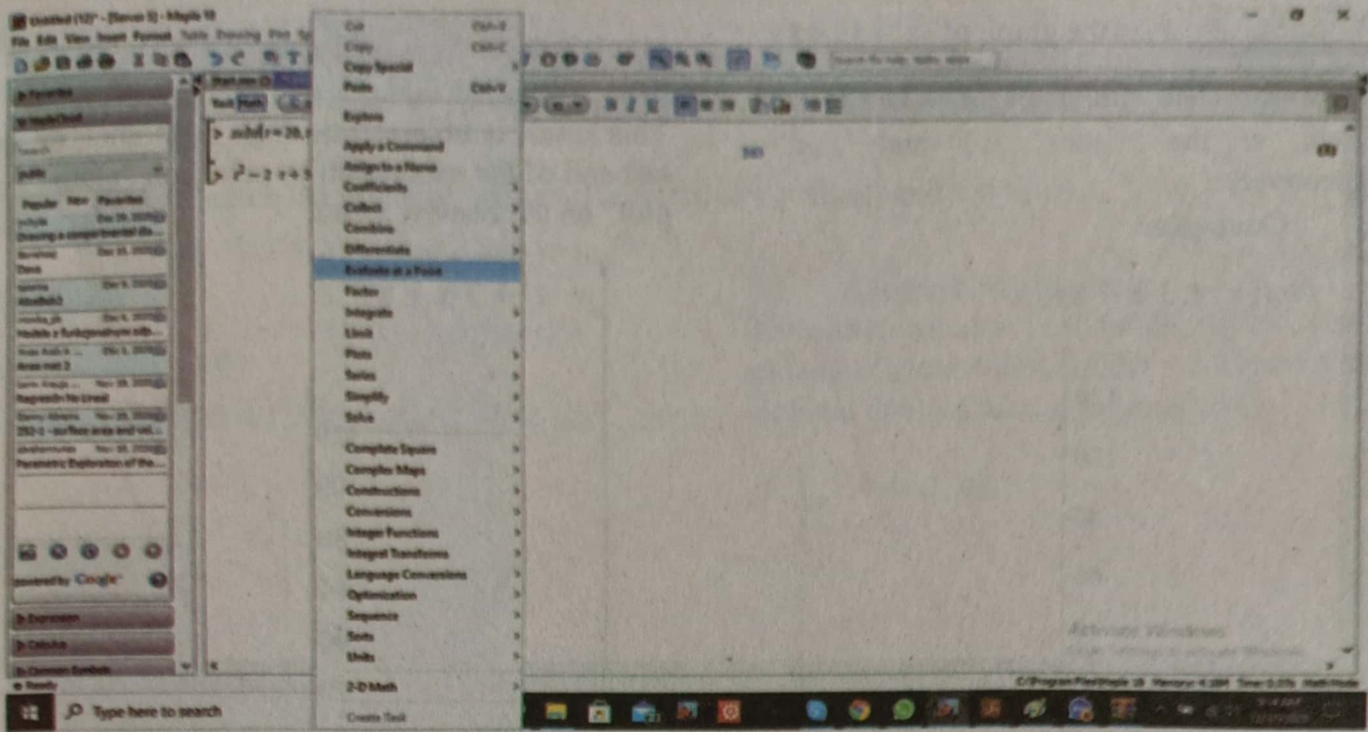


Figure 1.9

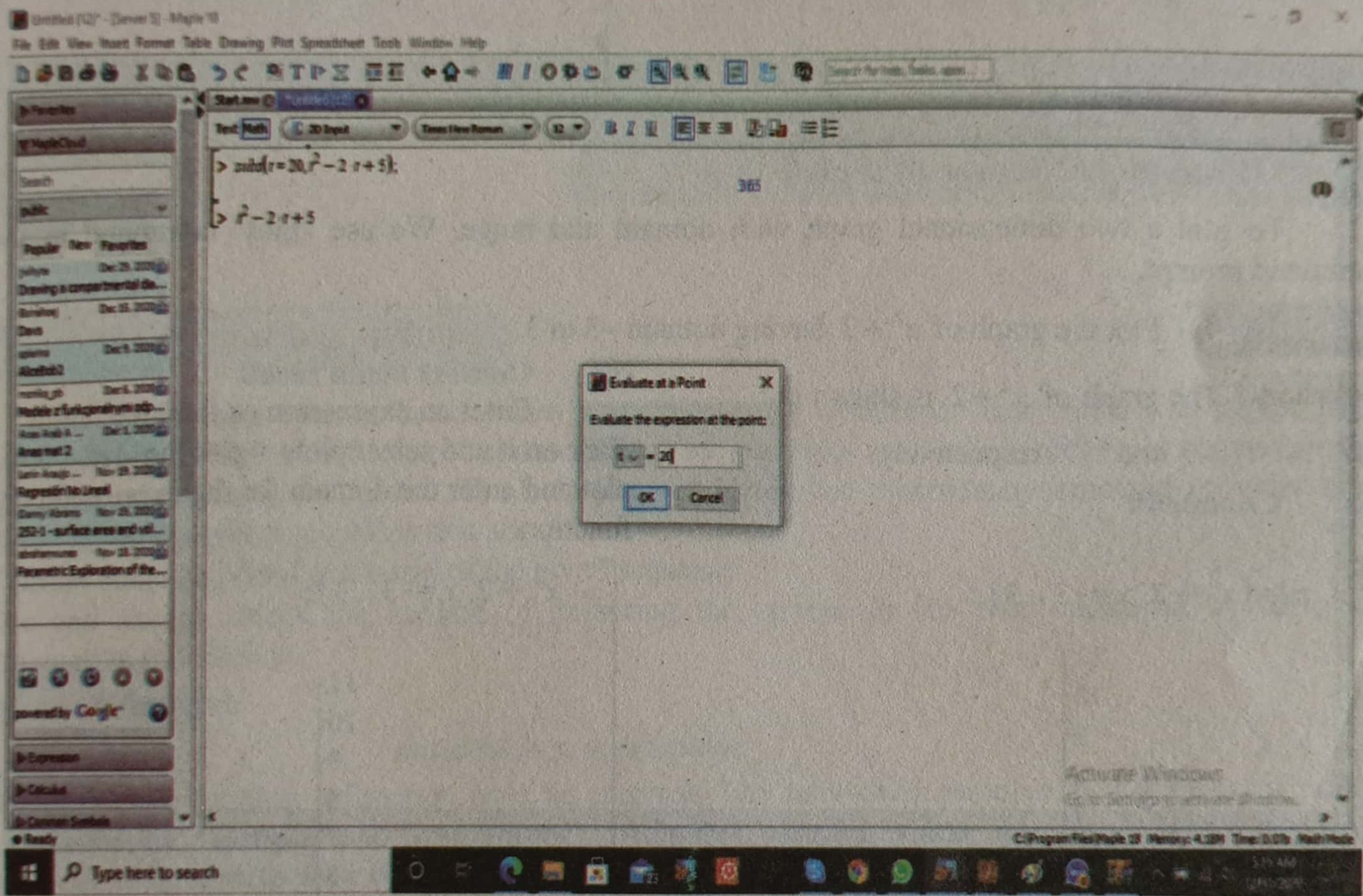


Figure 1.10

1.3 Graphics

1.3.1 Plot a two dimensional graph

To plot any two dimensional graph in MAPLE we use "plot" command in the command prompt.

Example 3 Plot the graph of x^2+3x+8.

Solution The graph of x^2+3x+8 is shown in the Figure 1.11 and 1.12 respectively.

(a) Command

```
> plot(x^2+3·x+8);
```

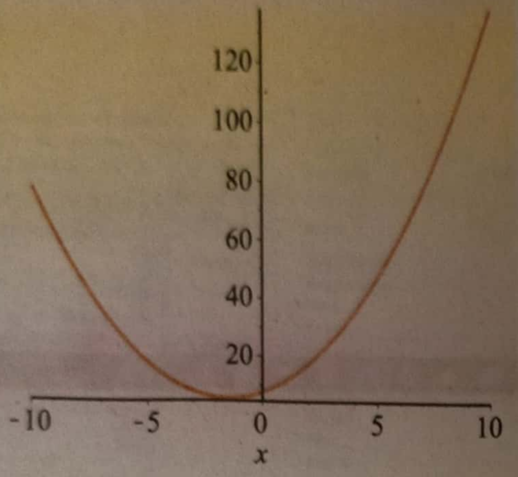


Figure 1.11

(b) Context menu result

This result is obtained through right click on the last end of the expression by selection "plots< 2-D plot" on the context menu.

```
> x^2+3·x+8
> smartplot ( x^2+3*x+8 );
```

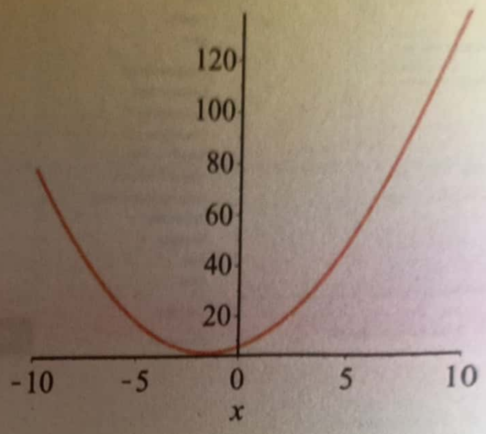


Figure 1.12

1.3.2 Domain and Range of a plot

To plot a two dimensional graph with domain and range. We use "plot" command in the command prompt.

Example 4 Plot the graph of x^2+2 having domain –3 to 3.

Solution The graph of x^2+2 is shown in the Figure 1.13 and 1.14 respectively.

(a) Command

```
> plot(x^2+2, x=-3..3);
```

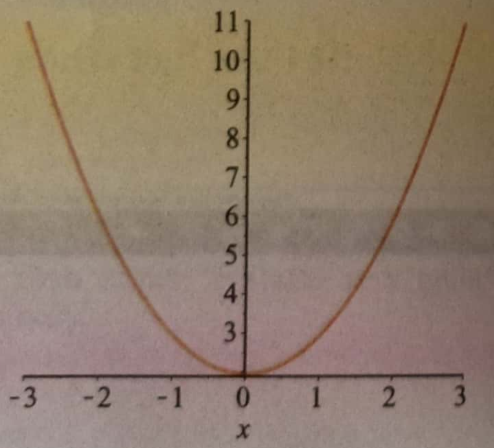


Figure 1.13

(b) Context menu result

Enter an expression or function, right click on it and select plots < plot builder < 2-D plot and enter the domain for the expression or function.

```
> x^2+2, x=-3..3);
> plot (x^2+2, x = -3 .. 3);
```

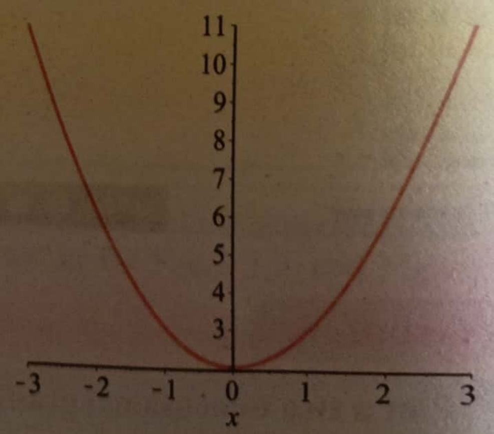


Figure 1.14

1.3.3 Sketch parametric equation

To sketch the parametric equation in MAPLE. We use MAPLE "plot" as discussed in the following example.

Example 5 Sketch the graph of parametric equation $x = \cos(t)$ and $x = \sin(t)$.

Where $-4 < t \leq 4$, $x = -2$ to 2 and $y = -2$ to 2

Solution

(a) Command

```
> plot([cos(t), sin(t), t=-4..4],-2..2,-2..2);
```

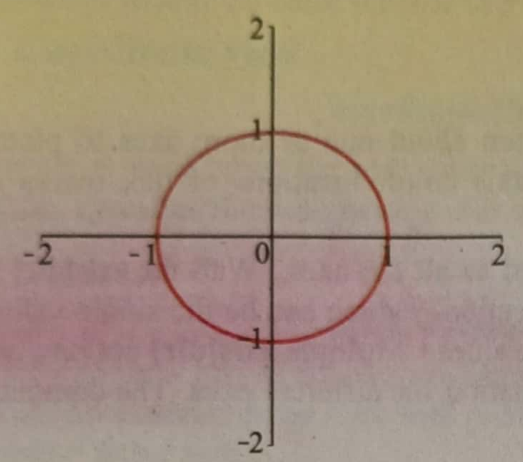


Figure 1.15

(b) Context menu result

Enter an expression or function, right click on it and select plots < plot builder < 2-D parametric plot and domain for the parametric t.

```
> cos(t), sin(t)
> plot ( [cos (t), sin (t), t = -4 .. 4] );
```

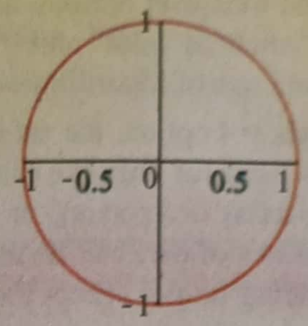


Figure 1.16

1.3.4 Know plotting options

The plotting options listed below can provide the commands that create 2-D plots. These options can be used with the "plot" command and are generally available to all Maple commands that generate two dimensional plots. The help page of Maple for a particular command provides more detail about the plotting options that accepts.

Options must be added at the end of the given sequence.

We can use an interactive method of exploring the options in the "plot" command by using the Interactive Plot Builder.

Adaptive

adaptive = n or truefalse

When we are plotting a function over an interval. The interval is sampled at a number of points, controlled by "**sample**" and "**numpoints**". Adaptive plotting, where necessary, subdivides these intervals to attempt to get a better representation of the given function. This subsampling can be turned off by setting the "**adaptive**" option to **false**. By default, this option is set to **true**, and intervals are subdivided at most six times in trying to improve the plot. By setting this option to a non-negative integer, we can control the maximum number of six times that subintervals are divided.

Annotation

annotation = t

The annotation option allows us to add descriptive text to a 2-D curve or point plot. The term "point plot" in this context means a collection of points that are treated as a single plot element, such that created by the "dataplot" command. The text which is included mathematical expressions, appears when the pointer hovers over the plot element to which it is associated. An annotation for a curve or a point, which appears whenever the pointer is placed over the element.

Axes

axes=f

Specifies the type of axes, one of: **boxed, frame, none,** or **normal.**

Axes font

axesfont = l

Font for the labels on the tick marks of the axes, specified in the same manner as **font**. This option overrides values specified for the **font**.

Axis

axis=t or axis[dir]=t

The **axis** and **axis[dir]** options allows us to provide information about one or more axes to plotting commands such as "plot" and " plot3d". For example, the axis color, locations of tick marks and gridlines, and use of logarithmic scales.

With the **axis** = **t** option, the information provided in **t** is applied to all the axes. With the **axis[dir]** = t option, **t** is used for only the direction(s) specified in **dir** (direction), which can be the single value at (*x*-axis), (*y*-axis) or (z-axis), or a sequence of two of these values. Multiple **axis[dir]** options, with different values of **dir**, can be used to specify different information for different axes. The commands for 2-D plotting do not accept the **axis[3]** option.

The axis information is given by **t**. The list **t** may contain one or more of the sub options.

Axis coordinates

axiscoordinates = t

Normally a coordinate system is used to display of the axes. The value **t** can be either cartesian of polar Cartesian axes are displayed by default. If **t** is **polar**, then radial and angular axes are generated. This option is used together with the **cords = polar** option.

Note

This option is only available in the Standard interface. In the Classic interface, the coordinates are always Cartesian.

Background

background = t

The value of **t** can be the name of an image file, as a string, a name, a **datatype = float,** Array as used with the Image Tools package, or a color. A plot can have a single background image or color.

If the size option is omitted then the dimensions of the plot can be determined by the dimensions of the image. If the **size** option is provided then the image is displayed with the dimensions of the plot.

A color may be given as a color tools [color] object or as a color string. If **t** is a string, then it is first interpreted as a filename. If the file **t** does not exist, it is then assumed to be a color.

Caption

caption = c

The value **c** can be an arbitrary expression. It can also be a list consisting of the caption followed by the **font** option. The default is no caption.

Caption font

captionfont = l

This option defines the font for the plot caption, specified in the same manner as **font**. This option over-rides values specified for **font**.

Color

color = n or colour = n

This allows us to specify the color of the curves to be plotted.

Color scheme

colorscheme = t or colourscheme = t

This allows us to apply a color scheme to a surface or set of points.

Coordinate view

coordinateview = $[r_1 ... r_2, a_1 ... a_2]$

This option is used when the **axis coordinates** option has the value **polar**. When that is the case, then $r_1 ... r_2$ specifies the radial range that is to be displayed and $a_1 ... a_2$ specifies the angular range.

Cords

cords = cname

The value **cname** is one of the choices listed on the coords help page. The cartesian axes are displayed by default. To generate polar axes with polar plots, we use **axiscoordinates = polar** option along with the **coords = polar** option.

Discount

discount = t

This allows us for detection of discontinuities.

Filled

filled = truefalse or list

If the **filled** option is set to **true**, the area between the curve and the x-axis is given by a solid color. The value of the **filled** option can also be a list containing one or more sub options (color, style or transparency). These options are applied only to the filled area, and not to the original curve itself. This option does not work with non-cartesian coordinate systems.

Filled regions

filledregions = truefalse

If the **filledregions** option is set to **true**, the regions defined by the curves are filled with different colors. This option is valid only by using the following commands:

"contourplot", "implicitplot" and "listcontplot". This option does not work with non-Cartesian coordinate systems.

Font

font = l

This option defines the font for the plot title, caption, axis tick mark labels, and axis labels if no values have been specified for the **axes font, captionfont, label font,** or **title font** options. The value **l** is a list of the form [**family, style, size**].

The value of **family** can be one of **Times, Courier, Helvetica,** or **Symbol**. It can also be any font name supported by your system, for example, **Times New Roman** and **Calibri** in Windows. The first letter of the **family** name must be capitalized.

The value of **style** can be omitted or one of **roman, bold, italic, bold italic, oblique,** or **bold oblique**. The **Symbol** family does not accept a style option. The final value, **size**, is the point size to be used.

Gridline

gridlines=truefalse

When **gridlines = true** or **gridlines** is provided, default gridlines are drawn. The default is **gridlines = false**.If the **axis** option is also provided and contains a **gridlines** sub option, then that option over rides this **gridlines** option.

Labels

labels = [x, y]

This option specifies labels for the axes. By default labels are the names of the variables in the original function to be plotted, if these are available otherwise, no labels will be used.

Label directions

labeldirections = [x,y]

This option specifies the direction in which labels are printed along the axes. The value of **x** and **y** should be **horizontal** or **vertical**. The default direction of any labels is **horizontal**.

Label font

labelfont = l

The font of the labels on the axes of the plot, specified in the same manner as **font**. This option overrides values specified for **font**.

Legend

legend = s

If the plot command is being used to plot multiple curves, then **s** can be a list containing a legend entry for each curve.

Legend style

legendstyle = s

Since the value **s** is a list consisting of one or more sub options. The sub options are available for the **legendstyle** option include **font = f** and **location = loc**. The **location = loc** sub option allow us **top, bottom, right** and **left** for **loc**.

Line style

linestyle = t

It controls the line style of curves. The **line style** value **t** can be one of the following names: **solid, dot, dash, dash dot, long dash, space dash,** or **space dot**. The default value of **t** is **solid**. The value **t** can also be an integer from 1 to 7, where each integer represents a line style as given in the order above.

Num points

numpoints = n

Specifies the minimum number of points to be generated. The default number of points is 200.

Remember

Plot employs an adaptive plotting scheme which automatically does more work where the function values do not lie close to a straight line.

Resolution

resolution = n

This sets the horizontal display resolution of the device in pixels. The default resolution is n=800. The value of **n** is used to determine when the adaptive plotting scheme terminates.

Sample

sample = [l]

A list of numerical values which is to be used for the initial sampling of the function. Normally, the function is sampled at additional points. To restrict sampling to only these values we include the **adaptive** = **false** option.

Scaling

scaling = s

It controls scaling of the graph. The value of **s** is **unconstrained** by default, which means the plot is scaled to fit the plot window. The **constrained** value causes all axes to use the same scale.

Size

size = [w, h]

We use this to specify the size of the plot window. We can set the size of the plot window by specifying the number of pixels, a proportion of worksheet width, or a ratio, such as a square, the golden ratio, or a custom ratio.

Smart view

smartview = truefalse

This is used to determine an appropriate view of the plot data. The plot command generates data based on the range provided by us or on a default range if this is not provided. When the **smarview = true** option is provided, a view that tries to present the important regions of the data is computed. To show all data computed, use the **smartview = false** option. The default setting of **smartview** is **true**. This option is available for the plot command and only applies to curves not points, polygons or text.

Style

style=s

The plot style should be one of **line**, **point**, **point line**, **polygon**, or **polygon outline**. The names in parentheses are aliases for the option values. The styles **line**, **polygon**, and **polygon outline** all draw curves by interpolating between the sample points. The **point** style results in a plot of the points only. The default style, **polygon outline**, draws any polygons as filled with an outline. The **polygon** style shows the polygons with no outline, whereas **line** draws the polygons as outlines only. The **point line** style is a combination of the **point** and **line** styles.

Symbol

symbol=s

Symbol for points in the plot, where the value **s** is one of **asterisk, box, circle, cross, diagonalcross, diamond, point, solidbox, solidcircle, soliddiamond**

Symbol size

symbolsize = n

The size of a symbol used in plotting can be given by a natural number. This does not affect the symbol **POINT**. The symbol size is **10** by default.

Thickness

thickness = n

This option specifies the thickness of lines in the plot. The **thickness** = **n** must be a non-negative number. A value with 0 produces the thinnest line. By default the value is 1.

Tick marks

Tickmarks = [m, n]

The values **m** and **n** specify the tick mark placement for the *x-axis* and *y -axis* respectively and can take an integer specifying the number of tick marks, a list of values specifying locations, a list of equations each having the form **location = label,** a name, or a **spacing** structure.

Title

title=t

We can give a title to the plot. The value t can be an arbitrary expression. The value **t** can also be a list consisting of the title followed by the font option. There is no title by default for a plot.

Title font

titlefont = l

It specified in the same manner as font. This option overrides values specified for **font**.

Transparency

transparency = t

This option specifies the transparency of the plot surface. The **transparency** = **t** must evaluate to a floating-point number in the range [0, 1]. 0 means "not transparent" but 1 means "fully transparent."

Use units

useunits = t

This option, with **t** set to **true**, indicates that units are part of the function and should be included in the axes labels. The value **t** can also be a list of units.

View

view = [xmin...xmax, ymin...ymax]

View option indicates the minimum and maximum coordinates of the curve to be displayed on the screen. By default it is determined by the **smartview** option: if **smartview** = **false** is given, then all the plot data will be displayed; if **smartview** = **true** or the **smartview** option is not given, then the plot structure will be analyzed to determine a reasonable view of the data which allows you to see the significant features of the data.

Remember

i. If the same option is provided more than once, with different values, then the final value specified is generally the one used.

ii. All above options are available for the Standard Worksheet interface.

Example 6 Plot the graph of sec(x) and give the title as;

title="Graph\n of Secant Function "

titlefont = ["Times New Roman", 20]

Solution

```
> plot(sec(x), x=-2π..2π, title=Graph of Secant Function", titlefont=["Times New Roman",
    20]);
```

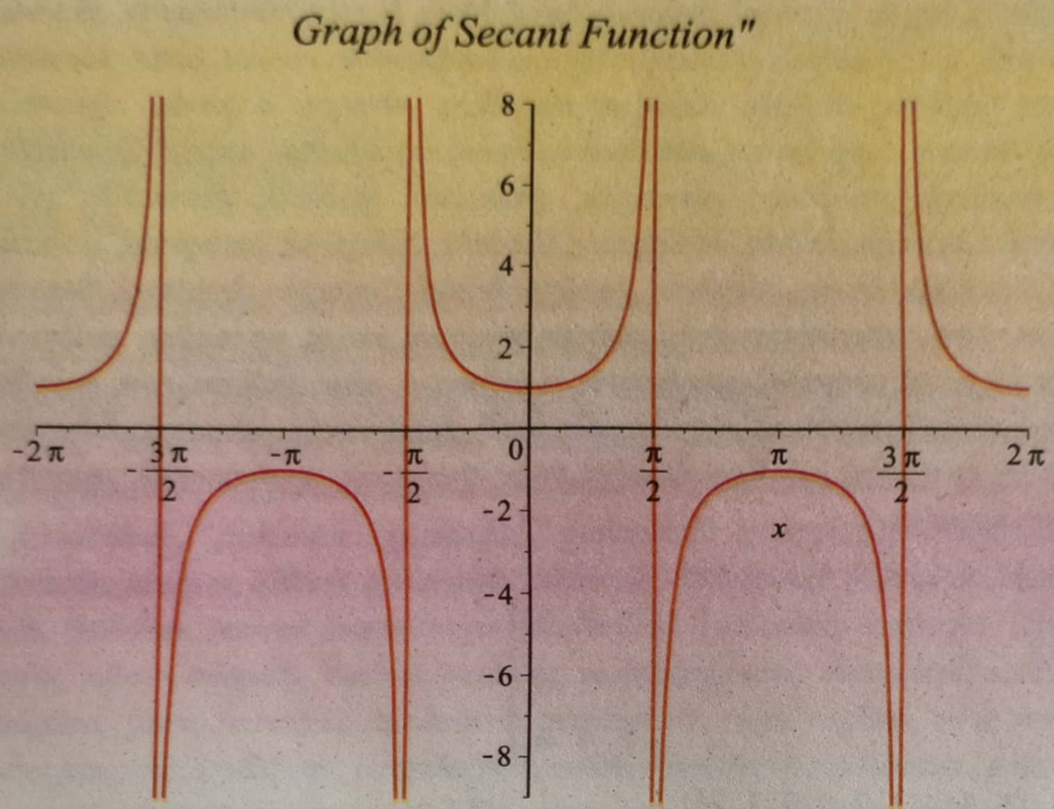


Figure 1.17

This example uses the following axes and graph properties:

title="Graph\n of Secant function"

axes = framed

style = point

symbol = asterisk

symbolsize = 20

tickmarks = [spacing (Pi), default]

1.4 Matrices

The command will show you full information about matrices on line by typing: [> *?matrices*

1.4.1 Recognition of matrix and vector entry arrangement

(a) Command

```
> with(linalg);
```

[BlockDiagonal, GramSchmidt, JordanBlock, LUdecomp, QRdecomp, Wronskian, addcol, addrow, adj, adjoint, angle, augment, backsub, band, basis, bezout, blockmatrix, charmat, charpoly, cholesky, col, coldim, colspace, colspan, companion, concat, cond, copyinto, crossprod, curl, definite, delcols, delrows, det, diag, diverge, dotprod, eigenvals, eigenvalues, eigenvectors, eigenvects, entermatrix, equal, exponential, extend, ffgausselim, fibonacci, forwardsub, frobenius, gausselim, gaussjord, geneqns, genmatrix, grad, hadamard, hermite, hessian, hilbert, htranspose, ihermite, indexfunc, innerprod, intbasis, inverse, ismith, issimilar, iszero, jacobian, jordan, kernel, laplacian, leastsqrs, linsolve, matadd, matrix, minor, minpoly, mulcol, mulrow, multiply, norm, normalize, nullspace, orthog, permanent, pivot, potential, randmatrix, randvector, rank, ratform, row, rowdim, rowspace, rowspan, rref, scalarmul, singularvals, smith, stackmatrix, submatrix, subvector, sumbasis, swapcol, swaprow, sylvester, toeplitz, trace, transpose, vandermonde, vecpotent, vectdim, vector, wronskian] (1)

```
> X=matrix(3, 3, [1, 4, 2, 2, 3, 4, 5, 4, 6]);
```

$$X=\begin{bmatrix}1 & 4 & 2\\ 2 & 3 & 4\\ 5 & 4 & 6\end{bmatrix} \tag{2}$$

```
> Y=matrix(3, 3, [5, 3, 4, 1, 2, 0, 9, 3, 7]);
```

$$Y=\begin{bmatrix}5 & 3 & 4\\ 1 & 2 & 0\\ 9 & 3 & 7\end{bmatrix} \tag{3}$$

```
> Z=matrix(3, 3, [1, 0, 1, 2, 3, 1, 0, 5, 1]);
```

$$Z=\begin{bmatrix}1 & 0 & 1\\ 2 & 3 & 1\\ 0 & 5 & 1\end{bmatrix} \tag{4}$$

```
> c := ⟨1, 2, 3⟩;
```

$$c:=\begin{bmatrix}1\\ 2\\ 3\end{bmatrix} \tag{5}$$

```
> r := ⟨1|2|3⟩;
```

$$r:=\begin{bmatrix}1 & 2 & 3\end{bmatrix} \tag{6}$$

```
>
```

(b) Using Palettes

Use cursor button to select matrix palette. Click- "matrix" <click-choose(for the umber of rows and columns of a required matrix) <click-data type (to select integers entries of the rows and columns of a required matrix), then finally click-"insert matrix" and press ENTER key to obtain a required matrix.

> $\begin{bmatrix} 1 & 4 & 2 \\ 2 & 3 & 4 \\ 5 & 4 & 6 \end{bmatrix}$

$$\begin{bmatrix} 1 & 4 & 2 \\ 2 & 3 & 4 \\ 5 & 4 & 6 \end{bmatrix} \quad (7)$$

1.4.2 Applying matrix operations

(a) Matrix addition

> *with(linalg);*

[BlockDiagonal, GramSchmidt, JordanBlock, LUdecomp, QRdecomp, Wronskian, addcol, addrow, adj, adjoint, angle, augment, backsub, band, basis, bezout, blockmatrix, charmat, charpoly, cholesky, col, coldim, colspace, colspan, companion, concat, cond, copyinto, crossprod, curl, definite, delcols, delrows, det, diag, diverge, dotprod, eigenvals, eigenvalues, eigenvectors, eigenvects, entermatrix, equal, exponential, extend, ffgausselim, fibonacci, forwardsub, frobenius, gausselim, gaussjord, geneqns, genmatrix, grad, hadamard, hermite, hessian, hilbert, htranspose, ihermite, indexfunc, innerprod, intbasis, inverse, ismith, issimilar, iszero, jacobian, jordan, kernel, laplacian, leastsqrs, linsolve, matadd, matrix, minor, minpoly, mulcol, mulrow, multiply, norm, normalize, nullspace, orthog, permanent, pivot, potential, randmatrix, randvector, rank, ratform, row, rowdim, rowspace, rowspan, rref, scalarmul, singularvals, smith, stackmatrix, submatrix, subvector, sumbasis, swapcol, swaprow, sylvester, toeplitz, trace, transpose, vandermonde, vecpotent, vectdim, vector, wronskian] (1)

> $X := Matrix([[1, 2, 3], [1, 3, 0], [1, 4, 3]]);$

$$X := \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 0 \\ 1 & 4 & 3 \end{bmatrix} \quad (2)$$

> $Y := Matrix([[1, 2, 5], [6, 3, 0], [1, 4, 3]]);$

$$Y := \begin{bmatrix} 1 & 2 & 5 \\ 6 & 3 & 0 \\ 1 & 4 & 3 \end{bmatrix} \quad (3)$$

> $matadd(X, Y);$

$$\begin{bmatrix} 2 & 4 & 8 \\ 7 & 6 & 0 \\ 2 & 8 & 6 \end{bmatrix} \quad (4)$$

>

(b) Matrix multiplication

```
> with(linalg);
```

[BlockDiagonal, GramSchmidt, JordanBlock, LUdecomp, QRdecomp, Wronskian, addcol, addrow, adj, adjoint, angle, augment, backsub, band, basis, bezout, blockmatrix, charmat, charpoly, cholesky, col, coldim, colspace, colspan, companion, concat, cond, copyinto, crossprod, curl, definite, delcols, delrows, det, diag, diverge, dotprod, eigenvals, eigenvalues, eigenvectors, eigenvects, entermatrix, equal, exponential, extend, ffgausselim, fibonacci, forwardsub, frobenius, gausselim, gaussjord, geneqns, genmatrix, grad, hadamard, hermite, hessian, hilbert, htranspose, ihermite, indexfunc, innerprod, intbasis, inverse, ismith, issimilar, iszero, jacobian, jordan, kernel, laplacian, leastsqrs, linsolve, matadd, matrix, minor, minpoly, mulcol, mulrow, multiply, norm, normalize, nullspace, orthog, permanent, pivot, potential, randmatrix, randvector, rank, ratform, row, rowdim, rowspace, rowspan, rref, scalarmul, singularvals, smith, stackmatrix, submatrix, subvector, sumbasis, swapcol, swaprow, sylvester, toeplitz, trace, transpose, vandermonde, vecpotent, vectdim, vector, wronskian] (1)

```
> X := Matrix([[1, 2, 3], [1, 3, 0], [1, 4, 3]]);
```

$$X := \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 0 \\ 1 & 4 & 3 \end{bmatrix} \qquad (2)$$

```
> Y := Matrix([[1, 2, 5], [6, 3, 0], [1, 4, 3]]);
```

$$Y := \begin{bmatrix} 1 & 2 & 5 \\ 6 & 3 & 0 \\ 1 & 4 & 3 \end{bmatrix} \qquad (3)$$

```
> multiply(X, Y);
```

$$\begin{bmatrix} 16 & 20 & 14 \\ 19 & 11 & 5 \\ 28 & 26 & 14 \end{bmatrix} \qquad (4)$$

```
>
```

(c) Using Palettes

To obtain the result simply right click on the last "matrix" then click simplify or press ENTER.

$$> \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 0 \\ 1 & 4 & 3 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 5 \\ 6 & 3 & 0 \\ 1 & 4 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 4 & 8 \\ 7 & 6 & 0 \\ 2 & 8 & 6 \end{bmatrix} \qquad (1)$$

```
> simplify( Matrix (%id = 18446744074594159542) +Matrix (%id =
18446744074594159662) );
```

$$\begin{bmatrix} 2 & 4 & 8 \\ 7 & 6 & 0 \\ 2 & 8 & 6 \end{bmatrix} \qquad (2)$$

```
>
```

1.4.3 Inverse and transpose of a matrix

(a) Inverse of a matrix

> with(linalg);

[BlockDiagonal, GramSchmidt, JordanBlock, LUdecomp, QRdecomp, Wronskian, addcol, addrow, adj, adjoint, angle, augment, backsub, band, basis, bezout, blockmatrix, charmat, charpoly, cholesky, col, coldim, colspace, colspan, companion, concat, cond, copyinto, crossprod, curl, definite, delcols, delrows, det, diag, diverge, dotprod, eigenvals, eigenvalues, eigenvectors, eigenvects, entermatrix, equal, exponential, extend, ffgausselim, fibonacci, forwardsub, frobenius, gausselim, gaussjord, geneqns, genmatrix, grad, hadamard, hermite, hessian, hilbert, htranspose, ihermite, indexfunc, innerprod, intbasis, inverse, ismith, issimilar, iszero, jacobian, jordan, kernel, laplacian, leastsqrs, linsolve, matadd, matrix, minor, minpoly, mulcol, mulrow, multiply, norm, normalize, nullspace, orthog, permanent, pivot, potential, randmatrix, randvector, rank, ratform, row, rowdim, rowspace, rowspan, rref, scalarmul, singularvals, smith, stackmatrix, submatrix, subvector, sumbasis, swapcol, swaprow, sylvester, toeplitz, trace, transpose, vandermonde, vecpotent, vectdim, vector, wronskian] (1)

> $X := Matrix([[1, 2, 3], [1, 3, 0], [1, 4, 3]]);$

$$X := \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 0 \\ 1 & 4 & 3 \end{bmatrix} \tag{2}$$

> $Y := Matrix([[1, 2, 5], [6, 3, 0], [1, 4, 3]]);$

$$Y := \begin{bmatrix} 1 & 2 & 5 \\ 6 & 3 & 0 \\ 1 & 4 & 3 \end{bmatrix} \tag{3}$$

> $inv := inverse(X);$

$$inv := \begin{bmatrix} \frac{3}{2} & 1 & -\frac{3}{2} \\ -\frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{6} & -\frac{1}{3} & \frac{1}{6} \end{bmatrix} \tag{4}$$

> $inv := inverse(Y);$

$$inv := \begin{bmatrix} \frac{3}{26} & \frac{7}{39} & -\frac{5}{26} \\ -\frac{3}{13} & -\frac{1}{39} & \frac{5}{13} \\ \frac{7}{26} & -\frac{1}{39} & -\frac{3}{26} \end{bmatrix} \tag{5}$$

>

(b) Transport of a matrix

```
> with(linalg);
```

[BlockDiagonal, GramSchmidt, JordanBlock, LUdecomp, QRdecomp, Wronskian, addcol, addrow, adj, adjoint, angle, augment, backsub, band, basis, bezout, blockmatrix, charmat, charpoly, cholesky, col, coldim, colspace, colspan, companion, concat, cond, copyinto, crossprod, curl, definite, delcols, delrows, det, diag, diverge, dotprod, eigenvals, eigenvalues, eigenvectors, eigenvects, entermatrix, equal, exponential, extend, ffgausselim, fibonacci, forwardsub, frobenius, gausselim, gaussjord, geneqns, genmatrix, grad, hadamard, hermite, hessian, hilbert, htranspose, ihermite, indexfunc, innerprod, intbasis, inverse, ismith, issimilar, iszero, jacobian, jordan, kernel, laplacian, leastsqrs, linsolve, matadd, matrix, minor, minpoly, mulcol, mulrow, multiply, norm, normalize, nullspace, orthog, permanent, pivot, potential, randmatrix, randvector, rank, ratform, row, rowdim, rowspace, rowspan, rref, scalarmul, singularvals, smith, stackmatrix, submatrix, subvector, sumbasis, swapcol, swaprow, sylvester, toeplitz, trace, transpose, vandermonde, vecpotent, vectdim, vector, wronskian] (1)

```
> X := Matrix([[1, 2, 3], [1, 3, 0], [1, 4, 3]]);
```

$$X := \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 0 \\ 1 & 4 & 3 \end{bmatrix} \tag{2}$$

```
> Y := Matrix([[1, 2, 5], [6, 3, 0], [1, 4, 3]]);
```

$$Y := \begin{bmatrix} 1 & 2 & 5 \\ 6 & 3 & 0 \\ 1 & 4 & 3 \end{bmatrix} \tag{3}$$

```
> trans := transpose(X);
```

$$trans := \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 3 & 0 & 3 \end{bmatrix} \tag{4}$$

```
> trans := transpose(Y);
```

$$trans := \begin{bmatrix} 1 & 6 & 1 \\ 2 & 3 & 4 \\ 5 & 0 & 3 \end{bmatrix} \tag{5}$$

```
>
```

(c) Using Palettes

Inverse of a matrix

To obtain the result through context menu. Right-click on the last end of the matrix by selecting "Standard operations" and then right-click on the "inverse" to obtained the inverse of a matrix.

> $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

> LinearAlgebra: -MatrixInverse (Matrix (%id = 18446744074459087686)) ;

$$\begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix} \qquad (1)$$

Similarly,

Transpose of a matrix

To obtain the result through context menu. Right-click on the last end of the matrix by selecting "Standard operations" and then right-click on the "transpose" to obtained the inverse of a matrix.

> $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 2 & 6 \\ 3 & 5 & 1 \end{bmatrix}$

> LinearAlgebra: -Transpose (Matrix (%id = 18446744073883260558)) ;

$$\begin{bmatrix} 1 & 4 & 3 \\ 2 & 2 & 5 \\ 3 & 6 & 1 \end{bmatrix} \qquad (2)$$

Determinant of a matrix

To obtain the result through context menu. Right-click on the last end of the matrix by selecting "Standard operations" and then right-click on the "determinant" to obtained the inverse of a matrix.

> $\begin{bmatrix} 3 & 2 & 5 \\ 1 & 5 & 7 \\ 2 & 9 & 4 \end{bmatrix}$

> LinearAlgebra: -Determinant (Matrix (%id = 18446744073883262118)) ;

$$-114 \qquad (3)$$

>

Unit 2

FUNCTIONS AND LIMITS

By the end of this unit, the students will be able to:

2.1 Function

i. Identify through graph the domain and range of a function.

ii. Draw the graph of modulus function (i.e. $y = |x|$) and identify its domain and range.

2.2 Composition of functions

i. Recognize the composition of functions. ii. Find the composition of two given functions.

2.3 Inverse of composition of functions

i. Describe the inverse of composition of two given functions.

2.4 Transcendental functions

i. Recognize algebraic, trigonometric, inverse trigonometric, exponential, logarithmic, hyperbolic (and their identities), explicit and implicit functions, and parametric representation of functions.

2.5 Graphical representations

i. Display graphically:

- the explicitly defined functions like $y = f(x)$, where $f(x) = e^x$, a^x, $\log_a x$, $\log_e x$.
- the implicitly defined functions such as $x^2 + y^2 = a^2$ and $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and distinguish between graph of a function and of an equation.
- the parametric equations of functions such as $x = at^2$, $y = 2at$; $x = a\sec\theta$, $y = b\tan\theta$.
- the discontinuous functions of the type $y = \begin{cases} x & \text{when } 0 \le x \le 1, \\ x-1 & \text{when } 1 \le x \le 2 \end{cases}$

ii. Use MAPLE graphic commands for two-dimensional plot of:

- an expression (or a function),
- parameterized form of a function,
- implicit function

by restricting domain and range.

iii. Use MAPLE package plots for plotting different types of functions.

2.6 Limit of a function

i. Identify a real number by a point on the number line.

ii. Define and represent

- open interval,
- closed interval,
- half open and half closed intervals, on the number line.

iii. Explain the meaning of phrase:

- x tends to zero ($x \to 0$),
- x tends to a ($x \to a$),
- x tends to infinity ($x \to \infty$)

iv. Define limit of a sequence. v. Find the limit of a sequence whose nth term is given.

vi. Define limit of a function.

vii. State the theorems on limits of sum, difference, product and quotient of functions and demonstrate through examples.

2.7 Important limits

i. Evaluate the limits of functions of the following types:

- $\frac{x^n - a^n}{x - a}, \frac{x-a}{\sqrt{x} - \sqrt{a}}$ when $x \to a$,
- $\left(1 + \frac{1}{x}\right)^x$ when $x \to \infty$,
- $(1+x)^{\frac{1}{x}}, \frac{\sqrt{x+a} - \sqrt{a}}{x}, \frac{a^x - 1}{x}$,
- $\frac{(1+x)^n - 1}{x}$, and $\frac{\sin x}{x}$ when $x \to 0$

ii. Evaluate limits of different algebraic, exponential and trigonometric functions.

iii. Use MAPLE command limit to evaluate limit of a function.

2.8 Continuous and discontinuous functions

i. Recognize left hand and right hand limits and demonstrate through examples.

ii. Define continuity of a function at a point and in an interval.

iii. Test continuity and discontinuity of a function at a point and in an interval.

iv. Use MAPLE command iscont to test continuity of a function at a point and in a given interval.

Introduction

The concept of function and its limit is fundamental idea to us, in the study of mathematics that distinguishes calculus from algebra and trigonometry. In this unit, we will revise the concept of function from unit-8 of grade-XI mathematics and then develop the concept of limit which is the fundamental building block on which all the calculus concepts are based.

2.1 Functions

The three animals Zurain owns before he started his collection.

Function are constantly encountered in mathematics and are essential for formulating physical relationships in science and technology. In our routine life function is very useful e.g. Zurain like all kind of animals. He started collecting them recently and already owns 3 animals. He plans on buying every month accordingly, of each type of animals.

Let 'x' be the number of months have past since Zurain started collecting animals. Let 'y' be the number of animals Zurain owns. How can we write a function in terms of x and y?

To write the function, at very beginning, when $x = 0$, Zurain has not bought any new animal, he owns 3 animals so, $y = 3$ animals

After the first month, when $x = 1$, Zurain owns 3 animals, plus the 1 animal he just bought. He now owns $y = 3 + 1$ animals

Similarly, after the second month, when $x = 2$, Zurain owns 3 animals, plus the 2 new animals he bought after he started animals. He now owns $y = 3 + 2$ animals

Therefore for x animals the function will be $y = 3 + x$ where, 'x' is *independent variable* and 'y' is *dependent variable.*

In mathematics, the word function is used in much the same way, but more restrictively. It is defined as:

History

Peter Gustav Lejeune Dirichlet (1805-1859)

Peter Dirichlet was a German Mathematician who made valuable contributions in the study of mathematics such as number theory, mechanics and analysis.

He was the first person who gave the modern definition of function in 1837.

"If a variable 'y' depends on a variable 'x' in such a way that each value of 'x' determines exactly one value of 'y' then we call it 'y' is a function of 'x'. e.g $y = f(x)$ "

2.1.1 Domain and range of a function through graph

The domain and range can be identified by the graph. Because domain refers to the set of all input values, so, "all the values shown on the x-*axis*. The range refers to the set of all output values. Which are shown on the y-*axis*. Consider the graph given in Figure 2.1.

$f(x) = \dfrac{1}{x+2}$

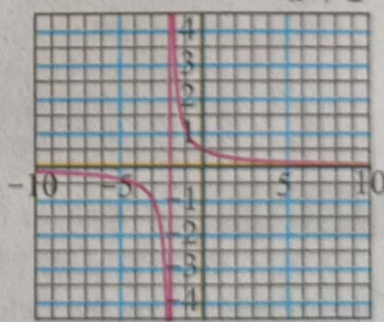


Figure 2.1

This is a graph of the function $f(x) = \dfrac{1}{x+2}$

Its domain is $(-\infty,-2)\cup(-2,\infty)$ and range is $(-\infty,0)\cup(0,\infty)$

Example 1 Find the domain and range of the function whose graph is shown in the Figure 2.2.

Solution Here the horizontal extent is -6 to 2. So, the domain of the function is $x \in [-6, 2]$ vertical extent of the graph is 0 to 4. So the range of the function is $y \in [0, 4]$. Shown in the Figure 2.3.

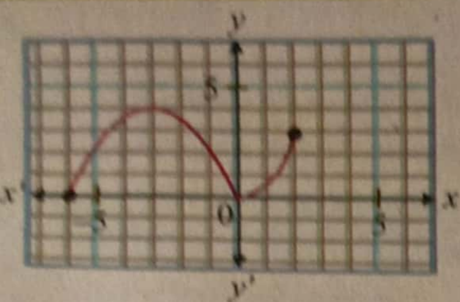


Figure 2.2

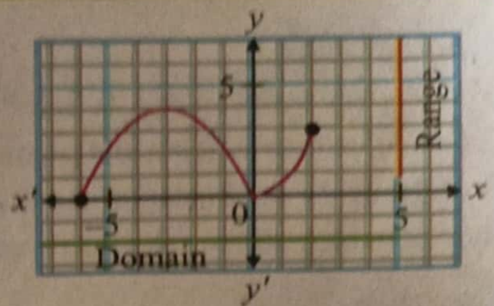


Figure 2.3

Remember

If the graph continuous beyond the given portion of the graph the domain and range can be greater than the given values.

2.1.2 Graph of modulus function (i.e. $y = |x|$) and its domain and range

"A function that contains an algebraic expression with in the absolute value symbols is called a absolute value function."

In our previous classes we have studied that the absolute value of a number is its distance from 0 on the number line.

The parent absolute value function can be written as $f(x) = |x|$ which is defined as $f(x) = \begin{cases} x & if \quad x \geq 0 \\ -x & if \quad x < 0 \end{cases}$

To graph the above absolute value functions simply choose some values of x and get the values of y then draw them accordingly.

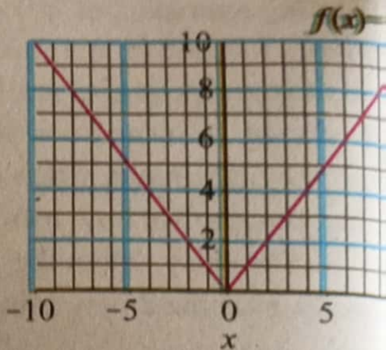


Figure 2.4

x	−4	−3	−2	−1	0	1	2	3	4	Domain values
y	4	3	2	1	0	1	2	3	4	Range values

Now, plot the points

In general, domain of $f(x) = |x|$ is $(-\infty, \infty)$ and range is $[0, \infty)$ but the graph is v-shaped grap

Figure 2.4 shows the graph of $f(x) = |x|$.

Remember

In any absolute value function for vertical translation of $f(x) = |x|$ you can use the function $g(x) = f(x) + h$

(i) When $h > 0$ the graph of $f(x)$ translate 'h' unit up to get $g(x)$.
(ii) When $h < 0$ the graph $f(x)$ translate 'h' units down to get $g(x)$.

This is called vertical translation. For horizontal translation of $f(x) = |x|$ you can use the function $g(x) = f(x - k$

(i) When $k > 0$ the graph of $f(x)$ will translate 'k' units to the right to get $g(x)$.
(ii) When $k < 0$ the graph of $f(x)$ will translate 'k' units to the left to get $g(x)$.
This is called horizontal translation.

Example 2 Graph the following absolute value functions and identify its domain and range: $f(x) = |3x - 4|$

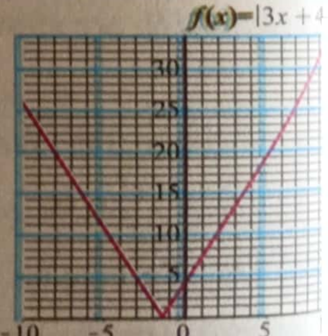


Figure 2.5

Solution From the definition of absolute value function, the given function is

$$f(x) = \begin{cases} 3x+4 & if \quad 3x+4 \geq 0 \\ -(3x+4) & if \quad 3x+4 < 0 \end{cases}$$

The inequality $3x + 4 \geq 0$ is satisfied whenever $x \geq -\dfrac{4}{3}$, and $3x + 4 < 0$

is satisfied whenever $x<-\frac{4}{3}$. If $x=-\frac{4}{3}$, $y=0$, so the graph will consist of two lines that meet at $\left(-\frac{4}{3},0\right)$. Use the tabular form to obtain the graph of a function:

$f(x)=3x+4,\ 3x+4\geq 0$

x	$-\frac{4}{3}$	0
y	0	4

$f(x)=-(3x+4),\ 3x+4<0$

x	$-\frac{4}{3}$	-3
y	0	5

This function has a domain set $(-\infty,\infty)$ and range set is $[0,\infty)$.

2.2 Composition of Functions

Composition of function can be described as a progression of "getting" and "dropping" "off". A function gets 'x' does something for it and drops it off. Then another function goes along and gets the drop off, does something for it, and drop it off once more. This pattern may proceed more than a few functions. Suppose a composition as a progression of car rides. 'x' boy is picked up by the first car function transported to a required location and dropped off. Then another car function come and pick up the x boy at this new location transports x boy to another location and drops x boy off.

2.2.1 Recognition of composite functions

Consider the function $h(x)$ whose rule is $h(x)=\sqrt{x^3}$. To compute h(4), you first need to find $x^3=4^3=64$ and then take the square root to obtain $\sqrt{x^3}=\sqrt{64}=8$. So the rule of $h(x)$ may be rephrased as:

$$h(x)=f(g(x)) \qquad \text{(i)}$$

Here $g(x)=x^3$ and $f(x)=\sqrt{x^3}$. We may think of the functions $f(x)$ and g(x) as being "composed" to create the function h.

In other words, when the output from one function is used as the input to another function, we form what is known as a composite function.

"If f(x) and g(x) are the two functions, then, a composite function or composition of g and f is the function whose values are given by g(f(x)) for all x in the domain of f(x) such that f(x) is in the domain of g(x)."

Example 3 Let $f(x)=2x-1$ and $g(x)=\sqrt{3x+5}$. Find each of the following:

(a). $g(f(4))$ (b). $f(g(4))$ (c). $f(g(-2))$

Solution

a. The function $f(x)$ for $x=4$ is used to obtain $f(4)=2(4)-1=7$. Use $f(4)$ to obtain:
$g(f(4))=g(7)=\sqrt{3(7)+5}=\sqrt{26}$

b. The function g(x) for $x=4$ is used to obtain $g(4)=\sqrt{3(4)+5}=\sqrt{17}$. Use g (4) to obtain:
$f(g(4))=f(\sqrt{17})=2\sqrt{17}-1$

c. $f(g(-2))$ does not exist, since -2 is not in the domain of g.

2.2.2 Composition of two given functions

Example 4 Let $f(x)=4x+1$ and $g(x)=2x^2+5x$. Find each of the following:

(a). $g(f(x))$ (b). $f(g(x))$

Solution a. Using the given functions to obtain:

$$g(f(x))=g(4x+1)=2(4x+1)^2+5(4x+1)=2(16x^2+8x+1)+20x+5$$
$$=32x^2+16x+2+20x+5=32x^2+36x+7$$

b. Using the given functions to obtain:

$$f(g(x))=f(2x^2+5x)=4(2x^2+5x)+1=8x^2+20x+1.$$

This example shows that $f(g(x))$ is not usually equal to $g(f(x))$.

Example 5 Air pollution is a problem for many metropolitan areas. Suppose that carbon monoxide is measured as a function of the number of people according to the following information:

Number of People	Daily Carbon Monoxide Level (in parts per million)
100,000	1.41
200,000	1.83
300,000	2.43
400,000	3.05
500,000	3.72

Further studies show that a refined formula for the average daily level of carbon monoxide in the air is $L(p)=0.7\sqrt{p^2+3}$

Further assume that the population of a given metropolitan area is growing according to the formula $p(t)=1+0.02t^3$, where t is the time from now (in years) and p is the population (in hundred thousands). Based on these assumptions, what level of air pollution should be expected in 4 years?

Solution The level of pollution at time t is given by the composite function:

$$L(p(t))=L(1+0.02t^3)=0.7\sqrt{(1+0.02t^3)^2+3}, \quad \text{(i)} \quad \because p(t)=1+0.02t^3$$

The air pollution expected in 4 years is obtained by putting $t=4$ in equation (i):

$$L(p(t))=L(1+0.02t^3)=0.7\sqrt{[1+0.02(4)^3]^2+3}\approx 2.0\ ppm$$

2.3 Inverse of composition of functions

*"Let y = f (x) be a function of x. This function takes a dependent variable y in response of independent variable x. The function that takes x as dependent variable in response of y as the independent is then called the **inverse function** of f (x)".*

The inverse function is denoted by $x=f^{-1}(y)$ (i)

The symbol $f^{-1}(y)$ means the inverse of f and does not mean $\frac{1}{f}$.

> **Note**
> If $f(x)$ is not one–to–one, then $f(x)$ does not have an inverse function.

For example, if $y=f(x)$ is one–to–one function, then the inverse of $y=f(x)$ is the function $x=f^{-1}(y)$ formed by interchanging the independent and dependent variables x and y for $y=f(x)$. Thus, if (a, b) is a point on the graph of $f(x)$, then (b, a) will be a point on the graph of the inverse of $f(x)$. The domain and range of $y=f(x)$ are also valid for its inverse function $x=f^{-1}(y)$.

2.3.1 Inverse of the composition of two given functions

Example 6 Find the inverse function of $f(t) = 3t - 8$.

Solution The function $f(t)$ takes an output $3t - 8$ in response of input t. The inverse function must take an output t in response of input $3t - 8$:

$$f^{-1}(3t-8) = t \qquad \text{(i)}$$

If $z = 3t - 8$ say, then $t = \dfrac{z+8}{3}$. Use these values in equation (i) to obtain: $f^{-1}(z) = \dfrac{z+8}{3}$

Put t as its argument instead of z to obtain the inverse function of $f(t)$=$3t - 8$: $f^{-1}(t) = \dfrac{t+8}{3}$

Example 7 Let $f(x) = 2x + 3$ and $g(x) = 3x$ and $h(x) = f(g(x))$.
Write expressions for the following functions

(a). $h(x)$ (b). $f^{-1}(x)$ (c). $g^{-1}(x)$ (d). $h^{-1}(x)$

Solution

a. In response of $f(x)$ and $g(x)$, the function $h(x)$ is:

$$h(x) = f(g(x))$$
$$= f(3x) \qquad \because g(x) = 3x$$
$$= 2(3x) + 3$$
$$= 6x + 3$$

b. In response of $f(x)$, the inverse of $f(x)$ is:

$$x = f^{-1}(2x+3)$$
$$x = f^{-1}(z), \qquad z = 2x+3 \Rightarrow x = \frac{(z-3)}{2}$$
$$\frac{z-3}{2} = f^{-1}(z)$$
$$\frac{x-3}{2} = f^{-1}(x) \qquad \textit{insert x instead of z}$$

c. In response of $g(x)$, the inverse of $g(x)$ is:

$$x = g^{-1}(3x)$$
$$x = g^{-1}(z), \qquad z = 3x \Rightarrow x = \frac{z}{3}$$
$$\frac{z}{3} = g^{-1}(z)$$
$$\frac{x}{3} = g^{-1}(x) \qquad \textit{insert x instead of z}$$

d. In response of $h(x)$, the inverse of $h(x)$ is:

$$x = h^{-1}(6x+3)$$
$$x = h^{-1}(z), \qquad z = 6x+3 \Rightarrow x = \frac{(z-3)}{6}$$
$$\frac{z-3}{6} = h^{-1}(z)$$
$$\frac{x-3}{6} = h^{-1}(x) \qquad \textit{insert x instead of z}$$

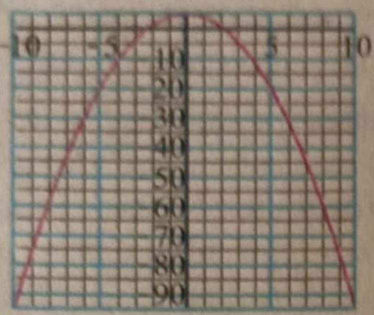
Facts about Calculus

Calculus was discovered as a tool of problem solving. Before the development of calculus, there were a wide range of issues that could not be addressed using the simple mathematics that was available .e.g. people did not know how to measure the speed of different objects when it was changing ever time. Another effective method was desired to calculate the area under the curve. Algebra, geometry, trigonometry and statistics were well understood, but they could not provide necessary tools to address these important issues. Some of the mathematicians of history give the credit to the ancient Greeks for discovering the calculus. But most of the scholars and mathematicians recognize Gottfried Wilhelm von Leibniz and sir, Isaac Newton developed its modran concepts in 17th century. According to the university of Laws Leibniz and Newton held different concept, while Leibniz introduced that the variables of x and y composing "sequences of infinitely close values". But Newton viewed them as variables that change with time. Leibniz considered calculus as a mathematical science for analysis but Newton took it being geometrical science.

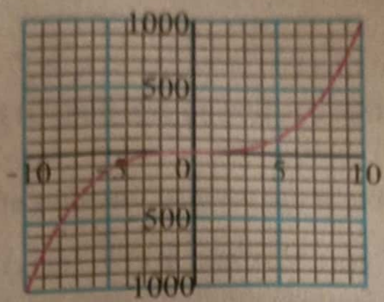
Exercise 2.1

1. Read the graphs and write the function, domain and range of f.

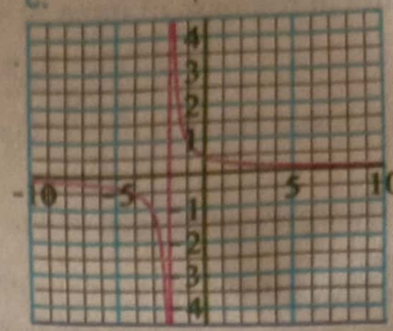
a.



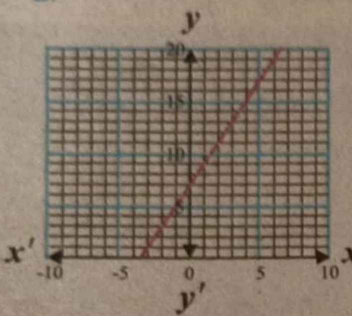
b.



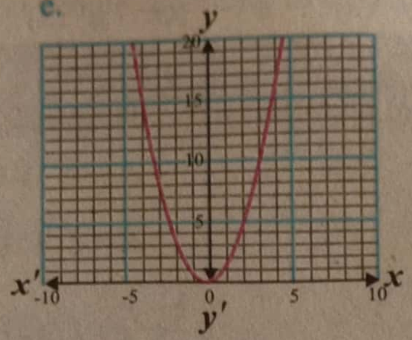
c.



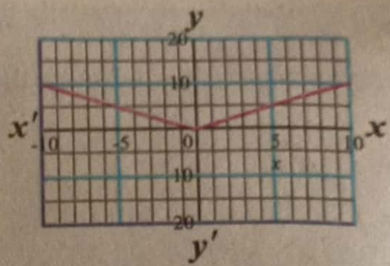
d.



e.



f.



2. Draw the graphs of the following functions.

a. $2|x-3|$

b. $4|x-3|+3$

c. $5|3t+7|-2$

d. $2|4x+3|+1$

3. Find the composite functions $f(g(x))$ and $g(f(x))$ of the following functions:

a. $f(x)=x^2+1,\ g(x)=2x$

b. $f(x)=\sin x,\ g(x)=1-x^2$

c. $f(x)=\dfrac{x-1}{x+1},\ g(x)=\dfrac{x+1}{1-x}$

d. $f(x)=\sin x,\ g(x)=2x+3$

4. Determine the inverse function of $f(g(x))$ and $g(f(x))$ for the following functions:

a. $f(x)=x+5,\ g(x)=x-4$

b. $f(x)=2x+7,\ g(x)=2x$

c. $f(x)=2(x-4),\ g(x)=\dfrac{x+5}{2}$

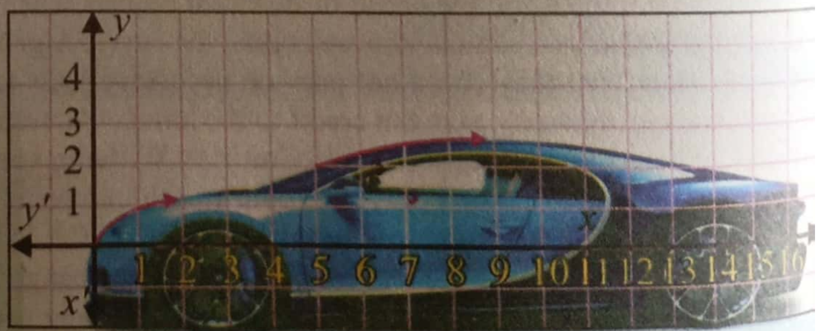
d. $f(x)=\dfrac{x+4}{2},\ g(x)=2x-4$

Project

Observe the following graphs and identify their domain and range.

i.

ii.



2.4 Transcendental Functions

"Functions that are not algebraic are called transcendental functions."

The functions, such as all trigonometric functions, hyperbolic functions, exponential functions and logarithmic functions are called transcendental functions.

Do You Know?

A polynomial $P_n(x)$ is a function of the form $f(x)=P_n(x)=a_n x^n+a_{n-1}x^{n-1}+....+a_2x^2+a_1x+a_0$ (i)
with n is a nonnegative integer and $a_n, a_{n-1}, ..., a_2, a_1, a_0$ are constants. If $a_n \neq 0$, then, the integer n is called the **degree** of the polynomial.
The constant a_n is called the **leading coefficient** and the constant a_0 is called the **constant term** of the polynomial function. In particular, the polynomial (i) is going to be a

constant function by putting $n=0$: $f(x)=a_0$
linear function by putting $n=1$: $f(x)=a_1x+a_0$
quadratic function by putting $n=2$: $f(x)=a_2x^2+a_1x+a_0$
cubic function by putting $n=3$: $f(x)=a_3x^3+a_2x^2+a_1x+a_0$

2.4.1 Recognition of algebraic, trigonometric, inverse trigonometric, exponential, logarithmic, hyperbolic (and their identities), explicit and implicit functions, and parametric representation of functions

i. Algebraic functions

A function $f(x)$ is called an algebraic function if it can be constructed using algebraic operations (such as adding, subtracting, multiplying, dividing or taking roots) starting with polynomials. Any rational function is an algebraic function e.g.

$$f(x)=x+5, \quad g(x)=3x^2+4x-7 \quad \text{and} \quad h(t)=\frac{8t^3+5t-9}{t^4+1}$$

ii. Trigonometric functions

"Trigonometric function are the functions that describe the relationship between the sides and angles of a right triangle".

Any trigonometric function include one or more of the following 6 trigonometric ratios.

(i). $\sin(x)$ (ii). $\cos(x)$ (iii). $\tan(x)$
(iv). $\csc(x)$ (v). $\sec(x)$ (vi). $\cot(x)$

These function has completely discussed in grade (XI) Mathematics.

iii. Inverse trigonometric functions

"Inverse trigonometric function are simply defined as the inverse functions of the basic trigonometric function."

These functions are used to get the angle with any of the trigonometric ratios. Inverse trigonometric functions are also known as "Arc functions" particularly these are 6 functions such as:

(i) Arc sine$(x)=\sin^{-1}(x)$ where $x\in[-1, 1]$
(ii). Arc cosine$(x)=\cos^{-1}(x)$ where $x\in[-1, 1]$
(iii). Arc tangent$(x)=\tan^{-1}(x)$ where $x\in R$
(iv). Arc cosecant$(x)=\csc^{-1}(x)$ where $x\in \geq 1$ or $x\leq -1$
(v). Arc secant$(x)=\sec^{-1}(x)$ where $x\in \geq 1$ or $x\leq -1$
(vi). Arc cotangent$(x)=\cot^{-1}(x)$ where $x\in R$

Do You Know?

Inverse trigonometric functions are also termed as, cyclometric functions, arcus functions and anti trigonometric functions.

Inverse trigonometric functions are widely used in the field of physics, engineering, geometry and navigations.

iv. Exponential Functions

The exponential function has widespread application in many areas of science and engineering. Areas which utilize the exponential function include expansion of materials, laws of cooling, radioactive decay and the discharge of a capacitor.

"An equation of the form $f(x)=b^x$, $b>0$, $b\neq 1$, b is a positive constant, defines an exponential function for each different constant b, called the base. The domain of f(x) is the set of all real numbers, and the range of f(x) is the set of all positive real numbers."

We require the base to be positive and to avoid imaginary numbers such as $(-2)^{\frac{1}{2}}=\sqrt{-2}=i\sqrt{2}$. We conclude $b=1$ as a base, since $f(x)=1^x=1$ is a constant function.

Remember

Exponent laws

If a and b are positive real numbers, $a\neq 1$ and $b\neq 1$, then,

i. $a^x a^y=a^{x+y}, \frac{a^x}{b^x}=a^{x-y}, (a^x)^y=a^{xy}, (ab)^x=a^x b^x, \left(\frac{a}{b}\right)^x=\frac{a^x}{b^x}$

ii. $a^x=a^y$ if and only if $x=y$

iii. For $x\neq 0$, $a^x=b^x$, if and only if $a=b$

Base e Exponential Functions

. Of all possible bases b, it can use for the exponential function $y=b^x$, which ones are the most useful? If you look at the keys on a scientific calculator, you will likely see 10^x and e^x. It is clear why base 10 would be important, because our number system is a base 10 system. But what is e, and why is it included as a base? It turns out that base e is used more frequently than all other bases combined. The reason for this is that certain formulas and the results of certain processes found in calculus and more advanced mathematics take on their simplest form if this base is used. This is why you will see e used extensively in expressions and formulas that model real–world phenomena. In fact, its use is so prevalent that you will often hear people refer to $y=e^x$ as the exponential function. The base ***e*** is an irrational number (like π) it cannot be represented exactly by any finite decimal fraction. However, ***e*** can be approximated as closely as we like by evaluating the expression

$$\left(1+\frac{1}{x}\right)^x \qquad \text{(i)}$$

for sufficiently large x. What happens to the value of expression (i) as x increases without bound? The results are summarized in the following table:

X	$\left(1+\frac{1}{x}\right)^x$
1	2
10	2.59374...
100	2.70481...
1000	2.71692...
10000	2.71814...
100000	2.71827...
1000000	2.71828...

Challenge

Use binomial theorem to find the value of e.

Interestingly, the value of expression (i) is never close to 1, but seems to be approaching a number close to 2.7183. In fact, as x increases without bound, the value of expression (i) approaches an irrational number that we call e. The irrational number e to twelve decimal places is:

$$e = 2.718\ 281\ 828\ 459$$

Example 8 Cholera, an intestinal disease, is caused by a cholera bacterium that multiplies exponentially by cell division as given approximately by

$$N = N_0 e^{1.386t},$$

with N is the number of bacteria present after t hours and N_0 is the number of bacteria present at the start ($t = 0$). If we start with 25 bacteria, how many bacteria (to the nearest unit) will be present in

(a). 1 hour? (b). 3 hours? (c). 4 hours? (d). Interpret

Solution Use the amount of initial bacteria $N_0 = 25$ in the given equation to obtain: $N = 25e^{1.386t}$ (i), $N_0 = 25$

a. The bacteria at a time $t = 1$ hour is obtained by putting $t = 1$ in equation (i):
$N = 25e^{1.386(1)} = 99.97$ *bacteria*

b. The bacteria at a time $t = 3$ hours is obtained by putting $t = 3$ in equation (i):
$N = 25e^{1.386(3)} = 1599$ *bacteria*

c. The bacteria at a time $t = 4$ hours is obtained by putting $t = 4$ in equation (i):
$N = 25e^{1.386(4)} = 6392$ *bacteria*

d. Thus, we conclude that the population of bacteria is growing when time t increases.

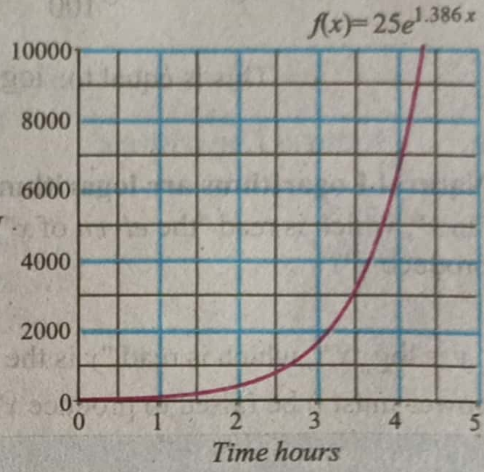


Figure 2.6

v. Logarithmic Functions

Logarithms are an alternative way of writing expressions which involve powers or indices. They are used extensively in the study of sound. The decimals used in defining the intensity of sound, is based on a logarithmic scale.

Until the development of computers and calculators, logarithms were the only effective tool for large scale numerical computations. They are no longer needed for this, but it still plays a crucial role in many applications.

For illustration, if we start with the exponential function $y = f(x)$ defined by $y = 2^x$ then the interchange of the variables is giving the inverse of $y = 2^x$:

$x = 2^y$

We call this inverse exponential function, the logarithmic function with base 2, and write this as: $y = \log_2 x$ if and only if $x = 2^y$ (i)

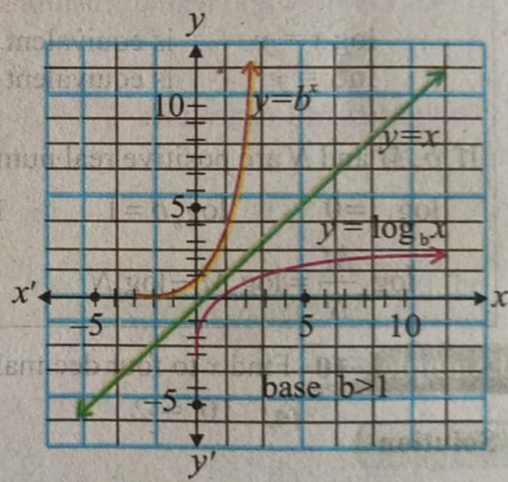


Figure 2.7

"The inverse of an exponential function is called a logarithmic function. For $b > 0$ and $b \neq 1$, the logarithmic function is: $y = \log_b x$, which is equivalent to $x = b^y$"

The log to the base b of x is the exponent to which b must be raised to obtain x. The domain of the logarithmic function is the set of all positive real numbers, which is also the range of the corresponding exponential function. Obviously, the range of the logarithmic function is the set of all real numbers, which is also the domain of the corresponding exponential function.

Typical graphs of an exponential function and its inverse, a logarithmic are shown in the Figure 2.7.

a. Common Logarithms

Common Logarithms are logarithms with base 10: $y = \log_{10} x$ means $10^y = x$

"Log x", which is read "the logarithm of x", is the answer to the question" to what exponent must 10 be raised to produce x ?

Example 9 Evaluate the following logarithmic functions:.

(a). $\log 10000$ *(b).* $\log .01$ *(c).* $\log\sqrt{10} = \frac{1}{2}$

Solution a. This is equal to: $\log 10000 = \log 10^4 = 4\log 10 = 4$

b. This is equal to:

$$\log .01 = \log\frac{1}{100} = \log(10)^{-2} = -2\log 10 = -2$$

c. This is equal to: $\log\sqrt{10} = \log(10)^{\frac{1}{2}} = \frac{1}{2}\log 10 = \frac{1}{2}$

Do You Know ?

The common logarithm is also called Briggsian logarithms and the natural logarithm is also called Napierian logarithms.

b. Natural Logarithms

Natural Logarithms are logarithms with base e: $y = \ln x$ means $e^y = x$

"$\ln x$", which is read "the *el–en* of x", is the answer to the question" to what exponent must e be raised to produce x"?

$$y = \log_b x \text{ means } b^y = x$$

"$y = \log_b x$", which is read "y is the logarithm of x to the base b", is the answer to the question" to what power must b be raised to produce x?

Logarithmic Notation

Common logarithmic: $\log x = \log_{10} x$ Natural logarithmic: $\ln x = \log_e x$

Logarithmic–Exponential Relationships

$\log x = y$ is equivalent to $x = 10^y$

$\ln x = y$ is equivalent to $x = e^y$

Properties of Logarithms

If b, M and N are positive real numbers $b \neq 1$, and p and x are also any positive real numbers, then:

i. $\log_b 1 = 0$ ii. $\log_b b = 1$ iii. $\log_b b^x = x$ iv. $b^{\log_b x} = x, x > 0$ v. $\log_b MN = \log_b M + \log_b N$

vi. $\log_b \frac{M}{N} = \log_b M - \log_b N$ vii. $\log_b M^p = p\log_b M$ viii. $\log_b M = \log_b N,\ M = N$

Example 10 Find x to four decimal places for the following indicated exponential functions:

(a). $10^x = 2$ *(b).* $e^x = 3$

Solution

a. $10^x = 2$

$\log 10^x = \log 2$, log of both sides

$x\log 10 = \log 2$, $\because \log 10 = 1$

$x = 0.3010$

b. $e^x = 3$

$\ln e^x = \ln 3$, ln of both sides

$x\ln e = \ln 3$, $\because \ln e = 1$

$x = 1.0986$

Example 11 Two people with the covid-19 positive visited the campus of Peshawar University. The number of days T that it took for the corona virus to infect n people is given by

$$T(n) = -1.43\ln\left(\frac{10{,}000 - n}{4998n}\right)$$

How many days will it take for the virus to infect a. 500 people? *b.* 5000 people?

Solution The number of days T that will take for the flu virus to infect n people is given by

$$T(n) = -1.43\ln\left(\frac{10{,}000-n}{4998n}\right) \qquad \text{(i)}$$

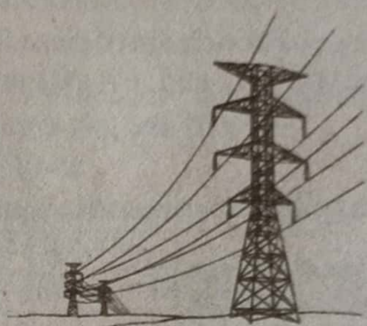
The number of days that will take for the virus to infect 500 people, is obtained by putting $n = 500$ in equation (i):

a. $$T(500) = -1.43\ln\left(\frac{10{,}000-500}{4998(500)}\right) = -1.43\ln\left(\frac{9500}{2499000}\right) = -1.43\ln(0.00380)$$
$$= -1.43\,(-5.57275) = 7.96903 \approx 8\ days$$

b. $$T(5000) = -1.43\ln\left(\frac{10{,}000-5000}{4998(5000)}\right) = -1.43\ln\left(\frac{5000}{24990000}\right) = -1.43\ln\left(\frac{1}{4998}\right) = 12.17 \approx 12\ days$$

vi. Hyperbolic functions and their identities

In physics, it is shown that a heavy, flexible cable (for example a power line) that is suspended between two points at the same height assumes the shape of a curve called a **catenary**, with an equation of the form $y = \frac{a}{2}\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right)$ (i)



Hanging cable

Figure 2.8

This is one of several important applications that involve combinations of exponential functions. In certain ways, the functions we shall study are analogous to be trigonometric functions, and they have essentially the same relationship to the hyperbola that the trigonometric functions have to the circle. For this reason, these functions are called hyperbolic functions. Three basic functions are the hyperbolic sine (denoted "sinh$x$" and pronounced "cinch"), the hyperbolic cosine (cosh$x$; pronounced "kosh") and the hyperbolic tangent (tanh$x$; pronounced "tansh"). They are listed as under:

$$\sinh x = \frac{e^x - e^{-x}}{2},\ \cosh x = \frac{e^x + e^{-x}}{2},\ \tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}},\ \coth x = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$

$$\cosh^2 x - \sinh^2 x = 1$$
$$\cosh(x+y) = \cosh x\cosh y + \sinh x\sinh y$$
$$\sinh(x+y) = \sinh x\cosh y + \cosh x\sinh y$$

The name "hyperbolic functions" comes from the fact that the functions sinht and cosht play the same role in the parametric representation of the hyperbolic $x^2 - y^2 = 1$, as the trigonometric functions sint and cos$t$, do in the parametric representation of the circle $x^2 + y^2 = 1$.

Eliminating the parameter t from the parametric equations $x = \cos t$, $y = \sin t$ to obtain the equation of the circle: $x^2 + y^2 = \cos^2 t + \sin^2 t = 1$

Similarly, the equations $x = \cosh t$, $y = \sinh t$ are the parametric equations of the hyperbola. Squaring these equations and subtracting the second from the first to obtain the equation of hyperbola: $x^2 - y^2 = \cosh^2 t - \sinh^2 t = 1$

vii. Explicit and Implicit Functions

So far we have met many functions of the form $y = f(x)$: $y = x^2 + 3$, $y = \sin x$, $y = e^{3x} - 2x$ (i)

If y is equated to an expression involving only x terms, then we say that y is expressed **explicitly** in terms of x that is in equation (i).

Sometimes we have an equation connecting x and y but it is impossible to write it in the form of $y = f(x)$:

$$y = x^2 - y^3 + \sin x - \cos y = 1,\quad \sin(x+y) + e^x + e^{-y} = x^3 + y^3 \qquad \text{(ii)}$$

In these cases we say that y is expressed **implicitly** in terms of x.

Remember

The following curves are modeled through implicit functions:

a. The Bifolium curve: $(x^2+y^2)^2=4x^2y$

b. Lemniscate curve: $(x^2+y^2)^2=\frac{25}{3}(x^2-y^2)$

c. Folium of Descartes: $x^3+y^3-\frac{9}{2}xy=0$

d. Cissoid of Dioceles: $y^2(6-x)=x^3$

e. Cardioid curve: $(x^2+y^2)^{\frac{3}{2}}=\sqrt{x^2+y^2}+x$

f. Ellipse curve: $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

g. Hyperbola curve: $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

viii. Parametric representation of functions

It is sometimes useful to define the variables x and y in the ordered pair (x, y), so that they are each functions of some other variable, say t:

$$x=f(t) \qquad \text{and} \qquad y=g(t) \qquad \text{(i)}$$

The domain of these functions $f(t)$ and $g(t)$ is some interval D. The variable t is called a parameter and $x=f(t)$ and $y=g(t)$ are called the parametric equations.

"If $f(t)$ and $g(t)$ are continuous functions of parameter t on an interval D, then the equations

$$x=f(t) \qquad \text{and} \qquad y=g(t)$$

are called the parametric equations for the plane curve generated by the set of ordered pairs in the plane:

$$(x, y)=(x(t), y(t))=(f(t), g(t)) \qquad \text{(ii)"}$$

Example 12 Sketch the graph of the parametric functions $(x(t), y(t))=(3-t, 2t)$ for all t.

Solution The graph is the collection of all points (x, y) with $x=3-t$, $y=2t$ for different real values of t:

$t=0 \Rightarrow (x(t), y(t))=(3,0)$

$t=1 \Rightarrow (x(t), y(t))=(2,2)$

$t=2 \Rightarrow (x(t), y(t))=(1,4)$

The plot of the position vectors $t_0=(3,0)$, $t_1=(2,2), t_2=(1,4)$ in the Figure 2.9 developed a straight line parallel to the direction vector $u=(-1,2)$ and passing through the point $p\ (3,0)$.

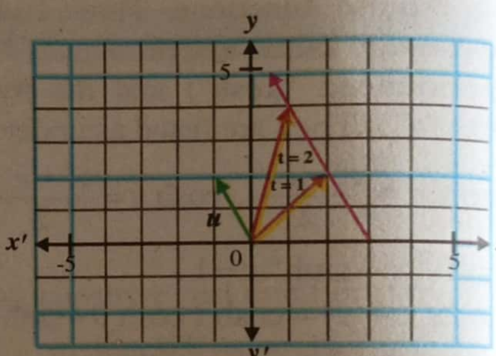


Figure 2.9

2.5 Graphical Representation

In our previous classes we have learnt that graphical representations refers to the use of intuitive charts to clearly visualize and simplify the given data sets. The data is ingested into the graphical representation of software and then represented by the different symbols. Like, curves, bars and slices on the chart.

(a) Graphical display of explicit defined functions like $y=f(x)$, where $f(x)=e^x, a^x, \log_a x, \log_e x$

i. Graphically representation of $f(x)=e^x$

Example 13 Draw the graphs of

$$y=e^x \text{ and } y=e^{-x}.$$

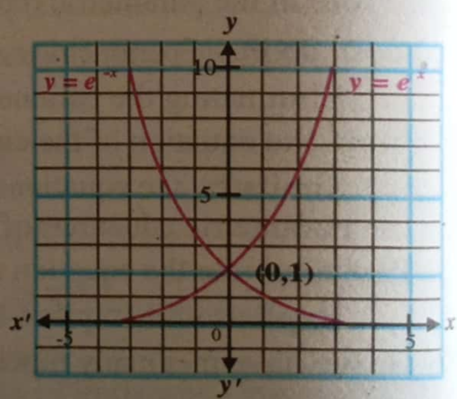


Figure 2.10

Solution Use a scientific calculator to create the table of points. Plot these points and then join them to obtain the graphs of smooth curves in the Figure 2.10. The domain set is $(-\infty,\infty)$, while the range set is $(0,\infty)$.

ii. Graphically representation of $f(x) = a^x$

Example 14 Sketch the graph of $y = 2^x$.

Solution To hand sketch graphs of equations such as $y = 2^x$ or $y = 2^{-x}$, simply make a tables by assigning integers to x, plot the resulting points, and then join these points with a smooth curve as shown in Figure 2.11.

$y = 2^x$						
x	–5	–1	0	1	2	3
y	0.03	0.5	1	2	4	8

$y = 2^{-x}$						
x	–3	–2	–1	0	1	5
y	8	4	2	1	0.5	0.03

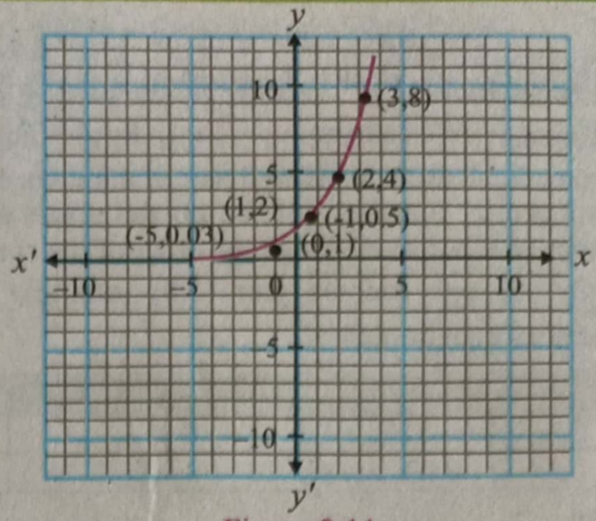


Figure 2.11

It is useful to compare the graphs $y = 2^x$ and $y = 2^{-x}$ by plotting both on the same set of coordinate axes as shown in Figures 2.12. The graph of $f(x) = b^x$, $b > 1$ shown in Figure 2.13 looks very much like the graph of $y = 2^x$, and the graph of $f(x) = b^{-x}$, $0 < b < 1$ in Figure 2.13 looks very much like the graph of $y = 2^{-x}$.

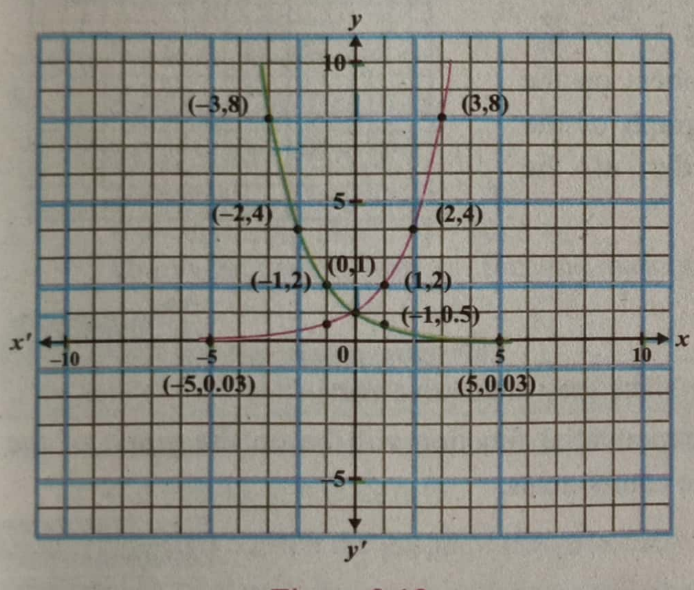


Figure 2.12

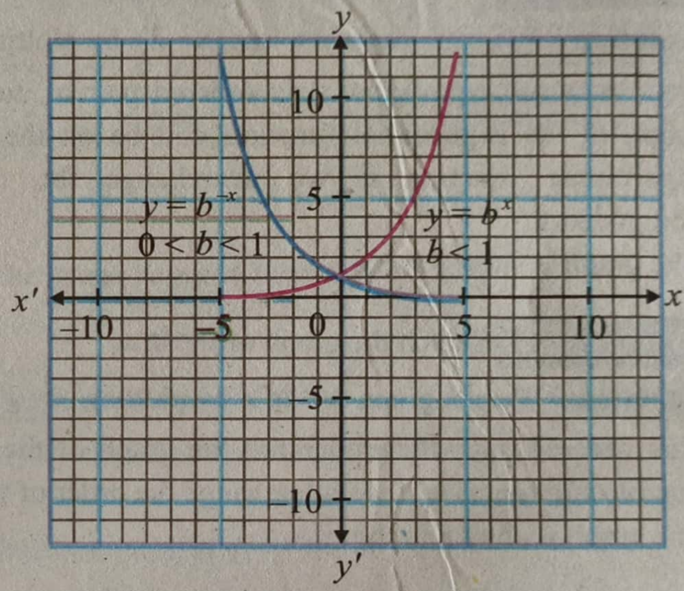


Figure 2.13

The graphs in Figures 2.12 and Figures 2.13 suggest the following important general properties of exponential functions that are summarized in the box below:

Remember

Basic properties of the graph of $f(x) = b^x, b > 0, b \neq 1$

1. All graphs will pass through the point (0,1).
2. All graphs are continuous curves, with no holes or jumps.
3. If $b > 1$, then b^x increases as x increases.
4. If $0 < b < 1$, then b^x decreases as x increases.

iii. Graphically representation of $\log_a x$ and $\log_e x$

Example 15 Sketch the graph of $y=\log_2 x$.

Solution We can graph $y=\log_2 x$ by plotting $x=2^y$, since they are equivalent. Any ordered pair of numbers on the graph of the exponential function will be on the graph of the logarithmic function if we interchange the order of the components.
For example, ordered pair (3,8) satisfies $y=2^x$ and (8,3) satisfies equation $x=2^y$.

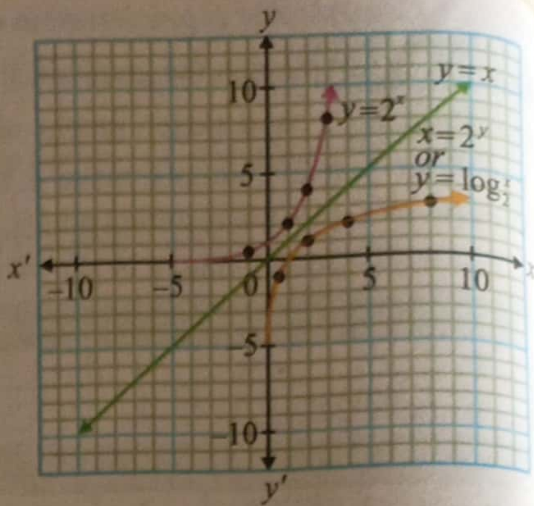


Figure 2.14

$y=2^x$						
x	−5	−1	0	1	2	3
y	0.03	0.5	1	2	4	8

and

$x=2^y$ or $y=\log_2 x$						
y	0.03	0.5	1	2	3	8
x	−5	1	0	1	2	3

The graphs of $y=2^x$ and $y=\log_2 x$ are shown in the Figure 2.14.

Example 16 Sketch the graph of $y=\log_2(x-2)$.

Solution We can graph $y=\log_2(x-2)$ by plotting $x-2=2^y$, since they are equivalent. Any ordered pair of numbers on the graph of the exponential function will be on the graph of the logarithmic function if we interchange the order of the components.
The graph is shown in the Figure 2.15.

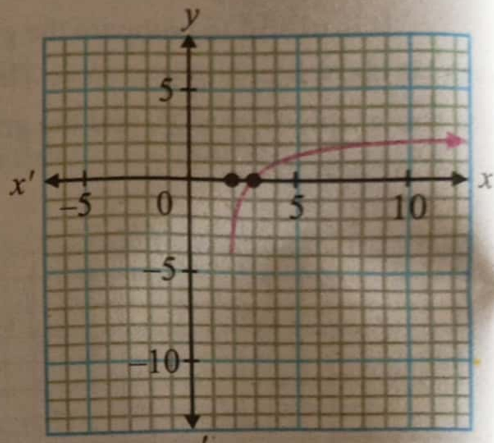


Figure 2.15

Example 17 Sketch the graph of $y=\ln x$.

Solution We can graph $y=\ln x$ by plotting $x=e^y$, since they are equivalent.
Any ordered pair of numbers on the graph of the exponential function will be on the graph of the logarithmic function if we interchange the order of the components.
The graph is shown in the Figure 2.16.

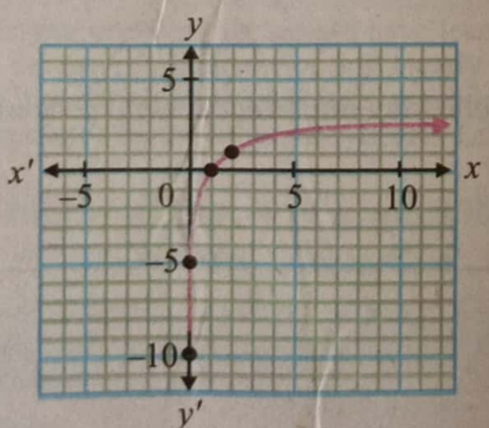


Figure 2.16

Do You Know

If we fold the paper along the dashed line $y=x$, the two graphs match exactly.
The line $y=x$ is a line of symmetry for the two graphs.

(b) Graphical display of implicit defined function such as $x^2+y^2=a^2$ and $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and distinguish between graph of a function and of an equation

Example 18 Sketch the graph of $x^2+y^2=9$.

Solution This equation $x^2+y^2=9$ is an equation of circle having radius 3 with centred at origin (0, 0).

Rewrite the equation $x^2+y^2=9$ as

$$(x-0)^2+(y-0)^2=3^2$$

The standard form of equation of circle is $(x-h)^2+(y-k)^2=r^2$

With centre $(h, k) = (0, 0)$ and radius $r=3$

So, this is a circle of radius 3 centred at origin (0, 0).

Figure 2.17 is showing the graph of $x^2+y^2=9$.

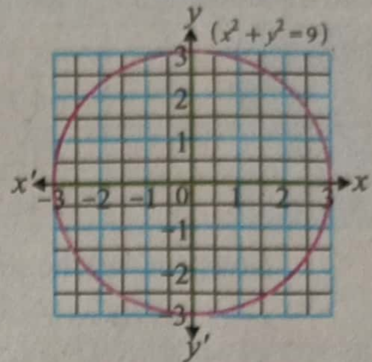


Figure 2.17

Example 19 Sketch the graph of $\frac{x^2}{16}+\frac{y^2}{9}=1$.

Solution The equation $\frac{x^2}{16}+\frac{y^2}{9}=1$ is an equation of ellipse.

Compare it with the standard equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ $\quad\therefore c^2=a^2-b^2$

where, $h=0, k=0$ $a=\sqrt{16}=4$, $b=\sqrt{9}=3$ and $c=\sqrt{16-9}=\sqrt{7}$

So, **Center:** $(h, k) = (0, 0)$

Foci: $(h\pm c, k) = (\pm\sqrt{7}, 0)$

Vertices: $(h\pm a, k) = (\pm 3, 0)$

Co-vertices: $(h, k\pm b) = (0, \pm 4)$

Graph the center, vertices, foci and axes on the graph paper as shown in Figure 2.18.

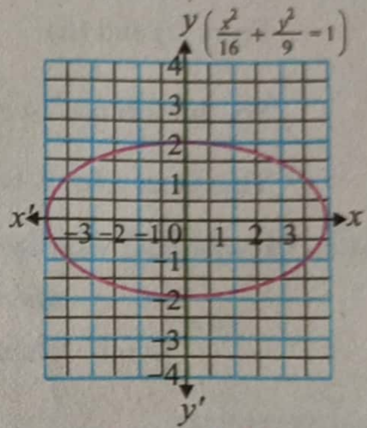


Figure 2.18

Note

The students will learn more about conic in unit 8 and 9.

(c) Graphical display of parametric equation functions such as $x=at^2, y=2at$; $x=a\sec\theta, y=b\tan\theta$

Example 20 Sketch the graph of the parametric function $(x(t), y(t)) = (3t^2, 4t+3)$.

Solution To sketch the graph of the parametric equation, Let's make a table to get the idea of the shape and direction of the graph.

t	–6	–5	–4	–3	–2	–1	0	1	2	3	4	5	6
$x(t)$	108	75	48	27	12	3	0	3	12	27	48	75	108
$y(t)$	–21	–17	–13	–9	–5	–1	3	7	11	15	19	23	27

$$y=4t+3 \Rightarrow t=\frac{y-3}{4} \quad \ldots \quad (i)$$

$$x=3t^2 \quad \ldots \quad (ii)$$

$$x=3\left(\frac{y-3}{4}\right)^2$$

$$x=\frac{1}{16}(y^2-6y+9) \Rightarrow x=\frac{3}{8}(y^2-6y+9)$$

This is a right opening parabola, its graph is shown in Figure 2.19.

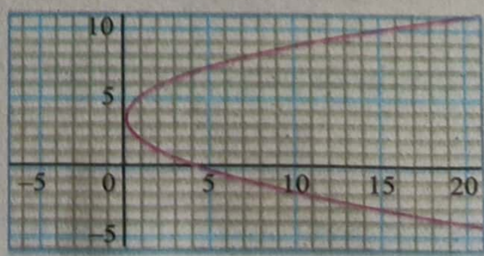


Figure 2.19

Example 21 Sketch the graph of the parametric function $(x(t), y(t)) = (3.\cos(t), 2.\sin(t))$ for $0 \le t \le 2\pi$.

Solution To sketch the graph of the parametric equation. Let's make a table to get the idea of the shape and direction of the graph.

t	0	$\left(\frac{\pi}{4}\right)$	$\left(\frac{\pi}{2}\right)$	$\left(\frac{3\pi}{4}\right)$	π	$\left(\frac{5\pi}{4}\right)$	$\left(\frac{3\pi}{2}\right)$	$\left(\frac{7\pi}{4}\right)$	2π
x(t)	3	2.12	0	–2.12	–3	–2.12	0	2.12	3
y(t)	0	1.41	2	1.41	0	–1.41	–2	–1.41	0

Now, convert the standard form by eliminating the parameter. e.g.

$x = 3\cos(t) \Rightarrow x^2 -9\cos^2(t)$ ... (i)

$y = 2\sin(t) \Rightarrow y^2 = 4\sin^2(t)$ ... (ii)

From (i) and (ii). $\frac{x^2}{9} + \frac{y^2}{4} = \cos^2(t) + \sin^2(t) = 1$

So, the formula of ellipse is $\frac{x^2}{9} + \frac{y^2}{4} = 1$ where $a = 3, b = 2$.

This is an ellipse, which will be discussed in detail in unit-9.

(d) Graphical display of discontinuous functions of the type

$$y = \begin{cases} x & \text{when } 0 \le x < 1 \\ x-1 & \text{when } 1 \le x \le 2 \end{cases}$$

Example 22 Graph the compound function:

$$f(x) = \begin{cases} 3-x & \text{if } x < -2 \\ x+2 & \text{if } -2 \le x < 2 \\ 1 & \text{if } x \ge 2 \end{cases}$$

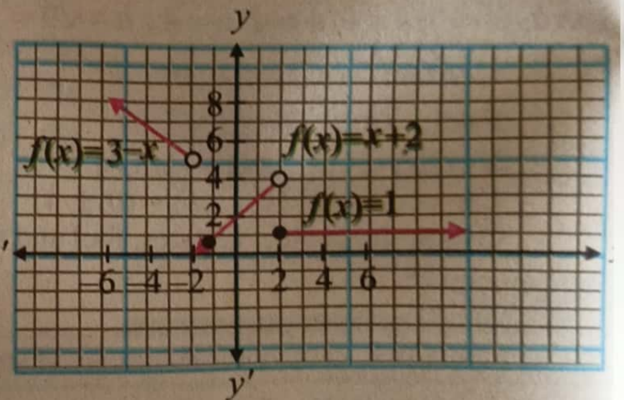


Figure 2.20

Solution

a. Use the function $f(x) = 3 - x$ for $x < -2$ to obtain a set of points:

x	–3	–2
$f(x)$	6	5

b. Use the function $f(x) = x + 2$ for $-2 \le x < 2$ to obtain a set of points:

x	–2	2
$f(x)$	0	4

Do You Know?

A function that defined by more than one equation is called compound function.

c. Use the function $f(x) = 1$ for $x \ge 2$ to obtain a set of points:

x	2	4
$f(x)$	1	1

Use these tabular points to obtain the graph of a compound function in Figure 2.20.

2.5.2 MAPLE graphic commands for two-dimensional plot of
- an expression (or a function)
- parameterized form of a function
- implicit function by restricting domain and range

Look at the following example, the procedure to use the maple graphic commands is illustrated.

Example 23 Use maple commands to draw the graphs of the given function.

(a). Function $f(x)=-(x+2)^2$, with domain $[0,-4]$.

(b). Parametric function $(x(t), y(t)=\cos(t).\sin(t)$ for $t=-3.5$ to 3.5, x from -1.5 to 1.5 and y from -1.5 to 1.5.

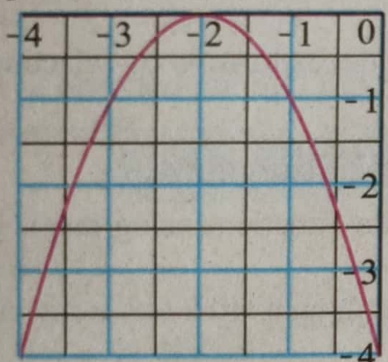
(c). An implicit function $x^2-y^2=1$ x from -5 to 5 and y from -5 to 5.

Solution The command below will show you full detail of plotting expressions/ functions on line by typing:

> *?plots*

a. Command

> $plot(-(x+2)^2, x=0..-4);$



>

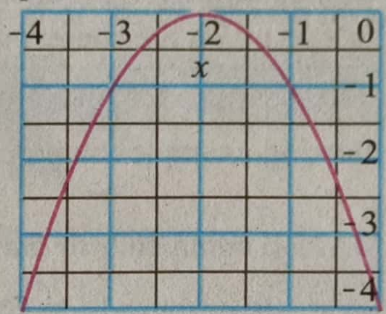
Figure 2.21

Context Menu

> $-(x+2)^2$

$-(x+2)^2$ (1)

> $plot(-(x+2)^2, x=0..-4)$

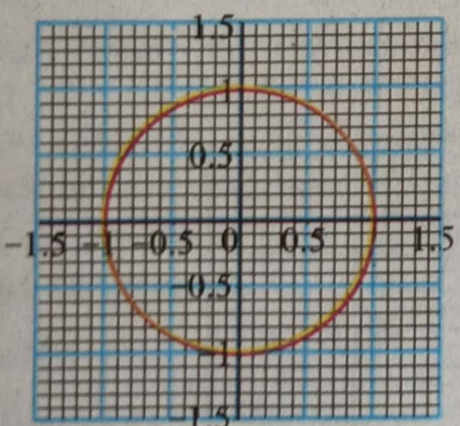


>

Figure 2.22

b. Command

> $plot([\cos(t), \sin(t), t=-3.5..3.5], -1.5..1.5, -1.5..1.5);$



>

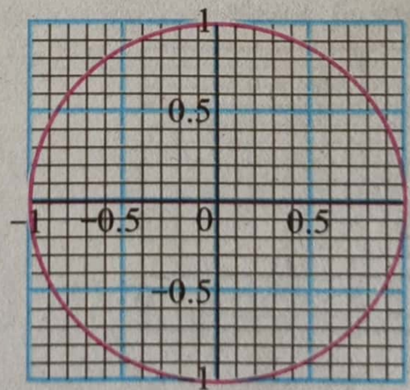
Figure 2.23

Context Menu

> $\sin(t), \cos(t)$

$\sin(t), \cos(t)$ (1)

> $plot([\sin(t), \cos(t), t=-3.5..3.5])$



>

Figure 2.24

This graph is obtained through right-click on the last end of the expression by selecting "Plots < Plot Builder < 2D Parametric Plot" on the context menu.

c. **Command**

> $plots[implicitplot](x^2 - y^2 - 1, x = -5..5, y = -5..5);$

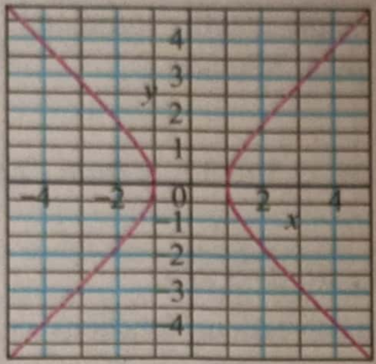


Figure 2.25

Context Menu

> $x^2 - y^2 = 1$

$$x^2 - y^2 = 1 \qquad (1)$$

> $smartplot[x, y](x^2 - y^2 = 1)$

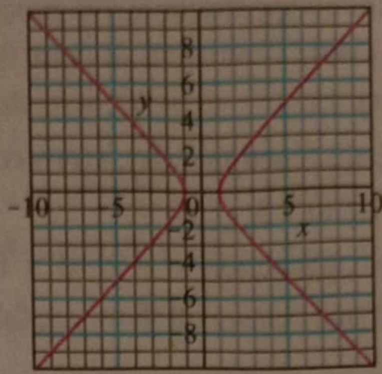


Figure 2.26

This graph is obtained through right-click on the last end of the expression by selecting " Plots < 2D–Implicit Plot < x, y " on the context menu.

2.5.3 MAPLE package plots for plotting different types of functions

Look at the following example the procedure of plotting the functions using maple package i illustrated.

Example 24 Use MAPLE commands to draw the following functions.

Solution *(a).* $f(x) = x^2 - b, x$ from –1– to 10 and y from –5 to 5.

(b). $(x(t), y(t)) = \cos(tx)\sin(ty)), t$ from –1 to 2, x from $-\pi$ to π, y from $-\pi$ to π.

The command below will show you full detail of plotting packages on line by typing

> ? $plots[animate]$

a. **Command**

> $plots[animate](plot, [x^2 - b, x = -10..10], b = -5..5);$

$b = -5.$

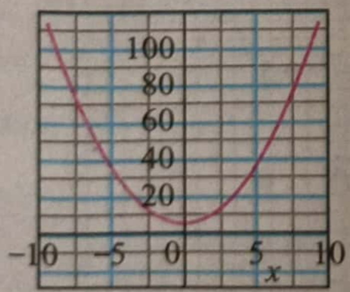


Figure 2.27

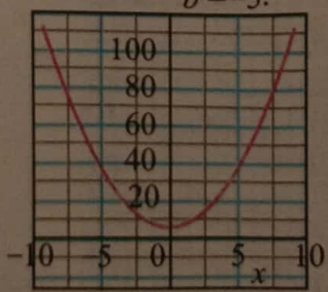
Context Menu

> $x^2 - b$

$$x^2 - b \qquad (1)$$

> $plots[:-animate]('plot', [x^2 - b, x = -10..10, labels = [x, ""]], b = -5..5)$

$b = -5.$



1. > Figure 2.28

This graph is obtained through right-click on the last end of the expression by selecting "Plots < Plot Builder < Animation (choose 2D–Implicit Plot)" on the context menu.

b. Command

```
> plots[animate]
('plot3d', [cos(t*x)*sin(t*y),
x=-π..π, y=-π..π], t=1..2);
```

$t = 1.$

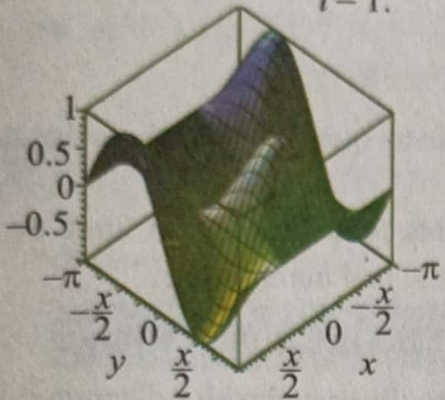


Figure 2.29

Context Menu

```
> plots[:-animate]('plot3d', [cos(t*x)
*sin(t*y), x=-π..π, y=-π..π,
labels=[x, y, ""]], t=1..2)
```

$t = 1.$

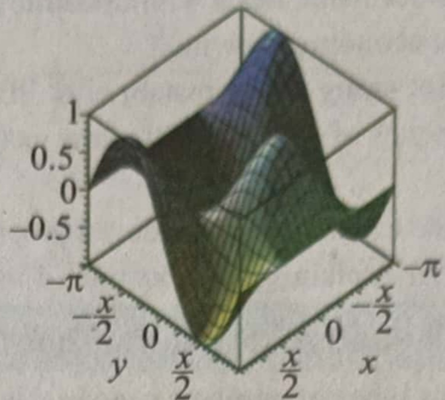


Figure 2.30

This graph is obtained through right-click on the last end of the expression by selecting "Plots < Plot Builder < Animation (choose 3D-Implicit Plot + Parameter) " on the context menu.

Exercise 2.2

1. Recognize and write the type for each of the following functions:
   a. $y = a\sin^2 x + x$ b. $y = 7x^4 + 3x^2 - 4x + 5$ c. $y = \text{arc}\tan x - 7$
   d. $y = \log_2 16 + 7\log_2(x)$ e. $x^2 + y^2 = 36$ f. $y = \dfrac{e^{3x} + e^{-3x}}{e^{3x} - e^{-3x}}$ g. $y = \sqrt{x-8}$
2. A sealed box contains radium. The number of grams present at time t is given by $Q(t) = 100e^{-0.00043t}$ where t is measured in years. Find the amount of radium in the box at the following times:
   a. $t = 0$ b. $t = 800$ c. $t = 1600$ d. $t = 5000$
   e. How did you guess from the above results?
3. Using a calculator and point–by–point to plot the following exponential functions:
   a. $h(x) = (2^x); [-5, 0]$ b. $m(x) = (3^{-x}); [0, 3]$ c. $N = e^t; [0, 5]$ d. $N = e^{-t}; [0, 5]$
4. Using a calculator and point–by–point to plot the following logarithmic functions:
   a. $y = \ln x$ b. $u = -\ln x$ c. $y = 2\ln(x+2)$
   d. $y = 4\ln(x-3)$ e. $y = 4\ln x - 2$
5. Sketch the following parametric curves:
   a. $(x(t), y(t)) = (3-t, 2t)$, t is real number. b. $(x(t), y(t)) = (4\cos t, -3\sin t)$
6. Sketch the graph of:
   a. e^{-2x} b. $\dfrac{2}{3}e^{3x}$ c. 4^x d. 4^{-x}
   e. $\log_2(x+5)$ f. $\log_2 x^2$ g. $\log_e(2x-5)$
7. Sketch the graph of:
   a. $x^2 + y^2 = 4$ b. $x^2 + y^2 = 16$ c. $\dfrac{x^2}{25} + \dfrac{y^2}{9} = 1$ d. $\dfrac{x^2}{36} + \dfrac{y^2}{9} = 1$
8. Use maple commands to plot the graphs of the functions given in Q.1.

2.6 Limit of Function

The algebraic problems considered in earlier sections dealt with static situations:
What is the revenue when x items are sold?
How much interest is earned in 2 years?
Calculus, on the other hand, deals with dynamic situations:
At what rate is the economy growing?
How fast is a rocket going at any instant after lift–off?

The techniques of calculus will allow us to answer many questions like these that deal with rates of change.

The key idea underlying the development of calculus is the concept of limit. So we begin by studying limits after explaining the location of intervals on the real number line.

2.6.1 Identification of a real number by a point on the number line

The various types of numbers used in this book can be illustrated with a diagram called a number line. Each real number corresponds to exactly one point on the line and vice–versa. A number line with several sample numbers located on it is shown in Figure 2.31:

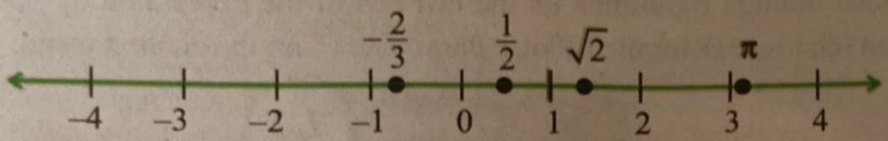


Figure 2.31

2.6.1 Representation of open interval, closed interval, half open and half closed intervals on the number line

"A set that consists of all the real numbers between two points is called an interval."

A special notation will be used to indicate an interval on the real number line.

For example, the interval including all numbers x, where $-2 < x < 3$ is written as $(-2, 3)$. The parentheses indicate that the number –2 and 3 are not included.
If –2 and 3 are to be included in the interval, square brackets are used, as in $[-2, 3]$.
The chart below shows several typical intervals, where $a < b$:

Inequality	Interval Notation	Explanation
o $a \le x \le b$	$[a,b]$: Closed	Both a and b are included.
o $a \le x < b$	$[a,b)$: Half open/Closed	a is included, b is not.
o $a < x \le b$	$(a,b]$: Half Open/Closed	b is included, a is not.
o $a < x < b$	(a,b): Open	Both a and b are not included.

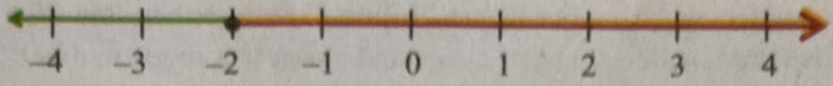
Interval notation is also used to describe sets such as the set of all numbers x, with $x \ge -2$. This interval is written $[-2,\infty)$.

Example 24 Represent the following intervals on number line.

(a). $[-2,\infty)$ (b). $[4,\infty)$ (c). $[-2,1]$

Solution

a. Start at –2 and draw a heavy line to the right, as in graph. Use a solid hole at –2 to show that –2 is itself a part of the graph.
The symbol ∞, read "infinity" does not represent a number. It simply indicates that all number greater than –2 are in the interval. Similarly, the notation $(-\infty,2)$ indicates the set of all real numbers with $x < 2$.

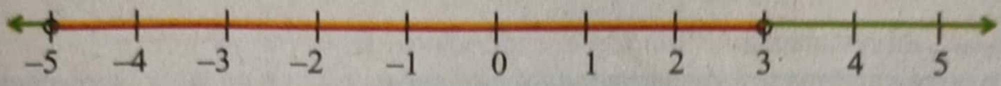


b. The graph of the interval $[4, \infty)$ is as under:

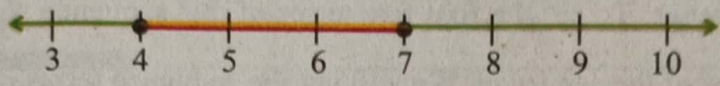
c. The graph of the interval $[-2, 1]$ is as under:

Example 25 Use number line to indicate the interval notation:

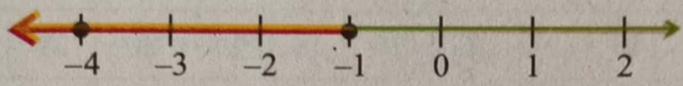
(a).



(b).



(c).



Solution The given graphs indicate the following intervals:

a. $(-5, 3)$ *b.* $[4, 7]$ *c.* $(-\infty, -1]$

2.6.3 Explanation of phrase

- **x tends to zero ($x \to 0$)**
- **x tends to a ($x \to a$)**
- **x tends to infinity ($x \to \infty$)**

i. x tends to zero ($x \to 0$)

The answer to the phrase x tends to "0" is easy to see that the value of a function $y = f(x) = \dfrac{x^2 - 4}{x - 2}$ gets closer and closer to a single real number "2"on both left and right sides of "2", when x is a number very close to "0" on both left and right sides of "0". In this situation, we are in position to say that x approaches to "0" or x tends to "0" and is denoted by $x \to 0$, when $f(x)$ tends to a single number "say $L = 2$".

ii. x tends to a ($x \to a$)

The answer to the phrase x tends to "a" (a is any real number) is easy to see that the value of a function

$$y = f(x) = \frac{x^2 - a^2}{x - a},$$

gets closer and closer to a single real number "$2a$"on both left and right sides of "$2a$", when x is a number very close to "a" on both left and right sides of "a". In this situation, we say that x approaches to "a" or x tends to "a" and is denoted by $x \to a$, when $f(x)$ tends to a single number "say $L = 2a$".

iii. x tends to infinity ($x \to \infty$)

The answer to the phrase x tends to "infinity" is easy to see that the function $f(x) = \dfrac{3x+2}{x+1}$

gets smaller and smaller, when x approaches "infinity" from either side of a number say 3. In this situation, we say that the function $f(x)$ gets closer and closer to a single number "say $L = 3$" when $x \to$ from either side.

2.6.4 Limit of a sequence

The number L is the limit of the sequence $\{S_n\}$ if

$$\text{given } \varepsilon > 0,\ S_n \approx L \quad for\ n \geq 1 \qquad \text{(i)}$$

If such an L exist, we say $\{S_n\}$ converges, or convergent.
If 'L' does not exist, $\{S_n\}$ diverges or divergent. There are two notations which we use to show the limit of a sequence. $\lim\limits_{n\to\infty}\{S_n\} = L,\quad S_n \to L \quad as\ n \to \infty$

These notations are abbreviated as $\lim S_n = L$ or $S_n \to L$

2.6.5 Limit of a sequence whose n[th] term is given

Let $S_n = \dfrac{n+2}{n^2}$ be a sequence. To get the first few terms of this sequence we need to plug in values of n into the general form of the sequence. We will get the sequence terms considering n as positive integer.

n	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
$\frac{n+2}{n^2}$	$\frac{1}{121}$	$\frac{7}{72}$	$\frac{15}{169}$	$\frac{4}{49}$	$\frac{17}{225}$	$\frac{9}{128}$	$\frac{19}{289}$	$\frac{5}{81}$	$\frac{21}{361}$	$\frac{1}{200}$	$\frac{23}{441}$	$\frac{6}{121}$	$\frac{25}{529}$	$\frac{1}{288}$	$\frac{27}{625}$

Similarly, we can get more terms by using this process and write the above sequence in the following notation.

$$S_n = \left\{\frac{n+2}{n^2}\right\}_{n=1}^{\infty} = 3, 1, \frac{5}{9}, \frac{3}{8}, \frac{7}{25}, \frac{2}{9}, \frac{9}{49}, \ldots$$

In the above sequence we treated it as a function that can only have integers plugged into them. This is an important idea which allows us to do many things with sequences that we can not compute by using other methods. To graph the sequence $\{S_n\}$ we plot the points (n, S_n) as n ranges over all possible values on the graph representing first 25 terms of the given sequence. From the graph we noticed that as n increases the terms of the sequence get closer and closer to zero, but not exactly equal to zero. We then say that zero is limiting value of the sequence and it can be written as

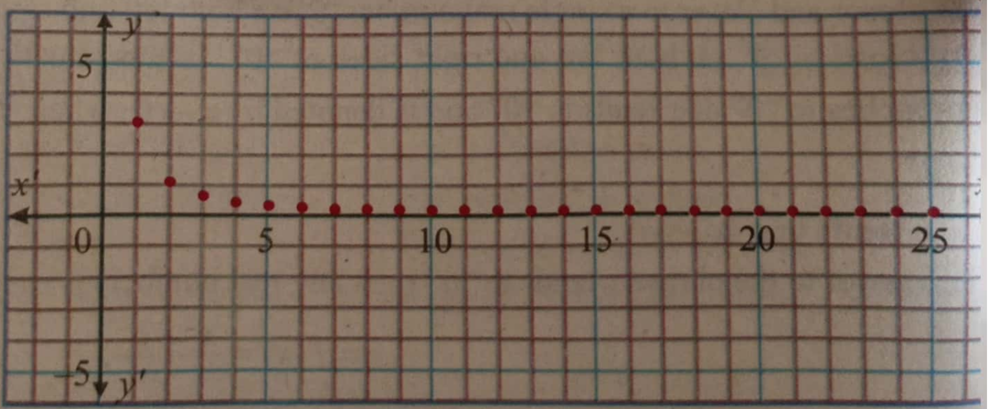


Figure 2.32

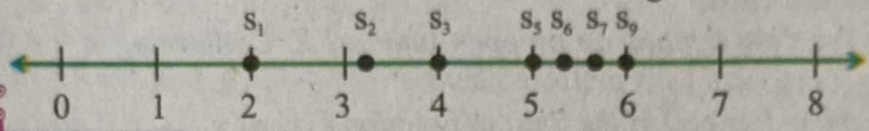
$$\lim_{n\to\infty} S_n = \lim_{n\to\infty}\frac{n+2}{n^2} = 0$$

Example 26 Represent the sequence on one dimensional space whose nth term is $s_n = \dfrac{8n}{n+3}$.

Solution The sequence $\{s_n\}$ in terms of function notation is $s(n) = \dfrac{8n}{n+3}$, whose domain is the set of non-negative integers. The functional values of $s\,(n)$ develop

n: Integers	Function: $s(n)$
1	$s_1 = s(1) = 2.0$
2	$s_2 = s(2) = 3.2$
3	$s_3 = s(3) = 4.0$
4	$s_4 = s(4) = 4.57$

The one dimensional view on a real number line is shown in figure



Remember

Limit theorem of sequence

If $\lim\limits_{n\to\infty} a_n = L$ and $\lim\limits_{n\to\infty} b_n = M$, then the limit exist are the following:

(i). $\lim\limits_{n\to\infty}(ra_n + sb_n) = rL + sM$ Linearity rule

(ii). $\lim\limits_{n\to\infty}(a_n b_n) = LM$ Product rule

(iii). $\lim\limits_{n\to\infty}\dfrac{a_n}{b_n} = \dfrac{L}{M},\ M \neq 0$ Quotient rule

(iv). $\lim\limits_{n\to\infty}\sqrt[m]{a_n} = \sqrt[m]{L}$ Root rule

Example 27 Find the limit of each of these convergent/divergent sequences.

(a). $\left\{\dfrac{8n}{n+3}\right\}$ (b). $\left\{\dfrac{n^5+n^3+2}{7n^4+n^2+3}\right\}$ (c). $\{(-1)^n\}$

Solution

a. Let $\{a_n\} = \dfrac{8n}{n+1}$

$$\lim_{n\to\infty}\{a_n\} = \lim_{n\to\infty}\frac{8n}{n+1}$$

$$= \lim_{n\to\infty}\frac{n(8)}{n\left(1+\frac{1}{n}\right)} = \lim_{n\to\infty}\frac{8}{1+\frac{1}{n}}$$

$$= \frac{8}{1+\frac{1}{\infty}} = \frac{8}{1} = 8 \qquad \because \frac{1}{\infty} = 0$$

Hence, the sequence is converging to 8.

Remember

Some of the sample points near n = infinity are:

n	$\lim\limits_{n\to\infty}(-1)^n$
1	1
10	1
100	1
1000	1
10000	1
100000	1

b. Let $a_n = \dfrac{n^5+n^3+2}{7n^4+n^2+3}$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{n^5+n^3+2}{7n^4+n^2+3} = \lim_{n\to\infty}\frac{n^5\left(1+\frac{1}{n^2}+\frac{2}{n^5}\right)}{n^5\left(\frac{7}{n}+\frac{1}{n^3}+\frac{3}{n^5}\right)}$$

$$= \lim_{n\to\infty} \frac{1+\frac{1}{n^2}+\frac{2}{n^5}}{\frac{7}{n}+\frac{1}{n^3}+\frac{3}{n^5}} = \frac{1+\frac{1}{(\infty)^2}+\frac{2}{(\infty)^5}}{\frac{7}{\infty}+\frac{1}{(\infty)^3}+\frac{3}{(\infty)^5}} = \frac{1}{0}$$

In the above expression the numerator tends to 1 as $n \to \infty$, but the denominator approaches to 0. So, the quotient increases without bound. Hence, the sequence is divergent.

c. Let $a_n = (-1)^n$

$\lim_{n\to\infty} a_n = \lim_{x\to\infty} (-1)^n$

The sequence does not approaches to any specific number. So it is divergent sequence by oscillation. The nth term is always either 1 or -1. It is 1 when n is even and -1 when n is odd.

2.6.6 Limit of a function

"Let $f(x)$ be a function defined on an open interval X. Containing $x = c$ (the value $f(c)$ do not needs to be defined)

The specific number L is called the limit of function $f(x)$ as $x \to c$ if and only if, for every $\varepsilon > 0$ there exist $\delta \to 0$ such that

$$|f(x)-L| < \varepsilon \text{ when ever, } 0 < |x-c| < \delta\text{"}$$

This definition is also known as the couchy definition for limit.

Usually limit of a function is written as $\lim_{x\to c} f(x) = L$ and read as "Lim of $f(x)$ as $x \to c$ is L"

This is neither desirable nor practicable to find the limit of a function by numerical approach. You must be able to evaluate a limit in some mechanical way.

2.6.7 Theorems on limits of sum, difference, product and quotient of functions and demonstrate through examples

Let $f(x)$ and $g(x)$ be two functions, for which $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = M$

i. **The limit of the sum of two functions is equal to the sum of their limits**

$\lim_{x\to c}[f(x)+g(x)] = \lim_{x\to c} f(x) + \lim_{x\to c} g(x) = L + M$

Example 28 If $f(x) = x^2+2x+3$ and $g(x) = x-4$ then calculate $\lim_{x\to 2}[f(x)+g(x)]$

Solution Since, $f(x) = x^2+2x+3$ (i)

$g(x) = x-4$ (ii)

By adding (i) and (ii) $f(x)+g(x) = x^2+3x-1$ (iii)

Applying $\lim_{x\to 2}$ on both sides of (iii).

$\lim_{x\to 2}[f(x)+g(x)] = \lim_{x\to 2}(x^2+3x-1)$

$= \lim_{x\to 2} x^2 + 3\lim_{x\to 2}(x) - \lim_{x\to 2}(1) = 4+6-1 = 9$

Hence, $\lim_{x\to 2}[f(x)+g(x)] = 9$

Remember

If $f(x) = K$, where K is any constant than

$\lim_{x\to c} f(x) = \lim_{x\to c} K = K$ (constant rule)

$\lim_{x\to c} K f(x) = K \lim_{x\to c} f(x) = K.L$ (multiple rule)

ii. **The limit of the difference of two functions is equal to the difference of their limits**

$\lim_{x\to c}[f(x)-g(x)] = \lim_{x\to c} f(x) - \lim_{x\to c} g(x) = \text{L} - \text{M}$

Example 29 If $f(x) = x-7$ and $g(x) = x^2+3x+2$ then calculate $\lim_{x\to 4}[f(x)-g(x)]$

Solution Since, $f(x) = x-7$ (i)

$g(x) = x^2+3x+2$ (ii)

By subtracting (ii) from (i)

$$f(x)-g(x)=(x-7)-(x^2+3x+2)=x-7-x^2-3x-2$$
$$=-x^2-2x-9=-(x^2+2x+9) \qquad \text{(iii)}$$

By applying $\lim_{x\to 1}$ on both sides of equation (iii).

$$\lim_{x\to 1}[f(x)-g(x)]=\lim_{x\to 1}-(x^2+2x+9)=-\lim_{x\to 1}\left(x^2+2x+9\right)$$
$$=-\left[\lim_{x\to 1}(x^2)+2\lim_{x\to 1}x+\lim_{x\to 1}(9)\right]=-[1+2+9]=-12$$

Hence, $\lim_{x\to 1}[f(x)-g(x)]=-12$

iii. The limit of the product of the functions is equal to the product of their limits

$$\lim_{x\to c}[f(x).g(x)]=\left[\lim_{x\to c}f(x)\right]\left[\lim_{x\to c}g(x)\right]=LM$$

Example 30 If $f(x)=x+5$ and $g(x)=2x-4$ then calculate $\lim_{x\to 3}[f(x).g(x)]$

Solution Since, $f(x)=x+5 \qquad \text{(i)}$

$g(x)=2x-4 \qquad \text{(ii)}$

By multiplying equation (i) and equation (ii).

$$f(x).g(x)=(x+5)(2x-4)=2x^2-4x+10x-20=2x^2+6x-20 \qquad \text{(iii)}$$

By applying $\lim_{x\to 3}$ on both sides of equation (iii).

$$\lim_{x\to 3}[f(x).g(x)]=\lim_{x\to 3}(2x^2+6x-20)=2\lim_{x\to 3}(x^2)+6\lim_{x\to 3}(x)-\lim_{x\to 3}(20)$$
$$=2(9)+6(3)-20=18+18-20=16$$

Hence, $\lim_{x\to 3}[f(x).g(x)]=16$

iv. The limit of the quotient of the functions is equal to the quotient of their limits provided the limit of the denominator is non-zero

$$\lim_{x\to c}\left[\frac{f(x)}{g(x)}\right]=\frac{\lim_{x\to c}f(x)}{\lim_{x\to c}g(x)}=\frac{L}{M} \text{ where } g(x)\neq 0 \text{ and } M\neq 0.$$

Example 31 If $f(x)=x-4$ and $g(x)=x^2+3$ then calculate $\lim_{x\to -2}\left[\frac{f(x)}{g(x)}\right]$

Solution Since, $f(x)=x-4 \qquad \text{(i)}$

$g(x)=x^2+3 \qquad \text{(ii)}$

By using equation (i) and equation (ii). $\frac{f(x)}{g(x)}=\frac{x-4}{x^2+3} \qquad \text{(iii)}$

By applying $\lim_{x\to -2}$ on both sides of equation (iii).

$$\lim_{x\to -2}\frac{f(x)}{g(x)}=\lim_{x\to -2}\left[\frac{x-4}{x^2+3}\right]=\frac{\lim_{x\to -2}(x-4)}{\lim_{x\to -2}(x^2+3)}=\frac{-2-4}{4+3}=-\frac{6}{7}$$

Hence, $\lim_{x\to -2}\left[\frac{f(x)}{g(x)}\right]=-\frac{6}{7}$

2.7 Important Limits

2.7.1 Limits of the functions of the following types

i. $\dfrac{x^n-a^n}{x-a}, \dfrac{x-a}{\sqrt{x}-\sqrt{a}}$ when $x \to a$

a. $\lim\limits_{x\to a}\left(\dfrac{x^n-a^n}{x-a}\right)=na^{(n-1)}$

Proof: In this situation, we need to divide out the numerator by denominator to obtain:

$$\frac{x^n-a^n}{x-a}=x^{n-1}+ax^{n-2}+a^2x^{n-3}+\dots\dots+a^{n-2}x+a^{n-1} \qquad \text{(i)}$$

Being a polynomial, the function to the right of the above expression (i) is continuous for all values of and as such its limit, when $x \to a$ must equal to its value at $x=a$. Thus, the limit of the expression (i) when x tends to a is:

$$\lim_{x\to a}\frac{x^n-a^n}{x-a}=\lim_{x\to a}(x^{n-1}+ax^{n-2}+a^2x^{n-3}+\dots\dots+a^{n-2}x+a^{n-1})$$
$$=a^{n-1}+aa^{n-2}+a^2a^{n-3}+\dots\dots+aa^{n-2}+a^{n-1}$$
$$=a^{n-1}+a^{n-1}+a^{n-1}+\dots\dots+a^{n-1}+a^{n-1}=na^{n-1}$$

b. $\lim\limits_{x\to a}\left(\dfrac{x-a}{\sqrt{x}-\sqrt{a}}\right)=2\sqrt{a}$

Proof: When $x \to a$, the limit of a function is of the form $\left(\dfrac{0}{0}\right)$, which is undefined. In this situatic

we need to rationalize the given function to obtain the required limit:

$$\lim_{x\to a}\left(\frac{x-a}{\sqrt{x}-\sqrt{a}}\right)=\lim_{x\to a}\left(\frac{x-a}{\sqrt{x}-\sqrt{a}}\right)\times\left(\frac{\sqrt{x}+\sqrt{a}}{\sqrt{x}+\sqrt{a}}\right)=\lim_{x\to a}\frac{(x-a)(\sqrt{x}+\sqrt{a})}{x-a}$$
$$=\lim_{x\to a}(\sqrt{x}+\sqrt{a})=\lim_{x\to a}\sqrt{x}+\lim_{x\to a}\sqrt{a}=\sqrt{a}+\sqrt{a}=2\sqrt{a}$$

ii. $\left(1+\dfrac{1}{x}\right)^x$ when $x \to \infty$

Prove that $\lim\limits_{x\to\infty}\left(1+\dfrac{1}{x}\right)^x=e$

Proof: The base "e" is an irrational number (like π), it cannot be represented exactly by any fin decimal fraction. However, e can be approximated as closely as we like by evaluating the expressi

$$\left(1+\frac{1}{x}\right)^x \qquad \text{(i)}$$

for sufficiently large x. What happens to the value of the expression as x increases without bound? Th results are summarized in the following table:

x	$\left(1+\frac{1}{x}\right)^x$
1	2
10	2.59374...
100	2.70481...
1000	2.71692...
10000	2.71814...
100000	2.71827...
1000000	2.71828...

Activity

Use binomial theorem to generate e.

Interestingly, the value of expression (i) is never close to 1, but seems to be approaching a number close to 2.7183. In fact, as x increases without bound, the value of expression (i) approaches an irrational number that we call e. The irrational number e to twelve decimal places is:

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = 2.718281828459 = e \qquad \text{(ii)}$$

From result (ii), the new result deduced is:

$$\lim_{x\to-\infty}\left(1+\frac{1}{x}\right)^x = 2.718281828459 = e$$

iii. $(1+x)^{\frac{1}{x}}$, $\dfrac{\sqrt{x+a}-\sqrt{a}}{x}$, $\dfrac{a^x-1}{x}$, $\dfrac{(1+x)^n-1}{x}$ and $\dfrac{\sin x}{x}$ when $x\to 0$

a. Show that $\lim\limits_{x\to\infty}(1+x)^{\frac{1}{x}} = e$

Proof: If we put $y=\dfrac{1}{x}$, then $y\to\infty$, when $x\to 0$, and the left-hand side of the limit thus gives the right-hand side:

$$\lim_{x\to 0}(1+x)^{\frac{1}{x}} = \lim_{y\to\infty}\left(1+\frac{1}{y}\right)^y = e$$

b. Prove that $\lim\limits_{x\to 0}\dfrac{a^x-1}{x} = \ln a,\ a>0.$

Proof: If we put $a^x-1=y$, then x is obtained by taking log of both sides:

$$a^x-1=y \quad \Rightarrow a^x = 1+y$$
$$\log(a^x)=\log(1+y) \quad \Rightarrow x\log a = \log(1+y)$$
$$x=\frac{\log(1+y)}{\log a}$$

Use this x in the expression $\dfrac{a^x-1}{x}$ to obtain: $\Rightarrow \dfrac{a^x-1}{x} = \dfrac{y}{\dfrac{\log(1+y)}{\log a}}$

Taking limit $y \to 0$, when $x \to 0$:

$$\lim_{x\to 0}\frac{a^x-1}{x}=\lim_{y\to 0}\frac{y}{\frac{\log(1+y)}{\log a}}=\lim_{y\to 0}\frac{1}{\frac{1}{y}\log(1+y)\left(\frac{1}{\log a}\right)}=\lim_{y\to 0}\frac{\log a}{\log(1+y)^{\frac{1}{y}}}$$

$$=\log a\frac{1}{\log\lim_{y\to 0}(1+y)^{\frac{1}{y}}}=\log a\frac{1}{\log e}=\log_e a=lna$$

By replacing 'a' with 'e' the following result can be deduced.

$$\lim_{x\to 0}\frac{e^x-1}{x}=\ln e=1$$

c. **Prove that** $\lim_{x\to 0}\frac{\sqrt{x+a}-\sqrt{a}}{x}=\frac{1}{2\sqrt{a}}$

Proof: We have to show that: $\lim_{x\to 0}\frac{\sqrt{x+a}-\sqrt{a}}{x}=\frac{1}{2\sqrt{a}}$

Take L.H.S $\lim_{x\to 0}f(x)=\lim_{x\to 0}\frac{\sqrt{x+a}-\sqrt{a}}{x}=\frac{\sqrt{0+a}-\sqrt{a}}{0}=\frac{0}{0}$ *which is undefined.*

Now, rationalize the function $f(x)$.

$$f(x)=\left(\frac{\sqrt{x+a}-\sqrt{a}}{x}\right)\left(\frac{\sqrt{x+a}+\sqrt{a}}{\sqrt{x+a}+\sqrt{a}}\right)$$

$$=\frac{x+a-a}{x\left(\sqrt{x+a}+\sqrt{a}\right)}=\frac{x}{x\left(\sqrt{x+a}+\sqrt{a}\right)}$$

$$\Rightarrow \lim_{x\to 0}f(x)=\lim_{x\to 0}\frac{1}{\sqrt{x+a}+\sqrt{a}}$$

$$=\frac{1}{\sqrt{0+a}+\sqrt{a}}=\frac{1}{2\sqrt{a}}=\text{R.H.S}$$

d. **Prove that** $\lim_{x\to 0}\frac{(1+x)^n-1}{x}=n$

Proof: If we put $(1+x)^n-1=z$, then: $(1+x)^n-1=z$

$(1+x)^n=(1+z)$ Taking log of both sides to obtain:

$$\log(1+x)^n=\log(1+z)$$

$$n\log(1+x)=\log(1+z)$$

Use these expressions in the left-hand side of the limit to obtain the right-hand side:

$$\frac{(1+x)^n-1}{x}=\frac{z}{x}=\frac{z}{x}\times\frac{\log(1+z)}{\log(1+z)}=\frac{z}{\log(1+z)}\times\frac{n\log(1+x)}{x}=\frac{1}{\log(1+z)^{\frac{1}{z}}}\times n\log(1+x)^{\frac{1}{x}}$$

Taking limit $z\to 0$, when $x\to 0$:

$$\lim_{x\to 0}\frac{(1+x)^n-1}{x}=\lim_{z\to 0}\frac{1}{\log(1+z)^{\frac{1}{z}}}\times\lim_{x\to 0}n\log(1+x)^{\frac{1}{x}}=\frac{1}{\log e}\times n\log e=n$$

Do You Know?

The sandwich theorem: This is a theorem that is used in calculus to evaluate a limit of a function. It is particularly useful to evaluate limits where other techniques might be unnecessarily complicated. To define sandwich theorem.

"Let $f(x)$, $g(x)$ and $h(x)$ be functions such that $f(x)\le g(x)\le h(x)$ for all x in some open interval containing "c", except possibly at c itself. If $\lim_{x\to c} f(x)=L$ *and* $\lim_{x\to c} h(x)=L$ *then* $\lim_{x\to c} g(x)=L$

c. **Prove that** $\lim_{x\to 0}\frac{\sin x}{x}=1$ if angle x is measured in radian.

Proof: Take a positive acute central angle of a circle with radius $r=1$ as shown in the figure.

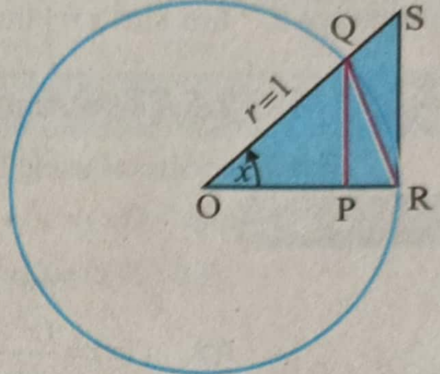


Figure 2.33

Given that ΔOPQ, $\sin x=\frac{|QP|}{|QO|}=|PQ|$ $|QO|=1$ (radius of unit circle)

In $\Delta ORS, \tan x=\frac{|SR|}{|RO|}=|RS|$

In term of x, the areas are expressed as produce QO to S so, that $SR\perp RO$ join QR.

(i) Area of $\Delta ORQ=\frac{1}{2}|OR||PQ|=\frac{1}{2}(1)(\sin x)=\frac{1}{2}\sin x$

(ii) Area of sector $ORQ=\frac{1}{2}r^2x=\frac{1}{2}(1)(x)=\frac{x}{2}$

(iii) Area of $\Delta ORS=\frac{1}{2}|OR||RS|=\frac{1}{2}(1)(\tan x)=\frac{1}{2}\tan x$

From the figure we observed

Area of $\Delta QRO<$ Area of sector $QRO<$ Area of Δ SRO.

$\Rightarrow\frac{1}{2}\sin x<\frac{1}{2}x<\frac{1}{2}\tan x$

As $\sin x$ is positive, so, dividing by $\frac{1}{2}\sin x$ we get

$0<\frac{x}{\sin x}<\frac{1}{\cos x}$ $\left(1<x<\frac{\pi}{2}\right)$ i.e. $1>\frac{\sin x}{x}>\cos x$ *or* $\cos x<\frac{\sin x}{x}<1$

When $x\to 0,\ \cos x\to 1$

Since $\frac{\sin x}{x}$ is sandwiched between 1 and a quantity approaches 1 itself.

Therefore, by the sandwich theorem, it must also approach 1 i.e., $\lim_{x\to 0}\frac{\sin x}{x}=1$

Remember

- The result is also true for $-\frac{\pi}{2}<x<0$
- The sandwich theorem is also known as squeeze theorem, pinching theorem.

2.7.2 Limits of different algebraic, exponential and trigonometric functions

The idea of limits in the above situations is illustrated in the following examples:

Example 32 Evaluate $\lim_{x\to 1}\frac{x^3-3x+7}{5x^2+9x+6}$.

Solution $\lim_{x\to 1}\frac{x^3-3x+7}{5x^2+9x+6}=\frac{\lim_{x\to 1}(x^3-3x+7)}{\lim_{x\to 1}(5x^2+9x+6)}=\frac{1-3+7}{5+9+6}=\frac{5}{20}=\frac{1}{4}$

Example 33 Evaluate $\lim_{x\to -2} \sqrt[3]{(x^2-3x-2)}$.

Solution

$$\lim_{x\to -2} \sqrt[3]{x^2-3x-2} = \lim_{x\to -2}(x^2-3x-2)^{\frac{1}{3}} = [\lim_{x\to -2}(x^2-3x-2)]^{\frac{1}{3}} = [(-2)^2-3(-2)-2]^{\frac{1}{3}} = (8)^{\frac{1}{3}} = 2$$

Example 34 Evaluate the limits

(a). $\lim_{x\to 0}\sin^2 x$

(b). $\lim_{x\to 0}(1-\cos x)$, when $\lim_{x\to 0}\sin x = 0$ and $\lim_{x\to 0}\cos x = 1$.

Solution a. $\lim_{x\to 0}\sin^2 x = \left[\lim_{x\to 0}\sin x\right]^2 = 0$

b. $\lim_{x\to 0}(1-\cos x) = \lim_{x\to 0}1 - \lim_{x\to 0}\cos x = 1-1 = 0$

2.7.3 MAPLE Command to evaluate limit of a function

The procedure of using MAPLE command 'limit' is illustrated in the following example.

Example 35 (a). $f(x)=x^2+2x+2$, when x tends to 2.

(b). $f(x)=(2x^3)(x-4)$, when x tends to 3.

(c). $f(x)=\dfrac{(x^3-a^3)}{(x-a)}$, when x tends to a.

Solution This will show you all commands about the limits.

a. Command

> $\text{limit}((x^2)+(2x+2), x=2);$

$$10$$

Using Palettes: Use cursor button to select limit palette. Click-the required limit palette and replace a by 2. Click $(a+b)$ (for sum rule of a function), then press "Enter" key to obtain the required limit:

> $\lim_{x\to 2}((x^2)+(2\cdot x+2))$

$$10$$

Remember

If the command for the required lin of a function is not known to yc then, easily on line, call the comma by typing:

> *limit.*

b. Command

> $\text{limit}((2x^3)(x-4), x=3);$

$$-10$$

Using Palettes: Use cursor button to select limit palette. Click-the required limit, and replace a b Click-(a*b) (for product rule of a function), then "Enter" key to obtain the required limit:

> $\lim_{x\to 3}((2\cdot x^3)\cdot(x-4))$

$$-54$$

c. Command

> $limit\left(\dfrac{(x^3-a^3)}{(x-a)}, x=a\right);$

$$3a^2$$

Using Palettes: Use cursor button to select limit palette. Click-the required limit and replace a b

Click-$\left(\dfrac{a}{b}\right)$ (the quotient rule of a function), then "Enter" key to obtain the required limit:

> $\lim_{x\to a}\dfrac{x^3-a^3}{x-a}$

$$3a^2$$

Exercise 2.3

1. Evaluate the following limits:

a. $\lim\limits_{x\to 4}\left(\dfrac{3}{x}+\dfrac{1}{x-5}\right)$ b. $\lim\limits_{x\to 1}\left(\dfrac{x^2+3x+2}{x^2+x+2}\right)^2$ c. $\lim\limits_{x\to 1}\dfrac{\sqrt{x}-1}{x-1}$

d. $\lim\limits_{x\to 1}\dfrac{\dfrac{1}{x}-1}{x-1}$ e. $\lim\limits_{x\to 0}\dfrac{1-\sin x}{\cos^2 x}$ f. $\lim\limits_{x\to 0}\dfrac{\tan x}{x}$

g. $\lim\limits_{x\to 0}\dfrac{\sec x-1}{x\sec x}$ h. $\lim\limits_{x\to 0}\dfrac{\sin^2(7x)}{7x}$

2. Use algebra and the rules of limits to evaluate the following limits:

a. $\lim\limits_{x\to 4}\dfrac{-6}{(x-4)^2}$ b. $\lim\limits_{x\to 0}\dfrac{\left(\dfrac{1}{(x+3)}\right)-\dfrac{1}{3}}{x}$ c. $\lim\limits_{x\to 5}\dfrac{\sqrt{x}-\sqrt{5}}{x-5}$

3. Find the limit of the convergent of the following sequences:

a. $\left\{\dfrac{5n}{n+7}\right\}$ b. $\left\{\dfrac{4-7n}{8+n}\right\}$ c. $\dfrac{(-1)^n}{n^2}$

4. Weekly sales (in rupees) at big store x weeks after the end of an advertising campaign are given by: $S(x)=5000+\dfrac{3600}{x+2}$

Find the sale for the indicated weeks limits:

a. $S(5)$ b. $\lim\limits_{x\to 5} S(x)$ c. $\lim\limits_{x\to 16} S(x)$

5. Use MAPLE command "limit" to evaluate the limit of all parts of Q.1.

6. Use algebraic techniques to evaluate the following.

a. $\lim\limits_{h\to 0}\dfrac{\sqrt{x+h}-\sqrt{x}}{h}$ b. $\lim\limits_{x\to a}\dfrac{x^n-a^n}{x^m-a^m}$ c. $\lim\limits_{0\to \theta}\dfrac{1-\cos p\theta}{1-\cos q\theta}$

d. $\lim\limits_{0\to \theta}\dfrac{\tan\theta-\sin\theta}{\sin^3\theta}$ e. $\lim\limits_{n\to +\infty}\left(1-\dfrac{1}{n}\right)^n$ f. $\lim\limits_{x\to 0}(1+2x^2)^{\frac{1}{x^2}}$

g. $\lim\limits_{x\to \infty}\left(\dfrac{x}{1+x}\right)^x$

Project

What type of function are represented by the curves drawn on each image?

i.

ii.

2.8 Continuous and Discontinuous Functions

Before discussion about continuous and discontinuous function we will revise the concept of limit of a function, which we have done in previous Section 2.6.

2.8.1 Recognition of left and right hand limits

It is a value the function approaches as the x-values approach the limit from one side only i.e. the left side limit and the right side limit.

i. The left hand limit of a function

A given function $f(x)$ has a left hand limit if $f(x)$ can be made as close to the number. 'L' as we please for all values of $x < c$ e.g. $\lim_{x\to c^-} f(x) = L$

ii. The right hand limit of a function

A function $f(x)$ has a right hand limit if $f(x)$ can be made as to the number 'L' as we please for all values of $x > c$. $\lim_{x\to c^+} f(x) = L$

"In general, the function has a limit as x approaches c if both the left hand and right hand limit at c exist and are equal. $\lim_{x\to c^+} f(x) = \lim_{x\to c^-} f(x) = \lim_{x\to c} f(x) = L$ "

Example 36 Determine whether $\lim_{x\to 3} f(x)$ and $\lim_{x\to 5} f(x)$ exist, if

$$f(x) = \begin{cases} 3x+4 & if\ 0 \le x < 3 \\ 16-x & if\ 3 \le x < 12 \\ x & if\ 12 \le x < 14 \end{cases}$$

Solution

(a). $\lim_{x\to 3^-} f(x) = \lim_{x\to 3} f(x) = \lim_{x\to 3}(3x+4) = 3(3)+4 = 13$

$\lim_{x\to 3^+} f(x) = \lim_{x\to 3} f(x) = \lim_{x\to 3}(16-x) = 16-3 = 13$

Since $\lim_{x\to 3^-} f(x) = \lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} f(x) = 13$

$\Rightarrow \lim_{x\to 3} f(x)$ exist and equal to 13.

(b). $\lim_{x\to 5^-} f(x) = \lim_{x\to 5} f(x) = \lim_{x\to 5}(16-x) = 16-5 = 11$

$\lim_{x\to 5^+} f(x) = \lim_{x\to 5} f(x) = \lim_{x\to 5}(x) = 5$

Since $\lim_{x\to 5^-} f(x) \ne \lim_{x\to 5^+} f(x)$ Therefore, $\lim_{x\to 5} f(x)$ does not exist.

Here, we observed that sometimes $\lim_{x\to c} f(x) = f(x)$ and sometime it does not and also sometim f(c) is not defined whereas $\lim_{x\to c} f(x) = f(x)$ exist.

2.8.2 Continuity of a function at a point and in an interval

i. Continuity of a function at a point

A function is said to be a continuous function at a point if two sided limit at that point exist ar equal to the function's value e.g. Consider a function $f(x)$, it is continuous at the point $x = c$ if.

$$\left.\begin{array}{ll} \text{(i)} & f(c)\ is\ exist. \\ \text{(ii)} & \lim_{x\to c^-} f(x) = \lim_{x\to c^+} f(x)\ exist\ or \lim_{x\to c} f(x)\ exist. \\ \text{(iii)} & \lim_{x\to c} f(x) = f(c). \end{array}\right\} \quad \text{..........(i)}$$

If any one of the above condition does not satisfied then the function is not continuous.

Example 37 Discuss the continuity of $f(x)=x^3-2x^2-3x+5$ at $x=1$

Solution

a. $f(x)=x^3-2x^2-3x+5$ For $x=1$

$f(1)=(1)^3-2(1)^2-3(1)+5$

$=1-2-3+5=1$

b. $\lim\limits_{x\to 1^-} f(x)=\lim\limits_{x\to 1^-}\left(x^3-2x^2-3x+5\right)$

$=(1)^3-2(1)^2-3(1)+5=1$

$\lim\limits_{x\to 1^+} f(x)=\lim\limits_{x\to 1^+}\left(x^3-2x^2-3x+5\right)$

$=(1)^3-2(1)^2-3(1)+5$

$=1-2-3+5=1$

c. $\Rightarrow \lim\limits_{x\to 1} f(x)=f(1)$

$\therefore$ $f(x)$ is continuous at $x=1$.

Now, look at the following graph of function.

Remember

A function is said to be a discontinuous function at a point 'c' if one of the three conditions of continuity does not satisfy.

It can be seen that the side limits consider with the value of the function with the point.

ii. Continuity of a function in an interval

"A function is said to be a continuous function in an interval when the function is defined at every point in that interval and no jumps or breaks involves."

If some functions $f(x)$ satisfies these criteria from $x=a$ to $x=b$ and we say that $f(x)$ is continuous on the interval $[a,b]$.

Note

$f(x)$ is continuous over the closed interval $[a, b]$ if it is continuous on the (a, b) interval.

2.8.3 Test of continuity and discontinuity of a function at a point and in an interval

Example 38 Discuss the continuity of $f(x)=\dfrac{x^2-4}{x+2}$, at $x=-2$.

Solution In order to check the continuity of the function $f(x)$ at $x=2$. We will have to check the function for all three conditions as we have done in Example 37.

$$f(x)=\frac{x^2-4}{x+2}$$

$$f(-2)=\frac{(-2)^2-4}{(-2)+2}=\frac{4-4}{-2+2}=\frac{0}{0}=\text{Undefined}$$

Hence, $f(-2)$ is not defined. We know that if any of the three conditions of continuity does not satisfy, the function will discontinuous.

Therefore, $f(x)$ is discontinuous function at $x=-2$.

However, if we try to find the limit of $f(x)$, we conclude that $f(x)$ is continuous on all the values other than -2.

$$\lim_{x\to -2} f(x)=\lim_{x\to -2}\frac{x^2-4}{x+2}=\lim_{x\to -2}(x-2)=-4$$

This implies that $f(x)$ is continuous at all the values of x other than -2.

Example 39 First–class postage in 1995 was $\$0.32$ for the first ounce and \$0.23 for each additional ounce up to 11 ounces. If $p(x)$ is the amount of postage for a letter weighing in x ounces, then we write:

$$P(x)=\begin{cases} \$0.32, & \text{if } 0<x\le 1 \\ \$0.55, & \text{if } 1<x\le 2 \\ \$0.78, & \text{if } 2<x\le 3 \\ \text{and so on} & \end{cases}$$

(a). Graph p(x) for $0<x\le 5$

(b). Find $\lim_{x\to 1^-} p(x)$, $\lim_{x\to 1^+} p(x)$ and $p(1)$.

(c). Find $\lim_{x\to 4.5} p(x)$ and $p(4.5)$

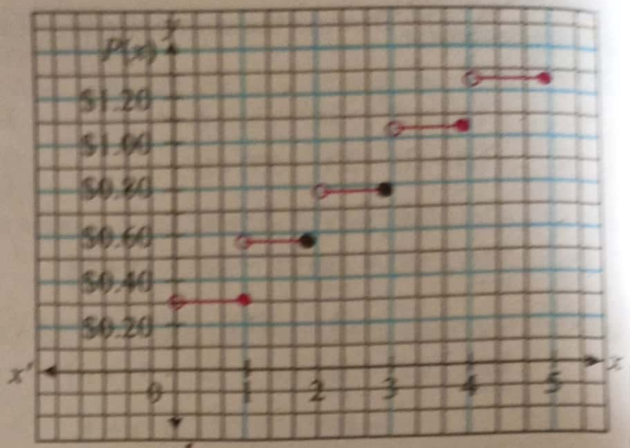


Figure 2.34

Solution

a. The graph of p(x) is shown in the Figure 2.34

b. From the graph of a function, the left, right limits and the value of a function at $x = 1$ are:

$\lim_{x\to 1^-} p(x)=0.32$, $\lim_{x\to 1^+} p(x)=0.55$ and $p(1)=0.32$.

c. From the graph of a function, the limit and the value of a function are equal:

$\lim_{x\to 4.5} p(x)=1.24$, $p(4.5)=1.24$

Thus the function is continuous at $x = 4.5$.

Note

Sometimes functions need to be defined in pieces, because they have a split domain. These functions require more than one formula to define the function, and therefore these types of functions are called **piecewise continuous functions**.

2.8.4 MAPLE command iscont to test continuity of a function at a point and in a given interval

Look at the following example. The procedure of using the maple command for continuity of a function is illustrated in this example.

Example 40 Use maple command "iscount" to check the continuity of function

$$f(x)=x^2+4 \text{ in:}$$

(a). Internal from 0 to 1.

(b). Closed interval [0, 1].

(c). Open interval (0, 1).

Solution

> $iscont(x^2+4, x=0..1);$ **true**

> $iscont(x^2+4, x=0..1,'closed');$ **true**

> $iscont(x^2+4, x=0..1,'open');$ **true**

Project

Create at least five functions randomly then use MAPLE command "iscont" to check their continuity on interval (0,1)

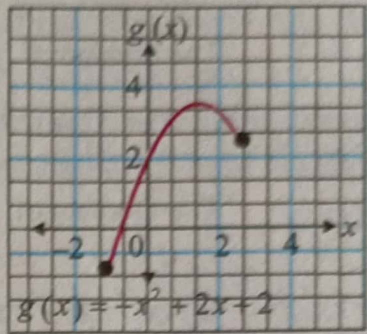
Exercise 2.4

1. Use properties of continuous function to test the continuity and discontinuity of the following functions:

   a. $f(x)=2x-3$ b. $h(x)=\dfrac{2}{x-5}$ c. $g(x)=\dfrac{x-5}{(x-3)(x+2)}$

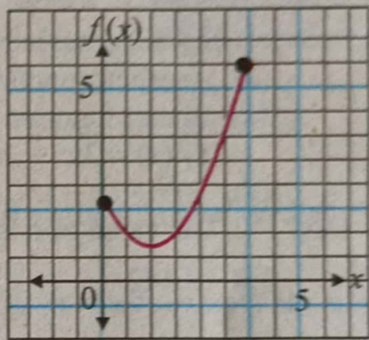
2. Show that function $f(x)=\begin{cases}\dfrac{\sin x}{x} & \text{if } x\neq 0\\ 1 & \text{if } x=0\end{cases}$ is continuous at $x=0$.

3. Use the graph of the function $g(x)$ to answer the following questions:



   a. Is $g(x)$ continuous on the open interval $(-1, 2)$?
   b. Is $g(x)$ continuous from the right at $x=-1$?
   Is $\lim\limits_{x\to -1^+} g(x)=g(-1)$?
   c. Is $g(x)$ continuous from the left at $x=2$?
   Is $\lim\limits_{x\to 2^-} g(x)=g(2)$?
   d. Is $g(x)$ continuous on the closed interval $[-1, 2]$?

4. Use the graph of the function $f(x)$ to answer the following questions:



   a. Is $f(x)$ continuous on the open interval $(0, 3)$?
   b. Is $f(x)$ continuous from the right at $x=0$?
   Is $\lim\limits_{x\to 0^+} f(x)=f(0)$?
   c. Is $f(x)$ continuous from the left at $x=3$?
   Is $\lim\limits_{x\to 3^-} f(x)=g(3)$?
   d. Is $g(x)$ continuous on the closed interval $[0, 3]$?

5. Graph and locate all points of discontinuity of the following piecewise functions:

   a. $f(x)=\begin{cases}1+x, & \text{if } x<1\\ 5-x, & \text{if } x\geq 1\end{cases}$ b. $f(x)=\begin{cases}-x, & \text{if } x<0\\ 1, & \text{if } x=0\\ x, & \text{if } x>0\end{cases}$

6. Personal computer salesperson receives a base salary of \$1,000 per month and a commission of 5% of all sales over \$10,000 during the month. If the monthly sales are \$20,000 or more, the salesperson is given an additional \$500 bonus. Let $E(s)$ represents the person's earnings during the month as a function of the monthly sales.

   a. Graph E(s) for $0\leq s\leq 30{,}000$ b. Find $\lim\limits_{s\to 10{,}000} E(s)$ and E(10,000).
   c. Find $\lim\limits_{s\to 20{,}000} E(s)$ and E(20,000) d. Is E continuous at $s=10{,}000$? At $s=2{,}000$

7. Use MAPLE command "iscont" to test the continuity of $f(x)=\dfrac{x^2+1}{x^3+2.7}$ in closed interval $[-5, 5]$.

Review Exercise 2

1. Choose the correct option.

i. The independent variable in the function $y=\frac{x^2+4x-3}{(x+3)^3}$ is:

(a). x (b). x^2 (c). x^3 (d). y

ii. If $f(x)=\frac{3x^2-2}{3x+9}$ then $f(-3)$ is:

(a). $\frac{25}{18}$ (b). $\frac{29}{18}$ (c). $\frac{3}{1}$ (d). undefined

iii. The domain of $f(x)=\frac{3x^2-2}{3x+9}$ is:

(a). $(-\infty,-3)\cup(-3,\infty)$ (b). $(-\infty,-3)\cup(-3,\infty)$
(c). $[-\infty,-3]$ (d). $(-\infty,3)$

iv. The Domain of $f(x)=\frac{3x-2}{3x^2+9}$ is:

(a). $x\geq-3$ (b). $x\leq-3$ (c). $\infty\leq x\leq\infty$ (d). $-\infty<x<\infty$

v. If $f(x)=2x+3$ then $f^{-1}(5)$ is:

(a). 13 (b). -13 (c). 1 (d). -1

vi. If y is expressed in term of x as $y=f(x)$ then y is called:

(a). implicit function (b). explicit function
(c). linear function (d). identity function

vii. If $f(x)=3x^2+2x-1$ and $g(x)=x+1$ then $f\{g(x)\}$ is:

(a). $3x^2+2x+1$ (b). $3x^2+8x+4$ (c). $3x^2+8x-1$ (d). $3x^2-8x-4$

viii. The value of e is:

(a). 3.142 (b). $\frac{22}{7}$ (c). 2.71 (d). 3.8

ix. If $5^x=7$ then the value of x is:

(a). $\frac{\ln(7)}{\ln(5)}$ (b). $\frac{\ln(5)}{\ln(7)}$ (c). $\frac{\ln(x)}{\ln(5)}$ (d). $\frac{\ln(7)}{\ln(x)}$

x. $\operatorname{Cot} h\, x=$ __________

(a). $\frac{e^x-e^{-x}}{2}$ (b). $\frac{e^x+e^{-x}}{2}$ (c). $\frac{e^x+e^{-x}}{e^x-e^{-x}}$ (d). $\frac{e^x-e^{-x}}{e^x+e^{-x}}$

xi. $\ln(m)-\ln(n)=$ __________

(a). $\ln(mn)$ (b). $\ln(m+n)$ (c). $\ln(m-n)$ (d). $\ln\frac{m}{n}$

xii. The function $f(x)=\frac{2}{x^2-9}$ is discontinuous at point:

(a). $-1, 1$ (b). $-2, 2$ (c). $-3, 3$ (d). $4, 4$

xiii. $\lim\limits_{\theta\to o}\frac{\sin 7\theta}{7\theta}=$

(a). 1 (b). 2 (c). 3 (d). 90

xiv. If $f(\theta)=\theta.\sec\theta$ then $f(0)=$

(a). 0 (b). 1 (c). $\sqrt{2}$ (d). -1

xv. The inverse function of $\frac{e^x-e^{-x}}{e^x+e^{-x}}$ is:

(a). $\frac{1}{2}\log\left(\frac{2-x}{2+x}\right)$ (b). $\frac{1}{2}\log\left(\frac{2+x}{2-x}\right)$ (c). $\frac{1}{2}\log\left(-\frac{1-x}{x-1}\right)$ (d). $\frac{1}{2}\log\left(\frac{-1-x}{1-x}\right)$

Summary

- A function $y = f(x)$ is a rule that assigns for each value of the independent variable x a unique value of the dependent variable y: $y = f(x)$
- A function that defined by more than one equation is called a **compound function**.
- The graph of a function $f(x)$ consists of all points whose coordinates (x, y) satisfy a function $y = f(x)$, for all x in the domain of $f(x)$.
- Let $y = f(x)$ be a function of x. This function takes an dependent variable y in response of independent variable x. The function that takes x as dependent variable in response of y as the independent is then called the **inverse function** of $f(x)$ and is denoted by: $x = f^{-1}(y)$
- A function $f(x)$ is called **algebraic** if it can be constructed using algebraic operations (such as adding, subtracting, multiplying, dividing, or taking roots) starting with polynomials. Any rational function is an **algebraic function.**
- Functions that are not algebraic are called **transcendental functions.**
- If a function is defined by an equation of the form $y = f(x)$, one says that the function is defined explicitly or is explicit. The terms **"explicit function"** and **"implicit function"** do not characterize the nature of the function but merely the way it is defined. Every explicit function $y = f(x)$ may also be represented as an implicit function $y = f(x) = 0$.
- If $f(t)$ and $g(t)$ are continuous functions of parameter t on an interval D, then the equations $x = f(t)$ and $y = g(t)$

  are called the **parametric equations** for the plane curve C generated by the set of ordered pairs in plane: $(x, y) = (x(t), y(t)) = (f(t), g(t))$
- If $f(x)$ is a function of x, and c, L are the real numbers, then L is the limit of a function $f(x)$ as x approaches c: $\lim_{x \to c} f(x) = L$
- A function $f(x)$ is said to be a continuous at $x = c$, if all three of the following conditions are satisfied:
  - The function is defined at $x = c$; that is, $f(c)$ exists.
  - The function approaches a definite limit as x approaches c; that is $\lim_{x \to c} f(x)$ exists.
  - The limit of a function is equal to the value of a function when $x = c$; that is, $\lim_{x \to c} f(x) = f(c)$.

History

H. Steinhaus was polish mathematician and educator. He earned his ph.D degree from his mutable contribution to functional analysis through the bancach-steinhaus theorem. He is also one of the early founders of probability and game theory. He also proposed sandwich theorem in 1938 first time specifically n=3 case of bisecting 3-solids with a plane.

Hugo Steinhaus (1887-1972)

Unit 3

DIFFERENTIATION

By the end of this unit, the students will be able to:

3.1 Derivative of a Function

i. Distinguish between independent and dependent variables.
ii. Estimate corresponding change in the dependent variable when independent variable is incremented (or decremented).
iii. Explain the concept of a rate of change.
iv. Define derivative of a function as an instantaneous rate of change of a variable with respect to another variable.
v. Define derivative or differential coefficient of a function.
vi. Differentiate $y = x^n$, where $n \in Z$ (the set of integers), from first principles (the derivation of power rule).
vii. Differentiate $y = (ax+b)^n$, when $n = \frac{p}{q}$ and p, q are integers such that $q \neq 0$, from first principles.

3.2 Theorems on differentiation

i. Prove the following theorems for differentiation.
- the derivative of a constant is zero.
- the derivative of any constant multiple of a function is equal to the product of that constant and the derivative of the function.
- the derivative of a sum (or difference) of two functions is equal to the sum (or difference) of their derivatives.
- the derivative of a product of two functions is equal to (the first function) × (derivative of the second function) plus (derivative of the first function × (the second function).
- the derivative of a quotient of two functions is equal to denominator times the derivative of the numerator, minus the numerator times the derivative of the denominator, all divided by the square of the denominator.

3.3 Application of Theorems on differentation

i. Differentiate:
- constant multiple of x^n,
- sum (or difference) of functions,
- polynomials,
- product of functions,
- quotient of two functions.

3.4 Chain rule

i. Prove that $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$ when $y = f(u)$ and $u = g(x)$
ii. Show that $\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$
iii. Use chain rule to show that $\frac{d}{dx}[f(x)]^n = n[f(x)]^{n-1} f'(x)$.
iv. Find derivative of implicit function.

3.5 Differentiation of trigonometric and inverse trigonometric functions

i. Differentiate:
- trigonometric functions ($\sin x$, $\cos x$, $\tan x$, $\operatorname{cosec} x$, $\sec x$, and $\cot x$) from first principles.
- inverse trigonometric functions ($\arcsin x$, $\arccos x$, $\arctan x$, $\operatorname{arccosec} x$, $\operatorname{arcsec} x$ and $\operatorname{arccot} x$) using differentiation formulae.

3.6 Differentiation of Exponential and Logarithmic Functions

i. Find the derivative of e^x and a^x from first principles.
ii. Find the derivative of $\ln x$ and $\log_a x$ from first principles.
iii. Use logarithmic differentiation to find derivative of algebraic expressions involving product, quotient and power.

3.7 Differentiation of Hyperbolic and Inverse Hyperbolic Functions

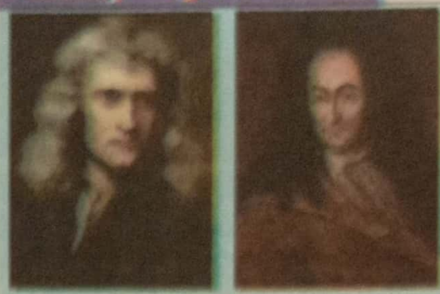
i. Differentiate:
- hyperbolic functions ($\sinh x$, $\cosh x$, $\tanh x$, $\operatorname{cosech} x$, $\operatorname{sech} x$ and $\coth x$).
- inverse hyperbolic functions ($\sinh^{-1} x$, $\cosh^{-1} x$, $\tanh^{-1} x$, $\operatorname{cosech}^{-1} x$, $\operatorname{sech}^{-1} x$ and $\coth^{-1} x$).
- Use MAPLE command diff to differentiate a function.

Introduction

The derivative is one of the main tools of calculus. It is instantaneous rate of change of a function at a point in the domain. It is same like the gradient or slope of the tangent line to the graph of the function at that point. Before going to the definition we need to revise the concept of limit introduced in previous unit of this book.

In this unit, we will start by defining derivative, which is the central concept of differential calculus. Then we need to develop a list of rules and formulas for finding the derivative of a variety of expressions, including polynomial functions, rational functions, exponential functions, logarithmic functions, trigonometric functions and hyperbolic functions. *"The process of finding the derivative is known as differentiation. But the inverse process of differentiation is known as integration."* We will discuss in details about integration in Unit-6 of this book.

History



Isaac Newton G.W Leibniz

In the sense of a tangent line the concept of derivative is very old in the study of mathematics. This is familiar to Greek geometers. But the modern development of a calculus credited to Isaac Newton and G.W Leibniz. Who provided the independent and unique approaches to the derivatives and differentiation.

3.1 Derivative of a Function

The derivative of a function at some point is known as the rate of change of the function at the point. We can estimate the rate of change by calculating the ratio of change of the function Δy to the change of the independent variable Δx. In the definition of derivative, the ratio is considered in the limit as $\Delta x \to 0$.

3.1.1 Independent and dependent variable

To understand the origin of the concept of variables, some real-life situations in which one numerical quantity depends on, corresponds to, or determines another are considered. For example,

1. The amount of income tax (output/dependent variable) you pay on the amount of your income (input/independent variable). The way in which the income determines the tax is given by the tax law (rule).
2. A person in business wants to know how profit (output/dependent variable) changes with respect to advertising (input/independent variable).
3. A person in medicine wants to know how a patient's reaction to a drug (output/dependent variable) changes with respect to dose (input/independent variable).

In each case, the change in dependent variable requires the definite change in independent variable through a definite rule which is called a **function.**

3.1.2 Estimation of corresponding change in the dependent variable, when independent variable is incremented (or decremented)

A familiar situation related to change in dependent with respect to change in independent is that a driver makes the run of 120, mile trip from Peshawar to Islamabad, in 2 hours. The table shows how far the driver has traveled from Peshawar at various times:

Time	0	0.5	1.0	1.5	2.00
Distance	0	24	54	88	120

If f is the function whose rule is $f(t)$ = distance from Peshawar at time t, then, the table shows that $f(1.0)=54, f(1.5)=88$ and $f(2.0)=120$ miles. So the distance traveled from time $t=1.5\ to\ t=2.0$ is $f(2)-f(1.5)=120-88=32,$ the change in dependent variable (change in distance) in response of **incremented** independent variable t, while the distance traveled from time $t=1.5\ to\ t=1.0$ is $f(1.5)-f(1.0)=88-54=35,$ the change in dependent variable (change in distance) in response of **decremented** independent variable t.

3.1.3 Concept of a rate of change

The idea of average rate of change is something we encounter every day. For example, if a car accelerates from 0 to 96 km/h in 8.0 s, then we say that it accelerates at an average rate of 12 km/h. If a spaceship climbs from 0 to 10,000 m in 2.5 s, then we say that the ship climbs at an average velocity of 4000 m/s. If corn grows a total of 28 inches in 2 weeks, then it grows an average of 2 inches per day.

In these examples, the indicated **average rate of change** is obtained by dividing the change in the **dependent variable** by the change in the **independent variable**.

Let us examine the process of finding the average rate of change of a function $y = f(x)$. If we select any value of x and increase it by an amount Δx, then a new value of the independent variable is $x+\Delta x$. As x changes from x to $x+\Delta x$, y will change to a corresponding amount of $y+\Delta y$. The ordered pairs $P(x, y)$ and $Q(x+\Delta x, y+\Delta y)$ developed must satisfy the function $y = f(x)$. This is shown in the Figure 3.1

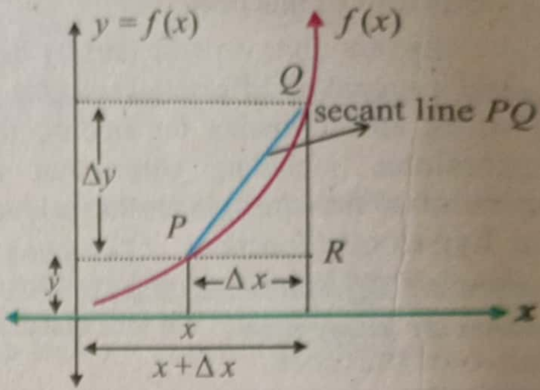


Figure 3.1

If the function value at a point $P(x, y)$ is $y = f(x)$ (i)

then, the function value at a point Q is $y+\Delta y = f(x+\Delta x)$ (ii)

The difference of equations (i) and (ii) gives the change in y;

$(y+\Delta y)-y = f(x+\Delta x)-f(x) \Rightarrow \Delta y = f(x+\Delta x)-f(x)$ (iii)

The change in x is $\Delta x = x+\Delta x - x$ (iv)

The average rate of change in y per unit change in x is the **slope of the secant line PQ**, obtained by taking the division of equation (iii) by equation (iv):

$$\frac{\Delta y}{\Delta x} = \frac{f(x+\Delta x)-f(x)}{x+\Delta x - x} = \frac{f(x+\Delta x)-f(x)}{\Delta x}, \quad y = f(x)$$

The average rate of change y per unit change in x is given by:

$$\frac{\Delta y}{\Delta x} = \frac{f(x+\Delta x)-f(x)}{\Delta x} \quad \text{(v)}$$

Remember

The slope of the secant is the average rate of change which measures always "the approximate rate of change in phenomena."

Example 1 Determine the average rate of change of y per unit change in x for $y = x^2-6x+5$ as x increases from 1 to 3.

Solution According to the definition of the average rate of change:

$$\frac{\Delta y}{\Delta x} = \frac{f(x+\Delta x)-f(x)}{\Delta x} \qquad \because y = f(x) = x^2-6x+6$$

$$= \frac{\{(x+\Delta x)^2-6(x+\Delta x)+5\}-(x^2-6x+5)}{\Delta x} = \frac{\cancel{x^2}+2x.\Delta x+\Delta x^2-\cancel{6x}-6\Delta x+5-\cancel{x^2}+\cancel{6x}-5}{\Delta x}$$

$$= \frac{2x.\Delta x+\Delta x^2-6\Delta x}{\Delta x} = \frac{\cancel{\Delta x}(2x+\Delta x-6)}{\cancel{\Delta x}} = 2x-6+\Delta x$$

$$\frac{\Delta y}{\Delta x} = 2x-6+\Delta x$$

As x increases from 1 to 3 then $x=1$ and $\Delta x = 2$.

$$\frac{\Delta y}{\Delta x} = 2(1)-6+2 = -2$$

Example 2 The height h of a certain brand of corn with respect to t days $(t \geq 1)$ after the seed germinates is $h(t) = \sqrt{t}-1$.

(a). Find the average growth rate $\dfrac{\Delta h}{\Delta t}$.

(b). Find the average growth rate between days 4 and 9.

Solution

a. The average growth rate through definition (v) is:

$$\frac{\Delta h}{\Delta t}=\frac{h(t+\Delta t)-h(t)}{\Delta t}=\frac{\sqrt{t+\Delta t}-1-(\sqrt{t}-1)}{\Delta t}=\frac{\sqrt{t+\Delta t}-\sqrt{t}}{\Delta t} \quad \text{(i)}$$

b. The average growth rate (i) is used for t = 4 and $\Delta t = 5$ to obtain the average growth between days 4 and 9:

$$\frac{\Delta h}{\Delta t}=\frac{\sqrt{t+\Delta t}-\sqrt{t}}{\Delta t}=\frac{\sqrt{4+5}-\sqrt{4}}{5}=\frac{3-2}{5}=\frac{1}{5}$$

Thus, the average rate of change of the height of the corn with respect to time (between days 4 and 9) is $\frac{1}{5}$(1 unit change in height for each 5 units change in time). The graph is shown in Figure 3.2. The average rate of change is of course helpful in understanding the instantaneous rate of change.

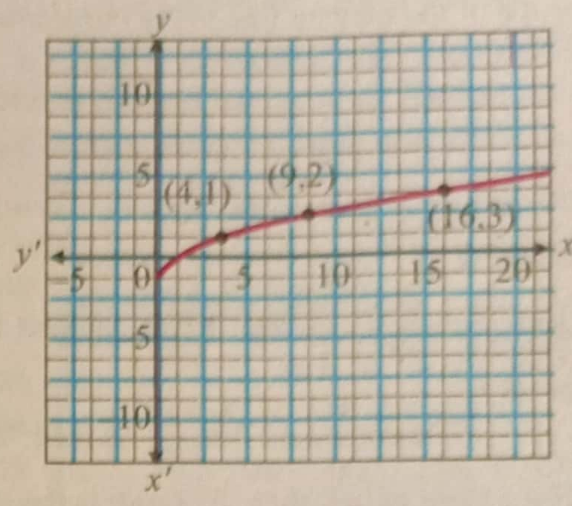


Figure 3.2

3.1.4 Derivative of a function as an instantaneous rate of change of a variable with respect to another variable

In the previous sub-section, we discussed the average rate of change, and learned that the average rate of change is the slope of the secant line joining two points on the curve $y=f(x)$. More commonly, we are asked to determine the exact or instantaneous rate of change at a particular time. For example, for an aeroplane, what is the instantaneous rate of change of the distance that occurs at a specific time? This can be dealt by the slope of a tangent line to a curve $y=f(x)$ at a specific point?

To illustrate this idea, let us examine the graph of a function $y=x^2$ at a particular point P (0.5, 0.25) with different secant lines $PQ_1, PQ_2,...$ that developed from the secant line PQ:

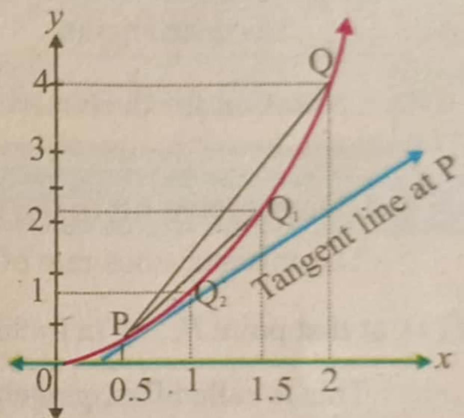


Figure 3.3

P	Q	Δx	Δy	$\frac{\Delta y}{\Delta x}$
P (0.5, 0.25)	$Q(2,4)$	1.5	3.75	2.5
P (0.5, 0.25)	$Q_1(1.5,2.25)$	1.0	2.00	2.0
P (0.5, 0.25)	$Q_2(1,1)$	0.5	0.75	1.5
P (0.5, 0.25)	$Q_3(0.8,0.64)$	0.3	0.39	1.3

The tabular form contains coordinates for the points P, Q, the change Δx in x, the change Δy in y and $\frac{\Delta y}{\Delta x}$, the slope of the secant lines PQ, PQ_1, PQ_2,..... Notice that the slope of the secant line PQ is 2.5 $\left(\frac{\Delta y}{\Delta x}=\frac{3.75}{1.5}=2.5\right)$. If we take values of Q closer to P (i.e, to Q_1, Q_2, Q_3.....), then, Δx gets smaller, and smaller, and tends to zero.

The tabular form clearly shows that, as Q approaches P, Δx approaches 0, and the slope of the secant line approaches the slope of the tangent line at a particular point P (0.5, 0.25) which is 1.

Geometrically, the slope of the tangent line to a curve at a particular point P is the instantaneous (or exact) rate of change at that particular point.

This terminology develops the idea that the slope of the secant line becomes a better approximation for the slope of the tangent line to the curve at a particular point P. From our discussion on limit, it follows that the exact/actual slope of the tangent line to a curve $y=f(x)$ at a particular point P corresponds to the instantaneous rate of change at that point. That is, $\frac{\Delta y}{\Delta x}=$ slope of the secant line PQ

$\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=\frac{dy}{dx}$ = slope of the tangent line at a particular point P.

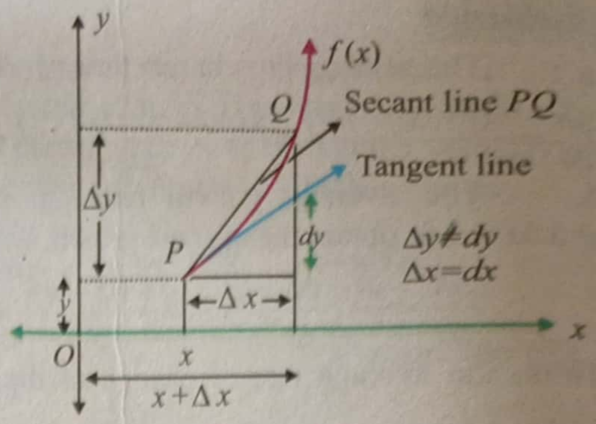


Figure 3.4

The statement $\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}$ is read "the limit as delta x approaches zero of delta y divided by delta x."
If the limit exists, then the result is the slope of tangent line or the instantaneous rate of change of with y respect to x which we call the derivative of function.

Note

Different mathematicians used different notations to write derivative.

Mathematician	Newton	Leibniz	Euler	Lagrange
Notation for derivative	$\dot{y}$ or $\dot{f}$	$\frac{dy}{dx}$ or $\frac{df}{dx}$	D. $f(x)$ or Dy	$f'(x)$

3.1.5 Derivative or differential coefficient of a function

The instantaneous rate of change of a function $f(x)$ at a point P is the derivative of a function $f(x)$ at that point P, $f'(x)=\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}$, **if this limit exists** (i)

This is called **first principle rule** of derivative of a function $f(x)$ with respect to x.

If $y=f(x)$ is a function, then its **derivative** or **differential coefficient** is denoted by f' or y'. If x is a number in the domain of $y=f(x)$ such that $y'=f'(x)$ is defined, then the function f is said to be **differentiable** at x. The process that produces the function f' from the function f is called **differentiation.**

Example 3 Determine the derivative of a function $f(x)=x^2-6x+5$ by first principle rule at a point $P(4,-3)$.

Solution The derivative of a given function by first principle rule (i) is:

$$f'(x)=\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x},\quad y=f(x)=x^2-6x+5$$

$$=\lim_{\Delta x\to 0}\frac{[(x+\Delta x)^2-6(x+\Delta x)+5]-(x^2-6x+5)}{\Delta x}$$

$$=\lim_{\Delta x\to 0}\frac{x^2+2x\Delta x+(\Delta x)^2-6x-6\Delta x+5-x^2+6x-5}{\Delta x}=\lim_{\Delta x\to 0}\frac{2x\Delta x+(\Delta x)^2-6\Delta x}{\Delta x}$$

$$=\lim_{\Delta x\to 0}\frac{\Delta x(2x+\Delta x-6)}{\Delta x}=\lim_{\Delta x\to 0}(2x+\Delta x-6)=2x-6$$

The result $f'(x)=2x-6$ represents the slope of the tangent line at any point $P(x, y)$ on the curve $f(x)=x^2-6x+5$. Thus, the slope of the tangent line at a particular point, say $P(4,-3)$ on a given curve is: $f'(x)=2x-6$

$$f'(4)=2(4)-6=2,\ at\ P(4,-3)$$

From this problem, we conclude that

- **the slope of the secant line (the average rate of change) is called the approximate rate of change.**
- **the slope of the tangent line (the instantaneous rate of change) is called the exact rate of change.**

First Principle Rule
If $f(x)$ is any function, then the derivative by first principle rule is

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}, \quad y = f(x)$$

The process used for finding the derivative of a function in example 3 is called the differentiation and the result 2 is the differential coefficient of a function $f(x) = x^2 - 6x + 5$ at a particular point P(4, 3).

Symbol $f'(x)$ is used to indicate the derivative of $f(x)$ with respect to x. Sometimes other symbols are used to indicate the derivative. Each of the symbols in the following box indicates the derivative of the dependent variable y with respect to the independent variable x:

"The tangent line to the graph of a function $y = f(x)$ at the point $(x, f(x))$ is the line through this point having slope

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \qquad \text{(ii)}$$

provided this limit exists. If this limit does not exist, then there is no tangent (no derivative) at the point.

Do You Know

i. $f'(x)$: read "f prime of x" (derivative of $f(x)$ with respect to x)

ii. $\frac{dy}{dx}$: read " dee y, dee x" (the derivative of y with respect to x)

iii. f': read "f prime" (the derivative of the function $f(x)$ with respect to x)

iv. $D_x\, y$: read " D sub x, y" (the derivative of y with respect to x)

v. y': read "y prime" (the derivative of y with respect to x)

The slope of the tangent line is the instantaneous rate of change, gives "the exact rate of change in the phenomena."

Example 4 The function is $f(x) = x^2$.

(a). Find the derivative of a function at a point $P(3,9)$

(b). Find the tangent line on a given curve $y = x^2$ at a point $P(3,9)$.

(c). View the slope of the tangent line on a curve $y = x^2$ at a point $P(3,9)$ graphically.

Solution a. By first principle rule, the derivative of a given function is:

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \qquad \therefore y = f(x) = x^2$$

$$= \lim_{\Delta x \to 0} \frac{(x+\Delta x)^2 - (x)^2}{\Delta x} = \lim_{\Delta x \to 0} \frac{x^2 + 2x\Delta x + (\Delta x)^2 - x^2}{\Delta x}$$

$$= \lim_{\Delta x \to 0} (2x + \Delta x) = 2x \qquad \text{(i)}$$

b. Result (i) is used to obtain the slope of the tangent line at a $P(3,9)$ on a curve $y = x^2$:

$$f'(3) = 2(3) = 6 \qquad \text{(ii)}$$

The tangent line on a curve $y = x^2$ at a point $P(3,9)$ develops a nonhomogeneous line:

$y - y_1 = f'(x)(x - x_1)$, Point from of the line

$$y - 9 = 6(x - 3), \qquad P(3,9)$$

$$6x - y - 9 = 0$$

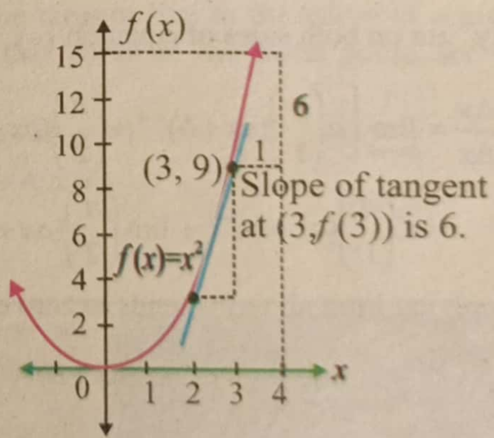


Figure 3.5

c. The graphical view of the slope of the tangent line is represented in Figure 3.5.

3.1.6 Differentiate of $y=x^n$ from first principles rule

If $f(x)=x^n, n$ is any integer, then, by first principle rule, the derivative of $f(x)=x^n$ w.r.t. x is:

$$f'(x)=\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x} \qquad \therefore y=f(x)=x^n$$

$$=\lim_{\Delta x\to 0}\frac{(x+\Delta x)^n-x^n}{\Delta x}, \text{ by binomial expansion}$$

$$=\lim_{\Delta x\to 0}\frac{x^n+nx^{n-1}\Delta x+\frac{n(n-1)}{2}x^{n-2}(\Delta x)^2+\ldots-x^n}{\Delta x}=\lim_{\Delta x\to 0}\frac{1}{\Delta x}\left[nx^{n-1}\Delta x+\frac{n(n-1)}{2}x^{n-2}(\Delta x)^2+\ldots\right]$$

$$=\lim_{\Delta x\to 0}\frac{\Delta x}{\Delta x}\left[nx^{n-1}+\frac{n(n-1)}{2}x^{n-2}\Delta x+\ldots\right]=\lim_{\Delta x\to 0}[nx^{n-1}]+\lim_{\Delta x\to 0}\left[\frac{n(n-1)}{2}x^{n-2}\Delta x+\ldots\right]$$

$$=nx^{n-1}+0=nx^{n-1}$$

3.1.7 Differentiation of $y=(ax+b)^n$ from first principle

Proof: Let $y=(ax+b)^n$ (i). Where n is an integer

$y+\Delta y=\{a(x+\Delta x)+b\}^n$ (ii). By using the binomial theorem

$$y+\Delta y=(ax+b)^n+\binom{n}{1}(ax+b)^{n-1}(a\Delta x)+\binom{n}{2}(ax+b)^{n-2}(a\Delta x)^2+\binom{n}{3}(ax+b)^{n-3}(a.\Delta x)^3+\ldots+(a\Delta x)^n \quad \text{(i}$$

Subtracting equation (i) from equation (iii)

$$y+\Delta y-y=(ax+b)^n+\binom{n}{1}(ax+b)^{n-1}(a.\Delta x)+\binom{n}{2}(ax+b)^{n-2}(a\Delta x)^2+\binom{n}{3}(ax+b)^{n-3}(a.\Delta x)^3+\ldots+(a.\Delta x)^n-(ax+$$

$$\Delta y=\binom{n}{1}(ax+b)^{n-1}(a\Delta x)+\binom{n}{2}(ax+b)^{n-2}(a.\Delta x)^2+\binom{n}{3}(ax+b)^{n-3}(a.\Delta x)^3+\ldots+(a.\Delta x)^n \quad \text{(i}$$

Dividing equation (iv) by Δx

$$\frac{\Delta y}{\Delta x}=\frac{\Delta x\left\{a\binom{n}{1}(ax+b)^{n-1}+\binom{n}{2}(ax+b)^{n-2}(a.\Delta x)^2+\binom{n}{3}(ax+b)^{n-3}(a.\Delta x)^3+\ldots+a^n(\Delta x)^{n-1}\right\}}{\Delta x} \quad \text{(v}$$

Apply $\lim_{\Delta x\to 0}$ on both sides of equation (v)

$$\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=\lim_{\Delta x\to 0}\left\{a\binom{n}{1}(ax+b)^{n-1}+\binom{n}{2}(ax+b)^{n-2}(a.\Delta x)^2+\binom{n}{3}(ax+b)^{n-1}(a\Delta x)^2+\ldots+a^n.(\Delta x)^{n-1}\right\}$$

$$=a.\binom{n}{1}(ax+b)^{n-1}+\lim_{\Delta x\to 0}\binom{n}{2}(ax+b)^{n-2}(a.\Delta x)^2+\lim_{\Delta x\to 0}\binom{n}{3}(ax+b)^{n-3}+\ldots+\lim_{\Delta x\to 0}a^n.(\Delta x)^{n-1}$$

By applying limit all terms tends to zero except first term so,

$$\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=a.\binom{n}{1}(ax+b)^{n-1}$$

Hence,

$$f'(x)=\frac{d}{dx}(ax+b)^n=n(ax+b)^{n-1}.a$$

This is generalized power rule of differentiation.

Exercise 3.1

1. Find the average rate of change of the following functions over the indicated intervals:

   a. $y=x^2+4$ from $x=2$ to $x=3$ b. $y=x^2+\frac{1}{3}x$ from $x=-3$ to $x=3$

   c. $s=2t^3-5t+7$ from $t=1$ to $t=3$ d. $h=\sqrt{2t}+4$ from $t=8$ to $t=8.5$

2. Use definition for the rate of change to find out the average rate of change over the specified interval for the following functions:

   a. $s=2t-3$ from $t=2$ to $t=5$ b. $y=x^2-6x+8$ from $x=3$ to $x=3.1$

   c. $A=\pi r^2$ from $r=2$ to $r=2.1$ d. $h=\sqrt{t}-9$ from $t=9$ to $t=16$

3. A ball is thrown straight up. Its height after t seconds is given by the formula $h=-16t^2+80t$. Use definition for the rate of change to determine the average velocity $\frac{\Delta h}{\Delta t}$ for the specified intervals:

   a. From $t=2$ to $t=2.1$. b. From $t=2$ to $t=2.01$.

4. The rate of change of price is called inflation. The price p in rupees after t years is $p(t)=3t^2+t+1$. Use definition for the rate of change to determine the average rate of change of inflation from $t=3$ to $t=5$ years. What the rate of change means? Explain.

5. A farmer plants x acres of sugar beets. The profit generated is $f(x)=1800x-9x^2$. Determine the average rate of change of the profit, when the planted area is in between $x=20$ acres and $x=50$ acres. What the rate of change means? Explain.

6. Use first principle rule to determine the derivative of the following functions:

   a. $f(x)=3x$ b. $f(x)=(5x+6)^{\frac{1}{2}}$ c. $f(x)=x^2+1$

   d. $f(x)=12-x^2$ e. $f(x)=16x^2-7x$ f. $f(x)=\frac{7}{x}$

7. Use function $f(x)=x^2-7x+6$ to do the following:

   a. Find the derivative of a function at point P(5, –4).

   b. Find the tangent line on the curve $y=x^2-7x+6$ at point P(5, –4).

   c. View the slope of the tangent line on the curve at P(6, 0)?

8. Use definition of derivative to determine the slope of the tangent line to the curve at a given point and then find out the tangent line equation on that curve at the same point, for the following curves:

   a. $f(x)=-x^2+7x, x=3$ b. $f(x)=6x^2-11x-10, x=1$

   c. $f(x)=3x^2-6x-10, x=0$ d. $f(x)=2x^2+3x-4, x=1$

Do You Know?

There is always an odometer and a speedometer in an automobile. These two things work in tandem and allow the driver to determine the speed of his/her vehicle and the distance he/she has traveled.
Electronic versions of these two gauges simply use derivatives to transform the data sent to the electronic motherboard from the tyres to miles per hour (MPH) and distance (KM).

3.2 Theorems on Differentiation

In previous section, the derivative of a function $f(x)$ is defined:

$$f'(x)=\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x} \qquad \text{(i)}$$

We learned that the derivative is found by applying the first principle rule. Now, after doing the exercise for the previous section, you may be wondering whether there is a shorter way of finding the derivative. In this and the next several sections, the discussion on the theorem that provides easier ways of finding derivatives.

3.2.1 Proof of differentiation theorem

Theorem -1: The derivative of a constant is zero

Proof: If $f(x)=c$, where c is any constant, then, by first principle rule, the derivative of a constant function is: $f'(x)=\lim\limits_{\Delta x\to 0}\dfrac{f(x+\Delta x)-f(x)}{\Delta x}=\lim\limits_{\Delta x\to 0}\dfrac{c-c}{\Delta x}=0,$

This calculation develops the rule that the derivative of a constant function is zero.

In general: If $f(x)=c$, where c is any constant, then: $f'(x)=0$

Example 5 Differentiate the following constant functions:

(a). $f(x)=13$ (b). $f(x)=3$ (c). $f(x)=7\pi$ (d). $f(x)=\sqrt{149}$

Solution The graphs of the functions are horizontal lines parallel to *x-axis*, since all function are constant. The derivative in each case is therefore going to be zero.

Theorem-2: The derivative of any constant multiple of a function is equal to the product of that constant and the derivative of the function.

Proof: If $f(x)=c.g(x)$, where c is any constant, then by the first principle rule, the derivative of a constant multiple function is: $f(x)=y=c.g(x)$ (i)

$$y+\Delta y=c.g(x+\Delta x) \qquad \text{(ii)}$$

Subtracting equation (i) from equation (ii)

$$y+\Delta y-y=c.g(x+\Delta x)-c.g(x)$$

$$\Delta y=c.g(x+\Delta x)-c.g(x) \qquad \text{(iii)}$$

Dividing equation (iii) by Δy

$$\frac{\Delta y}{\Delta x}=\frac{c.g(x+\Delta x)-c.g(x)}{\Delta x}$$

$$\frac{\Delta y}{\Delta x}=c\left\{\frac{g(x+\Delta x)-g(x)}{\Delta x}\right\} \qquad \text{(iv)}$$

Applying $\lim\limits_{\Delta x\to 0}$ on both sides of equation (iv)

$$\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=c.\lim_{\Delta x\to 0}\frac{g(x+\Delta x)-g(x)}{\Delta x}$$

Hence, $f'(x)=c.g'(x)$

This calculation develops the rule that the derivative of a constant multiple function is the product of the constant function and the derivative of a function $f(x)$.

In general: If $g(x)=x^n$ and $f(x)=cg(x)$, c is any constant, then: $f'(x)=cg'(x)=cnx^{n-1}$

Example 6 Differentiate the following functions: (a). $f(x)=4x^3$ (b). $0.555x^6$

Solution

a. If $f(x)=4x^3$, then, the derivative of a given function is: $f'(x)=4(3)x^{3-1}=12x^2$

b. If $f(x)=0.555x^6$, then, the derivative of a given function is: $f'(x)=0.555(6)x^{6-1}=3.33x^5$

Theorem-3: The derivative of a sum (or difference) of two functions is equal to the sum (or difference) of their derivatives.

Proof: To determine the derivative of a polynomial, such as the derivative of the sum or difference of two or more functions, we need to develop a rule that could be used in the determination of a derivative like $f(x)=3x^5+2x^2+3$. In this situation, if $h(x)=f(x)+g(x)$, then, our task is to determine $h'(x)$ by first principle rule of differentiation:

$$h(x)=y=f(x)+g(x) \qquad \text{(i)}$$

$$y+\Delta y=f(x+\Delta x)+g(x+\Delta x) \qquad \text{(ii)}$$

By subtraction equation (i) from equation (ii)

$$\cancel{y}+\Delta y-\cancel{y}=f(x+\Delta x)+g(x+\Delta x)-f(x)-g(x)$$

$$\Delta y=f(x+\Delta x)-f(x)+g(x+\Delta x)-g(x) \qquad \text{(iii)}$$

Dividing equation (iii) by Δx then we have

$$\frac{\Delta y}{\Delta x}=\frac{f(x+\Delta x)-f(x)+g(x+\Delta x)-g(x)}{\Delta x}$$

$$=\frac{f(x+\Delta x)-f(x)}{\Delta x}+\frac{g(x+\Delta x)-g(x)}{\Delta x} \qquad \text{(iv)}$$

Now, apply $\lim\limits_{\Delta x\to 0}$ on both sides of equation (iv)

$$\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}+\lim_{\Delta x\to 0}\frac{g(x+\Delta x)-g(x)}{\Delta x}$$

Hence, $h'(x)=f'(x)+g'(x)$

We can say that the derivative of a sum of two functions is the sum of the derivatives of two functions. The difference of two functions $f(x)-g(x)$ can be written as the sum of $f(x)-g(x)=f(x)+[-g(x)]$. Thus, the derivative of the difference of two functions is the difference of their derivatives.

In general: If $u=f(x)$ and $v=g(x)$, then, the sum rule can be restated using the notations:

$$\frac{d}{dx}(u\pm v)=\frac{du}{dx}\pm\frac{dv}{dx}$$

This rule generalizes to the sum and difference of any given number of functions.

Example 7 If $f(x)=3x^2+4x$ and $g(x)=7x-2$ the differentiate $f(x)+g(x)$ and $f(x)-g(x)$

Solution Since, $f(x)=3x^2+4x$ and $g(x)=7x-2$

$$f(x)+g(x)=(3x^2+4x)+(7x-2)=3x^2+11x-2$$

$$\frac{d}{dx}[f(x)+g(x)]=\frac{d}{dx}(3x^2+11x-2)=\frac{d}{dx}(3x^2)+\frac{d}{dx}(11x)-\frac{d}{dx}(2)=6x+11$$

Now, $$f(x)-g(x)=(3x^2+4x)-(7x-2)=3x^2-3x+2$$

$$\frac{d}{dx}[f(x)-g(x)]=\frac{d}{dx}(3x^2-3x+2)=\frac{d}{dx}(3x^2)-\frac{d}{dx}(3x)+\frac{d}{dx}(2)=6x-3$$

Theorem–4: The derivative of a product of two functions is equal to (the first function)×(derivative of the second function) plus (derivative of the first function)×(the second function).

Proof: If $h(x)=f(x)\,g(x)$, and $f(x)$ and $g(x)$ are differentiable functions of x, then by first principle rule,

$$h(x)=y=f(x)g(x) \qquad \text{(i)}$$

$$y+\Delta y=f(x+\Delta x).g(x+\Delta x) \qquad \text{(ii)}$$

Subtracting equation (i) from equation (ii)

$$y+\Delta y-y=f(x+\Delta x).g(x+\Delta x)-f(x).g(x)$$

$$\Delta y=f(x+\Delta x).g(x+\Delta x)-f(x).g(x) \qquad \text{(iii)}$$

The addition and subtraction of $f(x+\Delta x).g(x)$ to the right side of equation

$$\Delta y = f(x+\Delta x).g(x+\Delta x) - f(x+\Delta x).g(x) + f(x+\Delta x).g(x) - f(x).g(x)$$

$$= f(x+\Delta x)\{g(x+\Delta x) - g(x)\} + g(x)\{f(x+\Delta x) - f(x)\} \quad \text{(iv)}$$

Now, dividing equation (iv) by Δx

$$\frac{\Delta y}{\Delta x} = \frac{f(x+\Delta x)\{g(x+\Delta x)-g(x)\} + g(x)\{f(x+\Delta x)-f(x)\}}{\Delta x} \quad \text{(v)}$$

Now, apply $\lim\limits_{\Delta x \to 0}$ on both sides of equation (v)

$$\lim_{\Delta x \to 0}\frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0}\frac{f(x+\Delta x)\{g(x+\Delta x)-g(x)\} + g(x)\{f(x+\Delta x)-f(x)\}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} f(x+\Delta x).\lim_{\Delta x \to 0}\frac{g(x)\{f(x+\Delta x)-f(x)\}}{\Delta x} + g(x).\lim_{\Delta x \to 0}\frac{f(x+\Delta x)-f(x)\}}{\Delta x}$$

Hence, $h'(x) = f(x).g'(x) + g(x).f'(x)$

This calculation develops the idea that the derivative of a product of two functions is the first function times the derivative of the second, plus the second function times the derivative of the first.

In general: If $y = f(x)\,g(x) = u\,v$ with $u = f(x)$ and $v = g\,(x)$, then the product rule can be restated using the notations: $\dfrac{dy}{dx} = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$

Theorem-5: The derivative of a quotient of two functions is equal to denominator times the derivative of the numerator, minus the numerator times the derivative of the denominator all divided by the square of the denominator.

Proof: If $h(x) = \dfrac{f(x)}{g(x)}$, $g(x) \neq 0$ and $f\,(x)$ and $g\,(x)$ are differentiable functions of x, then, the derivative of $h(x)$ can be found by first principle rule:

$$h(x) = y = \frac{f(x)}{g(x)} \quad \text{(i)}$$

$$y + \Delta y = \frac{f(x+\Delta x)}{g(x+\Delta x)} \quad \text{(ii)}$$

Subtracting equation (i) om equation (ii)

$$y + \Delta y - y = \frac{f(x+\Delta x)}{g(x+\Delta x)} - \frac{f(x)}{g(x)}$$

$$\Delta y = \frac{f(x+\Delta x).g(x) - g(x+\Delta x).f(x)}{g(x+\Delta x).g(x)} \quad \text{(iii)}$$

The addition and subtraction of $f(x).g(x)$ to the numerator of equation (iii)

$$\Delta y = \frac{f(x+\Delta x).g(x) - f(x).g(x) - f(x)g(x+\Delta x) + f(x).g(x)}{g(x+\Delta x).g(x)}$$

$$\Delta y = \frac{g(x)\{f(x+\Delta x)-f(x)\} - f(x)\{g(x+\Delta x)-g(x)\}}{g(x+\Delta x).g(x)} \quad \text{(iv)}$$

Now, divide equation (iv) by Δx. Then

$$\frac{\Delta y}{\Delta x} = \frac{g(x)\left\{\dfrac{f(x+\Delta x)-f(x)}{\Delta x}\right\} - f(x)\left\{\dfrac{g(x+\Delta x)-g(x)}{\Delta x}\right\}}{g(x+\Delta x).g(x)} \quad \text{(v)}$$

Apply $\lim\limits_{\Delta x \to 0}$ on both sides of equation (v)

$$\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=\frac{g(x)\left\{\lim\limits_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}\right\}-f(x)\left\{\lim\limits_{\Delta x\to 0}\frac{g(x+\Delta x)-g(x)}{\Delta x}\right\}}{\lim\limits_{\Delta x\to 0}g(x+\Delta x).g(x)}$$

Hence, $h'(x)=\dfrac{g(x).f'(x)-f(x).g'(x)}{[g(x)]^2}$

In general: If $y=\dfrac{f(x)}{g(x)}=\dfrac{u}{v}$, with $u=f(x)$ and $v=g(x)$, then the quotient rule can be restated using the notations:

$$\frac{dy}{dx}=\frac{\frac{du}{dx}v-\frac{dv}{dx}u}{v^2}$$

Example 8 Differentiate the following functions:

(a). $y=4x^3-2x^2+5x$ (b). $y=(x^2-2x)(x^3-3)$ (c). $y=\dfrac{x^2+13x+9}{x^2+11x+3}$

Solution

a. If $y=u+v+w$ then $\dfrac{dy}{dx}=\dfrac{d}{dx}(u+v+w)=\dfrac{du}{dx}+\dfrac{dv}{dx}+\dfrac{dw}{dx}$ so, the derivative of a given function is:

$$\frac{dy}{dx}=\frac{d}{dx}[(4x^3)-(2x^2)+(5x)]=\frac{d}{dx}(4x^3)-\frac{d}{dx}(2x^2)+\frac{d}{dx}(5x)=12x^2-4x+5$$

b. If $y=u\cdot v$ then $\dfrac{dy}{du}=u\cdot\dfrac{dv}{dx}+v\cdot\dfrac{du}{dx}$ so, the derivative of the given function is:

$$\frac{dy}{dx}=(x^2-2x)\cdot\frac{d}{dx}(x^3-3)+(x^3-3)\cdot\frac{d}{dx}(x^2-2x)$$

$$=(x^2-2x)\cdot\left\{\frac{d}{dx}(x^3)-\frac{d}{dx}(3)\right\}+(x^3-3)\cdot\left\{\frac{d}{dx}(x^2)-2\frac{d}{dx}(x)\right\}$$

$$=(x^2-2x)\cdot(3x^2-0)+(x^3-3)(2x-2)=3x^2(x^2-2x)+x^3(2x-2)-3(2x-2)$$

$$=3x^4-6x^3+2x^4-2x^3-6x+6=5x^4-8x^3-6x+6$$

c. If $y=\dfrac{u}{v}$ then $\dfrac{dy}{dx}=\dfrac{v\cdot\frac{du}{dx}-u\cdot\frac{dv}{dx}}{v^2}$ so, the derivative of a given function is:

$$\frac{dy}{dx}=\frac{(x^2+11x+3)\frac{d}{dx}(x^2+3x+9)(-x^2+3x+9)\frac{d}{dx}(x^2+11x+3)}{(x^2+11x+3)^2}$$

$$\frac{dy}{dx}=\frac{(x^2+11x+3)(2x+13)-(x^2+13x+9)(2x+11)}{(x^2+11x+3)^2}$$

$$\frac{dy}{dx}=\frac{(2x^3+22x^2+6x+13x^2+143x+39)-(2x^3+26x^2+18x+11x^2+143x+99)}{(x^2+11x+3)^2}$$

$$\frac{dy}{dx}=\frac{\cancel{2x^3}+35x^2+149x+39-\cancel{2x^3}-39x^2-161x-99}{(x^2+11x+3)^2}$$

$$\frac{dy}{dx}=\frac{-4x^2-12x-60}{(x^2+11x+3)^2}$$

$$\frac{dy}{dx}=\frac{-4(x^2+3x+15)}{(x^2+11x+3)^2}$$

3.3 Application of Theorems on Differentiation

Calculus is used in both applied mathematics and pure mathematics, the biological and medicine, physical sciences, computer science, engineering, statics, economics, artificial intelligence and many more areas of other fields.

Few simple examples of applications of differentiation are given in this section.

3.3.1 Differentiation of

- Constant multiple of x^n
- Sum (or difference) of functions
- Polynomials
- Product of functions
- Quotient of two functions

Example 9 The cost in (million) dollars to produce x units of wheat is given by $C(x)=5000+20x+10\sqrt{x}$. Find the marginal cost, when

(a). $x=9$ units (b). $x=16$ units

(c). $x=25$ units (d). As more wheat is produced, what happens to the marginal cost?

Solution

a. If $C(x)=5000+20x+10\sqrt{x}$, then the marginal cost is the derivative of $C(x)$ with respect to x:

$$C'(x)=20+10\left(\frac{1}{2}\right)(x^{-\frac{1}{2}})=20+\frac{5}{\sqrt{x}}$$

The marginal cost at $x=9$ units is obtained by inserting $x=9$ in $C'(x)$:

$$C'(9)=20+\frac{5}{\sqrt{9}}=20+\frac{5}{3}=\frac{65}{3}\approx \$21.67$$

b. The marginal cost at $x=16$ units is obtained by inserting $x=16$ in $C'(x)$:

$$C'(16)=20+\frac{5}{\sqrt{16}}=20+\frac{5}{4}=\frac{85}{4}\approx \$21.25$$

c. The marginal cost at $x=25$ units is obtained by inserting $x=25$ in $C'(x)$:

$$C'(25)=20+\frac{5}{\sqrt{25}}=20+\frac{5}{5}\approx \$21$$

d. It decreases and approaches \$20.

Marginal Analysis:

In business and economics the rates of change of such variables as cost, revenue and profit are most important. Economists use the word **marginal** to refer to rates of change. For example, the marginal cost refers to the rate of change of cost. Since the derivative of a function gives the rate of change of the function, a marginal cost (or revenue of profit) function is found by taking the derivative of the cost (or revenue of profit) function. The marginal cost at some level of production x is the cost to produce the $(x+1)$st item (i.e., one more item).

Do You Know?

Does a feather fall more slowly than a rock? An Italian mathematician astronomer and physicist raise this question before 400 years ago. He theorized that the rate of falling objects depends on the air resistance, not on mass. It is believed that he tested his idea by dropping spheres of different masses but the same diameter from the top of the Leaning tower of Pisa in Italy. The result was exactly as he predicted they fell at the same rate.

Galileo Galilei (1564-1642)

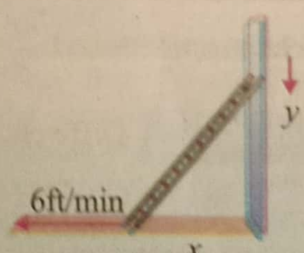
In 1971 during the Apollo 15 lunar landing, David Scott (commander) performed a demonstration on live television show. This is because of the surface of the moon is essentially a vacuum, a hammer and a feather fell at the same rate.

Exercise 3.2

1. Differentiate $f(x)+g(x)$ and $f(x)-g(x)$ if:
   a. $f(x)=3x+7$ and $g(x)=6x^2+2x-3$ b. $f(x)=17x^2-15$ and $g(x)=\frac{1}{3}x^2+\frac{1}{2}x-5$
   c. $f(x)=x^3-\frac{3}{2}$ and $g(x)=3x^3-4x^2+2$ d. $f(x)=4x^3-5x$ and $g(x)=\frac{3}{5}x^2-2x$
2. Use the product rule to find out the derivative of the following functions:
   a. $y=(x^2-2)(3x+1)$ b. $y=(7x^4+2x)(x^2-4)$
   c. $y=(2x-3)(\sqrt{x}-1)$ d. $y=(-3\sqrt{x}+6)(4\sqrt{x}-2)$
3. Use the quotient rule to find out the derivative of the following functions:
   a. $y=\frac{3x-5}{x-4}$ b. $y=\frac{-x^2+6x}{4x^3+1}$ c. $y=\frac{5x+6}{\sqrt{x}}$ d. $f(p)=\frac{(2p+3)(4p-1)}{(3p+2)}$
4. Find an equation of a tangent line to the graph of the function at the particular point in the following problems:
   a. $f(x)=3x-7$ at $(3, 2)$ b. $f(x)=x^3$ at $x=\frac{-1}{2}$
   c. $f(x)=\frac{1}{x+3}$ at $x=2$ d. $f(x)=\frac{x}{x-2}$ at $(3,3)$
5. For a thin lens of constant focal length P, the object distance x and the image distance y are related by the formula $\frac{1}{x}+\frac{1}{y}=\frac{1}{P}$
   a. Solve the above equation for y in terms of x and P.
   b. Determine the rate of change of y with respect to x.

Project

At your home. Take a ladder measure its length and put if with the wall as shown in the figure. Pull it away from the wall at the constant rate of 6ft/min. Calculate how fast is the top of the ladder moving down the wall when the bottom of the ladder is 6 feet from the wall.



3.4 Chain Rule

"The chain rule is a rule which we use to differentiate the composite functions". We have learnt about composition of functions in unit-2 that a function is a composite functions of the two similar functions $f(x)$ and $g(x)$ if it is written as $f[g(x)]$. In other words it is a function of a function. For example $\sin(x^2)$ is a composite functions because of we consider $f(x) = \sin(x)$ and $g(x) = x^2$ then $f[g(x)] = \sin(x^2)$. Generally, we write chain rule as;

$$\frac{d}{dx}[f(g(x))] = f'[g(x)].g'(x) \qquad \text{(i)}$$

3.4.1 Prove that $\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}$ when $y = f(u)$ and $u = g(x)$

Proof: For our convenience, we set $\quad y = F(x) = f[g(x)] \qquad$ (i)

Let $u = g(x)$ the equation (i) will be $\quad y = F(x) = f[g(x)] = f(u) \qquad$ (ii)

$$y + \Delta y = F(x + \Delta x) \qquad \text{(iii)}$$

Now, subtracting equation (i) from equation (iii)

$$y + \Delta y - y = F(x + \Delta x) - F(x)$$

$$\Delta y = F(x + \Delta x) - F(x) \qquad \text{(iv)}$$

Equation (iv) can be written as

$$\Delta y = f[g(x + \Delta x)] - f[g(x)] \qquad \text{(v)}$$

Where,

$$\Delta u = g(x + \Delta x) - g(x)$$

$$\Rightarrow g(x + \Delta x) = \Delta u + g(x) \qquad \text{(vi)}$$

Substitute the value of $g(x + \Delta x)$ from equation (vi) to equation (v)

$$\Delta y = f[\Delta u + g(x)] - f[g(x)]$$

$$F(x + \Delta x) - F(x) = f(u + \Delta u) - f(u) \qquad \because u = g(x)$$

Divide equation (iv) by Δx then we have

$$\frac{\Delta y}{\Delta x} = \frac{F(x + \Delta x) - F(x)}{\Delta x} = \frac{f(u + \Delta u) - f(u)}{\Delta x} \qquad \text{(vii)}$$

Multiply and divide the right side of equation (vii) by Δu so,

$$\frac{\Delta y}{\Delta x} = \frac{f(u + \Delta u) - f(u)}{\Delta u}\cdot\frac{\Delta u}{\Delta x}$$

$$= \frac{f(u + \Delta u) - f(u)}{\Delta u}\cdot\frac{g(x + \Delta x) - g(x)}{\Delta x} \qquad \text{(viii)}$$

Apply $\lim\limits_{\Delta x \to 0}$ and $\lim\limits_{\Delta u \to 0}$ on equation (viii)

We have $\quad \lim\limits_{\Delta x \to 0}\frac{\Delta y}{\Delta x} = \lim\limits_{\Delta u \to 0}\frac{f(u + \Delta u) - f(u)}{\Delta u}\cdot\lim\limits_{\Delta x \to 0}\frac{g(x + \Delta x) - g(x)}{\Delta x}$

Hence, $\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}$ as required.

It can also written as $\frac{d}{dx}[f(g(x))] = f'(u).g'(x) = f'[g(x)].g'(x)$

Example 10 Differentiate the following functions w.r.t. x:

(a). $f(x) = (4x - 3)^3$ $\qquad$ *(b).* $f(x) = \sqrt{15x^2 + 1}$

Solution

a. If $y = f(x) = (4x-3)^3 = u^3$ with $u = 4x-3$, then, the first derivative w.r.t. x by chain rule is:

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \frac{d}{du}(u^3)\frac{d}{dx}(4x-3) = (3u^2)(4) = 12u^2 = 12(4x-3)^2, u = (4x-3)$$

b. If $y = f(x) = \sqrt{15x^2+1} = \sqrt{u} = u^{\frac{1}{2}}$ with $u = 15x^2+1$, then, the first derivative by chain rule is:

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} \qquad \therefore u = (15x^2+1)$$

$$= \frac{d}{du}(u^{\frac{1}{2}})\frac{d}{dx}(15x^2+1) = \frac{1}{2}(u^{\frac{1}{2}-1})(30x) = \frac{15x}{\sqrt{u}} = \frac{15x}{\sqrt{15x^2+1}}$$

Example 11 The revenue realized by a small city from the collection of fines from parking tickets is given by $R(x) = \dfrac{8000x}{x+2}$ where x is the number of work hours each day that can be devoted to parking patrol. At the outbreak of a flu epidemic, 30 work hours are used daily in parking patrol, but during the epidemic that number is decreasing at the rate of 6 work hours per day. How fast is revenue from parking fines decreasing during the epidemic?

Solution We need to find $\dfrac{dR}{dt}$, the change in revenue with respect to time t. The chain rule is used to obtain $\dfrac{dR}{dt}$.

$$\frac{dR}{dt} = \frac{dR}{dx}\frac{dx}{dt} \qquad \text{(i)}$$

First find $\dfrac{dR}{dx}$ as follows.

$$R'(x) = \frac{dR}{dx} = \frac{(x+2)((8000) - 8000x(1))}{(x+2)^2} = \frac{16000}{(x+2)^2} \Rightarrow R'(30) = \frac{16000}{(30+2)^2} = 15.625, \quad at\ x = 30$$

$\dfrac{dR}{dx} = 15.625$ and $\dfrac{dx}{dt} = -6$ are used in equation (i) to obtain: $\dfrac{dR}{dt} = \dfrac{dR}{dx}\dfrac{dx}{dt} = (15.625)(-6) = -93.75$

This tells us that the revenue is being lost at the rate of approximately \$94 per day.

3.4.2 Show that $\dfrac{dy}{dx} = \dfrac{1}{\frac{dx}{dy}}$

If $y = f(x)$ is any differential function of x, then it admits an inverse function $x = g(y)$. Suppose y is changed by a small amount Δy. This will cause x to change by an amount Δx. The increment Δx in x corresponds to the increment Δy in y is determined from

$x = g(y)$, $\quad x = g(y)$ is inverse of $y = f(x)$

That gives: $\quad 1 = \dfrac{\Delta y}{\Delta x}\dfrac{\Delta x}{\Delta y}$

By letting $\Delta x \to 0$ to obtain: $\quad 1 = \lim\limits_{\Delta x \to 0} \dfrac{\Delta y}{\Delta x}\dfrac{\Delta x}{\Delta y}$

Do You Know?

Leibniz was the first person who mentioned the composite of the square root function and the function on 1676. The common notation of the chain rule is also due Leibniz.

$$1=\frac{dy}{dx}\frac{dx}{dy} \quad \Rightarrow \frac{1}{\frac{dx}{dy}}=\frac{dy}{dx} \qquad \text{(i)}$$

Thus $\frac{dy}{dx}$ and $\frac{dx}{dy}$ are reciprocal to each other.

Example 12 Verify result $\frac{dy}{dx}=\frac{1}{\frac{dx}{dy}}$ for the following problems:

(a). $f(x)=(4x-3)^3$ *(b).* $f(x)=\sqrt{15x^2+1}$

Solution

a. The derivative of $y=(4x-3)^3$ is $\frac{dy}{dx}=12(4x-3)^2$. This result agrees to result (i):

The derivative of $\frac{dy}{dx}=\frac{1}{\frac{dx}{dy}}=\frac{1}{\frac{1}{12(4x-3)^2}}=(1)\left(\frac{12(4x-3)^2}{1}\right)=12(4x-3)^2$

b. The derivative of $y=\sqrt{15x^2+1}$ is $\frac{dy}{dx}=\frac{15x}{\sqrt{15x^2+1}}$.

This result agrees to result (i):

$$\frac{dy}{dx}=\frac{1}{\frac{dx}{dy}}=\frac{1}{\frac{1}{\frac{\sqrt{15x^2+1}}{15x}}}=(1)\left(\frac{15x}{\sqrt{15x^2+1}}\right)=\frac{15x}{\sqrt{15x^2+1}}$$

3.4.3 Use of chain rule to show that $\frac{d}{dx}[f(x)]^n=n[f(x)]^{n-1}f'(x)$

Proof: Let $y=[f(x)]^n$ and $u=f(x)$ then $y=u^n$ and $\frac{dy}{du}=nu^{n-1}$ (by power rule)

But $\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}=nu^{n-1}\cdot\frac{du}{dx}$

Or $\frac{d}{dx}[f(x)]^n=n[f(x)]^{n-1}f'(x)$ $\qquad \because \frac{du}{dx}=f'(x)$

In words, if $f(x)$ is equal to an expression in x raised to a power of n, then $f'(x)$ is equal to product of n times the expression to the n-1 power times the derivative of the expression with respect the variable. The statement is known as the **general power rule.**

Example 13 Differentiate the following functions:

(a). $f(x)=(11x^2-7)^8$ *(b).* $f(x)=\sqrt{2x^3+11}$

Solution

a. If $y=f(x)=(11x^2-7)^8$ then, the first derivative of a given function is:

$$\frac{dy}{dx}=\frac{d}{dx}(11x^2-7)^8=8(11x^2-7)^{8-1}\frac{d}{dx}(11x^2-7)=8(11x^2-7)^7(22x)=176x(11x^2-7)^7$$

b. If $y = f(x) = \sqrt{2x^3+11}$, then, the first derivative of a given function is:

$$\frac{dy}{dx} = \frac{d}{dx}(2x^3+11)^{\frac{1}{2}} = \frac{1}{2}(2x^3+11)^{\frac{1}{2}-1}\frac{d}{dx}(2x^3+11) = \frac{1}{2}(2x^3+11)^{-\frac{1}{2}}(6x^2) = \frac{3x^2}{\sqrt{2x^3+11}}$$

Note

If two differential functions $x = f(t)$ and $y = g(t)$ of parameter t. If $t = h(x)$ is an inverse function of $x = f(t)$, then $y = g[h(x)]$ is a function of x.

By chain rule, the differentiation of $y = g[h(x)]$ w.r.t. x is $\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{dy}{dt}\frac{1}{\frac{dx}{dt}} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{g'(t)}{f'(t)}$ (A)

Find $\frac{dy}{dx}$, when $x = at^2$ and $y = 2at$.

Result (A) for the assumptions $\frac{dx}{dt} = 2at, \frac{dy}{dt} = 2a$ is used to obtain:

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = (2a)\left(\frac{1}{2at}\right) = \frac{1}{t}$$

3.4.4 Derivative of implicit function

Whether y is expressed explicitly or implicitly in terms of x, we can still differentiate to find the derivative $\frac{dy}{dx}$. If y is expressed **explicitly** in terms of x, then $\frac{dy}{dx}$ will also be expressed explicitly in terms of x. If y is expressed **implicitly** in terms of x, then $\frac{dy}{dx}$ will be expressed in terms of x and y.

Fortunately, there is a simple technique based on the chain rule that allows us to find $\frac{dy}{dx}$ without first solving the equation for y explicitly. This technique is known as **implicit differentiation**. It consists differentiation of the both sides of the equation with respect to x and then solving the resultant equation algebraically for $\frac{dy}{dx}$.

Example 14 Differentiate the implicit equation $x^2y + 2y^3 = 3x + 2y$.

Solution The implicit equation is $x^2y + 2y^3 = 3x + 2y$. (i)

The implicit differentiation of (i) is obtained by differentiating both sides w.r.t. x:

$$\frac{d}{dx}\left(x^2y + 2y^3\right) = \frac{d}{dx}(3x + 2y)$$

$$\frac{d}{dx}(x^2y) + \frac{d}{dx}(2y^3) = \frac{d}{dx}(3x) + \frac{d}{dx}(2y)$$

$$2xy + x^2\frac{dy}{dx} + 6y^2\frac{dy}{dx} = 3 + 2\frac{dy}{dx}$$

$$(x^2+6y^2-2)\frac{dy}{dx}=3-2xy$$

$$\frac{dy}{dx}=\frac{3-2xy}{x^2+6y^2-2}$$

Example 15 Find the slope of a tangent line to the circle $x^2+y^2=5x+4y$ at a particular point P(5,4).

Solution The slope of a tangent line to the given curve is $\frac{dy}{dx}$ that can be found by taking the derivative of $x^2+y^2=5x+4y$ with respect to x:

$$\frac{d}{dx}(x^2+y^2)=\frac{d}{dx}(5x+4y)$$

$$2x+2y\frac{dy}{dx}=5+4\frac{dy}{dx}$$

$$(2y-4)\frac{dy}{dx}=5-2x$$

$$\frac{dy}{dx}=\frac{5-2x}{2y-4}$$

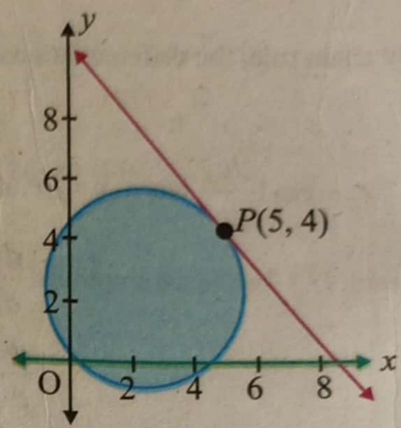


Figure 3.6

At a point P(5, 4), the slope of the tangent line is:

$$\frac{dy}{dx}=\frac{5-2(5)}{2(4)-4}=\frac{-5}{4}$$

Note that the expression is undefined at $y=2$. This makes sense, when you see that the tangent is vertical there.

Exercise 3.3

1. Find the derivative of the following functions w.r.t involved independent variable:

a. $w=4(x^3-4x+2)^5$ b. $y=\frac{(4-x^3)^{11}}{5}$

c. $u=\sqrt[3]{1-3t^2}$ d. $s=\frac{1}{(3t+1)^7}$

2. Determine the derivative $f'(x)$ in each case:

a. $f(x)=(2x-5)^3(5x-7)$ b. $f(x)=\frac{(x+2)^2}{x-1}$

c. $f(x)=\left(\frac{2x-5}{x-4}\right)^4$ d. $f(x)=x\sqrt{2x^2+11}$

3. Find $\frac{dy}{dx}$ of the following function in terms of parameter t:

a. $x=1+t^2, y=t^3+2t^2+1$ b. $x=3at^2+2, y=6t^4+9$

c. $x=\frac{a(1-t^2)}{1+t^2}, y=\frac{2bt}{1+t^2}$ d. $x=\frac{3at}{1+t^3}, y=\frac{3at^2}{1+t^3}$

4. At a certain factor, the total cost of manufacturing q units during the daily production run is $C(q)=0.2q^2+q+900$ dollars. From experience, it has been determined that approximately $q(t)=t^2+100t$ units are manufactured during the first t hours of a production run. Compute the rate at which the total manufacturing cost is changing with respect to time one hour after production begins.

5. Use implicit differentiation to perform $\frac{dy}{dx}$ for the following functions:

a. $x^2+y^2=25$ b. $xy(2x+3y)=2$ c. $(x+y)^3+3y=3$ d. $\frac{1}{y}+\frac{1}{x}=1$

6. Arrange the following functions explicitly and implicitly to perform $\frac{dy}{dx}$:

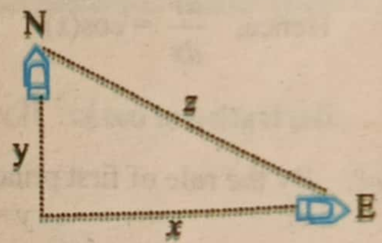
a. $x^2y^3+y^3=1$ b. $xy+2y=x^2$ c. $x+\frac{1}{y}=5$ d. $xy-x=y+2$

7. Let $\frac{u^2}{a^2}+\frac{v^2}{b^2}=1$, where a and b are nonzero constants. Find:

a. $\frac{du}{dv}$ b. $\frac{dv}{du}$

8. Determine the slope of the tangent line to the curve $3x^2-7y^2+14y=27$ at the point $P(-3,0)$.

9. Suppose two motor boats leave from the same point at the same time. If one boat travels north at 15 miles per hour and the other boat travels east at 20 miles per hour. How fast will the distance between them be changing after 2 hours?



3.5 Differentiation of Trigonometric and inverse Trigonometric Functions

To understand this section we need to know about trigonometric function. For differentiating all trigonometric functions we use the basic rule of differentiation that we have already learnt e.g. We will use product, quotient and chain rules to differentiation functions that are the combination of the trigonometric function.

3.5.1 Differentiation of trigonometric functions (sin$x$, cos$x$, tan$x$, cosec$x$, sec$x$ and cot$x$) from first principle

i. **Derivative of sin x:** If $y=\sin x$ then the derivative of $y=\sin x$ is $\frac{dy}{dx}=\cos x$.

Proof: By the rule of first principle.

Let $y=\sin(x)$ (i)

$y+\Delta y=\sin(x+\Delta x)$ (ii) $\because y=f(x)$

Subtracting equation (i) from equation (ii).

$y+\Delta y-y=\sin(x+\Delta x)-\sin(x)$

$\Delta y=\sin(x+\Delta x)-\sin(x)$

$=2\cos\left(\frac{x+\Delta x+x}{2}\right).\sin\left(\frac{x+\Delta x-x}{2}\right)$ $\because \sin\alpha-\sin\beta=2\cos\left(\frac{\alpha+\beta}{2}\right).\sin\left(\frac{\alpha-\beta}{2}\right)$

$$\Delta y = 2\cos\left(x+\frac{\Delta x}{2}\right).\sin\left(\frac{\Delta x}{2}\right) \quad \text{(iii)}$$

Now, divide equation (iii) by the Δx.

$$\frac{\Delta y}{\Delta x} = \frac{2\cos.\left(x+\frac{\Delta x}{2}\right).\sin\left(\frac{\Delta x}{2}\right)}{\Delta x} \quad \text{(iv)}$$

Now, apply $\lim\limits_{\Delta x\to 0}$ on both side of equation (iv).

$$\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x} = \lim_{\Delta x\to 0}\left[\cos\left(x+\frac{\Delta x}{2}\right).\frac{\sin\left(\frac{\Delta x}{2}\right)}{\frac{\Delta x}{2}}\right]$$

$$= \lim_{\Delta x\to 0}\cos\left(x+\frac{\Delta x}{2}\right).\lim_{\Delta x\to 0}\frac{\sin\left(\frac{\Delta x}{2}\right)}{\frac{\Delta x}{2}}$$

$$= \cos(x+0)\cdot 1$$

$$= \cos(x)$$

$$\because \lim_{\Delta x\to 0}\frac{\sin\left(\frac{\Delta x}{2}\right)}{\frac{\Delta x}{2}} = 1$$

$$\because \frac{\Delta x}{2}\to 0 \text{ as } \Delta x\to 0$$

Hence, $\frac{dy}{dx} = \cos(x)$

ii. Derivative of $\cos x$: If $y = \cos x$, then the derivative of $y = \cos x$ is $\frac{dy}{dx} = -\sin x$.

Proof: By the rule of first principle.

Let $y = \cos x$ (i)

$y + \Delta y = \cos(x+\Delta x)$ (ii) $\quad \because y = f(x)$

Subtracting equation (i) from equation (ii).

$$y+\Delta y - y = \cos(x+\Delta x) - \cos(x)$$

$$\Delta y = \cos(x+\Delta x) - \cos(x) \quad \text{(iii)}$$

$$\Delta y = -2\sin\left(\frac{x+\Delta x+x}{2}\right)\cdot\sin\left(\frac{x+\Delta x+x}{2}\right)$$

$$\Delta y = -2\sin\left(\frac{x+\Delta x}{2}\right)\cdot\sin\left(\frac{\Delta x}{2}\right) \quad \because \cos\alpha - \cos\beta = -2\sin\left(\frac{\alpha+\beta}{2}\right)\cdot\sin\left(\frac{\alpha-\beta}{2}\right)$$

$$\Delta y = -2\sin\left(x+\frac{\Delta x}{2}\right)\cdot\sin\left(\frac{\Delta x}{2}\right) \quad \text{(iv)}$$

Now, divide equation (iv) by the Δx.

$$\frac{\Delta y}{\Delta x} = -2\sin\left(x+\frac{\Delta x}{2}\right)\frac{\sin\left(\frac{\Delta x}{2}\right)}{\Delta x}$$

$$\frac{\Delta y}{\Delta x} = -\sin\left(x+\frac{\Delta x}{2}\right)\cdot\frac{\sin\left(\frac{\Delta x}{2}\right)}{\frac{\Delta x}{2}} \quad \text{(v)}$$

Now, apply $\lim\limits_{\Delta x \to 0}$ on both sides of equation (v).

$$\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x} = -\lim_{\Delta x\to 0}\sin\left(x+\frac{\Delta x}{2}\right).\lim_{\Delta x\to 0}\frac{\sin\left(\frac{\Delta x}{2}\right)}{\frac{\Delta x}{2}} \quad \because \lim_{\Delta x\to 0}\frac{\sin(\Delta x)}{\Delta x}=1$$

$$= -\sin(x+0)\cdot 1$$

$$= -\sin x$$

Hence, $\dfrac{dy}{dx} = -\sin(x)$

iii. Derivative of tan x: If $y=\tan x$, then the derivative of $y=\tan x$ is $\dfrac{dy}{dx}=\sec^2 x$.

Proof: By the rule of first principle.

Let $\quad y=\tan(x) \quad$ (i) and

$y+\Delta y=\tan(x+\Delta x) \quad$ (ii)

Subtracting equation (i) from equation (ii).

$$y+\Delta y-y=\tan(x+\Delta x)-\tan(x)$$

$$\Delta y=\frac{\sin(x+\Delta x)}{\cos(x+\Delta x)}-\frac{\sin(x)}{\cos(x)} \quad \text{(iii)}$$

Divide equation (iii) by Δx

$$\frac{\Delta y}{\Delta x}=\frac{1}{\Delta x}\left\{\frac{\sin(x+\Delta x)}{\cos(x+\Delta x)}-\frac{\sin(x)}{\cos(x)}\right\} \quad \text{(iv)}$$

Now, apply $\lim\limits_{\Delta x \to 0}$ on both sides of equation (iv).

$$\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=\lim_{\Delta x\to 0}\frac{\frac{\sin(x+\Delta x)}{\cos(x+\Delta x)}-\frac{\sin(x)}{\cos(x)}}{\Delta x}=\lim_{\Delta x\to 0}\frac{\sin(x+\Delta x).\cos(x)-\sin(x).\cos(x+\Delta x)}{\Delta x.\cos(x).\cos(x+\Delta x)}$$

$$=\lim_{\Delta x\to 0}\frac{\sin(x+\Delta x-x)}{\Delta x.\cos(x).\cos(x+\Delta x)}=\lim_{\Delta x\to 0}\left(\frac{1}{\cos(x+\Delta x)}.\frac{1}{\cos(x)}\frac{\sin\Delta x}{\Delta x}\right)$$

$$=\lim_{\Delta x\to 0}\frac{1}{\cos(x+\Delta x)}\cdot\lim_{\Delta x\to 0}\frac{1}{\cos(x)}\cdot\lim_{\Delta x\to 0}\frac{\sin\Delta x}{\Delta x}=\frac{1}{\cos(x)}\cdot\frac{1}{\cos(x)}\cdot(1)=\frac{1}{\cos^2(x)}=\sec^2(x)$$

Hence, $\dfrac{dy}{dx}=\sec^2(x)$

iv. Derivative of sec x: If $y=\sec x$, then the derivative of $y=\sec(x)$ is $\dfrac{dy}{dx}=\sec(x).\tan(x)$.

Proof: By the rule of first principle.

Let $\quad y=\sec(x) \quad$ (i)

and $\quad y+\Delta y=\sec(x+\Delta x) \quad$ (ii)

Subtracting equation (i) from equation (ii).

$$y+\Delta y-y=\sec(x+\Delta x)-\sec(x)$$

$$\Delta y=\sec(x+\Delta x)-\sec(x) \quad \text{(iii)}$$

Now, divide equation (iii) by the Δx

$$\frac{\Delta y}{\Delta x}=\frac{\sec(x+\Delta x)-\sec(x)}{\Delta x} \quad \text{(iv)}$$

Do You Know?

The traffic police officers uses radar guns to take the advantage of the easy use of derivatives. When a radar gun is pointed and fired at a car on the motorway. The gun is able to determine the time and distance at which the radar was able to hit a certain section of the car with the use of derivative it is able to calculate the speed at which the car was going and also report the distance that the car was from the radar gun.

Now, apply $\lim_{\Delta x\to 0}$ on both sides of equation (iv).

$$\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=\lim_{\Delta x\to 0}\frac{\frac{1}{\cos(x+\Delta x)}-\frac{1}{\cos(x)}}{\Delta x}=\lim_{\Delta x\to 0}\frac{\cos(x)-\cos(x+\Delta x)}{\Delta x.\cos(x).\cos(x+\Delta x)}$$

$$=\lim_{\Delta x\to 0}\frac{-2\sin\left(\frac{x+x+\Delta x}{2}\right).\sin\left(\frac{x-x-\Delta x}{2}\right)}{\Delta x.\cos(x).\cos(x+\Delta x)}=\lim_{\Delta x\to 0}\frac{-2\sin\left(\frac{2x+\Delta x}{2}\right).\sin\left(-\frac{\Delta x}{2}\right)}{\Delta x.\cos(x).\cos(x+\Delta x)}$$

$$=\lim_{\Delta x\to 0}\frac{2\sin\left(x+\frac{\Delta x}{2}\right).\sin\left(\frac{\Delta x}{2}\right)}{\Delta x.\cos(x).\cos(x+\Delta x)}=\lim_{\Delta x\to 0}\sin\left(x+\frac{\Delta x}{2}\right).\lim_{\Delta x\to 0}\frac{1}{\cos(x+\Delta x)}.\lim_{\Delta x\to 0}\frac{1}{\cos(x)}.\lim_{\Delta x\to 0}\frac{\sin\frac{\Delta x}{2}}{\frac{\Delta x}{2}}$$

$$=\sin(x+0)\cdot\frac{1}{\cos x}\cdot\frac{1}{\cos(x)}\cdot(1)=\frac{\sin(x)}{\cos(x)}\cdot\frac{1}{\cos x}=\sec(x)\cdot\tan(x)$$

Hence, $\frac{dy}{dx}=\sec(x)\cdot\tan(x)$

v. **Derivative of cosec x:** If $y=\text{cosec } x$, then the derivative of $y=\text{cosec}(x)$ is $\frac{dy}{dx}=-\cot(x).\text{cosec}(x)$.

Proof: By the rule of first principle.

Let $y=\text{cosec}(x)$ (i)

and $y+\Delta y=\text{cosec}(x+\Delta x)$ (ii)

Subtracting equation (i) from equation (ii).

$y+\Delta y-y=\text{cosec}(x+\Delta x)-\text{cosec}(x)$

$\Delta y=\text{cosec}(x+\Delta x)-\text{cosec}(x)$ (iii)

Now, divide equation (iii) by the Δx

$$\frac{\Delta y}{\Delta x}=\frac{\text{cosec}(x+\Delta x)-\text{cosec}(x)}{\Delta x} \quad \text{(iv)}$$

Now, apply $\lim_{\Delta x\to 0}$ on both sides of equation (iv).

$$\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=\lim_{\Delta x\to 0}\frac{\text{cosec}(x+\Delta x)-\text{cosec}(x)}{\Delta x}=\lim_{\Delta x\to 0}\frac{\frac{1}{\sin(x+\Delta x)}-\frac{1}{\sin(x)}}{\Delta x}$$

$$=\lim_{\Delta x\to 0}\frac{\sin(x)-\sin(x+\Delta x)}{\Delta x.\sin(x).\sin(x+\Delta x)}=\lim_{\Delta x\to 0}\frac{2\cos\left(\frac{x+x+\Delta x}{2}\right).\sin\left(\frac{x-x-\Delta x}{2}\right)}{\Delta x.\sin(x).\sin(x+\Delta x)}$$

$$=\lim_{\Delta x\to 0}\frac{2.\cos\left(x+\frac{\Delta x}{2}\right).\sin\left(-\frac{\Delta x}{2}\right)}{\Delta x.\sin(x).\sin(x+\Delta x)}=-\lim_{\Delta x\to 0}\cos\left(x+\frac{\Delta x}{2}\right).\lim_{\Delta x\to 0}\left(\frac{1}{\sin(x).\sin(x+\Delta x)}\right).\lim_{\Delta x\to 0}\frac{\sin\frac{\Delta x}{2}}{\frac{\Delta x}{2}}$$

$$=-\cos(x+0).\frac{1}{\sin(x).\sin(x+0)}.\frac{1}{\cos(x)}.(1)=-\frac{\cos(x)}{\sin(x)}.\frac{1}{\sin(x)}=-\cot(x).\text{cosec}(x)$$

Hence, $\frac{dy}{dx}=\cot(x).\text{cosec}(x)$

vi. **Derivative of cot x:** If $y=\cot x$, then the derivative of $y=\cot x$ is $\frac{dy}{dx}=-\text{cosec}^2(x)$.

Proof: By the rule of first principle.

Let $\quad y=\cot(x) \qquad$ (i)

and $\quad y+\Delta y=\cot(x+\Delta x) \qquad$ (ii)

Subtracting equation (i) from equation (ii).

$$\cancel{y}+\Delta y-\cancel{y}=\cot(x+\Delta x)-\cot(x)$$

$$\Delta y=\cot(x+\Delta x)-\cot(x) \qquad \text{(iii)}$$

Now, divide equation (iii) by the Δx.

$$\frac{\Delta y}{\Delta x}=\frac{\cot(x+\Delta x)-\cot(x)}{\Delta x} \qquad \text{(iv)}$$

Now, apply $\lim\limits_{\Delta x\to 0}$ on both sides of equation (iv).

$$\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=\lim_{\Delta x\to 0}\frac{\cot(x+\Delta x)-\cot x}{\Delta x}=\lim_{\Delta x\to 0}\frac{\frac{\cos(x+\Delta x)}{\sin(x+\Delta x)}-\frac{\cos x}{\sin x}}{\Delta x}=\lim_{\Delta x\to 0}\frac{\sin x\cos(x+\Delta x)-\cos x\sin(x+\Delta x)}{\Delta x\sin x\sin(x+\Delta x)}$$

$$=\lim_{\Delta x\to 0}\frac{\sin(x-x-\Delta x)}{\Delta x\sin x\sin(x+\Delta x)}=\lim_{\Delta x\to 0}\frac{1}{\sin(x+\Delta x)}\cdot\frac{1}{\sin x}\cdot\frac{-\sin\Delta x}{\Delta x}, \qquad \sin(-\Delta x)=-\sin\Delta x$$

$$=-\lim_{\Delta x\to 0}\frac{1}{\sin(x+\Delta x)}\lim_{\Delta x\to 0}\frac{1}{\sin x}\lim_{\Delta x\to 0}\frac{\sin\Delta x}{\Delta x}=-\frac{1}{\sin x}\frac{1}{\sin x}(1)=-\frac{1}{\sin^2 x}=-\cos ec^2 x,$$

The trigonometric formulae are listed below.

i. $\frac{d}{dx}(\sin x)=\cos x$ ii. $\frac{d}{dx}(\cos x)=-\sin x$ iii. $\frac{d}{dx}(\tan x)=\sec^2 x$

iv. $\frac{d}{dx}(\cos ecx)=-\cot x\cos ecx$ v. $\frac{d}{dx}(\sec x)=\tan x\sec x$ vi. $\frac{d}{dx}(\cot x)=-\cos ec^2 x$

The chain rule can be used to derive the generalization of the power rule and the rules for differentiating the trigonometric functions, as summarized in the box:

i. $\frac{d}{dx}(\sin u)=\cos u\frac{d}{dx}(u)$ ii. $\frac{d}{dx}(\cos u)=-\sin u\frac{d}{dx}(u)$

iii. $\frac{d}{dx}(\tan u)=\sec^2 u\frac{d}{dx}(u)$ iv. $\frac{d}{dx}(\cos ecu)=-\cot u\cos ecu\frac{d}{dx}(u)$

v. $\frac{d}{dx}(\sec u)=\tan u\sec u\frac{d}{dx}(u)$ vi. $\frac{d}{dx}(\cot u)=-\cos ec^2 u\frac{d}{dx}(u)$

Example 16 Differentiate the following trigonometric functions:

(a). $p(t)=(t^2+t)\sin t$ (b). $f(x)=\dfrac{1+\sin x}{2-\cos(x)}$

Solution

a. If the given function is $p(t)=(t^2+t)\sin t$, then the product rule of differentiation w.r.t. t is used to obtain: $\frac{dp}{dt}=(t^2+t)\frac{d}{dt}(\sin t)+(\sin t)\frac{d}{dt}(t^2+t)=(t^2+t)\cos t+\sin t(2t+1)$

b. If the given function is $f(x)=\dfrac{1+\sin x}{2-\cos x}$, then the quotient rule of differentiation w.r.t. x is used to obtain:

$$\frac{df}{dx}=\frac{d}{dx}\left(\frac{1+\sin(x)}{2-\cos(x)}\right)=\frac{(2-\cos x)\frac{d}{dx}(1+\sin x)-(1+\sin x)\frac{d}{dx}(2-\cos x)}{(2-\cos x)^2}=\frac{(\cos x)(2-\cos x)-(\sin x)(1+\sin x)}{(2-\cos x)^2}$$

$$=\frac{2\cos x-\cos^2x-\sin x-\sin^2x}{(2-\cos x)^2}$$

$$=\frac{2\cos x-\sin x-(\cos^2x+\sin^2x)}{(2-\cos x)^2}=\frac{2\cos x-\sin x-1}{(2-\cos x)^2} \qquad \because \sin^2x+\cos^2x=1$$

Example 17 Differentiate the following trigonometric functions:

(a). $f(x)=\sec x\tan x$ *(b).* $f(x)=\dfrac{x^2+\tan x}{3x+2\tan x}$

Solution

a. If the given function is $f(x)=\sec x\tan x$, then the product rule of differentiation w.r.t. x is used to obtain:

$$\frac{df}{dx}=\frac{d}{dx}\sec x\tan x=\sec x\frac{d}{dx}(\tan x)+(\tan x)\frac{d}{dx}(\sec x)=\sec x(\sec^2 x)+\tan x(\sec x\tan x)=\sec^3 x+\sec x\tan^2 x$$

b. If the given function is $f(x)=\dfrac{x^2+\tan x}{3x+2\tan x}$, then the quotient rule of differentiation w.r.t. x is used to

obtain: $$\frac{df}{dx}=\frac{d}{dx}\left(\frac{x^2+\tan(x)}{3x+2\tan(x)}\right)=\frac{(3x+2\tan x).\frac{d}{dx}(x^2+\tan x)-(x^2+\tan x)\frac{d}{dx}(3x+2\tan x)}{(3x+2\tan(x))^2}$$

$$=\frac{(2x+\sec^2 x)(3x+2\tan x)-(3+2\sec^2 x)(x^2+\tan x)}{(3x+2\tan x)^2}=\frac{3x^2+(4x-3)\tan x+x(3-2x)\sec^2 x}{(3x+2\tan x)^2}$$

3.5.2 Differentiation of inverse trigonometric functions

i. Derivative of $\sin^{-1}x$: If $y=\sin^{-1}x$, then $x=\sin y$.

The differentiation of $x=\sin y$ w.r.t. y is: $\dfrac{dx}{dy}=cosy$

Take its reciprocal to obtain the derivative of y w.r.t. x:

$$\frac{dy}{dx}=\frac{1}{\cos y}=\pm\frac{1}{\sqrt{1-\sin^2 y}}=\pm\frac{1}{\sqrt{1-x^2}} \qquad \because \sin^2y+\cos^2y=1,\ \sin y=x$$

Here, the sign of the radical is the same as that of $\cos y$. By definition of $\sin^{-1}x$:

$$-\frac{\pi}{2}\le\sin^{-1}x\le\frac{\pi}{2} \ \ or \ \ -\frac{\pi}{2}\le y\le\frac{\pi}{2},$$

Hence, $\cos y$ is positive: $\dfrac{d}{dx}(\sin^{-1}x)=\dfrac{1}{\sqrt{1-x^2}}$

ii. Derivative of $\cos^{-1}x$: If $y=\cos^{-1}x$, then $x=\cos y$.

The differentiation of $x=\cos y$ w.r.t. y is: $\dfrac{dx}{dy}=-\sin y$

Take its reciprocal to obtain the derivative of y w.r.t. x:

$$\frac{dy}{dx}=-\frac{1}{\sin y}=\pm\frac{-1}{\sqrt{1-\cos^2 y}}=\pm\frac{-1}{\sqrt{1-x^2}} \qquad \because \sin^2y+\cos^2y=1,\ \cos y=x.$$

Here, the sign of the radical is the same as that of $\sin y$. By definition of $\cos^{1-}x$:

$$0\le\cos^{-1}x\le\pi \ \ or \ \ 0\le y\le\pi,$$

Also, if y lies between 0 and π, then, $\sin y$ is necessarily positive. Hence $\dfrac{d}{dx}(\cos^{-1}x)=\dfrac{-1}{\sqrt{1-x^2}}$

iii. **Derivative of $\tan^{-1}x$:** If $y=\tan^{-1}x$, then $x=\tan y$.

The differentiation of $x=\tan y$ w.r.t. y is: $\dfrac{dx}{dy}=\sec^2 y$

Take its reciprocal to obtain the derivative of y w.r.t. x:

$$\frac{dy}{dx}=\frac{1}{\sec^2 y}=\frac{1}{1+\tan^2 y}=\frac{1}{1+x^2} \qquad \because \sec^2 y=1+\tan^2 y,\ \tan y=x$$

iv. **Derivative of $\sec^{-1}x$:** If $y=\sec^{-1}x$, then $x=\sec y$.

The differentiation of $x=\sec y$ w.r.t. y is: $\dfrac{dx}{dy}=\sec y\tan y$

Take its reciprocal to obtain the derivative of y w.r.t. x:

$$\frac{dy}{dx}=\frac{1}{\sec y\tan y}=\pm\frac{1}{\sec y\sqrt{\sec^2 y-1}}=\pm\frac{1}{x\sqrt{x^2-1}} \qquad \because 1+\tan^2 y=\sec^2 y,\ \sec y=x$$

We take + sign before the radical sign to obtain: $\dfrac{d}{dx}(\sec^{-1}x)=\dfrac{1}{x\sqrt{x^2-1}}$

v. **Derivative of $\operatorname{cosec}^{-1}x$** If $y=\operatorname{cosec}^{-1}x$, then $x=\operatorname{cosec} y$.

The differentiation of $x=\operatorname{cosec} y$ w.r.t. y is: $\dfrac{dx}{dy}=-\operatorname{cosec}(y)\cot(y)$

Take its reciprocal to obtain the derivative of y w.r.t. x:

$$\frac{dy}{dx}=\frac{-1}{\operatorname{cosec}y\cot y}=\pm\frac{-1}{\operatorname{cosec}y\sqrt{\operatorname{cosec}^2 y-1}} \qquad \because 1+\cot^2 y=\operatorname{cosec}^2 y,\ \operatorname{cosec} y=x$$

$$\Rightarrow \frac{dy}{dx}=\pm\frac{-1}{x\sqrt{x^2-1}}$$

We take + sign before the radical sign to obtain: $\dfrac{d}{dx}(\operatorname{cosec}^{-1}x)=\dfrac{-1}{x\sqrt{x^2-1}}$

vi. **Derivative of $\cot^{-1}x$:** If $y=\cot^{-1}x$, then $x=\cot y$.

The differentiation of $x=\cot y$ w.r.t. y is: $\dfrac{dx}{dy}=-\operatorname{cosec}^2 y$

Take its reciprocal to obtain the derivative of y w.r.t. x:

$$\frac{dy}{dx}=-\frac{1}{\operatorname{cosec}^2 y}=\frac{-1}{1+\cot^2 y}=\frac{-1}{1+x^2} \qquad \because \operatorname{cosec}^2 y=1+\cot^2 y,\ \cot y=x$$

These inverse trigonometric formulas are listed in the box:

i. $\dfrac{d}{dx}(\sin^{-1}x)=\dfrac{1}{\sqrt{1-x^2}}$ ii. $\dfrac{d}{dx}(\cos^{-1}x)=\dfrac{-1}{\sqrt{1-x^2}}$ iii. $\dfrac{d}{dx}(\tan^{-1}x)=\dfrac{1}{1+x^2}$

iv. $\dfrac{d}{dx}(\operatorname{cosec}^{-1}x)=\dfrac{-1}{x\sqrt{x^2-1}}$ v. $\dfrac{d}{dx}(\sec^{-1}x)=\dfrac{1}{x\sqrt{x^2-1}}$ vi. $\dfrac{d}{dx}(\cot^{-1}x)=\dfrac{-1}{1+x^2}$

The chain rule can be used to derive the generalization of the power rule and the rules for differentiating the inverse trigonometric functions, as summarized in the box:

i. $\dfrac{d}{dx}(\sin^{-1}u)=\dfrac{1}{\sqrt{1-u^2}}\dfrac{d}{dx}(u)$ ii. $\dfrac{d}{dx}(\cos^{-1}u)=\dfrac{-1}{\sqrt{1-u^2}}\dfrac{d}{dx}(u)$

iii. $\dfrac{d}{dx}(\tan^{-1}u)=\dfrac{1}{1+u^2}\dfrac{d}{dx}(u)$ iv. $\dfrac{d}{dx}(\operatorname{cosec}^{-1}u)=\dfrac{-1}{u\sqrt{u^2-1}}\dfrac{d}{dx}(u)$

v. $\dfrac{d}{dx}(\sec^{-1}u)=\dfrac{1}{u\sqrt{u^2-1}}\dfrac{d}{dx}(u)$ vi. $\dfrac{d}{dx}(\cot^{-1}u)=\dfrac{-1}{1+u^2}\dfrac{d}{dx}(u)$

Example 18 Differentiate the following inverse trigonometric functions:

(a). $y=\tan^{-1}\sqrt{x}$ *(b).* $y=\cos^{-1}\left(\frac{x-x^{-1}}{x+x^{-1}}\right)$

Solution *a.* $y=\tan^{-1}\sqrt{x}$

$y=f(x)=\tan^{-1}\sqrt{x}$ (i)

Let $u=\sqrt{x}\Rightarrow\frac{du}{dx}=\frac{1}{2\sqrt{x}}$ (ii)

Now, $y=\tan^{-1}.(u)\Rightarrow\frac{dy}{du}=\frac{1}{1+u^2}$ (iii)

By using the chain rule $\frac{dy}{dx}=\frac{dy}{du}.\frac{du}{dx}$

From equation (ii) and equation (iii) $\frac{du}{dx}=\left(\frac{1}{1+u^2}\right)\left(\frac{1}{2\sqrt{x}}\right)\Rightarrow\frac{dy}{dx}=\frac{1}{2\sqrt{x}(1+x)}$

b. Given $y=\cos^{-1}\left(\frac{x-x^{-1}}{x+x^{-1}}\right)$

Let $u=\frac{x-x^{-1}}{x+x^{-1}}$

$u=\frac{x^2-1}{x^2+1}$

$$\frac{du}{dx}=\frac{d}{dx}\left(\frac{x^2-1}{x^2+1}\right)$$

$$=\frac{(x^2+1)\frac{d}{dx}(x^2-1)-(x^2-1)\frac{d}{dx}(x^2+1)}{(x^2+1)^2}$$

$$=\frac{(x^2+1)(2x)-(x^2-1)(2x)}{(x^2+1)^2}$$

$$=\frac{2\cancel{x^3}+2x-2\cancel{x^3}+2x}{(x^2+1)^2}=\frac{4x}{(x^2+1)^2}$$

Now, $y=\cos^{-1}u$ where $u=\frac{x-x^{-1}}{x+x^{-1}}=\frac{x^2-1}{x^2+1}$

$$\Rightarrow\frac{dy}{du}=\frac{d}{du}\cos^{-1}u$$

$$\frac{dy}{du}=\frac{-1}{\sqrt{1-u^2}}$$

By chain rule

$$\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}$$

$$=\frac{-1}{\sqrt{1-u^2}}\times\frac{4x}{(x^2+1)^2}=\frac{-1}{\sqrt{1-\left(\frac{x^2-1}{x^2+1}\right)^2}}\times\frac{4x}{(x^2+1)^2}$$

$$\frac{dy}{dx}=\frac{-4x}{\sqrt{\frac{(x^2+1)^2-(x^2-1)^2}{(x^2+1)^2}}\times(x^2+1)^2}$$

$$=\frac{-4x}{\sqrt{\frac{4x^2}{(x^2+1)^2}}\times(x^2+1)^2}=\frac{-4x}{2x(x^2+1)}$$

$$\frac{dy}{dx}=\frac{-2}{(x^2+1)}$$

Exercise 3.4

1. Use of first principle rules to differentiate the following functions.
   a. $y=\sin(2x)$ b. $y=\cot(3x)$ c. $y=\cos(3x)+\tan(3x)$
   d. $y=\cot^2(x)$ e. $y=\tan\sqrt{x}$ f. $y=\sin^3(x)$
2. Differentiate the following trigonometric functions by using any suitable rule.
   a. $x^2\cot(3x)$ b. $y=(\sin 2x+\cot 3x)^2$ c. $y=4\operatorname{cosec}2x$
   d. $y=2\tan(x+3)^2$ e. $y=\frac{\sqrt{\tan(x)}}{\cos\sqrt{x}}$ f. $y=\frac{1+\tan 2x}{\operatorname{cosec}3x}$
3. Use any suitable rule of differentiation to perform $\frac{dy}{dx}$ for the following functions.
   a. $y=\cos^{-1}\left(\frac{x}{a}\right)$ b. $y=\tan^{-1}\left(\frac{x}{p}\right)$ c. $y=\cot^{-1}\left(\frac{a}{x}\right)$
   d. $y=\operatorname{cosec}^{-1}\sqrt{(1+x^2)}$ e. $y=\operatorname{cosec}^{-1}(t+3)$ f. $y=\frac{1}{x}.\tan^{-1}\left(\frac{x+1}{x-1}\right)$
4. Suppose profits on the sale of swimming suits in a departmental store are given approximately by
   $P(t)=5-5\cos\frac{\pi t}{26},\quad 0\le t\le 104$
   where P(t) is profit (in hundreds of dollars) for a week of sales t weeks after January first.
   a. What is the rate of change of profit t weeks after the first of the year?
   b. What is the rate of change of profit 8 weeks after the first of the year? 26weeks after the first of the year? 50 weeks after the first of the year?
5. A normal seated adult breathes in and exhales about 0.8 liter of air every 4 seconds. The volume of air $V(t)$ in the lungs t seconds after exhaling is given approximately by $V(t)=0.45-0.35\cos\frac{\pi t}{2}, 0\le t\le 8$
   a. What is the rate of flow of air t seconds afterexhaling?
   b. What is the rate of flow of air 3 seconds after exhaling? 4 seconds after exhaling? 5 seconds after exhaling?

3.6 Differentiation of Exponential and Logarithmic Functions

The goal of this section is to develop the differential calculus of logarithmic and exponential functions. We shall begin by deriving differentiation formulas for $\ln x$ and e^x. The derived formulas will be applied to a number of differentiation problems and applications.

3.6.1 Derivative of e^x and a^x from first principle

i. Derivative of e^x: If $y=e^x$, then the derivative of $y=e^x$ by first principle rule is:

$$\frac{dy}{dx}=\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x} \qquad \therefore y=f(x)=e^x$$

$$=\lim_{\Delta x\to 0}\frac{e^{(x+\Delta x)}-e^x}{\Delta x}=\lim_{\Delta x\to 0}\frac{e^x e^{\Delta x}-e^x}{\Delta x}$$

$$=\lim_{\Delta x\to 0}\frac{e^x(e^{\Delta x}-1)}{\Delta x}=\lim_{\Delta x\to 0}e^x\lim_{\Delta x\to 0}\frac{e^{\Delta x}-1}{\Delta x}=e^x(1)=e^x \qquad \therefore \lim_{\Delta x\to 0}\frac{e^{\Delta x}-1}{\Delta x}=1$$

ii. Derivative of a^x: If $y=a^x$, then the derivative of $y=a^x$ by first principle rule is:

$$\frac{dy}{dx}=\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x} \qquad \therefore y=f(x)=a^x$$

$$=\lim_{\Delta x\to 0}\frac{a^{(x+\Delta x)}-a^x}{\Delta x}=\lim_{\Delta x\to 0}\frac{a^x a^{\Delta x}-a^x}{\Delta x}$$

$$=\lim_{\Delta x\to 0}\frac{a^x(a^{\Delta x}-1)}{\Delta x}=\lim_{\Delta x\to 0}a^x\lim_{\Delta x\to 0}\frac{a^{\Delta x}-1}{\Delta x}=a^x\log_e a \qquad \therefore \lim_{\Delta x\to 0}\frac{a^{\Delta x}-1}{\Delta x}=\log_e a=\ln a$$

3.6.2 Derivative of $\ln x$ and $\log_a x$ from first principle

i. Derivative of $\ln x$: If $y=\ln x$, then the derivative of $y=\ln x$ by first principle rule is:

$$\frac{dy}{dx}=\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x} \quad \text{(i)} \qquad \therefore y=f(x)=\ln x$$

$$=\lim_{\Delta x\to 0}\frac{\ln(x+\Delta x)-\ln x}{\Delta x}$$

$$=\lim_{\Delta x\to 0}\frac{1}{\Delta x}\ln\left(\frac{x+\Delta x}{x}\right), \quad \textit{Lograithmic-rule}$$

$$=\lim_{\Delta x\to 0}\frac{x}{x}\frac{1}{\Delta x}\ln\left(1+\frac{\Delta x}{x}\right) \quad \textit{multipy and divide out by } x$$

$$=\lim_{\Delta x\to 0}\frac{1}{x}\left(\frac{x}{\Delta x}\right)\ln\left(1+\frac{\Delta x}{x}\right)=\lim_{\Delta x\to 0}\frac{1}{x}\ln\left(1+\frac{\Delta x}{x}\right)^{\frac{x}{\Delta x}}=\frac{1}{x}\ln e=\frac{1}{x} \qquad \because \lim_{s\to 0}(1+x)^{\frac{1}{x}}=e$$

ii. Derivative of $\log_a x$: If $y=\log_a x$, then the derivative of $y=\log_a x$ by first principle rule is:

$$\frac{dy}{dx}=\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x} \qquad \therefore y=f(x)=\log_a x$$

$$=\lim_{\Delta x\to 0}\frac{\log_a(x+\Delta x)-\log_a x}{\Delta x}=\lim_{\Delta x\to 0}\frac{1}{\Delta x}\log_a\left(\frac{x+\Delta x}{x}\right)=\lim_{\Delta x\to 0}\frac{1}{\Delta x}\log_a\left(1+\frac{\Delta x}{x}\right)$$

$$=\lim_{\Delta x\to 0}\frac{x}{x}\frac{1}{\Delta x}\log_a\left(1+\frac{\Delta x}{x}\right), \quad \textit{multiply and divide out by } x$$

$$=\frac{1}{x}\lim_{\Delta x\to 0}\frac{x}{\Delta x}\log_a\left(1+\frac{\Delta x}{x}\right)=\frac{1}{x}\lim_{\Delta x\to 0}\log_a\left(1+\frac{\Delta x}{x}\right)^{\frac{x}{\Delta x}}=\frac{1}{x}\log_a e \qquad \because \lim_{\Delta x\to 0}(1+x)^{\frac{1}{x}}=e$$

These exponential and logarithmic formulas are listed in the box:

i. $\frac{d}{dx}(e^x)=e^x$ *ii.* $\frac{d}{dx}(a^x)=a^x\log_e a$ *iii.* $\frac{d}{dx}(\ln x)=\frac{1}{x}$ *iv.* $\frac{d}{dx}(\log_a x)=\frac{1}{x}\log_a e$

The chain rule can be used to derive the generalization of the power rule and the rules for differentiating the exponential and logarithmic functions, as summarized in the box:

i. $\frac{d}{dx}(e^u)=e^u\frac{d}{dx}(u)$ *ii.* $\frac{d}{dx}(a^u)=a^u\log_e a\frac{d}{dx}(u)$

iii. $\frac{d}{dx}(\ln u)=\frac{1}{u}\frac{d}{dx}(u)$ *iv.* $\frac{d}{dx}(\log_a u)=\frac{1}{u}\log_a e\frac{d}{dx}(u)$

Example 19 Differentiate the following functions:

(a). $f(x)=7^{(4-3x^5)}$ *(b).* $f(x)=\log_{10}\sqrt{x^2-7x}+x^3$ *(c).* $f(x)=\ln(e^{mx}+e^{-mx})$ *(d).* $f(x)=\frac{e^{2x}}{\ln x}$

Solution

a. If the given function is $f(x)=7^{(4-3x^5)}$, then the derivative of the given function w.r.t. x is

$$\frac{dy}{dx}=\frac{d}{dx}\left[7^{(4-3x^5)}\right]$$

Let $u=4-3x^5$ then $\frac{du}{dx}=-15x^4$

Now, $y=7^u \Rightarrow \frac{dy}{du}=\frac{d}{du}(7^u)=7^u.\log_e(7)$

By using chain rule $\frac{dy}{dx}=\frac{dy}{du}.\frac{du}{dx}$

$$\frac{dy}{dx}=7^u.\log_e(7).(-15x^4)\Rightarrow\frac{dy}{dx}=-15x^4.7^{(4-3x^5)}.\log_e(7) \qquad \because u=4-3x^5$$

b. If the given function is $f(x)=\log_{10}\sqrt{(x^2-7x)}+x^3$, then the derivative of a given function w.r.t. x is:

$$\frac{df}{dx}=\frac{d}{dx}\left[\log_{10}\sqrt{(x^2-7x)}+x^3\right]=\frac{d}{dx}[\log_{10}\sqrt{(x^2-7x)}]+\frac{d}{dx}(x^3)=\frac{1}{u}\log_{10}e\frac{d}{dx}(u)+3x^2,\quad u=\sqrt{(x^2-7x)}$$

$$=\frac{1}{x^2-7x}\log_{10}e\frac{d}{dx}(x^2-7x)+3x^2=\frac{2x-7}{x^2-7x}\log_{10}e+3x^2$$

c. If the given function is $f(x)=\ln(e^{mx}+e^{-mx})$, then the derivative of a given function w.r.t. x is:

$$\frac{df}{dx}=\frac{d}{dx}[\ln(e^{mx}+e^{-mx})]=\frac{d}{du}(\ln u)\frac{d}{dx}(e^{mx}+e^{-mx}) \qquad \because u=e^{mx}+e^{-mx}$$

$$=\frac{1}{\ln u}(me^{mx}-me^{-mx})=\frac{1}{\ln(e^{mx}+e^{-mx})}(m)(e^{mx}-e^{-mx})=\frac{m(e^{mx}-e^{-mx})}{\ln(e^{mx}+e^{-mx})}$$

d. If the given function is $f(x)=\frac{e^{2x}}{\ln x}$, then the derivative of a given function w.r.t. x is:

$$\frac{df}{dx}=\frac{d}{dx}\left(\frac{e^{2x}}{\ln x}\right)=\frac{\frac{d}{dx}[e^{2x}].\ln(x)-e^{2x}.\frac{d}{dx}[\ln(x)]}{(\ln x)^2}=\frac{e^{2x}.\frac{d}{dx}(2x).\ln(x)-\frac{e^{2x}}{x}}{(\ln x)^2}$$

$$=\frac{2e^{2x}\ln x-\frac{1}{x}e^{2x}}{(\ln x)^2}=\frac{2xe^{2x}\ln x-e^{2x}}{x(\ln x)^2}=\frac{e^{2x}(2x\ln x-1)}{x(\ln x)^2}$$

3.6.3 Use of logarithmic differentiation to algebraic expressions involving product, quotient and power

Logarithmic differentiation is a procedure in which logarithms are used to trade the task of differentiating products and quotients for that of differentiating sums and differences. It is especially valuable as a means for handling complicated product or quotient functions and power functions where variables appear in both the base and the exponent.

Example 20 Differentiate the following functions

(a). $y=\ln\left[\dfrac{x(x^2-3)^2}{\sqrt{(x^2-4)}}\right]$ (b). $y=x^{\sin x}$

Solution a. $y=\ln\left[\dfrac{x(x^2-3)^2}{\sqrt{(x^2-4)}}\right]$

If the given function after simplification is

$$y=\ln\left[\frac{x(x^2-3)^2}{(x^2-4)^{\frac{1}{2}}}\right]=\ln[x(x^2-3)^2]-\ln[(x^2-4)^{\frac{1}{2}}], \quad \text{logarithms rules}$$

$$=\ln x+\ln(x^2-3)^2-\ln(x^2-4)^{\frac{1}{2}}=\ln x+2\ln(x^2-3)-\frac{1}{2}\ln(x^2-4),$$

then the derivative of y w.r.t. x is

$$\frac{dy}{dx}=\frac{d}{dx}\left[\ln x+2\ln(x^2-3)-\frac{1}{2}\ln\left(x^2-4\right)\right]=\frac{d}{dx}[\ln(x)]+2\frac{d}{dx}[\ln(x^2-3)]-\frac{1}{2}\frac{d}{dx}\ln(x^2-4)$$

$$=\frac{1}{x}+2\frac{1}{x^2-3}(2x)-\frac{1}{2}\frac{1}{x^2-4}(2x)=\frac{4x^4-20x^2+12}{x(x^2-3)(x^2-4)}=\frac{4(x^4-5x^2+3)}{x(x^2-3)(x+2)(x-2)}$$

b. $y=x^{\sin x}$:

$\ln y=\ln(x^{\sin x})$, taking ln of both sides

$\ln y=\sin x\ln x$

then on differentiation w.r.t. x. It becomes;

$$\frac{d}{dx}(\ln y)=\frac{d}{dx}(\sin x\ln x)$$

$$\frac{1}{y}\frac{dy}{dx}=\sin x\frac{d}{dx}(\ln x)+\ln x\frac{d}{dx}(\sin x)$$

$$\frac{1}{y}\frac{dy}{dx}=\frac{\sin x}{x}+\ln x\cos x$$

$$\frac{dy}{dx}=y\left[\frac{\sin x}{x}+\ln x\cos x\right]=x^{\sin(x)}\left[\frac{\sin x}{x}+\ln x\cos x\right]$$

3.7 Differentiation of Hyperbolic and Inverse Hyperbolic Functions

The concept of hyperbolic functions is completely discussed in Unit-2. The differentiation of hyperbolic functions can be found as follows:

3.7.1 Differentiation of the hyperbolic functions

i. Derivative of sinhx: If $y=\sin hx=\frac{e^x-e^{-x}}{2}$, then on differentiation w.r.t. x, it becomes

$$\frac{dy}{dx}=\frac{d}{dx}\left[\frac{e^x-e^{-x}}{2}\right]=\frac{1}{2}\left[\frac{d}{dx}(e^x)-\frac{d}{dx}(e^{-x})\right]=\frac{1}{2}(e^x+e^{-x})=\cos hx$$

ii. Derivative of coshx: If $y=\cos hx=\frac{e^x+e^{-x}}{2}$, then on differentiation w.r.t. x, it becomes

$$\frac{dy}{dx}=\frac{d}{dx}\left[\frac{e^x+e^{-x}}{2}\right]=\frac{1}{2}\left[\frac{d}{dx}(e^x)+\frac{d}{dx}(e^{-x})\right]=\frac{1}{2}(e^x-e^{-x})=\sin hx$$

iii. Derivative of tanhx: If $y=\tan hx=\frac{\sin hx}{\cos hx}$, then on differentiation w.r.t. x through quotient rule, it becomes

$$\frac{dy}{dx}=\frac{\cos hx\frac{d}{dx}(\sin hx)-\sin hx\frac{d}{dx}(\cos hx)}{\cos h^2x} \qquad \because \cos h^2x-\sin h^2x=1$$

$$=\frac{\cos hx\cos hx-\sin hx\sin hx}{\cos h^2x}=\frac{\cos h^2x-\sin h^2x}{\cos h^2x}=\frac{1}{\cos h^2x}=\sec h^2x$$

iv. Derivative of sechx: If $y=\text{sech}\,x=\frac{1}{\cosh x}$, then on differentiation w.r.t. x through quotient rule, it becomes

$$\frac{dy}{dx}=\frac{\cos hx\frac{d}{dx}(1)-(1)\frac{d}{dx}(\cos hx)}{\cos h^2x}=\frac{\cos hx(0)-\sin hx}{\cos h^2x}=\frac{-\sin hx}{\cos h^2x}=-\frac{\sin hx}{\cos hx}\frac{1}{\cos hx}=-\tan hx\sec hx$$

v. Derivative of cosech$x$: If $y=\text{cosech}x=\frac{1}{\sin hx}$, then on differentiation w.r.t. x through quotient rule, it becomes

$$\frac{dy}{dx}=\frac{\sin hx\frac{d}{dx}(1)-(1)\frac{d}{dx}(\sin hx)}{\sin h^2x}=\frac{\sin hx(0)-\cos hx}{\sin h^2x}=\frac{-\cos hx}{\sin h^2x}$$

$$=-\frac{\cos hx}{\sin hx}\frac{1}{\sin hx}=-\cot hx\ \ \cos echx$$

vi. Derivative of cothx: If $y=\cot hx=\frac{\cos hx}{\sin hx}$, then on differentiation w.r.t. x through quotient rule, it becomes

$$\frac{dy}{dx}=\frac{\sinh x\frac{d}{dx}(\cosh x)-\cosh x\frac{d}{dx}(\sinh x)}{\sinh^2x} \qquad \because \cos h^2x-\sin h^2x=1$$

$$=\frac{\sinh x\sinh x-\cosh x\cosh x}{\sinh^2x}=\frac{-(\cosh^2x-\sinh^2x)}{\sinh^2x}=-\frac{1}{\sinh^2x}=-\cos ech^2x$$

The hyperbolic formulas are listed below:

i. $\frac{d}{dx}(\sinh x)=\cosh x$ ii. $\frac{d}{dx}(\cosh x)=\sinh x$ iii. $\frac{d}{dx}(\tanh x)=\sec h^2x$

iv. $\frac{d}{dx}(\text{cosech}x)=-\coth x\ \text{cosech}x$ v. $\frac{d}{dx}(\sec hx)=-\tanh x\sec hx$ vi. $\frac{d}{dx}(\cot x)=-\text{cosec}^2x$

The chain rule can be used to derive the generalization of the power rule and the rules for differentiating the hyperbolic functions, as summarized in the box:

i. $\frac{d}{dx}(\sinh u) = \cosh u \frac{d}{dx}(u)$	ii. $\frac{d}{dx}(\cosh u) = \sinh u \frac{d}{dx}(u)$
iii. $\frac{d}{dx}(\tanh u) = \sec h^2 u \frac{d}{dx}(u)$	iv. $\frac{d}{dx}(\operatorname{cosech} u) = -\coth u \operatorname{cosech} u \frac{d}{dx}(u)$
v. $\frac{d}{dx}(\sec hu) = -\tanh u \sec hu \frac{d}{dx}(u)$	vi. $\frac{d}{dx}(\coth u) = -\operatorname{cosec}^2 u \frac{d}{dx}(u)$

Example 21 Differentiate the following functions: *(a).* $y = \cos h(2x^2 - 1)$ *(b).* $y = \sec h\left(\frac{1-x}{1+x}\right)$

Solution *a.* If the given function is $y = \cos h(2x^2 - 1)$, then the derivative of y w.r.t. x is:

$$\frac{dy}{dx} = \frac{d}{dx}[\cos h(2x^2 - 1)] = \sin h(2x^2 - 1).\frac{d}{dx}(2x^2 - 1)$$

$$= \sin h(2x^2 - 1)\left(2.\frac{d}{dx}[x^2] + \frac{d}{dx}[-1]\right) = \sin h(2x^2 - 1)(2(2x) + 0) = 4x.\sin h(2x^2 - 1)$$

b. If the given function is $y = \sec h\left[\frac{1-x}{1+x}\right]$, then the derivative of y w.r.t. x is:

$$y' = \frac{d}{dx}\left[\sec h\left(\frac{1-x}{1+x}\right)\right] = \frac{d}{du}(\sec hu)\frac{d}{dx}(u) \qquad \because u = \frac{1-x}{1+x}$$

$$= -\tanh u \sec hu \frac{d}{dx}\left(\frac{1-x}{1+x}\right) = -\tan hu \sec hu\left[\frac{(-1)(1+x) - (1-x)(1)}{(1+x)^2}\right]$$

$$= \frac{2}{(1+x)^2}\tan h\left(\frac{1-x}{1+x}\right)\sec h\left(\frac{1-x}{1+x}\right)$$

3.7.2 Differentiation of inverse hyperbolic functions

i. **Derivative of $\sinh^{-1}x$:** If $y = \sinh^{-1}x$, then $x = \sinh y$, the differentiation of $x = \sin h y$ w.r.t y is:

$\frac{dx}{dy} = coshy$ Take its reciprocal to obtain the derivative of y w.r.t. x:

$$\frac{dy}{dx} = \frac{1}{\cos hy} = \pm\frac{1}{\sqrt{1 + \sin h^2 y}} = \pm\frac{1}{\sqrt{1+x^2}} \qquad \therefore \sin hy = x, \quad \cos h^2 y - \sin h^2 y = 1$$

Here, the sign of the radical is the same as that of cos hy which we know is always positive.

Hence, $\frac{d}{dx}(\sin h^{-1}x) = \frac{1}{\sqrt{1+x^2}}$

ii. **Derivative of $\cosh^{-1}x$:** If $y = \cosh^{-1}x$, then $x = \cosh y$,

then the differentiation of $x = \cos h\ y$ w.r.t. y is: $\frac{dx}{dy} = \sin hy$

Take its reciprocal to obtain the derivative of y w.r.t. x:

$$\frac{dy}{dx} = \frac{}{\sin hy} = \pm\frac{1}{\sqrt{\cos h^2 y - 1}} = \pm\frac{1}{\sqrt{x^2 - 1}} \qquad \because \cos hy = x, \ \cos h^2 y \div \sin h^2 y = 1$$

Here, the sign of the radical is the same as that of cos hy which we know is always positive.

Hence, $\frac{d}{dx}(\cos h^{-1}x) = \frac{1}{\sqrt{x^2-1}}$

iii. **Derivative of $\tanh^{-1} x$:** If $y = \tan h^{-1}x$, then $x = \tanh y$,

then the differentiation of $x = \tan h\ y$ w.r.t. y is: $\dfrac{dx}{dy} = \sec h^2 y$

Take its reciprocal to obtain the derivative of y w.r.t. x:

$$\frac{dy}{dx} = \frac{1}{\sec h^2 y} = \frac{1}{1-\tan h^2 y} = \frac{1}{1-x^2},\ \sec h^2 y = 1-\tan h^2 y, |x|<1, \tan hy = x$$

iv. **Derivative of $\operatorname{sech}^{-1}x$:** If $y = \sec h^{-1}x$, then $x = \sec hy$.

The differentiation of $x = \sec hy$ w.r.t. y is: $\dfrac{dx}{dy} = -\sec hy \tan hy$

Take its reciprocal to obtain the derivative of y w.r.t. x:

$$\frac{dy}{dx} = \frac{-1}{\sec hy \tan hy} = \pm\frac{-1}{\sec hy\sqrt{1-\sec h^2 y}} = \pm\frac{-1}{x\sqrt{1-x^2}} \qquad \because 1-\sec h^2 y = \tan h^2, y \sec hy = x$$

Here, the sign of the radical is the same as that of tanhy but we know that $\sec h^{-1}x$ is always positive, so that tanhy is always positive. Hence, $\dfrac{d}{dx}(\sec h^{-1}x) = \dfrac{-1}{x.\sqrt{1-x^2}}$

v. **Derivative of $\operatorname{cosech}^{-1}x$:** If $y = \operatorname{cosech}^{-1}x$, then $x = \operatorname{cosech}y$.

The differentiation of $x = \operatorname{cosec}hy$ w.r.t. y is: $\dfrac{dx}{dy} = -\operatorname{cosec}hy \coth y$

Take its reciprocal to obtain the derivative of y w.r.t. x:

$$\frac{dy}{dx} = \frac{-1}{\operatorname{cosech}y \cot hy} = \pm\frac{-1}{\operatorname{cosech}y\sqrt{\operatorname{cosech}^2 y+1}} \qquad \because \cot h^2 y = \operatorname{cosech}^2 y + 1$$

$$= \pm\frac{-1}{x\sqrt{x^2+1}}, \qquad \operatorname{cosech}y = x$$

Here, the sign of the radical is the same as that of coth$y$ which. Here coth$y$ is positive or negative according as x is positive or negative.

$$\frac{d}{dx}(\operatorname{cosech}^{-1}x) = \frac{-1}{x\sqrt{x^2+1}} \text{ if } x>0 \text{ and } \frac{d}{dx}(\operatorname{cosech}^{-1}x) = \frac{-1}{-x\sqrt{x^2+1}} \text{ if } x<0.$$

Thus $\dfrac{d}{dx}(\operatorname{cosech}^{-1}x) = \dfrac{-1}{|x|\sqrt{x^2+1}}$ for all values of x.

vi. **Derivative of $\coth^{-1}x$:** If $y = \cot h^{-1}x$, then $x = \coth y$.

The differentiation of $x = \cot hy$ w.r.t. y is: $\dfrac{dx}{dy} = -\operatorname{cosech}^2 y$

Take its reciprocal to obtain the derivative of y w.r.t. x:

$$\frac{dy}{dx} = \frac{-1}{\operatorname{cosech}^2 y} = \frac{1}{\cot h^2 y - 1} = \frac{-1}{x^2-1} \qquad \because \operatorname{cosech}^2 y = \cot h^2 y - 1, |x|>1,\ \coth y = x$$

The inverse hyperbolic formulas are listed below:

i. $\frac{d}{dx}(\sin h^{-1}x)=\frac{1}{\sqrt{1+x^2}}$ ii. $\frac{d}{dx}(\cos h^{-1}x)=\frac{1}{\sqrt{x^2-1}}$ iii. $\frac{d}{dx}(\tan h^{-1}x)=\frac{1}{1-x^2}$

iv. $\frac{d}{dx}(\operatorname{cosech}^{-1}x)=\frac{-1}{|x|\sqrt{x^2+1}}$ v. $\frac{d}{dx}(\sec h^{-1}x)=\frac{-1}{x\sqrt{1-x^2}}$ vi. $\frac{d}{dx}(\cot h^{-1}x)=\frac{-1}{x^2-1}$

The chain rule can be used to derive the generalization of the power rule and the rules for differentiating the inverse hyperbolic functions, as summarized in the box:

i. $\frac{d}{dx}(\sin h^{-1}u)=\frac{1}{\sqrt{1+u^2}}\frac{d}{dx}(u)$ ii. $\frac{d}{dx}(\cos h^{-1}u)=\frac{1}{\sqrt{u^2-1}}\frac{d}{dx}(u)$

iii. $\frac{d}{dx}(\tan h^{-1}u)=\frac{1}{1-u^2}\frac{d}{dx}(u)$ iv. $\frac{d}{dx}(\operatorname{cosech}^{-1}u)=\frac{-1}{|u|\sqrt{u^2+1}}\frac{d}{dx}(u)$

v. $\frac{d}{dx}(\sec h^{-1}u)=\frac{-1}{u\sqrt{1-u^2}}\frac{d}{dx}(u)$ v. $\frac{d}{dx}(\cot h^{-1}u)=\frac{-1}{u^2-1}\frac{d}{dx}(u)$

Example 22 Differentiate the following functions: *(a).* $y=\sin h^{-1}(x^3)$ *(b).* $y=\frac{\sin h^{-1}x}{\cos h^{-1}x}$

Solution

a. If the given function is $y=\sin h^{-1}(x^3)$, then the derivative of y w.r.t. x is:

$$\frac{dy}{dx}=\frac{d}{dx}[\sin h^{-1}(x^3)] \qquad \because \frac{d}{dx}[\sin h^{-1}(x)]=\frac{1}{\sqrt{1+x^2}}$$

$$=\frac{1}{\sqrt{1+(x^3)^2}}.\frac{d}{du}(x^3)$$

$$=\frac{3x^2}{\sqrt{1+x^6}}$$

b. If the given function is $y=\frac{\sin h^{-1}(x)}{\cos h^{-1}(x)}$, then the derivative of y w.r.t. x is:

$$\frac{dy}{dx}=\frac{d}{dx}\left(\frac{\sin h^{-1}(x)}{\cos h^{-1}(x)}\right)=\frac{\cos h^{-1}(x).\frac{d}{dx}[\sin h^{-1}(x)]-\sin h^{-1}(x).\frac{d}{dx}[\cos h^{-1}(x)]}{[\cos h^{-1}(x)]^2}$$

$$=\frac{\cos h^{-1}(x)\frac{1}{\sqrt{x^2+1}}-\sin h^{-1}(x).\frac{1}{\sqrt{x^2-1}}}{[\cos h^{-1}(x)]^2}=\frac{\frac{\cos h^{-1}(x)}{\sqrt{x^2+}}-\frac{\sin h^{-1}(x)}{\sqrt{x^2-1}}}{[\cos h^{-1}(x)]^2}$$

$$=\frac{1}{\sqrt{x^2+1}.\cos h^{-1}(x)}-\frac{\sin h^{-1}(x)}{\sqrt{x^2-1}.[\cos h^{-1}(x)]^2}$$

3.8 MAPLE Command 'diff' to differentiate a function

The procedure to use the MAPLE command 'diff' to differentiate a function is illustrated in the following example.

Example 23 Use MAPLE command 'diff' to differentiate

(a). $f(x)=x^5+7x+2$ w.r.t. variable x. *(b).* $f(x)=\frac{(x^4+2x+16)}{(x^3+3x-2)}$ w.r.t. variable x.

(c). $f(x)=(x^3+\sin(x)^2+\arccos x$ w.r.t. variable x.

(d). $f(x)=x^2\cosh x+\operatorname{arcsinh} x$ w.r.t. variable x.

Solution

a. Command:

$> diff(x^5 + 7\cdot x + 2, x);$

$5x^4 + 7$

Context Menu:

$> x^5 + 7\cdot x + 2$

$> diff(x\hat{}5 + 7*x + 2, x)$

$5x^4 + 7$

This result is obtained through right-click on the last end of the expression by selecting " Differentiate < x " on the context menu.

b. Command:

$> diff\left(\dfrac{(x^4 + 2\cdot x + 16)}{(x^3 + 3\cdot x - 2)}, x\right);$

$$\frac{4x^3+2}{x^3+3x-2} - \frac{(x^4+2x+16)(3x^2+3)}{(x^3+3x-2)^2}$$

Context Menu:

$> \dfrac{(x^4 + 2\cdot x + 16)}{(x^3 + 3\cdot x - 2)}$

$> diff((x\hat{}4 + 2*x + 16)/(x\hat{}3 + 3*x - 2), x)$

$$\frac{4x^3+2}{x^3+3x-2} - \frac{(x^4+2x+16)(3x^2+3)}{(x^3+3x-2)^2}$$

c. Command:

$> diff(x^3 + \sin(x)^2 + \arccos(x), x);$

$$3x^2 + 2\sin(x)\cos(x) - \frac{1}{\sqrt{1-x^2}}$$

Context Menu:

$> x\hat{}3 + \sin(x)\hat{}2 + \arccos(x)$

$> diff(x\hat{}3 + \sin(x)\hat{}2 + \arccos(x), x)$

$$3x^2 + 2\sin(x)\cos(x) - \frac{1}{\sqrt{1-x^2}}$$

d. Command:

$> diff(x^2\cdot\cosh(x) + \operatorname{arcsinh}(x), x);$

$$2x\cosh(x) + x^2\sinh(x) + \frac{1}{\sqrt{1+x^2}}$$

Context Menu:

$> x^2\cdot\cosh(x) + \operatorname{arcsinh}(x)$

$> diff(x\hat{}2*\cosh(x) + \operatorname{arcsinh}(x), x)$

$$2x\cosh(x) + x^2\sinh(x) + \frac{1}{\sqrt{1+x^2}}$$

Exercise 3.5

1. Use the rule of first principle to find the derivative of the following functions:

   a. $f(x)= e^{2x}$ b. $f(x)=\dfrac{1}{3}e^{3x}$ c. $f(x)=\dfrac{5}{8}e^{x^2}+1$ d. $f(x)=2^x$

   e. $f(x)= 4^{x+4}$ f. $f(x)= \log(x+1)$ g. $f(x)= \log_a(x^2)$ h. $f(x)= \sinh 2x$

2. Find $f'(x)$ if $f(x)$ is:

   a. $11^{(3-4x^5)}$ b. $e^{\sqrt{x}-5}$ c. $x^5.e^{\frac{1}{x}}$

   d. $\dfrac{e^{2x}}{e^{-2x}+1}$ e. $\ln(e^{mx}-e^{-mx})$ f. $\dfrac{e^{ax}-e^{-ax}}{e^{ax}+e^{-ax}}$

3. Find $\dfrac{dy}{dx}$ by using any suitable rule of differentiation.

   a. $y=x^3.\ln\sqrt{x}$ b. $y=x^2\sqrt{\ln(x)}$ c. $y=\ln\sqrt{\dfrac{x^2+1}{x^2-1}}$

   d. $y=\ln(x-\sqrt{x^2+1})$ e. $y=e^{-2x}.\cos(2x)$ f. $y=x^2.e^{\sin(x)}$

4. Differentiate the following functions.

   a. $y=\log(x+2)^3$ b. $y=\cos h(3x)$ c. $y=\sin h^{-1}(\cos x)$

   d. $y=\tan h^{-1}\left(\dfrac{x}{2}\right)$ e. $y=\ln(\cot h(x))$ f. $y=x.\cos h^{-1}(x)-\sqrt{x^2-1}$

5. A research group (used hospital records) developed the approximate mathematical model related to systolic blood pressure and age is: $p(x) = 40 + 25\ln(x+1),\ 0 \le x \le 65$
where $p(x)$ is the pressure measured in millimeters of mercury and x is age in years. What is the rate of change of pressure at the end of 10 years? at the end of 30 years? at the end of 60 years?

6. A single cholera bacterium divides every 0.5 hour to produce two complete cholera bacteria. If we start with a colony of 5,000 bacteria, then after t hours there will be a $A(t) = 5000\, 2^{2t}$ bacteria. Find $A'(t)$, $A'(1)$ and $A'(5)$. Interpret the results.

7. Use MAPLE command "diff" to differentiate all the functions given in Q.3 and Q.4.

Review Excercise 3

1. **Choose the correct option:**

i. If $f(t) = 2t^2 + 3t + 2$ then $f(-3)$ is:
(a). 9 (b). 11 (c). 21 (d). 29

ii. The average rate of change for $f(x) = x^2 - 6x + 5$ is_______ if x increase at $x \in [1,3]$.
(a). 1 (b). −1 (c). 2 (d). −2

iii. If $y = f(x)$ then $f'(x) =$
(a). $\lim\limits_{\Delta x \to 0} \dfrac{f(x) - f(\Delta x)}{\Delta x}$ (b). $\lim\limits_{\Delta x \to 0} \dfrac{f(x+\Delta x) - f(x)}{\Delta x}$
(c). $\lim\limits_{\Delta x \to 0} \dfrac{f(x-\Delta x) + f(x)}{\Delta x}$ (d). $\lim\limits_{\Delta x \to 0} \dfrac{f(x-\Delta x) + f(x)}{f(x)}$

iv. If $f(t) = 2t^3 - 3t^2 + 4$ then $f'(3)$ is
(a). 54 (b). 45 (c). 36 (d). 27

v. If $f(x) = \dfrac{h(x)}{g(x)}$ then $f'(x) =$
(a). $\dfrac{h'(x) - g'(x)}{[g(x)]^2}$ (b). $\dfrac{g(x).h'(x) - h(x)g'(x)}{[g(x)]^2}$
(c). $\dfrac{f'(x).g(x) + g(x)f'(x)}{[g(x)]^2}$ (d). $\dfrac{f'(x).g'(x) - f'(x)g'(x)}{[g(x)]^2}$

vi. If $h(t) = \sqrt{t^3}$ then $h'(t)$ is:
(a). $\dfrac{3}{2}\sqrt{t}$ (b). $\dfrac{2}{3}\sqrt{t}$ (c). $\dfrac{3t}{2\sqrt{t^2}}$ (d). $\dfrac{3t^2}{2\sqrt{t^3}}$

vii. If $f(x) = x\tan x$ then $f'(x) =$
(a). $\tan x + x\sec^2 x$ (b). $x\tan x - x\sec^2 x$ (c). $\sec^2 x$ (d). $\tan(x) + x[1 + \tan(x)^2]$

viii. $\dfrac{d}{dy}\cos^{-1} y =$
(a). $\dfrac{1}{\sqrt{y^2 - 1}}$ (b). $\dfrac{x}{\sqrt{x^2 - y^2}}$ (c). $\dfrac{-1}{\sqrt{1 - y^2}}$ (d). $\dfrac{x}{\sqrt{1 + y^2}}$

ix. $-\dfrac{1}{|x|\sqrt{x^2 + 1}} = \dfrac{d}{dx}(\underline{\qquad\qquad})$
(a). $\sec^{-1}(x)$ (b). $\cos^{-1}(x)$ (c). $\operatorname{cosec}^{-1}(x)$ (d). $\tan^{-1}(x)$

x. If $f(x) = \ln(x)$ then $f'(x)$
(a). $\dfrac{1}{\ln(x)}$ (b). $\dfrac{2}{x}$ (c). $\dfrac{1}{x^2}$ (d). $\dfrac{1}{x}$

Project

Take a spherical balloon or a ball. It must be inflated.

- Find the general formula for instantaneous rate of change of the volume 'V' w.r.t, radius r. given that $V = \frac{4}{3}\pi r^3$
- Find its rate of change of V, w.r.t, r at the instant when $r = 3$.

Summary

- The **average rate of change** $y = f(x)$ per unit change in x is given by:

$$\frac{\Delta y}{\Delta x} = \frac{f(x+\Delta x) - f(x)}{\Delta x}, \quad y = f(x)$$

- The slope of the secant is the average rate of change which measures **"the approximate rate of change in phenomena."**
- The **instantaneous rate of change** of a function $y f(x)$ at a particular point $P(x, f(x))$ is the **derivative** of a function $y = f(x)$ at that point, $f'(x) = \lim\limits_{\Delta x \to 0} \frac{f(x+\Delta) - f(x)}{\Delta x}$, $y = f(x)$ provided this limit exists. This is named by **first principle rule** of derivative of a function $f(x)$.
- The **tangent line to the graph of a function** $y = f(x)$ at the point $P(x, f(x))$ is the line through this point having slope $f'(x) = \lim\limits_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$, $y = f(x)$ if this limit exists. If this limit does not exist, then there is no tangent at the point.
- The slope of the tangent is the instantaneous rate of change which measures **"the exact rate of change in phenomena."**
- In business terminology,
  - the instantaneous rate of change of cost is the **marginal cost** which counts "as the approximate rate of change in business phenomena".
  - the average rate of change is the exact rate of change which counts "as the actual rate of change in business phenomena"
- For any real number n, if $f(x) = x^n$, then: $f'(x) = nx^{n-1}$
- The **chain rule** is a rule which we use to differentiate the composite function. It is generally written as $\frac{d}{dx}[f(g(x))] = f'[g(x)].g'(x)$

Unit 4

HIGHER ORDER DERIVATIVES AND APPLICATIONS

By the end of this unit, the students will be able to:

4.1 **Higher Order Derivatives**

i. Find higher order derivatives of algebraic, trigonometric, exponential and logarithmic functions.
ii. Find the second derivative of implicit, inverse trigonometric and parametric functions.
iii. Use MAPLE command diff repeatedly to find higher order derivative of function.

4.2 **Maclaurin's and Taylor's Expansions**

i. State Maclaurin's and Taylor's theorems (without remainder terms). Use these theorems to expand $\sin x$, $\cos x$, $\tan x$, a^x, e^x, $\log_a(1+x)$ and $\ln(1+x)$.
ii. Use MAPLE command **taylor** to find Taylor's expansion for a given function.

4.3 **Application of Derivatives**

i. Give geometrical interpretation of derivative.
ii. Find the equation of tangent and normal to the curve at a given point.
iii. Find the angle of intersection of the two curves.
iv. Find the point on a curve where the tangent is parallel to the given line.

4.4 **Maxima and Minima**

i. Define increasing and decreasing functions.
ii. Prove that if $f(x)$ is a differentiable function on the open interval (a,b) then
   - $f(x)$ is increasing on (a,b) if $f'(x)>0, \forall\, x\in(a,b)$,
   - $f(x)$ is decreasing on (a,b) if $f'(x)<0, \forall\, x\in(a,b)$,
iii. Examine a given function for extreme values.
iv. State the second derivative rule to find the extreme values of a function at a point.
v. Use second derivative rule to examine a given function for extreme values.
vi. Solve real life problems related to extreme values.
vii. Use MAPLE command maximize (minimize) to compute maximum (minimum) value of a function.

Introduction

The higher order derivatives has useful physical interpretation. If $y=f(t)$ is the position of an object at time 't' then $\frac{dy}{dt}=f'(t)$ is its velocity at time 't' and $\frac{d^2y}{dt^2}$ is its acceleration at time 't'. According to the Newton's law of motion *"The acceleration of an object is proportional to the total force acting on it"*. So, the second order derivatives has importance in mechanics. The second order derivatives is also important to graph the functions. Now, in this unit we will learn in details about higher order differentiation and its applications.

4.1 Higher order derivatives

If a function $y=f(x)$ has a first derivative y', then the derivative of y', if it exists, is the **second derivative** of $y=f(x)$, written as y''. The derivative of y'', if it exists, is called the **third derivative** of $y=f(x)$, written as y'''. By continuing this process, we can find **fourth derivative** and other **higher derivatives**. For example, if $f(x)=x^4+2x^3+3x^2-5x+7$, then the higher derivatives are the following:

$$y'=f'(x)=\frac{dy}{dx}=4x^3+6x^2+6x-5, \qquad \text{first derivative of } y$$

$$y''=f''(x)=\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)=12x^2+12x+6, \qquad \text{second derivative of } y$$

$$y'''=f'''(x)=\frac{d^3y}{dx^3}=\frac{d}{dx}\left(\frac{d^2y}{dx^2}\right)=24x+12, \qquad \text{third derivative of } y$$

Example 1 Find the second derivative of the following functions:

(a). $f(x)=8x^3-9x^2+6x+4$ (b). $f(x)=\dfrac{4x+2}{3x-1}$

Solution

a. If the given function is $f(x)=8x^3-9x^2+6x+4$, then, the first and second derivatives of the given function through linearity property are the following:

$$f'(x)=\frac{dy}{dx}=\frac{d}{dx}(8x^3-9x^2+6x+4)=24x^2-18x+6$$

$$f''(x)=\frac{d^2y}{dx^2}=\frac{d}{dx}(24x^2-18x+6)=48x-18$$

Remember

The second derivative of $y=f(x)$ can be written with any of the following notations:

$\dfrac{d^2y}{dx^2}$, y'', $f''(x)$, $D^2_x[f(x)]$

The third derivative can be written in a similar way. For derivative $n\geq 4$, the derivative holds the notation $f^{(n)}(x)$, $n=4,5,\ldots$

b. If the given function is $f(x)=y=\dfrac{u}{v}=\dfrac{4x+2}{3x-1}$, then the first and second derivatives of the given function through quotient rule are the following:

$$\frac{dy}{dx}=\frac{d}{dx}\left(\frac{4x+2}{3x-1}\right)=\frac{(3x-1)\frac{d}{dx}(4x+2)-(4x+2)\frac{d}{dx}(3x-1)}{(3x-1)^2} \qquad \therefore \frac{d}{dx}\left(\frac{u}{v}\right)=\frac{\frac{du}{dx}v-\frac{dv}{dx}u}{v^2}$$

$$=\frac{(4)(3x-1)-(3)(4x+2)}{(3x-1)^2}=\frac{-10}{(3x-1)^2}$$

$$f''(x)=\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{(0)(3x-1)^2-(-10)(2)(3x-1)(3)}{(3x-1)^4}=\frac{60(3x-1)}{(3x-1)^4}=\frac{60}{(3x-1)^3}$$

In the previous unit, we saw that the first derivative of a function represents the rate of change of the function. The second derivative, then, represents the rate of change of the first derivative. If a function describes the position of a moving object at time t, then the first derivative gives the **velocity** of the object. That is, if $y=s(t)$ describes the position of the object at time t, then $v(t)=s'(t)$ gives the velocity at a time t.

The rate of change of velocity is called **acceleration**. Since the second derivative gives the rate of change of the first derivative, the acceleration is the derivative of the velocity. Thus, if a(t) represents the acceleration at time t, then

$$a(t)=\frac{dv}{dt}=\frac{d^2s}{dt^2}=s''(t)$$

Example 2 An object is moving along a straight line with its position s(t) (in feet) at time t (in seconds): $s(t)=t^3-2t^2-7t+9$

(a). Find the velocity at any time t. (b). Find the acceleration at any time t.

(c). The object stops when velocity is zero. For $t\geq 0$, when does that occur?

Solution

a. The velocity at any time t is the first derivative of $s(t)$ w.r.t. t: $v=\dfrac{ds}{dt}=3t^2-4t-7$

b. The acceleration at any time t is the first derivative of $v(t)$ w.r.t. t: $a=\dfrac{dv}{dt}=6t-4$

c. Use $v(t)=0$ to obtain the time: $3t^2-4t-7=0$

$$(3t-7)(t+1)=0,\ t=-1,\ \frac{7}{3}$$

The object will stop at $\frac{7}{3}$ seconds, since we want time $t\geq 0$.

4.1.1 Higher order derivatives of algebraic, trigonometric, exponential and logarithmic functions

The successive derivatives of some functions are gathered to obtain the general form of nth derivatives in the following cases:

i. **The nth derivative of $f(x)=(ax+b)^m$**

If $f(x)=(ax+b)^m$, m is positive integer, then the successive derivatives of the given function developed a general term for the nth derivative of a function:

$$f(x)=(ax+b)^m$$

$$f'(x)=ma(ax+b)^{m-1}$$

$$f''(x)=m(m-1)a^2(ax+b)^{m-2}$$

$$\vdots$$

$$f^n(x)=m(m-1)(m-2)...(m-n+1)(a^n)(ax+b)^{m-n} \qquad \text{(i)}$$

$$=\frac{m!}{(m-n)!}a^n(ax+b)^{m-n},\ \textit{if } m \textit{ is positive integer}$$

If $m=-1$, then the nth derivative of $f(x)=\dfrac{1}{(ax+b)}$ is obtained by inserting $m=-1$ in equation (i):

$$f^n(x)=(-1)(-2)(-3)...(-n)(a^n)(ax+b)^{-1-n}=\frac{(-1)^n\ n!a^n}{(ax+b)^{n+1}} \qquad \text{(ii)}$$

ii. **The nth derivative of $f(x)=\ln(ax+b)$**

If $f(x)=\ln(ax+b)$, then the successive derivatives developed a general term for the nth derivative of a function:

$$f(x)=\ln(ax+b)$$

$$f'(x)=\frac{a}{ax+b}$$

$$f''(x)=\frac{(-1)a^2}{(ax+b)^2}$$

$$f'''(x)=\frac{(-1)(-2)a^3}{(ax+b)^3}$$

$$\vdots$$

$$f^n(x)=(-1)(-2(-3)(-4).....(-(n-1))(a^n)(ax+b)^{-n}=\frac{(-1)^{n-1}(n-1)!a^n}{(ax+b)^n} \qquad \text{(iii)}$$

iii. **The nth derivative of $f(x)=a^{mx}$**

If $f(x)=a^{mx}$, then the successive derivatives developed a general term for the nth derivative:

$$f(x)=a^{mx}$$

$$f'(x)=a^{mx}\log a\frac{d}{dx}(mx)=ma^{mx}\log a$$

$$f''(x)=m\log a\frac{d}{dx}(a^{mx})=m\log a(a^{mx})\log a\frac{d}{dx}(mx) \qquad \text{(iv)}$$

$$= m^2 a^{mx} (\log a)^2$$

$$\vdots$$

$$f^n(x) = m^n a^{mx} (\log a)^n$$

If $a = e$, then the nth derivative of $f(x) = e^{mx}$ is obtained by inserting $a = e$:

$$f(x) = e^{mx}$$

$$f'(x) = me^{mx}$$

$$f''(x) = m(m)e^{mx} = m^2 e^{mx}$$

$$\vdots$$

$$f^n(x) = m^n e^{mx} \qquad \text{(v)}$$

iv. The nth derivative of $f(x) = \sin(ax + b)$:

If $f(x) = \sin(ax + b)$, then the successive derivatives developed a general term for the nth derivative of a function:

$$f(x) = \sin(ax + b)$$

$$f'(x) = a\cos(ax + b) = a\sin\left(ax + b + \frac{\pi}{2}\right)$$

$$f''(x) = a^2\cos\left(ax + b + \frac{\pi}{2}\right) = a^2\sin\left(ax + b + \frac{2\pi}{2}\right)$$

$$f'''(x) = a^3\cos\left(ax + b + \frac{2\pi}{2}\right) = a^3\sin\left(ax + b + \frac{3\pi}{2}\right)$$

$$\vdots$$

$$f^n(x) = a^n \sin\left[ax + b + \frac{n\pi}{2}\right] \qquad \text{(vi)}$$

v. The nth derivative of $f(x) = \cos(ax + b)$:

If $f(x) = \cos(ax + b)$, then the successive derivatives developed a general term for the n–th derivative:

$$f(x) = \cos(ax + b)$$

$$f'(x) = -a\sin(ax + b) = a\cos\left(ax + b + \frac{\pi}{2}\right)$$

$$f''(x) = -a^2\sin\left(ax + b + \frac{\pi}{2}\right) = a^2\cos\left(ax + b + \frac{2\pi}{2}\right)$$

$$f'''(x) = -a^3\sin\left(ax + b + \frac{2\pi}{2}\right) = a^3\cos\left(ax + b + \frac{3\pi}{2}\right)$$

$$\vdots$$

$$f^n(x) = a^n \cos\left[ax + b + \frac{n\pi}{2}\right] \qquad \text{(vii)}$$

Example 3 Find the 5th derivatives of the following functions:

(a). $f(x) = (6x + 4)^9$ (b). $f(x) = \dfrac{1}{4x + 3}$ (c). $f(x) = \ln(4x + 7)$

(d). $f(x) = 6^{4x}$ (e). $f(x) = e^{4x}$ (f). $f(x) = \sin(5x + 7)$

Solution

a. If $f(x)=(6x+4)^9$ with $a=6$, $b=4$ and $m=9$, then the 5^{th} derivative of the given function is obtained by inserting $n=5$ in equation: $f^n(x)=\dfrac{m!}{(m-n)!}a^n(ax+b)^{m-n}$

$$f^5(x)=\frac{9!}{(9-5)!}6^5(6x+4)^{9-5}=\frac{6^5 9!}{4!}(6x+4)^4=6^5(9)(8)(7)(6)(5)(6x+4)^4=6^5\left(15120\left(6x+4\right)^4\right)$$

b. If $f(x)=\dfrac{1}{(4x+3)}$ with $a=4$ and $b=3$, then the 5^{th} derivative of the given function is obtained by inserting n=5 in equation :

$$f^n(x)=\frac{(-1)^n n!a^n}{(ax+b)^{n+1}}$$

$$f^5(x)=\frac{(-1)^5 5!4^5}{(4x+3)^6}=\frac{-4^5 5!}{(4x+3)^6}$$

c. If $f(x)=\ln(4x+7)$ with $a=4$ and $b=7$, then the 5^{th} derivative of the given function is obtained by inserting $n=5$ in equation:

$$f^n(x)=\frac{(-1)^{n-1}(n-1)!a^n}{(ax+b)^n}$$

$$f^5(x)=\frac{(-1)^4 4!4^5}{(4x+7)^5}=\frac{4^5 4!}{(4x+7)^5}$$

d. If $f(x)=6^{4x}$ with $a=6$ and $m=4$, then the 5^{th} derivative of the given function is obtained by inserting $n=5$ in equation:

$$f^n(x)=m^n a^{mx}(\log a)^n$$

$$f^5(x)=4^5 6^{4x}(\log 6)^5$$

e. If $f(x)=e^{4x}$ with $m=4$, then the 5^{th} derivative of the given function is obtained by inserting $n=5$ in equation:

$$f^n(x)=m^n e^{mx}$$

$$f^5(x)=4^5 e^{4x}$$

f. If $f(x)=\sin(5x+7)$ with $a=5$ and $b=7$ then the 5th derivative of the given function is obtained by inserting $n=5$ in equation:

$$f^n(x)=a^n\sin\left[ax+b+\frac{n\pi}{2}\right]$$

$$f^5(x)=5^5\sin\left[5x+7+\frac{5\pi}{2}\right]$$

4.1.2 Second derivative of implicit, inverse trigonometric and parametric functions

Example 4 Find the second derivative of $x^2y+2y^3=3x+2y$.

Solution The equation is $x^2y+2y^3=3x+2y$ (i)

The first implicit derivative of (i) w.r.t. x is:

$$\frac{d}{dx}\left(x^2y+2y^3\right)=\frac{d}{dx}(3x+2y)$$

$$\frac{d}{dx}(x^2y)+\frac{d}{dx}(2y^3)=\frac{d}{dx}(3x)+\frac{d}{dx}(2y)$$

$$2xy+x^2\frac{dy}{dx}+6y^2\frac{dy}{dx}=3+2\frac{dy}{dx}$$

$$(x^2+6y^2-2)\frac{dy}{dx}=3-2xy \qquad \text{(ii)}$$

The second implicit derivative of first implicit derivative (ii) w.r.t. x is: $\frac{d}{dx}\left[(x^2+6y^2-2)\frac{dy}{dx}\right]=\frac{d}{dx}(3-2xy)$

$$\frac{d}{dx}[(x^2+6y^2-2)]\frac{dy}{dx}+(x^2+6y^2-2)\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d}{dx}(3)-\frac{d}{dx}(2xy)$$

$$(2x+12y\frac{dy}{dx}-0)\frac{dy}{dx}+(x^2+6y^2-2)\frac{d^2y}{dx^2}=0-2y-2x\frac{dy}{dx}$$

$$(2x+12y\frac{dy}{dx})\frac{dy}{dx}+(x^2+6y^2-2)\frac{d^2y}{dx^2}=-2y-2x\frac{dy}{dx}$$

$$\left(x^2+6y^2-2\right)\frac{d^2y}{dx^2}=-2y-(2x+2x)\frac{dy}{dx}-12y\left(\frac{dy}{dx}\right)^2$$

$$\frac{d^2y}{dx^2}=\frac{-2\left(y+2x\frac{dy}{dx}+6y\left(\frac{dy}{dx}\right)^2\right)}{\left(x^2+6y^2-2\right)}=\frac{-2\left(y+2x\left(\frac{3-2xy}{x^2+6y^2-2}\right)+6y\left(\frac{3-2xy}{x^2+6y^2-2}\right)^2\right)}{x^2+6y^2-2} \quad \text{Putting values of } \frac{dy}{dx}$$

$$=\frac{-2\left(y+\frac{6x-4x^2y}{x^2+6y^2-2}+6y\left(\frac{9+4x^2y^2-12xy}{(x^2+6y^2-2)^2}\right)\right)}{x^2+6y^2-2}$$

$$=\frac{-2\left(y+\frac{6x-4x^2y}{x^2+6y^2-2}+\frac{54y+24x^2y^3-72xy^2}{(x^2+6y^2-2)^2}\right)}{(x^2+6y^2-2)}$$

$$=\frac{\frac{-2(y(x^2+6y^2-2)^2+6x-4x^2y(x^2+6y^2-2)+54y+24x^2y^3-72xy^2}{(x^2+6y^2-2)^2}}{(x^2+6y^2-2)}$$

$$=\frac{-2(y(x^4+36y^4+4+12x^2y^2-24y^2-4x^2)+6x^3+36y^2x-12x-4x^4y-24x^2y^3+8x^2y+54y+24x^2y^3-72xy^2}{(x^2+6y^2-2)^3}$$

$$=\frac{-2(x^4y+36y^5+4y+12x^2y^3-24y^3-4x^2y+6x^3+36y^2x-12x-4x^4y-24x^2y^3+8x^2y+54y+24x^2y^3-72xy^2}{(x^2+6y^2-2)^3}$$

$$\frac{d^2y}{dx^2}=-\frac{2(36y^5+12y^3x^2-24y^3-36y^2x-3yx^4+4yx^2+58y+6x^3-12x)}{(6y^2+x^2-2)^3}$$

Example 5 Find the second derivative of $\cos^{-1}y+y=2xy$.

Solution The given equation is $\cos^{-1}y+y=2xy \qquad \text{(i)}$

The first implicit derivative of (i) w.r.t. x is:

$$\frac{d}{dx}\left(\cos^{-1}y+y\right)=\frac{d}{dx}(2xy)$$

$$\frac{d}{dx}(\cos^{-1}y)+\frac{d}{dx}(y)=2\frac{d}{dx}(xy)$$

$$\frac{-1}{\sqrt{1-y^2}}\frac{dy}{dx}+\frac{dy}{dx}=2\left(\frac{d}{dx}(x)y+x\frac{d}{dx}(y)\right)$$

$$\left(\frac{-1}{\sqrt{1-y^2}}+1\right)\frac{dy}{dx}=2y+2x\frac{dy}{dx} \qquad \text{(ii)}$$

Remember

Equation (ii) can be written in simplified form for 1[st] order derivative as

$$\frac{dy}{dx}=-\frac{2y\sqrt{1-y^2}}{1-\sqrt{1-y^2}+2x\sqrt{1-y^2}}$$

The second implicit derivative of first implicit derivative (ii) w.r.t. x is:

$$\frac{d}{dx}\left[\left(\frac{-1}{\sqrt{1-y^2}}+1\right)\frac{dy}{dx}\right]=\frac{d}{dx}\left(2y+2x\frac{dy}{dx}\right)$$

$$\frac{d}{dx}\left(\frac{-1}{\sqrt{1-y^2}}+1\right)\frac{dy}{dx}+\left(\frac{-1}{\sqrt{1-y^2}}+1\right)\frac{d^2y}{dx^2}=2\frac{dy}{dx}+2\frac{dy}{dx}+2x\frac{d^2y}{dx^2}$$

$$\left(\frac{-2y}{2(1-y^2)^{\frac{3}{2}}}+0\right)\frac{dy}{dx}+\left(\frac{-1}{\sqrt{1-y^2}}+1\right)\frac{d^2y}{dx^2}=4\frac{dy}{dx}+2x\frac{d^2y}{dx^2}$$

Equating coefficients of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ to obtain: $\left(\frac{-1}{\sqrt{1-y^2}}+1-2x\right)\frac{d^2y}{dx^2}=\left(\frac{y}{(1-y^2)^{\frac{3}{2}}}+4\right)\frac{dy}{dx}$

$$\frac{d^2y}{dx^2}=\frac{\left(\frac{y}{(1-y^2)^{\frac{3}{2}}}+4\right)\frac{dy}{dx}}{\left(\frac{-1}{\sqrt{1-y^2}}+1-2x\right)}=\frac{\left(\frac{y}{(1-y^2)^{\frac{3}{2}}}+4\right)\left(\frac{2y\sqrt{1-y^2}}{1-\sqrt{1-y^2}+2x\sqrt{1-y^2}}\right)}{\left(\frac{-1}{\sqrt{1-y^2}}+1-2x\right)}$$

$$\Rightarrow\frac{d^2y}{dx^2}=\frac{2y\left(y+4(-y^2+1)^{\frac{3}{2}}\right)}{\left(2x\sqrt{1-y^2}+1-\sqrt{1-y^2}\right)\left(-1+\sqrt{1-y^2}-2x\sqrt{1-y^2}\right)\left(\sqrt{1-y^2}\right)}$$

Example 6 Find the second derivative $\frac{d^2y}{dx^2}$, when the parametric functions are:

$$x(t)=1+t^2,\ y(t)=t^3+2t^2+1$$

Solution The first derivative of the parametric functions $x=x(t)$ and $y=y(t)$ w.r.t. x is:

$$\frac{dy}{dx}=\frac{dy}{dt}\frac{dt}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}} \qquad \text{(i)}$$

The second derivative of the parametric functions is obtained by taking the derivative of equation (i): $\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right)=\frac{d}{dx}\left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right)\frac{dt}{dt}$, Multiply and divide it by dt

$$= \frac{d}{dt}\left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right)\frac{dt}{dx} \qquad \text{(ii)}$$

The quotient rule of differentiation is used to simplify the right hand side of equation (ii):

$$\frac{d}{dt}\left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right) = \frac{\frac{dx}{dt}\frac{d}{dt}\left(\frac{dy}{dt}\right) - \frac{dy}{dt}\frac{d}{dt}\left(\frac{dx}{dt}\right)}{\left(\frac{dx}{dt}\right)^2} = \frac{\frac{dx}{dt}\frac{d^2y}{dt^2} - \frac{dy}{dt}\frac{d^2x}{dt^2}}{\left(\frac{dx}{dt}\right)^2} \qquad \text{(iii)}$$

Use (ii) in (iii) to obtain the general term for second derivative of parametric functions $x(t)$ and $y(t)$:

$$\frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right)\frac{dt}{dx} = \frac{\frac{dx}{dt}\frac{d^2y}{dt^2} - \frac{dy}{dt}\frac{d^2x}{dt^2}}{\left(\frac{dx}{dt}\right)^2}\frac{dt}{dx} = \frac{\frac{dx}{dt}\frac{d^2y}{dt^2} - \frac{dy}{dt}\frac{d^2x}{dt^2}}{\left(\frac{dx}{dt}\right)^3} \qquad \text{(iv)} \qquad \therefore \text{ replace } \frac{dt}{dx} = \frac{1}{\frac{dx}{dt}}$$

In light of result (iv), the first and second derivatives of the parametric functions $x(t) = 1 + t^2, y(t) = t^3 + 2t^2 + 1$ with $\frac{dx}{dt} = 2t, \frac{d^2x}{dt^2} = 2, \frac{dy}{dt} = 3t^2 + 4t, \frac{d^2y}{dt^2} = 6t + 4$

are the following: $\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{3t^2+4t}{2t} = \frac{t(3t+4)}{2t} = \frac{3t+4}{2}$

$$\frac{d^2y}{dx^2} = \frac{\frac{dx}{dt}\frac{d^2y}{dt^2} - \frac{dy}{dt}\frac{d^2x}{dt^2}}{\left(\frac{dx}{dt}\right)^3} = \frac{(2t)(6t+4) - (3t^2+4t)(2)}{8t^3} = \frac{12t^2 + 8t - 6t^2 - 8t}{8t^3} = \frac{6t^2}{8t^3} = \frac{3}{4t}$$

4.1.3 MAPLE command diff repeatedly to find higher order derivative of a function

The procedure to use of MAPLE command diff is illustrated in the following example.

Example 7 Differentiate $f(x) = x^4 + x^2 \sin x + x + 2$ w.r.t. 'x'

Solution

Command

```
> diff(x^4 + x^2·sin(x) + x + 2, x, x);
```

$$12x^2 + 2\sin(x) + 4x\cos(x) - x^2\sin(x)$$

For second derivative, after command, press the "Enter" key two times to obtain the second derivative of a given function above result.

Exercise 4.1

1. Find the indicated higher derivatives of the following functions:
   a. $f(x)=3x^3+4x+5,\ f''(x)$
   b. $f(x)=x+\dfrac{1}{x},\ f'''(x)$
   c. $s(t)=\sqrt{5t+7},\ s''(t)$
   d. $y=\dfrac{x+1}{x-1},\ y''$
2. Use implicit rule to find out the second derivative of the following functions:
   a. $b^2x^2+a^2y^2=a^2b^2$
   b. $x^2+y^2=r^2$
   c. $y^2-2xy=0$
   d. $e^x+x=e^y+y$
3. Use parametric differentiation to find out $\dfrac{d^2y}{dx^2}$ for the following parametric functions $x\ (t)$ and $y\ (t)$:
   a. $x=4t^2+1,\ y=6t^3+1$
   b. $x=3at^2+2,\ y=6t^4+9$
   c. $x=a\cos 2t,\ y=b\sin 2t$
   d. $x=\dfrac{3at}{1+t^3},\ y=\dfrac{3at^2}{1+t^3}$
4. Find the indicated higher order derivative of the following function.
   a. $f(x)=(x^3+4x-5)^4,\ f^{iv}(x)$
   b. $f(x)=\tan^2(x),\ f^{iii}(x)$
   c. $f(x)=\dfrac{1}{2}e^{9x},\ f^{iii}(x)$
   d. $f(x)=x^4.\ln|x^2|,\ f^{iv}(x)$
5. Use MAPLE command "diff" to find the indicated higher order derivative of the following functions.
   a. $f(x)=\sqrt{\sec(2x)},\ f'''(x)$
   b. $f(x)=\sin(\sin x),\ f'''(x)$
6. Find the indicated derivative of the following by using rule.
   a. $y=(3x+7)^{11}$, 7^{th} derivative
   b. $f(x)=\ln(2x-4)$, 10^{th} derivative
   c. $g(x)=4\cos(3x+8)$, 6^{th} derivative
   d. $h(x)=7e^{5x+4}$, 12^{th} derivative

Project

By using the chain rule and other differential rules, some of the derivative computations can be radius to perform. For complicated derivatives mathematicians, scientists and engineers use computer softwares. Such as mathematica, maple and Matlab, use computer software to compute.

- $\dfrac{d}{dx}\left[\dfrac{(x^2+4)^{10}\sin^5(\sqrt{x})}{\sqrt{1+\cos(x)}}\right]$
- $\dfrac{d}{dx}\left[\dfrac{\sqrt{1+\csc(x)}}{(x^2+4)^{10}\sin^5(\sqrt{x})}\right]$

Although we have all mathematical tools to compute above type of problems by hand. But the computation involving computer software may be more efficient.

4.2 Maclaurin's and Taylor's Expansions

Often the value of a function and the values of its derivatives are known at a particular point and from this information it is desired to obtain values of the function around that particular point. The Taylor polynomials and Taylor series allow us to make such estimates.

4.2.1 Maclaurin's and Taylor's theorems. Using these theorems to expand sinx, cos x, tan x, a^x, e^x, loga(1+x) and ln(1+x)

A. Taylor's Theorem

If $f(x)$ and its n derivatives at $x = x_0$ are $f'(x_0)$, $f''(x_0)$,..., $f^n(x_0)$, then the nth order Taylor polynomial $p_n(x)$ may be written as:

$$p_n(x) = f(x_0) + (x - x_0)f'(x_0) + \frac{(x - x_0)^2}{2!}f''(x_0) + ... + \frac{(x - x_0)^n}{n!}f^n(x_0) \qquad \text{(i)}$$

This polynomial provides an approximation to $f(x)$. The polynomial and its n derivatives are very much matched with the values of $f(x)$ and its first n derivatives evaluated at $x = x_0$:

$$p_n(x_0) = f(x_0),\ p_n'(x_0) = f'(x_0),\ p_n''(x_0) = f''(x_0),\ .....,\ p_n^n(x_0) = f_n^n(x_0)$$

Example 8 The function $y = f(x) = e^x$ and its derivatives evaluated at $x_0 = 0$ are known by $f(0) = 1$, $f^{i}(0) = 1$, $f^{ii}(0) = 1$, $f^{iii}(0) = 1$, $f^{iv}(0) = 1$. Use fourth order Taylor polynomial about $x_0 = 0$ to estimate $f(0.2)$ at $x = 0.2$.

Solution The fourth order Taylor polynomial $p_4(x)$ is obtained by terminating the Taylor polynomial (i) after fourth order derivative term:

$$p_4(x) = f(x_0) + (x - x_0)f^{i}(x_0) + \frac{(x - x_0)^2}{2!}f^{ii}(x_0) + \frac{(x - x_0)^3}{3!}f^{iii}(x_0) + \frac{(x - x_0)^4}{4!}f^{iv}(x_0) \qquad \text{(ii)}$$

Insert $x_0 = 0$ in (ii) to obtain:

$$p_4(x) = f(0) + f^{i}(0)x + f^{ii}(0)\frac{x^2}{2!} + f^{iii}(0)\frac{x^3}{3!} + f^{iv}(0)\frac{x^4}{4!}$$

$$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} \qquad \text{(iii)}$$

The Taylor polynomial (iii) is used to obtain approximation of a function $y = f(x) = e^x$ at $x = 0.2$:

$$p_4(x) = 1 + x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!}$$

$$p_4(0.2) = 1 + 0.2 + \frac{(0.2)^2}{2!} + \frac{(0.2)^3}{3!} + \frac{(0.2)^4}{4!}$$

$$= 1 + 0.2 + 0.02 + 0.00133 + 0.00007 = 1.2214$$

Notice that the Taylor polynomial approximation equals the actual function value $y = f(0.2) = e^{0.2} = 1.2214$ at $x = 0.2$.

Remember

Taylor's and Maclaurin's theorems are also known as Taylor's and Maclaurin's series.

Remember

Taylor's Series: The Taylor polynomials have been used to estimate the values of $y = f(x)$ at various x values. It is reasonable to ask:

i. How accurate Taylor polynomials generated by $y = f(x)$ at x_0 to approximate $y = f(x)$ at values of x other than x_0?

ii. If more and more terms are used in the Taylor polynomial, then this will produce a better and better approximation to $y=f(x)$.

To answer these questions, we introduce the Taylor series. As more and more terms are included in the Taylor polynomial, we obtain an infinite series, known as a **Taylor series**:

$$p(x)=f(x_0)+(x-x_0)f'(x_0)+\frac{(x-x_0)^2}{2!}f''(x_0)+......+\frac{(x-x_0)^n}{n!}f^n(x_0) \quad (i)$$

For some Taylor series, the value of the series equals the value of the function for every value of x. That is, the Taylor series approximations of e^x, $\sin x$ and $\cos x$ equal the values of e^x, $\sin x$ and $\cos x$ for every value of x. However, some functions have a Taylor series which equals the function only for a limited range of x values. For example, the value of a function $f(x)=\frac{1}{(1+x)}$ which equals its Taylor series only when $-1<x<1$.

B. Maclaurin's Series

A special case of a Taylor series occurs, when the function $y=f(x)$ is known only at the origin $x_0=0$. This special condition imposed on Taylor series, develops the **Maclaurin's series**:

$$p(x)=f(0)+xf'(0)+\frac{(x)^2}{2!}f''(0)+...+\frac{(x)^n}{n!}f^n(0) \qquad (i)$$

The Taylor and Maclaurin's series of $y=f(x)$ about a particular point x_0 are of course:

$$f(x)=f(x_0)+(x-x_0)f'(x_0)+\frac{(x-x_0)^2}{2!}f''(x_0)+\frac{(x-x_0)^3}{3!}f'''(x_0)+....+\frac{(x-x_0)^n}{n!}f^n(x_0) \quad (ii)$$

$$f(x)=f(0)+xf'(0)+\frac{(x)^2}{2!}f''(0)+\frac{(x)^3}{3!}f'''(0)......+\frac{(x)^n}{n!}f^n(0) \qquad (iii)$$

If we use $x-x_0=h$, then equations (ii) and (iii) take the popular notation for the Taylor and Maclaurin's series of order n:

$$f(x_0+h)=f(x_0)+hf'(x_0)+\frac{h^2}{2!}f''(x_0)+\frac{h^3}{3!}f'''(x_0)+....+\frac{h^n}{n!}f^n(x_0) \quad (iv)$$

$$f(x_0+h)=f(0)+hf'(0)+\frac{h^2}{2!}f''(0)+\frac{h^3}{3!}f'''(0)+....+\frac{h^n}{n!}f^n(0) \quad (v)$$

The graphical view of a function $y=f(x)$ at $x=x_0$ is shown in the Figure 4.1.

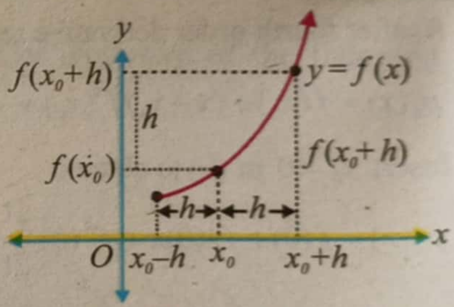


Figure 4.1

Note

The popular notation for the Taylor & Maclaurin's series of order *n* are:

i. $$f(x_0+h)=f(x_0)+hf'(x_0)+\frac{h^2}{2!}f''(x_0)+\frac{h^3}{3!}f'''(x_0)+..\frac{h^n}{n!}f^n(x_0)$$

ii. $$f(x_0+h)=f(0)+hf'(0)+\frac{h^2}{2!}f''(0)+\frac{h^3}{3!}f'''(0)+..\frac{h^n}{n!}f^n(0)$$

If a function $y=f(x)$ is known at a particular point $x_0\neq 0$, then the Taylor series (iv) at a forward or backward point $x=x_0\pm h$ of a function $y=f(x)$ are:

$$f(x_0+h)=f(x_0)+hf'(x_0)+\frac{h^2}{2!}f''(x_0)+\frac{h^3}{3!}f'''(x_0)+... \qquad x=x_0+h$$

$$f(x_0-h)=f(x_0)-hf'(x_0)+\frac{h^2}{2!}f''(x_0)-\frac{h^3}{3!}f'''(x_0)+.... \qquad x=x_0-h$$

Now, look at the following examples the procedure to the use of Taylor and Maclaurin's Theorem is illustrated in these examples.

Example 9 Use Taylor's series to approximate the value of a function $f(x)=e^x$ at a point $x_0=2$.

Solution The function and its derivatives at $x_0=2$

$f(x)=e^x$, $f(2)=e^2=7.3891$, $f'(x)=e^x$, $f'(2)=e^2=7.3891$, $f''(x)=e^x$, $f'''(2)=e^2=7.3891$

are used in Taylor series (ii) to obtain the Taylor series approximation of e^x at a point $x_0=2$:

$$e^x=f(2)+(x-2)f'(2)+\frac{(x-2)^2}{2!}f''(2)+\frac{(x-2)^3}{3!}f'''(2)+......$$

$$=7.3891+7.3891(x-2)+7.3891\frac{(x-2)^2}{2!}+7.3891\frac{(x-2)^3}{3!}+......$$

D. Maclaurin's theorem for the functions of the type $f(x)=a^x$

Example 10 Use Maclaurin's series to approximate the value of a function $f(x)=a^x$ at a point $x_0=0$.

Solution The function and its derivatives at $x_0=0$

$f(x)=a^x$, $f(0)=1$, $f'(x)=a^x\log_e a$, $f'(0)=\log_e a$

$f''(x)=a^x(\log_e a)^2$, $f''(0)=(\log_e a)^2$

are used in Maclaurin series (iii) to obtain the Maclaurin series approximation of a^x at a point $x_0=0$:

$$a^x=f(0)+x\,f'(0)+\frac{(x)^2}{2!}f''0)+\frac{(x)^3}{3!}f'''(0)+.....$$

$$=1+x\log_e a+\frac{x^2}{2!}(\log_e a)^2+\frac{x^3}{3!}(\log_e a)^3+.....$$

E. Maclaurin's theorem for the functions of the type $f(x)=e^x$

Example 11 Use Maclaurin's series to approximate the value of a function $f(x)=e^x$ at a point $x_0=0$.

Solution The function and its derivatives at $x_0=0$

$f(x)=e^x$, $f(0)=1$, $f'(x)=e^x$, $f'(0)=1$, $f''(x)=e^x$; $f'''(0)=1$

are used in Maclaurin series (iii) to obtain the Maclaurin's series approximation of e^x at a point $x_0=0$:

$$e^x=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+...$$

F. Maclaurin's theorem for the function of the type $f(x)=\sin(x)$

Example 12 Use Maclaurin's series to approximate the value of a function $f(x)=\sin(x)$ at a point $x_0=0$.

Solution The function and its derivatives at $x_0=0$

$f(x)=\sin x$, $f(0)=\sin(0)=0$, $f'(x)=\cos x$, $f'(0)=\cos(0)=1$

$f''(x)=-\sin x$, $f''(0)=-\sin(0)=0$, $f'''(x)=-\cos x$, $f'''(0)=-\cos(0)=-1$

are used in Maclaurin series (iii) to obtain the Maclaurin series approximation of $\sin x$ at a point $x_0=0$:

$$\sin x=f(0)+(x)f'(0)+\frac{(x)^2}{2!}f''(0)+\frac{(x)^3}{3!}f'''(0)+...$$

$$=0+x-0-\frac{(x)^3}{3!}+...=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+...$$

G. Maclaurin's theorem for the function of the type $f(x)=\cos(x)$

Example 13 Use Maclaurin's series to approximate the value of a function $f(x)=\cos x$ at a point $x_0=0$.

Solution The function and its derivatives at $x_0=0$.

$$f(x)=\cos x,\ f(0)=\cos 0=1,\quad f'(x)=-\sin x,\ f'(0)=-\sin 0=0$$
$$f''(x)=-\cos x,\ f''(0)=-\cos 0=-1,\quad f'''(x)=\sin x,\ f'''(0)=\sin 0=0$$

are used in Maclaurin series (iii) to obtain the Maclaurin series approximation of a function $\cos x$ at a point $x_0=0$: $\cos x = f(0)+xf'(0)+\frac{x^2}{2!}f''(0)f'''(0)+...$

$$=1+x(0)+\frac{x^2}{2!}(-1)+\frac{x^3}{3!}(0)+...=1-\frac{x^2}{2!}+\frac{x^4}{4!}+...$$

H. Maclaurin's theorem for the function of the type $f(x)=\tan(x)$

Example 14 Use Maclaurin's series to approximate the value of a function $f(x)=\tan x$ at a point $x_0=0$.

Solution The function and its derivatives at $x_0=0$

$$f(x)=\tan x,\ f(0)=\tan 0=0,\ f'(x)=\sec^2 x,\ f'(0)=\sec^2 0=1$$
$$f''(x)=2\sec^2 x\tan x,\ f''(0)=2(1)(0)=0,\ f'''(x)=2\sec^4 x+4\tan^2 x\sec^2 x$$
$$f'''(0)=2\sec^4 0+4\tan^2 0\sec^2 0=2$$

are used in Maclaurin's (iii) to obtain the Maclaurin's series approximation of a function $\tan x$ at a point $x_0=0$: $\tan x=f(0)+x f'(0)+\frac{x^2}{2!}f''(0)+\frac{x^3}{3!}f'''(0)+...$

$$=0+x(1)+\frac{x^2}{2!}(1)+\frac{x^3}{3!}(2)........=x+2\frac{x^3}{3!}+...$$
$$=x+\frac{x^2}{2}+\frac{x^3}{3}+.........$$

I. Maclaurin's theorem for the function of the type $f(x)=\log_a(1+x)$

Example 15 Use Maclaurin's series to approximate the value of a function $f(x)=\log_a(1+x)$ at a point $x_0=2$.

Solution The function and its derivatives at $x_0=0$

$$f(x)=\log_a(1+x),\ f(0)=\log_a(1)=0$$
$$f'(x)=\frac{1}{1+x}\log_a e,\ f'(0)=\log_a e=\log_a e$$
$$f''(x)=-\frac{1}{(1+x)^2}\log_a e,\ f''(0)=-\log_a e$$
$$f'''(x)=\frac{2}{(1+x)^3}\log_a e,\ f'''(0)=2\log_a e$$

are used in Maclaurin's series (iii) to obtain the Maclaurin's series approximation of a function $f(x)=\log_a(1+x)$ at a point $x_0=0$:

$$\log_a(1+x)=f(0)+(x)f'(0)+\frac{(x)^2}{2!}f''(0)+\frac{(x)^3}{3!}f'''(0)+...$$
$$=0+x\log_a e-\frac{(x)^2}{2!}\log_a e+2\frac{x^3}{3!}\log_a e-...$$
$$=x\log_a e-\frac{x^2}{2!}\log_a e+\frac{2x^3}{3!}\log_a e-...$$

$$\log_a(1+x)=x\log_a e-\frac{x^2}{2}\log_a e+\frac{x^3}{3}\log_a e-...$$

J. Maclaurin's theorem for the function of the type $f(x)=\ln(1+x)$

Example 16 Use Maclaurin's series to approximate the value of a function $f(x)=\ln(1+x)$ at a point $x_0=0$.

Solution The function and its derivatives at $x_0=0$

$$f(x)=\ln(1+x),\ f(0)=\ln(1)=0,\quad f'(x)=\frac{1}{(1+x)},\ f'(0)=1$$

$$f''(x)=-\frac{1}{(1+x)^2},\ f''(0)=-1,\quad f'''(x)=\frac{2}{(1+x)^3},\ f'''(0)=2$$

are used in Maclaurin's series (iii) to obtain the Maclaurin's series approximation of $\ln(1+x)$ at a point $x_0=0$: $\ln(1+x)=f(0)+x f'(0)+\frac{x^2}{2!}f''(0)+\frac{x^3}{3!}f'''(0)+...$

$$=0+x(1)+\frac{x^2}{2!}(-1)+\frac{x^3}{3!}(2)+...=x-\frac{x^2}{2!}+2\frac{x^3}{3!}-...=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+...$$

Challenge

Use Taylor's theorem to compute the series of the following functions at $x_0=3$.

i. $f(x)=\sin x$ ii. $f(x)=\cos x$ iii. $f(x)=\tan x$ iv. $f(x)=a^x$
v. $f(x)=e^x$ vi. $f(x)=\log_a(1+x)$ vii. $f(x)=\ln(1+x)$

4.2.2 MAPLE command "Taylor" to find Taylor's expansion for a given function

The use of MAPLE command 'Taylor' is illustrated in the following example.

Example 17 Use Maple command taylor for the function

(a). $f(x)=e^x$ by Taylor's series expansion to first four terms.
(b). $f(x)=sinx$ by Taylor's series expansion to first 5 terms.

Solution

a. Command:

> *taylor*$(e^x, x=0, 4)$;

$$1+\ln(e)\,x+\frac{1}{2}\ln(e)^2x^2+\frac{1}{6}\ln(e)^3x^3+0(x^4)$$

Context Menu:

> e^x
> *series*(e^x,x,4)

$$1+\ln(e)\,x+\frac{1}{2}\ln(e)^2x^2+\frac{1}{6}\ln(e)^3x^3+0(x^4)$$

This result is obtained through right click on the last end of the expression by selecting "Series $< x$" on the context menu.

b. Command:

> *taylor*$(\sin(x), x=0, 5)$;

$$x-\frac{1}{6}x^3+0(x^5)$$

Context Menu:

> $\sin(x)$
> *series*($\sin(x), x, 5$)

$$x-\frac{1}{6}x^3+0(x^5)$$

4.3 Application of Derivatives

In this section, we will see how to use derivatives to determine the tangent, and normal lines, the angles in between two curves, the maximum and minimum values of a function as well as the intervals where the function is increasing or decreasing.

4.3.1 Geometrical interpretation of derivative

Consider a function $y=f(x)$ as shown in the Figure 4.2.

Let $P(x_0, y_0)$ be a point on a curve $y=f(x)$.

The change Δx in x develops a change Δy in y.

The coordinates of a point Q are therefore $Q(x_0+\Delta x, y_0+\Delta y)$. Notice that the slope of the secant line PQ is:

$$\frac{\Delta y}{\Delta x}=\frac{f(x_0+\Delta x)-f(x_0)}{\Delta x} \qquad \text{(i)}$$

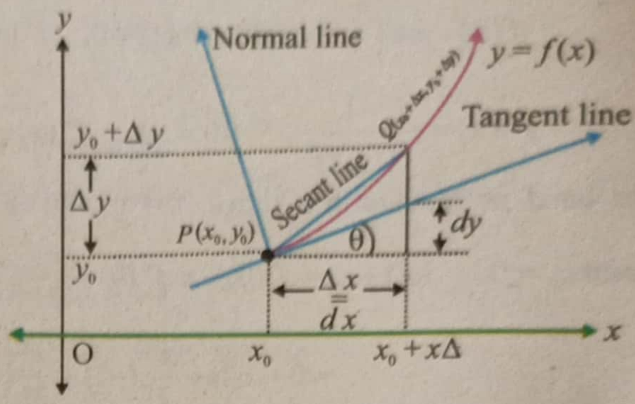


Figure 4.2

If we take values of Q closer to P, then Q approaches P, and Δx approaches 0 and the slope of the secant line PQ automatically approaches the slope of the tangent line at a particular point P and is denoted by:

$$\frac{dy}{dx}=\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=\lim_{\Delta x\to 0}\frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}=f'(x_0) \qquad \text{(ii)}$$

4.3.2 Equations of tangent and normal to the curve at a given point

If the slope of the tangent line on a curve $y=f(x)$ at a particular point $P(x_0, y_0)$ is $f'(x_0)$, then the tangent line on this curve at a particular point $P(x_0, y_0)$ is the nonhomogeneous line (developed from the definition of the point form of the straight line):

$$y-y_0=f'(x_0)(x-x_0)$$

$$y-y_0=m(x-x_0), \qquad m=f'(x_0) \qquad \text{(i)}$$

The normal line is the line perpendicular to the tangent line on this curve at a particular point $P(x_0, y_0)$ with slope $\dfrac{-1}{f'(x_0)}$:

$$(y-y_0)=\frac{-1}{f'(x_0)}(x-x_0)$$

$$(y-y_0)=-\frac{1}{m}(x-x_0), \qquad m=f'(x_0) \qquad \text{(ii)}$$

Example 18 Find the equations of the tangent and normal lines on a curve $y=x^2$ at a point $P(2,4)$.

Solution If the given curve is $y=x^2$, then, the slope of the tangent line is the first derivative of the given curve at a particular point $P(2,4)$:

$$f'(x)=2x$$

$$f(2)=2(2)=4=m, \textit{ say, at a point } P(2,4)$$

The tangent line (i) on the given curve at a particular point P(2,4) is:

$$y-y_0=m(x-x_0)$$

$$(y-4)=4(x-2)$$

$$-4x+y-4+8=0 \Rightarrow -4x+y+4=0 \Rightarrow 4x-y-4=0$$

The normal line (ii) on the given curve at a particular point P(2, 4) is:

$$(y-y_0)=\frac{-1}{m}(x-x_0)$$

$$(y-4)=\frac{-1}{4}(x-2)$$

$$4(y-4)=-(x-2)$$

$$x+4y-16-2=0 \Rightarrow x+4y-18=0$$

Example 19 Find the equations of the tangent and normal lines on the curve $y=9-x^2$ at a point, when y crosses the x– axis.

Solution The coordinates of a particular point P at which the given curve $y=9-x^2$ crosses the x– axis are $y=0$

Put y=0 in $y=9-x^2$ to obtain a set of points: $0=y=9-x^2 \Rightarrow x^2=9 \Rightarrow x=\pm 3 \Rightarrow (3,0), (-3,0)$

If the given curve is $y=9-x^2$, then, the slope of the tangent line is the first derivative of the given curve at a particular point $P(\pm 3,0)$:

$$f'(x)=-2x$$

$$f'(3)=-2(3)=-6=m, \quad at\ a\ point\ P(+3,0)$$

$$f'(-3)=-2(-3)=6=m, \quad at\ a\ point\ P(-3,0)$$

The tangent lines (i) on the given curve at the particular points are:

$$(y-0)=-6(x-3), \quad m=-6, \quad P(3,0)$$

$$6x+y-18=0$$

$$(y-0)=6(x+3), \quad m=6, \quad P(-3,0)$$

$$6x-y+18=0$$

The normal lines (ii) on the given curve at the particular points are:

$$(y-0)=\frac{-1}{-6}(x-3), \quad m=-6, \ P(3,0)$$

$$6y=x-3$$

$$x-6y-3=0$$

$$(y-0)=\frac{-1}{6}(x+3), \quad m=6, \ P(-3,0)$$

$$6y=-x-3 \Rightarrow x+6y+3=0$$

Remember

i. **The tangent equation at a point** $P(x_0, y_0)$ is $(y-y_0)=m(x-x_0)$.

ii. **The normal equation at a point** $P(x_0, y_0)$ is $(y-y_0)=\frac{-1}{m}(x-x_0)$.

4.3.3 Angle of intersection of the two curves

If m_1 is the slope of the first curve and m_2 is the slope of the second curve, then the angle of intersection in between these two curves at a point of intersection is the angle in between their tangents at that point. This angle takes the notation: $\tan\theta=\frac{m_1-m_2}{1+m_1m_2}$ (i)

Example 20 Find the angle of intersection in between the curves $y=x^3-2x+1$ and $y=x^2+1$ at the point of intersection (2, 5).

Solution The required angle of intersection in between the given two curves is: $\tan\theta=\frac{m_1-m_2}{1+m_1m_2}$ (i)

For point of intersection, solve the system of nonlinear equations for the unknowns x and y:

$$y=x^3-2x+1,\ y=x^2+1 \qquad \text{(ii)}$$

Using first equation of the nonlinear system (ii) in second equation to obtain:

$$x^3-2x+1=x^2+1$$

$$x^3-x^2-2x=0$$

$$x(x^2-x-2)=0 \Rightarrow x=0,-1,2$$

The set of x values is used in first equation of the nonlinear system (ii) to obtain a set of y values:

Put $x=0$ to obtain $y=x^3-2x+1=1$

Put $x=-1$ to obtain $y=x^3-2x+1=-1+2+1=2$

Put $x=2$ to obtain $y=x^3-2x+1=8-4+1=5$

This process developed a set of points of intersection: $(0,1)$, $(-1,2)$, $(2,5)$.

The slope of the first curve at a point (2, 5) is: $\frac{dy}{dx}=3x^2-2 \Rightarrow \left(\frac{dy}{dx}\right)_{(2,5)}=3(2)^2-2=10=m_1$, *say*

The slope of the second curve at a point (2, 5) is: $\frac{dy}{dx}=2x \Rightarrow \left(\frac{dy}{dx}\right)_{(2,5)}=2(2)=4=m_2$, *say*

The slopes m_1 and m_2 are used in (i) to obtain the angle of intersection in between the given two curves:

$$\tan\theta=\frac{m_1-m_2}{1+m_1m_2}=\frac{10-4}{1+(10)(4)}=\frac{6}{41}$$

$$\theta=\tan^{-1}\frac{6}{41}=0.1453$$

4.3.4 Point on a curve where the tangent is parallel to the given line

Look at the following example the procedure to find the point on a curve where tangent is parallel to the given line is illustrated in this example.

Example 21 Find all the points on the curve $y=2x^3+4x^2$ where tangent line is parallel to the line $y=8x-4$

Solution Since the given line is $y=8x-4$

Slope of the given line $=8$

Given curve $=y=2x^3+4x^2$ (i)

$$\frac{dy}{dx}=6x^2+8x$$

$$8=6x^2+8x$$

$$3x^2+4x-4=0$$

$$3x^2+6x-2x-4=0$$

$$3x(x+2)-2(x+2)=0$$

$$(x+2)(3x-2)=0$$

$$\Rightarrow x+2=0,\ 3x-2=0$$

$$x=-2 \ and \ x=+\frac{2}{3}$$

$y=2x^3+4x^2$ *when* $x=-2$

Then $y=2(-2)^3+4(-2)^2=-16+16=0$

$$y=2\left(\frac{2}{3}\right)^3+4\left(\frac{2}{3}\right)^2 \quad \text{when } x=\frac{2}{3}$$

$$=2\left(\frac{8}{27}\right)+4\left(\frac{4}{9}\right)$$

$$=\frac{16}{27}+\frac{16}{9}=\frac{16+48}{27}=\frac{64}{27}$$

$$y=0,\frac{64}{27}$$

Exercise 4.2

1. In each case, find the equation of the tangent line to the curve at the indicated value of x:
   a. $y=\sqrt{x+1},\ x=3$ b. $y=\sin(2x+\pi),\ x=0.$
   c. $y=x^2e^{-x},\ x=1$ d. $y=\dfrac{x}{x^2+1},\ x=1$
2. In each case, find the equation of normal to the curve at the indicated value of x:
   a. $y=xe^x,\ x=1$ b. $y=(2x+1)^6,\ x=0$
   c. $y=\cos(x-\pi),\ x=\dfrac{\pi}{2}$ d. $y=x^3\ln x,\ x=1$
3. a. Find an equation of the tangent line to the curve $x^2+y^2=13$ at $(-2,3)$.
   b. Find an equation of the tangent line to the curve $\sin(x-y)=xy$ at $(0,\pi)$.
   c. Find an equation of the normal line to the curve $x^2+2xy=y^3$ at $(1,-1)$.
   d. Find an equation of the normal line to the curve $x^2\sqrt{y-2}=y^2-3x-1$ at $(1,2)$.
4. a. Show that the first four terms in the Taylor series expansion of $f(x)=\tan x$ about $x=\dfrac{\pi}{4}$ are: $1+2\left(x-\dfrac{\pi}{4}\right)+2\left(x-\dfrac{\pi}{4}\right)^2+\dfrac{8}{3}\left(x-\dfrac{\pi}{4}\right)^3$
   b. Show that the first four terms in the Taylor series expansion of $f(x)=\sqrt{x}$ about $x=4$ are: $2+\dfrac{1}{4}(x-4)-\dfrac{1}{64}(x-4)^2+\dfrac{1}{512}(x-4)^3$
   c. Show that the first four terms in the Taylor series expansion of $f(x)=x+e^x$ about $x=1$ are: $(1+e)x+e\left[\dfrac{(x-1)^2}{2!}+\dfrac{(x-1)^3}{3!}+\dfrac{(x-1)^4}{4!}+\dots\right]$
5. Find the Maclaurin series expansion for the following functions:
   a. $f(x)=\dfrac{1}{1+x}$ b. $f(x)=\sin^2 x$ c. $f(x)=\cosh x$ d. $f(x)=\ln(1-4x)$
6. a. Use the Maclaurin series for e^x to show that the sum of the infinite series $1+\dfrac{1}{1!}+\dfrac{1}{2!}+\dfrac{1}{3!}+\dots$ is e.
   b. Use part (a) to find out the value of e that must be accurate to 4 decimal places.
7. Find the angle of intersection between the following curves:
   a. $x^2-y^2=a^2,\ x^2+y^2=a^2\sqrt{2}$ b. $y^2=ax,\ x^3+y^3=3axy$
8. Find the points on the curve $y=5x^3-4x^2$ where tangent line is parallel to the line $y=5x-3$.

History

B.Taylor was a British mathematician who is known by his invention of Taylor's theorem and the Taylor's series. In 1708 he obtained the solution of the problem of the "centre of oscillation" and published on 1714. Calculus of finite differences add to the branch of higher mathematics in 1715 with the name "Methodus Incrementorum Directa et Inversa". This word contain the well known layrange realized its importance and termed it as the main foundation of differential calculus.

Brook Taylor (1685–1731)

4.4 Maxima and Minima

Always the maximum and minimum values of a function can be read from its graphical view. For a quadratic function (whose graph is parabola), the maximum or minimum values can be determined without graphing by finding the vertex algebraically. For functions whose graphs are not known, other techniques are needed. In this unit, we shall see how to use derivatives to determine the maximum and minimum values of a function as well as the intervals where the function is increasing or decreasing.

4.4.1 Increasing and decreasing functions

Suppose an ecologist has determined the size of a population of a certain species as a function $f(t)$ of time t (months). If it turns out that the population is increasing until the end of the first year and decreasing thereafter. It is reasonable to expect the population to be maximized at time $t=12$ and for the population curve to have a high point at $t=12$ as shown in the Figure 4.3.

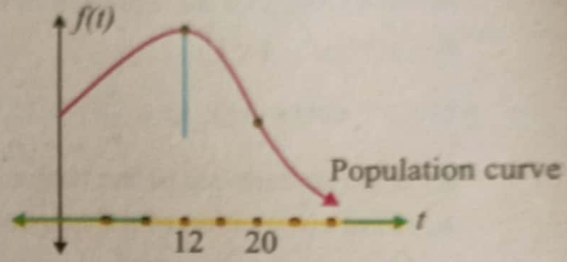


Figure 4.3

If the graph of a function $f(t)$, such as this population curve, is rising throughout the interval $0 < t < 12$, then we say that $f(t)$ is strictly increasing on that interval. Similarly, the graph of the function in Figure 4.3 is strictly decreasing on the interval $12 < t < 20$. These terms are defined more formally in the Figure 4.4.

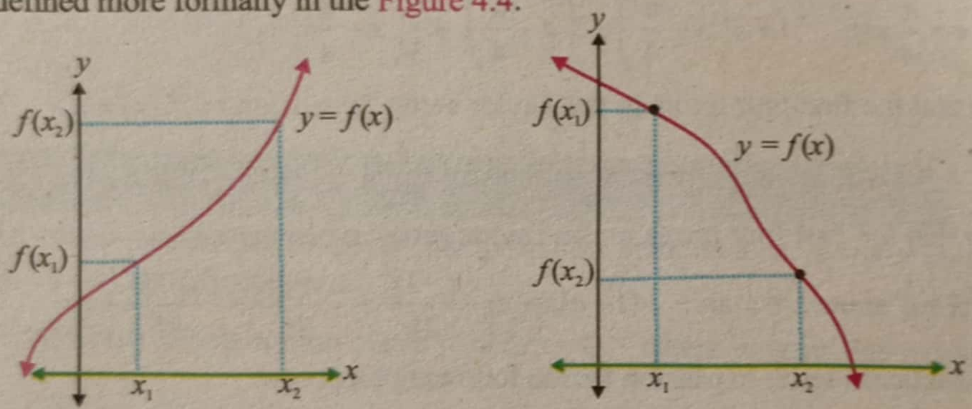


Figure 4.4

- The function $f(x)$ is **strictly increasing** on an interval (a, b), if $f(x_1) < f(x_2)$, whenever $x_1 < x_2$ for x_1 and x_2 on (a, b).
- The function $f(x)$ is **strictly decreasing** on an interval (a, b), if $f(x_1) > f(x_2)$, whenever $x_1 < x_2$ for x_1 and x_2 on (a, b).

Example 22 Find the intervals at which the function $f(x)=x^2$ is increasing or decreasing.

Solution The function $f(x)=x^2$ is a parabola passing through the origin. Take any two points x_1 and x_2 in the interval (a, b) for which: $f(x_2)-f(x_1)=x_2^2-x_1^2=(x_2-x_1)(x_2+x_1)$

If $x_1, x_2 \in (0,\infty)$ with condition $x_2 > x_1$, then the function $f(x)$ is increasing in the interval $(0,\infty)$:

$$f(x_2)-f(x_1)>0$$

$$f(x_2)>f(x_1),\quad \text{both } (x_2-x_1) \text{ and } (x_2+x_1) \text{ are } +ve, \text{ when } x_2>x_1$$

If $x_1, x_2 \in (-\infty,0)$ with condition $x_2 > x_1$, then the function $f(x)$ is decreasing in the interval $(-\infty,0)$:

$$f(x_2)-f(x_1)<0$$

$$f(x_2)<f(x_1),\quad (x_2-x_1) \text{ is } +ve \text{ while } (x_2+x_1) \text{ is } -ve, \text{ when } x_2>x_1$$

4.4.2 Prove that if $f(x)$ is a differentiable function on the open interval (a, b) then

- $f(x)$ is increasing on (a, b) if $f'(x) > 0,\ \forall x \in (a,b)$
- $f(x)$ is decreasing on (a, b) if $f'(x) < 0,\ \forall x \in (a,b)$

Proof: Let $x_1, x_2 \in (a,b)$ such that $x_2 > x_1$ then there exist a point c between x_1 and x_2 such that

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

$$(x_2 - x_1) f'(c) = f(x_2) - f(x_1)$$

For $f'(c) > 0$ and so, $x_2 - x_1$

Therefore, $\quad f(x_2) - f(x_1) > 0$ if $x_2 > x_1$

Or $\quad f(x_2) > f(x_1)$ if $x_2 > x_1$

Thus, f is an increasing function.

> **Remember**
>
> If a function f is continuous on [a, b] and differentiable on (a, b) then there exist a point $c \in$ (a, b) such that
>
> $$\frac{f(b) - f(a)}{b - a} = f'(c)$$

Similarly, the proof of part (ii) can be done which is left as an exercise for the reader.

Example 23 Determine the values of x at which the function $f(x) = x^2 + 2x - 3$ is increasing or decreasing. Also find the point at which the given function is neither increasing nor decreasing.

Solution For graphical view, the given function through completing square

$$f(x) = x^2 + 2x - 3 = x^2 + 2x + 1 - 1 - 3 = (x+1)^2 - 4$$

is compared with the general equation of parabola $f(x) = a(x-h)^2 + k$ to obtain a parabola with vertex (–1,–4) that opens upward (a =1is positive). The graph of a parabola through the points (–4,5) and (2,5) is shown in the Figure 4.5.

The derivative of a given function with respect to x is the slope of the parabola: $f'(x) = 2x + 2$

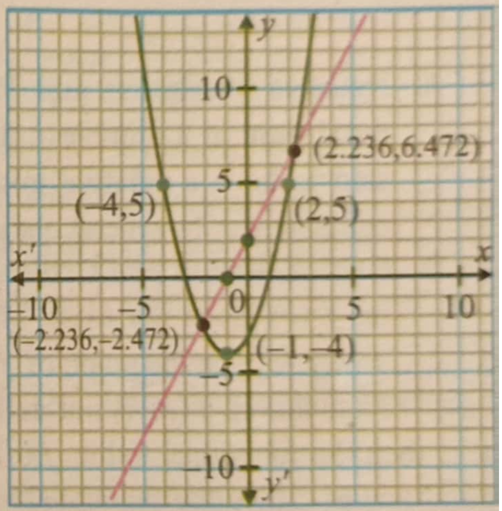


Figure 4.5

If the slope of parabola is $f'(x) > 0$ (positive), then it gives

$$f'(x) > 0$$

$$2x + 2 > 0 \quad \Rightarrow \quad 2x > -2 \quad \Rightarrow \quad x > -1$$

This shows that the given function $f(x)$ is increasing in the interval $(-1, \infty)$.

If the slope of parabola is $f'(x) < 0$ (negative), then it gives

$$f'(x) < 0$$

$$2x + 2 < 0 \quad \Rightarrow \quad 2x < -2 \quad \Rightarrow \quad x < -1$$

This shows that the given function f(x) is decreasing in the interval $(-\infty, -1)$.

If the slope of parabola is $f'(x) = 0$ (zero), then it gives

$$f'(x) = 0 \Rightarrow 2x + 2 = 0 \Rightarrow 2x = -2 \Rightarrow x = -1$$

This shows that the given function $f(x)$ is neither increasing nor decreasing at a vertex (–1,–4).

4.4.3 Examination of a given function for extreme values

Typically the extrema of a continuous function occur either at endpoints of the interval or at points where the graph has a "peak" or a "valley" (points where the graph is higher or lower than all nearby points). For example, the function $f(x)$ in Figure 4.6 has "peaks" at B and D and "valleys" at C and E. Peaks and valleys are what we call the **relative extrema.**

The exact location of a relative maximum or minimum rather than a graphic's approximation can normally be found by using derivatives. The concept developed is as under:

Let $f(x)$ be a function as a roller coaster track with a roller coaster car moving from left to right along the graph in the Figure 4.6. As the car moves up towards a peak, its floor tilts upward. At the instant the car reaches the peak, its floor is level, but then it begins to tilt downward as the car rolls down toward a valley. At any point along the graph, the floor of the car (a straight-line segment in the figure) represents the tangent line to the graph at that point. Using this analogy, we see that as the car passes through the peaks and valleys at A, B, C, the tangent line is horizontal and has slope 0. At peak D and valley E, however, a real roller coaster car would have trouble. It would fly off the track at peak D and be unable to make the 90^0 change of direction at valley E. There is no tangent line at D or E, because of the sharp corners.

Thus, the points where a peak or a valley occurs have this property: the tangent line is horizontal and has slope 0 there or no tangent line is defined there. The slope of the tangent line to the graph of the function $f(x)$ at a point $P(x, f(x))$ is the value of the derivative $f'(x)$.

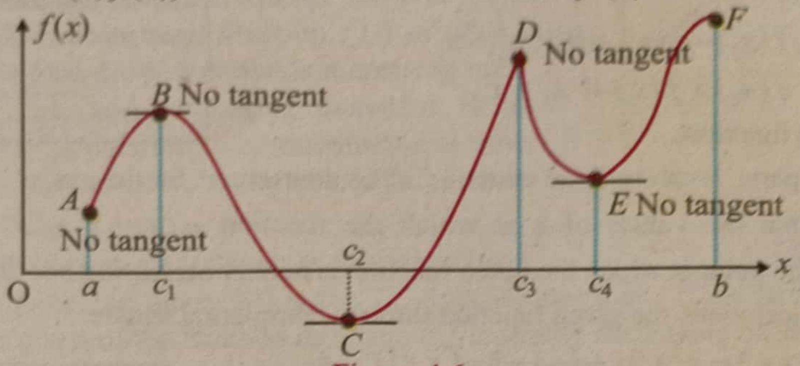


Figure 4.6

A. Relative Maximum and Relative Minimum: The function $f(x)$ is said to have a **relative maximum** at a number c if $f(c) \geq f(x)$ for all x in an open interval containing c. Also, $f(x)$ is said to have a **relative minimum** at a number d if $f(d) \leq f(x)$ for all x in an open interval containing d. In general, the relative maxima and relative minima are called **relative extrema**.

B. Critical Values and Critical Point: Suppose $f(x)$ is defined at a number c and either $f'(c)=0$ or $f'(c)$ does not exist. Then the number c is called a **critical value** of $f(x)$ and the point $P(c, f(c))$ on the graph of $f(x)$ is called a **critical point**.

Note that if $f(c)$ is not defined, then c cannot be a critical value. If there is a relative maximum at c, then the functional value $f(c)$ at that point is the maximum value. Similarly, if there is a relative minimum at c, then the functional value $f(c)$ at that point is the minimum value.

Example 24 Find the critical values for the following functions:

(a). $f(x)=4x^3-5x^2-8x+20$ (b). $f(x)=\dfrac{x^2}{x-2}$ (c). $f(x)=12x^{\frac{1}{2}}-2x^{\frac{3}{2}}$

(d). $f(x)=3x^{\frac{4}{3}}-12x^{\frac{1}{3}}$ (e). $f(x)=6x^{\frac{2}{3}}-4x$

Solution

a. The first derivative of the given function is: $f'(x)=12x^2-10x-8$

$f'(x)=12x^2-10x-8$ is defined for all values of x. Set $f'(x)=0$ to obtain the critical values:

$f'(x)=12x^2-10x-8=0 \Rightarrow 2(3x-4)(2x+1)=0 \Rightarrow x=\dfrac{4}{3}, \dfrac{-1}{2}$

b. The first derivative of the given function is: $f'(x)=\dfrac{x(x-4)}{(x-2)^2}$

The derivative is not defined at $x=2$, also the original function $f(x)$ is not defined at $x=2$. So $x=2$ is not a critical value. Set $f'(x)=0$ to obtain the other critical values:

$f'(x)=\dfrac{x(x-4)}{(x-2)^2}=0 \Rightarrow x(x-4)=0 \Rightarrow x=0,4$

c. The first derivative of the given function is: $f'(x)=6x^{\frac{-1}{2}}-3x^{\frac{1}{2}}$

The derivative is not defined at $x=0$, but the original function $f(x)$ at $x=0$ is $f(0)=12(0)^{\frac{1}{2}}-2(0)^{\frac{3}{2}}=0$ defined. So $x=0$ is a critical value. For other critical values, set $f'(x)=0$ to obtain: $f'(x)=6x^{\frac{-1}{2}}-3x^{\frac{1}{2}}=0 \Rightarrow 3x^{\frac{-1}{2}}(2-x)=0 \Rightarrow 2-x=0 \Rightarrow x=2$

Thus, the critical values are $x=0, 2$.

d. The derivative of a given function is: $f(x)=3x^{\frac{4}{3}}-12x^{\frac{1}{3}}$

$$f'(x)=3\left(\frac{4}{3}\right)x^{\frac{4}{3}-1}-12\left(\frac{1}{3}\right)x^{\frac{1}{3}-1}=4x^{\frac{1}{3}}-4x^{\frac{-2}{3}}=4x^{\frac{1}{3}}-\frac{4}{x^{\frac{2}{3}}}=\frac{4x-4}{x^{\frac{2}{3}}}$$

The derivative fails to exist when $x=0$, but the original function $f(x)$ is defined when $x=0$. So $x=0$ is a critical value of $f(x)$.

If $x \neq 0$, then $f'(x)$ is going to be 0 only, when the numerator $4x-4=0$ is zero for $x=1$. So $x=1$ is also the critical value of $f(x)$. Thus, the critical values of $f(x)$ are 0 and 1.

e. The derivative of a given function is: $f(x)=6x^{\frac{2}{3}}-4x$

$$f'(x)=6\left(\frac{2}{3}\right)x^{\frac{2}{3}-1}-4=4x^{\frac{1}{3}}-4=\frac{-4-4x^{\frac{1}{3}}}{x^{\frac{1}{3}}}$$

The derivative fails to exist when $x=0$, but the original function $f(x)$ is defined when $x=0$. So $x=0$ is a critical value of $f(x)$. If $x \neq 0$, then $f'(x)$ is going to be 0 only when the numerator $4-4x^{\frac{1}{3}}=0$ is zero for $x=1$. So $x=1$ is the critical value of $f(x)$. Thus, the critical values of $f(x)$ are 0 and 1.

Theorem 4.1: If a continuous function $f(x)$ has a relative extremum at c, then c must be a critical value of $f(x)$.

Example 25 The function $f(x)$ is defined by $f(x)=x^3-3x^2-9x+1$. Determine the intervals at which the function $f(x)$ is strictly increasing or decreasing.

Solution First, we need to find out the derivative of the given function, which is: $f'(x)=3x^2-6x-9$

For critical values, set $f'(x)=0$ to obtain: $3x^2-6x-9=0 \Rightarrow 3(x+1)(x-3)=0 \Rightarrow x=-1, 3$

These critical values divide the x– *axis* into three parts, as shown in the Figure 4.7. Next, we select a typical number from each of these intervals. For example, we select –2, 0 and 4, evaluate the derivative at these values and mark each interval as increasing or decreasing, according to whether the derivative is positive or negative respectively. This is shown in Figure 4.7.

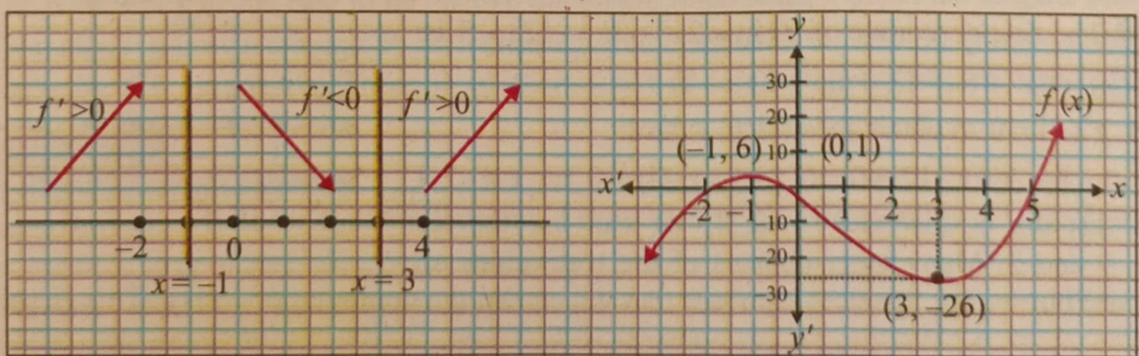


Figure 4.7

Thus, the function $f(x)$ increases in the intervals for $x<-1$ and $x>3$, but decreases in the interval $-1<x<3$.

Example 26 Draw the function $f(x)=x^3-3x^2-9x+1$ and its derivative $f'(x)=3x^2-6x-9$. Use these graphs to tell about the following questions:

(a). When $f'(x)$ is positive, what does that mean in terms of the graph of $f(x)$?

(b). When the graph of $f(x)$ is decreasing, what does that mean in terms of the graph of $f'(x)$?

Solution The graphs of $f(x)=x^3-3x^2-9x+1$ and $f'(x)=3x^2-6x-9$ are shown in Figure 4.8.

These graphs develop the idea that the critical values of $f(x)$ are always intercepts for the graph of $f'(x)=3x^2-6x-9$:

- If $f'(x)$ is positive, then $f(x)$ is increasing.
- If $f'(x)$ is negative, then $f(x)$ is decreasing.

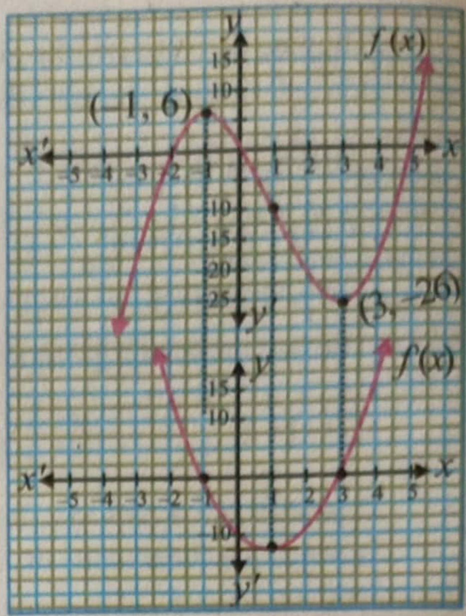


Figure 4.8

4.4.4 State the second derivatives rule to find the extreme values of a function at a point

The first derivative of a function can be used to determine whether the function is increasing or decreasing on a given interval. We shall use this information to develop a procedure called the **first derivative test** for classifying a given point as a relative maximum, a relative minimum, or neither.

The steps involved in first–derivative test for relative extrema are the following:

1. Find all critical values of $f(x)$. That is, find all numbers c such that $f(c)$ is defined and either $f'(c)=0$ or $f'(c)$ does not exist.
2. The point $(c, f(c))$ is a **relative maximum** if $f'(x)>0$ (rising) for all x in an open interval (a, c) to the left of c, and $f'(x)<0$ (falling) for all x in an open interval (c, b) to the right of c.
3. The point $(c, f(c))$ is a **relative minimum** if $f'(x)<0$ (falling) for all x in an en interval (a, c) to the left of c, and $f'(x)>0$ (rising) for all x in an open interval (c, b) to the right of c.
4. The point $(c, f(c))$ is not an **extremum** if the derivative $f'(x)$ has the same sign in open intervals (a, c) and (c, b) on both sides of c.

In light of first-derivative test, the function $f(x)=x^3-3x^2-9x+1$ (example 24) has the critical values –1 and 3. The function $f(x)$ is increasing when $x<-1$ and $x>3$ and decreasing when $-1<x<3$. The first derivative test tells us that there is a relative maximum of 6 at $x=-1$ and a relative minimum of –26 at $x=3$.

Example 27 Examine the function $f(x)=2x^3+3x^2-12x-5$ for the relative extrema using first–derivative test.

Solution The first derivative of $f(x)=2x^3+3x^2-12x-5$ is:

$$f'(x)=6x^2+6x-12=6(x+2)(x-1)$$

Set $f'(x)=0$ to obtain the critical values:

$$f'(x)=6x^2+6x-12=0=6(x+2)(x-1)=0 \Rightarrow x=-2, 1$$

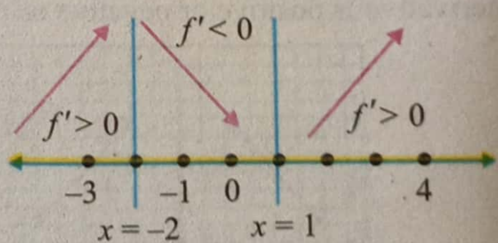


Figure 4.9

To test the critical values –2, 1, we can use the test values –3, 0 and 2. Many other choices of the test values are also possible, but we try to select numbers that will make the computations easy. This is shown in the Figure 4.9.

The test values –3 and 0 are used for the critical value $x=-2$ to obtain:

$$f'(-3)=6(-3+2)(-3-1)=24>0 \text{ (positive)}$$

$$f'(0)=6(0+2)(0-1)=-12<0 \text{ (negative)}$$

The value of the derivative is positive (rising) to the left of -2 and negative (falling) to the right of -2. Thus, $x = -2$ leads a **relative maximum point**

$$f(-2) = 2(-2)^3 + 3(-2)^2 - 12(-2) - 5 = -16 + 12 + 24 - 5 = 15.$$

The test values 0 and 2 are used for the critical value $x = 1$ to obtain:

$f'(0) = 6(0+2)(0-1) = -12$ (negative)

$f'(2) = 6(2+2)(2-1) = 24$ (positive)

The value of the derivative is negative (falling) to the left of 1 and positive (rising) to the right of 6. Thus, $x = 1$ leads a **relative minimum point**. $f(1) = 2(1) + 3(1) - 12(1) - 5 = -12$

Thus, the arrow pattern in the figure suggests that the graph of $f(x)$ has a relative maximum at $(-2, 15)$ and a relative minimum at $(1, -12)$.

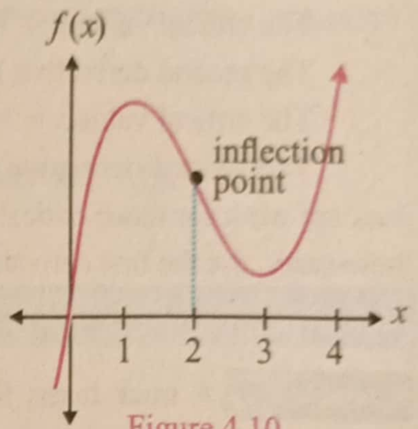
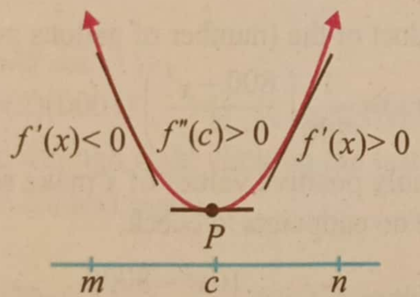


Figure 4.10

The Second–Derivative Rule: It is often possible to classify a critical point $P(c, f(c))$ on the graph of $f(x)$ by examining the sign of $f''(c)$. Specifically, if $f'(c) = 0$ and $f''(c) > 0$, then there is a horizontal tangent line at P and the graph of $f(x)$ is concave up in the neighborhood of P. This means that the graph of $f(x)$ is cupped upward from the horizontal tangent at P and to expect P to be a relative minimum, as shown in Figure 4.11.

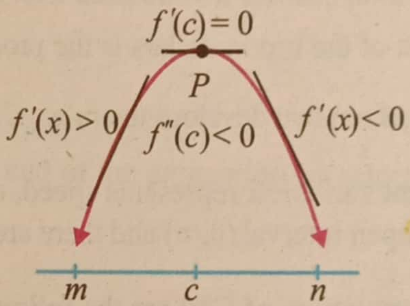
Similarly, we expect P to be a relative maximum, if $f'(c) = 0$ and $f''(c) < 0$, because the graph is cupped down beneath the critical point P, as shown in Figure 4.12.



$f'(c) = 0$ and $f''(c) > 0$

implies $f(c)$ is a relative minimum

Figure 4.11



$f'(c) = 0$ and $f''(c) < 0$

implies $f(c)$ is a relative maximum

Figure 4.12

In other words,

- The point $P(c, f(c))$ is said to be a relative maximum, if the slope $f'(c)$ of the tangent line from left to right along a curve through P, is decreasing from **positive to zero to negative** and the second derivative $f''(c)$ is negative.
- The point $P(c, f(c))$ is said to be relative minimum, if the slope $f'(c)$ of the tangent line from left to right along a curve through P, is increasing from **negative to zero to positive** and the second derivative $f''(c)$ is positive. These observations lead to the second–derivative test for relative extreme.

Remember

The Second Derivative Rule for Relative Extrema: Let $f(x)$ be a function such that $f'(c) = 0$ and the second derivative exists on an open interval (a, b) containing c.

1. If $f''(c) > 0$, then there is a **relative minimum at $x = c$** and the graph of $f(x)$ is **concave up** in the neighborhood of $P(c, f(c))$.

2. If $f''(c) < 0$, then there is a **relative maximum at $x = c$** and the graph of $f(x)$ is **concave down** in the neighborhood of $P(c, f(c))$.

3. If $f''(c) = 0$, then the second derivative test fails and gives no information.

Example 28 Use the second-derivative test to determine whether each critical value of the function $f(x)=3x^5-5x^3+2$ corresponds to a relative maximum, a relative minimum, or neither.

Solution The first and second derivatives of $f(x)$ are the following:

$f'(x)=15x^4-15x^2=15x^2(x-1)(x+1)$, $f''(x)=60x^3-30x=30x(2x^2-1)$

Put $f'(x)=15x^4-15x^2=15x^2(x-1)(x+1)=0$ to obtain the critical values 0, 1 and –1.

The second derivative $f''(x)$ at a critical point $x=0$ is: $f''(0)=30(0)(0-1)=0$

The critical value $x=0$ declares the failure of second derivative test.

The second derivative $f''(x)$ at a critical point $x=1$ is: $f''(1)=30>0$

The critical value $x=1$ leads to a relative minimum of $f(1)=3(1)-5(1)+2=0$.

The second derivative $f''(x)$ at a critical point $x=-1$ is: $f''(-1)=-30<0$

The critical value $x=-1$ gives a relative maximum of $f(-1)=-3-5(-1)+2=4$

The second derivative test works only for those critical values c that make $f'(c)=0$. This test does not work for those critical values c for which $f'(c)$ does not exist or that make $f''(c)=0$. In both of these cases, use the first derivative test to proceed the process of relative extrema.

4.4.6 Solve real-life problems related to extreme values

Example 29 A truck burns fuel at the rate of $G(x)=\frac{1}{200}\left(\frac{800+x^2}{x}\right)$, $x>0$ gallons per mile when traveling x miles per hour on a straight level road. If fuel costs \$2 per gallon, find the speed that will produce the minimum total cost for a 1000 mile trip. Find the maximum total cost.

Solution he total cost of the trip in dollars is the product of the (number of gallons per mile) (the number of miles) (the cost per gallon) that develops the rule: $C(x)=\frac{1}{200}\left(\frac{800+x^2}{x}\right)(1000)(2)=\frac{8000+10x^2}{x}$

The independent variable x represents speed, only positive values of x make sense here. Thus, the domain of $C(x)$ is the open interval $(0,\infty)$ and there are no endpoints to check.

The first and second derivatives of $C(x)$ are the following: $C'(x)=\frac{10x^2-8000}{x^2}$, $C''(x)=\frac{16000}{x^3}$

Put $C'(x)=0$ to obtain the critical values: $\frac{10x^2-8000}{x^2}=0 \Rightarrow 10x^2-8000=0 \Rightarrow x^2=8000 \Rightarrow x=\pm 28.3\,mph$

The only critical number in the domain is $x=28.3$. The second derivative test at a critical value $x=28.3$ is: $C''(28.3)=\frac{16000}{(28.3)^3}=0.72>0$

The second derivative test shows that the critical value $x=28.3$ leads to a minimum value. The minimum total cost is found by inserting $x=28.3$ in the cost function: $C(28.3)=\frac{8000+10(28.3)^2}{28.3}=565.69\,dollars$

Example 30 The supporting cable of a pipeline suspension system forms a parabolic arc between the supports, which is described by the equation $y=0.03125x^2-1.25x$. The distances are measured in meters. The origin of the axis system is at the point where the cable attaches to the left support tower. Where the point is on the and how far is it below the attachment point?

Solution For the low point of the cable, we need to find the first and second derivatives of the given function:

$y=0.03125x^2-1.25x$, $y'=0.0625x-1.25$, $y''=0.0625$ Set $y'=0$ to obtain the critical value: $0.0625x-1.25=0 \Rightarrow x=20$

Since the second derivative is positive for all values of x, the critical value $x=20$ will produce the minimum value on the curve.

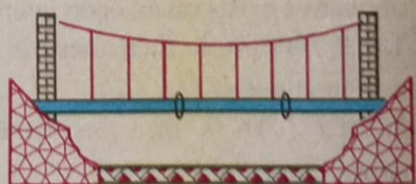


Figure 4.13

The low point on the cable occurs 20.0 m to the right of the left tower. Put the critical value for x in the original function to obtain the distance from the low point of the cable which is below the attachment point: $y = 0.03125(20)^2 - 1.25(20) = -12.5\ m$

Therefore, the low point of the cable is 20.0 m to the right and 12.5 m below its point of attachment to the support.

4.4.7 MAPLE command maximize (minimize) to compute maximum (minimum) value of a function

The procedure to the use of MAPLE command maximize (minimize) o complete maximum (minimum) value of a function is illustrated in the following example.

Example 31 Use MAPLE commands to compute

(a). $f(x) = \cos x$. (b). $f(x) = x^4 - 2x^2 + 3$ in the interval $[-1,2]$.

Solution

a. Command:

> $maximize(\cos(x));$

$$1$$

> $minimize(\cos(x));$

$$-1$$

Context Menu:

> $\cos(x)$

> $Optimization[Maximize](\cos(x))$

$$[1., [x = 5.58237824894110\,10^{-17}]]$$

This result is obtained through right click on the last end of the expression by selecting "Optimization < maximize local"on the context menu.

> $\cos(x)$

> $Optimization[Minimize](\cos(x))$

$$[-1., [x = 3.14159265358977]]$$

b. Command:

> $maximize(x^4 - 2x^2 + 3, x = -1..2);$

$$11$$

> $minimize(x^4 - 2x^2 + 3, x = -1..2);$

$$2$$

Exercise 4.3

1. Find the critical values of the given functions in the following problems and show where the function is increasing and where it is decreasing.

   a. $f(x) = x^3 + 3x^2 + 1$ b. $f(x) = x^3 + 35x^2 - 125x - 9.375$

2. Find the critical values of the following functions:

   a. $f(x) = 2x^3 - 3x^2 - 72x + 15$ b. $f(x) = \frac{1}{3}x^3 - x^2 - 15x + 6$

   c. $f(x) = 6x^{\frac{2}{3}} - 4x$ d. $f(x) = 3x^{\frac{4}{3}} - 12x^{\frac{1}{3}}$

3. Determine whether the given function has a relative maximum, a relative minimum, or neither at the given critical values for the following problems:

   a. $f(x)=(x^3-3x+1)^7=0$ at $x=1; x=-1$ b. $f(x)=(x^4-4x+2)^5$ at $x=1$

   c. $f(x)=(x^2-4)^4(x^2-1)^3$ at $x=1; x=2$ d. $f(x)=\sqrt[3]{x^3-48}$ at $x=4$

4. Find all critical points of the functions in the following problems, and determine where the graph of the function is rising, falling, concave up, or concave down. Sketch the graph.

   a. $f(x)=2(x+20)^2-8(x+20)+7$ b. $f(x)=\frac{1}{3}x^3-9x+2$

5. Find all relative extrema of the following functions:

   a. $f(x)=x^3-3x^2+1$ b. $f(x)=x^3+6x^2+9x+2$

6. a. Suppose $f(x)$ is a differential function with derivative $f'(x)=(x-1)^2(x-2)(x-4)(x+5)^4$ Find all critical values of $f(x)$ and determine whether each corresponds to a relative maximum, a relative minimum, or neither.

   b. Suppose $f(x)$ is a differential function with derivative $f'(x)=\frac{(2x-1)(x+3)}{(x-1)^2}$

   Find all critical values of $f(x)$ and determine whether each corresponds to a relative maximum, a relative minimum, or neither.

7. A company has found through experience that increasing its advertising also increases its sales up to a point. The company believes that the mathematical model connecting profit in hundreds of dollars P(x) and expenditures on advertising in thousands of dollars x is: $P(x)=80+108x-x^3,\ 0\le x\le 10$

   a. Find the expenditure on advertising that leads to maximum profit.
   b. Find the maximum profit.

8. The total profit P(x) (in thousands of dollars) from the sale of x hundred thousands of automobile tires is approximated by $P(x)=-x^3+9x^2+120x-400,\ 3\le x\le 15$

   Find the number of hundred thousands of tires that must be sold to maximize profit. Find the maximum profit.

9. The percent of concentration of a drug in the bloodstream x hours after the drug is administered is given by: $K(x)=\frac{4x}{3x^2+27}$

   a. On what time intervals is the concentration of the drug increasing?
   b. On what intervals is it decreasing?
   c. Find the time at which the concentration is a maximum.
   d. Find the maximum concentration.

10. A diesel generator burns fuel at the rate of $G(x)=\frac{1}{48}\left(\frac{300}{x}+2x\right)$ gallons per hour when producing x thousand kilowatt hours of electricity. Suppose that fuel costs $2.25 a gallon and find the value of x that leads to minimum total cost if the generator is operated for 32 hours. Find the minimum cost.

Review Exercise 4

1. **Choose the correct option.**

i. If $f(t)=3t^2+4t-5$ then $f'(t)$ is

(a). $3t^2-4t+5$ (b). $6t+4$ (c). 12 (d). $6t-5$

ii. If $f(t)=3t^2+4t-5$ then $f''(t)$ is

(a). $3t^2-4t+5$ (b). $6t-4$ (c). 6 (d). $6t+5$

iii. If $y=e^{mx}$ then $\frac{d^n y}{dx^n}=$

(a). $me^{mx}\log(e^x)$ (b). $m^n e^{mx}$ (c). me^{nx} (d). $me^n e^{mx}$

iv. The 5th derivative of $f(x)=e^x$ is

(a). e^x (b). e^{x+5} (c). $5e^x$ (d). e^{5x}

v. The 4th derivative of $\sin x$ is

(a). $\frac{d^4y}{dx^4}=\sin x$ (b). $\frac{d^4y}{dx^4}=\cos x$ (c). $\frac{d^4y}{dx^4}=-\sin x$ (d). $\frac{d^4y}{dx^4}=-\cos x$

vi. If $f(x)$ and its derivatives at $x=x_0$ are $f'(x_0)$, $f''(x_0),\ldots,f^n(x)$ then the nth order polynomial $f(x)$ will be equal to:

(a). $f(x)+f(x_0-x)f'(x)+\ldots+\frac{(x_0-x)^n}{n!}f^n(x)$ (b). $f(x_0)+(x-x_0)f'(x)+\ldots+\frac{(x-x_0)^n}{n!}f^n(x_0)$

(c). $f(x+h)+(x+h)f'(h)+\ldots+\frac{(x-h)}{n!}f'(h)$ (d). $f(x-h)+(x-h)f'(h)+\ldots+\frac{(x-h)}{n!}f'(h)$

vii. To calculate the first five terms of taylor's series for $f(x)=e^{2x}$, the MAPLE command is used as

(a). taylor $(x, e^{2x}=0)$ (b). taylor (e^{2x})

(c). taylor $(e^{2x}, x=0,5)$ (d). taylor $(e^{2x}, x=5)$

viii. The equation of normal at point (x_0, y_0) is:

(a). $x-x_0=\frac{1}{m}(y-y_0)$ (b). $(y-y_0)=\frac{1}{m}(x_0-x)$

(c). $(y-y_0)=m(x-x_0)$ (d). $(y-y_0)=-\frac{1}{m}(x-x_0)$

ix. The angle of intersection of the two curves can be calculated by using the formulas.

(a). $\theta=\sin^{-1}\frac{1+m_1m_2}{1-m_1m_2}$ (b). $\theta=\tan^{-1}\frac{1-m_1m_2}{1+m_1m_2}$

(c). $\theta=\tan^{-1}\frac{m_1-m_2}{1+m_1m_2}$ (d). $\theta=\sin^{-1}\frac{m_1+m_2}{1+m_1m_2}$

x. If $f(x)$ is differentiable on the open interval (a, b) the $f(x)$ is strictly increasing if:

(a). $f(x)>0$ (b). $f'(x)>0$ (c). $f''(x)>0$ (d). $f''(x)\angle 0$

Summary

- The second derivative of $y=f(x)$ can be written with any of the following notations:
  $\frac{d^2y}{dx^2}$, y'', $f''(x)$, $D_x^2[f(x)]$
  The third derivative can be written in a similar way. For $n \geq 4$, the nth derivative is written as $f^{(n)}(x)$.
- The second derivative of parametric functions $x(t)$ and $y(t)$ can be found as follows:
  $$\frac{d^2y}{dx^2}=\frac{d}{dt}\left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right)\frac{dt}{dx}=\frac{\frac{dx}{dt}\frac{d^2y}{dt^2}-\frac{dy}{dt}\frac{d^2x}{dt^2}}{\left(\frac{dx}{dt}\right)^3}, \quad Put\ \frac{dt}{dx}=\frac{1}{\left(\frac{dx}{dt}\right)}$$
- The popular notation for the Taylor theorem of order n is:
  $$f(x_0+h)=f(x_0)+f'(x_0)h+f''(x_0)\frac{h^2}{2!}+...+f^n(x_0)\frac{h^n}{n!}+...$$
- The popular notation for the Maclaurin's theorem of order n is:
  $$f(x_0+h)=f(0)+f'(0)h+f''(0)\frac{h^2}{2!}+...+f^n(0)\frac{h^n}{n!}+...$$
- i. If two lines are parallel, then their slopes are equal.
  ii. If two lines are perpendicular, then the product of their slopes equals -1.
  iii. The tangent equation at a point $P(x_0, y_0)$ is $(y-y_0)=m(x-x_0)$.
  iv. The normal equation at a point $P(x_0, y_0)$ is $(y-y_0)=\frac{-1}{m}(x-x_0)$.
- If $f(x)$ is differentiable on the open interval (a, b), then the function $f(x)$ is
  strictly increasing on (a, b) if $f'(x)>0$ for $a<x<b$.
  strictly decreasing on (a, b) if $f'(x)<0$ for $a<x<b$.
- If a continuous function $f(x)$ has a relative extremum at c, then c must be a critical value of $f(x)$.
- The graph of a function $f(x)$ is **concave upward** on an open interval (a, b), where $f''(x)>0$, and it is **concave downward** where $f''(x)<0$.
- If $y = f(x)$ is continuous on (a, b) and has an inflection point at $x = c$, then either $f''(c)=0$ or $f''(c)$ does not exist.
  i. A point $P(c, f(c))$ on the graph of a differential function $y = f(x)$ where the concavity changes is called a **point of inflection.**
  ii. If a function has a point of inflection $P(c, f(c))$ at a partition c and it is possible to differentiate the function twice, then $f''(c)=0$.

History

Omer Khayyam was a Persian mathematician, Astronomer and philosopher. He was born in Nishapur in north eastern Iran. He was most notable person in the history of mathematics because of his work on the classification and solution of cubic equation. Where he proved the geometric solution by the intersection of conics. He also contributed to the understanding of the parallel axiom. As an astronomer he designed the Jalali calendar, a Solor calander. He was the first person, who considered the three cases of acute, right and abtuse angle for summit angles of a Khayyam saccheri quadrilateral, three cases which are exhaustive and pairwise mutually exclusive.

Omer Khayyam (1048-1131)

Unit 5

DIFFERENTIATION OF VECTOR FUNCTIONS

By the end of this unit, the students will be able to:

5.1 Scalar and Vector Functions.

i. Define scalar and vector function.
ii. Explain domain and range of a vector function.

5.2 Limit and Continuity.

i. Define limit of a vector function and employ the usual technique for algebra of limits of scalar function to demonstrate the following properties of limits of a vector function.
- The limit of the sum (difference) of two vector functions is the sum (difference) of their limits.
- The limit of the dot product of two vector functions is the dot product of their limits.
- The limit of the cross product of two vector functions is the cross product of their limits.
- The limit of the product of a scalar function and a vector function is the product of their limits.

ii. Define continuity of a vector function and demonstrate through examples.

5.3 Derivative of Vector Function.

i. Define derivative of a vector function of a single variable and elaborate the result:
if $f(t) = f_1(t)i + f_2(t)j + f_3(t)k$, where $f_1(t)$, $f_2(t)$, $f_3(t)$ are differentiable functions of a scalar variable t, then $\frac{df}{dt} = \frac{df_1}{dt}i + \frac{df_2}{dt}j + \frac{df_3}{dt}k$.

5.4 Vector Differentiation.

i. Prove the following formulae of differentiation:
- $\frac{da}{dt} = 0,$
- $\frac{d}{dt}[f \pm g] = \frac{df}{dt} \pm \frac{dg}{dt},$
- $\frac{d}{dt}[\phi f] = \phi\frac{df}{dt} + \frac{d\phi}{dt}f,$
- $\frac{d}{dt}[f \cdot g] = f \cdot \frac{dg}{dt} + \frac{df}{dt} \cdot g,$
- $\frac{d}{dt}[f \times g] = f \times \frac{dg}{dt} + \frac{df}{dt} \times g,$
- $\frac{d}{dt}\left[\frac{f}{\phi}\right] = \frac{1}{\phi^2}\left[\phi\frac{df}{dt} - \frac{d\phi}{dt}f\right],$

where $\boldsymbol{a}$ is a constant vector function, f and $\boldsymbol{g}$ are vector functions, and ϕ is a scalar function of $\boldsymbol{t}$.

ii. Apply vector differentiation to calculate velocity and acceleration of a position vector $\boldsymbol{r}(t) = x(t)\boldsymbol{i} + y(t)\boldsymbol{j} + z(t)\boldsymbol{k}$

Introduction

In the same way that we studied numerical calculus after we learned numerical arithmetic. We can now study vectors calculus. Since we already studied vector arithmetic in unit-3 of grade-x Mathematics. Quite simply, we might have a vector quantity that varies with respect to another variable either a scalar or a vector. In this unit we shall study the vector functions and the applications of the differential calculus. We shall extend the basic concepts of calculus in a simple and natural way. The study of vector calculus makes the more useful in the geometrical, physical and engineering applications.

5.1 Scalar and Vector Functions

The relationship of calculus and vector methods forms what is called **vector calculus**. The key to use vector calculus is the concept of a vector function.

I. (a) Scalar Function

A function $f(x)$ is a rule which operates on an input x (x is any scalar quantity) and produces always just a single scalar output y. This gives a proper notation of a scalar function:

$$y = f(x) \qquad \text{(i)}$$

For example,

- $C(x) = 2x + 2$ is a cost function that depends on x number of units of items. Here x is the input, y is the output and $C(x) = 2x + 2$ is the rule which operates on an input x to produce a single output quantity y. In response of $x = 2$ items (2 is scalar), the cost (cost is also scalar) is: $C(2) = 2(2) + 2 = 6$ rupees. This function is then called a **scalar (single variable) function**; because it transforms one input x to produce just one output C.
- $A(x, y) = xy$ is the area of rectangle that depends on length x and width y. Here x, and y are the two inputs and the rule $A(x, y) = xy$ which operates on two inputs x and y to produce a single output quantity A. In response of $x = 2$ and $y = 1$ (2 and 1 are scalars), the area is (is also a scalar) $A(2, 1) = (2)(1) = 2$ square units.

  This function is then called a **scalar (double variables) function**, because it transforms two inputs x and y to produce just one output A.

This idea can easily be extended to define a scalar multivariate function.

The uniqueness of scalar function is to transform **scalar quantities in a single scalar quantity. Is** there any rule that will transform **scalar quantities in a vector quantity**? Yes, the rule is the vector functions. Vector functions are used to study curves in the plane and space.

(b) Vector Function

"A vector function $\overline{F} = (f(t), g(t), h(t))$ is a function of onevariable that has only one "input value". The "output" values are in two and three dimensional vector spaces instead of simple numbers. In other words we can say $\overline{F}$ is called a vector function of 't' $\overline{F} = \overline{F}(t)$.

If $\hat{i}, \hat{j}$ and $\hat{k}$ are the unit vectors associated with a rectangular coordinate system (discussed in details in unit-3 of grade-xi) then a vector function $\overline{F}(t)$ is written as

i. $\overline{F}(t) = f_1(t)\hat{i} + f_2(t)\hat{j}$ 2 spaces

ii. $\overline{F}(t) = f_1(t)\hat{i} + f_2(t)\hat{j} + f_3(t)\hat{k}$ 3 spaces

We can say that a function $\overline{F}(t)$ is defined if all its components f_1, f_2 and f_3 are defined.

Example 1 Find $\overline{F}\left(\frac{\pi}{2}\right)$ and $\overline{F}(\pi)$ if $\overline{F}(t) = \sin(t)\hat{i} + \cos(t)\hat{j}$

Solution We have given $\overline{F}(t) = \sin(t)\hat{i} + \cos(t)\hat{j}$

As, sin (t) and cos (t) are defined for all values of t, so, $\overline{F}(t)$ is defined for all t

For $t = \frac{\pi}{2}$, $F\left(\frac{\pi}{2}\right) = \sin\left(\frac{\pi}{2}\right)\hat{i} + \cos\left(\frac{\pi}{2}\right)\hat{j} = \hat{i}$

For $t = \pi$, $F(\pi) = \sin(\pi)\hat{i} + \cos(\pi)\hat{j} = -\hat{j}$

ii. Domain and range of a vector function

a. Domain

"The set of all t values used as input in $\overline{F}(t)$ is called the domain of a vector-valued function $\overline{F}(t)$".

b. Range

The set of $\overline{F}(t)$ values that the vector function $\overline{F}(t)$ takes as t varies, is called the range of a vector valued function $\overline{F}(t)$.

Example 2 Find the domain for the following vector functions:

(a). $\vec{F}(t)=2t\hat{i}-3t\hat{j}+t^{-1}\hat{k}$ (b). $\vec{F}(t)=\sin t\hat{i}+(1-t)^{-1}\hat{j}+\ln t\hat{k}$

Solution

a. The vector function is:

$\vec{F}(t)=(f_1(t),f_2(t),f_3(t))=2t\hat{i}-3t\hat{j}+t^{-1}\hat{k}$

The function $f_1(t)=2t$ is defined for all t; $f_2(t)=3t$ is defined for all values of t; $f_3(t)=t^{-1}$ is defined for all values of t except $t=0$. Thus, the domain of a function $\vec{F}(t)$ is $R-\{0\}$.

b. $\vec{F}(t)=(f_1(t),f_2(t),f_3(t))=\sin t\hat{i}+(1-t)^{-1}\hat{j}+\ln t\hat{k}$

The function $f_1(t)=\sin t$ is defined for all t; $f_2(t)=(1-t)^{-1}$ is defined for all values of t except $t\neq 1$; $f_3(t)=\ln t$ is defined for $t>0$. Thus, the domain of a function $\vec{F}(t)$ is $t>0,\ t\neq 1$. The range in each case is of course a vector quantity.

Do You Know?

Operations with vector functions

It follows from the definition of vector operations that vector functions can be added, subtracted, multiplied by a scalar function, and multiplied together e.g.

If $\vec{F}$ and $\vec{G}$ are vector functions of the real variable t, and h(t) is any scalar function, then $\vec{F}+\vec{G}$, $\vec{F}-\vec{G}$ and $\vec{F}\times\vec{G}$ are vector functions, and $\vec{F}\cdot\vec{G}$ is a scalar function.

5.2 Limit and Continuity

For the most part, vector limits behave like scalar limits. The proper definition of the limit of a vector function is given below.

5.2.1 Limit of a vector function and properties of limits of a vectors function

"Let a vector function $\vec{F}(t)$ be defined for all values of t in some neighbourhood about a point $t=t_0$ except possibly at to itself and let $\vec{L}$ be a constant vector called limit vector. The function $\vec{F}(t)$ is said to approach the limit vector $\vec{L}$ as "t approaches t_0" if for any given real number $\varepsilon>0$ such that $\left|\vec{F}(t)-\vec{L}\right|\angle\varepsilon$ whenever $0\angle\left|t-t_0\right|\angle\delta$ symbolically, it is written as $\lim_{t\to t_0}\vec{F}(t)=\vec{L}$

Now look at the following useful properties of vector valued functions.

i. The limit of the sum (difference of two vector functions is the sum (difference) of their limits.

If $\lim_{t\to t_0}\vec{F}(t)=\vec{L}$ and $\lim_{t\to t_0}\vec{G}(t)=\vec{M}$, where $\vec{L}$ and $\vec{M}$ are constant vector functions then:

a. $\lim_{t\to t_0}\left[\vec{F}(t)+\vec{G}(t)\right]=\lim_{t\to t_0}\vec{F}(t)+\lim_{t\to t_0}\vec{G}(t)=\vec{L}+\vec{M}$ b. $\lim_{t\to t_0}\left[\vec{F}(t)-\vec{G}(t)\right]=\lim_{t\to t_0}\vec{F}(t)-\lim_{t\to t_0}\vec{G}(t)=\vec{L}-\vec{M}$

ii. The limit of the dot product of two vector functions is the dot product of their limits.

If $\lim_{t\to t_0}\vec{F}(t)=\vec{L}$ and $\lim_{t\to t_0}\vec{G}(t)=\vec{M}$, where $\vec{L}$ and $\vec{M}$ are constant vector functions then:

$$\lim_{t\to t_0}\left[\vec{F}(t).\vec{G}(t)\right]=\lim_{t\to t_0}\vec{F}(t).\lim_{t\to t_0}\vec{G}(t)=\vec{L}.\vec{M}$$

iii. The limit of the cross product of two vector functions is the cross product of their limits.

If $\lim_{t\to t_0}\vec{F}(t)=\vec{L}$ and $\lim_{t\to t_0}\vec{G}(t)=\vec{M}$, where $\vec{L}$ and $\vec{M}$ are constant vector functions then:

$$\lim_{t\to t_0}\left[\vec{F}(t)\times\vec{G}(t)\right]=\lim_{t\to t_0}\vec{F}(t)\times\lim_{t\to t_0}\vec{G}(t)=\vec{L}\times\vec{M}$$

iv. The limit of the product of a scalar function and a vector function is the product of their limits. If $\lim_{t\to t_0}\vec{F}(t)=\vec{L}$ and $\lim_{t\to t_0}\vec{h}(t)=\vec{c}$, where $\vec{L}$ is a constant vector and $\vec{c}$ is a scalar constant then:

$$\lim_{t\to t_0}\left[\vec{h}(t)\times\vec{F}(t)\right]=\lim_{t\to t_0}\vec{h}(t)\times\lim_{t\to t_0}\vec{F}(t)=c\times\vec{L}$$

Example 3 Find $\lim_{t\to 2}F(t)$, when the vector function is $\vec{F}(t)=(t^2-3)\hat{i}+e^t\hat{j}+\sin\pi t\hat{k}$.

Solution $\lim_{t\to 2}\vec{F}(t)=\lim_{t\to 2}\left[\vec{f}_1(t)\hat{i}+\vec{f}_2(t)\hat{j}+\vec{f}_3(t)\hat{k}\right]$

$$=\left[\lim_{t\to 2}(t^2-3)\right]\hat{i}+\left[\lim_{t\to 2}(e^t)\right]\hat{j}+\left[\lim_{t\to 2}\sin\pi t\right]\hat{k}$$

$$=(4-3)\hat{i}+e^2\hat{j}+\sin 2\pi\hat{k}=\hat{i}+e^2\hat{j},\quad \sin 2\pi=0$$

5.2.2 Continuity of a vector function

A vector function $F(t)$ is said to be continuous at $t=t_0$ if

i. t_0 is in the domain of a vector function $F(t)$

ii. $\lim_{t\to t_0}F(t)=F(t_0)$

Do You Know?

A continuous vector value function is also continuous at every point in its domain.

Example 4 For what values of t is the vector function $F(t)=\left(sint,(1-t)^{-1}\right)$ continuous?

Solution The components of a vector function are: $f_1(t)=\sin t,\ f_2(t)=(1-t)^{-1},\quad t\in R$

The function $f_1(t)$ is continuous for all t; $f_2(t)$ is continuous where $1-t\neq 0\ (t\neq 1)$. Thus, $F(t)$ is continuous, when t is a real number other than 1.

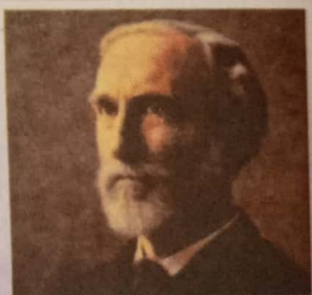
Example 5 For what values of t is $F(t)=\left(\sin t,(1-t)^{-1},\ln t\right)$ continuous?

Solution The components of a vector function are: $f_1(t)=\sin t,\ f_2(t)=(1-t)^{-1},\ f_3(t)=\ln t,\ t\in R$

The function $f_1(t)$ is continuous for all t; $f_2(t)$ is continuous where $1-t\neq 0$ (that is, where $t\neq 1$); $f_3(t)$ is continuous for $t>0$. Thus, F (t) is continuous function whenever t is any positive number other than 1. That is $t>0,\ t\neq 1$.

History

J. Willard Gibbs was an American scientist. He made his great contributions in the field of mathematics, physics and chemistry. He was the first American who obtained his doctorate degree in engineering after spending three years in Europe, he joined Yale university as professor of mathematical physics from 1871 to his death. He earned international reputation while working in relative isolation. A great scientist Albert Einstein praised him as "the greatest mind in American history". Together with Oliver Heaviside (Britain and American national) Gibbs developed vector analysis to express the new laws of electromagnetism.



Josiah Willard Gibbs (1839)-(1903)

Exercise 5.1

1. Find the domain for the following vector functions:
   a. $\vec{F}(t)=2t\hat{i}-3t\hat{j}+t^{-1}\hat{k}$
   b. $\vec{F}(t)=(1-t)\hat{i}+\sqrt{t}\hat{j}-(t-2)^{-1}\hat{k}$
   c. $\vec{F}(t=\sin t\hat{i}+\cos t\hat{j}+\tan t\hat{k}$
   d. $\vec{F}(t)=\cos t\hat{i}-\cot t\hat{j}+\operatorname{cosec} t\hat{k}$
2. Perform the operations of the following expressions with
   $\vec{F}(t)=2t\hat{i}-5\hat{j}+t^2\hat{k},\ \vec{G}(t)=(1-t)\hat{i}+\frac{1}{t}\hat{k},\ \vec{H}(t)=\sin t\hat{i}+e^t\hat{j}$:
   a. $2\vec{F}(t)-3\vec{G}(t)$ b. $3\vec{F}(t)+4\vec{G}(t)$ c. $\vec{G}(t)\cdot\vec{H}(t)$ d. $\vec{F}(t)\times\vec{H}(t)$
3. Evaluate the limits of the following expressions:
   a. $\lim_{t\to1}\left[3t\hat{i}+e^{2t}\hat{j}+\sin\pi t\hat{k}\right]$
   b. $\lim_{t\to1}\left[\frac{t^3-1}{t-1}\hat{i}+\frac{t^2-3t+2}{t^2+t-2}\hat{j}+(t^2+1)e^{t-1}\hat{k}\right]$
   c. $\lim_{t\to0}\left[\frac{\sin t}{t}\hat{i}+\frac{1-\cos t}{t}\hat{j}+e^{1-t}\hat{k}\right]$
   d. $\lim_{t\to0}\left[\frac{\sin(2t)}{2t}\hat{i}+\ln(4+t)\hat{j}\right]$
4. Test the continuity of the following expressions for all values of t:
   a. $\vec{F}(t)=t\hat{i}+3\hat{j}-(1-t)\hat{k}$
   b. $\vec{G}(t)=t\hat{i}-t^{-1}\hat{k}$
   c. $\vec{F}(t)=e^t\left(t\hat{i}+t^{-1}\hat{j}+3\hat{k}\right)$
   d. $\vec{G}(t)=\frac{t\hat{i}+\sqrt{t}\hat{j}}{\sqrt{t^2+t}}$

5.3 Derivative of Vector Function

A vector function $\vec{F}$ determines a curve in space as the collection of terminal points of the vectors $\vec{F}(t)$. If the curve is smooth, this is natural to ask whether $\vec{F}(t)$ has a derivative. Our experience with single variable calculus in previous units prompt us to wonder what the diferentiation of the vector valued function might be and what it might tell us. For now, let's recall some important ideas from unit 3 of this book. We defined the derivative of the scalar function $f(x)$. Which is the limit as $\Delta x\to0$ of the difference quotient $\frac{\Delta f}{\Delta x}$

e.g. Given a function $f(t)$ that measures the position of an object, moving along an axis its, derivative $f'(t)$ is defined as.

$$f'(t)=\lim_{\Delta t\to0}\frac{f(t+\Delta t)-f(t)}{\Delta t} \qquad \text{(i)}$$

and measure the instantanous rate of change of $f(t)$ with rspect to t in particular for a fixed value $t=a$, $f'(a)$ measure the velocity of the moving object as well as the slope of the tangent line to thecurve $y=f(t)$ at the point $(a, f(a))$. As we are working with vector valued functions, we will use the above ideas and perspectives into the context of curves in space and output that are vectors.

5.3.1 Derivative of a vector function of a single variable

If $f(t)=f_1(t)\hat{i}+f_2(t)\hat{j}+f_3(t)\hat{k}$, where $f_1(t), f_2(t), f_3(t)$ are differentiable functions of a scalar variable t, then

$$\frac{df}{dt}=\frac{df_1}{dt}\hat{i}+\frac{df_2}{dt}\hat{j}+\frac{df3}{dt}\hat{k}$$

The derivative of a vector-valued function $\vec{F}(t)$ is defined to be.

$$\vec{F}'(t)=\lim_{\Delta t\to0}\frac{\vec{F}(t+\Delta t)-\vec{F}(t)}{\Delta t} \qquad \text{(ii)}$$

for these values of t at which the limit exists, we can also use the Leibniz notation $\frac{dF}{dt}$, for derivative of $F(t)$, and $\frac{d}{dt}\left[\overline{F}(t)\right]$ The following theorem establishes a convenient method for computing the derivative of a vector function.

Theorem–1: The vector function $\overline{F}(t)=(f_1(t), f_2(t), f_3(t))=f_1(t)i+f_2(t)j+f_3(t)k$ is differentiable at a point $t=t_o$ whenever the component functions $f_1(t), f_2(t), f_3(t)$ of $\overline{F}(t)$ are all differentiable at a point $t=t_o$: i.e. $F'(t)=(f_1'(t), f_2'(t), f_3'(t))=f_1'(t)\hat{i}+f_2'(t)\hat{j}+f_3'(t)\hat{k}$

Proof: If a vector function $\overline{F}(t)$ is differentiable, then their component functions $f_1(t), f_2(t)$ and $f_3(t)$ exist, then the scalar derivatives $f_1'(t), f_2'(t)$ and $f_3'(t)$ by first-principle rule

$$\overline{F}'(t)=\lim_{\Delta t\to 0}\frac{\overline{F}(t+\Delta t)-\overline{F}(t)}{\Delta t}=\lim_{\Delta t\to 0}\frac{[f_1(t+\Delta t)\hat{i}+f_2(t+\Delta t)\hat{j}+f_3(t+\Delta t)\hat{k}]-[f_1(t)\hat{i}+f_2(t)\hat{j}+f_3(t)\hat{k}]}{\Delta t}$$

$$=\left[\lim_{\Delta t\to 0}\frac{f_1(t+\Delta t)-f_1(t)}{\Delta t}\right]\hat{i}+\left[\lim_{\Delta t\to 0}\frac{f_2(t+\Delta t)-f_2(t)}{\Delta t}\right]\hat{j}+\left[\lim_{\Delta t\to 0}\frac{f_3(t+\Delta t)-f_3(t)}{\Delta t}\right]\hat{k}$$

$$=f_1'(t)\hat{i}+f_2'(t)\hat{j}+f_3'(t)\hat{k}$$

In the Leibniz notation, the derivative of $\overline{F}(t)$ is denoted by: $\frac{d\overline{F}}{dt}=\frac{df_1}{dt}\hat{i}+\frac{df_2}{dt}\hat{j}+\frac{df_3}{dt}\hat{k}$ (iii)

Example 6 For what values of t is $G(t)=|t|\hat{i}+(\cos t)\hat{j}+(t-5)\hat{k}$ differentiable?

Solution The component functions $f_2(t)=\cos t$ and $f_3(t)=t-5$ are differentiable for all values of t, but $f_1(t)=|t|$ is not differentiable at $t=0$. Thus, the vector function $G(t)$ is differentiable for all $t\neq 0$.

Example 7 Find the derivative of the vector function $\overline{F}(t)=e^t\,\hat{i}+\sin t\,\hat{j}+(t^3+5t)\hat{k}$.

Solution Since, the given function is $\overline{F}(t)=e^t\hat{i}+\sin t\,\hat{j}+(t^3+5t)\,\hat{k}$.

Differentiate both sides w.r.t, "t"

$$\frac{d\overline{F}(t)}{dt}=\frac{d}{dt}\left[e^t\hat{i}+\sin t\,\hat{j}+(t^3+5t)\hat{k}\right]$$

$$\frac{d\overline{F}}{dt}=\frac{d}{dt}(e^t)\hat{i}+\frac{d}{dt}(\sin t)\hat{j}+\frac{d}{dt}(t^3+5t)\hat{k}=e^t\,\hat{i}+\cos t\,\hat{j}+(3t^2+5)\hat{k}$$

5.4 Vector Differentiation

Several rules for computing derivatives of vector functions are listed below, which can be proved by applying rules for limits of vector functions to appropriate theorems for scalar derivatives.

Remember

A vector $\overline{F}$ also written as ***F***.

5.4.1 Formula of differentiation

i. $\frac{da}{dt}=0$

ii. $\frac{d}{dt}[f\pm g]=\frac{df}{dt}\pm\frac{dg}{dt}$

iii. $\frac{d}{dt}[\varphi f]=\varphi\frac{df}{dt}+\frac{d\varphi}{dt}f$

iv. $\frac{d}{dt}[f.g]=f.\frac{dg}{dt}+\frac{df}{dt}.g$

v. $\frac{d}{dt}[f\times g]=f\times\frac{dg}{dt}+\frac{df}{dt}\times g$

vi. $\frac{d}{dt}\left[\frac{f}{\varphi}\right]=\frac{1}{\varphi^2}\left[\varphi\frac{df}{dt}-\frac{d\varphi}{dt}f\right]$

Where, a is a constant vector function, f and g are vector functions and φ is a scalar function of t.

I. $\frac{da}{dt}=0$

Proof: i. Let a be a constant vector function then $\frac{d}{dt}(a)=\frac{d}{dt}(a_1\hat{i}+a_2\hat{j}+a_3\hat{k})=\frac{d}{dt}a_1\hat{i}+\frac{d}{dt}a_2\hat{j}+\frac{d}{dt}a_3\hat{k}$

ii. $\frac{d}{dt}[f\pm g]=\frac{df}{dt}\pm\frac{dg}{dt}$

Proof: $\frac{da}{dt}[f\pm g]=\lim_{\Delta t\to 0}\frac{[f(t+\Delta t)\pm g(t+\Delta t)]-[f(t)\pm g(t)]}{\Delta t}=\lim_{\Delta t\to 0}\frac{f(t+\Delta t)-f(t)}{\Delta t}\pm\lim_{\Delta t\to 0}\frac{f(t+\Delta t)-g(t)}{\Delta t}$

$$=\frac{df}{dt}\pm\frac{dg}{dt}\quad or\quad \frac{d}{dt}[f+g]=\frac{df}{dt}\pm\frac{dg}{dt}$$

iii. $\frac{d}{dt}[\varphi f\pm g]=\varphi\frac{df}{dt}+\frac{d\varphi}{dt}$

Proof: $\frac{d}{dt}[\varphi f]=\lim_{\Delta t\to 0}\left[\frac{\varphi(t+\Delta t)f(t+\Delta t)-\varphi(t)f(t)}{\Delta t}\right]$

$$=\lim_{\Delta t\to 0}\left[\frac{\varphi(t+\Delta t)f(t+\Delta t)-\varphi(t+\Delta t)f(t)}{\Delta t}\right]+\lim_{\Delta t\to 0}\left[\frac{\varphi(t+\Delta t).f(t)-\varphi(t).f(t)}{\Delta t}\right]$$

$$=\lim_{\Delta t\to 0}\varphi(t+\Delta t)\left[\frac{f(t+\Delta t)-f(t)}{\Delta t}\right]+\lim_{\Delta t\to 0}\left[\frac{\varphi(t+\Delta t)-\varphi(t)}{\Delta t}\right]f(t)=\varphi\frac{df}{dt}+\frac{d\varphi}{dt}f(t)$$

Hence, $\frac{d}{dt}[\varphi f]=\varphi\frac{df}{dt}+\frac{d\varphi}{dt}.f$ or $\frac{d}{dt}[\varphi f]=\varphi\frac{df}{dt}+\frac{d\varphi}{dt}.f$

iv. $\frac{d}{dt}[f.g]=f.\frac{dg}{dt}+\frac{dg}{dt}.g$

Proof: $\frac{d}{dt}[f.g]=\lim_{\Delta t\to 0}\frac{[f(t+\Delta t).g(t+\Delta t)-f(t).g(t)]}{\Delta t}$

$$=\lim_{\Delta t\to 0}\frac{[f(t+\Delta t).g(t+\Delta t)-f(t+\Delta t).g(t)]}{\Delta t}+\lim_{\Delta t\to 0}\frac{[f(t+\Delta t).g(t)-f(t)g(t)]}{\Delta t}$$

$$=\lim_{\Delta t\to 0}f(t+\Delta t).\left[\frac{g(t+\Delta t)-g(t)}{\Delta t}\right]+\lim_{\Delta t\to 0}\left[\frac{f(t+\Delta t)-f(t)}{\Delta t}\right].g(t)=f\frac{dg}{dt}+\frac{df}{dt}g$$

Hence, $\Rightarrow\ =\frac{d}{dt}[f.g]=f.\frac{dg}{dt}+\frac{df}{dt}.g$

v. $\frac{d}{dt}[f\times g]=f\times\frac{dg}{dt}+\frac{df}{dt}+g$

Proof: $\frac{d}{dt}[f\times g]=\lim_{\Delta t\to 0}\frac{[f(t+\Delta t)\times g(t+\Delta t)]-[f(t)\times g(t)]}{\Delta t}$

$$=\lim_{\Delta t\to 0}\frac{[f(t+\Delta t)\times g(t+\Delta t)-f(t+\Delta t)\times g(t)]}{\Delta t}+\lim_{\Delta t\to 0}\frac{[f(t+\Delta t)\times g(t)-f(t)\times g(t)]}{\Delta t}$$

$$=\lim_{\Delta t\to 0}f(t+\Delta t)\times\frac{[g(t+\Delta t)-g(t)]}{\Delta t}+\lim_{\Delta t\to 0}\frac{[f(t+\Delta t)-f(t)]}{\Delta t}\times g(t)=f\times\frac{dg}{dt}+\frac{dg}{dt}\times g(t)$$

Hence, $\frac{d}{dt}[f\times g]=f\times\frac{dg}{dt}+\frac{df}{dt}\times g$

Example 8 Let $\vec{F}(t)=\hat{i}+t\hat{j}+t^2\hat{k}$ and $\vec{G}(t)=t\hat{i}+e^t\hat{j}+3\hat{k}$ are the vector functions. Verify the derivative: $\frac{d}{dt}(\vec{F}\times\vec{G})(t)=\frac{d\vec{F}}{dt}\times\vec{G}+\vec{F}\times\frac{d\vec{G}}{dt}$

Solution For verification, the L.H.S is:

$$L.H.S = \frac{d}{dt}(F \times G)(t) = \frac{d}{dt}\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & t & t^2 \\ t & e^t & 3 \end{vmatrix} = \frac{d}{dt}\left[\hat{i}\begin{vmatrix} t & t^2 \\ e^t & 3 \end{vmatrix} - \hat{j}\begin{vmatrix} 1 & t^2 \\ t & 3 \end{vmatrix} + \hat{k}\begin{vmatrix} 1 & t \\ t & e^t \end{vmatrix}\right]$$

$$= \frac{d}{dt}[(3t - t^2e^t)\hat{i} - (3 - t^3)\hat{j} + (e^t - t^2)\hat{k}] = \frac{d}{dt}(3t - t^2e^t)\hat{i} - \frac{d}{dt}(3 - t)\hat{j} + \frac{d}{dt}(e^t - t^2)\hat{k}$$

$$= (3 - 2te^t - t^2e^t)\hat{i} + 3t^2\hat{j} + (e^t - 2t)\hat{k}$$

Now, $\text{R.H.S} = \frac{d\vec{F}}{dt} \times \vec{G} + \vec{F} \times \frac{d\vec{G}}{dt}$

The expressions $\frac{d\vec{F}}{dt} = \hat{j} + 2t\hat{k}$, $\frac{d\vec{G}}{dt} = \hat{i} + e^t\hat{j}$

$$\Rightarrow \frac{d\vec{F}}{dt} \times \vec{G} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 1 & 2t \\ t & e^t & 3 \end{vmatrix} = (3 - 2te^t)\hat{i} - (-2t^2)\hat{j} + (-t)\hat{k}$$

$$\Rightarrow \vec{F} \times \frac{d\vec{G}}{dt} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & t & t^2 \\ 1 & e^t & 0 \end{vmatrix} = (-t^2e^t)\hat{i} - (-t^2)\hat{j} + (e^t - t)\hat{k}$$ are used in the RHS to obtain

$$R.H.S = \frac{d\vec{F}}{dt} \times \vec{G} + \vec{F} \times \frac{d\vec{G}}{dt} = (3 - 2te^t - t^2e^t)\hat{i} + (2t^2 + t^2)\hat{j} + (-t + e^t - t)\hat{k}$$

$= (3 - 2te^t - t^2e^t)\hat{i} + (3t^2)\hat{j} + (e^t - 2t)\hat{k}$ which is identical to the L.H.S.

Thus, the L.H.S = R.H.S.

Example 9 If $\vec{F}(t) = \hat{i} + e^t\hat{j} + t^2\hat{k}$ and $\vec{G}(t) = 3t^2\hat{i} + e^{-t}\hat{j} - 2t\hat{k}$ are the two vector functions and $h(t)$ is any scalar function, then evaluate the following derivatives: *(a).* $\frac{d}{dt}\left(2\vec{F} + t^3\vec{G}\right)$ *(b).* $\frac{d}{dt}\left(\vec{F}.\vec{G}\right)$

Solution

a. $$\frac{d}{dt}\left(2\vec{F} + t^3\vec{G}\right) = \frac{d}{dt}\left[2\left(\hat{i} + e^t\hat{j} + t^2\hat{k}\right) + t^3\left(3t^2\hat{i} + e^{-t}\hat{j} - 2t\hat{k}\right)\right] = \frac{d}{dt}\left[(2 + 3t^5)\hat{i} + (2e^t + t^3e^{-t})\hat{j} + (2t^2 - 2t^4)\hat{k}\right]$$

$$= 15t^4\hat{i} + \left(2e^t + 3t^2e^{-t} - t^3e^{-t}\right)\hat{j} + (4t - 8t^3)\hat{k} = 15t^4\hat{i} + \left(2e^t + t^2e^{-t}(3 - t)\right)\hat{j} + 4t(1 - 2t^2)\hat{k}$$

b. $$\frac{d}{dt}\left(\vec{F}.\vec{G}\right) = \frac{d}{dt}\left[(\hat{i} + e^t\hat{j} + t^2\hat{k}).(3t^2\hat{i} + e^{-t}\hat{j} - 2t\hat{k})\right] \qquad \therefore \hat{i}.\hat{i} = \hat{j}.\hat{j} = \hat{k}.\hat{k} = 1$$

$$= \frac{d}{dt}(3t^2 + 1 - 2t^3) = 6t + 0 - 6t^2 = -(6t^2 - 6t)$$

5.4.2 Applications of vector differentiation to calculate velocity and acceleration

In the calculus of single variable the velocity is defined as the derivative of the position function. For vector calculus we use the same definition.

i. Velocity

Let $\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$ be a differential vector valued function representing the position vector of a particle at time 't' then the velocity vector is the derivative of position vector.

$$v(t) = \vec{r}'(t) = x'(t)\hat{i} + y'(t)\hat{j} + z'(t)\hat{k} \quad \text{or} \quad \frac{d\vec{r}}{dt} = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k}$$

Example 10 Find the velocity of the particle whose position vector is $\vec{r} = \vec{r}(t) = 5t\hat{i} + 4t\hat{j} - \cos(t)\hat{k}$

Solution Since, $\vec{r}=5t\hat{i}+4t\hat{j}-\cos(t)\hat{k}$

$$\text{Velocity}=\frac{d}{dt}(\vec{r})=\frac{d}{dt}(5t\hat{i}+4t\hat{j}-\cos(t)\hat{k})\Rightarrow\frac{d\vec{r}}{dt}=5\hat{i}+4\hat{j}+\sin(t)\hat{k}$$

ii. Acceleration

In the calculus of single variable, we defined the acceleration of a particle as the second derivative of the position vector. There is no change for the vector calculus.

"Let $\vec{r}=\vec{r}(t)=x\hat{i}+y\hat{j}+z\hat{k}$ be a twice differentiable vector valued function, representing the position vector of a particle at time 't'. Then the acceleration vector is the second derivative of the position vector $\vec{r}(t)$

$$\vec{a}=\vec{r}''(t)=x''(t)\hat{i}+y''(t)\hat{j}+z''(t)\hat{k} \qquad \text{or} \qquad \vec{a}=\frac{d^2\vec{r}}{dt^2}=\frac{d^2x}{dt^2}\hat{i}+\frac{d^2y}{dt^2}\hat{j}+\frac{d^2z}{dt^2}\hat{k}$$

Example 11 Find the acceleration of the particle whose position vector is
$\vec{r}(t)=(3t^2+5)\hat{i}-(4t^2+2t-1)\hat{j}+\sin(t)\hat{k}$

Solution Since, $\vec{r}(t)=(3t^2+5)\hat{i}-(4t^2+2t-1)\hat{j}+\sin(t)\hat{k}$

$$\frac{d}{dt}(\vec{r})=\frac{d}{dt}\left[(3t^2+5)\hat{i}-(4t^2+2t-1)\hat{j}+\sin(t)\hat{k}\right]=\frac{d}{dt}(3t^2+5)\hat{i}-\frac{d}{dt}(4t^2+2t-1)\hat{j}+\frac{d}{dt}\sin(t)\hat{k}$$

$$\frac{d\vec{r}}{dt}=6t\hat{i}-(8t+2)\hat{j}+\cos(t)\hat{k}$$

$$\Rightarrow\frac{d}{dt}\left[\frac{d\vec{r}}{dt}\right]=\frac{d}{dt}\left[6t\hat{i}-(8t+2)\hat{j}+\cos(t)\hat{k}\right]$$

$$\Rightarrow\left[\frac{d^2\vec{r}}{dt^2}\right]=6\hat{i}-8\hat{j}-\sin\hat{k}$$

iii. Speed

In the calculus of single variable the speed was the absolute value of the velocity. In the vector calculus it is the magnitude of velocity vector.

Let $\vec{r}(t)$ a differentiable vector valued function representation the position ofa particle in time 't' the speed 's' of the particle is the magnitude of the velocity vector. Speed$=\vec{S}(t)=\left|\vec{V}(t)\right|=\left|\vec{r}'(t)\right|$

Remember

The direction of motion can be calculate by using $\dfrac{\vec{V}}{|\vec{V}|}$

Example 12 Fin the speed of particle whose position vector is $\vec{r}(t)=3t\hat{i}+4\hat{j}+\sin(t)\hat{k}$ after 30 seconds.

Solution Since, $\vec{r}(t)=3t\hat{i}+4\hat{j}+\sin(t)\hat{k}\ \Rightarrow\vec{V}(t)=\frac{d}{dt}(r)=\frac{d}{dt}(3t\hat{i}+4\hat{j}+\sin(t)\hat{k})$

$$\vec{V}(t)=3\hat{i}+\cos(t)\hat{k})\ \Rightarrow\vec{V}(30)=3\hat{i}+\cos(30)\hat{k})\ \Rightarrow\vec{V}=3\hat{i}+\frac{\sqrt{3}}{2}\hat{k}$$

$$\text{Speed }=\left|\vec{V}\right|=\sqrt{(3)^2+\left(\frac{\sqrt{3}}{2}\right)^2}=\sqrt{9+\frac{3}{4}}=\sqrt{\frac{36+3}{4}}=\frac{\sqrt{39}}{2}=3.12m/s$$

Example 13 A particle's position at time' t' is determined by the vector $\vec{r}(t)=\cos(t)\hat{i}+\sin(t)\hat{j}+t^3\hat{k}$. Find the particle's velocity, speed, direction and acceleration at a time $t=2$. Interpret the particle's motion.

Solution If the particle's position at a time t is , then $\vec{r}(t)=\cos t\,\hat{i}+\sin t\,\hat{j}+t^3\hat{k}$ then, the particle's velocity and acceleration are: $\vec{V}(t)=\frac{d}{dt}(\vec{r})=\frac{d}{dt}\left[\cos(t)\hat{i}+\sin(t)\hat{j}+t^3\hat{k}\right]=-\sin t\,\hat{i}+\cos t\,\hat{j}+3t^2\hat{k}$

$$a(t)=\frac{d\vec{V}}{dt}=\frac{d}{dt}\left(\frac{d\vec{r}}{dt}\right)=\frac{d}{dt}\left[-\sin(t)\hat{i}+\cos(t)\hat{j}+3t^2\hat{k}\right]=-\cos(t)\hat{i}-\sin(t)\hat{j}+6(t)\hat{k}$$

The velocity at a time $t=2$ is $\vec{V}(2)=-\sin(2)\hat{i}+\cos(2)\hat{j}+3(4)\hat{k}\approx-0.91\hat{i}-0.42\hat{j}+12\hat{k}$, *use radians*

The acceleration at a time t = 2 is $\vec{a}(2)=-\cos(2)\hat{i}-\sin(2)\hat{j}+6(2)\hat{k}\approx 0.42\hat{i}-0.91\hat{j}+12\hat{k}$

The speed is $|\vec{V}|=\sqrt{(-\sin t)^2+(\cos t)^2+(3t^2)^2}=\sqrt{1+9t^4}$ .At a time $t=2$,

The speed is $|\vec{V}|=\sqrt{2+9(2)^4}=\sqrt{145}$.

The direction of motion is: $\frac{\vec{V}}{|\vec{V}|}=\frac{1}{\sqrt{145}}\left[-\sin t\,\hat{i}+\cos t\,\hat{j}+3t^2\hat{k}\right]$

At a time t=2, the direction of motion is: $\frac{\vec{V}}{|\vec{V}|}=\frac{1}{\sqrt{145}}\left[-\sin 2\hat{i}+\cos 2\hat{j}+12\hat{k}\right]\approx-0.91\hat{i}-0.42\hat{j}+12\hat{k}$

Exercise 5.2

1. Find the vector derivative of the following vector functions:
   a. $\vec{F}(t)=t\hat{i}+t^2\hat{j}+(t+t^3)\hat{k}$
   b. $\vec{F}(s)=(s\hat{i}+s^2\hat{j}+s^2\hat{k})+(2s^2\hat{i}-s\hat{j}+3\hat{k})$
   c. $\vec{F}(\theta)=\cos\theta\left[\hat{i}+\tan\theta\,\hat{j}+3\hat{k}\right]$
2. Find the second order derivatives of the following vector valued functions.
   a. $\vec{F}(t)=t^2\hat{i}+3t^3\hat{j}-8t^2\hat{k}$
   b. $\vec{F}(s)=(3+s^2)\hat{i}-(s+1)^2\hat{j}+3s^4\hat{k}$
   c. $\vec{F}(x)=\ln x\hat{i}-x^2\hat{k}$
   d. $\vec{F}(\theta)=\sin^2\theta\,\hat{i}-\cos^2\theta\,\hat{j}$
3. Differentiate the following scalar functions:
   a. $\vec{f}(x)=[x\hat{i}+(x+1)\hat{j}].[2x\hat{i}-3x^2\hat{j}]$
   b. $\vec{g}(x)=\left|\sin x\hat{i}-2x\hat{j}+\cos x\hat{k}\right|$
4. Find the particle's velocity, acceleration, speed and direction of motion for the indicated value of t, when the position vector of a particle's in space at time t is $\vec{r}(t)$:
   a. $\vec{r}(t)=t\hat{i}+t^2\hat{j}+2t\hat{k}$ *at* $t=1$
   b. $\vec{r}(t)=\cos t\,\hat{i}+\sin t\,\hat{j}+3t\hat{k}$ *at* $t=\frac{\pi}{4}$
   c. $\vec{r}(t)=e^t\hat{i}+e^{-t}\hat{j}+e^{2t}\hat{k}$ *at* $t=\ln 2$
5. If $F(t)$ is a differentiable vector functions of t such that $F(t)\neq 0$, then show that

$$\frac{d}{dt}\frac{F(t)}{|F(t)|}=\frac{F'(t)}{|F(t)|}-\frac{[F(t).F'(t)]F(t)}{|F(t)|^3}$$

Review Exercise 5

1. Choose the correct option.

i. A quantity having magnitude and direction is called:
(a). vector (b). scalar (c). velocity (d). derivative

ii. A quantity having magnitude only and no direction is called:
(a). vector (b). scalar (c). velocity (d). derivative

iii. If $\vec{F} = t^2\hat{i} + t\hat{j} - \sin(t)\hat{k}$ and $\vec{G} = t\hat{i} + t^{-1}\hat{j} + 5\hat{k}$ then $\vec{F} \times \vec{G} =$

(a). $\left(5t^2 + \frac{\sin(t)}{t}\right)\hat{i} - (5t^2 + t\sin(t))\hat{j} - (t + t^2)\hat{k}$ (b). $\left(5t + \frac{\sin(t)}{t}\right)\hat{i} + (5t^2 + t\sin(t))\hat{j} + (t^2 - t)\hat{k}$

(c). $\left(5t + \frac{\sin(t)}{t}\right)\hat{i} - (5t^2 + t\sin(t))\hat{j} + (t - t^2)\hat{k}$ (d). $\left(5t^2 + \frac{\sin(t)}{t}\right)\hat{i} - (5t^2 + t\sin(t))\hat{j} + (t - t^2)\hat{k}$

iv. Let $\vec{F} = t^2\hat{i} + t\hat{j} - \sin(t)\hat{k}$ and $\vec{G} = t\hat{i} + t^{-1}\hat{j} + 5\hat{k}$ then $\vec{F} \cdot \vec{G} =$
(a). $t^3 + 1 + 5\sin(t)$ (b). $t^2 - 1 + 4\sin(t)$ (c). $3t^2 - 1 + 4\sin(t)$ (d). $t^3 + 1 - 5\sin(t)$

v. If $\vec{F}(t) = (t+3)\hat{i} + 2t^2\hat{j} - (1-t)\hat{k}$ then $\lim\limits_{x \to 5} \vec{F}(-2) =$ is:
(a). $8\hat{i} + 50\hat{j} + 4\hat{k}$ (b). $8\hat{i} + 10\hat{j} - 4\hat{k}$ (c). $8\hat{i} + 60\hat{j} - 4\hat{k}$ (d). $8\hat{i} + 50\hat{j} + 4\hat{k}$

vi. If $\vec{F} = t^3\hat{i} - (3+t^2)\hat{k}$ then $\vec{F}'(-2)$ is:
(a). $4\hat{i} - 12\hat{k}$ (b). $12\hat{i} - 4\hat{k}$ (c). $12\hat{i} + 4\hat{k}$ (d). $12\hat{i} - 8\hat{k}$

vii. If $\vec{r} = 2\hat{i} + 3\hat{j} + 4\hat{k}$ and $|\vec{r}| =$:
(a). $\sqrt{29}$ (b). $\sqrt{30}$ (c). 5 (d). 6

viii. If $\vec{r} = 5t^2\hat{i} + 3t\hat{j} + \hat{k}$ then velocity vector $\vec{v}$ is:
(a). $10t\hat{i} - 3\hat{j}$ (b). $5t\hat{i} + 3\hat{k}$ (c). $2t\hat{i} + 3\hat{j}$ (d). $10\hat{t} - 3\hat{k}$

ix. If velocity vector $\vec{v} = \sin(t)\hat{i} - 2\cos(t)\hat{j} + 4\hat{k}$ then acceleration $\vec{a} =$
(a). $-\sin(t)\hat{i} + 2\cos(t)\hat{j} - 4\hat{k}$ (b). $\cos(t)\hat{i} - 2\sin(t)\hat{j} + 4\hat{k}$
(c). $-\cos(t)\hat{i} + 2\sin(t)\hat{j}$ (d). $\cos(t)\hat{i} - 2\sin(t)\hat{j}$

x. $\underset{t \to t_0}{Lim}(C.\vec{F}(t)) =$
(a). $C \cdot \vec{F}(t)$ (b). $C \cdot F(t_0)$ (c). $C + \vec{F}(t_0)$ (d). $\dfrac{C}{\vec{F}(t_0)}$

Summary

- The **parametric equation** for the plane curve C generated by the set of ordered pairs in 2-space is: $(x, y)=(x(t), y(t))=(f(t), g(t))$
- The **parametric equation** for the plane curve C generated by the set of ordered triples in 3-space is: $(x, y, z)=(x(t), y(t), z(t))=(f(t), g(t), h(t))$
- A vector function $\vec{F}(t)$ F(t) is **continuous** at $t=t_o$ if t_o is in the domain of $F(t)$ $\lim_{t\to t_0} F(t)=F(t_0)$
- The derivative of a vector function $F(t)$ is the vector function $F'(t)$ determined by the limit

$$F'(t)=\lim_{\Delta t\to 0}\frac{\Delta\vec{F}}{\Delta t}=\lim_{\Delta t\to 0}\frac{\vec{F}(t+\Delta t)-\vec{F}(t)}{\Delta t},$$

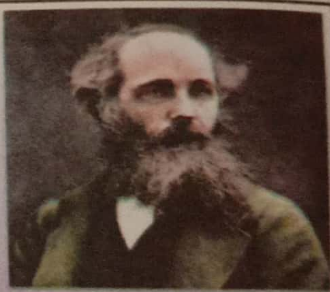
whenever this limit exists. In the Leibniz notation, the derivative of $F(t)$ is denoted by:

$$\frac{dF}{dt}=\lim_{\Delta t\to 0}\frac{\Delta F}{\Delta t}=\lim_{\Delta t\to 0}\frac{F(t+\Delta t)-F(t)}{\Delta t}$$

- If an object moves in such a way that its position at any time t is the **position vector** or displacement $R(t)$, then the
  - **Velocity is** $\vec{v}=\frac{d\vec{r}}{dt}=\vec{r}'(t)$
  - **Acceleration is** $\vec{a}=\frac{d\vec{v}}{dt}=\frac{d^2\vec{v}}{dt^2}=\vec{r}''(t)$
  - At any time t, **the speed is** $|\vec{v}|=|v_1\hat{i}+v_2\hat{j}+v_3\hat{k}|=\sqrt{v_1^2+v_2^2+v_3^2}$, the magnitude of the velocity and **the direction of motion is** $\frac{\vec{v}}{|\vec{v}|}$.

History

J.C. Maxwell was Scottish mathematician. He made a great contributions in the field of mathematical physics. He formulated the classical theory of electromagnetic radiation. Maxwell's equations for electromagnetism has been called the second great unification in physics. But the first unification in physics was raised by Sir Isaac Newton. In his publication "A Dynamical theory of electromagnetic field" he demonstrated electric and magnetic fields travel through space as waves moving at the speed of light. On the bases of his idea in electromagnetism Gibbs and Oliver developed vector analysis.



James Clark Maxwell
(1831)-(1879)

Project

Create an art on a chart paper by hand or use any technological mean. Your creation should demonstrate a topic from this unit.
Create something using your imagination or use the mathematical concepts discussed in this unit to create your real world object

Unit 6

INTEGRATION

By the end of this unit, the students will be able to:

6.1 Introduction

i. Demonstrate the concept of the integral as an accumulator.
ii. Know integration as inverse process of differentiation.
iii. Explain constant of integration.
iv. Know simple standard integrals which directly follow from standard differentiation formulae.

6.2 Rules of Integration

i. Recognize the following rules of integration.

- $\int \frac{d}{dx}[f(x)]dx = \frac{d}{dx}\int[f(x)dx] = f(x)+c$
- where c is a constant of integration.
- the integral of the product of a constant and a function is the product of the constant and the integral of the function.
- the integral of the sum of a finite number of functions is equal to the sum of their integrals.

ii. Use standard differentiation formulae to prove the results for the following integrals:

- $\int [f(x)]^n f'(x)dx$
- $\int \frac{f'(x)}{f(x)}dx$
- $\int e^{ax}[af(x)+f'(x)]dx$

6.3 Integration by substation

i. Explain the method of integration by substitution.
ii. Apply method of substitution to evaluate indefinite integrals.
iii. Apply method of substitution to evaluate integrals of the following types:

- $\int \frac{dx}{a^2-x^2}$, $\int \sqrt{a^2-x^2}\,dx$, $\int \frac{dx}{\sqrt{a^2-x^2}}$,
- $\int \frac{dx}{a^2+x^2}$, $\int \sqrt{a^2+x^2}\,dx$, $\int \frac{dx}{\sqrt{a^2+x^2}}$,
- $\int \frac{dx}{x^2-a^2}$, $\int \sqrt{x^2-a^2}\,dx$, $\int \frac{dx}{\sqrt{x^2-a^2}}$,
- $\int \frac{dx}{ax^2+bx+c}$, $\int \frac{dx}{\sqrt{ax^2+bx+c}}$,
- $\int \frac{px+q}{ax^2+bx+c}dx$, $\int \frac{px+q}{\sqrt{ax^2+bx+c}}dx$

6.4 Integration by parts

i. Recognize the formula for integration by parts.
ii. Apply method of integration by parts to evaluate integrals of the following types:

- $\int \sqrt{a^2-x^2}\,dx$, $\int \sqrt{a^2+x^2}\,dx$, $\int \sqrt{x^2-a^2}\,dx$,

iii. Evaluate integrals using integration by parts.

6.5 Integration using Partial Fractions

i. Use partial fractions to find $\int \frac{f(x)}{g(x)}dx$, where $f(x)$ and $g(x)$ are algebraic functions such that $g(x) \neq 0$.

6.6 Definite Integrals

i. Define definite integral as the limit of a sum.
ii. Describe the fundamental theorem of integral calculus and recognize the following basic properties:

- $\int_a^a f(x)dx = 0,$
- $\int_a^b f(x)dx = \int_a^b f(y)dy,$
- $\int_a^b f(x)dx = -\int_b^a f(x)dx,$
- $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx,\ a<c<b,$
- $\int_{-a}^{a} f(x)dx = \begin{cases} 2\int_0^a f(x)dx & \text{when } f(-x)=f(x), \\ 0 & \text{when } f(-x)=-f(x) \end{cases}$

iii. Extend techniques of integration using properties to evaluate definite integrals.
iv. Represent definite integral as the area under the curve.
v. Apply definite integrals to calculate area under the curve.
vi. Use MAPLE command in to evaluate definite and indefinite integrals.

6.1 Introduction

We have studied in the previous units 4 and 5 about the derivative of a function and various applications of derivatives. A material that belongs to the branch of calculus is called **differential calculus**. In this unit, we will study another branch of calculus, which is called **integral calculus**. Like the derivative of a function, the definite integral of a function is a special limit with many diverse applications. Geometrically, the derivative is related to the slope of the tangent line to the curve, while the reverse of derivative is the definite integral which is related to the area under a curve.

6.1.1 Concept of the integral as an accumulator

Functions used in applications in previous units have provided information about a total amount of a quantity, such as cost, revenue, profit, temperature, gallons of oil or distance. Derivatives of these functions provided information about the rate of change of these quantities and allowed us to answer important questions about the extrema of the functions. It is not always possible to find ready-made functions that provide information about the total amount of a quantity, but it is often possible to collect enough data to come up with a function that gives the rate of change of a quantity.

Accumulative point of view is that the derivative gives the rate of change when the total amount is known; the reverse process of derivative gives the total amount of a quantity, when the rate of change of a quantity is known. This reverse process of derivative or **antiderivative** or **antidifferentiation** (inverse process of differentiation) is the main topic of this unit.

6.1.2 Integration as inverse process of differentiation

If $F(x)$ is any unknown function and its derivative is $F'(x) = f(x)$, say, then the reverse process is to give a function $F(x)$ such that its derivative $F'(x)$ is equal to $f(x)$:

$$F'(x) = f(x) \qquad \text{(i)}$$

A function $F(x)$ is called an **antiderivative** of a function $f(x)$ on the interval $[a,b]$, if at all points of the interval, the identity $F'(x) = f(x)$ is true.

Example 1 Find the antiderivative of a function $f(x) = x^2$.

Solution From the definition of an antiderivative, it follows that the function $F(x) = \frac{x^3}{3}$ has an antiderivative of $f(x) = x^2$, since $F'(x) = f(x) = x^2$.

6.1.3 Constant of Integration

It is easy to see that if the given function $f(x)$ exists an antiderivative, then this antiderivative will not be the only one. In the example 1, we will take the following functions as antiderivatives

$$F(x) = \frac{x^3}{3} + 1,\quad F(x) = \frac{x^3}{3} - 7,\ F(x) = \frac{x^3}{3} + C, \text{ of a function } f(x) = x^2, \text{ since } F'(x) = f(x).$$

It may be proved that the functions of the form $\frac{x^3}{3} + C$ for any constant value of C exhaust all antiderivatives of the function $f(x) = x^2$. This develops a consequence of the following theorem:

Theorem 1: If $F_1(x)$ and $F_2(x)$ are two antiderivatives of a function $f(x)$ on an interval [a, b], then the difference between them is a constant quantity.

Proof: If $F_1'(x)$ and $F_2'(x)$ are the derivatives of $F_1(x)$ and $F_2(x)$, then by virtue of the definition of an antiderivative, we have

$$F_1(x) = f(x),\ F_2'(x) = f(x) \qquad \text{(i)}$$

If we put the difference of $F_1(x)$ and $F_2(x)$ by $F_1(x) - F_2(x) = \phi(x)$,
then by virtue of the definition of an antiderivative for any value of x on the interval $[a, b]$:

$$F_1'(x)-F_2'(x)=f(x)-f(x)$$

$$\frac{d}{dx}\left(F_1(x)-F_2(x)\right)=0 \Rightarrow \frac{d}{dx}\left(\phi(x)\right)=0 \Rightarrow \phi'(x)=0$$

From $\phi'(x)=0$, it follows that $\phi(x)=C$ is any constant quantity. It follows that if a given function $f(x)$ has an antiderivative $F(x)$, then any other antiderivative of $f(x)$ will be of the form $F(x)+C$, for any constant value of C.

The process of finding antiderivative is then called the **antidifferentiation** or **inverse of differentiation** or **integration.**

If the function $F(x)$ is an antiderivative of $f(x)$, then the expression $F(x)+C$ is of course the **indefinite integral** of the function $f(x)$ and is denoted by the symbol:

$$\int f(x)dx=F(x)+C \quad \text{if} \qquad F'(x)=f(x) \qquad \text{(ii)}$$

The symbol $\int$ is called an **integral sign** and the function $f(x)$ is called the **integrand.** The symbol ***dx*** indicates that the antiderivative is performed with respect to the variable x. The arbitrary constant C is called the **constant of integration**. Referring to the preceding discussion, we can write

$$\int 2xdx=x^2+C, \qquad \text{since} \quad \frac{d}{dx}(x^2+C)=2x$$

The variables other than x can also be used in indefinite integrals. For example,

$\int 2tdt=t^2+C$, since $\frac{d}{dt}(t^2+C)=2t$, $\int 2udu=u^2+C$, since $\frac{d}{du}(u^2+C)=2u$ etc.

6.1.4 Standard Integrals through standard differentiation formulae

If $f(x)$ is any function of x, then in different situations, the integral of $f(x)$ can be found directly in light of indefinite integral definition from the chart of integrals and its differentiation formulae. The truth of the integrals can easily be checked by differentiating the right side of the integral which always equals the integrand.

Integrals	Differentiation Formulae
1. $\int x^n dx=\frac{x^{n+1}}{n+1}+C$, $n\neq -1$	$\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}+C\right)=x^n$, $n\neq -1$
2. $\int \frac{dx}{x}=\ln\lvert x\rvert+C$	$\frac{d}{dx}(\ln x+C)=\frac{1}{x}$
3. $\int \cos kx\, dx=\frac{\sin kx}{k}+C$	$\frac{d}{dx}\left(\frac{\sin kx}{k}+C\right)=\cos kx$
4. $\int \sin kx\, dx=\frac{-\cos kx}{k}+C$	$\frac{d}{dx}\left(\frac{-\cos kx}{k}+C\right)=\sin kx$
5. $\int \frac{dx}{\cos^2 x}=\tan x+C$	$\frac{d}{dx}(\tan x+C)=\sec^2 x=\frac{1}{\cos^2 x}$
6. $\int \frac{dx}{\sin^2 x}=-\cot x+C$	$\frac{d}{dx}(-\cot x+C)=co\sec^2 x=\frac{1}{\sin^2 x}$
7. $\int \tan dx=-\ln\lvert\cos x\rvert+C$	$\frac{d}{dx}(-\ln\cos x+C)=\tan x$

8. $\int \cot x\,dx = \ln\lvert\sin\rvert + C$	$\frac{d}{dx}(\ln \sin x + C) = \cot x$
9. $\int e^{mx}\,dx = \frac{e^{mx}}{m} + C$	$\frac{d}{dx}\left(\frac{e^{mx}}{m} + C\right) = e^{mx}$
10. $\int a^x\,dx = \frac{a^x}{\ln a} + C$	$\frac{d}{dx}\left(\frac{a^x}{\ln a} + C\right) = a^x$
11. $\int \frac{dx}{1+x^2} = \tan^{-1} x + C$	$\frac{d}{dx}(\tan^{-1} x + C) = \frac{1}{1+x^2}$
12. $\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$	$\frac{d}{dx}\left(\frac{1}{a}\tan^{-1}\frac{x}{a} + C\right) = \frac{1}{a^2+x^2}$
13. $\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\ln\left\lvert\frac{a+x}{a-x}\right\rvert + C$	$\frac{d}{dx}\left(\frac{1}{2a}\ln\left\lvert\frac{a+x}{a-x}\right\rvert + C\right) = \frac{1}{a^2-x^2}$
14. $\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\ln\left\lvert\frac{x-a}{x+a}\right\rvert + C$	$\frac{d}{dx}\left(\frac{1}{2a}\ln\left\lvert\frac{x-a}{x+a}\right\rvert + C\right) = \frac{1}{x^2-a^2}$
15. $\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + C$	$\frac{d}{dx}(\sin^{-1} x + C) = \frac{1}{\sqrt{1-x^2}}$
16. $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a} + C$	$\frac{d}{dx}\left(\sin^{-1}\frac{x}{a} + C\right) = \frac{1}{\sqrt{a^2-x^2}}$
17. $\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \ln\left\lvert x + \sqrt{x^2 \pm a^2}\right\rvert + C$	$\frac{d}{dx}\left(\ln\left\lvert x + \sqrt{x^2 \pm a^2}\right\rvert + C\right) = \frac{1}{\sqrt{x^2 \pm a^2}}$

Example 2 Use the chart of integral formulae to evaluate the following integrals:

(a). $\int x^{\frac{1}{3}} dx$ (b). $\int \frac{1}{x^3} dx$

Solution

a. Formula serial number (1) for $n = \frac{1}{3}$ is used to obtain: $\int x^{\frac{1}{3}} dx = \frac{x^{\frac{1}{3}+1}}{\frac{1}{3}+1} + C = \frac{x^{\frac{4}{3}}}{\frac{4}{3}} + C = \frac{3}{4}x^{\frac{4}{3}} + C$

b. Formula (1) for $n = -3$ is used to obtain: $\int \frac{1}{x^3} dx = \frac{x^{-3+1}}{-3+1} + C = \frac{x^{-2}}{-2} + C = \frac{-1}{2x^2} + C$

6.2 Rules of Integration

6.2.1 Recognition of rules of integration

a. $\int \frac{d}{dx}[f(x)]dx = \frac{d}{dx}\left[\int f(x)dx\right] = f(x) + c$, **where c is constant of integration**

It is extremely important to recognize that differentiation is the inverse operation of integrati

$$\frac{d}{dx}\left[\int f(x)dx\right] = f(x) + C \qquad \text{(i)}$$

and the integration is the inverse operation of differentiation:

$$\int f'(x)dx = f(x) + C \qquad \text{(ii)}$$

The notations (i) and (ii) are illustrated by the following examples. Let us begin with equation If

If $f(x)=x^3$, then the integral of $f(x)$ is: $\int x^3 dx=\frac{x^4}{4}+C$

Now the derivative of the integral is: $\frac{d}{dx}\left[\frac{x^4}{4}+C\right]=\frac{4x^3}{4}=x^3$

This illustrates that the derivative of the integral of $f(x)$ is equal to $f(x)$. Now look at equation (ii). If

If $f(x)=x^3$ and $f'(x)=3x^2$, then the integral of $f'(x)$ is $\int 3x^2 dx=\frac{3x^3}{3}+C=x^3+C$, where c is a constant.

This illustrates that the derivative of integral of $f(x)$ is equal to $f(x)$ plus a constant quantity C. This is summarized in $f(x)$ box:

If all integrals exist, then:	1.	$\frac{d}{dx}\left[\int f(x)dx\right]=f(x)$
	2.	$\int f'(x)dx=f(x)+C$ Where c is constant of integration.

ii. The integral of the product of a constant and a function is the product of the constant and the integral of the function.

A constant factor may be taken outside the integral sign. If k is any real constant, then consider.

$$\int k\,f(x)\,dx=k\int f(x)dx \qquad \text{(i)}$$

To establish that the expressions to the right and left are the same in the sense indicated above, it is necessary to prove that their derivatives with respect to x are equal. Then the derivative of equation (i) w.r.t. x is:

$$\frac{d}{dx}\left[\int k\,f(x)\,dx\right]=\frac{d}{dx}\left[k\int f(x)\right] \Rightarrow kf(x)=kf(x) \qquad \text{(ii)}$$

The derivatives of the right and left sides of (ii) are equal, therefore, as in (i).

iii. The integral of the sum of a finite number of functions is equal to the sum of their integrals

The indefinite integral of an algebraic sum of two or more functions is equal to the algebraic sum of their integrals. So,

$$\int[f(x)+g(x)]dx=\int f(x)dx+\int g(x)dx \qquad \text{(i)}$$

The derivative of (i) w.r.t x is: $\frac{d}{dx}\left[\int[f(x)+g(x)]\right]dx=\frac{d}{dx}\left[\int f(x)dx+\int g(x)dx\right]$

$$f(x)+g(x)=f(x)+g(x) \qquad \text{(ii)}$$

The derivatives of the right and left sides of (ii) are equal, therefore as in (i).

These rules are summarized in the following box:

1. Constant Rule:	$\int kdx=kx+C,$ k is constant
2. Constant Multiple Rule:	$\int k\,f(x)dx=k\,f(x)+C$
3. Sum Rule:	$\int[f(x)+g(x)]dx=\int f(x)\,dx+\int g(x)\,dx$
4. Power Rule:	$\int x^n dx=\frac{x^{n+1}}{n+1}+C,\ n\neq-1$
5. Logarithmic Rule:	$\int\frac{1}{x}dx=\ln\lvert x\rvert+C,\ x\neq0$

Example 3 Evaluate the following integrals

(a). $\int(10+a)dx$ (b). $\int(10+a)x^2dx$ (c). $\int(5x^2+4x+2)dx$

Solution

a. The constant rule of integration is used to obtain: $\int(10+a)dx=(10+a)x+C$

b. The constant multiple rule of integration is used to obtain: $\int(10+a)x^2dx=(10+a)\frac{x^3}{3}+C=\frac{(10+a)x^3}{3}+C$

c. The sum rule of integration is used to obtain:

$$\int(5x^2+4x+2)dx=\int5x^2dx+\int4xdx+\int2dx+C=5\frac{x^3}{3}+4\frac{x^2}{2}+2x+C$$

$$=\frac{5x^3}{3}+\frac{4x^2}{2}+2x+C=\frac{5x^3}{3}+2x^2+2x+C$$

6.2.2 Standard differentiation formulae

A. $\int[f(x)]^nf'(x)dx=\frac{(f(x))^{n+1}}{n+1}+C,\ n\neq-1$

Proof: Let it will be required to find the integral

$$\int[f(x)]^n\,dx=? \qquad \text{(i)}$$

For this integral, we are not in position to directly select the antiderivative of $f(x)$, however the integral exists. In this situation, we need to change the variable x in the expression under the integral sign by putting: $x=u(t)$ and $\frac{dx}{dt}=u'(t)$

These are used in (i) to obtain:

$$\int[f(x)]^n\,dx=\int[f[u(t)]]^nu'(t)dt \qquad \text{(ii)}$$

To establish that the expressions to the right and left are the same in the sense indicated above, it is necessary to prove that their derivatives with respect to x are equal. Then the derivative of equation (ii) w.r.t. x is: $\frac{d}{dx}\left[\int[f(x)]^n\,dx\right]=\frac{d}{dx}\left[\int[f[u(t)]]^nu'(t)dt\right]$

$$[f(x)]^n=\frac{d}{dx}\left[\int[f[u(t)]]^nu'(t)dt\right]\times\frac{dt}{dt}=\frac{d}{dt}\left[\int[f[u(t)]]^nu'(t)dt\right]\times\frac{dt}{dx}$$

$$=[f[u(t)]]^nu'(t)\times\frac{1}{u'(t)},\quad dx/dt=u'(t)$$

$$=[f[u(t)]]^n=[f(x)]^n$$

The derivatives of the right and left sides of the expression are equal, therefore, as in equation (ii).

B. $\int\frac{f'(x)}{f(x)}dx=\ln|f(x)|+C$

Proof: We have to show that $\int\frac{f'(x)}{f(x)}dx=\ln|f(x)|+c...$ (i)

Take L.H.S of equation (i) $\int\frac{f'(x)}{f(x)}dx$

Let $f(x)=u\Rightarrow f'(x)dx=du$

$$\int\frac{f'(x)}{f(x)}dx=\int\frac{du}{u}=\ln|u|+c=\ln|f(x)|+c$$ where, c is a constant

C. $\int e^{ax}\left[af(x)+f'(x)\right]dx$

Proof: This proof can be done by using integration by parts. So, it is given in the Section 6.4.

6.3 Integration by Substitution

6.3.1 Method of integration by substitution

In previous section, we saw how to integrate a few simple functions. More complicated functions can sometimes be integrated by substitution. The technique depends on the idea of a differential. If $u = f(x)$, then, the differential of u, written du, is defined as $du = f'(x)dx$.

Differentials have many useful interpretations which are studied in more advanced courses. We shall only use them as a convenient notational device when finding an antiderivative such as

$$\int x^2\sqrt{x^3+1}\,dx \qquad \text{(i)}$$

The function $x^2\sqrt{x^3+1}$ is reminiscent of the chain rule and so we shall try to use differentials and the chain rule in reverse to find the antiderivative. Let $u = x^3 + 1$, then $du = 3x^2dx$. Now substitute u for $x^3 + 1$ and du for $3x^2dx$ in the indefinite integral (i) to obtain

$$\int x^2\sqrt{x^3+1}\,dx = \int \frac{\sqrt{u}}{3}du \qquad \therefore u = x^3+1, \frac{du}{dx} = 3x^2$$

$$= \frac{1}{3}\frac{u^{\frac{1}{2}+1}}{\frac{1}{2}+1}+C = \frac{1}{3}\left(\frac{2}{3}\right)u^{\frac{3}{2}}+C = \frac{2}{9}u^{\frac{3}{2}}+C = \frac{2}{9}\left(x^3+1\right)^{\frac{3}{2}}+C$$

This method of integration is called **integration by substitution.** As shown above, it is simply the chain rule for derivative in reverse. The results can always be verified by differentiation.

General Indefinite Integral Formulae:

If u is a function of x, where $u = f(x)$ and $du = f'(x)dx$, then

1. $\int [f(x)]^n f'(x)dx = \int [u(x)]^n du = \frac{[u(x)]^{n+1}}{n+1}+C,\ n \neq -1$

2. $\int \frac{f'(x)}{f(x)}dx = \int \frac{du}{u(x)} = \ln|u(x)|+C$

3. $\int e^{f(x)} f'(x)dx = \int e^{u(x)}du = e^{u(x)}+C$

If k is a real number, $k \neq 0$, then

1. $\int e^x dx = e^x + C$ 2. $\int e^{kx}dx = \frac{e^{kx}}{k}+C$ 3. $\int a^x dx = \frac{a^x}{\ln a}+C$

6.3.2 Method of substitution to evaluate the indefinite integrals

Example 4 Evaluate the following integrals:

(a). $\int (x^3-5x+7)^4(3x^2-5)dx$ (b). $\int \frac{dx}{9x+7}$ (c). $\int e^{(3x+2)}dx$

Solution

a. $\int (x^3-5x+7)^4(3x^3-5)dx \qquad$ (i)

We need to substitute a new variable $u(x)$:

$x^3-5x+7=u \Rightarrow \frac{d}{dx}(x^3-5x+7)=\frac{du}{dx} \Rightarrow 3x^2-5=\frac{du}{dx} \Rightarrow (3x^2-5)dx=du$ (ii)

Substitute values from (ii) in the integral (i)

$$\int (x^3-5x+7)^4(3x^2-5)dx = \int u^4 du = \frac{(u)^5}{5}+C$$

$$=\frac{1}{5}(x^3-5x+7)^5+C \quad \text{where, } u=x^3-5x+7$$

b. $\int \frac{dx}{9x+7}$ (i)

We need to substitute a new variable $u(x)$:

$9x+7=u=\frac{d}{dx}(9x+7)=\frac{du}{dx} \Rightarrow 9=\frac{du}{dx} \Rightarrow dx=\frac{du}{9}$ (ii)

Substitute values from (ii) in the integral (i).

$$\int \frac{dx}{9x+7}=\int \frac{du}{9u}=\frac{1}{9}\ln u+C=\frac{1}{9}\ln(9x+7)+C, \quad \text{where, } u=9x+7$$

c. $\int e^{(3x+2)}dx$ (i)

We need to substitute a new variable $u(x)$:

$3x+2=u, \frac{d}{dx}(3x+2)=\frac{du}{dx} \Rightarrow 3=\frac{du}{dx} \Rightarrow dx=\frac{du}{3}$ (ii)

Substitute values from (ii) in the integral (i)

$$\int e^{(3x+2)}dx=\frac{1}{3}\int e^u du=\frac{1}{3}e^u+C=\frac{1}{3}e^{(3x+2)}+C, \quad \text{where, } u=3x+2$$

Example 5 If the area A of a healing wound changes at a rate approximated by, $\frac{dA}{dt}=-4t^{-3}$, $1\le t\le 10$, 't' is time in days and the wound area is A=2 square centimeters on day t=1. What will be the area of the wound be in 10 days?

Solution The rate at which the wound area changes, is:

$$\frac{dA}{dt}=-4t^{-3} \quad \text{(i)}$$

The wound area A(t) is obtained by integrating equation (i) with respect to t

$$\int \frac{dA}{dt}=\int -4t^{-3}dt$$

$$A(t)=-4\int t^{-3}dt=-4\frac{t^{-3+1}}{-3+1}+C=\frac{2}{t^2}+C \quad \text{(ii)}$$

The fixed wound area C is obtained by inserting A= 2 and t = 1 in equation (ii):

$$2=\frac{2}{1}+C \Rightarrow C=0$$

Put $C=0$ in equation (ii) to obtain the total wound area:

$$A(t)=\frac{2}{t^2} \quad \text{(iii)}$$

The specific wound area with in 10 days is obtained by putting t = 10 in equation (iii):

$$A(10)=\frac{2}{10^2}=\frac{2}{100}=\frac{1}{50}=0.02 \text{ Square centimeters.}$$

Exercise 6.1

1. Evaluate the following.

a. $\int x^3 dx$ b. $\int (a+15)\,dx$ c. $\int \frac{1}{5x}\,dx$

d. $\int (3x^2+4x-5)\,dx$ e. $\int (1+3t)t^3 dt$ f. $\int \frac{1}{x^3}\,dx$

g. $\int e^{5x} dx$ h. $\int \frac{5}{1+x^2}\,dx$ i. $\int t^3\sqrt{t}\,dt$

2. Evaluate the following indefinite integrals by method of substitution.

a. $\int (3x+4)^8 dx$ b. $\int 3x^2(x^3-4)dx$ c. $\int (3x^2+7)(x^3+7x)^8 dx$

d. $\int \frac{3x^2-7}{(x^3-7x)^4}\,dx$ e. $\int \frac{6t}{\sqrt{3t^2-7}}\,dt$ f. $\int \frac{\sqrt{r^{\frac{1}{3}}+4}}{r^{\frac{2}{3}}}\,dr$

g. $\int \frac{x+3x^2}{\sqrt{x}}\,dx$ h. $\int \frac{x+1}{(x^2+2x+2)^2}\,dx$

3. Evaluate the following indefinite integrals by method of substitution:

a. $\int 6e^{6t} dt$ b. $\int xe^{(5x^2+1)} dx$ c. $\int (x^2-2)e^{(x^3-6x+4)} dx$ d. $\int 8^{(7-3x^2)}(-6x)dx$

4. Find the equation of the particular curve that has a slope $4x^3+6x^2$ at a point (1, 0).

5. A certain curve has a slope $x(2x^2-1)^2$ that passes through the point (3, 3). What is the equation of the specific curve?

6. If the rate of excretion of a biochemical compound is given by

$f'(t) = 0.01e^{-0.01t}$

a. Find an expression $f(t)$, the total amount excreted by time t (in minutes).

b. If $f(0)=0$ units are excreted at time $t=0$, how many units are excreted in10 minutes?

7. For an average person, the rate of change of weight W (in pounds) with respect to height h (in inches) is approximately by $\frac{dW}{dh} = 0.0015h^2$

a. Find $W(h)$, if the weight is $W = 108$ pounds in response of height $h = 60$ inches.

b. Find the weight W of a person who is 5 feet 10 inches (h) tall.

8. The rate of growth of the population $N(t)$ of a newly incorporated city t years after incorporation is estimated to be $\frac{dN}{dt} = 400 + 600\sqrt{t},\ 0 \le t \le 9$

If the population was 5,000 at the time incorporation, find the population 9 years later.

6.3.3 Method of substitution to evaluate the integral of the following types

1. $\int \frac{dx}{a^2-x^2}, \int \sqrt{a^2-x^2}\,dx, \int \frac{dx}{\sqrt{a^2-x^2}}$

a. **Evaluate** $\int \frac{1}{a^2-x^2} dx$

Solution $\int \frac{1}{a^2-x^2} dx$

$= -\int \frac{1}{x^2-a^2} du$, by applying linearity (i)

Now, Consider $\int \frac{1}{x^2-a^2} dx$

$= \int \frac{1}{(x-a)(x+a)} dx$, by factorize the denominator.

$= \int \frac{1}{2a(x-a)} - \frac{1}{2a(x+a)} dx$, by partial fraction decomposition.

$= \frac{1}{2a}\int \frac{1}{(x-a)} dx - \frac{1}{2a}\int \frac{1}{x+a} dx$, by applying linearity (ii)

For $\int \frac{1}{x+a} dx$

Substitute $u = x + a$

$$\frac{du}{dx} = 1$$

$$du = dx$$

So,

$\int \frac{1}{x+a} dx = \int \frac{1}{u} du = ln|u|$

Now, undo the substitution

$$u = x + a$$

$\int \frac{1}{x^2-a^2} = \int \frac{1}{(x+a)} dx = ln|x+a|$ (iii)

For $\int \frac{1}{x-a} dx$

Substitute $u = x - a$

$$\frac{du}{dx} = 1$$

$$du = dx$$

So,

$\int \frac{1}{x-a} dx = \int \frac{1}{u} du = ln|u|$

Now, undo the substitution

$$u = x - a$$

$\int \frac{1}{(x-a)} dx = ln|x-a|$ (iv)

Put the values from (iii) and (iv) to (ii) so,

$$\frac{1}{2a}\int \frac{1}{x-a} dx - \frac{1}{2a}\int \frac{1}{x+a} dx = \frac{ln|x-a|}{2a} - \frac{ln|x+a|}{2a} + C$$

Now, plug in the solved integrals.

$$-\int \frac{1}{x^2-a^2} dx = -\frac{ln|x+a|}{2a} - \frac{ln|x-a|}{2a} + C$$

Now, apply the absolute value function to argument of logarithm functions in order to extend the antiderivative domain.

$$\int\frac{1}{x^2-a^2}dx=-\frac{\ln|x+a|}{2a}+\frac{\ln|x-a|}{2a}+C \quad \Rightarrow \int\frac{1}{x^2-a^2}=\frac{\ln|x+a|-\ln|x-a|}{2a}+C$$

$$\int\frac{1}{a^2-x^2}=\frac{\ln|x-a|-\ln|x+a|}{2a}+C=\frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right|+C$$

b. Evaluate $\int\sqrt{a^2-x^2}\,dx$

Solution $\int\sqrt{a^2-x^2}\,dx$ .... (i)

Let $x=a.\sin(u)$ that is $x=\sin(u)$ for $-\frac{\pi}{2}<u<\frac{\pi}{2}$ then $\frac{dx}{du}=a.\cos(u)\Rightarrow dx=a.\cos(u).du$

$$\int\sqrt{a^2-x^2}\cdot dx=\int a.\cos(u).\sqrt{a^2-a^2.\sin^2(u)}.du$$

$$=\int a^2.\cos(u).\sqrt{1-\sin^2(u)}.du \qquad \sin^2(u)+\cos^2(u)=1$$

$$=a^2\int\cos(u).\sqrt{\cos^2(u)}.du=a^2\int\cos^2(u)du \text{ .... (ii)}$$

Now, use product to sum formula in equation (ii).

$$\cos\alpha.\cos\beta=\frac{1}{2}\left[\cos(\alpha+\beta)+\cos(\alpha-\beta)\right]$$

Here $\alpha=\beta=u$ then

$$\cos^2(u)=\frac{1}{2}\left[\cos(u+u)+\cos(u-u)\right]=\frac{1}{2}[\cos(2u)+1]$$

$$a^2\int\cos^2(u)\,du=a^2\int\frac{1}{2}[\cos(2u)+1]du=\frac{a^2}{2}\int(\cos(2u)+1)du=\frac{a^2}{2}\left[\int\cos(2u)\cdot du+\int 1du\right]$$

$$=\frac{a^2}{2}\left[\frac{1}{2}\cdot\sin(2u)+u\right]+C=\frac{a^2}{4}.\sin(2u)+\frac{a^2u}{2}+C$$

$$=\frac{a^2}{4}(2\cdot\sin(u)\cdot\cos(u))+\frac{a^2u}{2}+C=\frac{a^2}{2}(\sin(u)\cos(u))+\frac{a^2u}{2}+C$$

$$=\frac{a^2}{2}\sin(u)\sqrt{1-\sin^2(u)}+\frac{a^2u}{2}+C \qquad \text{(iii)}$$

Now, substitution returned in (iii)

$$a^2\int\cos^2(u)du=\frac{a^2}{2}\left(\frac{x}{a}\sqrt{1-\frac{x^2}{a^2}}\right)+\frac{a^2}{2}\cdot\sin^{-1}\left(\frac{x}{a}\right)+C$$

Hence, $\int\sqrt{a^2-x^2}\,dx=\frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)+\frac{x}{2}\sqrt{a^2-x^2}+C$

Challenge!

Use hyperbolic substitution to solve $\int\sqrt{a^2-x^2}dx$

c. Evaluate $\int\frac{dx}{\sqrt{a^2-x^2}}$

Solution $\int\frac{dx}{\sqrt{a^2-x^2}}$ (i)

Let $x=a\sin(u)$, that is $x=a\sin(u)$ for $-\frac{\pi}{2}<u<\frac{\pi}{2}$ then $\frac{dx}{du}=a.\cos(u)\Rightarrow dx=a.\cos(u).du$

Substitute the values in (i), then $\int\frac{dx}{\sqrt{a^2-x^2}}=\int\frac{a.\cos(u)}{\sqrt{a^2-a^2\sin^2(u)}}du=\int\frac{a\cos(u)du}{a\sqrt{1-\sin^2(u)}}$

$$=\int\frac{a\cos(u)}{a\sqrt{\cos^2(u)}}du=\int\frac{a\cos(u)}{a\cdot\cos(u)}du=\int 1\cdot du=u+C$$

Now, $x=a.\sin(u)\Rightarrow u=\sin^{-1}\left(\frac{x}{a}\right)$ Thus, $\int\frac{dx}{\sqrt{a^2-x^2}}=\sin^{-1}\left(\frac{x}{a}\right)+C$

ii. $\int\frac{dx}{a^2+x^2},\int\sqrt{a^2+x^2}dx,\int\frac{dx}{\sqrt{x^2+a^2}}$

a. Evaluate $\int\frac{dx}{a^2+x^2}$

Solution $\int\frac{dx}{a^2+x^2}$ (i)

Substitute $u=\frac{x}{a}\Rightarrow\frac{du}{dx}=\frac{1}{a}$ $\Rightarrow dx=adu$ in expression (i)

$\int\frac{1}{x^2+a^2}dx=\int\frac{a}{a^2u^2+a^2}du=\frac{1}{a}\int\frac{1}{u^2+1}du$ (ii)

From (ii) $\int\frac{1}{u^2+1}du$

$\int\frac{1}{u^2+1}du=tan^{-1}(u)$ (iii) Put the value from (iii) to (ii) $\frac{1}{a}\int\frac{1}{u^2+1}du=\frac{1}{a}tan^{-1}(u)$

Hence, $\int\frac{1}{a^2+x^2}dx=\frac{tan^{-1}\left(\frac{x}{a}\right)}{a}+C$ $u=\frac{x}{a}$

b. Evaluate $\int\sqrt{a^2+x^2}dx$

Solution $\int\sqrt{a^2+x^2}dx$ (i)

By using the trigonometric substitution.

Let $x=a\tan(u)\Rightarrow\frac{dx}{du}=a\cdot\sec^2(u)\Rightarrow dx=a.\sec^2(u)du$

Now, $\int\sqrt{a^2+x^2}\,dx=\int\sqrt{a^2+a^2\cdot\tan^2(u)}\cdot a\sec^2(u)\cdot du$

$$=\int a.\sec^2(u)\cdot\sqrt{a^2(1+\tan^2(u)}.du$$

$$=\int a^2.\sec^2(u)\sqrt{1+\tan^2(u)}\cdot du=\int a^2.\sec^2(u).\sec(u)\,du$$

$$=\int a^2\cdot\sec^3(u)\cdot du=a^2\int\sec^3(u)\cdot du$$

$$=\frac{a^2}{2}(\sec(u)\cdot\tan(u))+\frac{a^2}{2}\ln|\tan(u)+\sec(u)|+C \quad \text{(ii)}$$

Now, substitution returns as $u=\tan^{-1}\left(\frac{x}{a}\right)$ in equation (ii).

$$\int\sqrt{a^2+x^2}dx=\frac{a^2}{2}\left[\sec\left(\tan^{-1}\frac{x}{a}\right)\cdot\tan\left(\tan^{-1}\frac{x}{a}\right)\right]+\frac{a^2}{2}\ln\left|\tan\left(\tan^{-1}\frac{x}{a}\right)+\sec\left(\tan^{-1}\frac{x}{a}\right)\right|+C$$

$$=\frac{a^2}{2}\left[\frac{x}{a}\sqrt{\frac{x^2}{a^2}+1}\right]+\frac{a^2}{2}\ln\left|\frac{x}{a}+\sqrt{\frac{x^2}{a^2}+1}\right|+C$$

Hence,

$$\int\sqrt{a^2+x^2}\,dx=\frac{a}{2}\left[\frac{x\sqrt{x^2+a^2}}{|a|}+a\ln\left|a\sqrt{x^2+a^2}+|a|\,x\right|\right]+C$$

Challenge

Evaluate $\int\sqrt{x^2+a^2}\,dx$ by using hyperbolic substitution.

c. Evaluate $\int\frac{dx}{\sqrt{x^2+a^2}}$

Solution $\int\frac{dx}{\sqrt{x^2+a^2}}$ (i)

By using trigonometric substation.

Let $x=a\tan(u)\Rightarrow\frac{dx}{du}=a.\sec^2(u)\Rightarrow dx=a.\sec^2(u).du$

Now,

$$\int\frac{dx}{\sqrt{x^2+a^2}}=\int\frac{a.\sec^2(u)\cdot du}{\sqrt{(a^2+a^2\cdot\tan^2(u)}}=\int\frac{a.\sec^2(u)}{\sqrt{a^2(1+\tan^2(u)}}du$$

$$=\int\frac{a.\sec^2(u)}{a\sqrt{1+\tan^2u}}du$$

$$=\int\frac{\sec^2(u)}{\sec(u)}du=\int\sec(u).du$$

$$=\ln|\sec(u)+\tan(u)|+C \quad \text{(ii)}$$

$$=\ln\left|\sqrt{1+\tan^2(u)}+\tan(u)\right|+C$$

$\sec(u)=\sqrt{1+\tan^2(u)}$

$$=\ln\left|\sqrt{1+\left(\frac{x}{a}\right)^2}+\frac{x}{a}\right|+C$$

Hence, $\int\frac{dx}{\sqrt{x^2+a^2}}=\ln\left|\frac{\sqrt{x^2+a^2}+x}{a}\right|+C$

Challenge

Use hyperbolic substitution to evaluate $\int\frac{dx}{\sqrt{x^2+a^2}}$.

iii. $\int\frac{dx}{x^2-a^2},\int\sqrt{x^2-a^2}\,dx,\int\frac{dx}{\sqrt{x^2-a^2}}$

a. Evaluate $\int\frac{dx}{x^2-a^2}$

Solution $\int\frac{dx}{x^2-a^2}$ (i)

By using trigonometric substitutions

Let $x=a.\sec(u)\Rightarrow\frac{dx}{du}=a.\sec u\tan(u)\Rightarrow dx=a.\sec(u).\tan(u).du$

Now,

$$\int\frac{dx}{x^2-a^2}=\int\frac{a.\sec(u).\tan(u).du}{a^2.\sec^2(u)-a^2}$$

$$=\int\frac{a.\sec(u).\tan(u).dx}{a^2(\sec^2(u)-1)}=\int\frac{\sec.\tan(u)dx}{a\tan^2(u)}$$

$$=\frac{1}{a}\int\frac{\sec(u)}{\tan(u)}du=\frac{1}{a}\int\operatorname{cosec}(u).du$$

$$=-\frac{1}{2}\ln|\operatorname{cosec}(u)+\cot(u)|+C \qquad \text{(ii)}$$

Now, substitution returns $u=\sec^{-1}\left(\frac{x}{a}\right)$

$$\int\frac{dx}{x^2-a^2}=-\frac{1}{2}\ln\left|\operatorname{cosec}\left(\sec^{-1}\frac{x}{a}\right)+\cot\left(\sec^{-1}\left(\frac{x}{a}\right)\right)\right|+C$$

$$=-\frac{1}{2}\ln\left|\frac{x\sqrt{x^2-a^2}}{x^2-a^2}+\frac{a\sqrt{x^2-a^2}}{x^2-a^2}\right|+C$$

Hence, $\int\frac{dx}{x^2-a^2}=-\frac{1}{2}\ln\left(\frac{\left|1\sqrt{x^2-a^2}\right|}{|x-a|}\right)+C$

b. Evaluate $\int\sqrt{x^2-a^2}\,dx$

Solution $\int\sqrt{x^2-a^2}\,dx \qquad \text{(i)}$

By using the trigonometric substitutions.

Let $x=a\sec(u)\Rightarrow\frac{dx}{du}=a.\sec(u).\tan(u)\Rightarrow dx=a.\sec(u).\tan(u).du$

So,

$$\int\sqrt{x^2-a^2}\,dx=\int\sqrt{a^2\sec^2(u)-a^2}.a.\sec(u).\tan(u).dx$$

$$=\int a.\sec(u).\tan(u)\sqrt{a^2(\sec^2(u)-1)}\,du=\int a^2.\sec(u).\tan(u)\sqrt{\sec^2(u)-1}\,du$$

$$=\int a^2.\sec(u).\tan(u)\sqrt{\tan^2(u)}.du=\int a^2.\sec(u).\tan^2(u)\,du$$

$$=a^2\int\sec(u).\tan^2(u).du=a^2\int\sec(u).(\sec^2(u)-1)\,du \qquad \therefore\ \sec^2\theta-\tan^2\theta=1$$

$$=a^2\int\{\sec^3(u)-\sec(u)\}\,du=a^2\left\{\int\sec^3(u)du-\int\sec(u)\,du\right\}$$

$$=a^2\left\{\frac{1}{2}\sec(u).\tan(u)+\frac{1}{2}\int\sec(u)du-\ln|\tan(u)+\sec(u)|\right\}$$

$$=a^2\left\{\frac{1}{2}.\sec(u).\tan(u)+\frac{1}{2}\ln|\tan(u)+\sec(u)|-\ln|\tan(u)+\sec(u)|\right\}+C$$

$$=a^2\left\{\frac{1}{2}\sec(u).\tan(u)-\frac{1}{2}\ln|\tan(u)+\sec(u)|\right\}+C \qquad \text{(ii)}$$

Now, substitution returns as $u=\sec^{-1}\left(\frac{x}{a}\right)$ in (ii)

$$\int\sqrt{x^2-a^2}\,dx=a^2\left\{\frac{1}{2}\sec\left(\sec^{-1}\left(\frac{x}{a}\right)\right).\tan\left(\sec^{-1}\left(\frac{x}{a}\right)\right)-\frac{1}{2}\ln\left|\tan\left(\sec^{-1}\left(\frac{x}{a}\right)\right)+\sec\left(\sec^{-1}\left(\frac{x}{a}\right)\right)\right|\right\}+C$$

$$= a^2\left\{\frac{x}{2a}\left(\frac{\sqrt{x^2-a^2}}{a}\right)-\frac{1}{2}\ln\left|\frac{\sqrt{x^2-a^2}}{a}+\frac{x}{a}\right|\right\}+C$$

Hence, $\int\sqrt{x^2-a^2}\,dx=\frac{x}{2}\sqrt{x^2-a^2}-\frac{a^2}{2}\ln\left(\frac{\sqrt{x^2-a^2}+x}{|a|}\right)+C$

Challenge

Evaluate $\int\sqrt{x^2-a^2}\,dx$ by using hyperbolic substitution and use product and sum formula.

c. **Evaluate** $\int\frac{dx}{\sqrt{x^2-a^2}}$

Solution $\int\frac{dx}{\sqrt{x^2-a^2}}$ (i)

By using trigonometric substitutions

Let $x=a.\sec(u)\Rightarrow\frac{dx}{du}=a.\sec.\tan(u)\Rightarrow dx=a.\sec(u).\tan(u).du$

So, $\int\frac{dx}{\sqrt{x^2-a^2}}=\int\frac{a.\sec(u).\tan(u)}{\sqrt{a^2\sec^2(u)-a^2}}du$

$$=\int\frac{a.\sec(u).\tan(u)}{\sqrt{a^2(\sec^2(u)-1)}}du=\int\frac{a.\sec(u).\tan(u)}{a\sqrt{\sec^2u-1}}du$$

$$=\int\frac{\sec(u).\tan(u)}{\tan(u)}du=\int\sec(u).du$$

$$=\ln|\sec(u)+\tan(u)|+C \quad \text{(ii)}$$

Now, substitution returns as $u=\sec^{-1}\left(\frac{x}{a}\right)$

$$\int\frac{dx}{x^2-a^2}=\ln\left|\sec\left(\sec^{-1}\left(\frac{x}{a}\right)\right)+\tan\left(\sec^{-1}\left(\frac{x}{a}\right)\right)\right|+C$$

$$=\ln\left|\frac{x}{a}+\sqrt{\frac{x^2}{a^2}-1}\right|+C$$

Hence, $\int\frac{dx}{x^2-a^2}=\ln\left|\frac{x+\sqrt{x^2-a^2}}{a}\right|+C$ Or $\int\frac{dx}{\sqrt{x^2-a^2}}=\ln\left|x+\sqrt{x^2-a^2}\right|-\ln|a|+C$

d. $\int\frac{dx}{ax^2+bx+c}$, $\int\frac{dx}{\sqrt{ax^2+bx+c}}$

The procedure to solve this kind of integrals is illustrated in the following examples.

Example 6 Evaluate the integral $\int\frac{dx}{x^2+4x+5}$

Solution $\int\frac{1}{x^2+4x+5}dx$ (i)

By completing square in denominator $\int\frac{1}{x^2+4x+5}dx=\int\frac{1}{x^2+4x+4+1}dx=\int\frac{1}{(x+2)^2+1}$

Let $u=x+2\Rightarrow\frac{du}{dx}=1\Rightarrow du=dx$

$$\int\frac{1}{(x+2)^2+1}dx=\int\frac{1}{u^2+1}dx \quad \text{(ii)}$$

Now, use trigonometric substitutions in equation (ii) $\quad u=\tan\theta, \dfrac{du}{d\theta}=\sec^2\theta \Rightarrow du=\sec^2\theta.d\theta$

$$\int\frac{1}{u^2+1}du=\int\frac{\sec^2\theta.d\theta}{\tan^2\theta+1}=\int\frac{\sec^2\theta}{\sec^2\theta}.d\theta=\int 1.d\theta=\theta+C \qquad \text{(iii)}$$

Return substitutions in (iii)

$$\int\frac{1}{x^2+4x+5}dx=\tan^{-1}(u)+C=\tan^{-1}(x+2)+C$$

Example 7 Evaluate the integral $\displaystyle\int\frac{dx}{\sqrt{x^2+4x+5}}$

Solution $\displaystyle\int\frac{1}{\sqrt{x^2+4x+5}}dx \qquad \text{(i)}$

By completing square in denominator

$$\int\frac{1}{\sqrt{x^2+4x+5}}dx=\int\frac{1}{\sqrt{x^2+4x+4+1}}dx=\int\frac{1}{\sqrt{(x+2)^2+1}}dx$$

Let $u=x+2, \dfrac{du}{dx}=1\Rightarrow du=dx$

$$\int\frac{1}{\sqrt{(x+2)^2+1}}dx=\int\frac{1}{\sqrt{u^2+1}}du \qquad \text{(ii)}$$

Now, use trigonometric substitutions in equation (ii)

$u=\tan\theta, \dfrac{du}{d\theta}=\sec^2\theta\Rightarrow du=\sec^2\theta.d\theta$

$$\int\frac{1}{u^2+1}du=\int\frac{\sec^2\theta}{\sqrt{\tan^2\theta+1}}d\theta=\int\frac{\sec^2\theta}{\sqrt{\sec^2\theta}}d\theta=\int\frac{\sec^2\theta}{\sec\theta}d\theta=\int\sec d\theta=\ln|\sec\theta+\tan\theta|+C \qquad \text{(iii)}$$

Return substitutions in (iii)

$$\int\frac{1}{x^2+4x+5}dx=\ln\left|\sqrt{u^2+1}+u\right|+C$$

$$=\ln\left|\sqrt{(x+2)^2+1}+(x+2)\right|+C$$

e. $\displaystyle\int\frac{px+q}{ax^2+bx+c}dx, \int\frac{px+q}{\sqrt{ax^2+bx+c}}dx$

The procedure to solve these kind of integrals in illustrated in the following examples.

Example 8 Evaluate the integral $\displaystyle\int\frac{x+2}{x^2+4x+5}dx$

Solution $\displaystyle\int\frac{x+2}{x^2+4x+5}dx \qquad \text{(i)}$

By completing square in denominator

$$\int\frac{x+2}{x^2+4x+5}dx=\int\frac{x+2}{x^2+4x+4+1}dx=\int\frac{x+2}{(x+2)^2+1}dx$$

Let $u=x+2\Rightarrow\dfrac{du}{dx}=1\Rightarrow du=dx$ then $\displaystyle\int\frac{x+2}{(x+2)^2+1}dx=\int\frac{u}{u^2+1}du \qquad \text{(ii)}$

Now, use trigonometric substitutions in equation (ii)

$$u=\tan\theta, \frac{du}{d\theta}=\sec^2\theta \Rightarrow du=\sec^2\theta.d\theta$$

$$\int\frac{u}{u^2+1}du=\int\frac{\tan\theta.\sec^2\theta}{\tan^2\theta+1}.d\theta=\int\frac{\tan\theta.\sec^2\theta}{\sec^2\theta}\cdot d\theta=\int\tan\theta\, d\theta=\int\frac{\sin\theta}{\cos\theta}\cdot d\theta=-\ln|\cos\theta|+C \quad \text{(iii)}$$

Return substitutions in (iii)

$$\int\frac{x-2}{x^2+4x+5}dx=-\ln\left|\left(\cos(\tan^{-1}(u))\right)\right|+C$$

$$=-\ln\left|\frac{\sqrt{1+u^2}}{1+u^2}\right|+C$$

$$=-\ln\left|\frac{\sqrt{1+(x+2)^2}}{1+(x+2)^2}\right|+C$$

$$=-\ln\left|\frac{1}{\sqrt{1+(x+2)^2}}\right|+C$$

$$=-\ln\left|\{1+(x+2)^2\}^{-\frac{1}{2}}\right|+C=\frac{1}{2}\ln\left|1+(x+2)^2\right|+C$$

Hence, $$\int\frac{x-2}{x^2+4x+5}dx=\frac{1}{2}\ln\left|1+(x+2)^2\right|+C$$

$$=\frac{1}{2}\ln\left|x^2+4x+5\right|+C$$

Example 9 Evaluate the integral $\int\frac{x+2}{\sqrt{x^2+4x+5}}dx$

Solution $$\int\frac{x+2}{\sqrt{x^2+4x+5}}dx \quad \text{(i)}$$

By completing square in denominator

$$\int\frac{x+2}{\sqrt{x^2+4x+5}}dx=\int\frac{x+2}{\sqrt{x^2+4x+4+1}}dx=\int\frac{x+2}{\sqrt{(x+2)^2+1}}dx$$

Let $u=x+2, \frac{du}{dx}=1 \Rightarrow du=dx$

$$\int\frac{x+2}{\sqrt{(x+2)^2+1}}dx=\int\frac{u}{\sqrt{u^2+1}}du \quad \text{(ii)}$$

Now, use trigonometric substitutions in equation (ii)

$$u=\tan\theta, \frac{du}{d\theta}=\sec^2\theta \Rightarrow du=\sec^2\theta.d\theta$$

$$\int\frac{u}{\sqrt{u^2+1}}du=\int\frac{\tan\theta.\sec^2\theta}{\sqrt{\tan^2\theta+1}}d\theta=\int\frac{\tan\theta\sec^2\theta}{\sqrt{\sec^2\theta}}d\theta=\int\frac{\tan\theta.\sec^2\theta}{\sec\theta}d\theta=\int\tan\theta\cdot\sec\theta d\theta=\sec\theta+C \quad \text{(iii)}$$

Return substitutions in (iii)

$$\int \frac{x+2}{\sqrt{x^2+4x+5}}dx = \sec\left(\tan^{-1}(u)\right)+C$$

$$= \sqrt{1+u^2}+C = \sqrt{1+(x+2)^2}+C$$

Hence, $\int \frac{x+2}{\sqrt{x^2+4x+5}}dx = \sqrt{x^2+4x+5}+C$

6.4 Integration by Parts

In previous sections, we have learnt some of the basic techniques of integration to sol problems like $\int x^2 dx$ and $\int \sin x dx$. But, how do we evaluate an integral whose integrand is the produ of two functions such as $\int x\sin x dx$, $\int xe^x dx$, $\int x\ln x dx$

To solve integral of the type like that, we have a technique called **integration by parts.**

6.4.1 Recognition of integration by parts

For this technique, recall the differentiation of the product of two functions $f(x)$ and $g(x)$ w.r.t x:

$$\frac{d}{dx}[f(x)g(x)] = f(x)\frac{d}{dx}(g(x)+g(x)\frac{d}{dx}f(x) = f(x)g'(x)+g(x)f'(x)$$

$$f(x)g'(x) = \frac{d}{dx}[f(x)g(x)] - g(x)f'(x) \qquad \text{(i)}$$

The integral of (i) with respect to x is giving

$$\int f(x)g'(x)dx = \int \frac{d}{dx}[f(x)g(x)]dx - \int g(x)f'(x)dx = f(x)g(x) - \int g(x)f'(x)dx$$

The equation that can be transformed into more convenient form by substituting $u=f(x)$ ar $v=g(x)$, $du = f'(x)dx$ and $dv = g'(x)dx$: $\int u dv = uv - \int v du$ (ii)

This is the standard form of the integration by parts formula.

Example 10 Evaluate the integral $\int xe^x dx$.

Solution The integral rule (ii) with $u = x$ and $\frac{dv}{dx} = e^x$ is used to obtain:

$$\int xe^x dx = xe^x - \int e^x(1)dx, \quad \frac{du}{dx}=1 \qquad dv = e^x dx,\ v = e^x$$

$$= xe^x - e^x + C$$

6.4.2 Applying method of integration by parts to evaluate integrals of the followin types $\int\sqrt{a^2-x^2}dx$, $\int\sqrt{a^2+x^2}$, $\int\sqrt{x^2-a^2}$

i, Evaluate $\int\sqrt{a^2-x^2}dx$

Solution The given integral is

$$I = \int\sqrt{a^2-x^2}dx \qquad \text{(i)}$$

In this problem, we choose $u=\sqrt{a^2-x^2}$ and $\frac{dv}{dx}=1$ to integrate the integrand of (i):

$$I = \int\sqrt{a^2-x^2}dx = \sqrt{a^2-x^2}(x) - \int\frac{(x)(-x)}{\sqrt{a^2-x^2}}dx \qquad \therefore \frac{du}{dx} = \frac{1}{2}(a^2-x^2)^{\frac{1}{2}-1}(-2x),\ dv = 1dx,\ v = x$$

$$= x\sqrt{a^2-x^2} + \int\frac{x^2}{\sqrt{a^2-x^2}}dx = x\sqrt{a^2-x^2} + \int\frac{a^2-(a^2-x^2)}{\sqrt{a^2-x^2}}, \qquad \text{add and subtract } a^2$$

$$= x\sqrt{a^2-x^2}+\int\frac{a^2dx}{\sqrt{a^2-x^2}}-\int\sqrt{a^2-x^2}dx = x\sqrt{a^2-x^2}+a^2\int\frac{dx}{\sqrt{a^2-x^2}}-\mathrm{I}$$

$$2\mathrm{I} = x\sqrt{a^2-x^2}+a^2\sin^{-1}\frac{x}{a}+C, \qquad \int\frac{dx}{\sqrt{a^2-x^2}}=\sin^{-1}\frac{x}{a}$$

$$\Rightarrow \mathrm{I} = \frac{x\sqrt{a^2-x^2}}{2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}+\frac{C}{2}$$

ii. Evaluate $\int\sqrt{a^2+x^2}dx$

Solution $\mathrm{I}=\int\sqrt{a^2+x^2}dx$ (i)

In this problem, we choose $u=\sqrt{a^2+x^2}$ and $\frac{dv}{dx}=1$ to integrate the integrand of (i):

$$\mathrm{I}=\int\sqrt{a^2+x^2}dx$$

$$=\sqrt{a^2+x^2}(x)-\int\frac{(x)(x)}{\sqrt{a^2+x^2}}dx \qquad \therefore \frac{du}{dx}=\frac{1}{2}(a^2+x^2)^{\frac{1}{2}-1}(2x),\ dv=1dx,\ x=v$$

$$=x\sqrt{a^2+x^2}-\int\frac{x^2}{\sqrt{a^2+x^2}}dx = x\sqrt{a^2+x^2}-\int\frac{(a^2+x^2)-a^2}{\sqrt{a^2+x^2}}, \qquad \textit{add and subtract } a^2$$

$$=x\sqrt{a^2+x^2}-\int\frac{a^2dx}{\sqrt{a^2+x^2}}+\int\sqrt{a^2+x^2}dx = x\sqrt{a^2+x^2}-a^2\int\frac{dx}{\sqrt{a^2+x^2}}+\mathrm{I}$$

$$2\mathrm{I}=x\sqrt{a^2+x^2}-a^2\ln\left|x+\sqrt{a^2+x^2}\right|+C \qquad \therefore \int\frac{dx}{\sqrt{a^2+x^2}}=\ln\left|x+\sqrt{a^2+x^2}\right|$$

$$\mathrm{I}=\frac{x\sqrt{a^2+x^2}}{2}-\frac{a^2}{2}\ln\left|x+\sqrt{a^2+x^2}\right|+\frac{C}{2}$$

Thus, $\int\sqrt{a^2+x^2}dx=\frac{x}{2}\sqrt{x^2+x^2}-\frac{a^2}{2}\ln\left|x+\sqrt{a^2+x^2}\right|+C$

iii. Evaluate $\int\sqrt{x^2-a^2}dx$

Solution $\mathrm{I}=\int\sqrt{x^2-a^2}dx$ (i)

In this problem, we choose u = $\sqrt{x^2-a^2}$ and $\frac{dv}{dx}=1$ to integrate the integrand of (i)

$$\mathrm{I}=\int\sqrt{x^2-a^2}dx$$

$$=\sqrt{x^2-a^2}-\int\frac{(x)(x)}{\sqrt{x^2-a^2}}dx \qquad \therefore \frac{du}{dx}=\frac{1}{2}\left(x^2-a^2\right)^{\frac{1}{2}-1}2x,dv=1dx\Rightarrow v=x$$

$$=x\sqrt{x^2-a^2}-\int\frac{x^2}{\sqrt{x^2-a^2}}dx=x\sqrt{x^2-a^2}-\int\frac{x^2}{\sqrt{x^2-a^2}}dx$$

$$=x\sqrt{x^2-a^2}-\int\frac{a^2+\left(x^2-a^2\right)}{\sqrt{x^2-a^2}} \qquad \text{add and subtract } a^2$$

$$=x\sqrt{x^2-a^2}-\int\frac{a^2dx}{\sqrt{x^2-a^2}}-\int\sqrt{x^2-a^2}dx=x\sqrt{x^2-a^2}-a^2\int\frac{dx}{\sqrt{x^2-a^2}}-\mathrm{I}$$

$$I+I=x\sqrt{x^2-a^2}-a^2.\ln\left(\frac{\sqrt{x^2-a^2}+x}{|a|}\right)+C$$

$$I=\frac{x\sqrt{x^2-a^2}}{2}-\frac{a^2}{2}.\ln\left(\frac{\sqrt{x^2-a^2}+x}{|a|}\right)+\frac{C}{2}$$

Thus, $\int\sqrt{x^2-a^2}dx=\frac{x}{2}\sqrt{x^2-a^2}-\frac{a^2}{2}\ln\left(\frac{\left|\sqrt{x^2-a^2}-x\right|}{|a|}\right)+C$

6.4.3 Evaluation of integrals using integration by parts

Example 11 Evaluate the integral $\int x\ln x dx$.

Solution The integral rule $\int udv=uv-\int vdu$ with substitution $u=\ln x$ and $\frac{dv}{dx}=x$ is used to obtain:

$$\int x\ln x dx=\ln x\left(\frac{x^2}{2}\right)-\int\frac{x^2}{2}\left(\frac{1}{x}\right)dx \qquad \therefore \frac{du}{dx}=\frac{1}{x},\ \frac{dv}{dx}=x,\ v=\frac{x^2}{2}$$

$$=\frac{x^2}{2}\ln x-\frac{1}{2}\int xdx+C=\frac{x^2}{2}\ln x-\frac{1}{2}\left(\frac{x^2}{2}\right)+C=\frac{x^2}{2}\ln x-\frac{x^2}{4}+C$$

Example 12 Evaluate the integral $\int e^x\sin x\,dx$.

Solution The integral is

$$I=\int e^x\sin xdx \qquad \text{(i)}$$

The integral rule $\int udv=uv-\int vdu$ with substitution $u=e^x$ (let u be either e^x or $\sin x$) and $dv/dx=\sin x$ is used to obtain:

$$I=\int e^x\sin xdx=e^x(-\cos x)-\int(-\cos x)e^x dx \qquad \therefore \frac{du}{dx}=e^x,\ \frac{dv}{dx}=\sin x$$

$$=-e^x\cos x+\int e^x\cos xdx \qquad \text{(ii)}$$

It appears that we have not made any progress since we cannot evaluate the new integral. However, the form of the new integral prompts us to apply the technique a second time and see what happens.

Again, the integral of the integral part of equation (ii) with substitution $u=e^x$ and $\frac{dv}{dx}=\cos x$ is used in (ii) to obtain:

$$I=\int e^x\sin xdx=-e^x\cos x+\left[\int e^x\cos xdx\right]$$

$$=-e^x\cos x+\left[e^x\sin x-\int(\sin x)(e^x)dx\right]+C \qquad \therefore \frac{du}{dx}=e^x,\ \frac{dv}{dx}=\cos x$$

$$I=-e^x\cos x+e^x\sin x-\int e^x\sin xdx+C$$

$$I=-e^x\cos x+e^x\sin x-I+C$$

$$\Rightarrow 2I=e^x(\sin x-\cos x)+C$$

$$\Rightarrow I=\frac{e^x(\sin x-\cos x)}{2}+\frac{C}{2}$$

Thus, $\int e^x\sin xdx=\frac{e^x(\sin x-\cos x)}{2}+C$

Remember

Here's very helpful mnemonic for an order of priority for which factor the derivative must be passed to.

I	Inverse, trigonometric
L	Logarithms
A	Algebraic
T	Trigonometric
E	Exponential

Exercise 6.2

1. Evaluate the following indefinite integrals by method of substitution:

a. $\int \sin^4 x \cos x dx$ b. $\int \sqrt[3]{\sin^2 x} \cos x dx$ c. $\int \frac{\sin x \ln(\cos x)}{\cos x} dx$

d. $\int e^x \sin e^x dx$ e. $\int \frac{\cot \sqrt{x}}{\sqrt{x}} dx$ f. $\int \frac{\sin x - \cos x}{\sin x + \cos x} dx$

2. Use suitable substitutions and tables to evaluate the following indefinite integrals:

a. $\int \frac{dx}{x^2+16}$ b. $\int \frac{\sin x}{\cos^2 x+1} dx$ c. $\int \frac{dx}{\sqrt{e^{2x}-4}}$

d. $\int \frac{2x+5}{x^2+4x+5} dx$ e. $\int \frac{2+x}{\sqrt{4-2x-x^2}} dx$

3. Evaluate the following by using integration by parts.

a. $\int x^2 . e^{2x} dx$ b. $\int x \cos x \, dx$ c. $\int e^x \sin(x) dx$

d. $\int e^x \cos x \, dx$ e. $\int \sin^{-1}(x) dx$ f. $\int e^{3t} \sin(e^{3t}) dt$

History

Archimedes (207BC-212BC)

Archimedes was a Greek mathematician, physicist and astronomer. He was known as the leading scientist in classical antiquity. His mathematical work is to modern in technique that it is barely distinguishable from that of 17th century mathematicians. It was all done without the benefits of algebra or a convenient number system. He also developed general method for finding the areas and volumes. He used the method to find areas banded by parabolas and spirals and to find volume of cylinders, paraboloids and segments of spheres. Archimedes also gave a procedure to find approximating values of π and banded its value $\left[3\frac{10}{11}, 3\frac{1}{7}\right]$. He also invented a method to find the square roots and proposed another method based on the Greek myriad for representing numbers as large as one followed by 80 million billion zeros.

Archimedes was most proud of his discovery of a method for finding the volume of a sphere. He showed that the volume of a sphere is $\frac{2}{3}$ the volume of the cylinder. The method of mechanical theorems, which was the part of palimpsest found in the Constantinople in 1906. In that treatise Archimedes explain how he made some of his discoveries that are participating in the main idea of the integral calculus.

6.5 Integration using partial fractions

Partial fraction decomposition has great value as a tool for integration. This process may be thought of as the "reverse" of adding fractional algebraic expressions, and it allows us to break up rational expressions into simpler terms. Partial fraction decomposition is an algebraic procedure for expressing a reduced rational function as a sum of fractional parts. For example, the rational expression

$$f(x)=\frac{P(x)}{D(x)} \qquad \text{(i)}$$

can be decomposed into partial fractions only if $P(x)$ and $D(x)$ have no common factors and if the degree of $P(x)$ is less than the degree of $D(x)$. If the degree of $P(x)$ is greater than or equal to the degree of $D(x)$, then use division to obtain a polynomial plus a proper fraction. For example, the rational function after division is:

$$\frac{x^4+2x^3-4x^2+x-3}{x^2-x-2}=x^2+3x+1+\frac{8x-1}{x^2-x-2} \qquad \text{(ii)}$$

x^2+3x+1 is our polynomial term

$\frac{8x-1}{x^2-x-2}$ is our proper fraction (this is the part which requires decomposition into partial fractions).

In algebra, the theory of equations tells us that any polynomial $P(x)$ with real coefficients can be expressed as a product of linear and irreducible quadratic powers, some of which may be repeated. This fact can be used to justify the following general procedure for obtaining the partial fraction decomposition of a rational function.

Let $f(x)=\frac{P(x)}{D(x)}$, where $P(x)$ and $D(x)$ have no common factors and $D(x)\neq 0$.

The steps involved in decomposing the rational function are the following:

1. If the degree of $P(x)$ is greater than or equal to the degree of $D(x)$, use long division to express $\frac{P(x)}{D(x)}$ as the sum of a polynomial and a fraction $\frac{R(x)}{D(x)}$ in which the degree of the remainder polynomial $R(x)$ is less than the degree of the denominator polynomial $D(x)$.
2. Factorize the denominator $D(x)$ into the product of linear and irreducible quadratic powers.
3. Express $\frac{P(x)}{D(x)}$ as a cascading sum of partial fractions of the form $\frac{A_i}{(x-r)^n}$ and $\frac{A_j+B_k}{(x^2+sx+t)^m}$

Verify that the number of constants used is identical to the degree of the denominator.

6.5.1 Use of partial fraction to find $\int\frac{f(x)}{g(x)}$ where $f(x)$ and $g(x)$ are algebraic functions, such that $g(x)\neq 0$

Example 13 Evaluate the following integrals:

(a). $\int\frac{8x-1}{x^2-x-2}dx$ (b). $\int\frac{x^2-6x+3}{(x-2)^3}dx$ (c). $\int\frac{2x^3+x^2+2x+4}{(x^2+1)^2}dx$

Solution

a. $\int\frac{8x-1}{x^2-x-2}dx$

The integrand is a proper fraction, so we start by factoring the denominator $x^2-x-2=(x-2)(x+1)$

The denominator factors are the two distinct linear factors, so we can set the rational function equal to the sum of the two partial fractions

$$\frac{8x-1}{x^2-x-2}=\frac{A_1}{x-2}+\frac{A_2}{x+1} \qquad \text{(i)}$$

To determine the constants A_1 and A_2, we multiply both sides of the equation (i) by $(x-2)(x+1)$ to obtain:

$$8x-1 = A_1(x+1)+A_2(x-2) \quad \text{(ii)}$$

Set $x-2=0 \Rightarrow x=2$ in equation (ii) to obtain: $8(2)-1 = A_1(2+1)+A_2(2-2) \Rightarrow 15 = 3A_1 \Rightarrow A_1 = 5$

Set $x+1=0 \Rightarrow x=-1$ in equation (ii) to obtain :

$$8(-1)-1 = A_1(-1+1)+A_2(-1-2) \Rightarrow -9=-3A_2 \Rightarrow A_2 = 3$$

Use these constants values in equation (i) to obtain: $\dfrac{8x-1}{x^2-x-2} = \dfrac{A_1}{x-2}+\dfrac{A_2}{x+1} = \dfrac{5}{x-2}+\dfrac{3}{x+1}$

Use this decomposition instead of rational expression in the given integral to obtain:

$$\int \frac{8x-1}{x^2-x-2}dx = \int\left[\frac{5}{x-2}+\frac{3}{x+1}\right]dx = \int\frac{5}{x-2}dx+\int\frac{3}{x+1}dx = 5\ln(x-2)+3\ln(x+1)+\ln C$$

$$= \ln(x-2)^5+\ln(x+1)^3+\ln C = \ln C(x-2)^5(x+1)^3$$

b. $\displaystyle\int \frac{x^2-6x+3}{(x-2)^3}dx$

The integrand is a proper fraction, so we start by factoring the denominator

$$(x-2)^3 = (x-2)(x-2)(x-2)$$

The denominator factors are the three repeated linear factors, so we can set the rational function equal to the sum of the three partial fractions $\dfrac{x^2-6x+3}{(x-2)^3} = \dfrac{A_1}{x-2}+\dfrac{A_2}{(x-2)^2}+\dfrac{A_3}{(x-2)^3}$ (i)

To determine the constants A_1, A_2 and A_3, we multiply both sides of the equation (i) by $(x-2)^3$ to obtain: $x^2-6x+3 = A_1(x-2)^2+A_2(x-2)+A_3 = A_1(x^2-4x+4)+A_2(x-2)+A_3$ (ii)

Set $x-2=0 \Rightarrow x=2$ in equation (ii) to obtain:

$$(2)^2-6(2)+3 = A_1(2-2)^2+A_2(2-2)+A_3 \Rightarrow -5 = A_3 \Rightarrow A_3 = -5$$

For constants A_1, A_2, equate the coefficients of x^2 and x on each side of equation (ii) to obtain:

$1 = A_1$ $\quad x^2$ terms $\qquad -6 = -4A_1+A_2$ $\quad x$ terms

Solving this system of equations for the unknowns A_1 and A_2 to obtain $A_1 = 1$ and $A_2 = -2$

Use these constants values in equation (ii) to obtain:

$$\frac{x^2-6x+3}{(x-2)^3} = \frac{A_1}{x-2}+\frac{A_2}{(x-2)^2}+\frac{A_3}{(x-2)^3} = \frac{1}{x-2}-\frac{2}{(x-2)^2}-\frac{5}{(x-2)^3}$$

Use this decomposition instead of rational expression in the given integral to obtain:

$$\int\frac{x^2-6x+3}{(x-2)^3}dx = \int\left[\frac{1}{x-2}-\frac{2}{(x-2)^2}-\frac{5}{(x-2)^3}\right]dx = \ln(x-2)-2\frac{(x-2)^{-2+1}}{-2+1}-\frac{5(x-2)^{-3+1}}{-3+1}+C$$

$$= \ln(x-2)+\frac{2}{(x-2)}+\frac{5}{2(x-2)^2}+C$$

c. $\displaystyle\int\frac{2x^3+x^2+2x+4}{(x^2-1)^2}dx$

The intergand is a proper fraction and the denominator factors are the two repeated quadratic factors, so we can set the rational function equal to the sum of the two partial fractions:

$$\frac{2x^3+x^2+2x+4}{(x^2+1)^2} = \frac{A_1x+B_1}{(x^2+1)^2}+\frac{A_2x+B_2}{(x^2+1)} \quad \text{(i)}$$

To determine the constants values, the similar procedure is used to obtain $A_1 = 0, A_2 = 2, B_1 = 3, B_2 = 1$.

With these substitutions, the equation (i) becomes:

$$\frac{2x^3+x^2+2x+4}{(x^2+1)^2}=\frac{3}{(x^2+1)^2}+\frac{2x+1}{(x^2+1)} \quad \text{(ii)}$$

Integrate this decomposition to obtain:

$$\int\frac{2x^3+x^2+2x+4}{(x^2+1)^2}dx=\int\frac{3}{(x^2+1)^2}dx+\int\frac{2x+1}{(x^2+1)}dx$$

$$\int\frac{2x^3+x^2+2x+4}{(x^2+1)^2}dx=\int\frac{3}{(x^2+1)^2}dx+\int\frac{2x}{(x^2+1)}dx+\int\frac{1}{(x^2+1)}dx$$

Now the readers are in position, how to find the complete solution of the question.

Hint: $\int\frac{1}{(x^2+1)}dx=\tan^{-1}x$, $\int\frac{2x}{(x^2+1)}dx=\ln u, u=x^2+1$

$\int\frac{3}{(x^2+1)^2}dx=?$, $x=\tan\theta, dx=\sec^2\theta$

Exercise 6.3

1. Evaluate the indefinite integrals after decomposing the following rational functions into partial fractions:

a. $\int\frac{1}{x(x-3)}$ b $\int\frac{3x^2+2x-1}{x(x+1)}$ c. $\int\frac{4x^3+4x^2+x-1}{x^2(x+1)^2}$ d. $\int\frac{1}{x^3-1}$

e. $\int\frac{x^4-x^2+2}{x^2(x-1)}$ f. $\int\frac{dx}{x^2-1}$ g. $\int\frac{-x-3}{2x^2-x-1}dx$ h. $\int\frac{x^2-1}{x^2-2x-15}dx$

i. $\int\frac{x}{(x+1)(x^2+1)}dx$ j. $\int\frac{x^2+2}{(x^2+1)^2}dx$

2. The rate at which the body eliminates a drug (in milliliters per hour) is given by

$$\frac{R(t)}{dt}=\frac{60t}{(t+1)^2(t+2)}$$

where t is the number of hours since the drug was administered. If $R(0)=0$ is the current drug elimination, how much of the drug is eliminated during the first hour after it was administered? The fourth hour, after it was administered?

3. The rate of change of the voting population of a city with respect to time t (in years) is estimated to be

$$\frac{dN}{dt}=\frac{100t}{(1+t^2)^2}$$

where N(t) is in thousands. If N(0) is the current voting population, then how much will this population increase during the next 3 years?

4. An oil tanker aground on a reef is losing oil and producing an oil slick that is radiating outward at a rate approximated by

$$\frac{dr}{dt}=\frac{100}{\sqrt{t^2+9}},\ t\ge 0$$

where r is the radius (in feet) of the circular slick after t minutes. Find the radius of the slick after 4 minutes if the radius is $r=0$ when $t=0$.

6.6 Definite Integrals

"A definite integral is an integral that contains start and end value, say a and b, where interval [a, b] are limits or boundaries".

Look at the following figures, Figure 6.1 is showing the indefinite integral while the Figure 6.2 is showing definite integral.

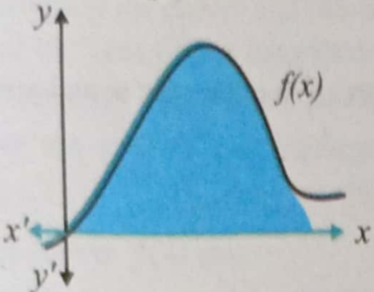


Figure 6.1

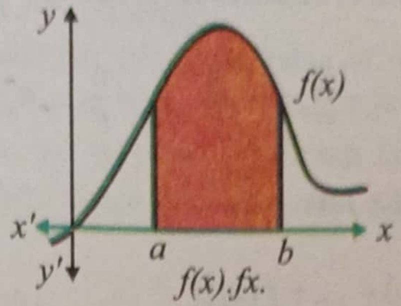


Figure 6.2

Definite integrals can be calculated in a same way as we have learnt in previous section for calculation of indefinite integral, but there is slight difference, to find definite integral simply calculate indefinite integral at point a and at point b, then subtract the result.

Example 14 Evaluate $\int_1^4 x^2 dx$

Solution Here $a = 1$, $b = 4$ and $n = 2$.

$$\int_1^4 x^2 dx = \left|\frac{x^{2+1}}{2+1}\right|_1^4 + C = \left|\frac{x^3}{3}\right|_1^4 + C \qquad \int_a^b x^n dx = \left|\frac{x^{n+1}}{n+1}\right|_a^b + C$$

For $a = 1, \frac{1}{3} + C$ (i)

For $b = 4, \frac{64}{3} + C$ (ii)

Subtract equation (i) from equation (ii)

Thus, $\int_1^4 x^2 dx = \left|\frac{x^3}{3}\right|_1^4 = \frac{64}{3} + \cancel{C} - \frac{1}{3} - \cancel{C} = 21$

Do You Know?

To calculate the approximate area of any mountain we use integration.

6.6.1 Definite integral as the limit of a sum

This limiting process is what we mean when we say the area is the **definite integral** of $f(x) = x^2$ from $x = 0$ to $x = 2$. It is written symbolically as

$$A = \int_{x=0}^{x=2} x^2 dx = \frac{8}{3} \qquad \text{(i)}$$

We read symbol as "the area A equals the integral from $x = 0$ to $x = 2$ of the function $f(x) = x^2$." The number 0 is called the **lower limit of integration**, the number 2 is called the **upper limit of integration**, the function $f(x) = x^2$ is called the **integrand** and the ***dx*** tells us that we are integrating the function $f(x) = x^2$ with respect to the variable x.

If $f(x)$ is continuous on the interval $[a, b]$ and $[a, b]$ is divided into n equal subintervals whose right-hand points are $x_1, x_2, \ldots, x_n$, then the definite integral of $f(x)$ from $x = a$ to $x = b$ is:

$$\int_{x=a}^{x=b} f(x)dx = \lim_{n\to\infty} \frac{(b-a)}{n}\left[f(x_1) + f(x_2) + \ldots + f(x_n)\right] \qquad \therefore \Delta x = \frac{b-a}{n}$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x, \quad i = 1, 2, 3, \ldots, n \qquad \text{(i)}$$

Example 15 Find the actual area of the region bounded by the curve $f(x)=x^2$ and the x-axis in the interval [0, 2].

Solution For n subintervals, the width of each rectangle is $\Delta x=\dfrac{b-a}{n}=\dfrac{2-0}{n}=\dfrac{2}{n}$

The right end points of the subintervals are $\dfrac{2}{n},\dfrac{4}{n},\dfrac{6}{n},......,2.$

Substitute $a=0, b=2,\ x_1=\dfrac{2}{n},\ x_2=\dfrac{4}{n},\ x_3=\dfrac{6}{n},...,x_n=\dfrac{2n}{n}$ in equation (i) to obtain the actual area:

$$A=\int_0^2 x^2dx=\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\,\Delta x$$

$$=\lim_{n\to\infty}\Delta x\left[f(x_1)+f(x_2)+.....+f(x_n)\right]$$

$$=\lim_{n\to\infty}\frac{2}{n}\left[f\left(\frac{2}{n}\right)+f\left(\frac{4}{n}\right)+f\left(\frac{6}{n}\right)+.....+f(2)\right] \qquad \therefore f(x)=x^2$$

$$=\lim_{n\to\infty}\frac{2}{n}\left[\frac{4}{n^2}+\frac{16}{n^2}+\frac{36}{n^2}+.....+\frac{4n^2}{n^2}\right] \qquad \therefore 4=\frac{4n^2}{n^2}$$

$$=\lim_{n\to\infty}\frac{8}{n^3}\left[1+2^2+3^2+.....+n^2\right]=\lim_{n\to\infty}\frac{8}{n^3}\left[\frac{n(n+1)(2n+1)}{6}\right]$$

$$=\lim_{n\to\infty}\frac{8}{n^3}\left[\frac{2n^3+3n^2+n}{6}\right]=\lim_{n\to\infty}\left[\frac{16}{6}+\frac{24}{6n}+\frac{8}{6n^2}\right]=\frac{8}{3}+0+0=\frac{8}{3}$$

6.6.2 Fundamental theorem of integral calculus

In previous section, we learned that we can determine the area of a region with a definite integral. However, with the tools available to us at this time, evaluating a definite integral using the summation process is rather tedious and time consuming. To provide us with a more efficient method of evaluating the definite integral, we now consider a very important theorem in calculus, the **"fundamental theorem of integral calculus"**. This explanation will show that the definite integral can be applied in a general manner and not only to the concept of area.

To help provide a better understanding of the meaning of the fundamental theorem of integral calculus, let us begin with area of a region using definite integral

$$Area=\int_{x=a}^{x=b} f(x)dx, \qquad \text{(i)}$$

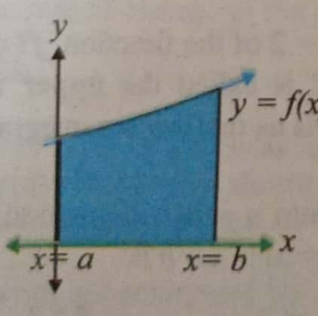


Figure 6.3

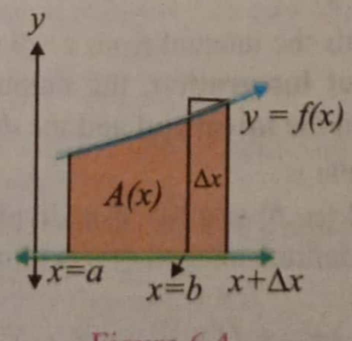


Figure 6.4

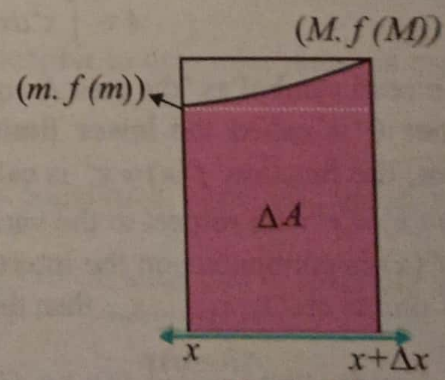


Figure 6.5

To develop the theorem, we need to introduce a new function called the area function $A(x)$. The function indicates the area of the region under the graph of the function from $x=a$ to $x=b$ in the Figure 6.3

The area function $A(x)$ is the area from a to x that must be continuous and non-negative on the interval $[a, b]$.

If we increase x by Δx, then the area $A(x)$ under the curve will increase by an amount that we call ΔA in Figure 6.4. We can see that ΔA is slightly bigger than the area of the inscribed rectangle and slightly smaller than the area of the circumscribed rectangle. In Figure 6.5, the smaller rectangle is inscribed (within the curve) and the large rectangle is circumscribed.

For the area of the inscribed rectangle, we take the minimum value of $f(x)$ within the closed interval $[x, x+\Delta x]$. We call this minimum value $f(m)$.

For the area of the circumscribed rectangle, we take the maximum value within the closed interval $[x, x+\Delta x]$. We refer to this value as $f(M)$. Hence the minimum area is $f(m)\Delta x$ and the maximum area is $f(M)\Delta x$.

Algebraically, we can write $\quad f(m)\Delta x \le \Delta A \le f(M)\Delta x$

$$f(m) \le \frac{\Delta A}{\Delta x} \le f(M), \quad \Delta x \ne 0 \qquad \text{(ii)}$$

If we take the limit as $\Delta x \to 0$, then $f(m)$ and $f(M)$ approach the same point on the curve and both approach $f(x)$

$$f(x) \le \lim_{\Delta x \to 0} \frac{\Delta A}{\Delta x} \le f(x)$$

which states that $\quad \dfrac{dA}{dx} = f(x), \quad \lim_{\Delta x \to 0} \dfrac{\Delta A}{\Delta x} = \dfrac{dA}{dx} \qquad$ (iii)

Integrating (iii) to obtain $\quad A(x) = F(x) + C \qquad$ (iv)

Here $F(x)$ is the antiderivative of $f(x)$. To determine a real value of $A(x)$, we must solve equation (iv) for C.

Put $x = a$ in (iv) to obtain: $\quad A(a) = F(a) + C \Rightarrow 0 = F(a) + C, \ A(a) = 0 \Rightarrow C = -F(a)$

Put $x = b$ in (iv) to obtain: $\quad A(b) = F(b) + C \Rightarrow A(b) = F(b) - F(a), \ C = -F(a) \qquad$ (v)

The last equation (v) tells us that if it is possible to find an antiderivative of $f(x)$, then we can evaluate the definite integral $\int_{x=a}^{x=b} f(x)dx$. This is nicely condensed in the fundamental theorem.

Statement: If a function $f(x)$ is continuous on the closed interval [a,b], then the definite integral of a function $f(x)$ in the interval $[a,b]$ is:

$$\int_{x=a}^{x=b} f(x)dx = \left|F(x)\right|_{x=a}^{x=b} = F(b) - F(a)$$

Proof: Here $F(x)$ is any function such that $F'(x) = f(x)$ for all x in $[a, b]$.

It is important to recognize that the fundamental theorem of integral calculus describes a means for evaluating a definite integral. It does not provide us with a technique for finding the antiderivative. To find the antiderivative of a definite integral, we use the same techniques we used to find the antiderivative of the indefinite integral. But what happens to the constant C? This constant C drops out as illustrated below:

$$\int_{x=a}^{x=b} f(x)dx = \left|F(x)+C\right|_{x=a}^{x=b} = \left[(F(b)+C) - (F(a)+C)\right]$$

$$= F(b) - F(a) + C - C = F(b) - F(a) \qquad \text{(vii)}$$

Basic properties of the definite integrals

In computations involving integrals, it is often helpful to use the seven basic properties related to fundamental theorem of calculus that are listed below:

i. $\int_a^a f(x)dx = 0$

Proof: By the definition of the definite integral

$$\int_a^a f(x)dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)(0)$$

$$\therefore \Delta x = \frac{a-a}{n} = 0$$

$$= \lim_{n\to 0}(0) = 0 \qquad \text{hence, } \int_a^a f(x)\,dx = 0$$

Challenge

Show that $\int_a^a f(x)dx = 0$ by using fundamental theorem of integral calculus.

ii. $\int_a^b f(x)dx = \int_a^b f(y)dy$

Proof:

Let $F(x) = f(x)$, $[a, b]$ be an interval then by the fundamental theorem of integral calculus.

$$\int_a^b f(x)dx = F(b) - F(a) \qquad \text{(i)}$$

$$\text{also, } \int_a^b f(y)dy = F(b) - F(a) \qquad \text{(ii)}$$

Hence, by (i) and (ii) $\int_a^b f(x)dx = \int_a^b f(y)dy$ (proved)

Challenge

Show that $\int_a^b f(x)dx = \int_a^b f(y)dy$ by using the definition of definite integral.

iii. $\int_a^b f(x)dx = -\int_b^a f(x)dx$

Proof: By using the definition of definite integrate

$$\int_a^b f(x)dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i) \qquad \therefore \Delta x = \frac{b-a}{n}$$

$$\text{and } \int_b^a f(x)dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x \qquad \therefore \Delta x = \frac{a-b}{n}$$

$$= \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\left(\frac{b-a}{n}\right) = \lim_{n\to\infty}\left(-\sum_{i=1}^{n} f(x_i)\frac{b-a}{n}\right)$$

$$= -\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\frac{b-a}{n} = -\int_b^a f(x)dx \text{ Hence, } \int_a^b f(x)dx = -\int_b^a f(x)dx$$

Challenge

Use fundamental theorem of integral calculus to show $\int_a^b f(x)dx = -\int_b^a f(x)dx$.

iv. $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx, a < c < b$

Proof: Let $F'(x) = f(x)$, $a \le x \le b$

By using the fundamental theorem of integral calculus.

$\int_a^b f(x)dx = F(b) - F(a)$ where f is continuous on $[a, b]$ then

$$\int_a^c f(x)dx = F(c) - F(a) \quad \text{(i)}$$

$$\int_c^b f(x)dx = F(b) - F(c) \quad \text{(ii)}$$

By adding (i) and (ii)

$$\int_a^c f(x)dx + \int_c^b f(x)dx = \cancel{F(c)} - F(a) + F(b) - \cancel{F(c)}$$

$$= F(b) - F(a) = \int_a^b f(x)dx$$

Hence, $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$

Challenge

Use definition of the definite integral to show that $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$.

v. $$\int_{-a}^{a} f(x)dx = \begin{cases} 2\int_0^a f(x)dx & \text{when } f(-x) = f(x) \\ 0 & \text{when } f(-x) = -f(x) \end{cases}$$

Proof: If $f(x)$ is integral on interval $[-a, a]$ w.r.t 'x', then for a number 0 in the interval $[-a, a]$, the definite integral of $f(x)$ from $-a$ to a is 2 times the definite integral of $f(x)$ from 0 to a:

$$\int_{-a}^{a} f(x)dx = \int_{-a}^{0} f(x)dx + \int_0^a f(x)dx = \int_a^0 f(-x)d(-x) + \int_0^a f(x)dx = -\int_a^0 f(-x)dx + \int_0^a f(x)dx$$

$$= \int_0^a f(-x)dx + \int_0^a f(x)dx = \int_0^a [f(-x) + f(x)]dx = \begin{cases} 2\int_0^a f(x)dx, & \text{when } f(-x) = f(x) \\ 0, & \text{when } f(-x) = -f(x) \end{cases}$$

6.6.3 Extend techniques of integration using properties to calculate definite integral

Example 16 Evaluate the following definite integrals:

(a). $\int_1^2 (2x^2+4x+1)dx$ (b). $\int_1^2 (2x^2+4x+1)dx = \int_1^2 (2y^2+4y+1)dy$ (c). $\int_0^2 (x^2+1)dx = -\int_2^0 (x^2+1)dx$

(d). $\int_0^2 (x^2+1)dx = \int_0^1 (x^2+1)dx + \int_1^2 (x^2+1)dx$ (e). $\int_{-1}^{+1} x^2dx = \int_{-1}^{0} x^2dx + \int_0^1 x^2dx = 2\int_0^1 x^2dx$

Solution a. $$\int_1^2 (2x^2+4x+1)dx = \left|\frac{2x^3}{3} + \frac{4x^2}{2} + x\right|_1^2$$

$$= \left(\frac{16}{3} + \frac{16}{2} + 2\right) - \left(\frac{2}{3} + \frac{4}{2} + 1\right) = \frac{92}{6} - \frac{11}{3} = \frac{70}{6} = \frac{35}{3}$$

b. $$\int_1^2 (2x^2+4x+1)\,dx = \int_1^2 (2y^2+4y+1)\,dy,$$

$$\Rightarrow \left|\frac{2x^3}{3} + \frac{4x^2}{2} + x\right|_1^2 = \left|\frac{2y^3}{3} + \frac{4y^2}{2} + y\right|_1^2 \Rightarrow \frac{35}{3} = \frac{35}{3}, \text{ since } f(x) = g(y)$$

c. $$\int_0^2 (x^2+1)dx = -\int_2^0 (x^2+1)dx,$$

$$\left|\frac{x^3}{3}+x\right|_0^2=-\left|\frac{x^3}{3}+x\right|_2^0 \Rightarrow \frac{8}{3}+2=-\left(0-\left(\frac{8}{3}+2\right)\right)\Rightarrow\frac{14}{3}=\frac{14}{3}$$

d. $\int_0^2 (x^2+1)dx=\int_0^1 (x^2+1)dx+\int_1^2 (x^2+1)dx,$

$$\left|\frac{x^3}{3}+x\right|_0^2=\left|\frac{x^3}{3}+x\right|_0^1+\left|\frac{x^3}{3}+x\right|_1^2 \Rightarrow \frac{14}{3}=\frac{4}{3}+\frac{8}{3}+2-\left(\frac{1}{3}+1\right)\Rightarrow\frac{14}{3}=\frac{14}{3}$$

e. $\int_{-1}^{+1} x^2dx=\int_{-1}^{0} x^2dx+\int_0^1 x^2dx=2\int_0^1 x^2dx,$

$$\left|\frac{x^3}{3}\right|_{-1}^1=\left|\frac{x^3}{3}\right|_{-1}^0+\left|\frac{x^3}{3}\right|_0^1=2\left|\frac{x^3}{3}\right|_0^1 \Rightarrow \frac{1}{3}+\frac{1}{3}=0+\frac{1}{3}+\frac{1}{3}=2\left(\frac{1}{3}\right)\Rightarrow\frac{2}{3}=\frac{2}{3}=\frac{2}{3}$$

Example 17 Evaluate the following definite integrals: (a). $\int_0^1 \frac{2e^{4x}-3}{e^{2x}}dx$ (b). $\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} x\sin x^2 dx$

Solution

a. $\int_0^1 \frac{2e^{4x}-3}{e^{2x}}dx=\int_0^1\left(2e^{2x}-3e^{-2x}\right)dx$

$$=\left|2\frac{e^{2x}}{2}-3\frac{e^{-2x}}{-2}\right|_0^1=\left|e^{2x}+\frac{3}{2}e^{-2x}\right|_0^1=\left(e^2+\frac{3}{2}e^{-2}\right)-\left(e^0+\frac{3}{2}e^0\right)=e^2+\frac{3}{2}e^{-2}-1=5.092$$

b. We need to substitute a new variable $u(x)$:

$$x^2=u,\ \frac{d}{dx}(x^2)=\frac{du}{dx}\Rightarrow 2x=\frac{du}{dx}\Rightarrow xdx=\frac{du}{2}$$

The lower and upper limit of $x=\frac{\pi}{3}$ and $x=\frac{\pi}{2}$ are used in $x^2=u$ to obtain the lower and upper limit of u

$$x=\frac{\pi}{3}: x^2=u\Rightarrow\left(\frac{\pi}{3}\right)^2=u\Rightarrow\frac{\pi^2}{9}=u$$

$$\Rightarrow x=\frac{\pi}{2}: x^2=u\Rightarrow\left(\frac{\pi}{2}\right)^2=u\Rightarrow\frac{\pi^2}{4}=u$$

Substitute all these in the given integral to obtain:

$$\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} x\sin x^2dx=\frac{1}{2}\int_{\frac{\pi^2}{9}}^{\frac{\pi^2}{4}}\sin u\,du=\left|\frac{-\cos u}{2}\right|_{\frac{\pi^2}{9}}^{\frac{\pi^2}{4}}=-\frac{1}{2}\left(\cos\frac{\pi^2}{4}-\cos\frac{\pi^2}{9}\right)$$

$$=\frac{-1}{2}[\cos(2.4647)-\cos(1.0966)]=\frac{-1}{2}(-0.7812-0.4566)=\frac{-1}{2}(-1.2378)=0.6189$$

Example 18 Evaluate the following definite integrals: (a). $\int_0^1 xe^x dx$ (b). $\int_0^1 e^x\sin x dx$

Solution a. The technique of integration by parts with $u=x$ and $\frac{dv}{dx}=e^x$ is used to obtain:

$$\int_0^1 xe^xdx=\left|xe^x\right|_0^1-\int_0^1 e^x1dx \qquad \therefore u=x, du=dx, \frac{dv}{dx}=e^x, v=e^x$$

$$=(e^1-0)-\left|e^x\right|_0^1=(e^1-0)-(e^1-e^0)=e^1-e^1+e^0=1$$

b. The integral is $I=\int_0^1 e^x \sin x\,dx$

The integration by parts rule with substitution $u=e^x$ and $\frac{dv}{dx}=\sin x$ is used to obtain:

$$I=\int_0^1 e^x \sin x\,dx$$

$$=\left|e^x(-\cos x)\right|_0^1-\int_0^1(-\cos x)e^x dx,\ u=e^x \qquad \therefore du=e^x dx,\ \frac{dv}{dx}=\sin x,\ v=-\cos x$$

$$=-(e^1\cos 1-e^0\cos 0)+\int_0^1 e^x\cos x\,dx$$

$$=-2.718(0.540)+1+\int_0^1 e^x\cos x\,dx=-0.468+\int_0^1 e^x\cos x\,dx \qquad \text{Use radians}$$

$$=-0.468+\int_0^1 e^x\cos x\,dx \qquad \text{Again integration by parts}$$

$$=-0.468+\left|e^x\sin x\right|_0^1-\int_0^1(\sin x)(e^x)dx \qquad \therefore u=e^x,\ \frac{dv}{dx}=\cos x,\ v=\sin x$$

$$=-0.468+(e^1\sin 1-e^0\sin 0)-\int_0^1 e^x\sin x\,dx$$

$$I=-0.468+\left(2.718(0.841)-(1)(0)\right)-I$$

$$2I=-0.468+2.287=1.819$$

$$I=\frac{1.819}{2}=0.91$$

6.6.4 Definite integral as the area under the curve

If $f(x)$ is continuous and $f(x)\geq 0$ on the closed interval $[a, b]$, then the area under a curve $y=f(x)$ on the interval $[a, b]$ is given by the definite integral of $f(x)$ on $[a, b]$:

$$Area=\int_a^b f(x)dx=F(b)-F(a) \qquad \text{(i)}$$

Area between a curve and the x-axis

The steps involved in finding the area between a curve and the x-axis are the following:

1. The definite integral $\int_a^b f(x)dx$ presents the sum of the signed areas between the graph of $y=f(x)$ and the x-axis from $x=a$ to $x=b$, where the area above the x-axis (peak) are counted positively and the areas below the x-axis (valley) are counted negatively. This is shown in the Figure 6.6.

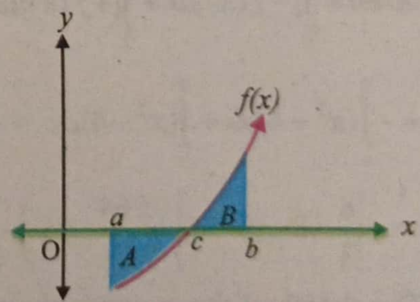


Figure 6.6

ii. If $f(x)$ is a continuous function over the interval $[a, b]$, then the area between $y = f(x)$ and the x-axis from $x = a$ to $x = b$ can be found using definite integrals as follows:

- For $f(x) \geq 0$ over $[a,b]$, the area is: $Area = \int_a^b [+f(x)]dx$
- For $f(x) \leq 0$ over $[a,b]$, the area is: $Area = \int_a^b [-f(x)]dx$

If $f(x)$ is positive for some values of x and negative for others on an interval (as in Figure 6.6), then, the area between the graph of and the x-axis can be found by (dividing the interval into subintervals over which f (x) is always positive or always negative) taking the sum of the areas of subregions over each subinterval:

$$Area = \int_a^b f(x)dx = \int_a^c [-f(x)dx] + \int_c^b [+f(x)dx] = -A + B \qquad \text{(ii)}$$

In Figure 6.6, A represents the area between $y = f(x)$ and the x-axis from $x = a$ to $x = c$, and B represents the area between $y = f(x)$ and the x-axis from $x = c$ to $x = b$. Both A and B are positive quantities. Since $f(x) \geq 0$ on the interval $[c, b]$, the area is

$\int_c^b [+f(x)]dx = B$ and $f(x) \leq 0$ on the interval $[a, c]$, the area is $\int_a^c [-f(x)dx] = -A$.

6.6.5 Application of definite integral as the area under a curve

Example 19 Find the area between the x-axis and the curve $f(x) = x^2 - 4$ from $x = 0$ to $x = 4$.

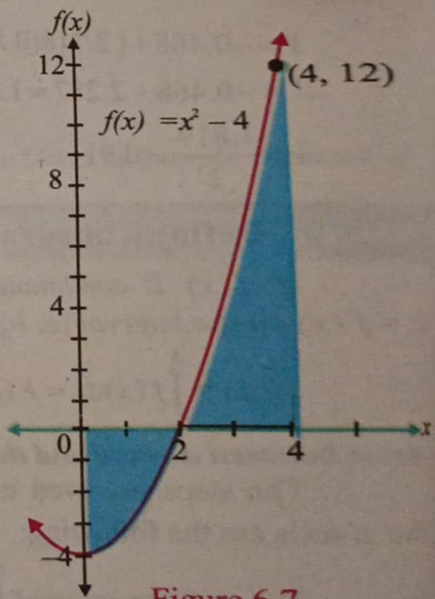


Figure 6.7

Solution First find out the x-intercepts of a curve $f(x) = x^2 - 4$ that can be found by solving the equation of a curve:

$x^2 - 4 = 0 \Rightarrow x = 2, -2$

The subintervals of the interval [0,4] are therefore [-2,0], [0,2]and [2,4]. The total area of the region in the required interval [0,4] is the sum of the areas of the sub regions in the subintervals [0,2] and [2,4]:

$$Area = \int_0^2 [-f(x)]dx + \int_2^4 [+f(x)]dx, \; f(x) \leq 0 \, in [0,2] \, and \, f(x) \geq 0 \, in [2,4]$$

$$= -\int_0^2 (x^2 - 4)dx + \int_2^4 (x^2 - 4)dx = -\left|\frac{x^3}{3} - 4x\right|_0^2 + \left|\frac{x^3}{3} - 4x\right|_2^4$$

$$= -\left|\frac{8}{3} - 8 - (0 - 0)\right| + \left(\frac{64}{3} - 16\right) - \left(\frac{8}{3} - 8\right) = \frac{16}{3} + \frac{16}{3} + \frac{16}{3} = 16 \text{ square units}$$

The sketch of the region is shown in the Figure 6.7.

The area over the entire interval [0,4] $\quad A = \int_0^4 (x^2 - 4)dx = x^3 - 4x\Big|_0^4 = \frac{16}{3}$

is not the correct area. This definite integral does not represent the area over the entire interval [0, 4], but is just a real number.

Example 20 Find the area between the x-axis and the curve $f(x)=x^2-2x$ from $x=-1$ to $x=3$.

Solution First find out the x-intercepts of a curve $f(x)=x^2-2x$ that can be found by solving the equation of a curve:

$$x^2-2x=0 \Rightarrow x=0,2$$

The subintervals of the interval $[-1,3]$ are therefore $[-1,0]$, $[0,2]$and $[2,3]$. The total area of the region in the required interval $[-1,3]$ is the sum of the areas of the sub regions in the subintervals $[-1,0]$, $[0,2]$and $[2,3]$:

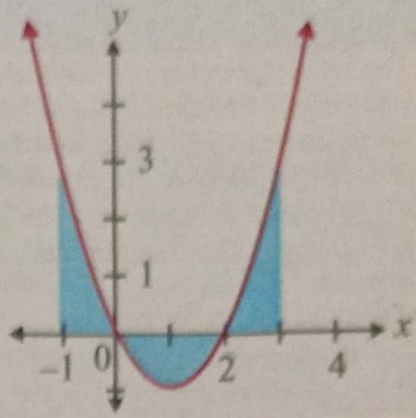


Figure 6.8

$$A=\int_{-1}^{0}[+f(x)]dx+\int_{0}^{2}[-f(x)]dx+\int_{2}^{3}[+f(x)]dx,\ f(x)\geq 0\, in[-1,0],[2,3]$$

$$=\int_{-1}^{0}(x^2-2x)dx-\int_{0}^{2}(x^2-2x)dx+\int_{2}^{3}(x^2-2x)dx=\left|\frac{x^3}{3}-\frac{2x^2}{2}\right|_{-1}^{0}-\left|\frac{x^3}{3}-\frac{2x^2}{2}\right|_{0}^{2}+\left|\frac{x^3}{3}-\frac{2x^2}{2}\right|_{2}^{3}$$

$$=(0-0)-\left(\frac{-1}{3}-1\right)-\left|\left(\frac{8}{3}-4\right)-(0-0)\right|+\left(\frac{27}{3}-9\right)-\left(\frac{8}{3}-4\right)=\frac{4}{3}+\frac{4}{3}+\frac{4}{3}=3\left(\frac{4}{3}\right)=4$$

The sketch of the region is shown in the Figure 6.8.

6.6.6 MAPLE command "*int*" to evaluate definite and indefinite integrals

The use of maple common '*int*' is illustrated in the following example.

Example 21 Use MAPLE command 'int' to solve.

(a). Indefinite integral of a function $f(x)=x^4+x^3+x^2+x+1$ w.r.t variable x.

(b). Definite integral of a function $f(x)=x^2$ w.r.t variable x.

(c). Definite integral of a function $f(x)=xe^x$ in the interval $[0,1]$.

Solution

a. **Command:**

> $int(x^4+x^3+x^2+x+1,x);$

$$\frac{1}{5}x^5+\frac{1}{4}x^4+\frac{1}{3}x^3+\frac{1}{2}x^2+x$$

Using Palettes: Use cursor button to select integral palette. Click-integral palette, insert the function required, then press "ENTER" key to obtain the integral of a given function:

> $\int x^4+x^3+x^2+x+1\,dx$

$$\frac{1}{5}x^5+\frac{1}{4}x^4+\frac{1}{3}x^3+\frac{1}{2}x^2+x$$

b. **Command:**

> $int(x^2,x=0..1);$

$$\frac{1}{3}$$

Using Palettes:

> $\int_0^1 x^2\,dx$

$$\frac{1}{3}$$

c. **Command:**

> $int(x\cdot\exp(x),x=0..1);$

$$1$$

Using Plalettes:

> $\int_0^1 x\cdot\exp(x)\,dx$

$$1$$

Do you know a 200 year old problem ?

The relationship between derivative and integrals as an inverse operation was noticed first time by Isaac barrow (1630-1677) in the 17th century. He was a teacher of Sir Isaac Newton. Newton and Leibniz are known as key inventor of calculus. They made the use of calculus as conjuctor, that is as a mathematical statement which is suspected to be true. But has not proven yet. The fundamental theorem of integral calculus was not officially proven in all its glory until Bernhard Riemann (1826-1866) demonstrated it in the 19th century. During this 200-years a lot of mathematic like real analysis had invented before Riemann could prove that derivatives and integrals are inverse.

Exercise 6.4

1. Evaluate the following definite integrals:

a. $\int_{3}^{4} 5x\,dx$ b. $\int_{12}^{20} x^3\,dx$ c. $\int_{1}^{2} (2x^{-2}-3)\,dx$ d. $\int_{1}^{4} 3\sqrt{x}\,dx$

e. $\int_{2}^{3} 12(x^2-4)^5 x\,dx$ f. $\int_{-1}^{1} \dfrac{e^{-x}-e^{x}}{(e^{-x}+e^{x})^2}\,dx$ g. $\int_{\frac{\pi}{2}}^{\pi} \cos\left(\dfrac{x}{2}+\pi\right)dx$ h. $\int_{0}^{\frac{\pi}{4}} \sec^2\theta\,d\theta$

2. Evaluate the following definite integrals:

a. $\int_{1}^{2} \dfrac{5t^2-3t+18}{t(9-t^2)}\,dt$ b. $\int_{1}^{2} \dfrac{4}{t^3+4t}\,dt$

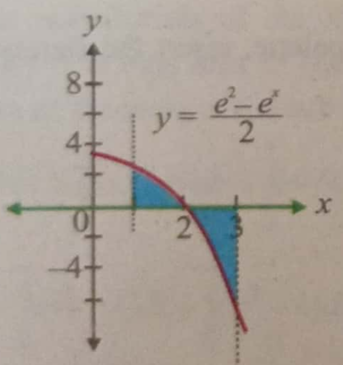
3. Use definite integral to find out the area between the curve f(x) and the x-axis over the indicated interval [a, b]:

a. $f(x)=4-x^2,\ [0,3]$ b. $f(x)=x^2-5x+6,\ [0,3]$

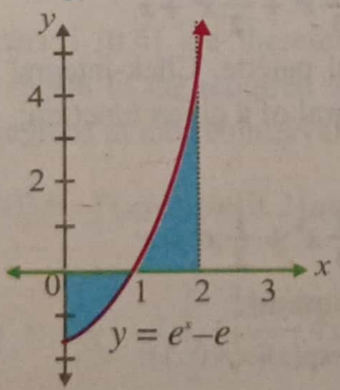
c. $f(x)=x^2-6x+8,\ [0,4]$ d. $f(x)=5x-x^2,\ [1,3]$

4. Setup definite integrals in problems **a** to **d** that represent the indicated shaded areas:

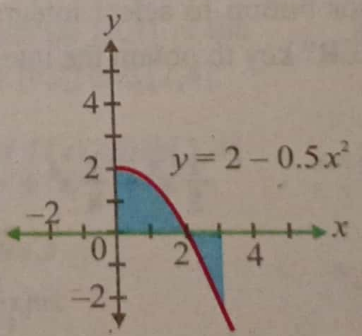
a.



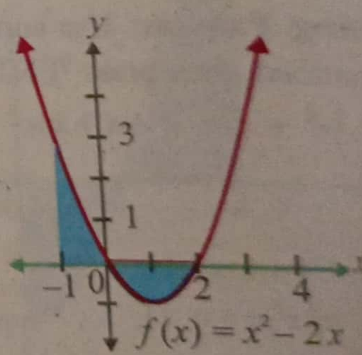
b.



c.



d.



5. An oil tanker is leaking oil at a rate given in barrels per hour by $\dfrac{dL}{dt}=\dfrac{80\ln(t+1)}{(t+1)}$

Where t is the time in hours after the tanker hits a hidden rock (when $t=0$).

a. Find the total number of barrels that the ship will leak on the first day.

b. Find the total number of barrels that the ship will leak on the second day.

c. What is happening over the long run to the amount of oil leaked per day?

6. Use MAPLE command *'int'* to evaluate

a. $f(x)=x^2+3x+1$ *w.r.t. 'x'* b. $f(x)=e^{2x}.\sin x$ *w.r.t. 'x'*

Review Exercise 6

Choose the correct option.

1.

i. The process of finding antiderivative is called.

(a). differentiation (b). integration (c). probability (d). linear equations

ii. $\int \tan\theta \, d\theta =$ ________

(a). $\ln|\sin\theta|+C$ (b). $\ln|\cos\theta|+C$ (c). $-\ln|\cos\theta|+C$ (d). $-\ln|\sin\theta|+C$

iii. $\int \frac{dx}{\sqrt{a^2-x^2}} =$ ________

(a). $\ln\left|x+\sqrt{x^2+a^2}\right|+C$ (b). $\sin^{-1}\frac{a}{x}+C$ (c). $\sin^{-1}\left(\frac{x}{a}\right)+C$ (d). $\cos^{-1}\left(\frac{x}{a}\right)+C$

iv. $\int \frac{1}{\sqrt{t^2-36}}dx =$ ________.

(a). $\frac{1}{2a}\ln\left|\frac{t-6}{t+6}\right|+C$ (b). $\frac{1}{2}\ln\left|\frac{t-6}{t+6}\right|+C$

(c). $-\frac{1}{2}\left(\ln\left|\frac{t}{6}+1\right|-\ln\left|\frac{t}{6}-1\right|+C\right)$ (d). $\frac{1}{6}\left(\ln\left|\frac{t}{6}+1\right|+6\right)+C$

v. $\int (x^3-4)dx =$ ________

(a). $\frac{x^4}{4}-4x+C$ (b). $\frac{x^3}{3}-4x+C$

(c). $-\frac{x^4}{4}+4x+C$ (d). $\frac{x^4}{4}+\frac{4x^3}{3}-\frac{4x^2}{2}-4x+C$

vi. $\int f(x)g'(x)dx =$ ________

(a). $f'(x)g(x)+\int g'(x)f(x)dx$ (b). $f(x)g(x)-\int g(x)f'(x)dx$

(c). $f'(x)'g(x)-\int g'(x)f'(x)dx$ (d). $f(x)g(x)-\int g(x)f(x)dx$

vii. $\int \tan^4(x)\,dx =$ ________

(a). $\frac{2}{3}\tan^2(x)+x-\tan(x)+C$ (b). $\frac{1}{3}\tan^3(x)+x-\tan(x)+C$

(c). $\frac{3}{4}\tan^2(x)-x+\tan(x)+C$ (d). $3\tan^3(x)+x+\tan(x)+C$

viii. $\int \frac{x+8}{x^2-64}dx =$

(a). $\ln|x-8|+C$ (b). $\frac{1}{2}\ln|x^2-64|+\frac{1}{2}\left|\frac{x}{8}+1\right|-\frac{1}{2}\ln\left|\frac{x}{2}-1\right|+C$

(c). $\frac{1}{2}\ln|x^2-64|-\frac{1}{2}\ln\left|\frac{x}{8}+1\right|+\frac{1}{2}\ln\left|\frac{x}{8}-1\right|+C$ (d). $\frac{1}{2}\ln\left|\frac{x}{8}+1\right|+\frac{1}{2}\ln\left|\frac{x}{8}-1\right|+C$

ix. $\int_0^2 e^{2x}dx =$ ________

(a). $\frac{e^4-1}{2}$ (b). $\frac{e^3-1}{2}$ (c). $\frac{e^2-1}{4}$ (d). $\frac{e^4+1}{2}$

x. $\int_0^2 x^3 dx =$ ________

(a). 1 (b). 2 (c). 3 (d). 4

Summary

- $F(x)$ is an **antiderivative** of $f(x)$ if $F'(x) = f(x)$.
- If $F'(x) = f(x)$, then $\int f(x)dx = F(x) + C$, for any real number C. It is called **indefinite integral**
- If $f(x)$ and $g(x)$ are integral functions w.r.t. x, then the integral of the product of $f(x)$ and $g(x)$ w.r.t. x is:

  $\int u dv = uv - \int v du$, $v = g(x)$, $du = f'(x)dx$ and $dv = g'(x)dx$:

- If $f(x)$ is continuous on the interval $[a, b]$ and $[a, b]$ is divided into n equal subintervals whose right-hand points are $x_1, x_2, \ldots, x_n$, then the **definite integral** of $f(x)$ from $x = a$ to $x = b$ is:

$$\int_{x=a}^{x=b} f(x)dx = \lim_{n \to \infty} \frac{(b-a)}{n}\left[f(x_1) + f(x_2) + \ldots + f(x_n)\right], \quad \Delta x = \frac{b-a}{n}$$

$$= \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i)\Delta x, \quad i = 1, 2, 3, \ldots, n$$

- The **definite integral** of the product of two functions u and v w.r.t. x is:

$$\int_a^b u dv = uv\Big|_a^b - \int_a^b v du$$

- If $f(x)$ is continuous and $f(x) \geq 0$ on the closed interval $[a, b]$, then the area under a curve $y = f(x)$ on $[a, b]$ is given by the **definite integral** of $f(x)$ on $[a, b]$:

$$Area = \int_a^b f(x)dx = F(b) - F(a)$$

- If a function $f(x)$ is continuous on the closed interval $[a, b]$, then

$$\int_{x=a}^{x=b} f(x)dx = \left|F(x)\right|_{x=a}^{x=b} = F(b) - F(a)$$

Where $F(x)$ is any function such that $F'(x) = f(x)$ for all x in $[a, b]$.

Unit 7

PLANE ANALYTIC GEOMETRY STRAIGHT LINE

By the end of this unit, the students will be able to:

7.1 **Division of a line segment**

i. Recall distance formula to calculate distance between two points given in Cartesian plane.
ii. Find coordinates of a point that divides the line segment in given ratio (internally and externally).
iii. Show that the medians and angle bisectors of a triangle are concurrent.

7.2 **Slope of a straight line**

i. Define the slope of a line.
ii. Derive the formula to find the slope of a line passing through two points.
iii. Find the condition that two straight lines with given slopes may be
- parallel to each other,
- perpendicular to each other

7.3 **Equation of a straight line parallel to Co-ordinate axes**

i. Find the equation of a straight line parallel to
- y-axis and at a distance a from it,
- x-axis and at a distance b from it

7.4 **Standard form of equation of a straight line**

i. Define intercepts of a straight line. Derive equation of a straight line in
- slope-intercept from,
- point-slope form,
- two-point form,
- intercepts form,
- symmetric form,
- normal form

ii. Show that a linear equation in two variables represents a straight line.
iii. Reduce the general form of the equation of a straight line to the other standard forms.

7.5 **Distance of a point from a line**

i. Recognize a point with respect to position of a line.
ii. Find the perpendicular distance from a point to the given straight lines.

7.6 **Angle between lines**

i. Find the angle between two coplanar intersecting straight lines.
ii. Find the equation of family of lines passing through the point of intersection of two given lines.
iii. Calculate angles of the triangle when the slopes of the sides are given.

7.7 **Concurrency of straight lines**

i. Find the condition of concurrency of three straight lines.
ii. Find the equation of median, altitude and right bisector of a triangle.
iii. Show that
- three right bisectors,
- three medians,
- three altitudes, of a triangle are concurrent.

7.8 **Area of a triangular region**

i. Find area of a triangular region whose vertices are given.

7.9 **Homogenous equation**

i. Recognize homogeneous linear and quadratic equations in two variables.
ii. Investigate that the 2nd degree homogeneous equation in two variables x and y represents a pair of straight lines through the origin and find acute angle between them.

Introduction

We are familiar about Cartesian coordinate system, we have learnt about it in our previous classes. This Cartesian coordinate system may be helpful to know the slope formula, Pythagoras theorem and distance formula. In this lesson we will learn in details and write the equations involving arbitrary points. Most of the geometric ideas can be expressed using algebraic equations. Analytic geometry is defined as:

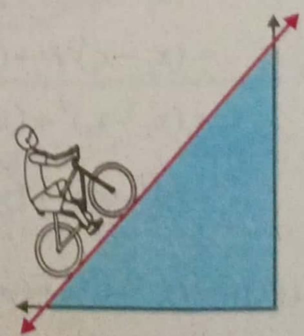


Figure 7.1

"The study of relationship between geometry and algebra is called analytic geometry".

For example to calculate the slope/gradient between two given points, the numerator is the difference in the y-coordinates some times called it "Rise" and the denominator is the difference between x-coordinates, some time called it "run" e.g.

$$\text{Slop between two points} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{Rise}{Run}$$

Do You Know?

Analytic Geometry was independent and simultaneous invention of Pierre De Fermat and Rene Descartes. The fundamental idea of Analytic Geometry and the representation of curved lines by algebraic equations relating two variables say, x and y was given in seventeenth century by them.

7.1 Division of a line segment

We are familiar with the set of real numbers as well as with several of its subsets, including natural numbers and real numbers. The real numbers can easily be visualized by using a one dimensional coordinate system call real number line.

7.1.1 Calculation of distance between two given points

The study of plane analytic geometry is greatly facilitated by the use of vectors. The distance between any two given points can be calculated by using the distance formula.

If $P(x_1, y_1)$ and $Q(x_2, y_2)$ are two points in the xy–plane and θ is the angle in between the positive directions of the x and y axes, then, PQ is the directed line segment associated to initial point $P(x_1, y_1)$ and terminal point $Q(x_2, y_2)$.

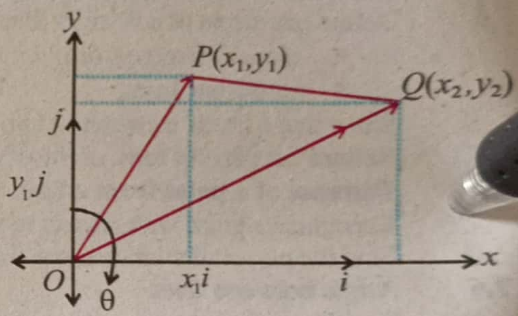


Figure 7.2

The components of the directed line segment PQ are:

$OP + PQ = OQ$

$PQ = OQ - OP$, position vectors

$= (x_2, y_2) - (x_1, y_1)$

$PQ = (x_2 - x_1, y_2 - y_1)$

$= (x_2 - x_1)i + (y_2 - y_1)j$

Pythagoras Theorem: If $P(x_1, y_1)$ and $Q(x_2, y_2)$ are the two points in the xy-plane, then the distance d between the given two points $P(x_1, y_1)$ and $Q(x_2, y_2)$ is obtained by applying the theorem of Pythagoras to triangle PQR:

$(PQ)^2 = (PR)^2 + (QR)^2$

$= (x_2 - x_1)^2 + (y_2 - y_1)^2$

$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = d, say$

Squaring both side of the directed line segment PQ to obtain

$(PQ)^2 = [(x_2 - x_1)i + (y_2 - y_1)j]^2 \quad \therefore (a+b)^2 = a^2 + b^2 + 2ab$

$= (x_2 - x_1)^2 i.i + (y_2 - y_1)^2 j.j + 2(x_2 - x_1)(y_2 - y_1)i.j$

$= (x_2 - x_1)^2 i.i + (y_2 - y_1)^2 j.j + 2(x_2 - x_1)(y_2 - y_1)|i||j|\cos\theta \quad \therefore i.i = j.j = 1,\ i.j = |i||j|\cos\theta,\ |i| = |j| = 1$

$= (x_2 - x_1)^2 + (y_2 - y_1)^2 + 2(x_2 - x_1)(y_2 - y_1)\cos\theta$

$= (x_2 - x_1)^2 + (y_2 - y_1)^2 + 2(x_2 - x_1)(y_2 - y_1)\cos\frac{\pi}{2} \quad \therefore \theta = \frac{\pi}{2}$

$(PQ)^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 \quad \therefore \cos\frac{\pi}{2} = 0$

$|PQ|^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 \quad \therefore (PQ)^2 = |PQ|^2$

$|PQ| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = d$, *say* (i)

This is the distance from point $P(x_1, y_1)$ to point $Q_2(x_2, y_2)$ in the Cartesian coordinate plane.

Note

- The distance from the origin O(0,0) to point $P(x_1, y_1)$ is obtained by inserting $x_2 = y_2 = 0$ in result (1): $d = |OP| = \sqrt{x_1^2 + y_1^2}$
- The distance from the origin O(0,0) to point $Q(x_2, y_2)$ is obtained by inserting $x_1 = y_1 = 0$ in result (1): $d = |OQ| = \sqrt{x_2^2 + y_2^2}$
- If the line segment PQ is horizontal, then the distance from the point $P(x_1, y_1)$ to point $Q(x_2, y_2)$ is obtained by inserting $y_1 = y_2$ in result (1): $d = |PQ| = \sqrt{(x_2 - x_1)^2}$
- If the line segment PQ is vertical, then the distance from point $P(x_1, y_1)$ to point $Q(x_2, y_2)$ is obtained by inserting $x_1 = x_2$ in result (1): $d = |PQ| = \sqrt{(y_2 - y_1)^2}$

Example 1 Find the distance between the two points P(3,–2) and Q(–1, –5).

Solution $P(x_1, y_1) = (3,-2)$, $Q(x_2, y_2) = (-1,-5)$ is used to obtain the distance d in between the two points P and Q:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{(-1-3)^2 + [(-5-(-2)]^2} = \sqrt{(-4)^2 + (-3)^2} = \sqrt{25} = 5$$

7.1.2 Co-ordinates of a point that divides the line segment in given ratio (Internally and externally)

Take $P(x_1, y_1)$ and $Q(x_2, y_2)$ are the initial and terminal points of a line segment PQ and $R(x, y)$ is a point that divides PQ in the ratio $m_1 : m_2$. If r_1, r_2 and r are the position vectors of P, Q and R, then

$r_1 = (x_1, y_1) = x_1 i + y_1 j, \quad r_2 = (x_2, y_2) = x_2 i + y_2 j, \quad r = (x, y) = xi + yj$

If $\dfrac{PR}{RQ} = \dfrac{m_1}{m_2}$, then, $PR = \dfrac{m_1}{m_1 + m_2} PQ = \dfrac{m_1}{m_1 + m_2}(OQ - OP) = \dfrac{m_1}{m_1 + m_2}(r_2 - r_1)$ $\quad \therefore OP + PQ = OQ$

If $OP + PR = r_1 + \dfrac{m_1}{m_1 + m_2}(r_2 - r_1)$, $\quad OP + PR = OR$

then the position vector of OR is:

$$OR = OP + PR = r_1 + \frac{m_1}{m_1 + m_2}(r_2 - r_1) = \frac{r_1 m_1 + r_1 m_2 + r_2 m_1 - r_1 m_1}{m_1 + m_2}$$

$$r = \frac{m_2 r_1 + m_1 r_2}{m_1 + m_2}, \qquad OR = r$$

$$(x, y) = \frac{m_2(x_1, y_1) + m_1(x_2, y_2)}{m_1 + m_2}, \quad \textit{components form}$$

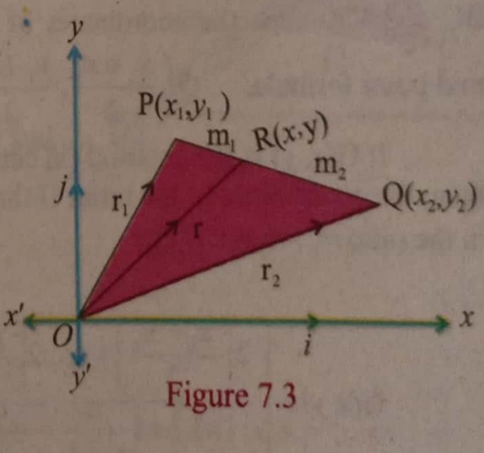


Figure 7.3

Equating x and y components to obtain the coordinates of R(x,y)

$$(x, y) = \left(\frac{m_1 x_2 + m_2 x_1}{m_1 + m_2}, \frac{m_1 y_2 + m_2 y_1}{m_1 + m_2}\right) \quad \text{(i)}$$

that divides the line segment PQ in the ratio $m_1 : m_2$.

Remember

- If R is the midpoint of the line segment PQ, then, $m_1 = m_2$ and the coordinates of the midpoint R of the line segment PQ are: $(x, y) = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$ (A)
- The coordinates of the point that divides the line segment PQ joining two points $P(x_1, y_1)$ and $Q(x_2, y_2)$ externally in the ratio $m_1 : m_2$ (m_1 or m_2 is negative) are: $(x, y) = \left(\frac{m_1x_2 - m_2x_1}{m_1 - m_2}, \frac{m_1y_2 - m_2y_1}{m_1 - m_2}\right)$ (B)

Example 2 Find the coordinates of the point which divides the line segment PQ joining the two points (a). P(1, 2) and Q(3, 4) in the ratio 5:7. (b). P(3, 4) and Q(–6, 2) in the ratio 3: –2.

Solution

a. If R(x, y) is a point that divides the line segment PQ in the ratio 5:7, then the coordinates of R(x, y) is obtained through result (B):

$$(x, y) = \left(\frac{m_1x_2 + m_2x_1}{m_1 + m_2}, \frac{m_1y_2 + m_2y_1}{m_1 + m_2}\right) \qquad \therefore m_1 = 5, m_2 = 7, P(1,2), Q(3,4)$$

$$= \left(\frac{5(3)+7(1)}{5+7}, \frac{5(4)+7(2)}{5+7}\right) = \left(\frac{11}{6}, \frac{17}{6}\right)$$

b. If R(x, y) is a point that divides the segment PQ in the ratio 3:–2, then the coordinates of R(x, y) is obtained through result (B):

$$(x, y) = \left(\frac{m_1x_2 + m_2x_1}{m_1 + m_2}, \frac{m_1y_2 + m_2y_1}{m_1 + m_2}\right)$$

$$= \left(\frac{3(-6)+(-2)(3)}{3-2}, \frac{3(2)+(-2)(4)}{3-2}\right) = (-24, -2) \qquad \therefore m_1 = 3,\ m_2 = -2$$

7.1.3 The medians and angle bisectors of a triangle are concurrent

i. The medians of a triangle are concurrent

Proof: If $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ are the vertices of a triangle ABC and P, Q and R are the midpoints of the sides AB, BC and CA, then the coordinates of the midpoint Q through mid point formula. $Q\left(\frac{x_2 + x_3}{2}, \frac{y_2 + y_3}{2}\right)$ $\therefore m_1 = m_2$

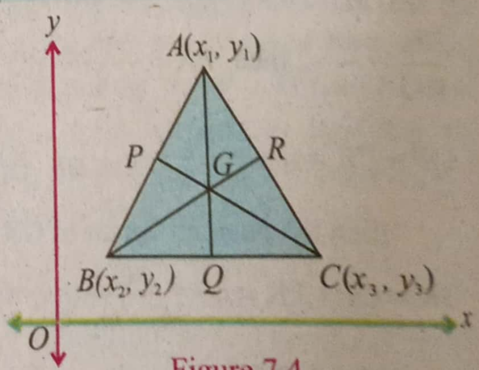


Figure 7.4

If G(x, y) is the centroid (in centre) of the triangle ABC, then, the coordinates of the point G that divides the median AQ in the ratio $m_1 : m_2 = 2:1$ are:

$$G(x, y) = \left(\frac{2\left(\frac{x_2 + x_3}{2}\right) + x_1}{2+1}, \frac{2\left(\frac{y_2 + y_3}{2}\right) + y_1}{2+1}\right) = \left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right) \quad \text{(i)}$$

Similarly, the coordinates of the point G(x, y) that divides the medians BR and CP each in the ratio 2: 1 are respectively:

$$G(x,y)=\left(\frac{2\left(\frac{x_1+x_3}{2}\right)+x_2}{2+1},\frac{\left(2\frac{y_1+y_3}{2}\right)+y_2}{2+1}\right)=\left(\frac{x_1+x_2+x_3}{3},\frac{y_1+y_2+y_3}{3}\right) \quad \text{(ii)}$$

$$G(x,y)=\left(\frac{2\left(\frac{x_1+x_2}{2}\right)+x_3}{2+1},\frac{2\left(\frac{y_1+y_3}{2}\right)+y_2}{2+1}\right)=\left(\frac{x_1+x_2+x_3}{3},\frac{y_1+y_2+y_3}{3}\right) \quad \text{(iii)}$$

Therefore, the point G(x, y) lies on each median and consequently the medians of the triangle ABC are concurrent.

Example 3 Find the centroid of the triangle ABC, whose vertices are A(3,–5), B(–7, 4) and C(10, –2).

Solution If A(3, –5), B(–7, 4) and C(10, –2) are the vertices of the triangle ABC, and P, Q and R are the midpoints of the sides AB, BC and CA, then, the coordinates of the midpoint Q through mid point formula:

$$Q=\left(\frac{x_2+x_3}{2},\frac{y_2+y_3}{2}\right)=\left(\frac{-7+10}{2},\frac{4-2}{2}\right)=\left(\frac{3}{2},1\right)$$

If G(x, y) is the centroid of the triangle ABC that divides each median in the ratio 2:1, then, the coordinates of the point G(x, y) (that divides the median AQ in the ratio $m_1:m_2=2:1$) through result (i) are:

$$G(x,y)=\left(\frac{x_1+x_2+x_3}{3},\frac{y_1+y_2+y_3}{3}\right)=\left(\frac{3-7+10}{3},\frac{-5+4-2}{3}\right)=(2,-1)$$

Similarly, the coordinates of the centroid G(x, y) that divides the medians BR and CP each in the ratio 2:1 are of course (2,–1).

ii. The bisectors of a triangle are concurrent

Proof: If ABC is a triangle with vertices $A(x_1,y_1), B(x_2,y_2)$ and $C(x_3,y_3)$, whose lengths are $|AB|=c, |BC|=a$ and $|CA|=b$, then, the position vectors of A, B and C are respectively:

$r_1=(x_1,y_1)=x_1i+y_1j,\ r_2=(x_2,y_2)=x_2i+y_2j,\ r_3=(x_3,y_3)=x_3i+y_3i$

Consider AD, BE and CF are the internal bisectors of the angles A, B and C that meet at centroid G. This is shown in Figure 7.5.

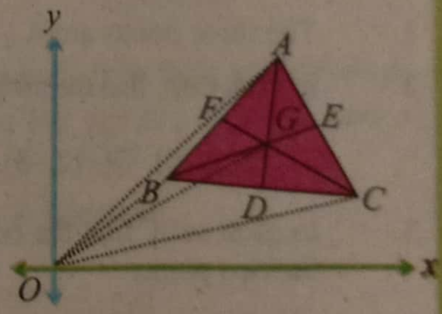


Figure 7.5

If AD is the internal bisector of angle A, then:

$$\frac{BD}{DC}=\frac{BA}{AC} \quad or \quad \frac{BD}{DC}=\frac{c}{b} \Rightarrow BD:DC=c:b \quad \text{(i)}$$

This means that D divides BC internally in the ratio c:b and the position vector of D is therefore: $\frac{cr_3+br_2}{c+b}$

$$\text{Again, } \frac{BD}{c}=\frac{DC}{b}=\frac{BD+DC}{c+b}=\frac{a}{c+b} \Rightarrow BD=\frac{ac}{c+b} \quad \text{(ii)}$$

If BG is the internal bisector of the angle B, then, $\frac{DG}{AG}=\frac{BD}{AB}=\frac{\frac{ac}{b+c}}{c}=\frac{a}{b+c} \Rightarrow DG:GA=a:(b+c)$

The position vector of G(x, y) is: $$r=\frac{ar_1+(b+c).\left(\frac{cr_3+br_2}{b+c}\right)}{a+b+c}=\frac{ar_1+br_2+cr_3}{a+b+c}=\frac{a(x_1,y_1)+b(x_2,y_2)+c(x_3,y_3)}{a+b+c} \quad \text{(iii)}$$

The coordinates of the centroid G(x, y) is obtained from equation (iii) by equating the x and y components:

$$G(x,y)=\left(\frac{ax_1+bx_2+cx_3}{a+b+c},\frac{ay_1+by_2+cy_3}{a+b+c}\right) \quad \text{(iv)}$$

Similarly, the internal bisector of the angle C also passes through the point G(x, y). Thus, the angle bisectors of a triangle ABC are concurrent and G(x, y) is the point of concurrency.

Example 4 Muhammad Ayaan has a triangular piece of backyard where he wants to build a swimming pool. How can he find the largest circular pool that can be built there?

Solution The largest possible circular pool would have the same size as the largest circle that can be inscribed in the triangular backyard. The largest circle that can be inscribed in a triangle is incircle. This can be determined by finding the point of concurrency of the angle bisectors of each corner of the backyard and then making a circle with this point as center and the shortest distance from this point to the boundary as radius.

Example 5 Find the length JO.

Solution Here, O is the point of concurrency of the three angle bisectors of ΔLMN and therefore is the incenter. The incenter is equidistant from the sides of the triangle. That is, JO = HO = IO.

We have the measures of two sides of the right triangle ΔHOL, so it is possible to find the length of the third side.

Use the Pythagorean Theorem to find the length HO.

$=\sqrt{(LO)^2-(HL)^2}=\sqrt{13^2-12^2}=\sqrt{169-144}=\sqrt{25}=5$

Since JO = HO, the length JO also equals 5 units.

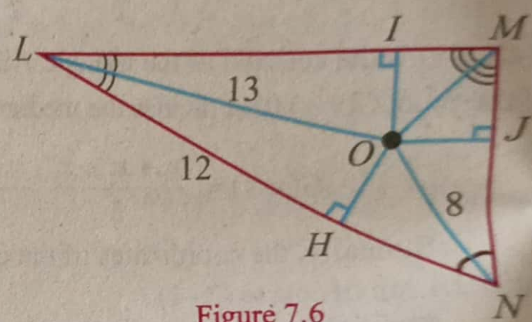


Figure 7.6

Exercise 7.1

1. The three points are A (–1,3), B (2,1) and C (5,–1). Show that $|AB|+|BC|=|AC|$.
2. In each case, find the midpoint of the line segment PQ joining the two points $P(x_1, y_1)$ and $Q(x_2, y_2)$:
   a. P(10,20), Q(–12,–8) b. P(a,–b), Q(–a, b) c. $P\left(\frac{1}{2},-\frac{1}{4}\right), Q\left(\frac{3}{5},\frac{4}{7}\right)$
3. In each case, find the coordinates of the point R(x,y) which divides the line segment PQ joining the two points
   a. P(1,2), Q(3,4) in the ratio 5:7. b. P(3,4), Q(–6,2) in the ratio 3:–2.
   c. P(–6,7), Q(5,–4) in the ratio $\frac{2}{7}:1$.
4. In each case, in what ratio is the line segment PQ (joining the two points $P(x_1, y_1)$ and $Q(x_2, y_2)$) divided by the point $R(x, y)$:
   a. P(8,10), Q(–12,6), $R\left(-\frac{4}{7},\frac{58}{7}\right)$. b. P(–2,4), Q(3,6), $R\left(\frac{4}{5},\frac{3}{5}\right)$.
5. Find the centroid of the triangle ABC, whose vertices are the following:
   a. A(4,–2), B(–2,4), C(5,5) b. A(3,5), B(4,6), C(3,–1) c. A(1,1), B(–2,–2), C(4,5)

7.2 Slope of a Straight Line

The slope of a line is a measure of the when "steepness" of the line, and whether it rises, or falls when moving from left to right. The line from A to B rises up, while the line from C to D goes down are depicted in the Figure 7.6:

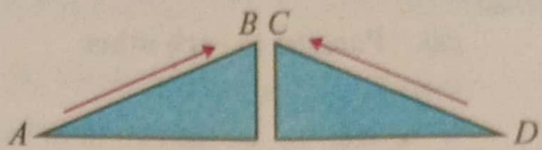


Figure 7.6

7.2.1 Slope of a line

The graph of a line can be drawn knowing only one point on the line if the "steepness" of the line is known, too.

"A number that measure the "steepness" of a line is called slope of a line."

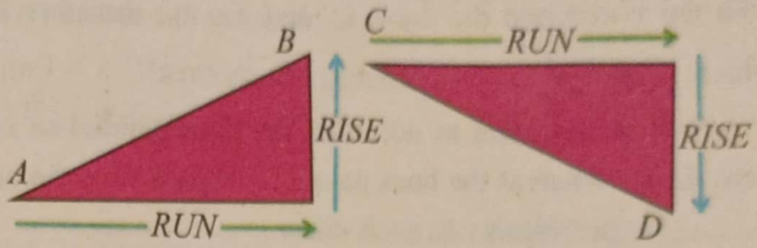


Figure 7.7

If move off the line horizontally to the right first or move up or down (vertically) to return to the line, then the slope of the line is the "steepness" defined as the ratio of the vertical rise to the horizontal run: $slope = \frac{rise}{run}$, the run is always a movement to the right

7.2.2 Formula to find the slope of a line passing through two points

Mathematically, if any two points on a line are available, then their join makes a constant angle with a fixed direction and the angle so formed is independent of the choice of the two points on the line. This is a precise way of saying that any line has a constant slope. It is customary to measure the angle θ which a line makes with the positive direction of the *x-axis*. The quantity $\tan\theta$ is defined to be the slope of the line and is denoted by $\boldsymbol{m}$. The slope of a line is also referred to **gradient** of the line.

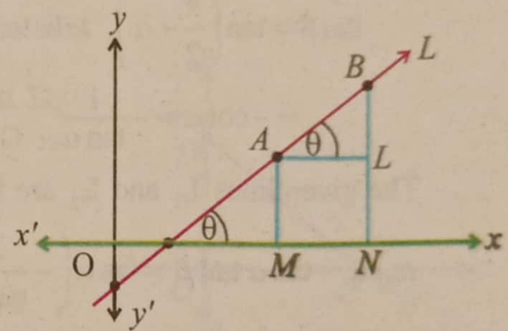


Figure 7.8

For illustration, if $A(x_1, y_1)$ and $B(x_2, y_2)$, where $x_1 \neq x_2$, are any two points, then their join develops a line L that makes a constant angle θ with the *x-axis*. Draw AM, and BN parallel to *y-axis* and AL parallel to *x*-axis.

The slope m of a line L through the two points $A(x_1, y_1)$ and $B(x_2, y_2)$, is therefore:

$$m = \tan\theta = \frac{LB}{AL} = \frac{NB-NL}{MN} = \frac{NB-AM}{ON-OM} = \frac{y_2-y_1}{x_2-x_1} \qquad \text{(i)}$$

Example 6 Find the slope m of the line L through the points

(a). E(2,4) and F(4,6) (b). M(3,1) and N(−1,3)

Remember

The standard equation of a line is $y = mx + c$ where m is a slope.

Solution

a. The given two points E(2,4) and F(4,6) form a line L, whose slope is: $m = \frac{y_2-y_1}{x_2-x_1} = \frac{6-4}{4-2} = \frac{2}{2} = 1$

b. The given two points M(3,1) and N(−1,3) is form a line L, whose slope is:

$$m = \frac{y_2-y_1}{x_2-x_1} = \frac{3-1}{-1-3} = \frac{2}{-4} = -\frac{1}{2}$$

7.2.3 Condition for two straight lines with given slopes are

(a). **Parallel to each other** *(b)*. **Perpendicular to each other**

a. Parallel to each other

If L_1 and L_2 are the two lines having slopes m_1 and m_2, then the lines L_1 and L_2 are **parallel** if they make the same angle with the *x-axis*, that means they have the same slope. Conversely, if two lines L_1 and L_2 have the same slope, then they will make the same angle with the *x-axis* and the lines L_1 and L_2 are therefore parallel for which: $m_1 = m_2$ (i)

It is important to note that the lines parallel to *x*-axis have zero slopes whereas the lines parallel to y-axis have the slope ∞.

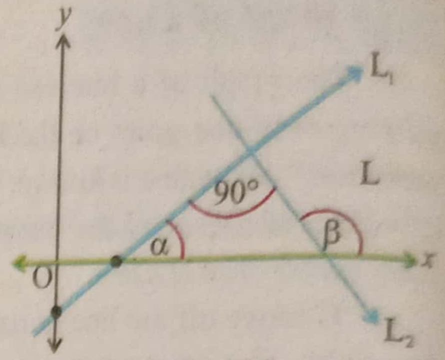


Figure 7.9

b. Perpendicular to each other

If L_1 and L_2 are the two perpendicular lines make the angles α and β with the *x*-axis, then the slopes of the lines L_1 and L_2 are respectively $m_1 = \tan\alpha$ and $m_2 = \tan\beta$. From the Figure 7.9, it is clear that

$$\frac{\pi}{2} = \beta - \alpha \Rightarrow \beta = \frac{\pi}{2} + \alpha$$

$$\tan\beta = \tan\left(\frac{\pi}{2} + \alpha\right), \textit{ take } \tan \textit{ of both sides}$$

$$= -\cot\alpha = -\frac{1}{\tan\alpha} \qquad \text{(ii)}$$

The given lines L_1 and L_2 are found perpendicular, since the product of their slopes equals -1:

$$m_1 m_2 = \tan\alpha\tan\beta = \tan\alpha\left(-\frac{1}{\tan\alpha}\right) = -1 \qquad \text{(iii)}$$

7.3 Equation of a Straight Line Parallel to Co-ordinate Axes

7.3.1 Equation of a straight line parallel to

- ***y*-axis and at distance '*a*' from it.**
- ***x*-axis and at a distance '*b*' from it.**

i. y-axis and at a distance '*a*' from it

Let PQ be a straight line parallel to *y-axis* at a distance '*a*' units from it see Figure 7.10. This is very clear, that all the points on the line PQ have the same ordinate say '*b*'. Therefore, PQ can be considered as the locus of a point at a distance '*a*' from y-axis and all points on the PQ satisfy the condition $x = a$ therefore, the equation of straight line is parallel to *y-axis* at a distance '*a*' from it. e.g $x = a$.

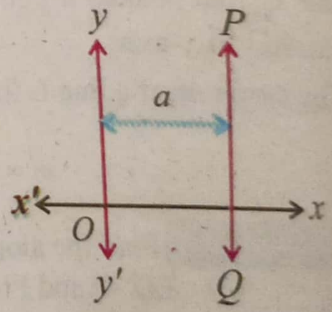


Figure 7.10

Remember

i. If a = 0, then the straight line considers with the *y*-axis and tis equation becomes $x = 0$.

ii. If *PQ* is parallel and to the left of y-axis at a distance '*a*', then its equation is $x = -b$.

Example 7 Find the equation of straight line parallel to y-axis at a distance 5 units on the right side of y-axis.

Solution Since, $x = a$ (i)

As, the distance is 5 units to right side of y-axis, so, equation (i) becomes $x = 5$

ii. x-axis and at a distance 'b' from it

Let PQ be a straight line parallel to x-axis at a distance 'b' units from it see Figure 7.11. This is every clear that all the points on the same ordinate say, 'b'. Therefore, PQ can be considered as the locus of a point at a distance 'b' from x-axis and all points on the PQ satisfy the condition $y = b$. Therefore, the equation of a straight line is parallel to x-axis at a distance b from it if e.g, $y = b$.

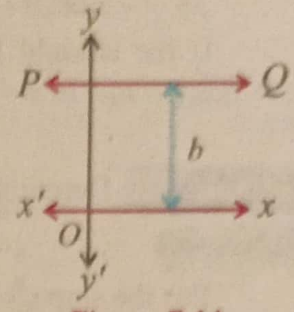


Figure 7.11

Remember

i. If $b = 0$, then the straight line considers with the x-axis and its equation becomes $y = 0$.

ii. If PQ is parallel and below the x-axis at a distance 'b', then its equation is $y = -b$.

7.4 Standard Form of Equation of a Straight Line

Because of their simplicity, linear equation (line) is used in many applications to describe relationships between two variables. We shall see some of these applications in this unit. First, we need to develop some standard forms that are related to linear equations.

(i) Intercepts of a straight line

"If a straight line AB intersects x–*axis* at C and y–*axis* at D, then OC is called the ***x*-intercept** of AB on the x–*axis* and OD is called the ***y*-intercept** of AB on the y–*axis*.

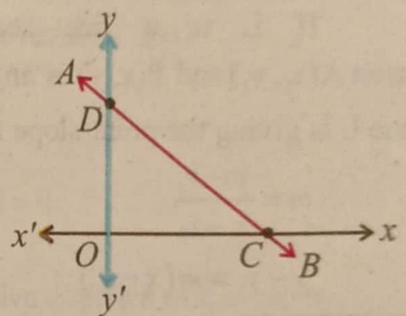


Figure 7.12

Example 8 Find the x and y intercepts of a line $2x+4y+6=0$.

Solution The x-intercept of a line is obtained by putting $y = 0$ in a line:

$$2x+4y+6=0$$

$$2x+4(0)=-6 \Rightarrow 2x=-6 \Rightarrow x=-3$$

The y-intercept of a line is obtained by putting $x = 0$ in a line:

$$2x+4y+6=0$$

$$2(0)+4y=-6 \Rightarrow 4y=-6 \Rightarrow y=-\frac{3}{2}$$

The general criteria are that a line in two dimensional space can be determined by specifying its slope and just one point.

(ii) Slope-Intercept Form

Let L be the line see Figure 7.13 develops the y-intercept c on the y-*axis*. The line L also makes an angle θ with the positive direction of the x-*axis* that develops a slope $m = \tan\theta$.

Let P(x,y) be any point on the line L. Draw PM parallel to y-*axis* and CN parallel to x-*axis* that give

$$CN = OM = x,$$

$$NP = MP - MN = MP - OC = y - c$$

In ΔPCN, the angle is $\angle PNC = 90^0$ and the slope of the line L is giving the slope-intercept form of the line L:

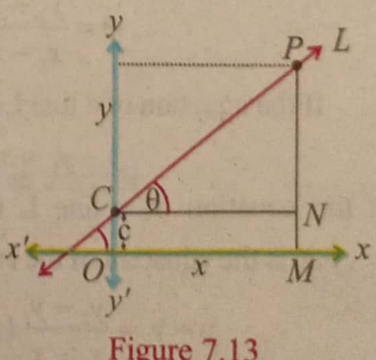


Figure 7.13

$$\frac{NP}{CN} = \tan\theta \Rightarrow \frac{y-c}{x} = \tan\theta$$

$$\Rightarrow y - c = x\tan\theta$$

$$\Rightarrow y = x\tan\theta + c = mx + c \qquad \text{(i)}$$

If the straight line L passes through the origin (0, 0), then $c = 0$ and the equation of line L becomes $y = mx$. In $y = mx + c$, m denotes the slope and c denotes the y-intercept of the line L on the axis of y.

Example 9 Determine the slopes of the following lines: (a). $x - y = 5$ (b). $2x + 3y = 6$

Solution

a. For the slope, solve the given line for y to obtain: $x - y = 5 \Rightarrow -y = -x + 5 \Rightarrow y = x - 5$
Thus, the slope of the line is the coefficient of x–term which is $m = 1$.

b. For the slope of the line, solve the given line for y to obtain:

$$2x + 3y = 6 \Rightarrow 3y = -2x + 6 \Rightarrow y = -\frac{2}{3}x + 2$$

Thus, the slope of the line is the coefficient of x–term which is $m = -\frac{2}{3}$.

Example 10 Find an equation of the line with slope 4, when the y–intercept is 6.

Solution Result (i) is used for the assumptions $m = 4$, $c = 6$ to obtain the required slope–intercept form of a line: $y = 4x + 6$

ii. Point–Slope Form

If L is a line see Figure 7.14 passing through the point $A(x_1, y_1)$ and $P(x, y)$ is any point on a line L, then the slope of the line L is giving the point-slope form of a line L:

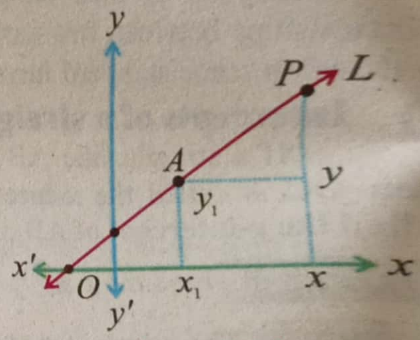


Figure 7.14

$$m = \frac{y - y_1}{x - x_1}$$

$$y - y_1 = m(x - x_1) \qquad \text{(ii)}$$

Example 11 Find an equation of a line with slope 4 and passes through the point (2,4).

Solution Result (ii) is used for the assumptions $m = 4$, $A(x_1, y_1) = A(2,4)$ to obtain the required point–slope form of a line: $y - 4 = 4(x - 2)$

$$-4x + y - 4 + 8 = 0 \Rightarrow -4x + y + 4 = 0 \Rightarrow 4x - y - 4 = 0$$

iii. Two–Point Form

If L is a line see Figure 7.15 passing through the two points $A(x_1, y_1)$ and $B(x_2, y_2)$, then the slope of the line L is:

$$m = \frac{y_2 - y_1}{x_2 - x_1} \qquad \text{(iii)}$$

If the equation of a line L through the $A(x_1, y_1)$ with slope m is

$$y - y_1 = m(x - x_1) \qquad \text{(iv)}$$

then the equation of a line L through the two points $A(x_1, y_1)$ and $B(x_2, y_2)$ is the equation of the **two-point form** of a line L:

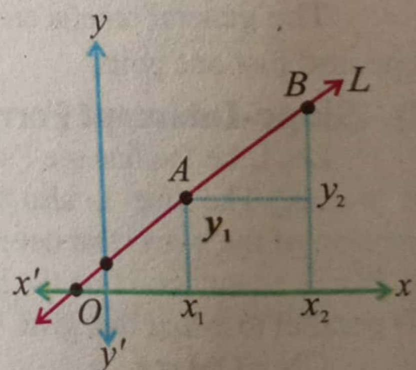


Figure 7.15

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1) \qquad \text{(v)}$$

Example 12 Find an equation of a line that passes through the two points $P(-1,-2)$ and $Q(-5,0)$.

Solution Result (v) is used for the assumptions $P(x_1,y_1)=P(-1,-2)$, $Q(x_2,y_2)=Q(-5,0)$ to obtain the required two-point form of a line:

$$y-(-2)=\frac{0-(-2)}{-5-(-1)}[x-(-1)] \Rightarrow y+2=\frac{2}{-4}(x+1)$$

$$\Rightarrow -4y-8=2x+2 \Rightarrow 2x+4y+10=0 \Rightarrow x+2y+5=0$$

iv. Double-Intercepts Form

If a line L intersects the *x-axis* and *y-axis* at points A and B, then $OA=a$ and $OB=b$ are the x and *y*-intercepts of the line L.

Let $P(x,y)$ be any point on the line L. Draw PM parallel to *y-axis* and PN parallel to *x-axis*. From the Figure 7.16, the comparison of similar triangles ΔBNP and ΔPMA is giving the equation of double-intercept form of a line L:

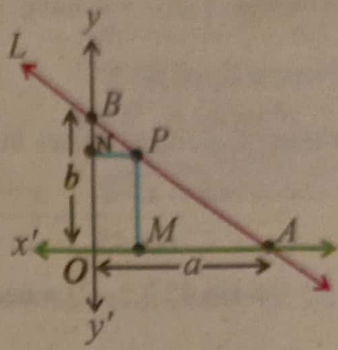


Figure 7.16

$$\frac{NB}{MP}=\frac{NP}{MA}$$

$$\frac{OB-ON}{ON}=\frac{OM}{OA-OM} \Rightarrow \frac{b-y}{y}=\frac{x}{a-x} \Rightarrow bx+ay=ab$$

$$\frac{bx}{ab}+\frac{ay}{ab}=1 \Rightarrow \frac{x}{a}+\frac{y}{b}=1 \qquad \text{(vi)}$$

Example 13 Find the equation of a line whose x and y intercepts are $(3,0)$ and $(0,4)$ respectively.

Solution Result (vi) is used for the assumptions $a=3$, $b=4$ to obtain the required line:

$$\frac{x}{3}+\frac{y}{4}=1$$

$$\frac{4x+3y}{12}=1 \Rightarrow 4x+3y=12 \Rightarrow 4x+3y-12=0$$

v. Symmetric Form

Let a line L through point $A(x_1,y_1)$ makes an angle θ with the positive direction of the *x-axis*.

If $P(x,y)$ is any point on the line L, then $AP=r$. If we allow r to vary with any positive or negative values, then P will take any position on the line L. Conversely, if P is given to be any point on the line L, then the unique value of r can be found which in fact is the distance of P from A. Thus, it follows that r serves as a parameter of point P.

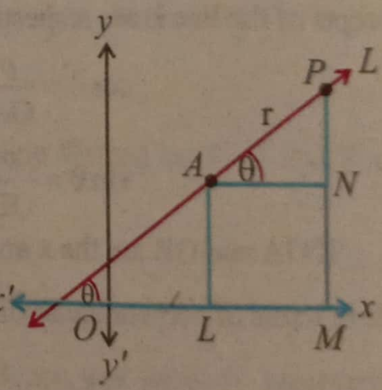


Figure 7.17

To find the coordinates of a point P in terms of the parameter r, let us draw AL and PM parallel to *y-axis* and AN parallel to *x-axis*, that with the following assumptions

$$OM=OL+LM=OL+AN$$

$$MP=MN+NP=LA+NP \qquad \text{(vii)}$$

dvelops the parametric equations of a line L through the point $A(x_1,y_1)$ at an angle θ:

$$OM=OL+LM \qquad MP=MN+NP$$

$$=OL+AN \qquad =LA+NP \qquad \text{(viii)}$$

$$x=x_1+r\cos\theta,\ \cos\theta=\frac{AN}{r} \qquad y=y_1+r\sin\theta,\ \sin\theta=\frac{NP}{r}$$

The parametric equations (viii) automatically give the symmetric form of a line L after simplification:

$$\left.\begin{aligned}\frac{x-x_1}{\cos\theta}=r\\ \frac{y-y_1}{\sin\theta}=r\end{aligned}\right\}\Rightarrow \frac{x-x_1}{\cos\theta}=\frac{y-y_1}{\sin\theta}=r \qquad \text{(ix)}$$

Example 14 Find the equation of a straight line with inclination 45° and passing through the point $(2,\sqrt{2})$.

Solution Here we have inclination $\alpha=45°$ and point $(x_1,y_1)=(2,\sqrt{2})$. The equation of line in its symmetric form is: $\frac{x-x_1}{\cos\alpha}=\frac{y-y_1}{\sin\alpha}$

Substitute the above values in the formula to get the equation of a straight line.

$$\frac{x-2}{\cos 45°}=\frac{y-\sqrt{2}}{\sin 45°}$$

$$\Rightarrow \sin 45°(x-2)=\cos 45°\left(y-\sqrt{2}\right)$$

$$\Rightarrow \frac{1}{\sqrt{2}}(x-2)=\frac{1}{\sqrt{2}}\left(y-\sqrt{2}\right)$$

$$\Rightarrow x-y-2+\sqrt{2}=0$$

vi. Normal Form

The normal form of a line is the equation of a line in terms of the length of the perpendicular on it from the origin and that perpendicular makes an angle with the *x-axis*.

If a line L intersects the *x*-axis and *y*-axis at points A and B, then OA and OB are the *x* and *y*-intercepts of the line L. Draw ON perpendicular to line L that provides the perpendicular distance *p* from the origin on the line L which is denoted by ON = *p*. If ON makes an angle θ with the positive direction of the *x-axis*, then the *x* and *y*-intercepts of the line L are respectively:

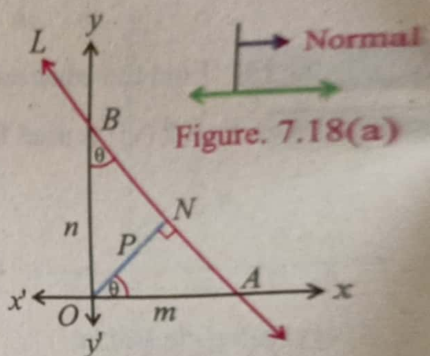


Figure. 7.18(a)

Figure 7.18(b)

$$\left.\begin{aligned}\cos\theta=\frac{p}{\text{OA}}\Rightarrow \text{OA}=p\sec\theta\\ \sin\theta=\frac{p}{\text{OB}}\Rightarrow \text{OB}=p\,\text{cosec}\theta\end{aligned}\right\} \qquad \text{(x)}$$

If OA and OB are the *x* and *y*-intercepts of a line L, then through result (x), the equation of a normal line L in terms of perpendicular distance *p* and angle θ is: $\frac{x}{\text{OA}}+\frac{y}{\text{OB}}=1$

$$\frac{x}{p\sec\theta}+\frac{y}{p\,\text{cosec}\,\theta}=1 \Rightarrow \quad x\cos\theta+y\sin\theta=p \qquad \text{(xi)}$$

The normal form of a line is also referred to **perpendicular form** of a line.

Example 15 Find the corresponding equation of a line, if the length of the perpendicular distance from the origin on a line is 3 units that makes an angle of 120^0.

Solution Result (xi) is used for the assumptions $p=3,\ \theta=120^0$ to obtain the required equation of a line:

$$x\cos 120^0+y\sin 120^0=3 \Rightarrow \frac{-1}{2}x+\frac{\sqrt{3}}{2}y=3\Rightarrow -x+\sqrt{3}y=6\Rightarrow x-\sqrt{3}y+6=0$$

7.4.2 A linear equation in two variables is a straight line

A first degree polynomial $p_1(x)=a_1x+a_0$ is rearranged to obtain an equation of the form

$$-a_1x+p_1(x)-a_0=0 \quad with\ -a_1=a, p_1(x)=y, -a_0=b$$
$$ax+by+c=0$$

is then called the **general equation of the straight line**. Here a, b and c are constants while x and y are variables.

Remember
The first degree polynomial $p_1(x)$ is also called the linear algebraic equation and is denoted by $p_1(x)=f(x)$.

Consider, If $P(x_1,y_1)$, $Q(x_2,y_2)$ and $R(x_3,y_3)$ are the three points on the locus represented by the straight line

$$ax+by+c=0 \qquad \text{(i)}$$

then, $P(x_1,y_1)$ on the line (i) gives: $\quad ax_1+by_1+c=0 \quad$ (ii)

$Q(x_2,y_2)$ on the line (i) gives: $\quad ax_2+by_2+c=0 \quad$ (iii)

$R(x_3,y_3)$ on the line (i) gives: $\quad ax_3+by_3+c=0 \quad$ (iv)

The three lines from equation (ii) to equation (iv) develops a homogeneous system of three linear equations in three unknowns a, b and c:

$$\begin{pmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow Ax=0 \qquad \text{(v)}$$

The homogeneous system of linear equations (v) defines a nontrivial solution only if the determinant of a coefficient matrix A of the system (v) is zero:

$$\det(A)=0$$

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}=0 \qquad \text{(vi)}$$

$x_1(y_2-y_3)-y_1(x_2-x_3)+(x_2y_3-x_3y_2)=0$ Equation (vi) is rearranged to obtain:

$$x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2)=0 \qquad \text{(vii)}$$

Multiply both sides of equation (vii) by $\frac{1}{2}$ to obtain the area of the triangle formed by P, Q and R that equals zero: $\quad \frac{1}{2}[x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2)]=0$

$$\textit{Area of the triangle}\ PQR=0 \qquad \text{(viii)}$$

Since the three points P, Q and R lying on the locus (i) are collinear. Hence, the locus (i) represents a straight line.

7.4.3 General form of a straight line is reducible to other standard forms

Any standard form of a line can also be determined from the general form of a line (i).

i. To reduce the general form (i) to the slope intercept form of a line, we need to involve the following steps:

$$ax+by+c=0$$
$$by=-ax-c$$

$$y = -\frac{a}{b}x - \frac{c}{b}$$

$$= mx + c_1, \ slope = m = -\frac{a}{b}, \ y\text{-intercept} = c_1 = -\frac{c}{b}$$

ii. To reduce the general form (i) to the double-intercept form, we need to involve the following steps:

$$ax + by + c = 0$$

$$ax + by = -c$$

$$\frac{ax}{-c} + \frac{by}{-c} = 1, \ \text{dividing by} -c$$

$$\frac{x}{-\frac{c}{a}} + \frac{y}{-\frac{c}{b}} = 1$$

$$\frac{x}{a_1} + \frac{y}{b_1} = 1, \quad x\text{-intercept} = a_1 = -\frac{c}{a}, \ y\text{-intercept} = b_1 = -\frac{c}{b}$$

iii. To reduce the general form (i) to the normal form, we need to involve the following steps:
From the Figure 7.18(b), the angles along the positive directions of the x and y-axis are the following:

$$\cos\theta = \frac{p}{m}, \ \sin\theta = \frac{p}{n}$$

The values of $\cos\theta$ and $\sin\theta$ are used in the trigonometric identity $\cos^2\theta + \sin^2\theta = 1$ to obtain p:

$$\cos^2\theta + \sin^2\theta = 1$$

$$\frac{p^2}{m^2} + \frac{p^2}{n^2} = 1$$

$$p^2\left(\frac{1}{m^2} + \frac{1}{n^2}\right) = 1 \qquad \text{(ix)}$$

$$p^2\left(\frac{m^2 + n^2}{m^2n^2}\right) = 1$$

$$p^2 = \frac{m^2n^2}{m^2 + n^2} \Rightarrow p = \pm\frac{mn}{\sqrt{m^2 + n^2}}$$

This p is the perpendicular distance from the origin to the line $\frac{x}{m} + \frac{y}{n} = 1$ (i.e. $nx + my - mn = 0$). Of course, the perpendicular distance from the origin to the line $ax + by + c = 0$ must be:

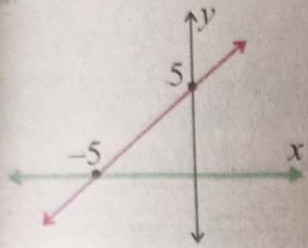
$$p = \frac{|c|}{\sqrt{a^2 + b^2}} \qquad \text{(x)}$$

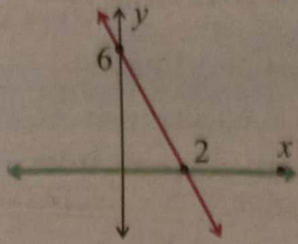
For converting the general form (i) to normal form, divide the line $ax + by + c = 0$ by $\frac{1}{\sqrt{a^2 + b^2}}$ to obtain the conversion of the general form (i) in the normal form:

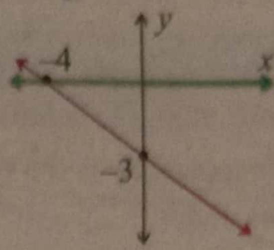
$$\frac{ax}{\sqrt{a^2 + b^2}} + \frac{by}{\sqrt{a^2 + b^2}} + \frac{c}{\sqrt{a^2 + b^2}} = 0 \qquad \text{(xi)}$$

Exercise 7.2

1. Find the equation of lines that are represented on the coordinate planes:

a.


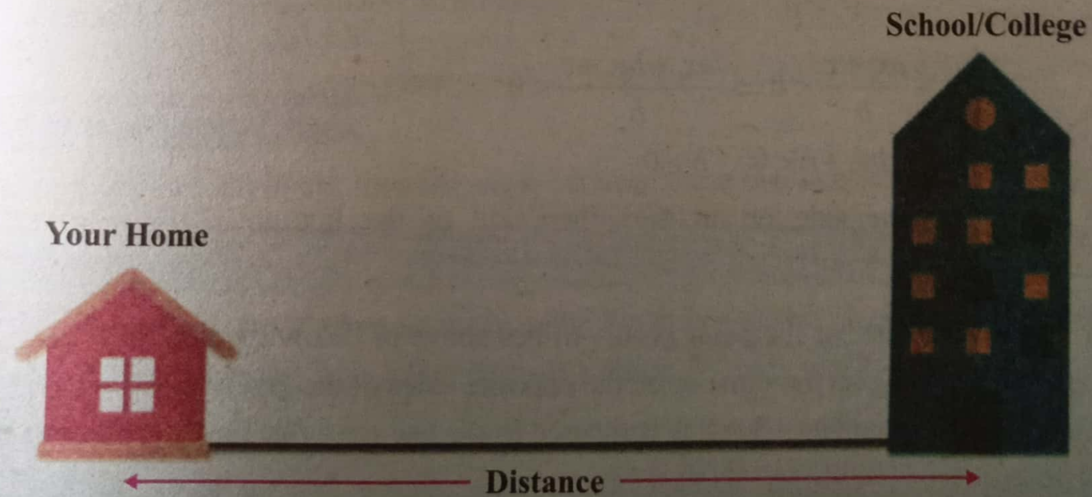
b.


c.


2. In each case, find the slope, if it is defined:
   a. $(-5, 4)$ and $(3,6)$ b. $4x+7y=1$
   c. Parallel to $2y-4x=7$ d. Perpendicular to $6x=y-3$
3. What are the x– and y–intercepts for each of the following lines?
   a. $y=2x+6$ b. $y=-3x+9$ c. $y=x+2$
4. In each case, show that the pair of lines are parallel or perpendicular or neither:
   a. $x-2y-6=0,\ 2x+y-5=0$ b. $3x+4y-8=0,\ x+3y-2=0$
   c. $2x-y-7=0,\ 4x-2y-5=0$ d. $2x-5y-7=0,\ 6x-15y+5=0$
5. In each case, find the equation of a line that passes through the pair of points:
   a. O(0,0) and A(2,6) b. E(1,0) and F(2,5) c. I(1,1) and J(3,3)
6. In each case, find the equation of a line that passes through the point $A(x_1, y_1)$ having slope m:
   a. $A(1,2),\ m=4$ b. $A(-1,-2),\ m=-\dfrac{1}{2}$ c. $A(-3,5),\ m=-3$ d. $A(7,-8),\ m=5$
7. In each case, find the equation of a line that exists the y–intercept c and slope m:
   a. $c=2,\ m=2$ b. $c=4,\ m=8$ c. $c=-4,\ m=\dfrac{1}{2}$
8. Transform the equation $7x-10y+13=0$ into:
   a. slop intercept from b. symmetric form c. normal form

Project

How can you calculate the midpoint between your home and school/college? Calculate this distance and write the procedure.



7.5 Distance of a point to a line

In Euclidean geometry, the distance from a point to a line is the shortest distance from a given point to any point on an infinite straight line. It is the perpendicular distance of the point to the line, the length of the line segment which joins the point to nearest point on the line.

7.5.1 Position of a point with respect to a line

To show that the point $P(x_1, y_1)$ is on one side or on the other side of the straight line $ax + by + c = 0$ according as the expression $ax_1 + by_1 + c < 0$ or $ax_1 + by_1 + c > 0$, the procedure developed is as under:

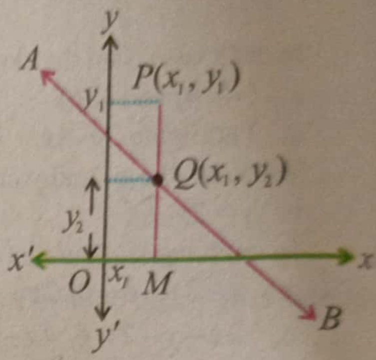


Figure 7.18

Let AB be the straight line $ax + by + c = 0$ and $P(x_1, y_1)$ is a point above the line AB Figure 7.18 and $P(x_1, y_1)$ is also a point below the line AB Figure 7.19. From P draw perpendicular PM on the x-axis that cuts the line AB at a point Q whose coordinates are $Q(x_1, y_2)$.

If $Q(x_1, y_2)$ lies on the line AB, then it give:

$$ax + by + c = 0$$

$$ax_1 + by_2 + c = 0, \quad Q(x_1, y_2) \text{ lies on AB}$$

$$by_2 = -(ax_1 + c)$$

$$y_2 = -\left(\frac{ax_1 + c}{b}\right) \quad \text{(i)}$$

If P lies above AB as in Figure 7.18, then:

$$MP - MQ > 0$$

$$y_1 - y_2 > 0$$

$$y_1 + \frac{ax_1 + c}{b} > 0 \Rightarrow \frac{ax_1 + by_1 + c}{b} > 0$$

$$ax_1 + by_1 + c > 0, \quad b > 0$$

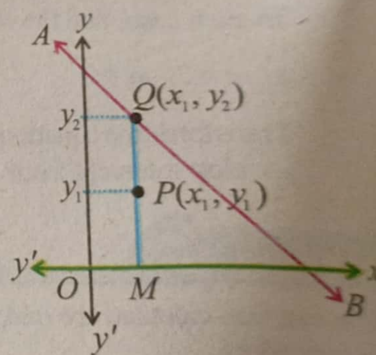


Figure 7.19

If P lies below AB as in Figure 7.19, then:

$$MP - MQ < 0$$

$$y_1 - y_2 < 0$$

$$y_1 + \frac{ax_1 + c}{b} < 0 \Rightarrow \frac{ax_1 + by_1 + c}{b} < 0$$

$$ax_1 + by_1 + c < 0, \quad b > 0$$

Hence P lies on one side or on the other side of the line $ax + by + c = 0$ according as $ax_1 + by_1 + c > 0$ or $ax_1 + by_1 + c < 0$.

Example 16 Determine whether the point P(10, –6) lies above or below the line $9x + 10y - 3 = 0$. Show that the point and the origin lie on the same or on the opposite sides of the given line.

Solution The given line $9x + 10y - 3 = 0$ is compared to the line $ax + by + c = 0$ to obtain the coefficient of y is $b = 10 > 0$:

i. The given point P(10, –6) is substituted in the given line to obtain:

$$9(10) + 10(-6) - 3 = 90 - 63 = 27 > 0$$

Thus, the point P(10, –6) lies above the given line $9x+10y-3=0$

ii. The given point P(10, –6) and the origin O(0,0) are substituted in the given line to obtain:

$$\left.\begin{aligned} 9(10)+10(-6)-3=90-63=27>0 \\ 9(0)+10(0)-3=-3<0 \end{aligned}\right\} \text{opposite in signs}$$

Hence, the point P(10, –6) and the origin O(0,0) lie on the opposite side of the given line $9x+10y-3=0$.

7.5.2 Perpendicular distance from a point to the given straight line

If $Q(x_1, y_1)$ is any point on a line

$$ax+by+c=0 \qquad \text{(i)}$$

and $n=(a,b)$ is a nonzero vector perpendicular to the line (i) at a point $Q(x_1, y_1)$, then the distance D is the scalar projection of a vector QP (associated to any point ($P(x_1, y_1)$)) onto n:

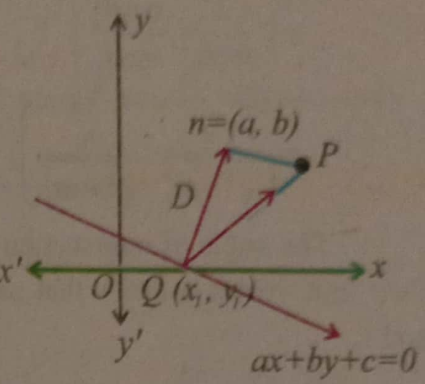


Figure 7.20

$$D = |proj_n \, QP|$$

$$= \frac{|QP.n|}{|n|}$$

$$d = \frac{|(x_0-x_1, y_0-y_1).(a,b)|}{\sqrt{a^2+b^2}} \qquad n=(a,b), |n|=\sqrt{a^2+b^2}$$

$$d = \frac{|a(x_0-x_1)+b(y_0-y_1)|}{\sqrt{a^2+b^2}}$$

$$d = \frac{|ax_0+by_0-ax_1-by_1|}{\sqrt{a^2+b^2}} = \frac{|ax_0+by_0+c)|}{\sqrt{a^2+b^2}} \qquad \text{(ii)}$$

the perpendicular distance from a line $ax+by+c=0$ to a point $P(x_1, y_1)$.

Example 17 Find the perpendicular distance from a line $7x+3y-9=0$ to a point P(2,3).

Solution Result (ii) is used for the assumptions $P(x_1, y_1)= P(2,3)$, $c=-9$, $a=7$, $b=3$ to obtain the perpendicular distance d from the line $7x+3y-9=0$ to the point $P(2,3)$:

$$d = \left|\frac{ax_1+by_1+c}{\sqrt{a^2+b^2}}\right| = \frac{7(2)+3(3)-9}{\sqrt{7^2+3^2}} = \frac{14}{\sqrt{58}}$$

7.6 Angle between Lines

If the two lines are available, then the angle between these two lines can found as follows:

7.6.1 The angle between two coplanar intersecting straight lines

The unit vectors are the vectors lie in the same directions of the given lines. The unit vectors along the line AB and CD are respectively $u=(\cos\theta_1, \sin\theta_1)$ and $v=(\cos\theta_2, \sin\theta_2)$.

The unit vector u of a line AB is:

$$u=(\cos\theta_1, \sin\theta_1) \qquad \cos\theta_1 = \frac{1}{\sec\theta_1}$$

$$=\left(\frac{1}{\sec\theta_1},\frac{\tan\theta_1}{\sec\theta_1}\right), \qquad |u|=1$$

$$=\left(\frac{1}{\sqrt{1+m_1^2}},\frac{m_1}{\sqrt{1+m_1^2}}\right), \qquad \sec^2\theta_1=1+\tan^2\theta_1=1+m_1^2$$

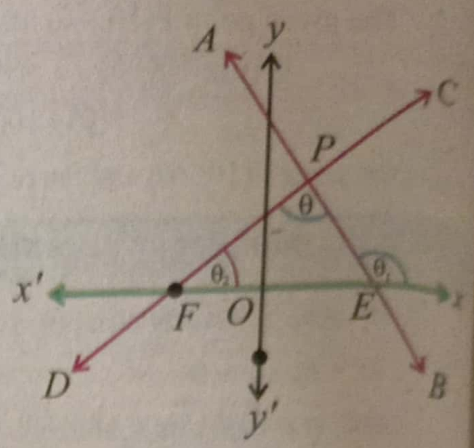


Figure 7.21

The unit vector v of a line CD is:

$$v=(\cos\theta_2,\sin\theta_2),|v|=1$$

$$=\left(\frac{1}{\sec\theta_2},\frac{\tan\theta_2}{\sec\theta_2}\right)$$

$$=\left(\frac{1}{\sqrt{1+m_2^2}},\frac{m_1}{\sqrt{1+m_2^2}}\right), \qquad \sec^2\theta_2=1+\tan^2\theta_2=1+m_2^2$$

The angle of intersection between the lines AB and CD is the angle of intersection in between their unit vector u and v that can be found by taking the dot product in between the unit vectors u and v:

$$\cos\theta=\frac{u.v}{|u||v|}, \quad |u|=|v|=1$$

$$=u.v$$

$$=\left(\frac{1}{\sqrt{1+m_1^2}},\frac{m_1}{\sqrt{1+m_1^2}}\right)\cdot\left(\frac{1}{\sqrt{1+m_2^2}},\frac{m_2}{\sqrt{1+m_2^2}}\right)$$

$$=\frac{1}{\sqrt{1+m_1^2}\sqrt{1+m_2^2}}+\frac{m_1m_2}{\sqrt{1+m_1^2}\sqrt{1+m_2^2}}$$

$$=\frac{1+m_1m_2}{\sqrt{1+m_1^2}\sqrt{1+m_2^2}}$$

The standard form of the angle is obtained if

$$\tan^2\theta=\sec^2\theta-1$$

$$=\frac{1}{\cos^2\theta}-1, \text{ use value of } \cos\theta$$

$$=\frac{(1+m_1^2)(1+m_2^2)}{(1+m_1m_2)^2}-1$$

$$=\frac{(1+m_1^2)(1+m_2^2)-(1+m_1m_2)^2}{(1+m_1m_2)^2}=\frac{(m_1-m_2)^2}{(1+m_1m_2)^2}$$

$$\tan\theta=\pm\sqrt{\frac{(m_1-m_2)^2}{(1+m_1m_2)^2}}$$

$$=\pm\frac{m_1-m_2}{1+m_1m_2} \qquad \text{(i)}$$

Remember

- If $\frac{m_1 - m_2}{1+m_1 m_2}$ is positive, then result (i) gives the acute angle between the lines AB and CD.
- If $\frac{m_1 - m_2}{1+m_1 m_2}$ is negative, then result (i) gives the obtuse angle between the lines AB and CD.
- If one of the given lines is parallel to the y-axis, then the angle θ is not possible to obtain by formula:

  $\tan\theta = \pm\frac{m_1 - m_2}{1+m_1 m_2}$

  Because 90° is the angle made by that line with the positive x-axis and $\tan 90^\circ = \infty$. In such a case, the angle between the lines will be calculated by drawing the figure.
- The lines are parallel, if the cross product in between the unit vector u and v is zero:

$$u \times v = 0$$

$$\left(\frac{1}{\sqrt{1+m_1^2}}, \frac{m_1}{\sqrt{1+m_1^2}}\right) \times \left(\frac{1}{\sqrt{1+m_2^2}}, \frac{m_2}{\sqrt{1+m_2^2}}\right) = 0$$

$$\frac{m_1 - m_2}{\sqrt{(1+m_1^2)(1+m_2^2)}} k = 0$$

$$\Rightarrow m_1 - m_2 = 0 \Rightarrow m_1 = m_2$$

where k is normal to the plane of the lines.
- The lines are perpendicular, if the dot product in between the unit vector u and v is zero:

$$u . v = 0$$

$$\left(\frac{1}{\sqrt{1+m_1^2}}, \frac{m_1}{\sqrt{1+m_1^2}}\right) . \left(\frac{1}{\sqrt{1+m_2^2}}, \frac{m_2}{\sqrt{1+m_2^2}}\right) = 0$$

$$\frac{1+m_1 m_2}{\sqrt{(1+m_1^2)(1+m_2^2)}} = 0$$

$$\Rightarrow 1 + m_1 m_2 = 0 \Rightarrow m_1 m_2 = -1$$

Example 18 Find the angle from the line $7x + 3y - 9 = 0$ to the line $5x - 2y + 2 = 0$.

Solution The slope of a line $7x + 3y - 9 = 0$ is $m_1 = -\frac{7}{3}$

The slope of a line $5x - 2y + 2 = 0$ is $m_2 = \frac{5}{2}$

If θ is the angle from first line to line second, then

$$\tan\theta = \frac{m_1 - m_2}{1+m_1 m_2} = \frac{-\frac{7}{3} - \frac{5}{2}}{1 + \frac{-7}{3}\cdot\frac{5}{2}} = \frac{-29}{6\left(1 - \frac{35}{6}\right)} = \frac{-29}{-29} = 1 \Rightarrow \theta = \tan^{-1}(1) = 45^\circ$$

The angle $\theta = 45^\circ$ is acute.

7.6.2 The equation of family of lines passing through the point of intersection of two given lines

Suppose, the two lines are

$L_1 : a_1x+b_1y+c_1=0$ (i)

$L_2 : a_2x+b_2y+c_2=0$ (ii)

and $P(x_1, y_1)$ is their point of intersection. The given lines L_1 and L_2 are used to obtain a first degree equation of a straight line in x and y: $(a_1x+b_1y+c_1)+\lambda(a_2x+b_2y+c_2)=0$, λ is constant (iii)

The coordinates of a point P will reduce each line in (iii) to zero, since, by hypothesis, P is the point of intersection, i.e., it lies on each line. Therefore P satisfies (iii) and represents the family of lines through the point of intersection of $L_1=0$ and $L_2=0$.

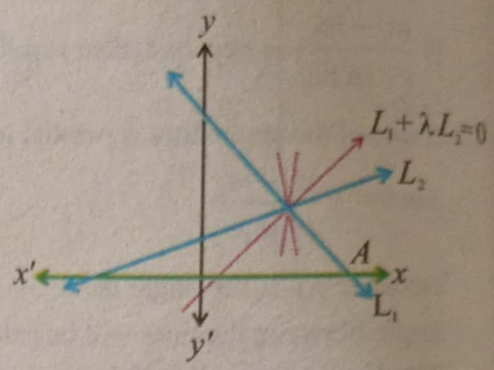


Figure 7.22

Example 19 Develop the family of lines through the point of intersection of the lines $2x-3y+4=0$ and $2x+y-1=0$. Find the line from the family of lines which is

(a). parallel to the line whose slope is $m_1=-\frac{2}{3}$. (b). perpendicular to the line $4x+3y-1=0$.

Solution Result (iii) is used for the lines $2x-3y+4=0$, $2x+y-1=0$ to obtain the family of lines:

$$\left.\begin{aligned}(2x-3y+4)+\lambda(2x+y-1)=0\\(2+2\lambda)x+(-3+\lambda)y+(4-\lambda)=0\end{aligned}\right\} \quad \text{(iv)}$$

The slope of the family of lines is:

$$(2+2\lambda)x+(-3+\lambda)y+(4-\lambda)=0 \Rightarrow (-3+\lambda)y=-(2+2\lambda)x-(4-\lambda)$$

$$y=\frac{-(2+2\lambda)}{-3+\lambda}x-\frac{4-\lambda}{-3+\lambda},\ m_2=\frac{-(2+2\lambda)}{-3+\lambda}$$

a. The family of lines (iv) is parallel to the line with slope $m_1=-\frac{2}{3}$ if and only if their slopes are equal:

$$\frac{-(2+2\lambda)}{-3+\lambda}=-\frac{2}{3} \Rightarrow 6+6\lambda=-6+2\lambda \Rightarrow 4\lambda=-12 \Rightarrow \lambda=-3$$

The value of $\lambda=-3$ is used in (iv) to obtain the particular line from the family of lines (iv):

$$(2x-3y+4)-3(2x+y-1)=0 \qquad \Rightarrow 2x-3y+4-6x-3y+3=0$$

$$\Rightarrow -4x-6y+7=0 \qquad \Rightarrow 4x+6y-7=0$$

b. The slope of the given line $4x+3y-1=0$ is $m_3=-\frac{4}{3}$. The family of lines (iv) is perpendicular to the line $4x+3y-1=0$, if and only if the product of their slopes equals -1:

$$\left[\frac{-(2+2\lambda)}{-3+\lambda}\right]\left(-\frac{4}{3}\right)=-1 \Rightarrow \frac{8+8\lambda}{-9+3\lambda}=-1 \Rightarrow 8+8\lambda=9-3\lambda \Rightarrow 11\lambda=1 \Rightarrow \lambda=\frac{1}{11}$$

The value of $\lambda=\frac{1}{11}$ is used in (iv) to obtain the particular line from the family of lines:

$$(2x-3y+4)+\frac{1}{11}(2x+y-1)=0 \Rightarrow 22x-33y+44+2x+y-1=0 \Rightarrow 24x-32y+43=0$$

7.6.3 The angels of the triangle when the slopes of the sides are given

If $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ are the vertices of a triangle ABC and the slopes of the sides AB, BC and CA of the triangle ABC are respectively:

$$m_1 = \frac{y_2 - y_1}{x_2 - x_1}, \textit{ slope of side AB},$$

$$m_2 = \frac{y_3 - y_2}{x_3 - x_2}, \textit{ slope of side BC}$$

$$m_3 = \frac{y_1 - y_3}{x_1 - x_3}, \textit{ slope of side CA}$$

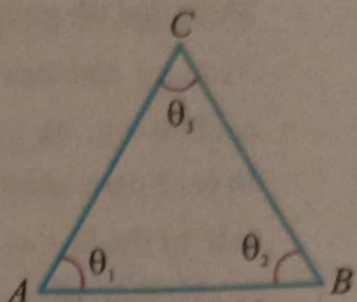


Figure 7.23

If θ_1, θ_2 and θ_3 are the angles in between their sides AB to AC, BC to BA and CB to CA respectively, then the angles can be found through results $\tan\theta = \pm\dfrac{m_1 - m_2}{1 + m_1 m_2}$:

The angles from the sides AB to AC, BC to BA, and CA to CB of a triangle ABC are respectively:

$$\tan\theta_1 = \frac{m_1 - m_3}{1 + m_1 m_3}, \qquad \tan\theta_2 = \frac{m_2 - m_1}{1 + m_1 m_2}, \qquad \tan\theta_3 = \frac{m_3 - m_2}{1 + m_2 m_3} \qquad \text{(v)}$$

Example 20 Find the angles of the triangle ABC, whose vertices are A(–2,–3), B(4,–1) and C(2,3).

Solution If *ABC* is a triangle and the slopes of their sides AB, BC and CA are respectively:

$$m_1 = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-1+3}{4+2} = \frac{2}{6} = \frac{1}{3}, \text{ slope of side AB}$$

$$m_2 = \frac{y_3 - y_2}{x_3 - x_2} = \frac{3+1}{2-4} = -\frac{4}{2} = -2, \text{ slope of side BC}$$

$$m_3 = \frac{y_1 - y_3}{x_1 - x_3} = \frac{-3-3}{-2-2} = \frac{-6}{-4} = \frac{3}{2}, \text{ slope of side CA}$$

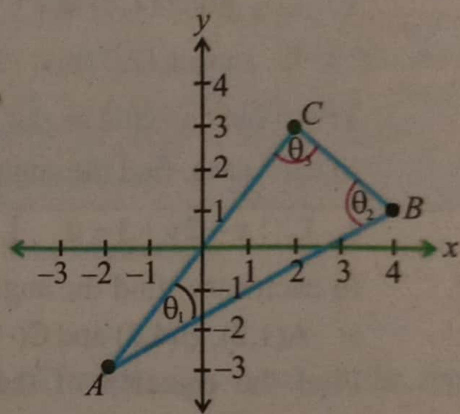


Figure 7.24

Result (v) to obtain the angle θ_1 from the sides AB to AC use:

$$\tan\theta_1 = \frac{m_1 - m_3}{1 + m_1 m_3} = \frac{\frac{1}{3} - \frac{3}{2}}{1 + \left(\frac{1}{3}\right)\left(\frac{3}{2}\right)} = \frac{2-9}{6\left(1 + \frac{1}{2}\right)} = -\frac{7}{6\left(\frac{3}{2}\right)} = -\frac{7}{9} \Rightarrow \theta_1 = \tan^{-1}\left(\frac{-7}{9}\right)$$

Result (v) to obtain the angle θ_2 from the sides BC to BA use:

$$\tan\theta_2 = \frac{m_2 - m_1}{1 + m_1 m_2} = \frac{-2 - \frac{1}{3}}{1 + (-2)\left(\frac{1}{3}\right)} = \frac{-7}{3\left(\frac{1}{3}\right)} = -7 \Rightarrow \theta_2 = \tan^{-1}(-7)$$

Result (v) to obtain the angle θ_3 from the sides CA to CB use: $\tan\theta_3 = \dfrac{m_3 - m_2}{1 + m_2 m_3}$

$$= \frac{\frac{3}{2} + 2}{1 + (-2)\left(\frac{3}{2}\right)} = \frac{7}{2(1-3)} = \frac{7}{-4} = \frac{7}{-4} \Rightarrow \theta_3 = \tan^{-1}\left(\frac{-7}{4}\right)$$

Exercise 7.3

1. Show that the point $P(x_1, y_1)$ lies above or below the line $ax + by + c = 0$. Also show that the point $P(x_1, y_1)$ and the origin lie on the same side or on the opposite side of the line $ax + by + c = 0$:

   a. $P(4,-5)$, $4x - 3y - 17 = 0$ b. $P(-3,8)$, $5x + 7y + 9 = 0$

2. In each case, show that the points $P(x_1, y_1)$ and $Q(x_2, y_2)$ are on the same side or on the opposite side of the line $ax + by + c = 0$:

   a. $P(-4,2)$, $Q(11,-3)$; $5x + 14y - 11 = 0$ b. $P(-3,5)$, $Q(1,-2)$; $2x - 3y - 10 = 0$

3. In each case, find the perpendicular distance from the line $ax + by + c = 0$ to a point $P(x_1, y_1)$:

   a. $P(3,-4)$, $4x - 3y + 6 = 0$ b. $P(5,8)$, $3x - 2y + 7 = 0$ c. $P(3,-1)$, $5x + 12y - 16 = 0$

4. In each case, find the angle θ from the line L_1 to line L_2, if the slopes of the lines L_1 and L_2 are the following:

   a. $L_1 : m_1 = \frac{1}{2}$, $L_2 : m_2 = 3$ b. $L_1 : m_1 = 2$, $L_2 : m_2 = 3$

5. In each case, find the angle θ from the line L_1 to the line L_2:

   a. L_1 : joins (1,2) and (7,–1), L_2 : joins (3,2) and (5,6).

   b. L_1 : joins (2,7) and (7,10), L_2 : joins (1,1) and (–5,3).

   Try to obtain acute angles.

6. In each case, find the angle θ from the line L_1 to the line L_2:

   a. $L_1 : x - 2y + 3 = 0$, $L_2 : 3x - y + 7 = 0$ b. $L_1 : 2x + 4y - 10 = 0$, $L_2 : 5x - 3y + 1 = 0$

7. In each case, find the angles of the triangle ABC whose vertices are the following:

   a. A(1,2), B(4,2) and C(–2,3) b. A(3,–4), B(1,5) and C(2,–4).

8. Find the equation of the straight line from the family of straight lines through the point of intersection of the lines

   a. $2x - 3y + 4 = 0$, $3x + 4y - 5 = 0$ and is perpendicular to the line $6x - 7y - 18 = 0$.

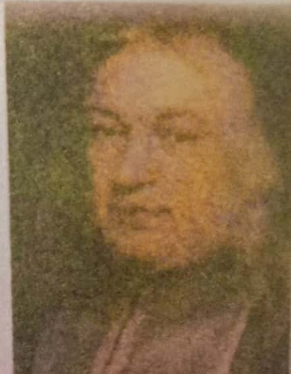
   b. $3x - 4y + 1 = 0$, $5x + y - 1 = 0$ and cuts off equal intercepts from the axes.

   c. $x - 2y = a$, $x + 3y = 2a$ and is parallel to the line $3x + 4y = 0$.

   d. $2x - y = 0$, $3x + 2y = 0$ and is perpendicular to the line $3x + y - 6 = 0$.

History

Pierre de Fermat was French lawyer and a mathematician. He was credited for the early development of calculus. In particular he was recognized because of his discovery of an original method of finding the greatest and smallest ordinates of curved lines which are very important for the differential calculus. He also made some contribution to number theory, but a magnificent contribution to analytical geometry, optics and probability. He became famous in the community of mathematics because of Fermat's principle for light propagation and his Fermat's last theorem in the field of number theory. Fermat's work in analytical geometry was circulated in manuscript form in 1636. He also developed a method for determining maxima, minima and tangents to various curves that was equivalent to differentiate calculus.



Pierre de Fermat
(1607-1665)

7.7 Concurrency of Straight Lines

Before to touch the concurrency of straight lines, we need to develop the concept of intersection of lines. Logically, the solution of the system of lines exists only, if the lines intersect.
For illustration, the two lines $x+y=1$ and $x-y=0$ is forming the system of two linear equations

$$x+y=1$$
$$x-y=0$$

that in matrix form is represented by $Ax=b$:

$$\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}1\\ 0\end{pmatrix}$$

The augmented matrix of the system $Ax=b$ is

$$A/b=\begin{pmatrix}1 & 1 & 1\\ 1 & -1 & 0\end{pmatrix}$$

reduced in an echelon form through row operations

$$A/b=\begin{pmatrix}1 & 1 & 1\\ 1 & -1 & 0\end{pmatrix}\sim\begin{pmatrix}1 & 1 & 1\\ 0 & -2 & -1\end{pmatrix}\text{ by } R_2+R_1(-1)$$

that gives the system of equations:

$$\left.\begin{matrix}x+y=1\\ -2y=-1\end{matrix}\right\}, y=\frac{1}{2}, x=\frac{1}{2}$$

The second equation is giving $y=\frac{1}{2}$ which is used in first equation to obtain $x=\frac{1}{2}$. The solution set $(x,y)=\left(\frac{1}{2},\frac{1}{2}\right)$ of the system of two linear equations is unique (one solution set). This unique solution is the unique point of intersection at which the given two lines intersect.

Remember

It is important to note that the system of two lines

- $x+y=1$ and $x-y=0$ is giving a **unique** solution set, since the lines are intersecting at just a single point.
- $x+y=1$ and $x+y=0$ is not giving a solution set, since the lines are not intersecting, because the line are **parallel**.
- $x+y=1$ and $2x+2y=2$ is giving an **infinite** set of solutions, since the lines are intersecting more than one points, because the lines make a sense of coincident lines.

7.7.1 Condition of concurrency of three straight lines

The condition of concurrency of three straight lines is the point of intersection at which the three straight lines intersect. For illustration, if the given three lines are

$$\left.\begin{matrix}a_1x+b_1y+c_1=0\\ a_2x+b_2y+c_2=0\\ a_3x+b_3y+c_3=0\end{matrix}\right\}\qquad \text{(i)}$$

then, the three lines develop a homogeneous system of three linear equations

$$Ax=0$$

$$\begin{pmatrix}a_1 & b_1 & c_1\\ a_2 & b_2 & c_2\\ a_3 & b_3 & c_3\end{pmatrix}\begin{pmatrix}x\\ y\\ 1\end{pmatrix}=\begin{pmatrix}0\\ 0\\ 0\end{pmatrix}\qquad \text{(ii)}$$

In homogeneous system of three linear equations lines (ii), the homogeneous coordinates are used:

$$\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}x\\ y\\ 1\end{pmatrix}\qquad \text{(iii)}$$

Concurrency means that the three lines must intersect at a point $G(x,y)$, say, that can be found by solving the system of linear equations (i). The system (ii) has a nontrivial solution if the determinant of the coefficient matrix A of the system (ii) is zero:

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0 \qquad \text{(iv)}$$

This is the condition of concurrency of three lines.

Remember

For required point of concurrency follow the steps given below:

- Choose any two lines from the given three system of linear equations (i).
- Develop the system of these two linear equations.
- Develop the augmented matrix $\frac{A}{b}$ of the system of two linear equations and reduce it in an echelon form to obtain the point of intersection.
- Substitute the developed point of intersection in the remaining third line. If the point of intersection satisfies the remaining third line, then that point of intersection should be taken as the point of concurrency of the given three lines.

Example 21 Show that the three lines $x+4y+3=0$, $5x-4y-5=0$ and $2x+2y+1=0$ are concurrent. If the lines are concurrent, then find out the point of concurrency.

Solution The condition of concurrency (iv) in light of the given three lines is going to be zero:

$$\begin{vmatrix} 1 & 4 & 3 \\ 5 & -4 & -5 \\ 2 & 2 & 1 \end{vmatrix} = 1(-4+10)-4(5+10)+3(10+8)=6-60+54=0$$

The given three lines are concurrent. For the point of concurrency $G(x,y)$, choose the first two lines

$x+4y=-3,$

$5x-4y=5$

that develops the system of two linear equations, whose augmented matrix A/b is reduced in an echelon form

$$A/b=\begin{pmatrix} 1 & 4 & -3 \\ 5 & -4 & 5 \end{pmatrix} \sim \begin{pmatrix} 1 & 4 & -3 \\ 0 & -24 & 20 \end{pmatrix} \quad by\ R_{21}(-5)=R_2+R_1(-5)$$

to obtain the reduced system of linear equations:

$x+4y=-3,$

$-24y=20$

The second equation is giving $y=-\frac{5}{6}$ which is used in first equation to obtain $x=\frac{1}{3}$. The third line with substitution of the point of intersection $(x,y)=\left(\frac{1}{3},-\frac{5}{6}\right)$ is going to be zero:

$$2x+2y+1=0 \Rightarrow 2\ (2\left(\frac{1}{3}\right)+2\left(-\frac{5}{6}\right)+1=\left(\frac{2}{3}\right)-\left(\frac{5}{3}\right)+1=0$$

Thus, the given three lines are concurrent at a point $G\left(\frac{1}{3},-\frac{5}{6}\right)$.

7.7.2 Equation of median, altitude and right bisector of a triangle

A. Equation of median of a triangle

A median is a line segment from an interior angle of a triangle to the mid point of the opposite side. Look at the following example, the procedure to find the equation of median of triangle is illustrated in this example.

Example 22 Find the equation of the median of a triangle having vertices are $A(8, -5)$, $B(6,5)$ and $C(-6,9)$.

Solution If C' is midd point of side $\overline{AB}$ of ΔABC

Then its co-ordinates are given as $\left(\frac{8+6}{2}, \frac{-5+5}{2}\right) = (7,0)$

Since, the median CC' passes through points C and C', using the two-point form of the equation of a straight line, the equation of median CC' can be found as

$$\frac{y-0}{9-0} = \frac{x-7}{-6-7} \Rightarrow \frac{y}{9} = \frac{x-7}{-13} \Rightarrow 13y-9x+63=0$$

If A' is the midpoint of side BC of ΔABC then its coordinates are given as

$$\left(\frac{6-6}{2}, \frac{5+9}{2}\right) = \left(\frac{0}{2}, \frac{14}{2}\right) = (0,7)$$

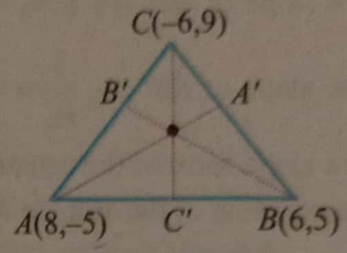


Figure 7.25

Since, the median AA' passes through the point A and A' respectively. By using the two point form of the equation of a straight line, the equation of median AA' can be found as

$$\frac{y-7}{7-(-5)} = \frac{x-0}{0-8} \Rightarrow \frac{y-5}{12} = \frac{x}{-8} \Rightarrow -8y-12x+40=0$$

If B' is the midpoint of side AC of ΔABC then its coordinated are given as

$$\left(\frac{8-6}{2}, \frac{-5+9}{2}\right) = \left(\frac{2}{2}, \frac{4}{2}\right) = (1,2)$$

Since, the median BB' passes through the point B and B' respectively. By using the two point form of the equation of a straight line, the equation of median BB' can be found as

$$\frac{y-2}{2-5} = \frac{x-1}{1-6} \Rightarrow \frac{y-2}{-3} = \frac{x-1}{-5} \Rightarrow 3x-5y+7=0$$

B. Equation of altitude of a triangle

Altitude of a triangle is a perpendicular drawn from the vertex of the triangle to the opposite side. This is also known as the height of the triangle. Mostly it is used to find the area of the triangle. Look at the following example the procedure to find the equation of altitude is illustrated in example 22.

Example 23 Find the equation of altitude of triangle ABC having vertices are $A(-7, 4)$, $B(9, 6)$ and $C(7, -10)$.

Solution First we find slope of side

$$AB = m_1 = \frac{6-4}{9+7} = \frac{2}{16} = \frac{1}{8}$$

The altitude CC' is perpendicular to side AB, so, the slope of

$$CC' = \frac{1}{m_1} = -8$$

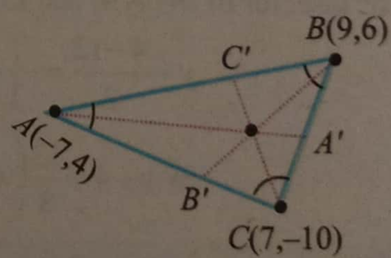


Figure 7.26

Since the altitude CC' passes through the point $C(7, -10)$, by using point slope form of the equation of a line, the equation of CC' is

$$y-(-10) = -8(x-7)$$

$$\Rightarrow y+10 = -8x+56 \Rightarrow 8x+y-46=0$$

Which is the equation of the altitude from C to AB.

The slope of side $\overline{BC} = m_2 = \frac{-10-6}{7-9} = \frac{-16}{-2} = 8$

The altitude AA' is perpendicular to side BC, so, the slope of $AA' = = \frac{1}{m_2} = \frac{1}{8}$

Since, the altitude passes through the point $A(-7, 4)$, by using the point slope form of the equation of a line, the equation of AA' is $y-4=-\frac{1}{8}(x+7) \Rightarrow x+8y-27=0$

Which is the equation of the altitude from A to BC

The slope of side $AC = m_3 = \frac{-10-(-7)}{7-4} = \frac{-3}{3} = -1$. The altitude BB' is perpendicular to side AC. So,

The slope of $BB' = -\frac{1}{m_3} = -\frac{1}{(-1)} = 1$, since, the altituded passes through the point $B(9,6)$. By using the pint slope form of the equation of a line, the equation of BB' is $y-6=1(x-9) \Rightarrow x-y-3=0$. Which is the equation of the altitude from B to AC.

C. **Equation of bisector of a triangle**

The bisector of a triangle is a line perpendicular to the side and passing through its midpoint. The three perpendicular bisectors of the sides of a triangle meet in a single point. The procedure to find the equation of a bisector is illustrate in the following example.

Example 24 Find the equation of the right bisector of a triangle having pints are $A(-7, 4)$, $B(10, 8)$ and $C(6, -12)$.

Solution Since, the equation of a perpendicular bisector is given as,

$$y-\frac{y_1+y_2}{2} = -\frac{x_2-x_1}{y_2-y_1}\left(x-\frac{x_1+x_2}{2}\right) \qquad \text{(i)}$$

Fore bisector of $A(-7, 4)$ and $B(10, 8)$ put the values in equation (i)

$$y-\frac{8+4}{2} = -\frac{10+7}{8-7}\left(x-\frac{10-7}{2}\right) \Rightarrow y-6=\frac{17}{4}\left(x-\frac{3}{2}\right) \Rightarrow 34x-8y-3=0$$

For bisector of $B(10, 8)$ and $C(6, -12)$, put the values in equation (i)

$$y-\frac{8-12}{2} = -\left(\frac{6-10}{-12-8}\right)\left(x-\frac{10+6}{2}\right)$$

$$\Rightarrow y+2=-\frac{1}{5}(x-8) \Rightarrow x+5y+2=0$$

For bisector of $A(-7, 4)$ and $C(6, -12)$ put the values in equation (i)

$$y-\frac{4-12}{2} = -\left(\frac{6+7}{-12+4}\right)\left(x-\frac{-7+6}{2}\right)$$

$$\Rightarrow y+4=\frac{13}{8}\left(x+\frac{1}{2}\right) \Rightarrow 26x-16y-51=0$$

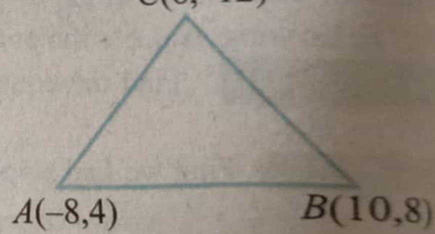


Figure 7.27

7.7.3 Show that, three right bisectors, three medians and three altitudes of a triangle are concurrent

i. **Three right bisectors of a triangle are concurrent**

To show the concurrency of the right bisectors of a triangle, the procedure developed is as under:

Let ABC be a triangle, whose vertices are $A(x_1, y_1), B(x_2, y_2)$ and $C(x_3, y_3)$, D, E, F are the midpoints of the sides BC, CA, AB of a triangle ABC whose coordinates are respectively:

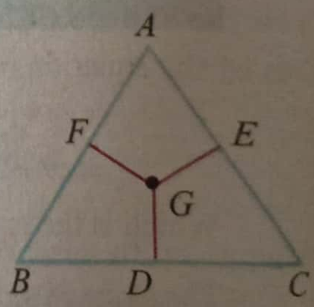


Figure 7.28

$$D\left(\frac{x_2+x_3}{2},\frac{y_2+y_3}{2}\right),\quad E\left(\frac{x_1+x_3}{2},\frac{y_1+y_3}{2}\right),\quad F\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2}\right)$$

If the slope of the side BC and the slope of the right bisector DG of the side BC are respectively:

$$m_1=\frac{y_3-y_2}{x_3-x_2},\quad m_2=\frac{-1}{m_1}=-\frac{x_3-x_2}{y_3-y_2}$$

then, the equation of the right bisector DG of side BC is obtained by point–slope form of a line:

$$\left(y-\frac{y_2+y_3}{2}\right)=-\frac{x_3-x_2}{y_3-y_2}\left(x-\frac{x_2+x_3}{2}\right)$$

$$(y_3-y_2)\left(y-\frac{y_2+y_3}{2}\right)=(x_2-x_3)\left(x-\frac{x_2+x_3}{2}\right)$$

$$x(x_2-x_3)+y(y_2-y_3)-\frac{1}{2}(y_2^2-y_3^2)-\frac{1}{2}(x_2^2-x_3^2)=0$$

Similarly, the equations of the right bisectors EG (of side CA), FG (of side AB) is respectively:

$$x(x_3-x_1)+y(y_3-y_1)-\frac{1}{2}(y_3^2-y_1^2)-\frac{1}{2}(x_3^2-x_1^2)=0$$

$$x(x_1-x_2)+y(y_1-y_2)-\frac{1}{2}(y_1^2-y_2^2)-\frac{1}{2}(x_1^2-x_2^2)=0$$

The right bisectors DG, EG and FG is concurrent, if the determinant of the coefficient matrix A of the related system of equations of the right bisectors DG, EG and FG equals zero:

$$\begin{vmatrix} x_2-x_3 & y_2-y_3 & -\frac{1}{2}(y_2^2-y_3^2)-\frac{1}{2}(x_2^2-x_3^2) \\ x_3-x_1 & y_3-y_1 & -\frac{1}{2}(y_3^2-y_1^2)-\frac{1}{2}(x_3^2-x_1^2) \\ x_1-x_2 & y_1-y_2 & -\frac{1}{2}(y_1^2-y_2^2)-\frac{1}{2}(x_1^2-x_2^2) \end{vmatrix}=0 \qquad \text{(i)}$$

The operation of addition of rows $R_1+R_2+R_3$ to row R_1 is used to obtain:

$$\sim\begin{vmatrix} 0 & 0 & 0 \\ x_3-x_1 & y_3-y_1 & -\frac{1}{2}(y_3^2-y_1^2)-\frac{1}{2}(x_3^2-x_1^2) \\ x_1-x_2 & y_1-y_2 & -\frac{1}{2}(y_1^2-y_2^2)-\frac{1}{2}(x_1^2-x_2^2) \end{vmatrix}=0 \qquad \text{(ii)}$$

The value of the determinant is zero. Hence, the right bisectors DG, EG and FG of a triangle ABC are concurrent.

Example 25 Let ABC be a triangle with vertices $A(0,0)$, $B(8,6)$ and $C(12,0)$. Show that the right bisectors DG, EG and FG of the triangle ABC are concurrent.

Solution The vertices $A(x_1,y_1)=A(0,0)$, $B(x_2,y_2)=B(8,6)$ and $C(x_3,y_3)=C(12,0)$ of the triangle ABC are used in the determinant (i) to obtain:

$$\begin{vmatrix} x_2-x_3 & y_2-y_3 & -\frac{1}{2}(y_2^2-y_3^2)-\frac{1}{2}(x_2^2-x_3^2) \\ x_3-x_1 & y_3-y_1 & -\frac{1}{2}(y_3^2-y_1^2)-\frac{1}{2}(x_3^2-x_1^2) \\ x_1-x_2 & y_1-y_2 & -\frac{1}{2}(y_1^2-y_2^2)-\frac{1}{2}(x_1^2-x_2^2) \end{vmatrix}=0 \Rightarrow \begin{vmatrix} -4 & 6 & 22 \\ 12 & 0 & -72 \\ -8 & -6 & 50 \end{vmatrix}=0$$

$$\Rightarrow -4(-432)-6(600-576)+22(-72)=0$$

$$\Rightarrow 1728-1728=0$$

The determinant (ii) equals zero. Hence, the right bisectors DG, EG and FG of the triangle ABC are concurrent.

ii. Three median of a triangle are concurrent

See topic 7.1.3 at page no 181.

iii. Three altitudes of a triangle are concurrent

Let ABC be a triangle, whose vertices are $A(x_1,y_1), B(x_2,y_2)$ and $C(x_3,y_3)$. The altitudes of the triangle ABC are AD, BE and CF.

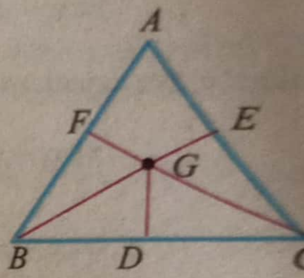


Figure 7.29

If the slope of the side BC and the slope of the altitude AD are respectively $m_1=\frac{y_3-y_2}{x_3-x_2}$, $m_2=-\frac{1}{m_1}=-\frac{x_3-x_2}{y_3-y_2}$,

then, the equation of the altitude AD is obtained by point–slope form of a line:

$$(y-y_1)=-\frac{x_3-x_2}{y_3-y_2}(x-x_1)$$

$$(y-y_1)(y_3-y_2)=(x_2-x_3)(x-x_1)x(x_2-x_3)+y(y_2-y_3)-x_1(x_2-x_3)-y_1(y_2-y_3)=0$$

Similarly, the equations of the altitudes BE and CF are respectively:

$$x(x_3-x_1)+y(y_3-y_1)-x_2(x_3-x_1)-y_2(y_3-y_1)=0$$

$$x(x_1-x_2)+y(y_1-y_2)-x_3(x_1-x_2)-y_3(y_1-y_2)=0$$

The altitudes AD, BE and CF are concurrent, if the determinant of the coefficient matrix A of the related system of equations of the altitudes AD, BE and CF equals zero:

$$\begin{vmatrix} x_2-x_3 & y_2-y_3 & -x_1(x_2-x_3)-y_1(y_2-y_3) \\ x_3-x_1 & y_3-y_1 & -x_2(x_3-y_1)-y_2(y_3-y_1) \\ x_1-x_2 & y_1-y_2 & -x_3(x_1-x_2)-y_3(y_1-y_2) \end{vmatrix}=0 \qquad \text{(iii)}$$

The operation of addition of rows $R_1+R_2+R_3$ to row R_1 is used to obtain:

$$\sim\begin{vmatrix} 0 & 0 & 0 \\ x_3-x_1 & y_3-y_1 & -x_2(x_3-y_1)-y_2(y_3-y_1) \\ x_1-x_2 & y_1-y_2 & -x_3(x_1-x_2)-y_3(y_1-y_2) \end{vmatrix}=0$$

The value of the determinant is zero. Therefore, the altitudes AD, BE and CF of a triangle are concurrent at a point G.

The conclusion drawn from the above results is that the three medians AD, BE and CF of a triangle ABC will also make concurrency at a point say, $G(x,y)$.

Example 26 Let ABC be a triangle with vertices $A(0,0)$, $B(8,6)$ and $C(12,0)$. Show that the altitudes AD, BE and CF of the triangle ABC are concurrent.

Solution The vertices $A(x_1,y_1)=A(0,0)$, $B(x_2,y_2)=B(8,6)$ and $C(x_3,y_3)=C(12,0)$ of the triangle ABC are used in the determinant (iii) to obtain:

$$\begin{vmatrix} x_2-x_3 & y_2-y_3 & -x_1(x_2-x_3)-y_1(y_2-y_3) \\ x_3-x_1 & y_3-y_1 & -x_2(x_3-y_1)-y_2(y_3-y_1) \\ x_1-x_2 & y_1-y_2 & -x_3(x_1-x_2)-y_3(y_1-y_2) \end{vmatrix}=0 \Rightarrow \begin{vmatrix} -4 & 6 & 0 \\ 12 & 0 & -96 \\ -8 & -6 & 96 \end{vmatrix}=0$$

$$-4(-576)-6(1152-768)=0 \Rightarrow 2304-6(384)=0 \Rightarrow 2304-2304=0$$

The determinant (iii) equals zero. Hence, the altitudes AD, BE and CF of the triangle ABC are concurrent.

7.8 Area of a Triangular Region

Let ABC be a triangle whose vertices are $P_1(x_1,y_1)$, $P_2(x_2,y_2)$ and $P_3(x_3,y_3)$. Project P_1A, P_2B and P_3C upon the x-axis that develops the trapezia P_1ACP_3, P_3CBP_2 and P_1ABP_2.

The area A of the triangular region $P_1P_2P_3$ is the sum of the areas of the trapezia P_1ACP_3, P_3CBP_2 and P_1ABP_2 minus the area of the trapezium P_1ABP_2.

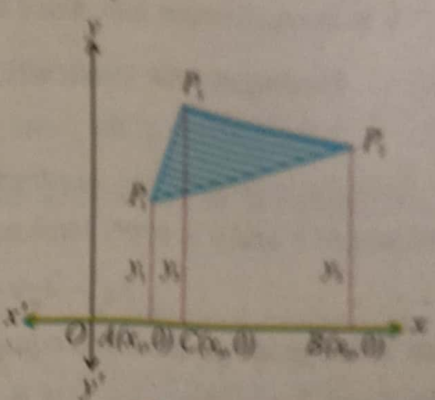


Figure 7.30

$$A=\frac{1}{2}[(y_1+y_3)(x_3-x_1)]+\frac{1}{2}[(y_3+y_2)(x_2-x_3)]-\frac{1}{2}[(y_1+y_2)(x_2-x_1)]$$

$$=\frac{1}{2}[x_3y_1-x_1y_1+x_3y_3-x_1y_3+x_2y_3-x_3y_3+x_2y_2-x_3y_2-x_2y_1+x_1y_1-x_2y_2+x_1y_2] \quad \text{(i)}$$

$$=\frac{1}{2}[x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2)]$$

$$=\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \quad \text{(ii)}$$

It is important to note that the area A of the triangular region $P_1P_2P_3$ equals zero, when the vertices of the triangular region are collinear points.

Example 27 Find the area of the triangular region $P_1P_2P_3$ whose vertices are $P_1(4,-5)$, $P_2(5,-6)$ and $P_3(3,1)$.

Solution Result (i) is used for points $P_1(4,-5)$, $P_2(5,-6)$, $P_3(3,1)$ to obtain the area of the triangular region $P_1P_2P_3$:

$$A=\frac{1}{2}\begin{vmatrix} 4 & -5 & 1 \\ 5 & -6 & 1 \\ 3 & 1 & 1 \end{vmatrix}, \text{ first row expansion}$$

$$=\frac{1}{2}[4(-6-1)+5(5-3)+(5+18)]=\frac{1}{2}[-28+10+23]=\frac{5}{2} \text{ square units}$$

7.9 Homogeneous Equation

In general, any line equation in two variables that passes through the origin is called a homogeneous equation.

7.9.1 Homogeneous linear and quadratic equation in two variables

i. Homogeneous linear equation in two variables

"An equation of the form in two variables x and y $ax+by+c=0$, $c\neq 0$, a, b and c are constants (i)
is called a nonhomogeneous equation of a line." For $c=0$, the nonhomogeneous equation (i) gives the homogeneous equation of the form $ax+by=0$ (ii)
that passes through the origin definitely. This also defines a homogeneous equation of degree 1, since the indices of x and y in every term of (ii) is the same, the degree being 1. For example, the equation of line $x+y=0$ is homogeneous line, since it defines a homogeneous equation of degree 1.

ii. Homogeneous quadratic equation in two variables

"An equation of the form $ax^2+2hxy+by^2=0$, $a\neq 0$, where a, b, c are constants (iii)
is called a homogeneous quadratic equation of second degree in variables x and y". Since the sum of the indices of x and y in every term are the same number "2". For example,

$$3x^2-4xy+5y^2=0 \quad \text{and} \quad lx^2+mxy+ny^2=0$$

are homogeneous quadratic equations of the second degree in x and y. On the other hand, the equation of the form $3xy^2-4xy+5y^2=0$ is not a homogeneous equation, since the sum of the indices of x and y are not the same in each and every term.

7.9.2 Second degree homogeneous equations represents a pair of straight lines through the origin

i. Standard form of second degree homogeneous equation

If $a_1x+b_1y+c_1=0$ and $a_2x+b_2y+c_2=0$ are the two straight lines, then the simple product of the given two nonhomogeneous lines defines a joint equation of a line:

$$(a_1x+b_1y+c_1)(a_2x+b_2y+c_2)=0 \qquad \text{(i)}$$

The joint equation of the homogeneous straight lines is obtained from (i) by putting $c_1=c_2=0$:

$$(a_1x+by_1)(ax_2+by_2)=0 \qquad \text{(ii)}$$

The product of homogeneous lines (ii) is giving the standard form of the second degree homogeneous equation:

$$(a_1x+b_1y)(a_2x+b_2y)=0$$

$$a_1a_2x^2+a_1b_2xy+b_1a_2xy+b_1b_2y^2=0$$

$$a_1a_2x^2+(a_1b_2+a_2b_1)xy+b_1b_2y^2=0 \qquad \text{(iii)}$$

If $a_1a_2=a$, $(a_1b_2+a_2b_1)=2h$, $b_1b_2=b$, then (iii) gives: $ax^2+2hxy+by^2=0$. (iv)

Any point $P(x, y)$ that satisfies first line $a_1x+b_1y=0$ or second line $a_2x+b_2y=0$ will also satisfies the joint homogeneous equation of (ii).

ii. **Representation as pair of straight lines**

The product of (iv) to constant quantity $\frac{a}{a}$ is giving the joint equation of the two first degree homogeneous equations in x and y:

Note

The lines are
- real and distant, if $h^2 - ab > 0$.
- real and coincident, if $h^2 - ab = 0$.
- imaginary, if $h^2 - ab < 0$.

$$\frac{a}{a}\left[ax^2 + 2hxy + by^2\right] = 0$$

$$a^2x^2 + 2ahxy + aby^2 = 0, \text{ Add and subtract } h^2y^2$$

$$(ax+hy)^2 - h^2y^2 + aby^2 = 0 \Rightarrow (ax+hy)^2 - y^2(h^2 - ab) = 0$$

$$(ax+hy)^2 - \left(y\sqrt{(h^2-ab)}\right)^2 = 0 \Rightarrow \left(ax + hy + y\sqrt{h^2-ab}\right)\left(ax + hy - y\sqrt{h^2-ab}\right) = 0$$

$$ax + hy + y\sqrt{h^2-ab} = ax + \left(h + \sqrt{h^2-ab}\right)y = 0 \qquad \text{(v)}$$

$$ax + hy - y\sqrt{h^2-ab} = ax + \left(h - \sqrt{h^2-ab}\right)y = 0 \qquad \text{(vi)}$$

The lines (v) and (vi) are therefore first degree equations in x and y.

Example 28 Find two first degree straight lines in x and y when the second degree homogeneous equation is $5x^2 + 3xy - 8y^2 = 0$.

Solution The Standard form of second degree homogenous equations (iv) is compared to the given second degree homogeneous equation $5x^2 + 3xy - 8y^2 = 0$ to obtain: $a = 5,\ 2h = 3 \Rightarrow h = \frac{3}{2},\ b = -8$

These values are used in the standard form of two first degree homogeneous lines (v) and (vi) to obtain the required two homogeneous lines:

$$ax + \left(h + \sqrt{h^2-ab}\right)y = 0$$

$$5x + \left[\frac{3}{2} + \sqrt{\frac{9}{4} - (5)(-8)}\right]y = 0$$

$$5x + \left[\frac{3}{2} + \sqrt{\frac{9}{4} + 40}\right]y = 0$$

$$5x + \left[\frac{3}{2} + \frac{13}{2}\right]y = 0$$

$$5x + 8y = 0$$

$$ax + \left(h - \sqrt{h^2-ab}\right)y = 0$$

$$5x + \left[\frac{3}{2} - \sqrt{\frac{9}{4} - (5)(-8)}\right]y = 0$$

$$5x + \left[\frac{3}{2} - \sqrt{\frac{9}{4} + 40}\right]y = 0$$

$$5x + \left[\frac{3}{2} - \frac{13}{2}\right]y = 0$$

$$5x - 5y = 0$$

$$x - y = 0$$

iii. **Angle between pair of straight lines**

If the standard form of the second degree homogeneous equation in two variables x and y

$$ax^2 + 2hxy + by^2 = 0 \qquad \text{(vii)}$$

is decomposed into the product of two homogeneous straight lines $y = m_1x$ and $y = m_2x$:

$$(y - m_1x)(y - m_2x) = 0$$

$$\Rightarrow y^2 - m_2xy - m_1xy + m_1m_2x^2 = 0$$

$$y^2 - (m_1 + m_2)xy + m_1m_2x^2 = 0 \qquad \text{(viii)}$$

The comparison of equations (vii) and (viii) gives:

$$b=1, -(m_1+m_2)=2h \Rightarrow (m_1+m_2)=\frac{-2h}{1}=\frac{-2h}{b},\ m_1m_2=a=\frac{a}{1}=\frac{a}{b}$$

The angle θ in between the given two straight lines is:

$$\tan\theta=\frac{m_1-m_2}{1+m_1m_2}=\frac{\sqrt{(m_1-m_2)^2}}{1+m_1m_2}, \qquad (m_1-m_2)^2=(m_1+m_2)^2-4m_1m_2$$

$$=\frac{\sqrt{(m_1+m_2)-4m_1m_2}}{1+m_1m_2}=\frac{\sqrt{\frac{4h^2}{b^2}-\frac{4a}{b}}}{1+\frac{a}{b}}=\frac{2\sqrt{h^2-ab}}{a+b} \qquad \text{(ix)}$$

Remember

Result (ix) developed the idea that:

- the given two straight lines are perpendicular, if the angle between them is 90^0 that makes $a+b=0$.
- the given two straight lines are coinciding if the angle between them is zero that makes $h^2=ab$.

Example 29 Find the angle in between the lines represented by the second degree homogeneous equation $x^2-xy-6y^2=0$

Solution The standard form of second degree homogeneous equation (iv) is compared to the given second degree homogeneous equation $x^2-xy-6y^2=0$ to obtain: $a=1, 2h=-1 \Rightarrow h=\frac{-1}{2}, b=-b$

These values are used in result (ix) to obtain the angle in between the two straight lines represented by $x^2-xy-6y^2=0$:

$$\tan\theta=\frac{2\sqrt{h^2-ab}}{a+b}=\frac{2\sqrt{\frac{1}{4}-(1)(-6)}}{1+(-6)}=\frac{2\sqrt{25/4}}{-5}=\frac{\sqrt{25}}{-5}=\frac{5}{-5}=-1$$

$$\theta=\tan^{-1}(-1)$$

$$\theta=\frac{3\pi}{4}=135^0$$

Example 30 Find the first degree straight lines in x and y when the second degree homogeneous equation is $35x^2-4xy-3y^2=0$. Show that the resultant lines are coincident or perpendicular.

Solution The standard form of second degree homogeneous equations (iv) is compared to the given second degree homogeneous equation $35x^2-4xy-3y^2=0$ to obtain: $a=3, 2h=-4 \Rightarrow h=-2, b=-3$

These values are used in the standard forms (v) and (vi) of the two first degree homogeneous lines to obtain the required two homogeneous lines:

$$ax+\left(h+\sqrt{h^2-ab)}\right)y=0 \qquad ax+\left(h-\sqrt{h^2-ab}\right)y=0$$

$$3x+\left[-2+\sqrt{(4+9)}\right]y=0 \qquad 3x+\left[-2-\sqrt{(4+9)}\right]y=0$$

$$3x+\left[-2+\sqrt{13}\right]y=0 \qquad 3x+\left[-2-\sqrt{13}\right]y=0$$

The two lines are perpendicular, since $a+b=3-3=0$ is zero for $a=3$ and $b=-3$.

Exercise 7.4

In each case, find the point of intersection $P(x, y)$ of the pair of lines:

a. $2x+4y-10=0,\ 5x-3y+1=0$ b. $2x+y-8=0,\ 3x+2y-2=0$

Show that the following lines are concurrent. If the lines are concurrent, then find out the point at which the given lines can make concurrency:

a. $x-y-2=0,\ 2x-y-5=0,\ 11x-5y-28=0$

b. $x+2y-3=0,\ 2x-y+4=0,\ x+4y-7=0$

c. $3x+2y-1=0,\ 2x-3y+4=0,\ x+y-2=0$

If ABC is a triangle with vertices A(0,0), B(8,6) and C(12,0), then show that

a. the right bisectors of the triangle ABC are concurrent.

b. the altitudes of the triangle ABC are concurrent.

c. the medians of the triangle ABC are concurrent.

Find the area of the triangular region whose vertices are the following:

a. $P_1(0,0), P_2(2,4), P_3(-2,2)$. b. $P_1(-1,-2), P_2(2,5), P_3(5,2)$.

c. $P_1(4,-5), P_2(5,-6), P_3(3,1)$.

Find the area bounded by the triangle ABC whose vertices are the following:

a. A(–3, 6), B(3, 2) C(6, 0) b. A(–2, 4), B(3, –6) C(1, –2)

c. Are the vertices in parts a and b of the triangle ABC collinear?

Find two first degree straight lines in x and y, when the second degree homogeneous equations are the following:

a. $3x^2-2xy-5y^2=0$ b. $4x^2-9xy+5y^2=0$

Show the two first degree straight lines in x and y are coincident, perpendicular or neither, when they are represented by the following second degree homogeneous equations:

a. $x^2+5xy-y^2=0$ b. $2x^2-xy-y^2=0$

Find a joint equation of the straight line that passes through the origin and

a. perpendicular to the lines represented by $3x^2-7xy+2y^2=0$.

b. perpendicular to the lines represented by $x^2-2\tan\theta xy-y^2=0$.

c. perpendicular to the lines represented by $ax^2+2hxy+by^2=0$.

Review Exercise 7

1. Choose the correct option.

i. The distance between point A(7,5) and B(–5,–7) is:
(a) $2\sqrt{37}$ (b) $12\sqrt{2}$ (c) $2\sqrt{2}$ (d) 0

ii. To find the mid point of a line we use the formula:
(a) $\frac{x_1-x_2}{2}, \frac{y_1-y_2}{2}$ (b) $\frac{m_1x_2-m_2x_1}{2}, \frac{m_1y_2-m_2y_1}{2}$
(c) $\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}$ (d) $\frac{m_1x_2+m_2x_1}{2}, \frac{m_1y_2+m_2y_1}{2}$

iii. For the points (a, b) and (5,7) the slope of line is:
(a) $\frac{7-b}{5-a}$ (b) $\frac{5}{7}$ (c) $\frac{2}{3}$ (d) $-\frac{2}{3}$

iv. y intercept of the line $2x+4y=-6$ is:
(a) $\frac{3}{2}$ (b) $-\frac{3}{2}$ (c) $\frac{2}{3}$ (d) $-\frac{2}{3}$

v. $\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}$ is a
(a) slope intercept form (b) two point form
(c) point slop form (d) double intercept form

vi. The normal form of an equation is:
(a) $x\cos\theta+y\sin\theta=p$ (b) $x\cos\theta-y\sin\theta=p$
(c) $x\sin\theta+y\tan\theta=0$ (d) $x\cos\theta-y\sin\theta=0$

vii. Two given line are perpendciula if there slopes m_1 and m_2 are
(a) $m_1+m_2=1$ (b) $m_2-m_1=0$ (c) $m_1m_2=1$ (d) $m_1m_2=-1$

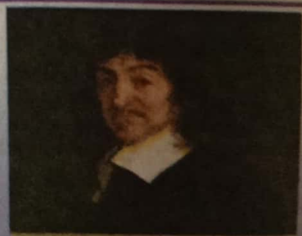
viii. $5x+7y=0$ is a
(a) homogeneous linear equation (b) non-homogeneous lines equation
(c) quadratic equation (d) only linear equation

ix. The area of a triangular region is A(4,5),B(–7, 4) and C(3,1) is:
(a) 31 (b) –31 (c) 47 (d) –47

x. The angle between two pair of lines can be calculated by using the formula:
(a) $\frac{2\sqrt{h^2+ab}}{a+b}$ (b) $\frac{2-\sqrt{a^2+hb}}{a+b}$ (c) $\frac{2\sqrt{h^2+pq}}{a-b}$ (d) $\frac{2\sqrt{h^2-ab}}{a+b}$

History

Rene Descartes was a French mathematician, Philosopher and scientist. He was a creative mathematician of first order. He developed the techniques that made possible algebraic or analytic geometry. In metaphysics, he provided arguments for the existence of God, to show that the essence of matter is extension and that the essence of mind is thought. He claimed early on to possess a special method, that was variously exhibited in mathematics.



Rene Descartes
(1596)-(1650)

Descartes has been called the father of analytic geometry. He laid the foundation for 17th century continental rationalism. Because of his contribution considered a well versed in mathematics. Descartes's was influence in mathematics is equally apparent the Cartesian coordinated system was named after him. He was also considered one of the key figure in the scientific revolution.

Summary

- The distance from point $P(x_1, y_1)$ to point $Q(x_2, y_2)$ in the coordinate plane is:
  $d = |PQ| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$
- If L_1 and L_2 are the two lines having slopes m_1 and m_2, then, these two lines are
  a. **parallel** if and only if they have the same slopes: $m_1 = m_2$.
  b. **perpendicular** if and only if the product of their slopes equals -1: $m_1m_2 = -1$
- The equation of a straight line parallel to the *x–axis* and at a distance a from it, is $y = a$. The equation of the *x–axis* is $y = 0$ and the vector equation of *x–axis* is $r.j = 0$.
- The equation of a straight line parallel to the *y–axis* and at a distance **b** from it, is $x = b$. The equation of the *y–axis* is $x = 0$ and the vector equation of *y–axis* is $r.i = 0$.
- The standard forms of the line are the following:
  a. $y = mx + c$, Slope–Intercept form
  b. $y - y_1 = m(x - x_1)$, Point–Slope form
  c. $y - y_1 = \dfrac{y_2 - y_1}{x_2 - x_1}(x - x_1)$ Two–Point form
  d. $\dfrac{x}{a} + \dfrac{y}{b} = 1$ Double–Intercept form
  e. $\dfrac{x - x_1}{\cos\theta} = \dfrac{y - y_1}{\sin\theta} = r$ Symmetric Form
  f. $x\cos\theta + y\sin\theta = p$ Normal Form
- The standard form of a line is $ax + by + c = 0$
- For $c = 0$, the line is homogeneous that passes through the origin.
- For $c \neq 0$, the line is nonhomogeneous that does not pass through the origin.
- The perpendicular distance from a line $ax + by + c = 0$ to a point $P(x_1, y_1)$ is: $d = \dfrac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}}$
- The angle between the two lines $y = m_1x + c_1$ and $y = m_2x + c_2$ is: $\tan\theta = \dfrac{m_1 - m_2}{1 + m_1m_2}$
- The general equation of a straight line that passes through the point of intersection of the lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ is: $(a_1x + b_1y + c_1) + \lambda(a_2x + b_2y + c_2) = 0$, λ constant
- The condition of concurrency of the three lines is: $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$
- An equation of the form $ax^2 + 2hxy + by^2 = 0$, $a \neq 0, b, c$ *are* constants is called a homogeneous equation of second degree in x and y when the sum of the indices of x and y in every term is the same, the sum being 2.
- The angle between two homogeneous straight lines $y = m_1x$ and $y = m_2x$ is:
  $\tan\theta = \dfrac{2\sqrt{h^2 - ab}}{a + b}$
  i. The given two straight lines are perpendicular, if the angle between them is 90^0 that makes $a + b = 0$.
  ii. The given two straight lines are coinciding if the angle between them is zero that makes $h^2 = ab$.

Unit 8 CONICS-I

By the end of this unit, the students will be able to:

8.1 Introduction

i. Define conics and demonstrate members of its family i.e. circle, parabola, ellipse and hyperbola.

8.2 Circle

i. Define circle and derive its equation in standard form i.e. $(x-h)^2+(y-k)^2=r^2$.

ii. Recognize general equation of a circle $x^2+y^2+2gx+2fy+c=0$ and find its centre and radius.

iii. Find the equation of a circle passing through
- three non-collinear points,
- two points and having its centre on a given line,
- two points and equation of tangent at one of these points is known,
- two points and touching a given line.

8.3 Tangent and Normal

i. Find the condition when a line intersects the circle.

ii. Find the condition when a line touches the circle.

iii. Find the equation of a tangent to a circle in slope form.

iv. Find the equations of a tangent and a normal to a circle at a point.

v. Find the length of tangent to a circle from a given external point.

vi. Prove that two tangent drawn to a circle from an external point are equal in length.

8.4 Properties of circle

i. Prove analytically the following properties of a circle.
- Perpendicular from the centre of a circle on a chord bisects the chord.
- Perpendicular bisector of any chord of a circle passes through the centre of the circle.
- Line joining the centre of a circle to the midpoint of a chord is perpendicular to the chord.
- Congruent chords of a circle are equidistant from its centre and its converse.
- Measure of the central angle of a minor arc is double the measure of the angle subtended by the corresponding major arc.
- An angle in a semi-circle is a right angle.
- The perpendicular at the outer end of a radial segment is tangent to the circle.
- The tangent to a circle at any point of the circle is perpendicular to the radial segment at that point.

Introduction

In geometry conic section is also called a conic. Any curve produced by the intersection of plane and right circle cone depending on the angle of plane and right circular cone. The 'conic section' comes from the fact that the principle type of conic section known as ellipses, hyperbolas and parabola are generated by cutting a cone with a plane. Most modern textbooks of calculus depart from this geometrical approach, instead conic section are defined as some types of loci and studied through analytic geometry. In this unit we will study an approach to conic section which are defined as the intersection of two cones. Then the vertices of two cones becomes the inherent foci of the conic section and directrix exists associated with each of the inherent foci. At the end of this unit we will learn about the properties through real life examples.

8.1 Conics and members of its family

A conic section (or simply conic) is a curve obtained as the intersection of the surface of a cone with a plane. The three types of conic sections are the hyperbola, the parabola, and the ellipse. The circle is type of ellipse, and is sometimes considered to be a fourth type of conic section.

Conic sections can be generated by intersecting a plane with a cone. A cone has two identically shaped parts called nappes. One nappe is what most people mean by "cone," and has the shape of a party hat.

Conic sections are generated by the intersection of a plane with a cone. If the plane is parallel to the axis of revolution (the y-axis), then the conic section is a hyperbola. If the plane is parallel to the generating line, the conic section is a parabola. If the plane is perpendicular to the axis of revolution, the conic section is a circle. If the plane intersects one nappe at an angle to the axis (other than 90^0), then the conic section is an ellipse.

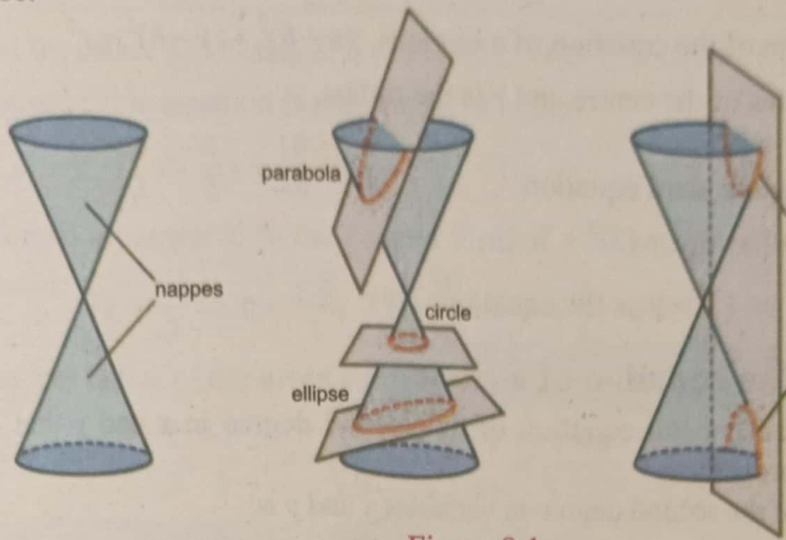


Figure 8.1

In this unit, we look at one of these conic sections that are the circles.

8.2 Circle

A circle is a shape that has a continuous and constant curve. Though it is always curving; it has an algebraic expression that describes its nature.

8.2.1 Circle and its equation in standard form

*"The set of all points in the plane in such a way that its distances from a fixed point in that plane **(called the center)** is equal to a fixed distance **(called the radius)** of the circle."*

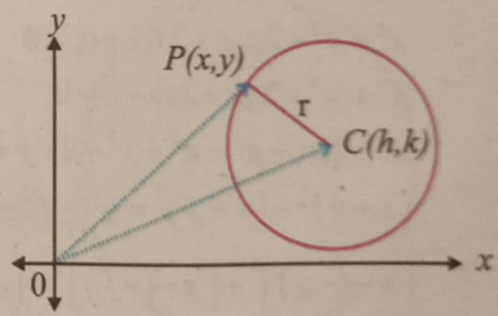


Figure 8.2

Derivation of Circle Equation: This definition helps us in developing a standard form of the equation of a circle.

Let C (h, k) be a fixed point at the center of the circle and r is the radius of the circle and $P(x, y)$ is any one of the collection of points on the circumference of the circle that gives the distance from the fixed point $C(h, k)$ which is called the radius of a circle. The position vectors of P and C relative to origin are respectively. $OP=(x,y),\ OC=(h,k)$ (i)

From the Figure 8.2, the distance from the center C to point P is the fixed distance equals the radius of the circle: $OC+CP=OP$

$$\Rightarrow \quad CP=OP-OC$$

$$\Rightarrow \quad CP=(x,y)-(h,k)=(x-h,y-k)$$

$$|CP|=\sqrt{(x-h)^2+(y-k)^2}, \qquad \text{distance formula}$$

$$|CP|^2=\left(\sqrt{(x-h)^2+(y-k)^2}\right)^2, \qquad \text{squaring both sides}$$

$$r^2=(x-h)^2+(y-k)^2, \qquad |CP|^2=(CP)^2=r^2$$

The standard form of a circle with radius r and center (h, k) is: $(x-h)^2+(h-k)^2=r^2$ (ii)

If the center of the circle is at the origin $(h, k)=(0,0)$, then the circle equation (ii) becomes:

$$x^2+y^2=r^2 \quad \text{(iii)}$$

Example 1 Determine the equation of a circle with center at $(-2,1)$ and radius $r=3$.

Solution The standard form of the equation of a circle is $(x-h)^2+(y-k)^2=r^2$

Where (h, k) are the coordinates of the centre and r is the radius.
In this equation $h=-2,\ k=1$ and $r=3$
substitute these values into the standard equation.

$$\Rightarrow (x-(-2))^2+(y-1)^2=(3)^2$$

$$\Rightarrow (x+2)^2+(y-1)^2=9 \text{ is the equation}$$

8.2.2 General form of an equation of a circle

The arrangement of the general equation of the second degree in x and y that may represent a circle through the following procedure:

The general equation of the second degree in variables x and y is:

$$ax^2+2hxy+by^2+2gx+2fy+c=0 \quad \text{(i)}$$

Divide out both sides of equation (i) by a to obtain:

$$x^2+\frac{2h}{a}xy+\frac{b}{a}y^2+\frac{2g}{a}x+\frac{2f}{a}y+\frac{c}{a}=0 \quad \text{(ii)}$$

$$x^2+\frac{2h}{a}xy+\frac{b}{a}y^2+2g_1x+2f_1y+c_1=0, \qquad g_1=\frac{g}{a}, f_1=\frac{f}{a}, c_1=\frac{c}{a}$$

The rearranged equation (ii) of the general equation of the second degree (i) in x and y gives the general equation of a circle if and only if $\frac{b}{a}=1$ and $\frac{2h}{a}=0$:

$$x^2+y^2+2g_1x+2f_1y+c_1=0$$

$$x^2+y^2+2gx+2fy+c=0 \quad \text{(iii)} \qquad g_1=g,\ f_1=f, c_1=c$$

$x^2+2gx+g^2-g^2+y^2+2fy+f^2-f^2+c=0$ Adding and subtracting g^2 and f^2 in equation (iii)

$$(x+g)^2+(y+f)^2=g^2+f^2-c$$

$$[x-(-g)]^2+[y-(-f)]^2=\left(\sqrt{g^2+f^2-c}\right)^2$$

The locus of a point (x, y) which moves in such a way that its distance from a fixed point $(-g,-f)$ is constant and equals $\sqrt{g^2+f^2-c}$. This of course represents a circle.

For general equation of circle $x^2+y^2+2gx+2fy+c=0$

- **Center and Radius:** The coordinates of the center are $(-g,-f)$ and the radius is $r=\sqrt{(-g)^2+(-f)^2-c}$.
- **Independent Constant:** The general equation contains three independent constants **g, f and c.** They can be determined from the three independent conditions.
- **Nature of the Circle:**

  If $g^2+f^2-c>0$, then, the circle is real and different from zero.

If $g^2+f^2-c=0$, then, the circle shrinks into a point $(-g,-f)$. It is called point circle.
If $g^2+f^2-c<0$, then, the circle is imaginary or virtual.

- The coefficients of x^2 is equal to the coefficients of y^2, and there is no term containing xy and the square of the radius $r^2 \geq 0$.

Example 2 Find the center and radius of a circle $45x^2+45y^2-60y+36y+19=0$.

Solution The given circle equation is rearranged to obtain:

$$x^2+y^2-\frac{4}{3}x+\frac{4}{5}y+\frac{19}{45}=0, \qquad \text{(i)} \qquad \text{Dividing by 45}$$

The circle equation (i) is compared to the general form of a an equation of circle to obtain the values of g, f and c:

$$2g=\frac{-4}{3} \Rightarrow g=\frac{-2}{3},\ 2f=\frac{4}{4} \Rightarrow f=\frac{2}{5},\ c=\frac{19}{45}$$

The center and radius of the given circle are therefore:

$$(-g,-f)=\left(\frac{2}{3},\frac{-2}{5}\right),\ r=\sqrt{\frac{4}{9}+\frac{4}{25}-\frac{19}{45}}=\sqrt{\frac{41}{225}}=\sqrt{\frac{41}{15}}$$

8.2.3 The equation of a circle passing through

A. three non-collinear points
B. two points and having its centre on a given line
C. two points and equation of tangent at one of these points is known
D. two points and touching a given line

A. Equation passing through three non-collinear points

Consider the general equation a circle is given by $x^2+y^2+2gx+2fy+c=0$

If the given circle is passing through three non-collinear points, say, $A(x_1,y_1), B(x_2,y_2)$ and $C(x_3,y_3)$ then these points must satisfy the general equation of a circle. Now put the above three points in the given equation of a circle, i.e.:

$$x_1^2+y_1^2+2gx_1+2fy_1+c=0 \qquad \text{(i)}$$
$$x_2^2+y_2^2+2gx_2+2fy_2+c=0 \qquad \text{(ii)}$$
$$x_3^2+y_3^2+2gx_3+2fy_3+c=0 \qquad \text{(iii)}$$

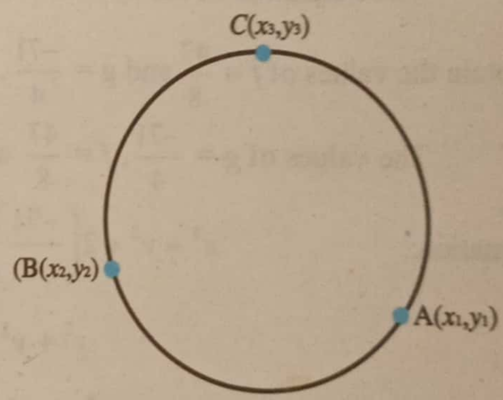


Figure 8.3

Example 3 Find the equation of a circle which passes through the three points A(1,0), B(0,–6) and C(3,4)

Solution The required equation of a circle is $x^2+y^2+2gx+2fy+c=0$ (i)
which passes through the three points A(1,0), B(0,–6) and C(3,4), which gives a system of three linear equations in three unknowns g, f and c:

$$\left.\begin{array}{lcl} 1+2g+c=0 & \Rightarrow & 2g+c=-1 \\ 36-12f+c=0 & \Rightarrow & -12f+c=-36 \\ 25+6g+8f+c=0 & \Rightarrow & 6g+8f+c=-25 \end{array}\right\} \qquad \text{(ii)}$$

The system of three linear equation (ii) in matrix form $\begin{pmatrix} 2 & 0 & 1 \\ 0 & -12 & 1 \\ 6 & 8 & 1 \end{pmatrix}\begin{pmatrix} g \\ f \\ c \end{pmatrix} = \begin{pmatrix} -1 \\ -36 \\ -25 \end{pmatrix}$, $Ax = b$

whose augmented matrix is: $A|b = \left(\begin{array}{ccc|c} 2 & 0 & 1 & -1 \\ 0 & -12 & 1 & -36 \\ 6 & 8 & 1 & -25 \end{array}\right)$

Reduce this augmented matrix in an echelon form to obtain:

$$R\left(\begin{array}{ccc|c} 2 & 0 & 1 & -1 \\ 0 & -12 & 1 & -36 \\ 0 & 8 & -2 & -22 \end{array}\right), -3R_{31} = R_3 + (-3)R_1$$

$$R\left(\begin{array}{ccc|c} 2 & 0 & 1 & -1 \\ 0 & -12 & 1 & -36 \\ 0 & 0 & \frac{-4}{3} & -46 \end{array}\right), \frac{2}{3}R_{32} = R_3 + \frac{2}{3}R_2$$

$$\left.\begin{aligned} 2g + c &= -1 \\ -12f + c &= -36 \\ \left(\frac{-4}{3}\right)c &= -46 \end{aligned}\right\} \quad \text{(iii)}$$

Third equation of the system (iii) is giving $c = \frac{69}{2}$ which is used in second and first equations to obtain the values of $f = \frac{47}{8}$ and $g = \frac{-71}{4}$.

The values of $g = \frac{-71}{4}, f = \frac{47}{8}$ and $c = \frac{69}{2}$ are used in equation (iii) to obtain the required circle equation:

$$x^2 + y^2 + 2\left(\frac{-71}{4}\right)x + 2\left(\frac{47}{8}\right)y + \frac{69}{2} = 0$$

$$x^2 + y^2 - \frac{71}{2}x + \frac{47}{4}y + \frac{69}{2} = 0$$

$$4x^2 + 4y^2 - 142x + 47y + 138 = 0$$

B. Equation of circle passing through two points and having its centre on a given line

Consider the general equation a circle is given by

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

If the given circle is passing through two points, say $A(x_1, y_1)$ and $B(x_2, y_2)$, then these points must satisfy the general equation of a circle. Now put these two points in the given equation of a circle, i.e.:

$$x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = 0 \quad \text{(i)}$$

$$x_2^2 + y_2^2 + 2gx_2 + 2fy_2 + c = 0 \quad \text{(ii)}$$

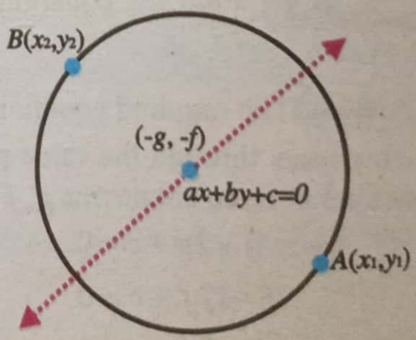


Figure 8.4

Also, the given straight line $ax + by + c_1 = 0$ passes through the center $(-g, -f)$ of the circle.

Example 4 Find the equation of a circle which passes through the points A(3,1) and B(2,2) having its center on the line $x+y-3=0$.

Solution: The required equation of a circle is $x^2+y^2+2gx+2fy+c=0$ (i)

which passes through the two points A(3,1) and B(2,2) that gives a system of two linear equations

$$\left.\begin{array}{lll} 10+6g+2f+c=0 & \Rightarrow & 6g+2f+c=-10 \\ 8+4g+4f+c=0 & \Rightarrow & 4g+4f+c=-8 \end{array}\right\} \quad \text{(ii)}$$

If the center $(-g,-f)$ of the circle lies on the line $x+y-3=0$, then the line $x+y-3=0$ becomes:

$-g-f-3=0 \Rightarrow g+f=-3$ (iii)

The combination of equations (ii) and (iii) is giving the system of three linear equations in three unknown g, f and c

$$\begin{pmatrix} 6 & 2 & 1 \\ 4 & 4 & 1 \\ 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} g \\ f \\ c \end{pmatrix}=\begin{pmatrix} -10 \\ -8 \\ -3 \end{pmatrix}, \qquad Ax=b$$

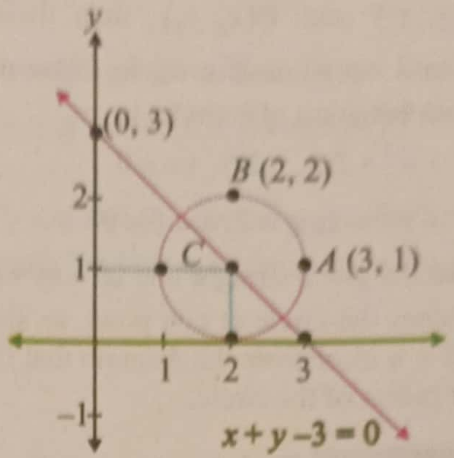


Figure 8.5

whose augmented matrix is: $A|b=\left(\begin{array}{ccc|c} 6 & 2 & 1 & -10 \\ 4 & 4 & 1 & -8 \\ 1 & 1 & 0 & -3 \end{array}\right)$

Reduce this augmented matrix in an echelon form to obtain the unknowns g, f and c:

$$R\left(\begin{array}{ccc|c} 6 & 2 & 1 & -10 \\ 0 & \frac{8}{3} & \frac{1}{3} & \frac{-4}{3} \\ 0 & \frac{2}{3} & \frac{-1}{6} & \frac{-4}{3} \end{array}\right) by\ R_2-\frac{2}{3}R_1,\ \ R_3-\frac{1}{6}R_1$$

$$R\left(\begin{array}{ccc|c} 6 & 2 & 1 & -10 \\ 0 & \frac{8}{3} & \frac{1}{3} & \frac{-4}{3} \\ 0 & 0 & \frac{-1}{4} & -1 \end{array}\right) by\left(R_3-\frac{1}{4}R_2\right)$$

$$\begin{aligned} 6g+2f+c&=-10 \\ \left(\frac{8}{3}\right)f+\left(\frac{1}{3}\right)c&=-\frac{4}{3} \\ \left(\frac{-1}{4}\right)c&=-1 \end{aligned} \quad \text{(iv)}$$

Third equation of the system (iv) is giving $c=4$ which is used in second and first equations to obtain the values of $f=-1$ and $g=-2$.

The values of $g=-2$, $f=-1$ and $c=4$ are used in equation (i) to obtain the required circle equation: $x^2+y^2-4x-2y+4=0$

C. The equation of a circle passing through two points and equation of tangent at one of these points is known

Consider the general equation a circle is given by

$$x^2+y^2+2gx+2fy+c=0 \qquad \text{(i)}$$

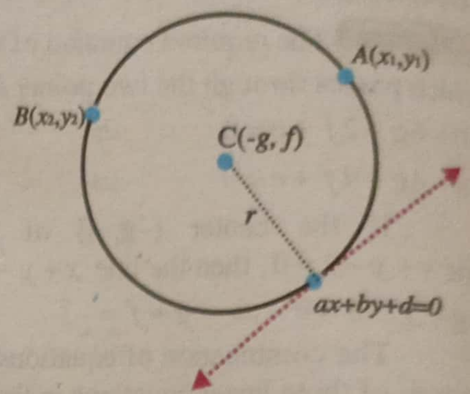


Figure 8.6

If the given circle is passing through two points, say $A(x_1,y_1)$ and $B(x_2,y_2)$, then these points must satisfy the general equation of a circle. Now put these two points in the given equation of a circle, i.e.:

$$x_1^2+y_1^2+2gx_1+2fy_1+c=0 \qquad \text{(ii)}$$

$$x_2^2+y_2^2+2gx_2+2fy_2+c=0 \qquad \text{(iii)}$$

Also the given straight line $ax+by+d=0$ (iv) touches the circle at one point, as shown in the given diagram, and it is clear from the diagram that the distance of a given point from the center $(-g,-f)$ must be equal to the radius of the circle.

Example 5 Find the equation of a circle which passes through the two points A(0,–1) and B(3,–3) and $3x-2y-2=0$ is the tangent line on the circle at a point A(0,–1).

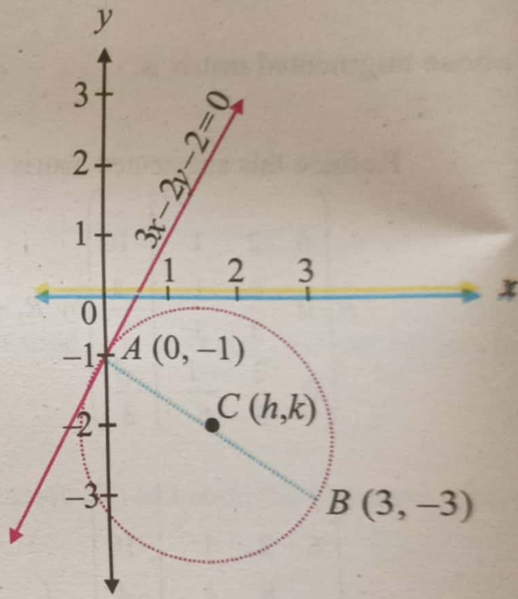


Figure 8.7

Solution Let C (h, k) be the center of the required circle. If A(0,–1) and B(3,–3) are the two points lie on the circle, then the square of the distance from C to A equals the square of the distance from C to B:

$$|CA|^2=|CB|^2, CA=(h-0,k+1), CB=(h-3,k+3)$$

$$(h-0)^2+(k+1)^2=(h-3)^2+(k+3)^2$$

$$h^2+k^2+2k+1=h^2-6h+9+k^2+9+6k$$

$$6h-4k-17=0 \qquad \text{(i)}$$

The slope of CA is

$$m_1=\frac{-1-k}{0-h}=\frac{k+1}{h}$$

and the slope of the tangent line $3x-2y-2=0$ is

$$3x-2y-2=0$$

$$-2y=-3x+2 \Rightarrow y=\frac{3}{2}x-1, m_2=\frac{3}{2}$$

If CA is perpendicular to the tangent line $3x-2y-2=0$, then the product of their slopes equals to (–1):

i.e. $m_1 m_2=-1$

$$\left(\frac{k+1}{h}\right)\left(\frac{3}{2}\right)=-1 \qquad \text{(ii)}$$

$$3k+3=-2h \Rightarrow 2h+3k+3=0$$

The equations (i) and (ii) are solved to obtain the values of $k=-2$ and $h=\frac{3}{2}$.

The required circle with center $(h, k)=\left(\frac{3}{2},-2\right)$ and radius

$$r=|CA|=|CB|=\sqrt{4^2+(k+1)^2}=\sqrt{\left(\frac{9}{4}\right)+1}=\frac{\sqrt{13}}{2}\text{ is:}$$

$$(x-h)^2+(y-k)^2=r^2$$

$$\left(x-\frac{3}{2}\right)^2+(y+2)^2=\frac{13}{4}\Rightarrow x^2+y^2-3x+4y+3=0$$

D. The equation of a circle passing through two points and touching a given line

Example 6 Find the equation of a circle which passes through the two points $A(0,0)$ and $B(4,0)$ and is touching a line $3x+4y+4=0$.

Solution Let $C(h, k)$ be the center of the required circle. If $A(0,0)$ and $B(4,0)$ are the two points lie on the circle, then the radius of the circle from center C to point A equals the radius of the circle from the center C to point B:

$$|\mathrm{CA}|^2=|\mathrm{CB}|^2, \mathrm{CA}=(0-4,0-1), \mathrm{CB}=(4-h,0-k)$$

$$\left(\sqrt{h^2+k^2}\right)^2=\left(\sqrt{(4-h)^2+k^2}\right)^2$$

$$h^2+k^2=16+h^2-8h+k^2\Rightarrow 8h=16\Rightarrow h=2$$

The radius of the required circle is $r=|\mathrm{CA}|=\sqrt{4+k^2}$ and the center is $\mathrm{C}(2, k)$.

For the values of k, the perpendicular distance from the center $(2, k)$ on the line $3x+4y+4=0$ equals the radius of the circle:

$$\frac{3(2)+4(k)+4}{\sqrt{9+16}}=\sqrt{4+k^2}$$

$$\frac{4k+10}{5}=\sqrt{4+k^2}$$

$$4k+10=5\sqrt{4+k^2},\ \textit{squaring both sides}$$

$$16k^2+100+80k=25\left(4+k^2\right)$$

$$-9k^2+80k=0\Rightarrow -k(9k-80)=0\Rightarrow k=0, k=\frac{80}{9}$$

The coordinates of the center are $(2,0)$ and $\left(2, \frac{80}{9}\right)$ and the radii are $r=\sqrt{4+0}=2$ and $r=\sqrt{4+\frac{80^2}{9^2}}=\sqrt{\frac{6724}{81}}=\frac{82}{9}$.

The equations of the circles with the above centers and radii are the following:

$$(x-2)^2+y^2=4,$$

$$(x-2)^2+\left(y-\frac{80}{9}\right)^2=\frac{6724}{81}$$

Exercise 8.1

1. In each case, find an equation of a circle, when the center and radius are the following:
   a. $(0,0)$, $r=4$
   b. $(3,2)$, $r=1$
   c. $(-4,-3)$, $r=4$
   d. $(-a,-b)$, $r=a+b$
2. In each case, determine the equation of a circle using the given information:
   a. $C(0,0)$, tangent to the line $x=-5$
   b. $C(0,0)$, tangent to the line $y=6$
   c. $C(6,-6)$, circumference passes through the origin.
   d. $C(-9,-6)$, circumference passes through the point $(-20,8)$.
   e. $C(-5,4)$, tangent to the x-axis.
   f. $C(5,3)$, tangent to the y-axis.
3. In each case, find the center $C(-g,-f)$ and radius $r=\sqrt{g^2+f^2-c}$ of the following:
   a. $x^2+y^2-8x-6y+9=0$
   b. $4x^2+4y^2+16x-12y-7=0$
   c. $x^2+y^2+4x-6y+13=0$
   d. $x^2+y^2-x-8y+18=0$
4. In each case, find an equation of a circle which passes through the three points:
   a. $(-3,0),(5,4),(6,-3)$
   b. $(7,-1),(5,3),(-4,6)$
   c. $(1,2),(3,-4),(5,-6)$
5. In each case, find an equation of a circle which
   a. contains the point $(2,6),(6,4)$ and has its center on the line $3x+2y-1=0$.
   b. contains the point $(4,1),(6,5)$ and has its center on the line $4x+y-16=0$.
6. Find an equation of a circle which passes through the points
   a. $(0,0),(0,3)$ and the line $4x-5y=0$ is tangent to it at $(0,0)$.
   b. $(0,-1),(3,0)$ and the line $3x+y=9$ is tangent to it at $(3,0)$.
7. Find an equation of a circle that is concentric to circle
   a. $2x^2+2y^2+16x-7y=0$ and is tangent to the *y-axis*.
   b. $x^2+y^2-8x+4=0$ and is tangent to the line $x+2y+6=0$.
   c. $x^2+y^2+6x-10y+33=0$ and is touching the *x-axis*.
8. Find equation of circle which passes through origin and whose intercepts are on the coordinate axis are:
   a. 3 and 4
   b. 2 and 4

8.3 Tangents and Normal

If a secant PQ of a circle is moved upward about one of its points of intersection P, then the second point of intersection Q is moving gradually along the curve that tends to coincide with P. The limiting position PT of PQ is then called the **tangent** to the circle at the point P.

The point of the circle at which a tangent meets the circle is called **point of contact** (Say P) of the tangent.

The **normal** at a contact point P to a circle (or conic) is the straight line PR perpendicular to the tangent PT to the circle (or conic) at that point P.

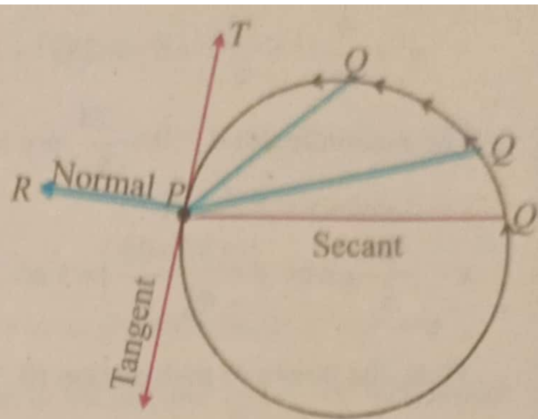


Figure 8.8

8.3.1 The condition when a line intersect the circle

The circle and the line are $\quad x^2+y^2=a^2 \qquad$ (i)

$y=mx+c \qquad$ (ii)

which develops a system of two nonlinear equations:

$$x^2+y^2=a^2$$
$$y=mx+c \qquad \text{(iii)}$$

The solution set $\{(x, y)\}$ of the nonlinear system of equations (iii) exists only, if the curves of the system (iii) are intersecting. That set of points of intersection $\{(x, y)\}$ is the solution set, can be found by solving the nonlinear system (iii) simultaneously.

The line (ii) is used in a circle (i) to obtain the quadratic equation in x:

$$x^2+(mx+c)^2=a^2$$
$$x^2(1+m^2)+2mcx+(c^2-a^2)=0 \qquad \text{(iv)}$$

The equation (iv) being a quadratic equation in x, gives a set of two values x_1 and x_2 of x which will be used in a line (ii) to obtain a set of two y values y_1 and y_2.

The solution set $\{(x_1,y_1),(x_2,y_2)\}$ of the system (iii) is of course a set of points of intersection of the line and circle.

The points of intersection of the system (iii) are real, coincident or imaginary, according as the roots of the quadratic equation (iv) are real, coincident or imaginary or according as the discriminate of the quadratic equation (iv):

$$disc=4m^2c^2-4\left[(1+m^2)(c^2-a^2)\right]>0, \qquad \text{real and distant}$$
$$disc=4m^2c^2-4\left[(1+m^2)(c^2-a^2)\right]=0, \qquad \text{coincident}$$
$$disc=4m^2c^2-4\left[(1+m^2)(c^2-a^2)\right]<0, \qquad \text{imaginary}$$

Example 7 Find the points of intersection of the line $3x-4y+20=0$ and the circle $x^2+y^2=25$.

Solution The equations of the line and circle are: $3x-4y+20=0 \Rightarrow y=\frac{3}{4}x+5 \qquad$ (i)

$$x^2+y^2=25 \qquad \text{(ii)}$$

The line (i) is used in a circle (ii) to obtain the x-coordinates of the points of intersection:

$$x^2+\left(\frac{3}{4}x+5\right)^2=25$$
$$x^2+\frac{9}{16}x^2+25+\frac{30}{4}x=25$$

$$x^2+\frac{9}{16}x^2+\frac{30}{4}x=0 \Rightarrow 25x^2+120x=0 \Rightarrow x=0,\frac{-24}{5}$$

The x-coordinates $x=0,\ \frac{-24}{5}$ are used in the line (i) to obtain the y-coordinates:

$x=0$ gives $y=5$

$x=\frac{-24}{5}$ gives $y=\frac{3}{4}\left(\frac{-24}{5}\right)+5=\frac{-72+100}{20}=\frac{28}{20}=\frac{7}{5}$

Thus, the points of intersection (0, 5) and $\left(\frac{-24}{5},\frac{7}{5}\right)$ are real and distant.

8.3.2 Condition when a line touches the circle $x^2+y^2=a^2$

To determine the position of a line with respect to the circle, we need to find its distance from centre of a circle and compare it with radius then:

i. If distance is less than the radius, the line will intersect at two points.
ii. If the distance is equal to the radius, then the line will touch the circle.
iii. If the distance is greater than the radius, the line will completely outside the circle.

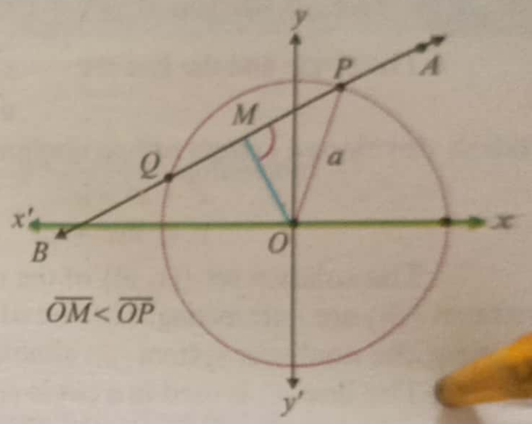


Figure 8.9

Let AB be the straight line $y=mx+c$ that intersects the circle $x^2+y^2=a^2$ at points P and Q respectively.

Join $\overline{OP}$ and put it by $\overline{OP}$ = a, which is the radius of a given circle. Draw $\overline{OM}$ perpendicular on $\overline{PQ}$. If $\overline{OM}$ is perpendicular to $\overline{PQ}$, then, the perpendicular distance $\overline{OM}$ from O(0,0) on a secant line $mx-y+c=0$ (line $\overline{PQ}$) is: $\overline{OM}=\frac{|m(0)-(0)+c|}{\sqrt{m^2+1}}=\frac{c}{\sqrt{1+m^2}}$

From the right-angled triangle OMP, it is known that:

$$|\overline{OP}|^2=|\overline{OM}|^2+|\overline{MP}|^2$$

$$|\overline{MP}|^2=|\overline{OP}|^2-|\overline{OM}|^2$$

$$=a^2-\frac{c^2}{1+m^2}=\frac{a^2(1+m^2)-c^2}{1+m^2}$$

$$|\overline{MP}|=\sqrt{\frac{a^2(1+m^2)-c^2}{1+m^2}}$$

The secant line $\overline{PQ}$ is 2 times of $\overline{OM}$, and the length of the intercept $\overline{PQ}$ is therefore:

$$|\overline{PQ}|=2|\overline{OM}|=2\sqrt{\frac{a^2(1+m^2)-c^2}{1+m^2}} \qquad \text{(i)}$$

Condition of Tangency: The line $y=mx+c$ touches the circle $x^2+y^2=a^2$, if the length of the intercept $\overline{PQ}$ is zero: $2\sqrt{\frac{a^2(1+m^2)-c^2}{1+m^2}}=0$

$$2\frac{\sqrt{a^2(1+m^2)-c^2}}{\sqrt{1+m^2}}=0$$

$\sqrt{a^2(1+m^2)-c^2}=0$, squaring both sides

$$a^2(1+m^2)-c^2=0 \Rightarrow c^2=a^2(1+m^2) \Rightarrow c=\pm a\sqrt{1+m^2} \qquad \text{(ii)}$$

The equation (ii) is the required condition at which the line $y=mx+c$ touches the circle $x^2+y^2=a^2$.

Example 8 Find the length of the chord joining the points P and Q on the line $\frac{x}{a}+\frac{y}{b}=1$ which cuts the circle $x^2+y^2=r^2$. Show that if the line touches the circle, then $a^{-2}+b^{-2}=r^{-2}$.

Solution The slope of a given line $\frac{x}{a}+\frac{y}{b}=1$

$$\frac{y}{b}=-\frac{x}{a}+1$$

$$y=-\frac{b}{a}x+b,\ m=-\frac{b}{a}$$

If PQ is the chord of a circle $x^2+y^2=r^2$, and PQ is 2 times of MP, then the length of the chord PQ through result (i) is:

$$|\overline{PQ}|=2|\overline{OM}|=2\sqrt{\frac{a^2(1+m^2)-c^2}{1+m^2}}$$

$$=2\sqrt{\frac{r^2\left[1+\left(\frac{-b}{a}\right)^2\right]-b^2}{1+\left(\frac{-b}{a}\right)^2}}, \qquad a^2=r^2,\ m=\frac{-b}{a},\ c=b$$

If the given line touches the circle $x^2+y^2=r^2$, then, the length of the chord $\overline{PQ}$ is going to be zero:

$$2\sqrt{\frac{r^2\left[1+\left(\frac{-b}{a}\right)^2\right]-b^2}{1+\left(\frac{-b}{a}\right)^2}}=0$$

$$4\frac{r^2\left[1+\left(\frac{-b}{a}\right)^2\right]-b^2}{1+\left(\frac{-b}{a}\right)^2}=0, \qquad \text{squaring both sides}$$

$$r^2\left[1+\left(\frac{-b}{a}\right)^2\right]-b^2=0$$

$$b^2=r^2\left[1+\frac{b^2}{a^2}\right]$$

$$b^2 = r^2\frac{a^2+b^2}{a^2} \Rightarrow \frac{a^2b^2}{a^2+b^2} = r^2 \Rightarrow r^{-2} = \frac{a^2+b^2}{a^2b^2} = \frac{1}{a^2}+\frac{1}{b^2} = a^{-2}+b^{-2}$$

Example 9 Find the coordinates of the middle point of the chord which the circle $x^2+y^2+4x-2y-3=0$ cuts off on the line $x-y+2=0$.

Solution The center of the given circle is $C(-g,-f)=C(-2,1)$ and the line $x-y+2=0$ (line AB) intersects the circle at points P and Q and $M(x_1,y_1)$ is the middle point of the chord PQ. Join C and M that develops a line $\overline{CM}$ perpendicular to chord PQ.

If M lies on line AB, then, the line equation $x-y+2=0$ becomes:

$$x_1-y_1+2=0 \qquad \text{(i)}$$

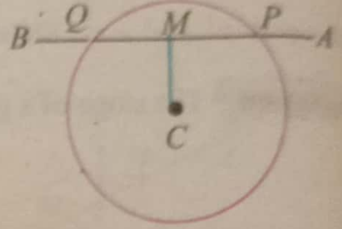


Figure 8.10

The slopes of the lines (i) and $\overline{CM}$ are respectively:

$$m_2 = 1, \textit{ coefficient of } x$$

$$m_1 = \frac{y_1-1}{x_1+2}, \textit{ slop of } \overline{CM}$$

If CM is perpendicular to $\overline{AB}$, then the product of their slopes equals -1:

$$\left(\frac{y_1-1}{x_1+2}\right)(1) = -1 \Rightarrow y_1-1 = -x_1-2 \Rightarrow x_1+y_1+1=0 \qquad \text{(ii)}$$

The equations (i) and (ii) are solved to obtain the coordinates of the middle point M:

$$\left.\begin{aligned} x_1-y_1+2=0 \\ x_1+y_1+1=0 \end{aligned}\right\} \qquad x_1 = \frac{-3}{2},\ y = \frac{1}{2}$$

Thus, the coordinates of the middle point is $M\left(-\frac{3}{2},\frac{1}{2}\right)$

8.3.3 The equation of a tangent to a circle in slope form

If m is the slope of the tangent line to the circle $x^2+y^2=a^2$ (i)
then the equation of that tangent line is of the form $y=mx+c$ (ii)

Here c is to be calculated from the fact that the line (ii) is tangent to the circle (i). The line (ii) is used in circle (i) to obtain the quadratic equation in x:

$$x^2+(mx+c)^2=a^2,\quad y=mx+c$$

$$x^2(1+m^2)+2mcx+(c^2-a^2)=0 \qquad \text{(iii)}$$

If the line (ii) touches the circle (i), then the quadratic equation (iii) has coincident roots for which the discriminant of the quadratic equation (iii) equals zero:

$$\text{Disc} = 0$$

$$4m^2c^2-4(1+m^2)(c^2-a^2)=0$$

$$4m^2c^2=4(1+m^2)(c^2-a^2)$$

$$m^2c^2=m^2c^2+c^2-a^2-a^2m^2$$

$$-c^2=-a^2(1+m^2) \Rightarrow c=\pm a\sqrt{1+m^2} \qquad \text{(iv)}$$

Equation (iv) is the **condition of tangency**. The value of c from equation (iv) is used in the line (ii) to obtain the required equation of the tangent:

$$y=mx+c=mx\pm a\sqrt{1+m^2} \qquad (v)$$

Remember

- The equation of any tangent to the circle $x^2+y^2=a^2$ in the slope form is: $y=mx\pm a\sqrt{1+m^2}$ (vi)
- The line $y=mx+c$ should touch the circle $x^2+y^2=a^2$ under condition: $c=\pm a\sqrt{1+m^2}$ (vii)
- The interpretation of result (v) is that the line $x+my+n=0$ should touch the circle $x^2+y^2=a^2$ under condition: $a^2(1^2+m^2)-n^2=0 \Rightarrow n=\pm a\sqrt{1^2+m^2}$ (viii)
- The interpretation of result (v) that the line $x+my+n=0$ should touch the circle $x^2+y^2+2gx+2fy+c=0$ under condition:
$$(c-f^2)^2+2fg\ m+(c-g^2)m^2-2n(g+fm)+n^2=0 \qquad (ix)$$
- Let $y=mx\pm a\sqrt{1+m^2}$ be a tangent to a circle (i) at a point (x_1, y_1), if the circle equation (i) is identical to $xx_1+yy_1=a^2$, then the coefficients of like terms of $y=mx\pm a\sqrt{1+m^2}$ and $xx_1+yy_1=a^2 \Rightarrow yy_1=-xx_1+a^2$ are compared to obtain the point of contact:
$$\frac{x_1}{-m}=\frac{y_1}{1}=\frac{a^2}{a\sqrt{1+m^2}}$$
$$\frac{x_1}{-m}=\frac{a}{\sqrt{1+m^2}} \Rightarrow x_1=\frac{-am}{\sqrt{1+m^2}}$$
$$\frac{y_1}{1}=\frac{a}{\sqrt{1+m^2}} \Rightarrow y_1=\frac{a}{\sqrt{1+m^2}}$$
Thus, the point of contact is $(x_1,y_1)=\left(\frac{-am}{\sqrt{1+m^2}},\frac{a}{\sqrt{1+m^2}}\right)$ (x)

Example 10 For what value of c, the line $x+y+c=0$ will touch the circle $x^2+y^2=64$? Use that value of c to find the tangent that should touch the given circle. Find also the contact point.

Solution The slope of the line $x+y+c=0$ is $m=-1$. The value of c at which the line $x+y+c=0$ will touch the given circle $x^2+y^2=64$ is: $c=\pm a\sqrt{1+m^2}$, result (vii)

$$=\pm 8\sqrt{1+(-1)^2}=8\sqrt{2}, \quad a=8, \quad m=-1$$

The required tangent line that should touch the given circle is:

$$y=mx\pm a\sqrt{1+m^2}=-x\pm 8\sqrt{2}, \text{ result (vi)}$$

The point of contact through result (x) is: $(x_1,y_1)=\left(\frac{-am}{\sqrt{1+m^2}},\frac{a}{\sqrt{1+m^2}}\right)=\left(\frac{8}{\sqrt{2}},\frac{8}{\sqrt{2}}\right)$

8.3.4 The equations of tangent and normal to a circle at a point

The equation of a circle is: $x^2+y^2+2gx+2fy+c=0$ (i)

If $A(x_1, y_1)$ is a point lying on the circle (i), then the circle (i) becomes:

$x_1^2+y_1^2+2gx_1+2fy_1+c=0$ (ii)

If r_1 and r_2 are the position vectors of A and the center $C(-g,-f)$ of the circle relative to origin $r_1=(x_1,y_1)=x_1 i+y_1 j,\ r_2=(-g,-f)=-gi-fj$ then, from the Figure 8.11:

$$OC+CA=OA$$

$$CA=OA-OC=r_1-r_2=(x_1+g,\ y_1+f)=(x_1+g)i+(y_1+f)j$$

Let $P(x,y)$ be any point on the tangent line AT, whose position vector is $OP=(x,y)$ which gives:

$$OA+AP=OP$$

$$\begin{aligned} AP &= OP-OA \\ &= r-r_1 \\ &= (x,y)-(x_1,y_1) \\ &= (x-x_1,y-y_1)=(x-x_1)i+(y-y_1)j \end{aligned}$$

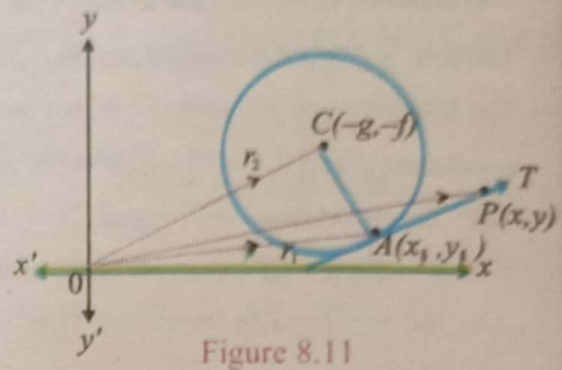


Figure 8.11

The equation of tangent to the circle (i) is obtained if AP is perpendicular to CA for which the dot product in between the vectors AP and AC equals zero:

$$AP.CA=0$$

$$(x_1+g,y_1+f).(x-x_1,y-y_1)=0$$

$$(x_1+g)(x-x_1)+(y_1+f)(y-y_1)=0$$

$$xx_1+yy_1+gx+fy-(x_1^2+y_1^2+gx_1+fy_1)=0$$

$$xx_1+yy_1+gx+fy=x_1^2+y_1^2+gx_1+fy_1$$

$$=-gx_1-fy_1-c \qquad \text{result} \qquad \text{(ii)}$$

$$xx_1+yy_1+g(x+x_1)+f(y+y_1)+c=0 \qquad \text{(iii)}$$

The tangent equation to the circle $x^2+y^2=a^2$ at a point $A(x_1,y_1)$ through result (iii) is:

$$xx_1+yy_1=a^2 \qquad \text{(iv)}$$

The procedure for the normal equation at a point (x_1,y_1) on the circle $x^2+y^2+2gx+2fy+c=0$ is as under:

If $C(-g,-f)$ is the center of the circle and $A(x_1,y_1)$ is a contact point, then the slope $\frac{y_1+f}{x_1+g}$ of the required normal line develops the normal line CA at $A(x_1,y_1)$:

$$y-y_1=\frac{y_1+f}{x_1+g}(x-x_1)$$

$$(y-y_1)(g+x_1)=(f+y_1)(x-x_1)$$

$$x(y_1+f)-y(x_1+g)+(gy_1-fx_1)=0 \qquad \text{(v)}$$

The normal equation to the circle $x^2+y^2=a^2$ at a point $A(x_1,y_1)$ through result (v) is:

$$xy_1-yx_1=0 \qquad \text{(vi)}$$

Example 11 Find the equations of the tangent and normal to the circle $x^2+y^2=25$ at a point (3, 4).

Solution Result (iv) is used to obtain the tangent equation to the given circle:

$$xx_1+yy_1=a^2$$
$$3x+4y=25,\quad a^2=25,\ (x_1,y_1)=(3,4)$$

Result (vi) is used to obtain the normal equation to the given circle:

$$xy_1-yx_1=0$$
$$4x-3y=0,\quad a^2=0,\ (x_1,y_1)=(3,4)$$

Example 12 Find the equations of the tangent and normal to the circle $x^2+y^2-2x+4y+3=25$ at a point (2,– 3).

Solution Result (iii) is used to obtain the tangent line to the given circle:

$$xx_1+yy_1+g(x+x_1)+f(y+y_1)+c=0$$
$$2x-3y+(-1)(x+2)+(2)(y-3)+3=0 \qquad 2g=-2,\ 2f=4,\ c=3$$
$$2x-3y-x-2+2y-6+3=0,\ (x_1,y_1)=(2,-3)$$
$$x-y-5=0$$

Result (v) is used to obtain the normal line to the given circle:

$$x(y_1+f)-y(x_1+g)+(gy_1-fx_1)=0$$
$$x(-3+2)-y(2-1)+(3-4)=0,\ 2g=-2,\quad 2f=4,\quad c=3$$
$$-x-y-1=0$$
$$x+y+1=0$$

8.3.5 Length of a tangents to a circle from a given external point

The procedure for finding the length of the tangent drawn from the external point $P(x_1,y_1)$ to the circle $x^2+y^2+2gx+2fy+c=0$ is as under:

Let $P(x_1,y_1)$ be the given external point and PT be one of the two tangents drawn from point P to the circle

$$x^2+y^2+2gx+2fy+c=0 \qquad \text{(i)}$$

Join $\overline{CP}$ and $\overline{CT}$. C($-g$, $-f$) is the center of the circle (i) and $\overline{CT}=\sqrt{g^2+f^2-c}$ is the radius of the circle (i).

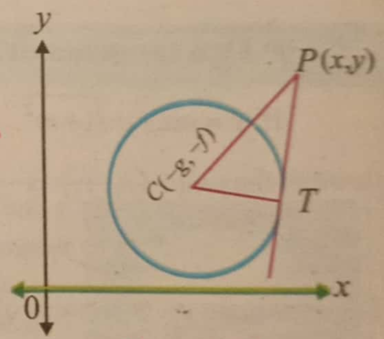


Figure 8.12

From the right-angled triangle PTC, the length of the tangent PT drawn from point P to the given circle is:

$$\left|\overline{PC}\right|^2=\left|\overline{CT}\right|^2+\left|\overline{PT}\right|^2$$
$$\left|\overline{PT}\right|^2=\left|\overline{PC}\right|^2-\left|\overline{CT}\right|^2$$
$$=(x_1+g)^2+(y_1+f)^2-(g^2+f^2-c)$$
$$=x_1^2+y_1^2+2gx_1+2fy_1+c$$
$$\left|\overline{PT}\right|=\sqrt{x_1^2+y_1^2+2gx_1+2fy_1+c} \qquad \text{(ii)}$$

Note

- The length of the tangent drawn from the point $P(x_1, y_1)$ to the circle $x^2+y^2=a^2$ is:

  $|\overline{PT}| = \sqrt{x_1^2+y_1^2-a^2}$ (iii)

- The lengths of the two tangents drawn from the point $P(x_1, y_1)$ on the given circle are equal.

Example 13 Find the length of the tangent drawn from the point P(3,4) on the circles

(a). $x^2+y^2=9$

(b). $x^2+y^2-2x-y-3=0$

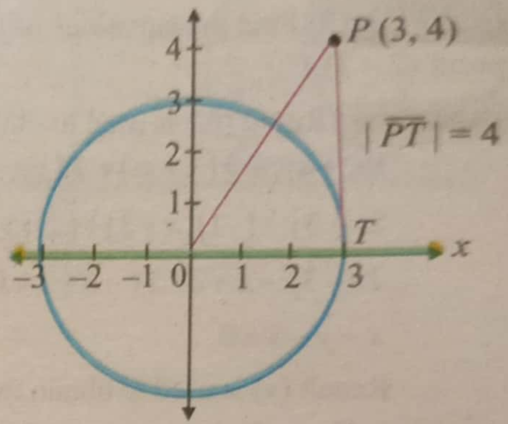


Figure 8.13

Solution

a. If $|\overline{PT}|$ is the tangent drawn from the point P(3,4) on the given circle, then, the length of the tangent PT on the given circle through result (iii) is:

$$|\overline{PT}| = \sqrt{x_1^2+y_1^2-9}$$

$$=\sqrt{9+16-9}=\sqrt{16}=4 \qquad (x_1, y_1)=(3,4),\ a^2=9$$

b. If PT is the tangent drawn from the point P(3,4) on the given circle, then, the length of the tangent PT on the given circle through result (ii) is:

$$|PT| = \sqrt{x_1^2+y_1^2+2gx_1+2f\,y_1+c}, \qquad 2g=-2,\ 2f=-1,\ c=-3, (x_1, y_1)=(3,4)$$

$$=\sqrt{9+16-2(3)-(4)-3}=\sqrt{12}$$

8.3.6 Two tangents drawn to a circle from an external point are equal in length

If $y=mx\pm a\sqrt{1+m^2}$ is any tangent to the circle $x^2+y^2=a^2$, then the tangent line that passes through the point (x_1, y_1) is $y_1=mx_1+a\sqrt{1+m^2}$

$$y_1-mx_1=a\sqrt{1+m^2} \qquad \text{(i)}$$

Taking square on both sides of equation (i) $(y_1-mx_1)^2=a^2(1+m^2)$

that gives the quadratic equation in m: $m^2(x_1^2-a^2)-2mx_1y_1+(y_1^2-a^2)=0$ (ii)

This quadratic equation (ii) gives two values of m that two values of m represent the slopes of the required two tangents on the given circle.

The tangents are real and different, real and coincident or imaginary or according as the discriminant of the quadratic equation (ii):

$disc=4x_1^2y_1^2-4(x_1^2-a^2)(y_1^2-a^2)>0,$ *real and different*

$disc=4x_1^2y_1^2-4(x_1^2-a^2)(y_1^2-a^2)=0,$ *real and coincident*

$disc=4x_1^2y_1^2-4(x_1^2-a^2)(y_1^2-a^2)<0,$ *imaginary*

or according as

$$x_1^2 + y_1^2 - a^2 > 0, \quad \textit{real and different}$$
$$x_1^2 + y_1^2 - a^2 = 0, \quad \textit{real and coincident}$$
$$x_1^2 + y_1^2 - a^2 < 0, \quad \textit{imaginary}$$

or according as the point $P(x_1, y_1)$ lies outside, on, or inside the circle $x^2 + y^2 = a^2$.

In general, two tangent can also be drawn from the point $P(x_1, y_1)$ to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$.

Example 14 Find the equations of the tangents drawn from the point (6,4) to the circle $x^2 + y^2 = 16$.

Solution If $y = mx \pm c = mx \pm a\sqrt{1+m^2}$ is any tangent to the circle $x^2 + y^2 = 16$, then the number of tangents through result (ii)

$$m^2\left(x_1^2 - a^2\right) - 2mx_1y_1 + \left(y_1^2 - a^2\right) = 0$$

$$m^2(36-16) - 2m(6)(4) + (16-16) = 0, \ (x_1, y_1) = (6,4), \ a^2 = 16$$

$$20m^2 - 48m = 0$$

$$m(20m - 48) = 0 \Rightarrow m = 0, \frac{12}{5}$$

can be found by putting m = 0 and $m = \frac{12}{5}$ in $y = mx \pm a\sqrt{1+m^2}$:

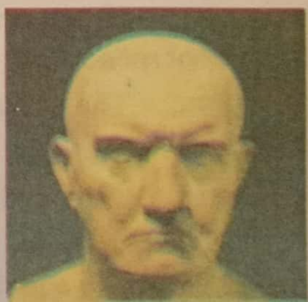
a. $y = mx \pm a\sqrt{1+m^2} = (0)x \pm 4\left(\sqrt{1+0}\right) = \pm 4 = 4, \ a = 4, \ m = 0$

b. $y = \left(\frac{12}{5}\right)x \pm 4\left(\sqrt{1+\frac{144}{25}}\right)$

$$= \frac{12}{5}x \pm 4\left(\frac{13}{5}\right) = \frac{12}{5}x \pm \frac{52}{5} = \frac{12}{5}x - \frac{52}{5}, \textit{ choose negative sign}$$

History

Menaechmus was a Greek mathematician. He was teacher or Alexander the Great and a friend of Ploto. He was the first person who introduced the conic section and investigate ellipse, parabola and hyperbola. He also gave the solution to the problem of doubling the cube. He introduced parabola as $y^2 = Lx$ where 'L' is a constant called the latus rectum although he was not acute of the fact that any equation in two unknowns determines a curve. He deliberately derived these properties of conic section and other properties also. By using these information it has not possible to find a solution to the problem of the duplication of the cube by solving for the point at which two parabolas intersect. Menaechmus's work an conic section is known as primary work for conic section.



Menaechmus
(380BC)-(320BC)

Exercise 8.2

1. In each case, find the tangent and normal equations
   a. at a point (1,2) to the circle $x^2+y^2=5$. b. at a point $(-3,-2)$ to the circle $x^2+y^2=13$.
   c. at a point (4,1) to the circle $x^2+y^2-4x+2y-3=0$.
2. In each case, find the tangent and normal equations
   a. at a point $(\cos 60°, \sin 60°)$ to the circle $36(x^2+y^2)=13$.
   b. at a point $(2\cos 45°, 2\sin 45°)$ to the circle $x^2+y^2=4$.
   c. at a point $(\cos 30°, \sin 30°)$ to the circle $x^2+y^2=1$.
3. For what value of n,
   a. the line $lx+my+n=0$ touches the circle $x^2+y^2=a^2$?
   b. the line $x+y+n=0$ touches the circle $x^2+y^2=9$?
   c. the line $2x+2y+n=0$ touches the circle $x^2+y^2=81$?
4. For what value of c
   a. the line $y=mx+c$ touches the circle $x^2+y^2=a^2$?
   b. the line $y=-x+c$ touches the circle $x^2+y^2=9$?
   c. the line $y=-x-\left(\frac{c}{2}\right)$ touches the circle $x^2+y^2=81$?
5. Find the condition at which the line $lx+my+n=0$ touches the circle $x^2+y^2+2gx+2fy+c=0$.
6. For what value of n,
   a. the line $3x+4y+n=0$ touches the circle $x^2+y^2-4x-6y-12=0$?
   b. the line $x-2y+n=0$ touches the circle $x^2+y^2+3x+6y-5=0$?
   c. the line $2x+y+n=0$ touches the circle $x^2+y^2-2x-10y+21=0$?
7. If the tangent length from the point P to the circle $x^2+y^2=a^2$ is equal to the perpendicular distance from P to the line $lx+my+n=0$, then find out the locus of P.
8. Find the locus of the point P,
   a. If the length of the tangent line from the point P to the circle $x^2+y^2=9$ is equal to the perpendicular distance form P to the line $3x+4y+3=0$.
   b. If the length of the tangent line from the point P to the circle $x^2+y^2=25$ is equal to the perpendicular distance form P to the line $4x+3y+3=0$.
9. a. The length of the tangent from (f,g) to the circle $x^2+y^2=6$ is twice the length of the tangent to the circle $x^2+y^2+3x+3y=0$. Prove that $f^2+g^2+4f+4g+2=0$.
   b. the length of the tangent from (f,g) to the circle $x^2+y^2=4$ is 4 times the length of the tangent to the circle $x^2+y^2+2x+2y=0$. Prove that $15f^2+15g^2+32f+32g+4=0$.
10. Find the equations of the tangents to the
    a. circle $x^2+y^2=4$ which are parallel to the straight line $x+2y+3=0$.
    b. circle $x^2+y^2=25$ which are parallel to the straight line $3x+4y+3=0$.
11. Prove that the lines
    a. $x=8$ and $y=7$ touch the circle $x^2+y^2-6x-4y-12=0$. Find also the contact points.
    b. $x+y-1=0$ and $x-y+1=0$ touch the circle $x^2+y^2-4x-2y+3=0$. Find also the contact points.
12. Find the equations of the tangents
    a. to the circle $x^2+y^2=2$, which make an angle of 45° with the *x-axis*.
    b. to the circle $3x^2+3y^2=1$, which make an angle of 30° with the *x-axis*.
    c. to the circle $x^2+y^2=4$, which make an angle of 60° with the *x-axis*.

8.4 Properties of circle

There are some properties of a circle that are listed as under.

8.4.1 Perpendicular from the center of a circle on a chord bisects the chord

Let the circle be $x^2+y^2+2gx+2fy+c=0$ (i)

and PQ be any chord of a circle, whose end points are $P(x_1,y_1)$ and $Q(x_2,y_2)$ respectively.

If PQ is a chord of the circle, then P and Q are the points lying on the circle: $x_1^2+y_1^2+2gx_1+2fy_1+c=0$

$$x_2^2+y_2^2+2gx_2+2fy_2+c=0$$

The subtraction of these two circles equations gives the slope of the chord $\overline{PQ}$:

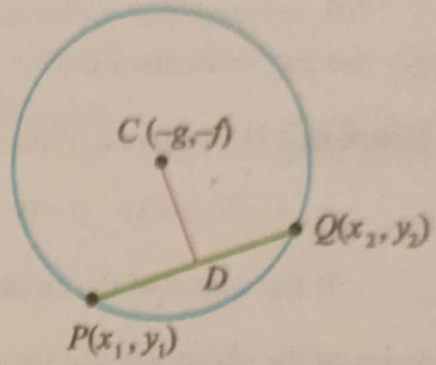


Figure 8.14

$$(x_2^2-x_1^2)+(y_2^2-y_1^2)+2g(x_2-x_1)+2f(y_2-y_1)=0$$

$$(x_2^2-x_1^2)+2g(x_2-x_1)+(y_2^2-y_1^2)+2f(y_2-y_1)=0$$

$$(x_2-x_1)(x_1+x_2+2g)+(y_2-y_1)(y_1+y_2+2f)=0$$

$$\frac{y_2-y_1}{x_2-x_2}=-\frac{x_1+x_2+2g}{y_1+y_2+2f}=m_1,\ say \qquad \text{(ii)}$$

If the center of the circle is C(– g,– f) and the midpoint of the chord $\overline{PQ}$ is $D\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2}\right)$, then the slope of the perpendicular line $\overline{CD}$ is:

$$m_2=\frac{\frac{y_1+y_2}{2}+f}{\frac{x_1+x_2}{2}+g}=\frac{y_1+y_2+2f}{x_1+x_2+2g} \qquad \text{(iii)}$$

From the Figure 8.14 the chord $\overline{PQ}$ and the line $\overline{CD}$ are perpendicular if and only if the product of their slopes equals –1:

$$m_1m_2=-\frac{x_1+x_2+2g}{y_1+y_2+2f}\cdot\frac{y_1+y_2+2f}{x_1+x_2+2g}=-1$$

Thus, $\overline{CD}$ is bisector of the chord $\overline{PQ}$.

Note

- the perpendicular bisector of any chord $\overline{PQ}$ of a circle passes through the center of the circle. This is our **second property**.
- the line joining the two points of the circle that touches the center of the circle is called the diameter of the circle. This diameter acts as the perpendicular bisector to the chord $\overline{PQ}$, if the diameter of a circle bisects the chord $\overline{PQ}$. This is our **third property**. The proof is similar to property first, but the graphical view is shown in the Figure 8.14.

Example 15 If $A(-3,4)$ and $B(1,5)$ are the end points of the chord $\overline{AB}$ of the circle $x^2+y^2+x-5y-2=0$, then show that

(a). the line from the center of the circle is perpendicular to $\overline{AB}$, also bisects the chord $\overline{AB}$.

(b). the line from the center of the circle to the midpoint of the chord $\overline{AB}$ is perpendicular to the chord $\overline{AB}$.

(c). the perpendicular bisector $\overline{CD}$ of the chord $\overline{AB}$ passes though the center of the given circle.

Solution The equation of the circle with center $C\left(\frac{-1}{2},\frac{5}{2}\right)$ is:

$$x^2+y^2+x-5y-2=0$$

If the center of the circle is $C\left(\frac{-1}{2},\frac{5}{2}\right)$ and the midpoint of the chord $\overline{AB}$ is $D\left(-1,\frac{9}{2}\right)$, then the slopes of the chord AB and the perpendicular line CD are respectively:

$$\text{slope of } \overline{AB}=m_1=\frac{5-4}{1+3}=\frac{1}{4}, \qquad \text{slope of } \overline{CD}=m_2=\frac{\frac{9}{2}-\frac{5}{2}}{-1+\frac{1}{2}}=-4$$

The chord $\overline{AB}$ and the line $\overline{CD}$ are perpendicular if and only if the product of their slopes equals -1:

$$m_1m_1=\frac{1}{4}.(-4)=-1$$

Therefore, $\overline{CD}$ is perpendicular bisector of the chord $\overline{AB}$. This result is automatically valid for parts *b* and *c*.

8.4.4 Congruent chords of a circle are equidistant from its center and its converse.

If the perpendicular distances form the center of a circle to its two chords are equal, then the chords are congruent.

Let the circle equation with center $C(-g,-f)$ is:

$$x^2+y^2+2gx+2fy+c=0 \qquad \text{(i)}$$

If $\overline{AB}$ and $\overline{DE}$ are the two chords of the circle (i), then the coordinates of the end points of the chord $\overline{AB}$ and $\overline{DE}$ are respectively:

$$A(x_1,y_1), D(x_2,y_2), B(x_3,y_3), E(x_4,y_4).$$

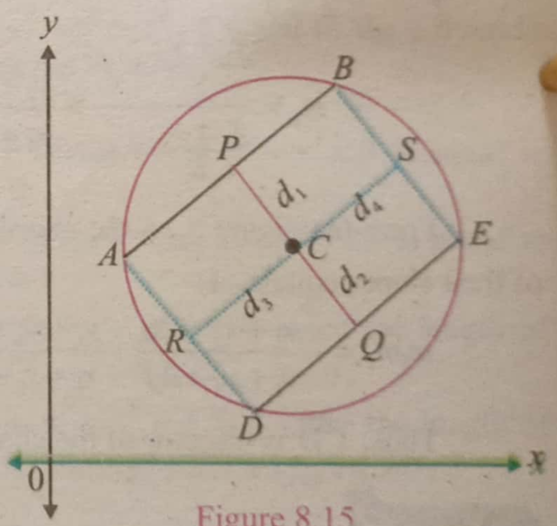


Figure 8.15

From the Figure 8.15, it is clear that the perpendicular distance $d_1=\overline{CP}$ from the center C on the chord $\overline{AB}$ equals the perpendicular distance $d_2=\overline{CQ}$ from C on the chord DE, if and only if the chords $\overline{AB}$ and $\overline{DE}$ are with equal lengths: $|\overline{AB}|=|\overline{DE}|$

Thus, the chords AB and DE are equidistant from C on the circle (i) if and only if $d_1=d_2$ (ii)

In similar manner, the chords $\overline{AB}$ (join A to D) and $\overline{BE}$ (join B to E) are congruent chords, if the perpendicular distance $d_3=\overline{CR}$ from C on the chord $\overline{AD}$ equals the perpendicular $d_4=\overline{CS}$ from C on the chord BE: $d_3=d_4$, $\quad |\overline{AD}|=|\overline{BE}|$ (iii)

Example 16 Show that the chords $\overline{AB}$ and $\overline{DE}$ are equidistant from the center C(0,0) of the circle $x^2+y^2=4$. The coordinates of the end points of the two chords are A(0,2), B(−2,0), D(0,−2) and E(2,0).

Solution The circle equation with center C(0,0) is: $x^2+y^2=4$ (v)

If $\overline{AB}$ and $\overline{DE}$ are the two chords of the circle (i), whose coordinates are respectively:

$A(0,2),\quad B(-2,0),\quad D(0,-2),\quad E(2,0)$

From the Figure 8.16, it is clear that the chords AB and DE are with equal length:

$$\overline{AB}=(-2-0,0-2),\quad |\overline{AB}|=\sqrt{(-2)^2+(-2)^2}=2\sqrt{2}$$

$$\overline{DE}=(2-0,0+2),\quad |\overline{DE}|=\sqrt{(2)^2+(2)^2}=2\sqrt{2}$$

Thus, the two chords $\overline{AB}$ and $\overline{DE}$ are equal. For equidistant, the procedure is as under:

The equations of the chords $\overline{AB}$ and $\overline{DE}$ (through two-point form of the line) are respectively:

$$\frac{y-y_1}{x-x_1}=\frac{y_2-y_1}{x_2-x_1}$$

$$\frac{y-2}{x-0}=\frac{0-2}{-2-0},\quad A(x_1,y_1)=A(0,2),\ B(x_2,y_2)=B(-2,0)$$

$$\frac{y-2}{x}=1\Rightarrow x-y+2=0$$

$$\frac{y-y_1}{x-x_1}=\frac{y_2-y_1}{x_2-x_1}$$

$$\frac{y+2}{x-0}=\frac{0+2}{2-0},\quad D(x_1,y_1)=D(0,-2),\ E(x_2,y_2)=E(2,0)$$

$$\frac{y+2}{x}=1\Rightarrow x-y-2=0$$

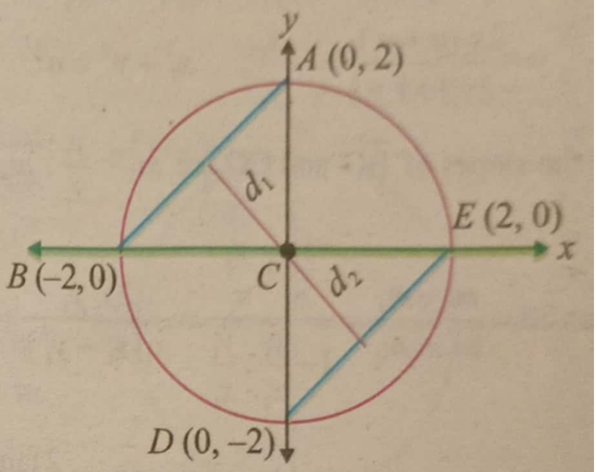


Figure 8.16

The perpendicular distance d_1 from C(0,0) on the chord AB is: $d_1=\left|\frac{0-0+2}{\sqrt{1+1}}\right|=\frac{2}{\sqrt{2}}$

The perpendicular distance d_2 from C(0,0) on the chord DE is: $d_2=\left|\frac{0-0-2}{\sqrt{1+1}}\right|=\frac{2}{\sqrt{2}}$

The perpendicular distance d_1 from C(0,0) on the chord AB is equal to the perpendicular distance d_2 form C(0,0) on the chord DE: $d_1=d_2=\frac{2}{\sqrt{2}}$

Thus, the chords AB and DE are equidistant from the center C(0,0) of the circle (i).

8.4.5 Measure of the central angle of a minor arc is double the measure of the angle subtended by the corresponding major arc

Let the circle be $x^2+y^2=a^2$ (i)

The arc BC is the minor arc of the circle (i), whose coordinates are $B(-x_1,-y_1)$ and $C(x_1,-y_1)$, and the minor arc BC subtended the angle from the center of the circle is $\angle BOC$.

If A(0, a) is a point on the major arc, then join AB and AC that develops the angle of the minor arc which is two times the angle subtended by the major arc: $\angle BOC=2\angle BAC$ (ii)

From the Figure 8.17, if $\angle BAC = \theta$ and $\angle BOC = 2\theta$, then, result (ii) can be verified as follows:
If the slopes of BA and AC are

$m_1 = \dfrac{a+y_1}{x_1}$, $m_2 = \dfrac{-y_1-a}{x_1} = \dfrac{-(a+y_1)}{x_1}$, then,

the angle $\angle BAC = \theta$ from $\overline{BA}$ to $\overline{AC}$ is:

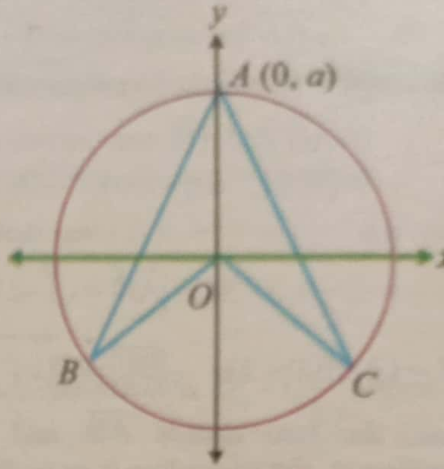


Figure 8.17

$$\tan\theta = \frac{m_1 - m_2}{1+m_1m_2} = \frac{\dfrac{a+y_1}{x_1} + \dfrac{a+y_1}{x_1}}{1-\dfrac{a+y_1}{x_1}\cdot\dfrac{a+y_1}{x_1}} = \frac{2(a+y_1)}{x_1}\cdot\frac{x_1^2}{x_1^2-(a+y_1)^2}$$

$$= \frac{2x_1(a+y_1)}{x_1^2 - a^2 - y_1^2 - 2ay_1} = \frac{2x_1(a+y_1)}{-2y_1^2 - 2ay_1}$$

$$= \frac{2x_1(a+y_1)}{-2y_1(a+y_1)} = -\frac{x_1}{y_1}, \quad x_1^2 + y_1^2 = a^2 \qquad \text{(iii)}$$

If the slopes of $\overline{BO}$ and CO are $m_3 = \dfrac{y_1}{x_1}$, $m_4 = \dfrac{y_1}{-x_1}$, then, the angle $\angle BOC = 2\theta$ from BO to OC is:

$$\tan 2\theta = \frac{m_3 - m_4}{1+m_3m_4} = \frac{\dfrac{y_1}{x_1} + \dfrac{y_1}{x_1}}{1-\dfrac{y_1}{x_1}\cdot\dfrac{y_1}{x_1}} = \frac{2x_1^2y_1}{x_1(x_1^2-y_1^2)} = \frac{2x_1y_1}{x_1^2 - y_1^2} \qquad \text{(iv)}$$

The trigonometric identity $\tan 2\theta = \dfrac{2\tan\theta}{1-\tan^2\theta} = \dfrac{\dfrac{-2x_1}{y_1}}{1-\dfrac{x_1^2}{y_1^2}} = \dfrac{-2x_1y_1}{y_1^2-x_1^2} = \dfrac{-2x_1y_1}{-(x_1^2-y_1^2)} = \dfrac{2x_1y_1}{x_1^2-y_1^2}$

is proving result (iv) with result (iii). Thus $\angle BOC = 2\angle BAC$.

Example 17 Show that the angle subtended by the minor arc BC of the circle $x^2+y^2=9$ is two times the angle subtended in the major arc. The coordinates of the minor arc are $B(2,\sqrt{5}), C(2,-\sqrt{5})$.

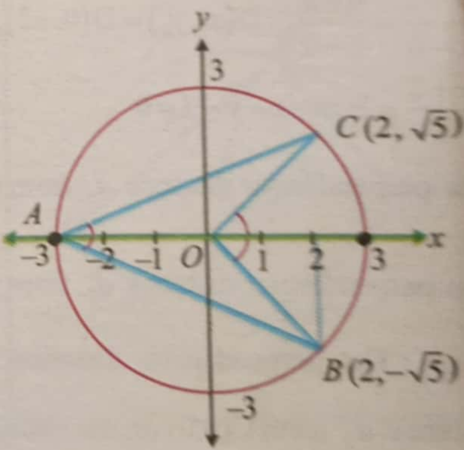


Figure 8.18

Solution The circle $x^2+y^2=9$, whose center is O(0,0). The arc $\overline{BC}$ is the minor arc of the given circle, whose coordinates are $B(2,\sqrt{5}), C(2,-\sqrt{5})$ and the minor arc $\overline{BC}$ subtended the angle from the center of the circle is $\angle BOC$.

If A(−3,0) is a point on the major arc, then join $\overline{AB}$ and $\overline{AC}$ that develops the angle of the minor arc which is two times the angle subtended by the major arc: $\angle BOC = 2\angle BAC$ (i)

From the Figure 8.18, if $\angle BAC = \theta$ and $\angle BOC = 2\theta$, then, result (v) can be verified as follows:

If the slopes of $\overline{BA}$ and $\overline{AC}$ are $m_1 = \dfrac{0+\sqrt{5}}{-3-2} = -\dfrac{1}{\sqrt{5}}$, $m_2 = \dfrac{\sqrt{5}-0}{2+3} = \dfrac{1}{\sqrt{5}}$,

then, the angle $\angle BAC = \theta$ from $\overline{BA}$ to $\overline{AC}$ is $\tan\theta = \dfrac{m_1-m_2}{1+m_1m_2} = \dfrac{-\dfrac{1}{\sqrt{5}} - \dfrac{1}{\sqrt{5}}}{1-\dfrac{1}{\sqrt{5}}\cdot\dfrac{1}{\sqrt{5}}} = \dfrac{-\dfrac{2}{\sqrt{5}}}{1-\dfrac{1}{5}} = \dfrac{\sqrt{5}}{2}$ (ii)

If the slopes of $\overline{BO}$ and $\overline{OC}$ are $m_3 = \frac{0+\sqrt{5}}{-2} = -\frac{\sqrt{5}}{2}$, $m_4 = \frac{\sqrt{5}-0}{2} = \frac{\sqrt{5}}{2}$

then, the angle $\angle BOC = 2\theta$ from OC to BO is $\tan 2\theta = \frac{m_4 - m_3}{1+m_3 m_4} = \frac{\frac{\sqrt{5}}{2}+\frac{\sqrt{5}}{2}}{1-\frac{\sqrt{5}}{2}\cdot\frac{\sqrt{5}}{2}} = \frac{\sqrt{5}}{1-\frac{5}{4}} = -4\sqrt{5}$ (iii)

The trigonometric identity $\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta} = \frac{2\left(\frac{\sqrt{5}}{2}\right)}{1-\frac{5}{4}} = -4\sqrt{5}$

is proving result (ii) with result (iii). Thus $\angle BOC = 2\angle BAC$.

8.4.6 An angle in a semi-circle is a right angle

Let the circle equation be $x^2 + y^2 = a^2$ (i)

If $P(x_1, y_1)$ is any point on the semicircle and $\overline{BA}$ is fixed as the diameter of the circle (i) on the x-axis, whose coordinates are A $(a, 0)$ and B $(-a, 0)$, then the point $P(x_1, y_1)$ lies on the circle (i) that changes the circle equation to: $x_1^2 + y_1^2 = a^2$

Join $\overline{PA}$ and $\overline{PB}$ that develops a right angle $\angle APB$. The angle $\angle APB$ is a right angle, if $\overline{AP}$ and $\overline{PB}$ are perpendicular to each other, for which the slopes of $\overline{AP}$ and $\overline{PB}$ are respectively:

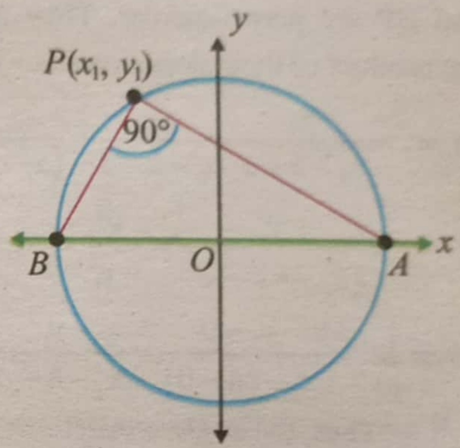


Figure 8.19

$$m_1 = \frac{y_1 - 0}{x_1 - a} = \frac{y_1}{x_1 - a}, \quad m_2 = \frac{y_1 - 0}{x_1 + a} = \frac{y_1}{x_1 + a}$$

The product of the slopes of $\overline{AP}$ and $\overline{BP}$ is

$$m_1 m_2 = \left(\frac{y_1}{x_1 - a}\right)\cdot\left(\frac{y_1}{x_1 + a}\right) = \frac{y_1^2}{x_1^2 - a^2} = \frac{y_1^2}{-y_1^2} = -1, \quad x_1^2 + y_1^2 = a^2$$

Thus, $\overline{PA}$ and $\overline{PB}$ are perpendicular and the angle $\angle APB = 90°$ is of course a right-angle.
If $\angle APB = 90°$, then P is a point lies on the semicircle, for which the Pythagorean rule

$$\left|\overline{PA}\right|^2 + \left|\overline{PB}\right|^2 = \left|\overline{AB}\right|^2 \quad \text{(iii)}$$

with substitution of $\overline{PA} = (a - x_1, 0 - y)_1 \quad \Rightarrow \left|\overline{PA}\right|^2 = \left[\sqrt{(a-x_1)^2 + y_1^2}\right]^2 = (a-x_1)^2 + y_1^2$

$$\overline{PB} = (-a - x_1, -0 - y_1) \Rightarrow \left|\overline{PB}\right|^2 = \left[\sqrt{(a+x_1)^2 + y_1^2}\right]^2 = (a+x_1)^2 + y_1^2$$

$$\overline{AB} = (-a - a, 0 - 0) \quad \Rightarrow \left|\overline{AB}\right|^2 = \left[\sqrt{(-2a)^2 + 0}\right]^2 = 4a^2$$

gives the locus of $P(x_1, y_1)$

$$\left|\overline{PA}\right|^2 + \left|\overline{PB}\right|^2 = \left|\overline{AB}\right|^2, \ \overline{AB} = (-a, 0) - (a, 0)$$

$$(a - x_1)^2 + y_1^2 + (a + x_1)^2 + y_1^2 = 4a^2$$

$$a^2 + x_1^2 - 2ax_1 + y_1^2 + a^2 + x_1^2 - 2ax_1 = 4a^2$$

$$2a^2 + 2x_1^2 + 2y_1^2 = 4a^2 \Rightarrow 2x_1^2 + 2y_1^2 = 2a^2 \Rightarrow x_1^2 + y_1^2 = a^2$$

which is a circle, P may lie on the upper or the lower semicircle.

Example 18 Show that the angle in the semicircle of the circle $(x-h)^2+y^2=a^2$ is a right-angle.

Solution The circle equation with center $C(h, 0)$ is : $(x-h)^2+y^2=a^2$

If $P(x_1, y_1)$ is any point on the semicircle and $\overline{AB}$ is fixed as the diameter of the given circle on the x-axis, then $(x_1-h)^2+y_1^2=a^2$

The coordinates of A and B are respectively:

$\overline{OA}=\overline{OC}-\overline{AC}=h-a \Rightarrow A(h-a, 0)$

$\overline{OB}=\overline{OA}+\overline{AB}=(h-a)+2a=h+a \Rightarrow B(h+a, 0)$

Join $\overline{PA}$ and $\overline{PB}$ that develops a right angle $\angle APB$. This angle $\angle APB$ is a right angle, if AP and BP are perpendicular. They are perpendicular, if the product of their slopes equals -1:

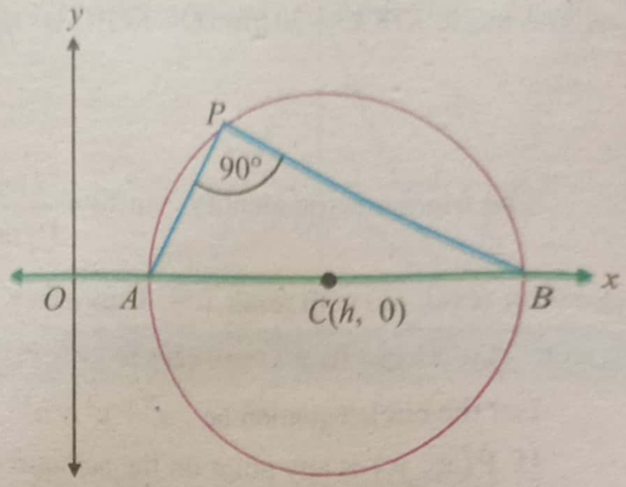


Figure 8.20

$$m_1m_2=\frac{y_1}{x_1-h+a}\cdot\frac{y_1}{x_1-h-a}=\frac{y_1^2}{(x_1-h+a)(x_1-h-a)}$$

$$=\frac{y_1^2}{(x_1-h)^2-a^2}=\frac{y_1^2}{-y_1^2}=-1, \qquad y_1^2=a^2-(x_1-h)^2=-\left[(x_1-h)^2-a^2\right]$$

when $m_1=\dfrac{y_1-0}{x_1-(h-0)}=\dfrac{y_1}{x_1-h+a}$ and $m_2=\dfrac{y_1-0}{x_1-(h+a)}=\dfrac{y_1}{x_1-h-a}$

Thus, the angle $\angle APB=90°$ is right angle.

8.4.7 The Perpendicular at the outer end of radial segment is tangent to the circle

The circle equation with center $C(-g,-f)$ is : $x^2+y^2+2gx+2fy+c=0$ (i)

If $P(x_1, y_1)$ is a point on the circle and $C(-g, -f)$ is the center of the circle (i), then $\overline{CP}$ is the radial segment of the circle.

The equation of the tangent line on the circle (i) at point P is

$xx_1+yy_1+g(x+x_1)+f(y+y_1)+c=0$

$x(x_1+g)+y(y_1+f)+gx_1+fy_1+c=0$

whose slope is $m_1=-\dfrac{x_1+g}{y_1+f}$ and the slope of $\overline{CP}$ is

$m_2=\dfrac{x_1+f}{y_1+g}$.

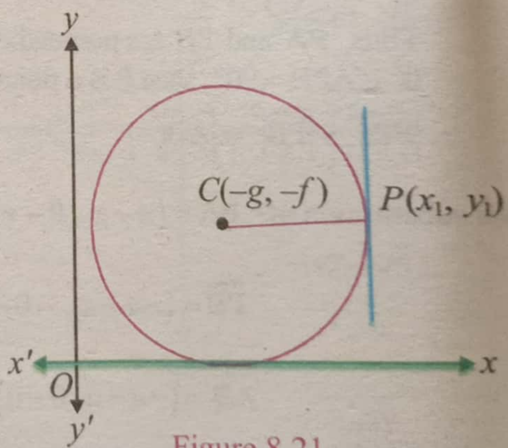


Figure 8.21

The perpendicular at the outer end P of the radial segment $\overline{CP}$ is tangent to the circle (i) if the product of the slopes of the radial segment $\overline{CP}$ and the line of the outer end of the radial segment $\overline{CP}$ is -1: $m_1m_2=-\dfrac{x_1+g}{y_1+f}\cdot\dfrac{y_1+f}{x_1+g}=-1$

Thus, the perpendicular at the outer end P of the radial segment is tangent to the circle (i).

Remember

- the tangent line is perpendicular to the radial segment, if the radial segment is the segment through the point of contact of the tangent and the center of the circle.
- if a line is perpendicular to the tangent of the circle at the point of contact, then it passes through the center of the circle:

Example 19 Show that the perpendicular at the outer end point P(1,1) of the radial segment is tangent to the circle $x^2+y^2-13x-5y+16=0$.

Solution The circle equation with center $C\left(\frac{13}{2},\frac{5}{2}\right)$ is: $x^2+y^2-13x-5y+16=0$

The equation of the tangent line on the given circle at a point $P(1,1)$ is

$$xx_1+yy_1+g(x+x_1)+f(y+y_1)+c=0$$

$$x(1)+y(1)-\frac{13}{2}(x+1)+\frac{5}{2}(y+1)+16=0,\quad P(x_1,\ y_1)=P(1,1)$$

$$11x+3y-16=0$$

whose slope is $m_1=\frac{-11}{3}$ and the slope of the radial segment $\overline{CP}$ is $m_2=\frac{1-\frac{5}{2}}{1-\frac{13}{2}}=\frac{3}{11}$

The product of the slopes of the tangent to the circle and the radial segment $\overline{CP}$ is $m_1m_2=\frac{-11}{3}\cdot\frac{3}{11}=-1$

Thus, the perpendicular at the outer end P of the radial segment is tangent to the given circle at point P.

Exercise 8.3

1. If A(2,2) and B(3,1) are the end points of the chord $\overline{AB}$ of the circle $x^2+y^2-4x-2y+4=0$, then show that
   a. the line from the center of the circle is perpendicular to $\overline{AB}$, also bisects the chord $\overline{AB}$.
   b. the line from the center of the circle to the midpoint of the chord $\overline{AB}$ is perpendicular to the chord $\overline{AB}$.
2. If A(0,0) and B(0,3) are the end points of the chord $\overline{AB}$ of the circle $x^2+y^2+4x-5y=0$, then show that
   a. the line from the center of the circle is perpendicular to $\overline{AB}$, also bisects the chord $\overline{AB}$.
   b. the line from the center of the circle to the midpoint of the chord $\overline{AB}$ is perpendicular to the chord $\overline{AB}$.
3. Show that the chords $\overline{AB}$ and $\overline{DE}$ are equidistant from the center C(0,0) of the circle
   a. $x^2+y^2=4$. The coordinates of the end points of the two chords $\overline{AB}$ and $\overline{DE}$ are A(–2,0), B(0,2), D(0,2 and E(2,0).
   b. $x^2+y^2=16$. The coordinates of the end points of the two chords $\overline{AB}$ and $\overline{DE}$ are A(–4,0), B(0,4), D(0,4) and E(4,0).
4. Show that the angle subtended by the minor arc $\overline{AB}$ of the circle
   a. $x^2+y^2=9$ is two times the angle subtended in the major arc. The coordinates of the minor arc $\overline{AB}$ are A $(2, \sqrt{5})$, B $(2, -\sqrt{5})$.

b. $x^2+y^2=4$ is two times the angle subtended in the major arc. The coordinates of the minor arc $\overline{AB}$ are $A(1,\sqrt{3})$, $B(1,-\sqrt{3})$.

5. Show that the angle in the semicircle of the circle

a. $(x-h)^2+y^2=a^2$, $h=1$, $a=2$ is a right-angle.

b. $(x-h)^2+y^2=a^2$, $h=3$, $a=4$ is a right-angle.

Note that the diameter of the circle (in each case) is considered to be $\overline{AB}$.

6. Show that the perpendicular at the outer end point

a. P(1,5) of the radial segment is tangent to the circle $x^2+y^2+x-5y-2=0$.

b. P(5,6) of the radial segment is tangent to the circle $x^2+y^2-22x-4y+25=0$.

Review Exercise 8

1. Choose the correct option.

i. If radius of a circle is 2*cm* then its equation will be:

(a) $x^2+y^2=2$ (b) $x^2+y^2=\sqrt{2}$ (c) $x^2+y^2=8$ (d) $x^2+y^2=4$

ii. In the general equation of circle the coordinates of centre are:

(a) (x,y) (b) $(-x,-y)$ (c) (f,g) (d) $(-f,-g)$

iii. For the general equation of the circle the radius can be calculated by r =

(a) $\sqrt{x^2+y^2-c^2}$ (b) $\sqrt{(-x)^2+(-y)^2-c^2}$

(c) $\sqrt{(-g)^2+(-f)^2-c^2}$ (d) $\sqrt{(-g)^2+(-f)^2+c^2}$

iv. A point of the circle at which a tangent meets the circle is called

(a) point of contact (b) normal (c) center point (d) none of these

v. If the discriminant of the quadratic equations $m^2(x_1^2-a_2)-2mx_1y_1+(y_1^2-a_2)=0$ zero than tangent are

(a) real and different (b) imaginary

(c) real an coincident (d) equal is greater than

vi. The length of the tangent drawn from the point p(3,4) on the circle $x^2+y^2-9=0$ is

(a) 1 (b) 2 (c) 3 (d) 4

vii. The angle subtended can be calculated by using the trigonometric identity.

(a) $\tan 2\theta=\dfrac{m_3-m_1}{1+m_3m_2}$ (b) $\tan 2\theta=\dfrac{2\tan\theta}{1-\tan^2\theta}$

(c) $\tan 2\theta=\dfrac{2\tan^2\theta}{1+\tan^2\theta}$ (d) $\tan 2\theta=\dfrac{\tan^2\theta}{1-2\tan^2\theta}$

viii. What is the contact point for the lines $x=7$ touches the $x^2+y^2-4x-6y-12=0$

(a) (2,8) (b) (8.2) (c) (3,7) (d) (7,3)

ix. For the condition of tangent the line $y=mx+c$ should touch the circle $x^2+y^2=r^2$ if

(a) $c=r\sqrt{1+m^2}$ (b) $c=-r\sqrt{1+m^2}$

(c) both option a & b (d) not option (b) nor option (c)

x. A line perpendicular to the contact point to a circle is called

(a) Tangent (b) Normal (c) Chord (d) Diameter

Summary

- Standard Form of a Circle: The standard form of a circle with radius r and center $C(h, k)$ is: $(x-h)^2+(y-k)^2=r^2$
- General Form of a Circle: The general form of a circle with radius $r=\sqrt{(-g)^2+(f)^2-c}$ and center $C(-g, -f)$ is: $x^2+y^2+2gx+2fy+c=0$
  The coefficient of x^2 is equal to the coefficient of y^2, and there is no term containing xy and the square of the radius is $r^2 \geq 0$.
- Nature of the circle:
  If $g^2+f^2-c>0$, then the circle is real and different from zero.
  If $g^2+f^2-c=0$, then the circle shrinks to a point $(-g, -f)$. It is called a point circle.
  If $g^2+f^2-c<0$, then the circle is imaginary or virtual.
- Condition of Tangency:
  The condition at which the line $y=mx+c$ should touch the circle $x^2+y^2=a^2$ is: $c=\pm a\sqrt{1+m^2}$
  The equation of any tangent to the circle $x^2+y^2=a^2$ in the slope-form is: $y=mx\pm a\sqrt{1+m^2}$
  The condition at which the line $lx+my+n=0$ should touch the circle $x^2+y^2=a^2$ is: $n=\pm a\sqrt{l^2+m^2}$
  The condition at which the line $lx + my + n = 0$ should touch the circle $x^2+y^2+2gx+2fy+c=0$ is: $(c-f^2)l^2+2fglm+(c-g^2)m^2-2n(gl+fm)+n^2=0$
- The tangent equation to the circle $x^2+y^2+2gx+2fy+c=0$ at a point $A(x_1, y_1)$ is: $xx_1+yy_1+g(x+x_1)+f(y+y_1)+c=0$
  The tangent equation to the circle $x^2+y^2=a^2$ at a point $A(x_1, y_1)$ is: $xx_1+yy_1=a^2$
- The normal equation to the circle $x^2+y^2+2gx+2fy+c=0$ at a point $A(x_1, y_1)$ is: $x(y_1+f)-y(x_1+g)+(gy_1-fx_1)=0$
  The normal equation to the circle $x^2+y^2=a^2$ at a point $A(x_1, y_1)$ is: $xy_1-yx_1=0$
- The length of the tangent drawn from the point $P(x_1, y_1)$ to the circle $x^2+y^2+2gx+2fy+c=0$ is: $|\overline{PT}|=\sqrt{x_1^2+y_1^2+2gx_1+2fy_1+c}$
  The length of the tangent drawn from the point $P(x_1, y_1)$ to the circle $x^2+y^2=a^2$ is: $|\overline{PT}|=\sqrt{x_1^2+y_1^2-a^2}$
  The lengths of the two tangents drawn from the point $P(x_1, y_1)$ on the given circle are equal.

Unit 9

CONICS-II

By the end of this unit, the students will be able to:

9.1 Parabola

i. Define a parabola and its elements (i.e. focus, directrix, eccentricity, vertex, axis, focal chord and latus rectum).
ii. General form of an equation of a parabola.
iii. Standard equations of parabola, sketch their graphs and find their elements.
iv. Find the equation of a parabola with the following given elements:
- focus and vertex,
- focus and directrix,
- vertex and directrix.

v. Recognize tangent and normal to a parabola.
vi. Find the condition when a line is tangent to a parabola at a point and hence write the equation of a tangent line in slope form.
vii. Find the equation of a tangent and a normal to a parabola at a point.
viii. Solve suspension and reflection problems related to parabola.

9.2 Ellipse

i. Define ellips and its elements (i.e. centre, foci, vertices, covertices, directories, major and minor axes, eccentricity, focal chord and latus rectum).
ii. Explain that circle is a special case of an ellipse.
iii. Derive the standard form of equation of an ellipse and identify its elements.
iv. Find the equation of an ellipse with the following given elements.
- major and minor axes,
- two points,
- foci, vertices or lengths of a latus rectum,
- foci, minor axes or length of a latus rectum.

v. Convert a given equation to the standard form of equation of an ellipse, find its elements and draw the graph.
vi. Recognize tangent and normal to an ellipse.
vii. Find points of intersection of an ellipse with a line including the condition of tangency.
viii. Find the equation of a tangent in slope form.
ix. Fine the equation of a tangent and a normal, to an ellipse at a point.

9.3 Hyperbola

i. Define hyperbola and its elements (i.e. centre, foci, vertices, directrices, transverse and conjugate axes, eccentricity, focal chord and latus rectum).
ii. Derive the standard form of equation of a hyperbola and identify its elements.
iii. Find the equation of a hyperbola with the following given elements:
- transverse and conjugate axes with centre at origin,
- two points,
- eccentricity, Latera recta and transverse axes,
- focus, eccentricity and centre,
- focus, centre and directrix.

iv. Convert a given equation to the standard form of equation of a hyperbola, find its elements and sketch the graph.
v. Recognize tangent and normal to a hyperbola.
vi. Find,
- points of intersection of a hyperbola with a line including the condition of tangency,
- the equation of tangent in slope form.

vii. Find the equation of a tangent and a normal to a hyperbola at a point.

9.4 Translation and rotation of axes

i. Define translation and rotation of axes and demonstrate through examples.
ii. Find the equations of transformation for
- translation of axes,
- rotation of axes.

iii. Find the transformed equation by using translation or rotation of axes.
iv. Find new origin and new axes referred to old origin and old axes.
v. Find the angle through which the axes be rotated about the origin so that the product term xy is removed form the transformed equation.

Introduction

In our previous unit of this book we have learnt that a conic section (or simply a conic) is a curve obtained as the intersection of the surface of a cone with a plane. In this unit we will study in details about the three types of conic sections that are parabola, hyperbola and the ellipse. The circle is a type of ellipse and same time considered to be the fourth type of conic section. We have already discussed in details about tangent and normal in previous section.

9.1 Parabola

When you kick a soccer ball (or shoot an arrow, fire a missile or throw a stone) it arcs up into the air and comes down again ...

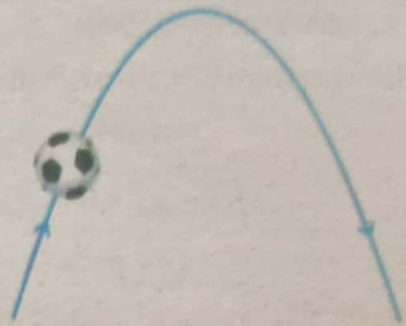


Figure 9.1(a)

A parabola is a curve where any point is at an **equal distance** from:

- a fixed point (the **focus**), and
- a fixed straight line (the **directrix**)

et a piece of paper, draw a straight line on it, then make a big dot for the focus (not on the line!).

Now play around with some measurements until you have another dot that is exactly the same distance from the focus and the straight line.

Keep going until you have lots of little dots, then join the little dots and you will have a parabola!

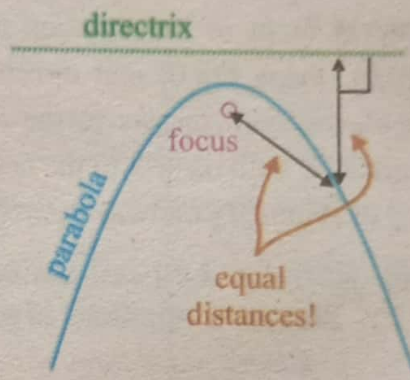


Figure 9.1(b)

In our study of quadratic functions, the graph of the general form of the quadratic equation $y=ax^2+bx+c$ (1) (with $a \neq 0$) is a parabola that opens upward if $a>0$ and downward if $a<0$.

Remember

The graph of a quadratic equation is always parabola. But all parabolas can not be represented by quadratic equation, because all parabolas are not graphs of the functions.

(i) Parabola and its elements (i.e focus, directrix, eccentricity, vertex, axis, focal chord and latus rectum

The **parabola** is the set of all points P in the plane such that the distance from a fixed point F **(focus)** and the distance from a fixed straight line **(directrix)** to a point are **equidistant**.

The line through the focus perpendicular to the directrix is called the **principal axis** of the parabola, and the point where the axis intersects the parabola is called the **vertex**. The line segment AB that passes through the focus perpendicular to the axis and with endpoints on the parabola is called the **focal chord** or its **latus rectum**. These terminologies are shown in the Figure 9.2.

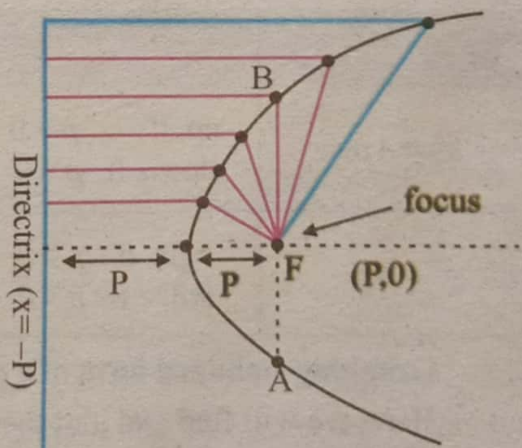


Figure 9.2

(ii) General form of an equation of a parabola

To obtain the general form of the parabola, let us assume a focus with coordinates $F(p,0)$ and a directrix $x=-p$, (p is any positive number) parallel to the y-axis. If $P(x, y)$ is any point on the curve and $p_1(-p, y)$ is a point on the directrix $x=-p$, then by the definition of parabola Figure 9.3.

$$\frac{\text{distance from P } to \text{ F}}{\text{distance from P } to \text{ P}_1}=e=1, \quad \text{for parabola } e=1$$

$$\text{distance from } P(x,y)\ to\ P_1 = \text{distance from } P(x,y)\ to\ F$$

$$d(P,P_1)=d(P,F)$$

$$\sqrt{(x+p)^2+0}=\sqrt{(x-p)^2+(y-0)^2}, \qquad (2)$$

$$(x+p)^2=(x-p)^2+y^2, \qquad \text{by squaring}$$

$$x^2+2px+p^2=x^2-2px+p^2+y^2 \Rightarrow 4px=y^2 \qquad (3)$$

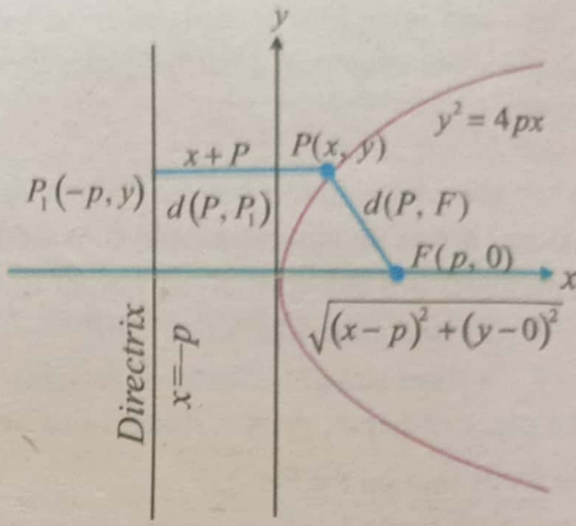


Figure 9.3

The result (3) is the **standard form** obtained from the **general form** of the equation of a parabola with vertex at $V(0,0)$, focus $F(p,0)$ and directrix $x=-p$. The parabola is symmetric with respect to the positive x–*axis* if $p>0$ and symmetric with respect to the negative x–*axis* if $p<0$. The vertex $V(0,0)$ of the parabola is on the principal axis of symmetry midway between the focus and the directrix.

(iii) Standard equations of parabola, sketch their graphs and find their element

a. Standard equations of parabola

"The standard form of the equation of a parabola that is symmetric with respect to the x–axis, with vertex $V(0,0)$, focus $F(p, 0)$ and directrix the line $x=-p$ is:

$$y^2=4px" \qquad (4)$$

"The standard form of an equation of a parabola that is symmetric with respect to the y–axis, with vertex $V(0,0)$, focus $F(0, p)$ and directrix the line $y=-p$ is:

$$x^2=4py" \qquad (5)$$

The parabolas that have their vertex at the origin and open upward, downward, to the left and to the right are summarized in the following table:

Parabola	Curve	Focus	Directrix	Vertex
$x^2=4py$	up, if $p>0$ down, if $p<0$	$F(0,p)$ $F(0,p)$	$y=-p$ $y=-p$	$V(0,0)$ $V(0,0)$
$y^2=4px$	right, if $p>0$ left, if $p<0$	$F(p,0)$ $F(p,0)$	$x=-p$ $x=-p$	$V(0,0)$ $V(0,0)$

b. Graphing standard form of a parabola

Here, we will find and plot the parabola by inspection and count out units from the vertex in the appropriate direction as determined by the form of the equation. Finally, it is shown in the problem set that the length of the focal chord (latus rectum) is $|4p|$. This number could be used in determination of the width of the parabola. This approach is employed in the following examples.

Example 1 Graph the parabola $y^2 - 8x = 0$ and indicate the vertex, focus, directrix and the focal chord.

Solution Rewrite the given parabola in the standard form

$$y^2 = 8x \qquad (6)$$

and is compared with the standard form of the parabola (2) to obtain:

$$8 = 4p \Rightarrow p = 2$$

Since $p > 0$, the parabola opens to the right. The vertex is V(0,0), the focus is F(2,0), the directrix is the line $x = -2$ and the length of the focal chord is $4p = 4(2) = 8$. The line of symmetry is the positive x–*axis*. This is shown in the Figure 9.4.

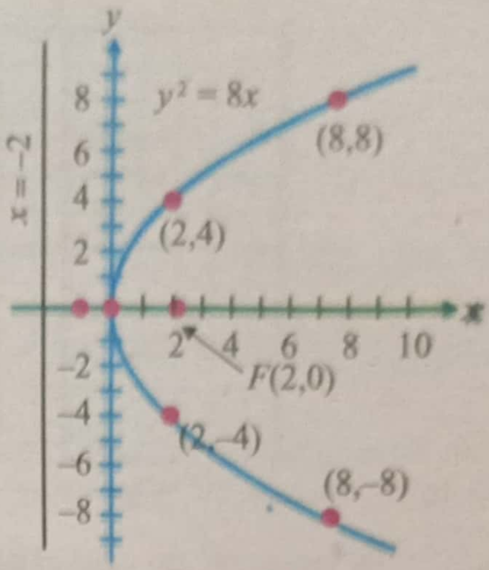


Figure 9.4

Example 2 Graph the parabola $x^2 + y = 0$ and indicate the vertex, focus, directrix and the focal chord.

Solution Rewrite the given parabola in the standard form

$$x^2 = -y \qquad (7)$$

and is compared with the standard form of the parabola (5) to obtain :

$$-1 = 4p \Rightarrow p = -\frac{1}{4}$$

Since $p < 0$, the parabola opens downward. The vertex is V(0,0), the focus is $F\left(0, \frac{-1}{4}\right)$, the directrix is the line $y = \frac{1}{4}$ and the length of the focal chord is $4p = 4\left(\frac{-1}{4}\right) = -1$. The line of symmetry is the negative y–*axis*. This is shown in the Figure 9.5.

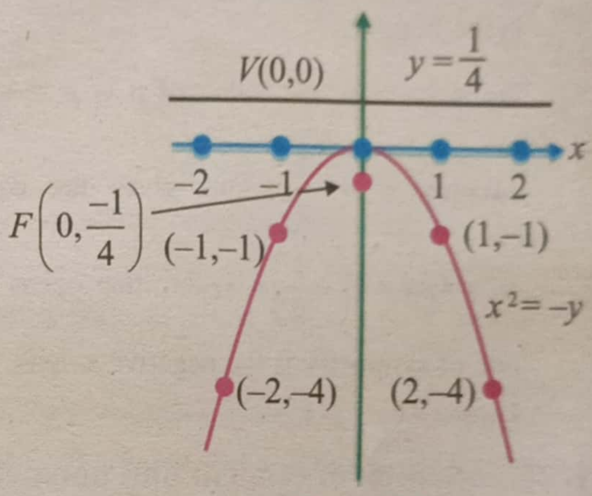


Figure 9.5

(iv) The equation of a parabola with the given elements

- focus and vertex
- focus and directrix
- fertex and directrix

Example 3 Find an equation of parabola with

(a). Focus F(0, –2) and directrix $y = 2$.

(b). Focus $\left(\frac{5}{8}, 0\right)$ and vertex (0,0).

(c). Vertex V(0,0) and directrix $x = \frac{1}{2}$.

Solution

a. By inspection, the value of p is $p = -2$ that satisfies the directrix $y = 2$. This gives the equation of parabola $x^2 = 4py = 4(-2)y = -8y$, that opens downward ($p < 0$) and the line of symmetry is the negative y–axis. This is shown in the Figure 9.6.

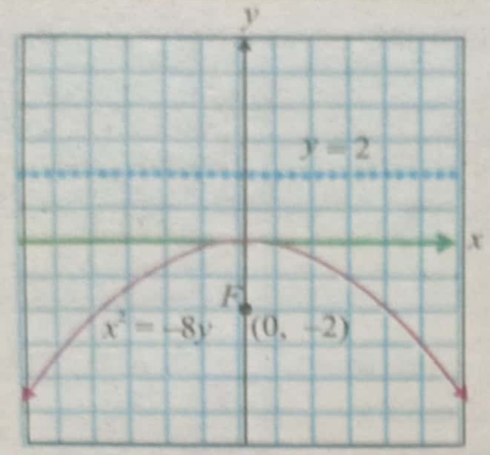


Figure 9.6

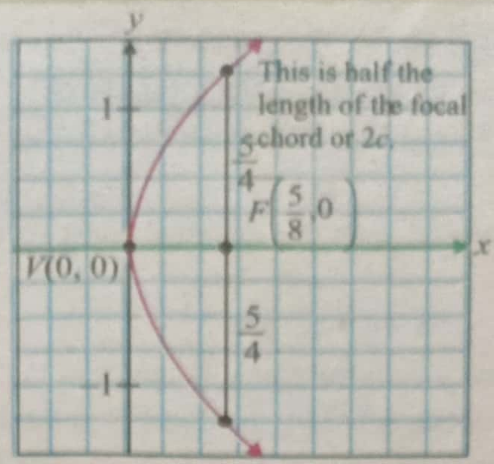


Figure 9.7

b. By inspection, the value of p is $p=\frac{5}{8}$ that satisfies the directrix $x=-\frac{5}{8}$. This gives the equation of parabola $y^2=4px=4\left(\frac{5}{8}\right)x=\frac{5}{2}x$, that opens right $(p>0)$ and the line of symmetry is the positive *x–axis*. This is shown in the Figure 9.7.

c. By inspection, the value of p is $p=-\frac{1}{2}$ that satisfies the directrix $x=\frac{1}{2}$. This gives the equation of parabola $y^2=4px=4\left(-\frac{1}{2}\right)x=-2x$, that opens left $(p<0)$ and the line of symmetry is the negative x–axis. This is shown in the Figure 9.8.

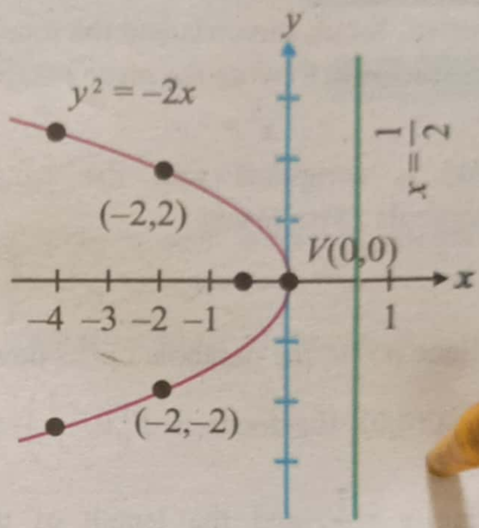


Figure 9.8

(v) Recognition of tangent and normal to a parabola

A line which is parallel to the axis of a parabola intersects the parabola in only one (finite) point; all other lines will cut the parabola in two real and distinct points, real and coincident points, or complex conjugate points. *"A line which meets a parabola in two coincident points is called a tangent."* A tangent to any curve at a point P is the limiting position of a secant line, cutting the curve in two points P and Q as $Q\to P$. The normal can easily be shown in the subsection of this section.

(vi) The condition at which a line is tangent to parabola at a point

The line is tangent to parabola, when the line intersects the parabola in **two real and coincident** points. The given parabola and line

$$y^2=4px \qquad (8)$$

$$y=mx+c \qquad (9)$$

develops a system of nonlinear equations:

$$\left.\begin{aligned} y^2&=4px\\ y&=mx+c\end{aligned}\right\} \qquad (10)$$

The solution set $\{x,y\}$ of nonlinear system of equations (10) exists only, if the curves of the system (10) are intersecting. That set of points of intersection $\{x,y\}$ (a solution set) can be found by solving the nonlinear system (10) simultaneously.

The line (9) is used in parabola (8) to obtain the quadratic equation in x:

$$(mx+c)^2 = 4px$$
$$m^2x^2 + 2mcx + c^2 = 4px$$
$$m^2x^2 + 2x(mc-2p) + c^2 = 0 \qquad (11)$$

The equation (11) being a quadratic equation in x, gives a set of two values x_1 and x_2 of x, which will be used in a line (9) to obtain a set of two y values y_1 and y_2.

Thus, a solution set $\{(x_1, y_1), (x_2, y_2)\}$ of the system (10) is of course a set of points of intersection of the system (10).

The points of intersection of the system (10) are real, coincident or imaginary, according as the roots of the quadratic equation (11) are real, coincident or imaginary or according as the discriminant of the quadratic equation (11):

$Disc = 4(mc-2p)^2 - 4m^2c^2 > 0$, *real and different*

$Disc = 4(mc-2p)^2 - 4m^2c^2 = 0$, *real and coincident*

$Disc = 4(mc-2p)^2 - 4m^2c^2 < 0$, *imaginary*

Example 4 For what condition the tangent line $4x - y - 4 = 0$ intersects the parabola $x^2 = y$?

Solution The equations of the line and parabola are:

$$4x - y - 4 = 0$$
$$y = 4x - 4 \qquad (12)$$
$$x^2 = y \qquad (13)$$

The line (12) is used in parabola (13) to obtain the y–coordinates of the points of intersection:

$$x^2 = y$$

$x^2 = 4x - 4$ Put the value of y from equation (12)

$$x^2 - 4x + 4 = 0 \Rightarrow (x-2)^2 = 0 \Rightarrow x = 2, 2$$

The x–coordinates are used in the line (12) to obtain the y–coordinates $y = 4, 4$

Thus, the set of two points of intersection (2,4) and (2,4) are real and different and the tangent line $4x - y - 4 = 0$ is of course intersecting the parabola (13) at two coincident points (2,4) and (2,4).

a. **The Equation of a tangent line in slope-form**

If m is the slope of the tangent to parabola $y^2 = 4px$ (14)

then the equation of that tangent line is of the form $y = mx + c$ (15)

Here c is to be calculated from the fact that the line (15) is tangent to parabola (14). The line (15) is used in parabola (14) to obtain the quadratic equation in x:

$$y^2 = 4px$$
$$(mx+c)^2 = 4px$$
$$m^2x^2 + c^2 + 2mcx = 4px$$
$$m^2x^2 + 2(mc-2p)x + c^2 = 0 \qquad (16)$$

If the line (15) touches the parabola (14), then the quadratic equation (16) has coincident roots for which the discriminant of the quadratic equation (16) equals zero:

$$4(mc-2p)^2 - 4(m^2)(c^2) = 0$$
$$4m^2c^2 + 16p^2 - 16mcp - 4m^2c^2 = 0$$
$$16p^2 - 16mcp = 0$$

$$16p(p-mc)=0 \Rightarrow p-mc=0 \Rightarrow c=\frac{p}{m} \qquad (17)$$

The equation (17) represents the **condition of tangency**. The value of c from equation (17) is used in the line (15) to obtain the required equation of tangent:

$$y=mx+\left[\frac{p}{m}\right] \qquad (18)$$

Remember

- the equation of any tangent to parabola $y^2=4px$ in the slope–form is:

  $y=mx+\left[\frac{p}{m}\right]$

- the line y=mx+c should touch the parabola $y^2=4px$ under condition:

  $y=mx+c=mx+\left[\frac{p}{m}\right]$ (19) $c=\frac{p}{m}, y^2=4px$

- the **condition of tangency** in case of parabola $x^2=4py$ and line $y=mx+c$ is:

  $y=mx+c=mx-pm^2, c=-pm^2, x^2=4py$ (20)

Example 5 For what value of c, the line $x-y+c=0$ will touch the parabola $x^2=8y$? Use that value of c to find the tangent line that should touch the given parabola.

Solution The value of c at which the line $x-y+c=0$ will touch the given parabola through result (20) is:

$$c=-pm^2=-2(1)=-2$$
$$x^2=4py=4(2)y$$
$$p=2, m=1$$

Here m is the slope of the line $x-y+c=0$, which is $m=1$. The required tangent line that should touch the parabola through (20) is: $y=mx+c$

$$=x-2 \Rightarrow x-y-2=0$$

(vii) The equation of a tangent and a normal to a parabola at a point

a. Equation of tangent to a parabola at a point

Let the equation of the tangent line at a point $p(x_1, y_1)$ to parabola $y^2=4px$ be:

$$y-y_1=m_1(x-x_1) \qquad (21)$$

Here m_1 is the slope of the tangent line to parabola $y^2=4px$ at a point $p(x_1, y_1)$ that can be found by differentiating $y^2=4px$ with respect to x:

$$y^2=4px$$

$$2y\frac{dy}{dx}=4p \Rightarrow \frac{dy}{dx}=\frac{2p}{y} \Rightarrow \left(\frac{dy}{dx}\right)_{(x_1,y_1)}=\frac{2p}{y_1}=m_1, \ say \qquad (22)$$

The substitution of (22) in (21) is giving the equation of the tangent line at a point $p(x_1, y_1)$ to parabola $y^2=4px$:

$$y-y_1=m_1(x-x_1)$$

$$y-y_1=\frac{2p}{y_1}(x-x_1)$$

$$yy_1 - y_1^2 = 2px - 2px_1$$

$$yy_1 - 4px_1 = 2px - 2px_1, \quad y_1^2 = 4px_1$$

$$yy_1 = 2px + 2px_1 \Rightarrow yy_1 = 2p(x + x_1) \qquad (23)$$

Note

- the equation of the tangent line at a point $p(x_1, y_1)$ to parabola $x^2 = 4py$ is:

$$xx_1 = 2p(y + y_1) \qquad (24)$$

- if the tangent line $y = mx + \left[\frac{p}{m}\right]$ to parabola $y^2 = 4px$ is identical to $yy_1 = 2p(x + x_1)$, then the coefficients of like terms of $y = mx + \left[\frac{p}{m}\right]$ and $yy_1 = 2p(x + x_1)$ are compared to obtain the contact point:

$$mx = \frac{2px}{y_1} \Rightarrow y_1 = \frac{2p}{m}$$

$$\frac{2px_1}{y_1} = \frac{p}{m} \Rightarrow 2x_1 = \frac{y_1}{m} \Rightarrow 2x_1 = \frac{2p}{m^2} \Rightarrow x_1 = \frac{p}{m^2}$$

Thus, the contact point is $p(x_1, y_1) = \left(\frac{p}{m^2}, \frac{2p}{m}\right)$ in case of parabola $y^2 = 4px$. (25)

- if the tangent line $y = mx - pm^2$ to parabola $x^2 = 4py$ is identical to $xx_1 = 2p(y + y_1)$, then the coefficients of like terms of $y = mx - pm^2$ and $xx_1 = 2p(y + y_1)$ are compared to obtain the point of contact:

$$mx = \frac{xx_1}{2p} \Rightarrow x_1 = 2pm$$

$$-pm^2 = -y_1 \Rightarrow y_1 = pm^2$$

Thus, the contact point is $p(x_1, y_1) = (2pm, pm^2)$ in case of parabola $x^2 = 4py$. (26)

Example 6 Find the equation of tangent line at a point $p(2,-4)$ to parabola $y^2 = 8x$. Show that $p(2,-4)$ is the point of contact in between the required tangent line and the given parabola.

Solution Result (23) is used to obtain the tangent line to the given parabola:

$$yy_1 = 2p(x + x_1)$$

$$y(-4) = 2(2)(x + 2) \qquad \therefore p(x_1, y_1) = (2,-4),\ 4p = 8$$

$$-4y = 4x + 8 \Rightarrow 4x + 4y + 8 = 0 \Rightarrow x + y + 2 = 0$$

The point of contact through result (25) is:

$$p(x_1, y_1) = \left(\frac{p}{m^2}, \frac{2p}{m}\right) = (2,-4),\ p = 2,\ m = -1 \text{ is the slope of the tangent line } x + y + 2 = 0$$

b. The Equation of a normal line to parabola at a point

The equation of the normal line at a point $p(x_1, y_1)$ to parabola $y^2 = 4px$ is:

$$y - y_1 = m_2(x - x_1) \qquad (27)$$

Here m_2 is the slope of the normal line to parabola $y^2=4px$ at a point $p(x_1,y_1)$ that can be found by differentiating $y^2=4px$ with respect to x:

$$y^2=4px$$

$$2y\frac{dy}{dx}=4p$$

$$\frac{dy}{dx}=\frac{2p}{y}\Rightarrow\left(\frac{dy}{dx}\right)_{(x_1,y_1)}=\frac{2p}{y_1}=m_1\Rightarrow m_2=\frac{-1}{m_1}=\frac{-y_1}{2p},\ say \quad (28)$$

The substitution of (28) in (27) is giving the normal equation at a point $p(x_1,y_1)$ to parabola $y^2=4px$:

$$y-y_1=m_2(x-x_1)$$

$$y-y_1=\frac{-y_1}{2p}(x-x_1) \quad (29) \quad \therefore m2=\frac{-y_1}{2p}$$

Example 7 Find the normal equation at a point $p(2,-4)$ to parabola $y^2=8x$.

Solution Result (29) is used to obtain the normal line to the given parabola:

$$y-y_1=\frac{-y_1}{2p}(x-x_1)$$

$$y-(-4)=\frac{4}{2(2)}(x-2),\quad P(x_1,y_1)=(2,-4),4p=8$$

$$y+4=x-2\Rightarrow x-2-y-4=0\Rightarrow x-y-6=0$$

(viii) Suspension and reflection problems related to parabola

The parabola is more than just a geometric concept. It has many uses in the physical world that are listed under:

1. Projectiles in the air, such as a ball, or a missile, or water sprayed from a hose, describe a parabolic path when acted on only by gravity.
2. Many arches of bridges or buildings are parabolic in shape. With this shape, the arch can support the structure above it.
3. Rotating a parabola about its line of symmetry, creates a bowl type surface called a paraboloid of revolution. A paraboloid has an important reflection property. Any ray or wave that originates at the focus and strikes the surface of the paraboloid is reflected parallel to the line of symmetry. See Figure 9.9.

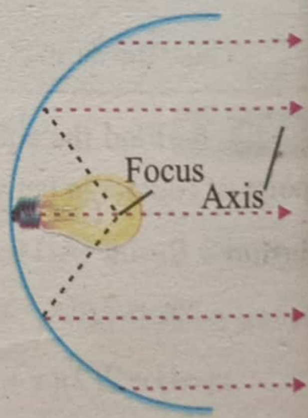


Figure 9.9

This forms the basic design of the reflectors for automobile headlights, flashlights, searchlights, telescopes, etc. This is also an excellent collecting device and is the basic design of TV, radar, and radio antennas.

Example 8 The cables of a bridge form a parabolic arc. The low point of the cable is 10ft above the roadway midway between two towers. The distance between the towers is 400 ft. The cable is attached to the towers 50ft above the roadway. Determine the equation of the parabola that describes the path of the cable. This is shown in the Figure 9.10:

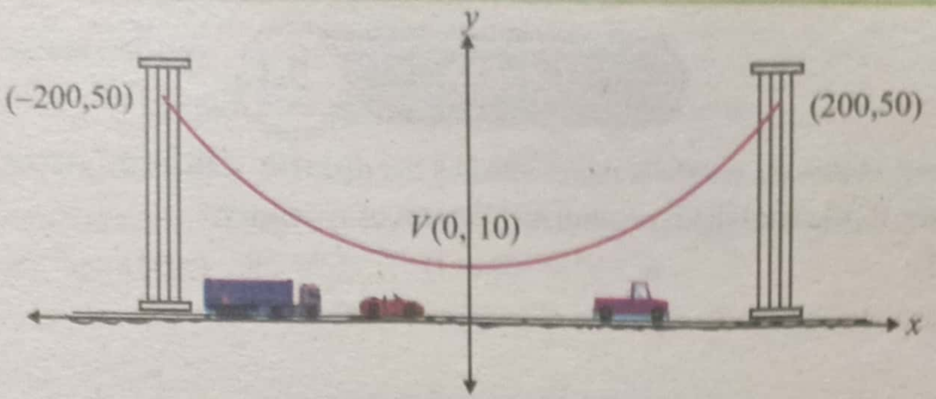


Figure 9.10

Solution The parabola is formed by the cable between the two towers. The low point on the cable is midway between the towers, and 10ft above the roadway. In order to write an equation, locate the *x–axis* and the *y–axis* in the *xy–plane*. Select the roadway as the *x–axis* and the line perpendicular to the roadway through the lowest point of the tower as the *y–axis*. The parabola opens up with the vertex at the point (0,10). Two other points on the parabola are (200,50) and (–200,50). The standard form for this equation is:

$$(x-h)^2 = 4p(y-k) \qquad (30)$$

The vertex V(0,10) and a point on the curve $(x, y) = (200,50)$ are used in (30) to obtain p:

$$(x-h)^2 = 4p(y-k), \text{ translate } h \text{ units on the } x-\text{axis}, \ k \text{ units on the } y-\text{axis}$$

$$(200-0)^2 = 4p(50-10)), \quad V(h,k) = V(0,10), (x,y) = (200,50)$$

$$40000 = 160p \Rightarrow p = 250$$

The substitution of $V(h, k) = (0,10)$ and $p = 250$ in equation (30) is giving the parabolic equation

$$(x-0)^2 = 4(250)(y-10)$$

$$x^2 = 1000(y-10) \Rightarrow x^2 - 1000y + 10000 = 0$$

that describes the path of the cable.

Example 9 A radar antenna is constructed so that a cross section along its axis is a parabola with the receiver at the focus. Find the focus if the antenna is 12 *m* across and its depth is 4 *m*. Find the equation of parabola that described the radar antenna. This is shown in the Figure 9.11.

Solution The parabola is formed by the radar antenna. In order to write an equation, locate the *x–axis* and the *y–axis* in the *xy–plane*. The axis of symmetry is the positive *x–axis*. The parabola opens to the right with the vertex at the origin *V*(0,0). The other point on the parabola is (4, 6). The standard form for this equation is:

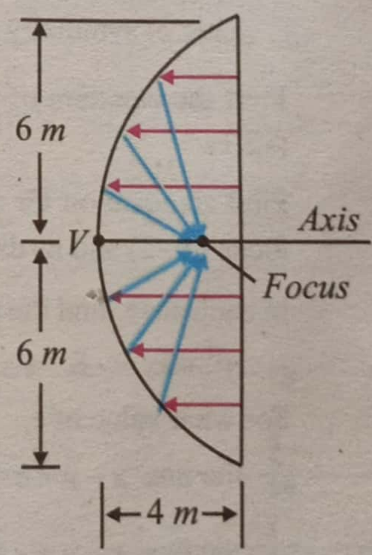


Figure 9.11

$$y^2 = 4px$$

$$36 = 4p(4), \quad (x,y) = (4,6) \qquad (31)$$

$$p = \frac{36}{16} = \frac{9}{4}$$

Thus, the parabolic equation that describes the radar antenna is obtained by putting $p = \frac{9}{4}$ in (31):

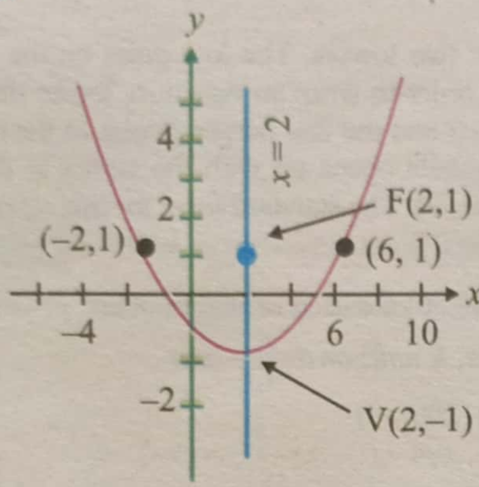
$$y^2 = 4px = 4\left(\frac{9}{4}\right)x = 9x$$

The focus is $F\left(\frac{9}{4}, 0\right)$ which is $\frac{9}{4}$ *m* from the vertex *V*(0,0).

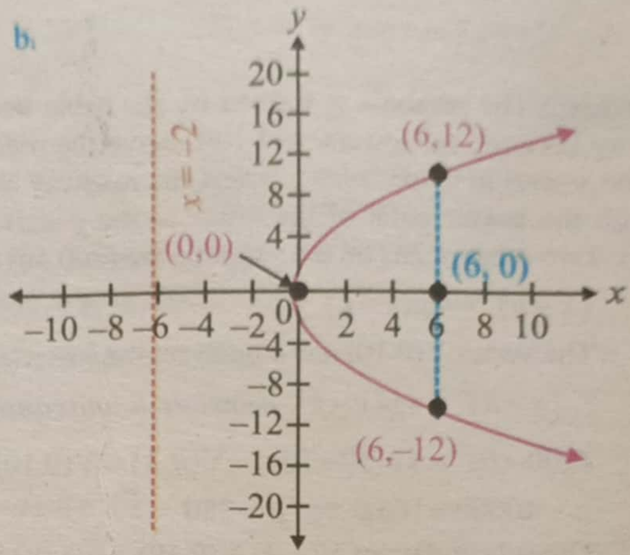
Exercise 9.1

1. In each case, sketch the parabola represented by the equation, indicate the vertex, the focus, the end points of the focal chord (latus rectum) and the axis of symmetry:
   a. $x^2 = 2y$ b. $y^2 = -3(x+1)$ c. $(y-3)^2 = x$
2. In each case, determine the equation of graphed parabola:

a.



b.



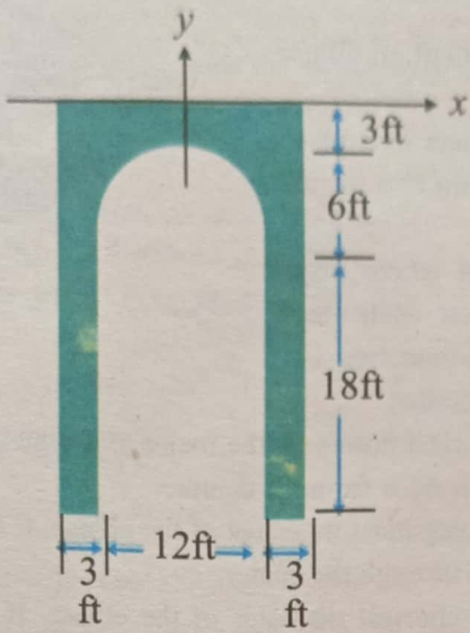
3. In each case, write the equation of parabola through the given information:
   a. Focus at $F(0,3)$, directrix $y=-3$. b. Focus at $F(4,0)$, directrix $x=-4$.
   c. Vertex at $V(0,0)$, *x–axis* is the line of symmetry, passes through (3,6).
   d. Vertex at $V(0,0)$, *y–axis* is the line of symmetry, passes through (–12,–3).
   e. Line of symmetry is vertical, passes through (–3,4), vertex at $V(5,1)$.
   f. Line of symmetry is horizontal, passes through (7,9), vertex at $V(3,-7)$.
4. Find the equation of the set of all points with distances from (4,3) that equal their distances from (–2,1).
5. Find an equation for a parabola whose focal chord has length 6, if it is known that the parabola has focus (4,–2) and its directrix is parallel to the *y–axis*.
6. In each case, find the points of intersection in between the line and the parabola:
   a. $y^2+3x=-8,\ x-y+2=0$ b. $x^2=2y,\ x-y-2=0$
7. For what value of c,
   a. the line $x-y+c=0$ will touch the parabola $y^2=9x$?
   b. the line $x-y+c=0$ will touch the parabola $x^2=\frac{2}{3}y$?
8. In each case, find the tangent equation and normal equation
   a. at a point (3,6) to parabola $y^2=12x$. b. at a point $\left(\frac{1}{2},\frac{-1}{3}\right)$ to parabola $x^2=\frac{-3y}{4}$.

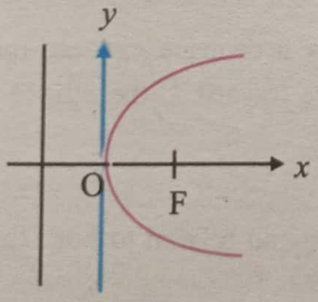
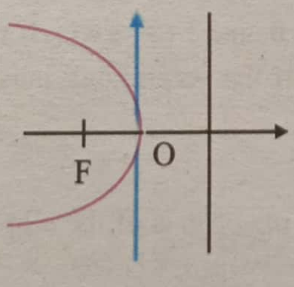
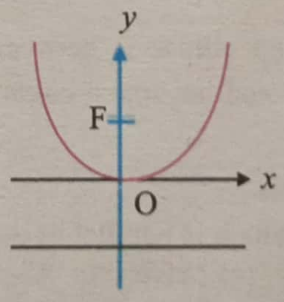
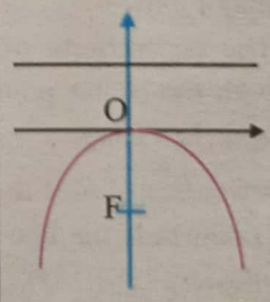
9. Find the tangent equation

a. to parabola $y^2 = x$, which makes an angle of $135°$ with the *x–axis*.

b. to parabola $x^2 = y$ which makes an angle of $60°$ with the *x–axis*.

10. Find the equation of the parabolic portion of the archway, if parabolic archway has the dimensions shown in the figure below:



Summary of standard Parabola				
Equation	$y^2 = 4ax$	$y^2 = -4ax$	$x^2 = 4ay$	$x^2 = -4ay$
Focus	$(a, 0)$	$(-a, 0)$	$(0, a)$	$(0, -a)$
Directrix	$x = -a$	$x = a$	$y = -a$	$y = a$
Vertex	$(0, 0)$	$(0, 0)$	$(0, 0)$	$(0, 0)$
Axis	$y = 0$	$y = 0$	$x = 0$	$x = 0$
Latus rectum	$x = a$	$x = -a$	$y = a$	$y = -a$
Graph				

9.2 Ellipse

In shape and in format, the ellipse is different from the parabola. Although the parabola is open at one end, the ellipse is entirely closed. The parabola has one focus and one vertex, while the ellipse has two foci (plural of focus) and two vertices.

(i) Ellipse and its elements

Ellipse

The second type of conic is called an ellipse, and is defined as follows.

An ellipse is the set of all points in a plane, the sum of whose distances from two distinct fixed points (foci) is constant.

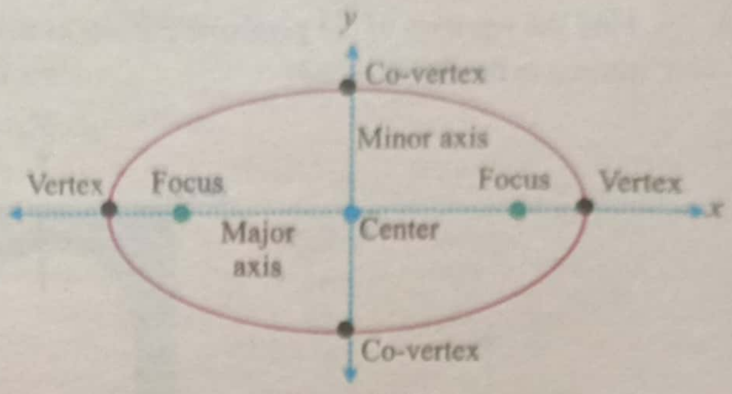


Figure 9.12

- **Center**– It is the point where major and minor axis intersects each other. The midpoint of the connecting two foci line segment is the center.
- **Focus** – There are two focal points on the major axis which defines the ellipse. These are at the same distance to the both sides from the center.
- **Major Axis** – It is the lengthiest diameter of the ellipse. It has the end points on the widest part of the ellipse and passes through the center.
- **Minor Axis** – It is the shortest diameter of the ellipse. It is the perpendicular bisector of the major axis. It has the end points on the narrow part of the ellipse and passes through the center.
- **Vertices** – The four points where the major and minor axis touches the ellipse are the vertices. The end points of major axis are generally called **Vertex** and the end points of minor axis are generally called **Co-vertex**.
- **Chord** – It is a line segment that has both the end points on the ellipse. Major axis is also the chord which is the longest one in an ellipse.

Eccentricity of an Ellipse

Eccentricity is the factor related to conic sections which shows how circular the conic section is. More eccentricity means less spherical and less eccentricity means more spherical. It is denoted by "e".

The eccentricity of an ellipse is showed by the ratio of the distance between the two foci, to the size of the major axis,

$$e = \frac{c}{a}.$$

where $e =$ Eccentricity, $c =$ The distance from the center to any one of the foci and $a =$ The semi major axis.

The eccentricity of an ellipse is between 0 and 1 $(0 < e < 1)$. If the eccentricity is zero the foci match with the center point and become a circle. If the eccentricity moves toward 1, the ellipse gets a more stretched shape.

Directrix of an Ellipse

Directrix is the line which is parallel to the minor axis of the ellipse and related to both the foci of the ellipse.

Directrix of an ellipse

Directrix is the line which is parallel to the minor axis of the ellipse and related to both the foci of the ellipse.

Latus rectum of an Ellipse

It is the line parallel to directrix and passes through any of the focus of an ellipse. It is denoted by "$2l$". In an ellipse, latus rectum is $2b^2/a$ (where a is one half of the major diameter and b is the half of the minor diameter).

The half of latus rectum till its intersection point with the major axis is the semi latus rectum. It is denoted by "l".

Note: General form of the ellipse

If $P(x, y)$ is any point on the ellipse, then the distances from the two foci $F_1(-c,0)$ and $F_2(c,0)$ to the point $P(x, y)$ are the following:

$$d(F_1,P)=(x+c,y-0)$$

$$|d(F_1,P)|=\sqrt{(x+c)^2+(y)^2}$$

$$d(F_2,P)=(x-c,y-0)$$

$$|d(F_2,P)|=\sqrt{(x-c)^2+(y)^2}$$

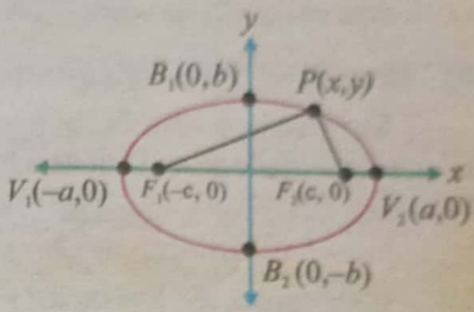


Figure 9.13

By definition of an ellipse, the general form of an ellipse is:

$$|d(F_1,P)|+|d(F_2,P)|=2a \qquad (32)$$

$$\sqrt{(x+c)^2+(y)^2}+\sqrt{(x-c)^2+y^2}=2a$$

$$\sqrt{(x+c)^2+y^2}=2a-\sqrt{(x-c)^2+y^2}$$

Squaring both sides to obtain

$$(x+c)^2+y^2=4a^2-4a\sqrt{(x-c)^2+y^2}+(x-c)^2+y^2$$

$$4a\sqrt{(x-c)^2+y^2}=4a^2-4cx,$$

$$a\sqrt{(x-c)^2+y^2}=a^2-cx \qquad (33)$$

Again squaring to obtain

$$a^2[(x-c)^2+y^2]=a^4-2a^2cx+c^2x^2$$

$$a^2\left(x^2-2cx+c^2+y^2\right)=a^4-2a^2cx+c^2x^2$$

$$a^2x^2-2a^2cx+a^2c^2+a^2y^2=a^4-2a^2cx+c^2x^2$$

$$a^2x^2-c^2x^2+a^2y^2=a^4-a^2c^2$$

$$(a^2-c^2)x^2+a^2y^2=a^2(a^2-c^2)$$

$$b^2x^2+a^2y^2=a^2b^2, \qquad a^2-c^2=b^2,\ a>0,b>0$$

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, \qquad \textit{divide out by } a^2b^2 \qquad (34)$$

(ii) Circle is a special case of an ellipse

The relative shape of an ellipse can be determined by its eccentricity e. The distance from the center of the ellipse to a foci is c, and the distance from the center to a vertex is a. The eccentricity is given by the equation:

$$e = \frac{\text{distance from center to focus}}{\text{distance from center to vertex}} = \frac{c}{a} \qquad (35)$$

The eccentricity of all ellipses are in a range between 0 and 1 $(0 < e < 1)$. This is shown in the Figure 9.14.

An ellipse with an eccentricity close to 1 is long and thin, and the foci are relatively far apart. If the eccentricity is small, close to 0, then the ellipse resembles a circle. It can be shown that the circle is a special case of the ellipse when $e = 0$.

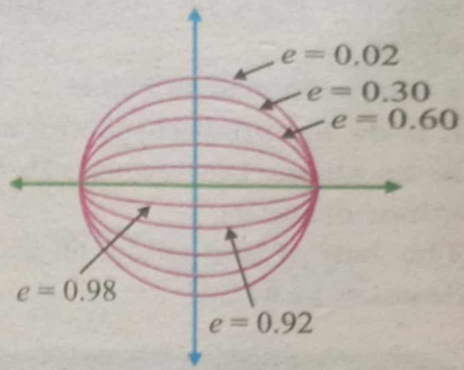


Figure 9.14

(iii) Standard form of equation of an ellipse

The standard form of the equation of an ellipse with center at the origin, length of the semimajor axis a, length of the semiminor axis b and major axis along the x–*axis* is shown in the Figure 9.15.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad (36)$$

If we replace the foci on the y–*axis*, center at the origin, and pick any point $P(x, y)$ on the plane, then we can develop the equation of the vertical ellipse given in the following definition.

"The standard form of the equation of an ellipse with center at the origin, length of the semimajor axis a, length of the semiminor axis b and major axis along the y–axis is shown in the Figure 9.16.

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1 \qquad (37)"$$

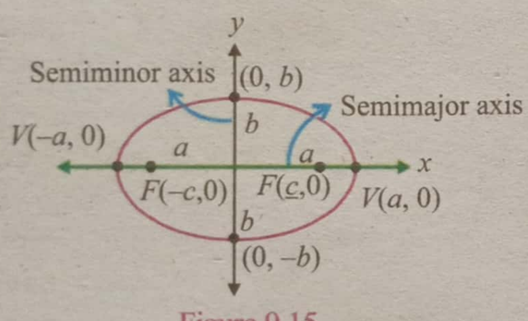


Figure 9.15

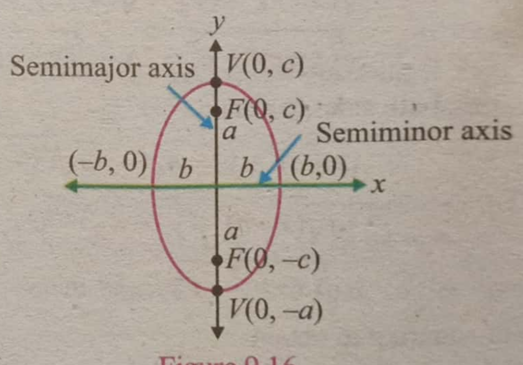


Figure 9.16

Graphing Ellipse: In order to sketch an ellipse, it is required to plot the center, the intercepts $\pm a$ on the major axis and $\pm b$ on the minor axis.

First, rewrite the equation of the ellipse in the standard form, so that there is a "1" on the right and the numerator coefficients of the square terms are also 1. The center is at (0,0) and plot the intercepts on the x–axis and y–axis. For the x–intercepts, plot $\pm$ the square root of the number a^2; for the y–intercepts, plot $\pm$ the square root of the number b^2, finally, and draw the ellipse using these intercepts. The longer axis is called the **major axis**. If this larger axis is horizontal, then the ellipse is called **horizontal**, and if the major axis is vertical, the ellipse is then called **vertical**.

The orientation of the ellipse equation with center $C(0,0)$, vertices/end points of the major axis and the end points of the semiminor axis are summarized in the box:

Orientation	Foci	Vertices/Semimajor axis	Semiminor axis
Horizontal:	$(-c,0),(c,0)$	$(-a,0),(a,0)$	$(0,b),(0,-b)$
Vertical:	$(0,c),(0,-c)$	$(0,-a),(0,a)$	$(b,0),(-b,0)$
Here $c^2=a^2+b^2$ or	$c^2=a^2-b^2$	$a>b>0$	

Example 10 Determine the vertices, end points of the minor axis and the coordinates of foci of the ellipse $9x^2+4y^2=36$. Sketch the ellipse.

Solution Rewrite the ellipse equation in the standard form:

$$9x^2+4y^2=36$$

$$\frac{9x^2}{36}+\frac{4y^2}{36}=1, \quad \textit{divide out by } 36$$

$$\frac{x^2}{4}+\frac{y^2}{9}=1 \qquad \text{(38-a)}$$

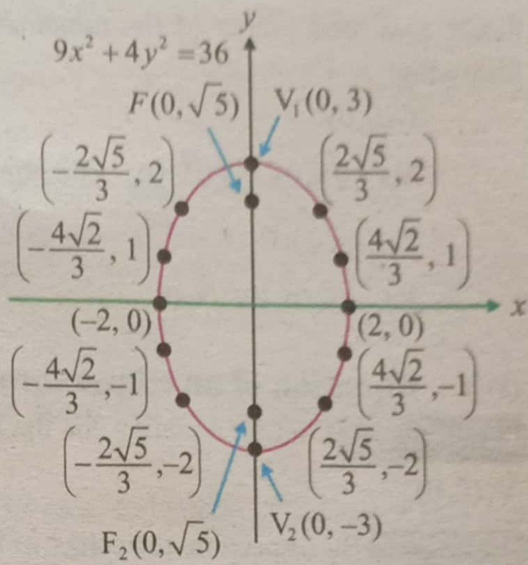


Figure 9.17

The equation (38-a) is related to the vertical standard form ellipse (37). The center of the ellipse is at the origin, but the vertices of the major axis are on the y–axis, since the larger numerical value is under y^2. Thus, $a^2=9$ or $a=3$, and $b^2=4$ or $b=2$ and $c^2=a^2-b^2=9-4=5$ or $c=\pm\sqrt{5}$.

The coordinates of the center, vertices/end points of the major axis, end points of the minor axis and the foci are the following:

$C(0,0)$	center
$V_1(0,3), V_2(0,-3)$	end points of the major axis
$B_1(2,0), B_2(-2,0)$	end points of the minor axis
$F_1\left(0,\sqrt{5}\right), F_2\left(0,-\sqrt{5}\right)$	foci

For some points on the ellipse,

when $y=1$, then $\frac{x^2}{4}+\frac{1}{9}=1 \Rightarrow x=\pm 4\frac{\sqrt{2}}{3}$

when $y=2$, then $\frac{x^2}{4}+\frac{4}{9}=1 \Rightarrow x=\pm 2\frac{\sqrt{5}}{3}$

The ellipse is symmetrical with respect to the major axis, minor axis. The center, vertices, foci, and the points

$$\left(\frac{4\sqrt{2}}{3},-1\right),\left(\frac{-4\sqrt{2}}{3},1\right),\left(\frac{-4\sqrt{2}}{3},-1\right),\left(\frac{2\sqrt{5}}{3},-2\right),\left(\frac{-2\sqrt{5}}{3},2\right),\left(\frac{-2\sqrt{5}}{3},-2\right)$$

are labeled to obtain the graph of the given ellipse in Figure 9.17.

Example 11 Determine the vertices, end points of the minor axis and the coordinates of foci of the ellipse $2x^2+5y^2=10$. Sketch the ellipse.

Solution Rewrite the ellipse equation in the standard form:

$$2x^2+5y^2=10$$

$$\frac{2x^2}{10}+\frac{5y^2}{10}=1, \quad \textit{divide out by } 10$$

$$\frac{x^2}{5}+\frac{y^2}{2}=1 \Rightarrow \frac{x^2}{(\sqrt{5})^2}+\frac{y^2}{(\sqrt{2})^2}=1 \qquad \text{(38-b)}$$

The ellipse (38-b) is related to the horizontal standard form ellipse (36). The center of the ellipse is at the origin, but the vertices of the major axis are on the x-axis, since the larger numerical value is under x^2. Thus, $a^2=5$ or $a=\sqrt{5}$ and $b^2=2$ or $b=\sqrt{2}$ and $c^2=a^2-b^2=5-2=3$ or $c=\pm\sqrt{3}$.

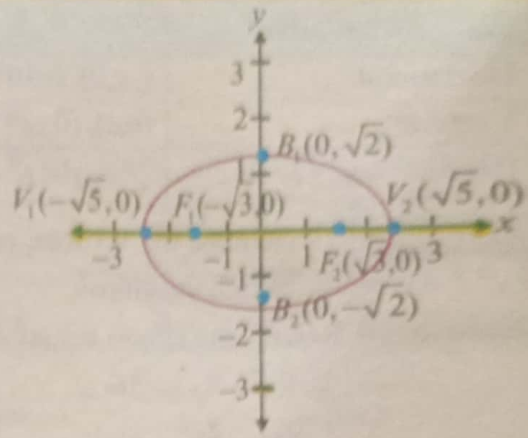


Figure 9.18

The coordinates of the center, vertices/end points of the major axis, end points of the minor axis and the foci are the following:

$C(0,0)$	center
$V_1(-\sqrt{5},0), V_2(\sqrt{5},0)$	end points of the major axis
$B_1(0,\sqrt{2}), B_2(0,-\sqrt{2})$	end points of the minor axis
$F_1(-\sqrt{3},0), F_2(\sqrt{3},0)$	foci

(iv) Equation of an ellipse through its elements

Example 12 Find an equation for the ellipse with foci $F_1(-1,0)$ and $F_2(1,0)$ and vertices $V_1(-2,0)$ and $V_2(2,0)$.

Solution By inspection, the center of the ellipse is at C(0,0) and the distance from the center to the vertex is $a=2$; and the distance to a focus is $c=1$. The value of b is obtained by inserting a and c in the equation:

$$b^2=a^2-c^2=4-1=3 \Rightarrow b=\pm\sqrt{3}$$

The values of a and b are used in the horizontal standard form ellipse (36) to obtain

$$\frac{x^2}{4}+\frac{y^2}{3}=1 \qquad (39)$$

(v) Standard form of equation of an ellipse

The standard form of the equation of an ellipse with center at $C(h, k)$, length of the semimajor axis a and semiminor axis b, and major axis parallel to the *x–axis* is:

$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1 \qquad (40)$$

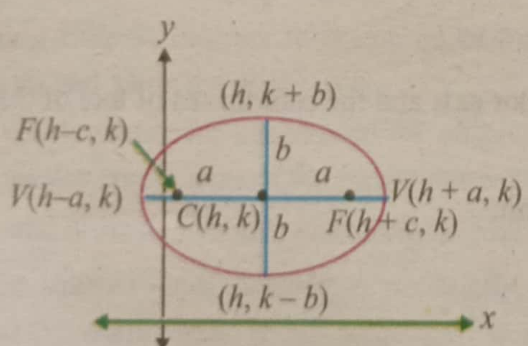


Figure 9.19

The standard form of the equation of an ellipse with center at $C(h, k)$, length of the semimajor axis a and semiminor axis b, and major axis parallel to the *y–axis* is:

$$\frac{(x-h)^2}{b^2}+\frac{(y-k)^2}{a^2}=1 \qquad (41)$$

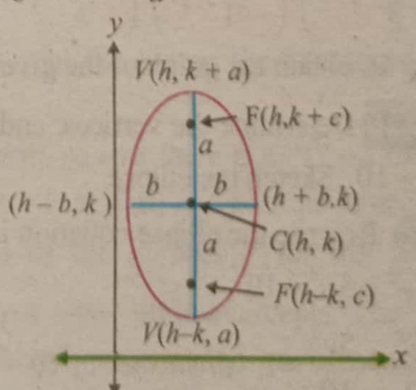


Figure 9.20

Example 13 Graph the ellipse whose equation is $4x^2+25y^2-8x+100y+4=0$. Indicate the center, vertices, foci and the end points of the minor axis.

Solution: Rewrite the given ellipse equation to the standard form by completing square:

$$4x^2-8x+25y^2+100y=-4$$

$$4(x^2-2x+1)+25(y^2+4y)=0$$

Add and subtract 100 to obtain

$$4\left(x^2-2x+1\right)+25\left(y^2+4y+4\right)=100$$

$$4(x-1)^2+25(y+2)^2=100$$

$$\frac{4(x-1)^2}{100}+\frac{25(y+2)^2}{100}=1 \Rightarrow \frac{(x-1)^2}{25}+\frac{(y+2)^2}{4}=1 \qquad (42)$$

The given ellipse (42) with substitution $X=x-h=x-1$ and $Y=y-k=y+2$, $h=1$, $k=-2$ gives the translated ellipse in the XY–plane:

$$\frac{X^2}{25}+\frac{Y^2}{4}=1 \qquad (43)$$

The center of the ellipse is at the origin. The major axis is horizontal and the vertices are on the x-axis. Thus, $a=5$, $b=2$ and $c=\pm\sqrt{21}=\pm4.58$.

The coordinates of the center, vertices/end points of the major axis, end points of the minor axis and the foci of the translated ellipse (43) are the following:

$C(0,0)$	center
$V_1(-5,0), V_2(5,0)$	end points of the major axis
$B_1(0,2), B_2(0,-2)$	end points of the minor axis
$F_1(-4.58,0), F_2(4.58,0)$, $\pm\sqrt{21}=\pm4.58$	foci

The coordinates of the center, vertices/end points of the major axis, the end points of the minor axis and foci of the given ellipse (42) are the following:

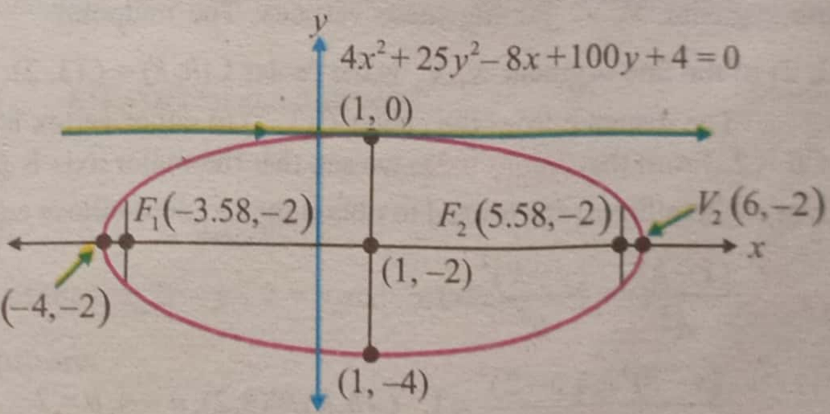


Figure 9.21

- The coordinates of the center $C(0,0)$ of the translated ellipse are $X=0$, $Y=0$. Put $X=0$ and $Y=0$ in (43) to obtain the coordinates of the center of the given ellipse (42):
  $X=x-1 \Rightarrow 0=x-1 \Rightarrow x=1$ and $Y=y+2 \Rightarrow 0=y+2 \Rightarrow y=-2$
  The center of the given ellipse (42) is $C(1,-2)$.
- The coordinates of the vertices $V_1(-5,0)$, $V_2(5,0)$ of the translated ellipse are $X=-5$, $Y=0$ (in case of V_1) and $X=-5$, $Y=0$ (in case of V_2). Put $X=-5$ and $Y=0$ in (43) to obtain the coordinates of the vertex V_1 of the given ellipse (42):
  $X=x-1 \Rightarrow -5=x-1 \Rightarrow x=-4$ and $Y=y+2 \Rightarrow 0=y+2 \Rightarrow y=-2$

The vertex V_1 of the given ellipse (42) is $V_1(-4,-2)$ and the vertex V_2 of the given ellipse (42) is of course $V_2(6,-2)$.

- The coordinates of the foci $F_1(-4.58,0)$, $F_2(4.58,0)$ of the translated ellipse are $X=-4.58$, $Y=0$ (in case of F_1) and $X=4.58$, $Y=0$ (in case of F_2). Put $X=-4.58$ and $Y=0$ in (43) to obtain the coordinates of the focus F_1 of the given ellipse (42):

  $X=x-1 \Rightarrow -4.58=x-1 \Rightarrow x=-3.58$ and $Y=y+2 \Rightarrow 0=y+2 \Rightarrow y=-2$

  The focus F_1 of the given ellipse (42) is $F_1(-3.58,-2)$ and the focus F_2 of the given ellipse (42) is of course $F_2(5.58,-2)$.

  The graph of the ellipse is shown in the Figure 9.21.

The orientation of the ellipse equation with center $C(h, k)$ are summarized in the boxes:

Orientation	Foci	Vertices/End Points of Major Axis	End Points of Minor Axis
Horizontal	$F_1(h-c,k), F_2(h+c,k)$	$V_1(h-a,k), V_2(h+a,k)$	$B_1(h,k+b), B_2(h,k-b)$
Vertical	$F_1(h,k+c), F_2(h,k-c)$	$V_1(h,k+a), V_2(h,k-a)$	$B_1(h+b,k), B_2(h-b,k)$

Note that $b^2=a^2-c^2$ or $c^2=a^2-b^2$ with $a>b>0$.

Example 14 Find the equation of the ellipse with vertices at (–1, 2) and (7, 2) and with 2 as the length of the semiminor axis.

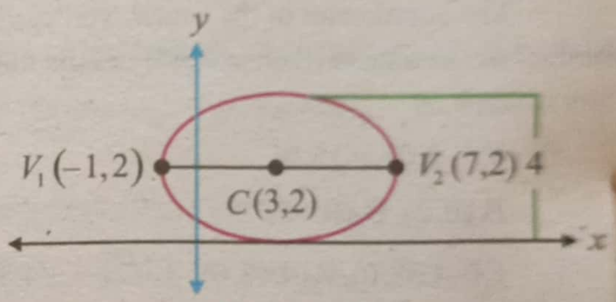


Figure 9.22

Solution With the vertices of the ellipse are at $V_1(-1,2)$ and $V_2(7,2)$, the center is at the midpoint of the line segment V_1V_2 joining these vertices. The midpoint (3, 2) of the line segment V_1V_2 is the center $C(h, k) = C(3, 2)$. This is shown in the Figure 9.22.

The distance from the center C(3, 2) to either vertex is $a=4$ units. The semiminor axis has a length of $b=2$. From the Figure 9.22, we see that the major axis is parallel to the x–axis. The horizontal standard form of the ellipse (40) is used to obtain the required ellipse equation:

$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$$

$$\frac{(x-3)^2}{16}+\frac{(y-2)^2}{4}=1,\quad C(h,k)=(3,2), a=4, b=2 \qquad (44)$$

(vi) Recognition of tangent and normal to an ellipse

A. Tangent to an ellipse

"A line that interacts the ellipse at a point is known as tangent at the ellipse" in the Figure 9.23, the line LM is tangent to the ellipse which is intersecting the ellipse at point 'P' as shown in Figure 9.23.

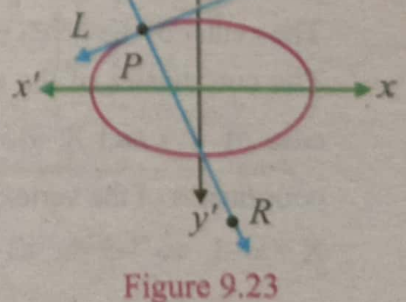


Figure 9.23

B. Normal to an ellipse

Normal to an ellipse is a line perpendicular to the tangent to curve through the point of contact. Line QR is normal to the ellipse which is perpendicular to the tangent LM at point 'P', as shown in Figure 9.23.

(vii) Point of Intersection of an ellipse and a line

The given line and ellipse

$$y = mx + c \quad (45)$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad (46)$$

develops a system of nonlinear equations:

$$\left.\begin{aligned} \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \\ y = mx + c \end{aligned}\right\} \quad (47)$$

Note

The angle between tangent to ellipse and normal is always a right angle.

The solution set $\{x, y\}$ of nonlinear system of equations (47) exists only, if the curves of the system (47) are intersecting. That set of points of intersection $\{x, y\}$ (a solution set) can be found by solving the nonlinear system (47) simultaneously.

The line (45) is used in ellipse (46) to obtain the quadratic equation in x:

$$\frac{x^2}{a^2} + \frac{(mx+c)^2}{b^2} = 1$$

$$x^2\left(a^2m^2 + b^2\right) + 2a^2mcx + a^2\left(c^2 - b^2\right) = 0 \quad (48)$$

The equation (48) being a quadratic equation in x, gives a set of two values x_1 and x_2 of x, which will be used in a line (45) to obtain a set of two values y_1 and y_2 of y.

Thus, a solution set $\{(x_1, y_1), (x_2, y_2)\}$ of the system (47) is of course a set of points of intersection of the system (47).

The points of intersection of the system (47) are real, coincident or imaginary, according as the roots of the quadratic equation (48) are real, coincident or imaginary, according as the discriminant of the quadratic equation (48)

$$Disc = 4a^4m^2c^2 - 4\left(a^2m^2 + b^2\right)\left(a^2\right)\left(c^2 - b^2\right) > 0, \text{ real and different}$$

$$Disc = 4a^4m^2c^2 - 4\left(a^2m^2 + b^2\right)\left(a^2\right)\left(c^2 - b^2\right) = 0, \text{ real and coincident}$$

$$Disc = 4a^4m^2c^2 - 4\left(a^2m^2 + b^2)\right)\left(a^2\right)\left(c^2 - b^2\right) < 0, \text{ imaginary}$$

Example 15 Find the points of intersection of the line $2x - y - 2 = 0$ and the ellipse $4x^2 + 9y^2 = 36$.

Solution The equations of the line and ellipse are:

$$2x - y - 2 = 0 \quad (49)$$

$$y = 2x - 2$$

$$4x^2 + 9y^2 = 36 \quad (50)$$

The line (49) is used in an ellipse (50) to obtain the x–coordinates of the points of intersection:

$$4x^2 + 9(2x-2)^2 = 36$$

$$4x^2 + 9(4x^2 + 4 - 8x) - 36 = 0$$

$$40x^2 - 72x = 0$$

$$\Rightarrow x = 0, \; x = \frac{9}{5}$$

The x–coordinates are used in the line (49) to obtain the y–coordinates: $x = 0, \frac{9}{5}$ give $y = -2, \frac{8}{5}$

Thus, the set of two points of intersection (0,–2) and $\left(\frac{9}{5}, \frac{8}{5}\right)$ are real and distant and the line $2x - y - 2 = 0$ intersects the ellipse (50) at points (0,–2) and $\left(\frac{9}{5}, \frac{8}{5}\right)$.

(viii) The equation of a tangent line in slope-form

If m is the slope of the tangent line to ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ (51)

then the equation of that tangent line is of the form $y = mx + c$ (52)

Here c is to be calculated from the fact that the line (52) is tangent to ellipse (51).

The line (52) is used in an ellipse (51) to obtain the quadratic equation in x:

$$\frac{x^2}{a^2} + \frac{(mx+c)^2}{b^2} = 1$$

$$x^2\left(a^2m^2 + b^2\right) + 2a^2mcx + a^2\left(c^2 - b^2\right) = 0 \quad (53)$$

If the line (52) touches the ellipse (51), then the quadratic equation (53) has coincident roots for which the discriminant of the quadratic equation (53) equals zero:

$$4a^4m^2c^2 - 4\left(a^2m^2 + b^2\right)\left(a^2\right)\left(c^2 - b^2\right) = 0$$

$$a^2m^2c^2 - \left(a^2m^2 + b^2\right)\left(c^2 - b^2\right) = 0, \textit{ divide out by } 4a^2$$

$$a^2m^2c^2 - a^2m^2c^2 + a^2m^2b^2 - b^2c^2 + b^4 = 0$$

$$a^2m^2b^2 - b^2c^2 + b^4 = 0$$

$$-b^2c^2 = -\left(a^2m^2b^2 + b^4\right)$$

$$c^2 = a^2m^2 + b^2$$

$$c = \pm\sqrt{a^2m^2 + b^2} \quad (54)$$

The equation (54) is the **condition of tangency**. The value of c from equation (54) is used in the line (52) to obtain the required equation of the tangent line: $y = mx + c = mx \pm \sqrt{a^2m^2 + b^2}$ (55)

Note

- the equation of any tangent to ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ in the slope–form is:

  $y = mx \pm \sqrt{a^2m^2 + b^2}$ (56)

- **Condition of Tangency:** The line $y = mx + c$ should touch the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ under condition: $c = \pm\sqrt{a^2m^2 + b^2}$ (57)

Example 16 For what value of c, the line $2x-y+c=0$ will touch an ellipse $\frac{x^2}{3}+\frac{y^2}{4}=1$. Use those values of c to find the tangent lines that should touch the given ellipse.

Solution The values of c at which the line $2x-y+c=0$ will touch the given ellipse through result (57) are: $c=\pm\sqrt{a^2m^2+b^2}=\pm\sqrt{3(2)^2+4}=\pm4$

Here $m=2$ is the slope of the line $2x-y+c=0$.

The required tangent lines that should touch the ellipse through result (56) is:

$$y=mx+c=2x\pm4,\ m=2$$

(ix) The equation of a tangent line to ellipse at a point

The equation of a tangent line at a point $P(x_1,y_1)$ to ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is: $y-y_1=m_1(x-x_1)$ (58)

Here m_1 is the slope of the tangent line to ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at a point $P(x_1,y_1)$ that can be found by differentiating $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ with respect to x:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

$$\frac{2x}{a^2}+\frac{2y}{b^2}\frac{dy}{dx}=0$$

$$\frac{dy}{dx}=\frac{-b^2x}{a^2y}$$

$$\left(\frac{dy}{dx}\right)_{(x_1,y_1)}=\frac{-b^2x_1}{a^2y_1}=m_1 \quad (59)$$

The substitution of (59) in (58) is giving the equation of the tangent line at a point $P(x_1,y_1)$ to ellipse:

$$y-y_1=m_1(x-x_1)$$

$$y-y_1=-\frac{b^2x_1}{a^2y_1}(x-x_1)$$

$$\frac{yy_1}{b^2}-\frac{y_1^2}{b^2}=-\frac{xx_1}{a^2}+\frac{x_1^2}{a^2}\Rightarrow\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}$$

$$\Rightarrow\frac{xx_1}{a^2}+\frac{yy_1}{-b^2}=1 \quad (60) \qquad \therefore\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}=1\ at\ (x_1,y_1)$$

Example 17 Find the equation of the tangent at a point $P\left(3,\frac{12}{5}\right)$ to ellipse $\frac{x^2}{25}+\frac{y^2}{9}=1$.

Solution Result (60) is used to obtain the tangent line to the given ellipse:

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$$

$$\frac{x(3)}{25}+\frac{y\left(\frac{12}{5}\right)}{9}=1 \qquad \therefore(x_1,y_1)=\left(3,\frac{12}{5}\right),a^2=25,b^2=9$$

$$\frac{3x}{25}+\frac{12y}{45}=1\Rightarrow27x+60y-225=0\Rightarrow9x+20y-75=0$$

(x) The equation of a normal line to ellipse at a point

The equation of a normal at a point $P(x_1, y_1)$ to ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is:

$$y-y_1=m_2(x-x_1) \qquad (61)$$

Here m_2 is the slope of the normal to ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at a point $P(x_1, y_1)$ that can be found by differentiating $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ with respect to x:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

$$\frac{2x}{a^2}+\frac{2y}{b^2}\frac{dy}{dx}=0$$

$$\frac{dy}{dx}=-\frac{b^2x}{a^2y}$$

$$\left(\frac{dy}{dx}\right)_{(x_1,y_1)}=\frac{-b^2x_1}{a^2y_1}=m_1$$

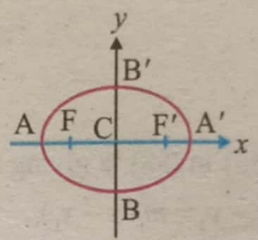
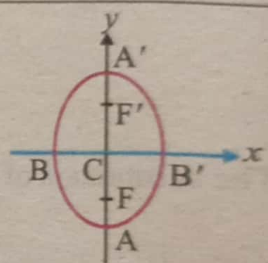
$$m_2=-\frac{1}{m_1}=\frac{a^2y_1}{b^2x_1} \qquad (62)$$

The substitution of (62) in (61) is giving the normal equation at a point $P(x_1, y_1)$ to ellipse:

$$y-y_1=m_2(x-x_1)$$

$$y-y_1=\frac{a^2y_1}{b^2x_1}(x-x_1),\ m_2=\frac{a^2y_1}{b^2x_1}$$

$$\frac{y-y_1}{\frac{y_1}{b^2}}=\frac{x-x_1}{\frac{x_1}{a^2}} \qquad (63)$$

Summary of standard Ellipses		
Equation	$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, a>b$ $c^2=a^2-b^2$	$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, a<b$ $c^2=a^2-b^2$
Focus	$(\pm C, 0)$	$(0, \pm C)$
Directrices	$x=\pm\frac{c}{e^2}$	$y=\pm\frac{c}{e^2}$
Major axis	$y=0$	$x=0$
Vertices	$(\pm a, 0)$	$(0, \pm a)$
Covertices	$(0, \pm b)$	$(\pm b, 0)$
Centre	$(0,0)$	$(0,0)$
Eccentricity	$e=\frac{c}{a}<1$	$e=\frac{c}{a}<1$
Graph		

Example 18 Find the normal equation at a point $P\left(3, \frac{12}{5}\right)$ to ellipse $\frac{x^2}{25}+\frac{y^2}{9}=1$.

Solution Result (63) is used to obtain the normal line to the given ellipse:

$$\frac{x-x_1}{\frac{x_1}{a^2}}=\frac{y-y_1}{\frac{y_1}{b^2}} \qquad \therefore (x_1, y_1)=\left(3, \frac{12}{5}\right), a^2=25, b^2=9$$

$$\frac{x-3}{\frac{3}{25}}=\frac{y-\frac{12}{5}}{\frac{\left(\frac{12}{5}\right)}{(9)}}$$

$$\frac{25(x-3)}{3}=\frac{3(5y-12)}{4}$$

$$100(x-3)=9(5y-12) \Rightarrow 100x-300-45y+108=0 \Rightarrow 100x-45y-192=0$$

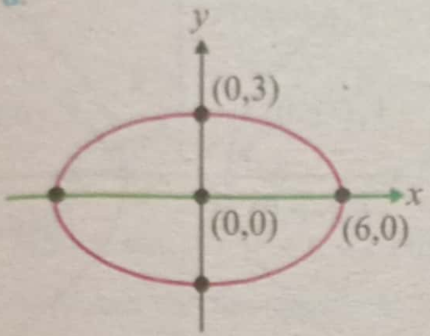
Exercise 9.2

1. In each case, sketch the ellipse represented by the equation. Indicate the center, foci, endpoints of the major axis and end points of the minor axis:

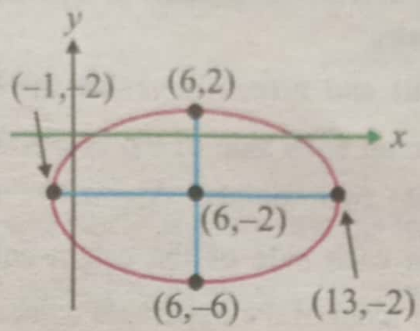
a. $\frac{x^2}{9}+\frac{y^2}{4}=1$ b. $\frac{x^2}{16}+\frac{y^2}{25}=1$ c. $\frac{(x+1)^2}{16}+\frac{(y-2)^2}{9}=1$

2. In each case, determine the equation of graphed ellipse:

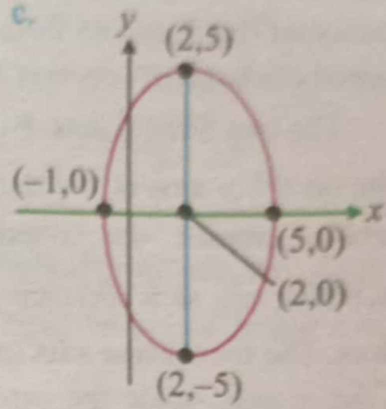
a.



b.



c.



3. In each case, write the equation of ellipse through the given information:.

a. Center is at $(-3, 2)$, $a=2$, $b=1$, major axis is horizontal.

b. Vertices are at $(4, 2)$ and $(12,2)$, $b=2$.

c. A focus is at $(-2, 3)$, a vertex is at $(6,3)$, length of minor axis is 6.

d. Vertices are at $(0,8)$ and $(0,2)$, $c=\sqrt{5}$.

4. The shape of an ellipse depends on the eccentricity of the ellipse $e=\frac{c}{a}$. Determine

a. the eccentricity of the ellipse $\frac{x^2}{25}+\frac{y^2}{16}=1$.

b. the equation of the ellipse with vertices are at $(-5,0)$, and $(5,0)$ and the eccentricity is $e=\frac{3}{5}$.

c. the eccentricity of the ellipse, if the length of the semimajor axis is $a=4$ and the length of the semiminor axis is $b=2$.

5. For what value of c,

a. the line $x-y+c=0$ will touch the ellipse $\frac{x^2}{4}+\frac{y^2}{1}=1$?

b. the line $2x-y+c=0$ will touch the ellipse $\frac{x^2}{3}+\frac{y^2}{4}=1$?

c. the line $x+y+c=0$ will touch the ellipse $\frac{x^2}{25}+\frac{y^2}{11}=1$?

6. In each case, find the tangent equation and normal equation

a. at a point $(1,2)$ to ellipse $\frac{x^2}{4}+\frac{y^2}{9}=1$? b. at a point $(3,6)$ to ellipse $\frac{x^2}{7}+\frac{y^2}{4}=1$?

c. at a point $(1,1)$ to ellipse $\frac{x^2}{4}+\frac{y^2}{1}=1$?

7. Find the tangent equation

a. to the ellipse $\frac{x^2}{4}+\frac{y^2}{9}=1$ which is perpendicular to the line $9x+8y-36=0$.

b. to the ellipse $\frac{x^2}{7}+\frac{y^2}{4}=1$ which is parallel to the line $6x+21y-14=0$.

9.3 Hyperbola

The last of the conic sections to be considered has a definition similar to that of the ellipse. In the previous section, the ellipse was expressed in terms of the **sum** of two distances being a constant. Exactly, the hyperbola is expressed in terms of the **difference** of two distances being a constant.

(i) Hyperbola and its center, foci, vertices, eccentricity, focal chord, transverse and conjugate axes, eccentricity, focal chord and latus recta)

A hyperbola is the set of all points in the plane such that the difference of the distances from two fixed points (foci) to a point on the hyperbola hold off constant quantity.

The two fixed points $F_1(-c,0)$ and $F_2(c,0)$ are called the **foci** that lie on the x–axis at a distance **c** on each side of the origin. The hyperbola crosses the x–axis at two points $V_1(-a,0)$ and $V_2(a,0)$ that are at a distance a on each side of the origin called **vertices**. The **transverse axis** (major axis) of the hyperbola coincides with the x–axis, while the **conjugate axis** (minor axis) of the hyperbola coincides with the y–axis. This is shown in the Figure 9.24.

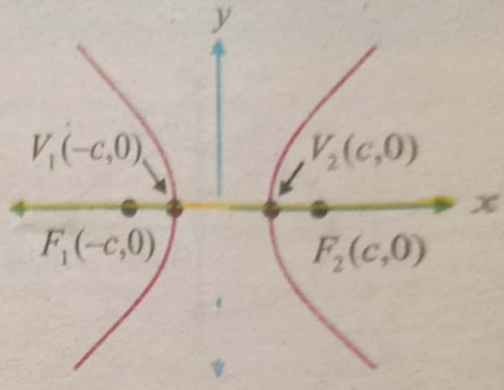


Figure 9.24

To check the definition, the absolute value of the difference of the distances from the two foci $F_1(-c,0)$, $F_2(c,0)$ to the point $V_2(a,0)=P$, *say*, is:

$$|d(F_1,V_2)-d(F_2,V_2)|=|(a+c,0)-(c-a,0)|=|2a| \qquad (64)$$

Since **a** is a measured distance and is always positive, the constant specified in the definition of a hyperbola equals 2a.

(ii) General form of the hyperbola

If $P(x,y)$ is any point on the hyperbola, then the distances from the two foci $F_1(-c,0)$ and $F_2(c,0)$ to the point $P(x,y)$ are the following:

$$F_1P=(x+c,y-0)$$
$$d(F_1,P)=\sqrt{(x+c)^2+(y)^2}$$
$$F_2P=(x-c,y-0)$$
$$d(F_2,P)=\sqrt{(x-c)^2+(y)^2}$$

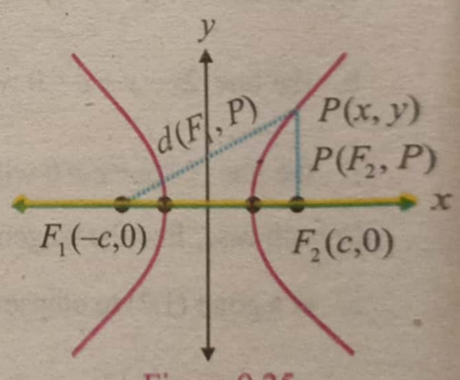


Figure 9.25

The definition of hyperbola is used to obtain the general form of hyperbola:

$$d(F_1,P)-d(F_2,P)=2a$$
$$\sqrt{(x+c)^2+(y)^2}-\sqrt{(x-c)^2+y^2}=2a$$
$$\sqrt{(x+c)^2+y^2}=2a+\sqrt{(x-c)^2+y^2}$$

Squaring both sides to obtain

$$(x+c)^2+y^2=4a^2+4a\sqrt{(x-c)^2+y^2}+(x-c)^2+y^2$$
$$4a\sqrt{(x-c)^2+y^2}=4cx-4a^2$$
$$a\sqrt{(x-c)^2+y^2}=cx-a^2$$

Again squaring to obtain

$$a^2[(x-c)^2+y^2]=a^4-2a^2cx+c^2x^2$$

$$a^2\left(x^2-2cx+c^2+y^2\right)=a^4-2a^2cx+c^2x^2$$

$$a^2x^2-2a^2cx+a^2c^2+a^2y^2=a^4-2a^2cx+c^2x^2$$

$$a^2x^2-c^2x^2+a^2y^2=a^4-a^2c^2$$

$$-(c^2-a^2)x^2+a^2y^2=-a^2(c^2-a^2)$$

$$-b^2x^2+a^2y^2=-a^2b^2,$$

$$b^2x^2-a^2y^2=a^2b^2, \qquad c^2=a^2+b^2$$

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1, \qquad \textit{divide out by } a^2b^2 \qquad (65)$$

Notice that $c^2=a^2-b^2$ for ellipse and that $c^2=a^2+b^2$ for the hyperbola. For ellipse, it is necessary that $a^2>b^2$, but for the hyperbola, there is no restriction on the relative sizes for a and b (but c is still greater than a for the hyperbola).

(iii) Standard form of the equation of hyperbola

The standard form of the equation of a hyperbola with center at the origin and the x–axis as the transverse axis is shown in the Figure 9.26.

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \qquad (66)$$

If we replace the foci on the y–axis, center at the origin, and pick any point P(x, y) on the plane, then we can develop the equation of the vertical hyperbola given below (standard form of vertical hyperbola).

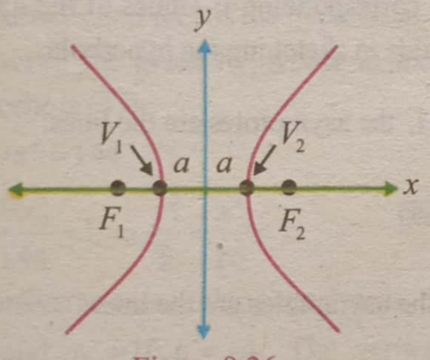


Figure 9.26

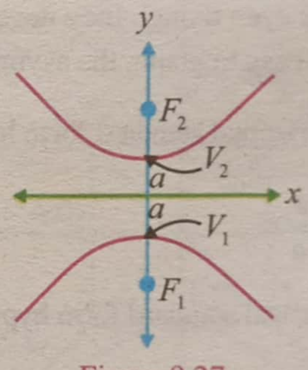


Figure 9.27

Standard form of vertical Hyperbola: The standard form of the equation of a hyperbola with center at the origin and the y–axis as the transverse axis is shown in the Figure 9.27.

$$\frac{y^2}{a^2}-\frac{x^2}{b^2}=1 \qquad (67)$$

Graphing standard form Hyperbola: As with the other conics, we shall sketch a hyperbola by determining some information about the curve directly from the equation by inspection. The end points of the transverse axis are the vertices of the hyperbola at $V_1(-a,0)$ and $V_2(a,0)$. The axis has a length of $2a$. The conjugate axis coincides with the y–axis and has its end points at (0,b) and (0,–b). If we set x–intercept $x=0$, then there is no real number y–intercept (but $y=\pm\sqrt{-b^2}$ is in complex conjugate). The question is, why should we be concerned about the conjugate axis or the length b? The significance of b is determined by solving the standard form of hyperbola for y:

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

$$-\frac{y^2}{b^2}=-\frac{x^2}{a^2}+1$$

$$y^2=b^2\left(\frac{x^2-a^2}{a^2}\right)=\frac{b^2}{a^2}(x^2-a^2)=\frac{b^2x^2}{a^2}\left(1-\frac{a^2}{x^2}\right)$$

$$y=\pm\frac{bx}{a}\sqrt{1-\frac{a^2}{x^2}} \qquad (68)$$

Let us examine the fraction $\frac{a^2}{x^2}$ (a is constant). If we substitute larger and larger values for x, then the fraction $\frac{a^2}{x^2}$ becomes smaller and smaller. In fact, the fraction eventually gets very close to zero. Thus, for large values of x, the term $1-\frac{a^2}{x^2}$ approaches 1. Therefore, for large values of x, the y values approaches the value $\pm\frac{b}{a}x$, and the value of the hyperbola gets closer and closer to the lines:

$$y=\pm\frac{b}{a}x.$$

These lines are called the **asymptotes** of the hyperbola. As x takes on values that are greater distances from the center of the hyperbola, the values of y (of the hyperbola) become closer and closer to the asymptotes even though they never actually reach the corresponding y–values of the asymptotes. Since these lines are easy to graph, the asymptotes are valuable aids in sketching the hyperbola.

- For horizontal standard form hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, the asymptotes are the lines:

$$y=\pm\frac{b}{a}x \qquad (69)$$

- For vertical standard form hyperbola $\frac{y^2}{a^2}-\frac{x^2}{b^2}=1$, the asymptotes are the lines

$$y=\pm\frac{a}{b}x \qquad (70)$$

Asymptotes:

If a straight line cuts a hyperbola in two points at an infinite distance from the origin and is itself at a finite distance from the origin is then called the **asymptotes**.

The orientation of the hyperbola with center C(0,0), vertices and foci are summarized in the box:

Orientation	Foci	Vertices
Horizontal	$F_1(-c,0), F_2(c,0)$	$V_1(-a,0), V_2(a,0)$
Vertical	$F_1(0,c), F_2(0,-c)$	$V_1(0,a), V_2(0,-a)$

Note that $c^2=a^2+b^2, c>0$.

Example 19 Sketch the hyperbola $\frac{x^2}{9}-\frac{y^2}{16}=1$, with center at the origin and the transverse axis is at the x-axis. Determine the vertices and foci of the hyperbola.

Solution The given hyperbola equation is in the standard form of the equation of a hyperbola (66) with transverse axis along the x–axis. This tells us that $a^2=9, a=\pm3$ and $b^2=16, b=\pm4$. The vertices of the hyperbola are $V_1(-3,0)$ and $V_2(3,0)$. The value of c for foci can be found by using the formula:

$c^2=a^2+b^2=9+16=25, c=\pm5$

The foci are therefore $F_1(-5,0)$ and $F_2(5,0)$. The asymptotes are the lines

$$y=\frac{b}{a}x=\frac{4}{3}x,\quad y=-\frac{b}{a}x=-\frac{4}{3}x$$

For sketching the hyperbola, the end points of the conjugate axis (0,–4) and (0,4) are located, then draw the lines through the points (0,–4) and (0,4) parallel to the x–axis. Similarly, draw the lines through the end points of the transverse axis $V_1(-3,0)$ and $V_2(3,0)$ parallel to y–axis to complete the rectangle. The resultant rectangle and the extended diagonals of the rectangle are the asymptotes of the hyperbola. The sketch of the hyperbola is shown in the Figure 9.28.

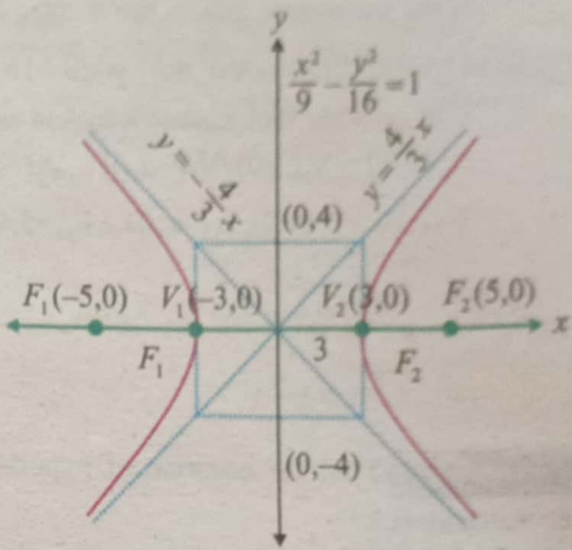


Figure 9.28

Example 20 Sketch the hyperbola $16y^2-9x^2=144$, with center at the origin and the transverse axis is at the y–axis. Determine the vertices and foci of the hyperbola.

Solution Rewrite the given hyperbola in the standard form of the hyperbola (67):

$$16y^2-9x^2=144$$

$$\frac{16y^2}{144}-\frac{9x^2}{144}=1 \Rightarrow \frac{y^2}{9}-\frac{x^2}{16}=1 \qquad (71)$$

If the transverse axis is along the y–axis, then select $a^2=9, a=\pm3$ and $b^2=16, b=\pm4$. The vertices of the hyperbola are $V_1(0,3)$ and $V_2(0,-3)$. The end points of the conjugate axis are (–4, 0) and (4, 0). The value of c for foci can be found by using the formula:

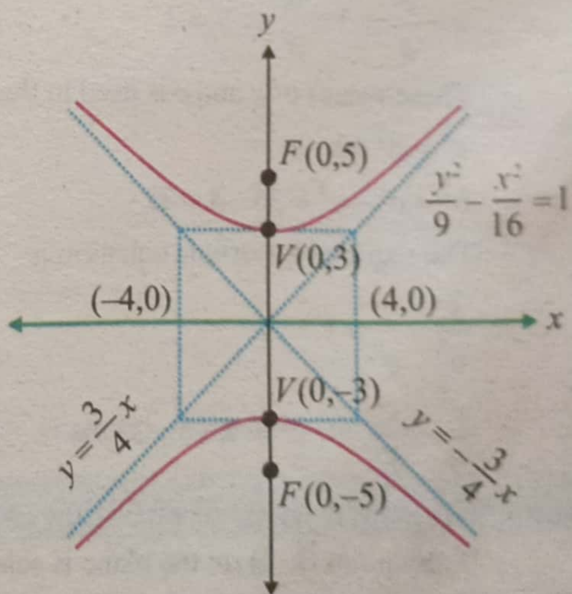


Figure 9.29

$$c^2=a^2+b^2=9+16=25, c=\pm5$$

The foci are therefore $F_1(0,5)$ and $F_2(0,-5)$. The asymptotes are the lines

$$y=\frac{a}{b}x=\frac{3}{4}x,\quad y=-\frac{a}{b}x=-\frac{3}{4}x$$

Sketch the rectangle formed by the points $(0,\pm3)$ and $(\pm4,0)$ and then sketch the asymptotes using the diagonals of the rectangle. With the asymptotes, vertices and foci, it is easy to sketch the hyperbola, as shown in the Figure 9.29.

(iv) Equation of hyperbola through its elements

Example 21 Find the vertices, foci, eccentricity and the asymptotes of the hyperbola $16x^2-9y^2=144$.

Solution Rewrite the given hyperbola in the standard form:

$$16x^2-9y^2=144$$

$$\frac{16x^2}{144}-\frac{9y^2}{144}=1 \Rightarrow \frac{x^2}{9}-\frac{y^2}{16}=1$$

If the transverse axis is the x–axis, then select $a^2=9$, $a=3$ and $b^2=16$, $b=4$. The value of c is obtained by formula $c=\sqrt{a^2+b^2}=\sqrt{9+16}=\pm 5$.

The vertices, foci, eccentricity and asymptotes of the given hyperbola are the following:

$V_1(a,0)=V_1(3,0),\ V_2(-a,0)=V_2(-3,0)$ vertices

$F_1(c,0)=F_1(5,0),\ F_2(-c,0)=F_2(-5,0)$ foci

$e=\frac{c}{a}=\frac{5}{3}>1$ eccentricity

$y=\pm\frac{b}{a}x=\pm\frac{4}{3}x$ asymptotes

Example 22 Find the equation of hyperbola, when one focus is at (0,6), center is at C(0,0) and the eccentricity is 3.

Solution Focus (0,6) gives $c=6$. This indicates that the transverse axis is the y–axis. The eccentricity 3 is giving the value of a:

$$e=\frac{c}{a}\Rightarrow 3=\frac{6}{a}\Rightarrow a=2$$

These values of a and c is used in the formula to obtain the value of b:

$$c^2=a^2+b^2$$

$$b^2=c^2-a^2=36-4=32$$

The required hyperbola equation is:

$$\frac{y^2}{a^2}-\frac{x^2}{b^2}=1$$

$$\frac{y^2}{4}-\frac{x^2}{32}=1,\ a^2=4,\ b^2=32$$

(v) Standard form of equation of Hyperbola

If any point (h, k) on the plane is selected as the center of the hyperbola and a major axis parallel to the x–axis or y–axis is selected, then with the geometrical definition, a new set of equations for hyperbola can be derived through translation of axes.

Translation of hyperbola horizontally:

The standard form of the equation of a hyperbola with center at C(h, k), vertices at $V_1(h+a,k)$ and $V_2(h-a,k)$, foci at $F_1(h-c,k)$ and $F_2(h+c,k)$ is:

$$\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1 \qquad (72)$$

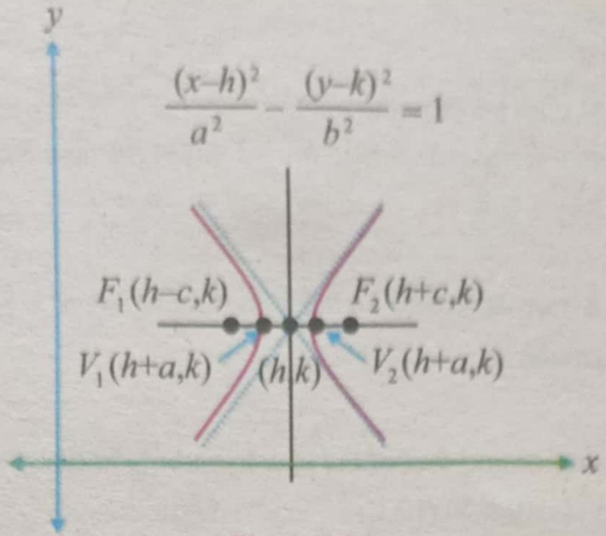


Figure 9.30

Translation of hyperbola vertically:

The standard form of the equation of a hyperbola with center at C(h, k), vertices at $V_1(h, k+a)$ and $V_2(h, k-a)$, foci at $F_1(h, k+c)$ and $F_2(h, k-c)$ is:

$$\frac{(y-k)^2}{a^2}-\frac{(x-h)^2}{b^2}=1 \qquad (73)$$

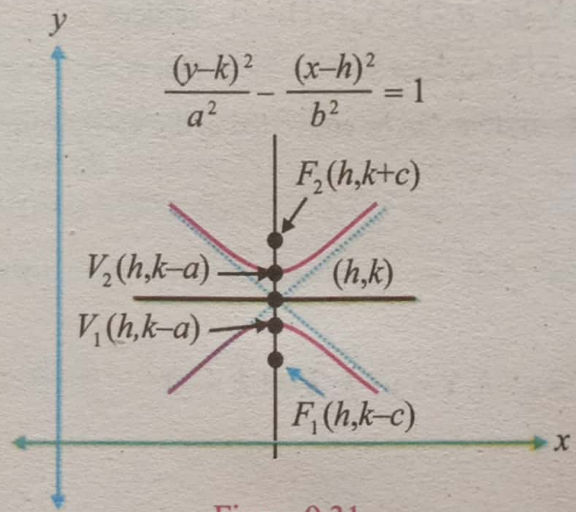


Figure 9.31

The orientation of the hyperbola equation with center C(h, k) are summarized in the box:

Orientation	Foci	Vertices
Horizontal	$F_1(h-c,k), F_2(h+c,k)$	$V_1(h-a,k), V_2(h+a,k)$
Vertical	$F_1(h,k+c), F_2(h,k-c)$	$V_1(h,k+a), V_2(h,k-a)$

Note that $c^2=a^2+b^2$, $c>0$.

Example 23 Sketch the hyperbola

$$\frac{(x-1)^2}{144}-\frac{(y+2)^2}{25}=1$$

with center at C(h, k) and the transverse axis is the x–axis. Determine the vertices and foci of the hyperbola.

Solution The given hyperbola is:

$$\frac{(x-1)^2}{144}-\frac{(y+2)^2}{25}=1 \qquad (74)$$

The given hyperbola (74) with substitution

$X=x-h=x-1$ and $Y=y-k=y+2,\ h=1,\ k=-2$ gives the new hyperbola in the XY–system:

$$\frac{X^2}{144}-\frac{Y^2}{25}=1 \qquad (75)$$

The hyperbola (75) has a transverse axis on the x–axis that give $a^2=144, a=12$ and $b^2=25, b=5$. The value of c is obtained by formula:

$c=\sqrt{a^2+b^2}=\sqrt{144+25}=13$

The coordinates of the vertices and foci of the new hyperbola in the XY–system are the following:

$V_1(-a,0)=V_1(-12,0), V_2(a,0)=V_2(12,0)$ vertices

$F_1(-c,0)=F_1(-13,0), F_2(c,0)=F_2(13,0)$ foci

The asymptotes in the XY–system are the lines:

$Y=\frac{b}{a}X=\frac{5}{12}X, \qquad Y=-\frac{b}{a}X=-\frac{5}{12}X$

The coordinates of the vertices and foci of the given hyperbola (74) in the xy–system are the following:

$V_1(h+a,k)=V_1(13,-2), V_2(h-a,k)=V_2(-11,-2)$ vertices

$F_1(h-c,k)=F_1(-12,-2), F_2(h+c,k)=V_2(14,-2)$ foci

The asymptotes in the XY–system can be converted in the xy–system to obtain:

$$Y=\frac{b}{a}X$$

$$=\frac{5}{12}X,\ a=12, b=5$$

$$(y-k)=\frac{5}{12}(x-h),\ X=x-h, Y=y-k,\ h=1, k=-2$$

$$y+2=\frac{5}{12}(x-1),$$

$$12y+24=5x-5 \Rightarrow 12y-5x+29=0$$

$$Y=-\frac{b}{a}X$$

$$=-\frac{5}{12}X$$

$$(y-k)=-\frac{5}{12}(x-h)$$

$$y+2=-\frac{5}{12}(x-1) \Rightarrow 12y+24=-5x+5 \Rightarrow 12y+5x+19=0$$

Having sketched the asymptotes and the vertices, we can sketch the hyperbola as shown in the Figure 9.32.

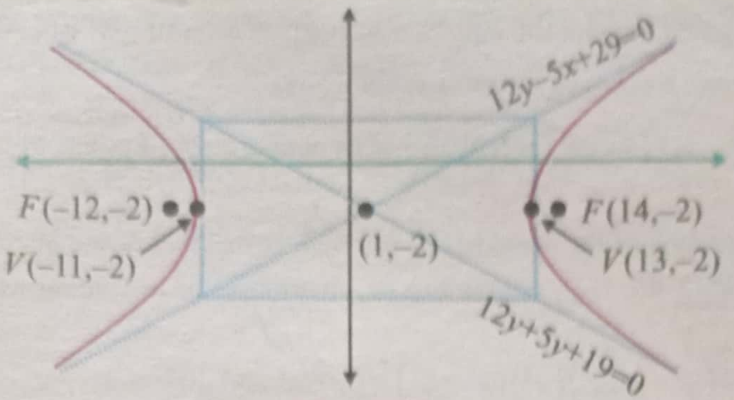


Figure 9.32

- For translating the hyperbola $\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$ horizontally, the asymptotes are the lines:

$$(y-k)=\pm\frac{b}{a}(x-h) \qquad (76)$$

- For translating the hyperbola $\frac{(y-k)^2}{a^2}-\frac{(x-h)^2}{b^2}=1$ vertically, the asymptotes are the lines:

$$(y-k)=\pm\frac{a}{b}(x-h) \qquad (77)$$

(vi) Recognition of tangent and normal to hyperbola

A line which intersects a hyperbola in two coincident points is a tangent. For the hyperbola, there will be two tangents [real and distanct, coincident (with an asymptote), or complex] with a given slope. The formulation for tangents to hyperbola will be discussed in the succeeding sections.

(vii) Point of intersection of hyperbola with a line including the condition of tangency

The given line and hyperbola

$$y=mx+c \qquad (78)$$

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \qquad (79)$$

develops a system of nonlinear equations:

$$\left.\begin{array}{l}\frac{x^2}{a^2}-\frac{y^2}{b^2}=1\\ y=mx+c\end{array}\right\} \qquad (80)$$

The solution set $\{x,y\}$ of nonlinear system of equations (80) exists only, if the curves of the system (80) are intersecting. That set of points of intersection $\{x,y\}$ (a solution set) can be found by solving the nonlinear system (80) simultaneously.

The line (78) is used in hyperbola (79) to obtain the quadratic equation in x:

$$\frac{x^2}{a^2}-\frac{(mx+c)^2}{b^2}=1$$

$$x^2\left(b^2-a^2m^2\right)-2a^2mcx+a^2\left(c^2+b^2\right)=0 \qquad (81)$$

The equation (81) being a quadratic equation in x, gives a set of two values x_1 and x_2 of x, which will be used in a line (78) to obtain a set of two values y_1 and y_2 of y.

The solution set $\{(x_1, y_1), (x_2, y_2)\}$ of the system (80) is of course a set of points of intersection of the system (80).

The points of intersection of the system (80) are real, coincident or imaginary, according as the roots of the quadratic equation (81) are real, coincident or imaginary, or according as the discriminant of the quadratic equation (81):

$Disc = 4a^4m^2c^2 + 4(b^2 - a^2m^2)(a^2)(c^2 + b^2) > 0$, real and different

$Disc = 4a^4m^2c^2 + 4(b^2 - a^2m^2)(a^2)(c^2 + b^2) = 0$, real and coincident

$Disc = 4a^4m^2c^2 + 4(b^2 - a^2m^2)(a^2)(c^2 + b^2) < 0$, imaginary

Example 24 Find the points of intersection of the line $x - y - 1 = 0$ and the hyperbola $4x^2 - y^2 = 4$.

Solution The equations of the line and hyperbola are :

$$x - y - 1 = 0 \qquad (82)$$

$$y = x - 1$$

$$4x^2 - y^2 = 4 \qquad (83)$$

The line (82) is used in hyperbola (83) to obtain the x–coordinates of the points of intersection:

$$4x^2 - (x-1)^2 = 4$$

$$4x^2 - (x^2 + 1 - 2x) - 4 = 0 \Rightarrow 3x^2 + 2x - 5 = 0 \Rightarrow x = 1, -\frac{5}{3}$$

The x–coordinates are used in the line (82) to obtain the y–coordinates:

$$x = 1, -\frac{5}{3} \text{ give } y = 0, -\frac{8}{3}$$

Thus, the set of two points of intersection (1,0) and $\left(-\frac{5}{3}, -\frac{8}{3}\right)$ are real and distinct and the line $x - y - 1 = 0$ intersects the hyperbola (83) at points (1,0) and $\left(-\frac{5}{3}, -\frac{8}{3}\right)$.

(viii) Equation of a tangent line in slope-form

If m is the slope of the tangent line to hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \qquad (84)$$

then the equation of that tangent line is of the form

$$y = mx + c \qquad (85)$$

Here c is to be calculated from the fact that the line (85) is tangent to hyperbola (84).

The line (85) is used in hyperbola (84) to obtain the quadratic equation in x:

$$\frac{x^2}{a^2} - \frac{(mx+c)^2}{b^2} = 1$$

$$x^2(b^2 - a^2m^2) - 2a^2mcx + a^2(c^2 + b^2) = 0 \qquad (86)$$

If the line (85) touches the hyperbola (84), then the quadratic equation (86) has coincident roots for which the discriminant of the quadratic equation (86) is going to be zero:

$$4a^4m^2c^2+4(b^2-a^2m^2)(a^2)(c^2+b^2)=0$$

$$a^2m^2c^2+(b^2-a^2m^2)(c^2+b^2)=0, \text{ divide out by } 4a^2$$

$$a^2m^2c^2-a^2m^2c^2-a^2m^2b^2+b^2c^2+b^4=0$$

$$-a^2m^2b^2+b^2c^2+b^4=0$$

$$-a^2m^2+c^2+b^2=0$$

$$c^2=(a^2m^2-b^2)$$

$$c^2=a^2m^2-b^2$$

$$c=\pm\sqrt{a^2m^2-b^2} \qquad (87)$$

The equation (87) is the **condition of tangency**. The value of c from equation (87) is used in the line (85) to obtain the required equation of the tangent line:

$$y=mx+c=mx\pm\sqrt{a^2m^2-b^2} \qquad (88)$$

Remember

- the equation of any tangent to hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ in the slope–form is:

$$y=mx\pm\sqrt{a^2m^2+b^2} \qquad (89)$$

- the line $y=mx+c$ should touch the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ under condition:

$$c=\pm\sqrt{a^2m^2-b^2} \qquad (90)$$

Example 25 For what value of c, the line $y=\frac{5}{2}x+c$ will touch the hyperbola $\frac{x^2}{4}-\frac{y^2}{9}=1$. Use those values of c to find the tangent lines that should touch the given hyperbola.

Solution The values of c at which the line $y=\frac{5}{2}x+c$ will touch the given hyperbola through result (87) are:

$$c=\pm\sqrt{a^2m^2-b^2}=\pm\sqrt{4\left(\frac{25}{4}\right)-9}=\pm 4 \qquad \because a^2=4, b^2=9, m=\frac{5}{2}$$

Here $m=\frac{5}{2}$ is the slope of the line $y=\frac{5}{2}x+c$. The required tangent lines that should touch the hyperbola through result (88) is:

$$y=mx+c=\frac{5}{2}x\pm 4 \qquad \because m=\frac{5}{2}$$

(ix) Equation of a tangent line to hyperbola at a point

The equation of the tangent at a point $P(x_1, y_1)$ to hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, is:

$$y-y_1=m_1(x-x_1) \qquad (91)$$

Here m_1 is the slope of the tangent line to hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ at a point $P(x_1,y_1)$ that can be found by differentiating $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ with respect to x:

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \qquad \text{differentiate w.r.t. } x$$

$$\frac{2x}{a^2}-\frac{2y}{b^2}\frac{dy}{dx}=0$$

$$\frac{dy}{dx}=\frac{b^2x}{a^2y}$$

$$\left(\frac{dy}{dx}\right)_{(x_1,y_1)}=\frac{b^2x_1}{a^2y_1}=m_1,\, say \qquad (92)$$

The substitution of (92) in (91) is giving the equation of the tangent line at a point $P(x_1,y_1)$ to hyperbola:

$$y-y_1=m_1(x-x_1)$$

$$y-y_1=\frac{b^2x_1}{a^2y_1}(x-x_1)$$

$$\frac{yy_1}{b^2}-\frac{y_1^2}{b^2}=\frac{xx_1}{a^2}-\frac{x_1^2}{a^2} \Rightarrow \frac{xx_1}{a^2}-\frac{yy_1}{b^2}=\frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}=1$$

$$\Rightarrow \frac{xx_1}{a^2}-\frac{yy_1}{b^2}=1 \qquad (93) \qquad \therefore \frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}=1\, at\, (x_1,y_1)$$

Example 26 Find the equation of the tangent at a point $P\left(5,\frac{16}{9}\right)$ to hyperbola $\frac{x^2}{9}-\frac{y^2}{16}=1$.

Solution Result (93) is used to obtain the tangent line to the given hyperbola:

$$\frac{xx_1}{a^2}-\frac{yy_1}{b^2}=1, \quad a^2=9, b^2=16$$

$$\frac{x(5)}{9}-\frac{y\left(\frac{16}{9}\right)}{16}=1 \qquad \therefore (x_1,y_1)=\left(5,\frac{16}{9}\right)$$

$$\frac{5x}{9}-\frac{16y}{144}=1$$

$$80x-16y=144 \Rightarrow 80x-16y-144=0 \Rightarrow 5x-y-9=0$$

(x) Equation of a normal line to hyperbola at a point

The equation of the normal at a point $P(x_1,y_1)$ to hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, is:

$$y-y_1=m_2(x-x_1) \qquad (94)$$

Here m_2 is the slope of the normal to hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, at a point $P(x_1,y_1)$ that can be found by differentiating $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ with respect to x:

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

$$\frac{2x}{a^2}-\frac{2y}{b^2}\frac{dy}{dx}=0$$

$$\frac{dy}{dx}=\frac{b^2x}{a^2y}$$

$$\left(\frac{dy}{dx}\right)_{(x_1,y_1)}=\frac{b^2x_1}{a^2y_1}=m_1,$$

$$m_2=-\frac{1}{m_1}=\frac{-a^2y_1}{b^2x_1}, say \qquad (95)$$

The substitution of (95) in (94) is giving the normal equation at a point $P(x_1, y_1)$ to hyperbola:

$$y-y_1=m_2(x-x_1)$$

$$y-y_1=\frac{-a^2y_1}{b^2x_1}(x-x_1) \qquad \therefore m_2=-\frac{a^2y_1}{b^2x_1}$$

$$\frac{y-y_1}{\frac{y_1}{b^2}}=\frac{-(x-x_1)}{\frac{x_1}{a^2}} \qquad (96)$$

Example 27 Find the normal equation at a point $P\left(5,\frac{16}{9}\right)$ to hyperbola $\frac{x^2}{9}-\frac{y^2}{16}=1.$

Solution Result (96) is used to obtain the normal line to the given hyperbola:

$$\frac{-(x-x_1)}{\frac{x_1}{a^2}}=\frac{y-y_1}{\frac{y_1}{b^2}},\ a^2=9, b^2=16$$

$$\frac{-(x-5)}{\frac{5}{9}}=\frac{y-\frac{16}{9}}{\frac{\left(\frac{16}{9}\right)}{16}} \qquad \therefore (x_1,y_1)=\left(5,\frac{16}{9}\right)$$

$$\frac{-9(x-5)}{5}=\frac{(9y-16)}{1}$$

$$\Rightarrow -9(x-5)(1)=5(9y-16)$$

$$-9x+45=45y-80$$

$$9x+45y=125$$

Exercise 9.3

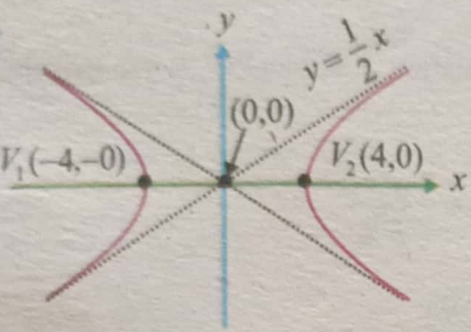
1. In each case, sketch the hyperbola represented by the equation. Indicate the center, vertices, foci and the equations of the asymptotes:

   a. $\dfrac{x^2}{4}-\dfrac{y^2}{9}=1$ b. $\dfrac{y^2}{25}-\dfrac{x^2}{4}=1$

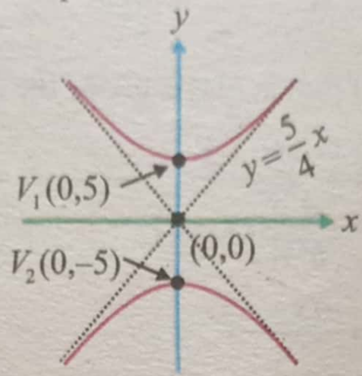
   c. $\dfrac{(x-2)^2}{9}-\dfrac{(y-3)^2}{16}=1$ d. $\dfrac{(y+1)^2}{16}-\dfrac{(x+3)^2}{25}=1$

2. In each case, determine the equation of graphed ellipse:

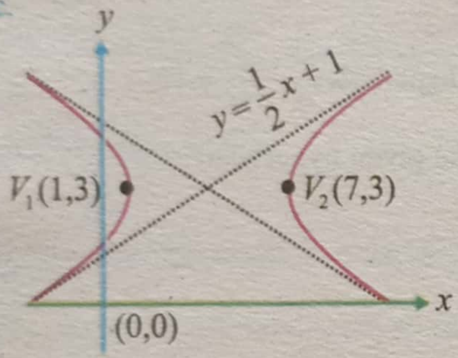
   a.



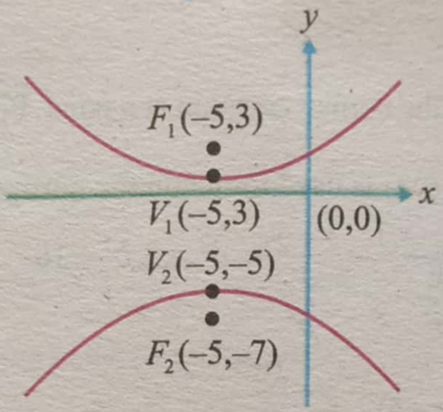
   b.



   c.



   d.



3. In each case, write the equation of hyperbola through the given information:

   a. Foci are at (0,3) and (0,–3), one vertex is at (0,–2).
   b. Vertices are at (5,0) and (–5,0), one focus is at (–7,0).
   c. Transverse axis is the x–axis, asymptotes are the lines $y=3x$ and $y=-3x$.
   d. Foci are at (5,0) and (–5,0), eccentricity is 5/3.
   e. Vertices are at (3,–1) and (–1,–1), asymptotes are the lines $y=(9/4)\,x-(13/4)$ and $y=(-9/4)\,x+(5/4)$.

4. Determine the path of a point that moves so that the difference of its distances from

   a. the points (–5,0) and (5,0) is 8.
   b. the points (0,–13) and (0,13) is 10.

5. Write the equation of the hyperbola

   a. with vertices at (2,–2), (–4,–2) and that passes through the point with coordinates (5,1).
   b. with vertices at (–3,1), (–3,3) and that passes through the point with coordinates (0,4).

6. In each case, sketch the rectangular hyperbola and identify the vertices, the foci and the asymptotes:

   a. $(x+1)^2-(y-2)^2=1$ b. $\dfrac{(x-3)^2}{4}-\dfrac{(y+1)^2}{4}=1$

7. In each case, find the points of intersection in between the line and the hyperbola:

a. $xy=4,\ \ y=x-3$ b. $\dfrac{y^2}{1}-\dfrac{x^2}{4}=1,\ \ x+4y-4=0$

8. For what value of c,

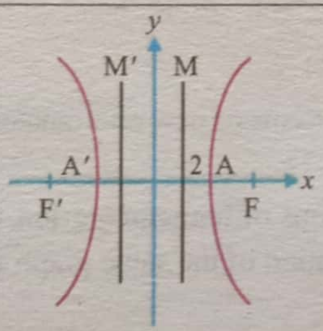
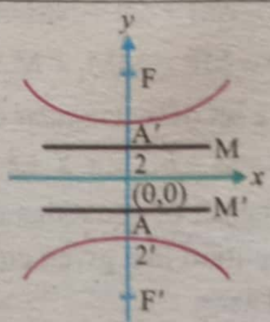
a. the line $y=x+c$ will touch the hyperbola $\dfrac{x^2}{16}-\dfrac{y^2}{4}=1$?

b. the line $y=-x+c$ will touch the hyperbola $\dfrac{x^2}{16}-\dfrac{y^2}{9}=1$?

9. In each case, find the tangent equation and normal equation

a. at a point $\left(-\sqrt{13},\dfrac{9}{2}\right)$ to hyperbola $\dfrac{x^2}{4}-\dfrac{y^2}{9}=1$?

b. at a point $\left(\dfrac{16}{3},5\right)$ to hyperbola $\dfrac{x^2}{9}-\dfrac{y^2}{16}=1$?

Summary of standard Ellipses		
Equation	$\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$ $c^2=a^2+b^2$	$\dfrac{y^2}{a^2}-\dfrac{x^2}{b^2}=1$ $c^2=a^2+b^2$
Focus	$(\pm C,0)$	$(0,\pm C)$
Directrices	$x=\pm\dfrac{c}{e^2}$	$y=\pm\dfrac{c}{e^2}$
Major axis	$y=0$	$x=0$
Vertices	$(\pm a,0)$	$(0,\pm a)$
Covertices	$(0,\pm b)$	$(\pm b,0)$
Centre	$(0,0)$	$(0,0)$
Eccentricity	$e=\dfrac{c}{a}>1$	$e=\dfrac{c}{a}>1$
Graph		

9.4 Translation and Rotation of Axes

If the coordinates of a point or the equation of a curve be given with reference to a system of axes, rectangular or oblique, then the coordinates of the same point or the equation of the same curve can be obtained with reference to another system of axes, rectangular or oblique. The process of so changing the coordinates of a point or the equation of a curve is called the **transformation of coordinates**.

(i) Translation and rotation of axes

In general, we come across to define three types of change of axes that are the following:

a. **Translation of Axes:** This will be used in changing the origin only and the new axes are parallel to the old ones.

b. **Rotation of Axes:** This will be used in changing the directions of the axes without changing the origin of the system.

c. **General Transformation:** This will be used, when the change of the direction and the origin of the axes both come together.

The relationship between the two sets of coordinate axes is called the **translation of axes.**

The rotational relationship between the two sets of coordinate axes is called the **rotation of axes.**

(ii) Equations of transformation for translation of axes

If o(0,0) is the old origin of the set of old rectangular coordinate axes ox and oy, then the coordinates of a point P with respect to the old axes are P(x, y).

If O(h, k) is the new origin of the set of new rectangular coordinate axes OX and OY parallel to the old rectangular coordinate axes, then the coordinates of a point P with respect to the new axes are P(X, Y).

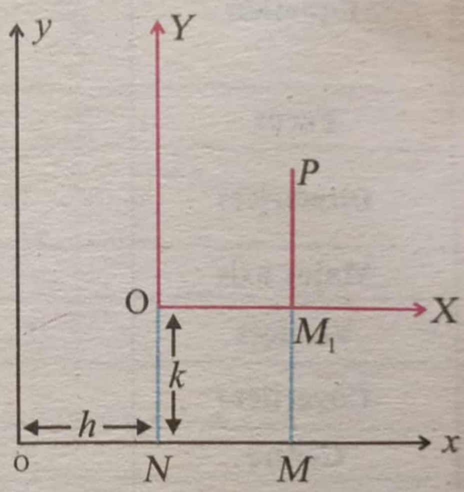


Figure 9.33

If PM and ON are perpendicular to old coordinate axis ox, where PM intersects the new coordinate axis OX at M_1, then, the following assumptions

$oN = h$, $NO = k$, $oM = x$, $MP = y$ also $OM_1 = X$, $M_1P = Y$

through old rectangular coordinate axes

$$x = oM = oN + NM = oN + OM_1 = h + X$$

$$y = MP = MM_1 + M_1P = NO + M_1P = k + Y$$

develops a set of rectangular coordinate axes in terms of new coordinates X, Y by means of the relation:

$$x = X + h, \quad y = Y + k \qquad (97)$$

The set of equations (97) are the equations of **transformation for translation of axes**. By making this substitution in a given equation, a new equation of the same graph is obtained, referred now to the new translated axes.

(iii) Equations of transformation for rotation of axes

If ox and oy is the set of old rectangular coordinate axes, then the set of new rectangular coordinate axes OX and OY is obtained by rotating the old rectangular coordinates through an angle $\theta, 0 < \theta < 90^0$ in anti-clockwise direction.

If the coordinates of a point P with respect to the old axes are P(x, y), then the coordinates of a point P with respect to the new axes are P(X, Y.)

If PM and PN are perpendicular to OX and OX and NN_1 and NM_1 are perpendicular to OX and PM, then, the following assumptions

$$oM = x,\ MP = y,\ ON = X,\ NP = Y,\ \angle M_1PN = \theta$$

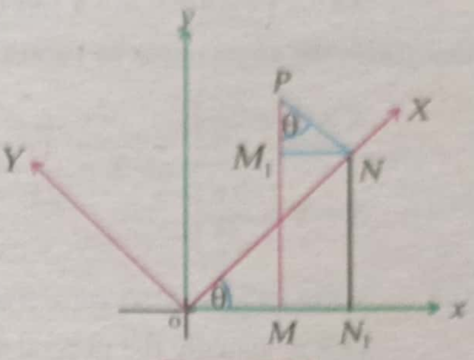


Figure 9.34

through old rectangular coordinate axes

$x = oM$

$= oN_1 - MN_1 = oN_1 - M_1N = ON\cos\theta - NP\sin\theta = X\cos\theta - Y\sin\theta$

$y = MP$

$= MM_1 + M_1P = N_1N + M_1P = ON\sin\theta + NP\cos\theta = X\sin\theta + Y\cos\theta$

develops a set of rectangular coordinate axes in terms of new coordinates X, Y by means of the relation:

$$x = X\cos\theta - Y\sin\theta,\quad y = X\sin\theta + Y\cos\theta \qquad (98)$$

An equivalent form of the relations (98) is:

$$X = x\cos\theta + y\sin\theta,\quad Y = -x\sin\theta + y\cos\theta \qquad (99)$$

(iv) Transformed equations through translation and rotation of axes

Example 28 Translate to parallel axes through the point (1, –2) the conic $4x^2 + 25y^2 - 8x + 100y + 4 = 0$.

Solution Substitute $x = X + h = X + 1$ ($h = 1$) and $y = Y + k = Y - 2$ ($k = -2$) in the given equation

$$4x^2 + 25y^2 - 8x + 100y + 4 = 0$$

$$4(X+1)^2 + 25(Y-2)^2 - 8(X+1) + 100(Y-2) + 4 = 0$$

which yields the standard form of ellipse in XY-plane by completing the square:

$$\frac{Y^2}{25} + \frac{Y^2}{4} = 1,\ XY-\textit{System} \qquad (100)$$

The standard form of the given conic (ellipse) equation in xy-plane is obtained by backward substitution of $X = x + 1$ and $Y = y - 2$:

$$\frac{(x-1)^2}{25} + \frac{(y+2)^2}{4} = 1,\ X = x - 1, = y + 2,\ xy-\textit{system}$$

Example 29 Transform to axes inclined at an angle 45^0 to the original axes of the conic $x^2 + y^2 - 8x + 4xy - 1 = 0$.

Solution The substitution of equation (98)

$x = X\cos\theta - Y\sin\theta,\ \theta = 45^0$

$= X\cos 45 - Y\sin 45 = \frac{\sqrt{2}}{2}X - \frac{\sqrt{2}}{2}Y = \frac{\sqrt{2}}{2}(X - Y) \qquad \therefore \sin 45 = \cos 45 = \frac{\sqrt{2}}{2}$

$y = X\sin\theta + Y\cos\theta$

$= X\cos 45 + Y\sin 45 = \frac{\sqrt{2}}{2}X + \frac{\sqrt{2}}{2}Y = \frac{\sqrt{2}}{2}(X + Y)$

in the given conic equation

$$\frac{2}{4}(X-Y)^2 + \frac{2}{4}(X+Y)^2 - 8\frac{\sqrt{2}}{2}(X-Y) + 4\left(\frac{2}{4}\right)(X-Y)(X+Y) - 1 = 0$$

to obtain the transformed equation of hyperbola

$$3X^2-Y^2-4\sqrt{2}(X-Y)-1=0$$

that yields the given conic by substituting:

$$\left.\begin{aligned}X+Y=\frac{2y}{\sqrt{2}}\\ X-Y=\frac{2x}{\sqrt{2}}\end{aligned}\right\}\Rightarrow X=\frac{1}{\sqrt{2}}(x+y),\ Y=\frac{1}{\sqrt{2}}(x-y)$$

(v) New origin and new axes with respect to old origin and old axes

This is actually the general transformation (third type) which requires both translation and rotation of axes. The procedure developed is as under:

If O(0,0) is the old origin of the set of old rectangular coordinate axes ox and *oy*, then the origin of the set of rectangular coordinate axes OX and OY parallel to the old rectangular coordinate axes is O(*h*, *k*). Further, the set of new rectangular coordinate axes OX' and OY' is obtained by rotating the rectangular coordinates axes OX and OY through an angle $\theta, 0<\theta<90^0$ in anti-clockwise direction.

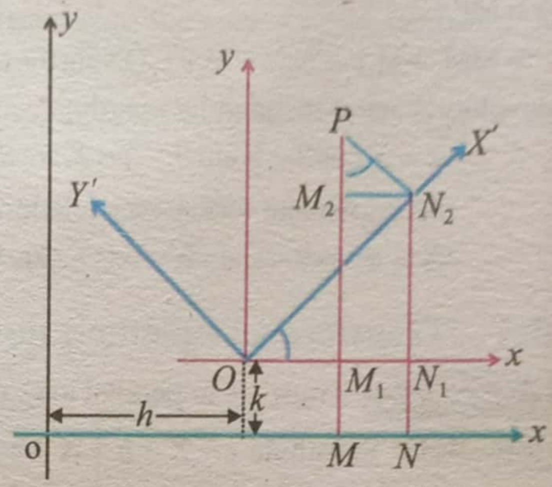


Figure 9.35

If the rectangular coordinates of a point P with respect to old rectangular coordinate axes are P(*x*,*y*), then the rectangular coordinates of a point P with respect to rectangular coordinates axes OX, OY and new rectangular coordinates axes OX', OY' are respectively P(X,Y) and $P(X',Y')$.

The following assumptions

$$oM=x,\ MP=y,\ OM_1=X,\ M_1P=Y,\ ON_2=X',\ N_2P=Y'$$

through old rectangular coordinate axes

$$\begin{aligned}x&=oM\\ &=oN-MN\\ &=oL+LN-MN\\ &=oL+ON_1-M_2N_2\\ &=oL+ON_2\cos\theta-N_2P\sin\theta=h+X'\cos\theta-Y'\sin\theta\end{aligned}$$

$$\begin{aligned}y&=MP\\ &=MM_1+M_1M_2+M_2P\\ &=OL+N_1N_2+M_2P=k+ON_2\sin\theta+N_2P\cos\theta\\ &=k+X'\sin\theta+Y'\cos\theta\end{aligned}$$

develops a set of rectangular coordinate axes in terms of new coordinates X' and Y' by means of the relation:

$$x=h+X'\cos\theta-Y'\sin\theta,\quad y=k+X'\sin\theta+Y'\cos\theta \qquad (101)$$

Example 30 Transform to new axes inclined at an angle 45^0 to the original axes of the conic $x^2+y^2-8x+4xy-1=0$ through (2, 3).

Solution The substitution of equation (101)

$$x = h + X'\cos\theta - Y'\sin\theta,\ \theta = 45^0$$
$$= h + X'\cos 45 - Y'\sin 45$$
$$= 2 + \frac{\sqrt{2}}{2}X' - \frac{\sqrt{2}}{2}Y' = 2 + \frac{\sqrt{2}}{2}(X' - Y') \qquad \therefore \sin 45 = \cos 45 = \frac{\sqrt{2}}{2}$$
$$y = k + X'\sin\theta + Y'\cos\theta$$
$$= k + X'\cos 45 + Y'\sin 45$$
$$= 3 + \frac{\sqrt{2}}{2}X' + \frac{\sqrt{2}}{2}Y' = 3 + \frac{\sqrt{2}}{2}(X' + Y')$$

in the given conic equation

$$\left[2+\frac{2}{\sqrt{2}}(X'-Y')\right]^2+\left[3+\frac{2}{\sqrt{2}}(X'+Y')\right]^2-8\left[2+\frac{\sqrt{2}}{2}(X'-Y')\right]+4\left[2+\frac{\sqrt{2}}{2}(X'-Y')\right]\left[3+\frac{\sqrt{2}}{2}(X'+Y')\right]-1=0$$

to obtain the transformed equation of hyperbola

$$3X'^2 - Y'^2 + 4\sqrt{2}(X' - Y') + 7\sqrt{2}(X' + Y') + 20 = 0$$

that yields the given conic by substituting:

$$\left.\begin{aligned} x &= 2+\frac{\sqrt{2}}{2}(X'-Y') \\ y &= 3+\frac{\sqrt{2}}{2}(X'+Y') \end{aligned}\right\} \Rightarrow X' = \frac{(x-2)}{\sqrt{2}} + \frac{(y-3)}{\sqrt{2}},\ Y' = \frac{(y-3)}{\sqrt{2}} - \frac{(x-2)}{\sqrt{2}}$$

(vi) Angle through which the axes be rotated about the origin so that the product term xy is removed from the transformed equation

The substitution of the equations of transformation (98)

$$x = X\cos\theta - Y\sin\theta, \quad y = X\sin\theta + Y\cos\theta$$

in the conic of the form $ax^2 + 2hxy + by^2$ is

$$ax^2 + 2hxy + by^2 = a(X\cos\theta - Y\sin\theta)^2 + 2h(X\cos\theta - Y\sin\theta)(X\sin\theta + Y\cos\theta) + b(X\sin\theta + Y\cos\theta)^2$$
$$= \left(a\cos^2\theta + 2h\sin\theta\cos\theta + b\sin^2\theta\right)X^2 + \left\{-2(a-b)\sin\theta\cos\theta + 2h(\cos^2\theta - \sin^2\theta)\right\}XY + \left\{a\sin^2\theta - 2h\sin\theta\cos\theta + b\cos^2\theta\right.$$

The expression $ax^2 + 2hxy + by^2$ will be of the form $aX^2 + bY^2$, if the coefficient of XY term on the right side of the above equation equals zero:

$$\left\{-2(a-b)\sin\theta\cos\theta + 2h(\cos^2\theta - \sin^2\theta)\right\} = 0$$
$$-(a-b)\sin 2\theta + 2h\cos 2\theta = 0$$
$$-(a-b)\sin 2\theta = -2h\cos 2\theta$$
$$\frac{\sin 2\theta}{\cos 2\theta} = \frac{2h}{a-b}$$
$$\tan 2\theta = \frac{2h}{a-b}$$
$$\theta = \frac{1}{2}\tan^{-1}\frac{2h}{a-b} \qquad (102)$$

Example 31 At what angle the axes are rotated about the origin so that the transformed equation of the conic $9x^2 + 4y^2 + 12xy - x - y = 0$ does not contain the term involving XY?

Solution If the axes of the given conic are rotated through an angle θ, then the angle θ can be found through result (102):

$$\theta = \frac{1}{2}\tan^{-1}\frac{2h}{a-b}, \quad a=9, b=4, h=6$$

$$= \frac{1}{2}\tan^{-1}\frac{2(6)}{9-4} = \frac{1}{2}\tan^{-1}\frac{12}{5} = \frac{1}{2}\tan^{-1}(2.4) = \frac{1}{2}(67^0) \approx 34^0$$

Exercise 9.4

1. Translate to parallel axes through the
   a. point (0,2), the equation $2x - y + 2 = 0$.
   b. point (–1,2), the conic $x^2 + y^2 + 2x - 4y + 1 = 0$.
   c. point (3,–4), the conic $x^2 + 2y^2 - 6x + 16y + 39 = 0$.
   d. point (–2,2), the conic $x^2 + y^2 - 3xy + 10x - 10y + 21 = 0$.
2. Transform to axes inclined at an angle
   a. 45^0 to the original axes of the conic $x^2 - y^2 = a^2$.
   b. 90^0 to the original axes of the conic $y^2 = 4px$.
   c. 45^0 to the original axes of the conic $x^2 + y^2 + 4xy - 1 = 0$.
   d. 45^0 to the original axes of the conic $x^2 - y^2 - 2\sqrt{2}x - 10\sqrt{2}y + 2 = 0$.
3. Transform to new axes inclined at an angle
   a. $\tan^{-1}(1/2)$ to the original axes of the conic
      $14x^2 + 11y^2 - 36x + 48y - 4xy + 41 = 0$ through (1,–2).
   b. $\tan^{-1}(-4/3)$ to the original axes of the conic
      $11x^2 + 4y^2 - 20x - 40y + 24xy - 5 = 0$ through (2,–1)
   c. $\tan^{-1}(3/4)$ to the original axes of the conic
      $3x^2 + 10y^2 + 6x + 52y - 24xy = 0$ through (3,1).
4. At what angle the axes are rotated about the origin so that the transformed equation of the conic
   a. $11x^2 + 4y^2 - 20x - 40y + 24xy - 5 = 0$ does not contain the term involving XY?
   b. $5x^2 + 7y^2 + 2\sqrt{3}xy - 16 = 0$. does not contain the term involving XY?

Summary

Parabola:

- The eccentricity of the conic is e = c/a. The conic is
  ellipse, if e < 1
  parabola, if e = 1
  hyperbola, if e > 1
- a. The standard form of the equation of a parabola that is symmetric with respect to the x–axis, with vertex V(0,0), focus F(p,0) and directrix the line $x=-p$ is: $y^2=4px$
  b. The standard form of the equation of a parabola that is symmetric with respect to the y–axis, with vertex V(0,0), focus F(0,p) and directrix the line $y=-p$ is: $x^2=4py$
- a. The standard form of the equation of a parabola that is symmetric with respect to the line $y=k$ and with vertex $V(h, k)$, focus $F(h+p, k)$ and directrix line $x=h-p$ is: $(y-k)^2=4p(x-h)$
  b. The standard form of the equation of a parabola that is symmetric with respect to the line x=h and with vertex $V(h, k)$, focus $F(h, k+p)$ and directrix line $y=k-p$ is: $(x-h)^2=4p(y-k)$
- The equation of the tangent line at a point $P(x_1, y_1)$ to parabola is: $yy_1=2p(x+x_1)$
- The equation of any tangent to parabola $y^2=4px$ in the slope–form is: $y=mx+\frac{p}{m}$
- The line $y=mx+c$ should touch the parabola $y^2=4px$ under condition: $y=mx+\frac{p}{m}$, $y^2=4py$
- The normal equation to parabola $y^2=4px$ at a point $P(x_1, y_1)$ is: $y-y_1=\frac{-y_1}{2p}(x-x_1)$

Ellipse:

- a. The standard form of the equation of an ellipse with center at the origin, length of the semimajor axis a, length of the semiminor axis b and major axis along the x–axis is: $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$
  b. The standard form of the equation of an ellipse with center at the origin, length of the semimajor axis a, length of the semiminor axis b and major axis along the y–axis is: $\frac{x^2}{b^2}+\frac{y^2}{a^2}=1$
- a. The standard form of the equation of an ellipse with center at $C(h, k)$, length of the semimajor axis a and semiminor axis b, and major axis parallel to the x–axis is: $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$
  b. The standard form of the equation of an ellipse with center at $C(h, k)$, length of the semimajor axis a and semiminor axis b, and major axis parallel to the y–axis is:
  $$\frac{(x-h)^2}{b^2}+\frac{(y-k)^2}{a^2}=1$$
- The equation of the tangent line to ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at a point $P(x_1, y_1)$ is:
  $$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1, \quad \frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}=1\ at\ (x_1, y_1)$$
- The equation of any tangent line to ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ in the slope–form is: $y=mx\pm\sqrt{a^2m^2+b^2}$
- The line $y=mx+c$ should touch the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ under condition: $c=\pm\sqrt{a^2m^2+b^2}$

- The normal equation to ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at a point $P(x_1, y_1)$ is: $\frac{y-y_1}{y_1/b^2}=\frac{x-x_1}{x_1/a^2}$

Hyperbola:

- a. The standard form of the equation of a hyperbola with center at the origin and the x–axis as the transverse axis is: $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

  b. The standard form of the equation of a hyperbola with center at the origin and the y–axis as the transverse axis is: $\frac{y^2}{a^2}-\frac{x^2}{b^2}=1$

- a. For horizontal standard form hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, the asymptotes are the lines: $y=\pm\frac{b}{a}x$

  b. For vertical standard form hyperbola $\frac{y^2}{a^2}-\frac{x^2}{b^2}=1$, the asymptotes are the lines: $y=\pm\frac{b}{a}x$

- a. The standard form of the equation of a hyperbola with center at C(h,k), vertices at $V_1(h+a,k)$ and $V_2(h-a,k)$, and foci at $F_1(h-c,k)$ and $F_2(h+c,k)$ is: $\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$

  b. The standard form of the equation of a hyperbola with center at C(h,k), vertices at $V_1(h,k+a)$ and $V_2(h,k-a)$, and foci at $F_1(h,k+c)$ and $F_2(h,k-c)$ is: $\frac{(y-k)^2}{a^2}-\frac{(x-h)^2}{b^2}=1$

- a. For translating the hyperbola $\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$, horizontally, the asymptotes are the lines:

  $(y-k)=\pm\frac{b}{a}(x-h)$

  b. For translating the hyperbola $\frac{(y-k)^2}{a^2}-\frac{(x-h)^2}{b^2}=1$, vertically, the asymptotes are the lines:

  $(y-k)=\pm\frac{a}{b}(x-h)$

- A hyperbola whose asymptotes are at right angles to each other is called a rectangular hyperbola.
- A function of the form $y=C/x$ is an inverse function called the rectangular hyperbola.
- The equation of the tangent line to hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ at a point $P(x_1, y_1)$ is: $\frac{xx_1}{a^2}-\frac{yy_1}{b^2}=1$
- The equation of any tangent line to hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ in the slope–form is:

  $y=mx\pm\sqrt{a^2m^2+b^2}$
- The line y=mx+c should touch the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ under condition: $c=\pm\sqrt{a^2m^2-b^2}$
- The normal equation to hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ at a point $P(x_1, y_1)$ is: $\frac{-(x-x_1)}{x_1/a^2}=\frac{y-y_1}{y_1/b^2}$

Unit 10

DIFFERENTIAL EQUATIONS

By the end of this unit, the students will be able to:

10.1 Introduction

i. Define ordinary differential equation (DE), order of a DE, degree of a DE, solution of a DE – general solution and particular solution.

10.2 Formation of differential equations

i. Demonstrate the concept of formation of a differential equation.

10.3 Solution of differential equation

i. Solve differential equations of first order and first degree of the form:
- separable variables,
- homogeneous equations,
- equations reducible to homogeneous form.

ii. Solve real life problems related to differential equations.

10.4 Orthogonal Trajectories

i. Find orthogonal trajectories (rectangular coordinates) of the given family of curves.

ii. Use MAPLE graphic commands to view the graphs of given family of curves and its orthogonal trajectories.

Introduction

The laws of the universe are written in the language of mathematics. Algebra is sufficient to solve many static problems, but the most interesting natural phenomena involve change and are describe only by equations that relates rates at which quantities change.

Suppose the solution of problems concerning the motion of objects, the flow of charged particles, heat transport, etc often involves discussion of relations of the form

$$\frac{dx}{dt}=f(x,t) \quad \text{or} \quad \frac{dq}{dt}=g(q,t)$$

In the first equation, x might represent distance. For this case, $\frac{dx}{dt}$ is the rate of change of distance with respect to time t that is speed. In the second equation, q might be a charge and $\frac{dq}{dt}$ is the rate of flow of charge that is current. These are examples of different equations, so called because these are equations involving the derivatives of various quantities. Such equations arise out of situations in which change is occurring.

In engineering, differential equations are most commonly used to model dynamic systems. These are the systems which change with time. Examples include an electronic circuit with time-dependent currents and voltages, a chemical production line in which pressure, tank levels, flow rates, etc, vary with time.

There is a wide variety of differential equations which occur in engineering applications, and consequently there is a wide variety of solution techniques available.

10.1 Ordinary Differential Equations

A differential equation is an equation that involves the derivatives of an unknown function (dependent variable) of one or more variables (independent variables).

*"If the unknown function depends on only one variable, then the derivative is an ordinary derivative, and the equation is then called an **ordinary differential equation.**"*

If the unknown function depends on more than one variable, then the derivative is partial derivative, and the equation is then called **partial differential equation.**

The following differential equations are the examples of ordinary differential equations with their corresponding unknown functions:

$$\frac{dy}{dx} = xy, \quad y(x) = ? \qquad \text{(i)}$$

$$\frac{dy}{dx} = x + y, \quad y(x) = ? \qquad \text{(ii)}$$

$$\frac{dy}{dx} = \frac{x+y}{x-y}, \quad y(x) = ? \qquad \text{(iii)}$$

$$\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} + y = 3, \quad y(x) = ? \qquad \text{(iv)}$$

$$\left(\frac{d^3y}{dx^3}\right)^2 - \frac{d^2y}{dx^2} + x^2\frac{dy}{dx} + y = 3, \quad y(x) = ? \qquad \text{(v)}$$

(a) Order of a differential equation

The order of a differential equation is the order of the highest-order derivative occurring in the equation e.g.

i. $\frac{dy}{dx} = xy$ is first order differential equation.

ii. $\frac{d^2y}{dx^2} + (x^2 + 2x)y = 7$ is second order differential equation.

(b) Degree of a differential equation

The degree of a differential equation is the power of the highest-order derivative occurring in the equation.

i. $\frac{d^3y}{dx^3} + \left(\frac{d^2y}{dx^2}\right)^2 - 8\frac{dy}{dx} + 2y = 8$ is an equation having degree is 1.

Example 1 Determine the order and degree of the following ordinary differential equations:

(a). $\frac{dy}{dx} = \frac{x+y}{x-y}$ (b). $\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} + y = 3$ (c). $\left(\frac{d^3y}{dx^3}\right)^2 - \frac{d^2y}{dx^2} + x^2\frac{dy}{dx} + y = 3$

Solution Differential equation (a) is an ordinary differential equation of order 1 and degree 1, since the highest ordinary derivative is of order 1 and the exponent of the highest ordinary derivative is 1. Differential equation (b) is an ordinary differential equation of order 2 and degree 1, while Differential equation (c) is an ordinary differential equation of order 3 and degree 2.

(c) Solution of a differential equation

A solution of an equation in a single variable is a **number** which satisfies the equation. In similar fashion, solutions of the differential equations are **functions**, rather than numbers, which satisfy the differential equation. The variables which appear in equations are called **"unknowns."** Exactly, the only dependent variable in differential equations is referred to as **"unknown."**

For illustration, a solution of the differential equation $\frac{dy}{dx} = 1$ is an expression of the unknown dependent variable y in terms of the independent variable x.

"A solution of an ordinary differential is any function $y = f(x)$ or $f(x, y)$ which when substituted in the differential equation , reduces the differential equation to an identity; that is, it satisfies the equation."

Example 2 Show that $y = x + A$ is a solution of the first order differential equation $\frac{dy}{dx} = 1$.

Solution The given function $y = x + A$ and its derivative $\frac{dy}{dx} = 1$ is used in the differential equation $\frac{dy}{dx} = 1$ to obtain:

$$\frac{dy}{dx} = 1$$

$$1 = 1, \textit{ identity left side} = \textit{right side}$$

This shows that $y = x + A$ is a solution of the ordinary differential equation $\frac{dy}{dx} = 1$.

(d) General and particular solution

The solution of a differential equation when depends on a single arbitrary constant quantity, is called the ***general solution*** *of the first order differential equation. If we give particular steps for value to a single arbitrary constant quantity, then the solution to obtain is called the* ***particular solution.***

Do You Know?

The particular solution is also known as specific solution or exact solution or actual solution.

Graphically,

- the general solution of a first order deferential equation represents a family of curves for any choice of arbitrary constant quantity.
- The particular solution of a first order differential equation is a particular curve chosen from a family of curves (general solution) for a particular value of a constant quantity.

Example 3 Graphically, show that $y = x + A$ is a general solution of the first order differential equation $\frac{dy}{dx} = 1$. Find a particular solution, when $x = 0$ and $y = 1$.

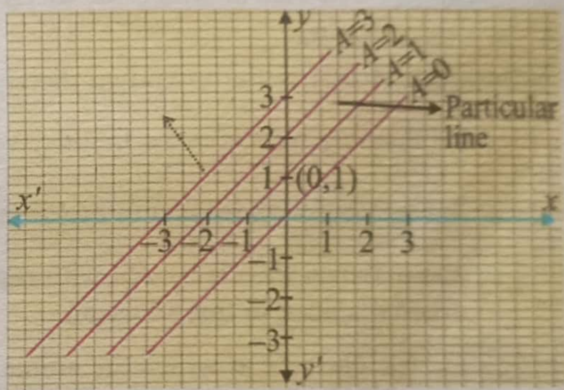


Figure 10.1

Solution The general solution $y = x + A$ of a first order differential equation $\frac{dy}{dx} = 1$, represents a family of parallel straight lines for different values of arbitrary constant quantities $A = 0, 1, 2, ...$

The particular value for the particular line that passes through a point $P(0, 1)$ can be found from the general solution $y = x + A$ by putting $x = 0$, $y = 1$:

$$y = x + A \Rightarrow 1 = 0 + A \Rightarrow A = 1$$

Use this particular value of $A = 1$ in general solution $y = x + A$ to obtain a particular solution (line) $y = x + 1$.

Note

If we are to determine the solutions of a differential equation subject to conditions on the unknown function and its derivatives specified for one value of the independent variable, the conditions are then called **conditions** and the related differential equation is called an **initial value problem** "IVP".

Thus, the problem of example 3 is the initial value problem that leads the notation:

$$\frac{dy}{dx} = 1, \quad y(0) = 1 \qquad \text{IVP}$$

Example 4 Determine a particular solution for the first order differential equation $\frac{ds}{dt} = -32\, ft/sec$ that satisfies the initial condition $s = 0$, when $t = 0$.

Solution This information develops the initial value problem $\frac{ds}{dt} = -32,\ s(0) = 0$

for which the solution is the unknown function $s(t)$ that can be found by integrating directly the first order differential equation with respect to t:

$$\frac{ds}{dt} = -32 \Rightarrow \int \frac{ds}{dt} = \int -32dt + c \Rightarrow s(t) = -32t + c$$

The general solution $s(t) = -32t + c$ at a point P(0, 0) is giving $c = 0$. Use this $c = 0$ in general solution to obtain the particular solution $s(t) = -32t$.

10.2 Formation of Differential Equation

In most of the physical situations, we can observe the process but can not worked out directly to the differential equation. As, a result, we have a general solution at our disposal before we know the equation of which it is the solution. Let's begin with the step for forming with differential equation.

i. Discover the differential equation that describes a specified physical situation.
ii. Find either exactly or approximately, the appropriate solution of that equation.
iii. Interpret the solution that is found.

Look at the following examples for the concept of formation of a differential equation.

Example 5 The rate at which the distance travels by Ali is 30 mph. Find the total distance travels by Ali at a time t hours.

Solution If S(t) is the unknown distance travel by you w.r.t. 't' number of hours, then, the rate at which the distance travels is the first derivative of S(t) with respect to t: $\frac{dS}{dt} = 30$, the per hour speed

Integrating with respect to t to obtain S(t) $\int \frac{dS}{dt} = \int 30dt + c \Rightarrow S(t) = 30t + c$,

the distance travels by Ali with respect to 't' number of hours and the constant quantity c is the fixed distance in this situation.

Example 6 The rate at which the animal population is growing at a constant rate 4%. The habitat will support no more than 10,000 animals. There are 3000 animals present now. Find an equation that gives the animal population y w.r.t. x number of years.

Solution If $P(x)$ is the unknown animal population w.r.t. x number of years, then, the rate at which the animal population grows is the first derivative of $P(x)$ w.r.t. x:

$$\frac{dP}{dx} = k(N - P)$$
$$= 0.04(10,000 - P)$$
$$\Rightarrow \frac{dP}{(10,000 - P)} = 0.04dx$$

Here $k = 0.04$ is the constant growth, $N = 10,000$, is the total size of animal population in that habitat. Integrating with respect to x to obtain P(x),

$$\int \frac{dP}{(10,000 - P)} = \int 0.04dx + c$$

the total population and c, the fixed population that depends on P = 3000 when $x = 0$. This problem is the IVP problem with the initial condition P(0) = 3000.

Exercise 10.1

1. Find the order and degree of each the following ordinary differential equations:

a. $\frac{dy}{dx}=x^2+y$ b. $\frac{d^2y}{dx^2}+3\frac{dy}{dx}+11y=3x$

c. $\frac{d^3y}{dx^3}+2\left(\frac{dy}{dx}\right)^3-y=0$ d. $y\frac{d^2y}{dx^2}+5x\frac{dy}{dx}=4$

2. In each case, show that the indicated function is a solution of the differential equation:

a. $y=e^x+e^{2x},\ \frac{d^2y}{dx^2}-3\frac{dy}{dx}+2y=0$ b. $y=x-x\ln x,\ x\frac{dy}{dx}+x-y=0$

c. $y=(x+c)e^{-x},\ \frac{dy}{dx}+y=e^{-x}$ d. $y=e^x+e^{-x},\ \frac{d^2y}{dx^2}-3\frac{dy}{dx}+2y=0$

3. In each case, use the initial condition and the general solution of the differential equation to determine a particular solution:

a. $xy=c,\ y(2)=1$ b. $y=x-x\ln x+c,\ y(1)=2$

c. $\sin(xy)+y=c,\ y\left(\frac{\pi}{4}\right)=1$ d. $\frac{y^2}{x}=\frac{x^2}{2}+c,\ y(1)=1$

4. Solve the following initial value problems:

a. $\frac{dy}{dx}=\cos x,\ y(0)=1$ b. $\frac{dy}{dx}=x^2,\ y(0)=1$

c. $\frac{dy}{dx}=\frac{1}{x^2},\ y(2)=0.$ d. $\frac{dy}{dx}=2xy^2,\ y(3)=-1$

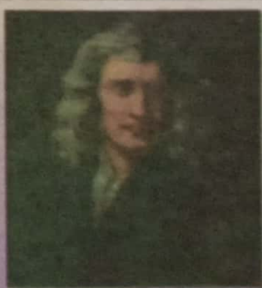
5. Suppose a student carrying Corona Virus returns to an isolated college campus of 1000 students. If it is considered that the rate at which the virus spreads is proportional not only to the number 'x' of infected students but also to the number of students not infected. Find the number of infected students after 6 days. If it is further observed after 4 days = 50.

History

Gottfried Wilhelm Leibniz
(1646-1716)

Leibniz was the German Mathematician and philosopher. He introduced and published the concept of differential equation in (1684). In most of the documents indicated that he knew how to solve the differential equations in 1666.

Sir Isaac Newton was an English Mathematician, Physicist and Astronomer. He never published his "Method of fluxions" but it is claimed that he discovered it in 1665 to 1667. Which is known as exact by the modern classification.



Isaac Newton
(1643-1727)

10.3 Solution of differential equations

If the solution of a first order differential equation is not possible by direct integration, then, the integral process (in case of difficulties) for obtaining the solution of a differential equation indicates the actual concept of a differential equation.

10.3.1 Solution of first order and first degree differential equations

We examine techniques for solving first order differential equations. For this unit, the recommended techniques for solving the differential equations are the separation of variables, reducible to separation form, homogeneous and equations reducible to homogeneous form.

i. **Separable variables**

If the solution of a differential equation is not possible by direct integration, then the integral technique called **separation of variables** will be used for solving the differential equation. Separation of variables is a technique commonly used to solve first order differential equations. It is so called because we try to rearrange the equation to be solved in such a way that all terms involving the dependent variable (y say) appear on one side of the equation, and all terms involving the independent variable (x, say) appear on the other side. It is not possible to rearrange all first order differential equations in this way so this technique is not always appropriate. Further, it is not always possible to perform the integration even if the variables are separable.

In general, a differential equation of the form $\dfrac{dy}{dx}=\dfrac{f(x)}{g(y)},\ g(y)\neq 0$ (i)

that by shifting x on one side and y on the other side $g(y)dy = f(x)dx$, SDE (ii)

is giving a **separable differential equation**. The solution to separable differential equation (ii) can be found by integrating left hand side w.r.t. y and right hand side w.r.t. x.

Example 7 Find the general solution of the linear differential equation $\dfrac{dy}{dx}=y$.

Solution The solution of the given differential equation is not possible by direct integration. The separable form of the given first order differential equation is obtained by shifting y on the left and x on the right: $\dfrac{1}{y}dy = dx$

Integrating both sides

$$\int \frac{1}{y}dy = \int dx \Rightarrow \ln y = x + c$$

$$\Rightarrow y = e^{x+c} = e^x e^c = c_1 e^x, \quad c_1 = e^c$$

is giving the general solution of the first order differential equation. This general solution represents a family of exponential functions as shown in the Figure 10.2.

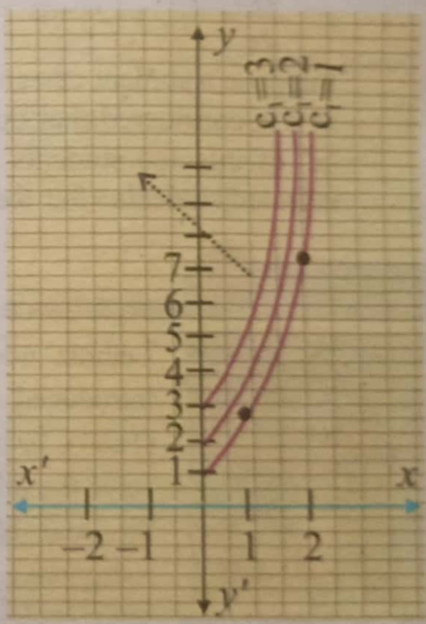


Figure 10.2

Remember

If the solution of the differential equation is not possible by separable form, then the given differential equation can be reduced in separable form by substitution. This substitution changes the dependent variable form y to a new variable, say, u and keeps x as the independent variable.

Example 8 Find the general solution of the non-linear differential equation $\frac{dy}{dx}=(x+y)^2$.

Solution The given non-linear differential equation is not separable differential equation, but can be reduced into separable form by substitution: $x+y=u(x)$

that on differentiation w.r.t. x is giving:

$$\frac{d}{dx}(x+y)=\frac{du}{dx}\Rightarrow 1+\frac{dy}{dx}=\frac{du}{dx}\Rightarrow\frac{dy}{dx}=\frac{du}{dx}-1$$

Use $x+y=u$ and $\frac{dy}{dx}=\left(\frac{du}{dx}\right)-1$ in the given differential equation to obtain separable differential equation in variable u and its derivative $\frac{du}{dx}$:

$$\frac{du}{dx}-1=u^2\Rightarrow\frac{du}{dx}=1+u^2\Rightarrow\frac{du}{1+u^2}=dx \qquad \text{(i)}$$

Integrating equation (i) to obtain the general solution of ordinary differential equation (i).

$$\int\frac{du}{1+u^2}=\int dx\Rightarrow\tan^{-1}u=x+c\Rightarrow u=\tan(x+c)$$

that by back substitution of $u=x+y$ is giving

$$x+y=\tan(x+c)\Rightarrow y=-x+\tan(x+c)$$

the general solution of the given ordinary differential equation that depends on a single arbitrary constant c.

6. Homogeneous equations

The homogeneous differential equations are related to homogeneous functions.

"A function $f(x,y)$ is homogeneous function of degree n in variables x and y if and only if for all values of the variables x, and y and for every positive value of λ, the identity is true."

$$f(\lambda x,\lambda y)=\lambda^n f(x,y),\quad n=1,2,3,... \qquad \text{(i)}$$

For illustration, the function $f(x,y)=x^2+y^2$ is homogeneous function of degree 2, since the identity (i) is true:

$$f(\lambda x,\lambda y)=(\lambda x)^2+(\lambda y)^2=\lambda^2(x^2+y^2)=\lambda^2 f(x,y) \qquad x=\lambda x,\ y=\lambda y$$

The identity (i) is not true for a function $f(x,y)=x^2+y^2+1$, since the function is not homogeneous.

"The differential equation $\frac{dy}{dx}=\frac{f(x,y)}{g(x,y)}$ (ii)

is called a homogenous differential equation, if it defines a homogenous function of degree zero."

The homogeneous differential equation (ii) can be reduced to separable form by introducing a new variable:

$$u(x)=\frac{y}{x}\ \text{or}\ y=ux\quad\text{and}\quad\frac{dy}{dx}=\frac{d}{dx}(ux)=u+x\frac{du}{dx} \qquad \text{(iii)}$$

The substitution of (iii) in equation (ii) automatically converts the homogeneous differential equation in separable differential equation.

Example 9 Find the general solution of the homogeneous differential equation: $\frac{dy}{dx}=\frac{y-x}{y+x}$

Solution The given differential equation defines a homogeneous function of degree zero, when the function on the right of the given differential equation defines a homogeneous function of degree zero:

$$\frac{dy}{dx}=\frac{y-x}{y+x}=\frac{\lambda y-\lambda x}{\lambda y+\lambda x}$$

$$=\frac{\lambda(y-x)}{\lambda(y+x)}=\frac{\lambda}{\lambda}\left[\frac{y-x}{y+x}\right]=\lambda^{1-1}\left[\frac{y-x}{y+x}\right]=\lambda^{0}\left[\frac{y-x}{y+x}\right]=\left[\frac{y-x}{y+x}\right],\ HDE$$

The given homogeneous differential equation is used for the assumptions

$y=ux,\ \frac{dy}{dx}=u+x\frac{du}{dx}$ to obtain a separable differential equation of the form:

$$u+x\frac{du}{dx}=\frac{ux-x}{ux+x}=\frac{u-1}{u+1}$$

$$\Rightarrow \quad x\frac{du}{dx}=\frac{u-1}{u+1}-u=\frac{u-1-u^2-u}{u+1}=\frac{-(u^2+1)}{u+1} \qquad \text{(i)}$$

$$\Rightarrow \quad -\frac{(u+1)}{u^2+1}du=\frac{dx}{x},\ \text{SDE} \qquad \text{(ii)}$$

Integrating *SDE* (ii) to obtain the general solution of the *SDE* (ii):

$$-\int\frac{(u+1)du}{u^2+1}=\int\frac{dx}{x} \Rightarrow -\int\frac{2(u+1)du}{2(u^2+1)}=\int\frac{dx}{x},\ \textit{Multiply and divide out by } 2$$

$$\Rightarrow -\frac{1}{2}\int\frac{2udu}{u^2+1}-\int\frac{du}{u^2+1}=\ln x+c,\ \Rightarrow -\frac{1}{2}\ln\left(u^2+1\right)-\tan^{-1}u=\ln x+\ln c$$

$$\Rightarrow -\ln\sqrt{u^2+1}-\tan^{-1}u=\ln cx$$

$$\Rightarrow -\tan^{-1}u=\ln\sqrt{u^2+1}+\ln cx=\ln cx\sqrt{u^2+1}$$

$$\tan^{-1}u=-\ln cx\sqrt{u^2+1} \qquad \text{(iii)}$$

The back substitution $u=\frac{y}{x}$ is used in equation (iii) to obtain the general solution of the given homogeneous differential equation:

$$\tan^{-1}\frac{y}{x}=-\ln cx\sqrt{\frac{y^2}{x^2}+1}$$

$$\tan^{-1}\frac{y}{x}=-\ln c\sqrt{y^2+x^2} \Rightarrow \frac{y}{x}=\tan\left[-\ln c\sqrt{y^2+x^2}\right]$$

iii. Equations reducible to homogenous form

A given differential equation of the form $\frac{dy}{dx}=\frac{ax+by+c}{a_1x+b_1y+c_1}$, where $\frac{a}{a_1}\neq\frac{b}{b_1}$ can be reduced to the homogeneous form by taking new variable x and y such that $x=X+h$ and $y=Y+k$, where h and k are constants to be choosen as to make the given equation homogeneous.

By using the above substitutions, we get $dx=dX$ and $dy=dY$ implies that $\frac{dy}{dx}=\frac{dY}{dX}$

Therefore, the given equation becomes, $\frac{dY}{dX}=\frac{a(X+h)+b(Y+k)+c}{a_1(X+h)+b_1(Y+k)+c_1}$

$$= \frac{aX + bY + (ah + bk + c)}{a_1X + b_1Y + (a_1h + b_1k + c_1)}$$

Now, by choosing h and k such that $ah + bk + c = 0$ and $a_1h + b_1k + c_1 = 0$

So, the differential equation becomes.

$$\frac{dY}{dX} = \frac{aX + bY}{a_1X + b_1Y}, \text{ which is a homogeneous equation.}$$

Example 10 Find the general solution of the differential equation $x\frac{dy}{dx} = x + y$.

Solution The given differential equation is not in the standard form of homogeneous differential equation, but it can be reduced in the standard form of homogenous differential equation by the following procedure:

Divide out by x to obtain the standard form of homogeneous differential equation:

$$\frac{dy}{dx} = \frac{x+y}{x}, \text{ HDE} \qquad \text{(i)}$$

Homogeneous differential equation (i) is used for the assumptions

$y = ux$, $\frac{dy}{dx} = u + x\frac{du}{dx}$ to obtain a separable differential equation of the form:

$$u + x\frac{du}{dx} = \frac{x + xu}{x} = 1 + u$$

$$x\frac{du}{dx} = 1 \Rightarrow du = \frac{dx}{x}, \text{ SDE} \qquad \text{(ii)}$$

Remember

If the differential equation is not homogeneous differential equation, then it might be a nonhomogeneous differential equation.

Integrating the SDE (ii) to obtain the general solution of the SDE (ii):

$$\int du = \int \frac{dx}{x} \Rightarrow u = \ln x + \ln c = \ln cx$$

that by back substitution $u = \frac{y}{x}$ is giving $\frac{y}{x} = \ln cx \quad \Rightarrow y = x \ln cx$

the general solution of the given homogeneous differential equation that depends on a single arbitrary constant c.

10.3.2 Solve real life problems related to differential equation

Example 11 A certain bacteria grows at a rate that is proportional to the number present at a particular time. If the number of bacterial at a time $t = 0$ is N_0 and at time $t = 1$ hour, the number of bacteria is $\frac{5N_0}{2}$. Determine the time necessary for the number of bacteria to be quadruple.

Solution If $N(t)$ is the unknown number of bacteria w.r.t time t hours, then, the rate at which bacterial rows, is represented by:

$$\frac{dN}{dt} \alpha N \Rightarrow \frac{dN}{dt} = kN$$

Reduce the differential equation to separable form $\frac{dN}{N} = k\,dt \qquad$ (i)

at on integration is giving the general solution of (i):

$$\int \frac{dN}{N} = \int k\;dt$$

$$\ln N = kt + c \qquad \text{(ii)}$$

The initial condition $N(0)=N_o$ is used in equation (ii) to obtain c:

$$\ln N_o = 0 + c \Rightarrow c = \ln N_o \qquad \text{(iii)}$$

Use c in equation (ii) to obtain a particular solution:

$$\ln N = kt + \ln N_o \Rightarrow \ln\frac{N}{N_o} = kt \Rightarrow \frac{N}{N_o} = e^{kt} \Rightarrow N = N_o e^{kt} \qquad \text{(iv)}$$

The condition $N(1)=\frac{5N_o}{2}$ is used in equation (iv) to obtain the value of k:

$$\frac{5}{2}N_o = N_o e^{k} \Rightarrow e^{k} = \frac{5}{2} \Rightarrow k = \ln\left(\frac{5}{2}\right) = 0.9163 \qquad \text{(v)}$$

Use the value of k in equation (iv) to obtain a particular solution (specific number of bacteria):

$$N = N_o e^{0.9163t} \qquad \text{(vi)}$$

The condition $N = 4N_o$ (when the bacterial have quadrupled) is used (vi) to obtain the time

$$4N_o = N_o e^{0.9163t}$$

$$4 = e^{0.9163t} \Rightarrow 0.9163t = \ln 4 \Rightarrow t = \frac{\ln(4)}{0.9163} = 1.51\ hr$$

at which the bacteria is four times of the original number of bacterial.

10.4 Orthogonal Trajectories

Our experience with first order differential equations has taught us that such equations often have general solutions containing a single arbitrary constant. Each such solution defines a corresponding set of integral curves. A nonempty set of plane curves defined by a differential equation involving just one parameter (single arbitrary constant) is commonly called a one-parameter family of curves. Of special importance in certain applications are those one-parameter families of curves which are orthogonal trajectories of one another.

"The curves of a family $F(x,y,c_1)$ are said to be orthogonal trajectories of curves of a family $G(x,y,c_2)$, if and only if each curve of either family is intersected by at least one curve of the other family and at every point of intersection of a curve of F with a curve of G, the two curves are perpendicular."

10.4.1 Orthogonal trajectories of the given family of curves

The two families of curves $F(x,y,c_1)$ and $G(x,y,c_2)$ are perpendicular at a point of intersection, if and only if their tangents are perpendicular at the point of intersection. If their tangent lines, say, L_1 and L_2, are perpendicular, then the product of their slopes equals -1:

$$m_1 m_2 = -1,\ m_1 \text{ and } m_2 \text{ are the slopes of the two tangent lines } L_1 \text{ and } L_2$$

$$m_1 = -\frac{1}{m_2} \Rightarrow \left(\frac{dy}{dx}\right)_G = -\frac{1}{\left(\frac{ay}{dx}\right)_F} \qquad \text{(i)}$$

This is called the **differential equation of orthogonal trajectories**. If one family of curves F is given, then the other family of curves G can be found by solving the differential equation of orthogonal trajectories (i).

Example 12 Determine the orthogonal trajectories of the family of curves (circles) $x^2+y^2=c$.

Solution To determine the orthogonal trajectories of the circles, we need to determine the slope (derivative) of the family of circles

$x^2+y^2=c$ with respect to x $\quad 2x+2y\dfrac{dy}{dx}=0$

$$\left(\frac{dy}{dx}\right)_F=-\frac{x}{y} \quad \text{(ii)}$$

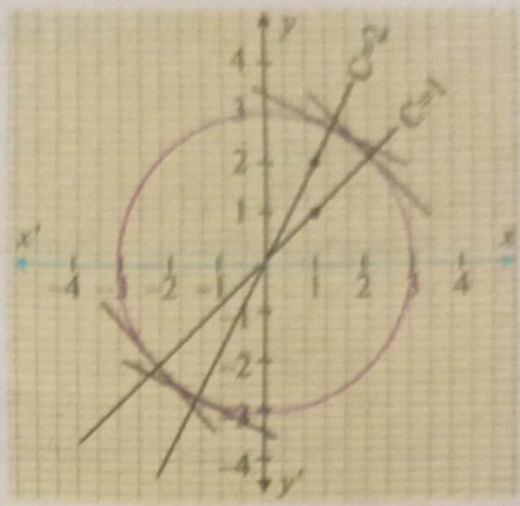


Figure 10.3

The differential equation of the orthogonal trajectories (i) with the slope of the given orthogonal trajectories (ii) is used to obtain the other family of curves G of orthogonal trajectories:

$$\left(\frac{dy}{dx}\right)_G=\frac{-1}{\left(\frac{dy}{dx}\right)_F}$$

$$\frac{dy}{dx}=\frac{-1}{-\frac{x}{y}}=\frac{y}{x} \Rightarrow \frac{dy}{y}=\frac{dx}{x}, SDE \Rightarrow \int\frac{dy}{y}=\int\frac{dx}{x}$$

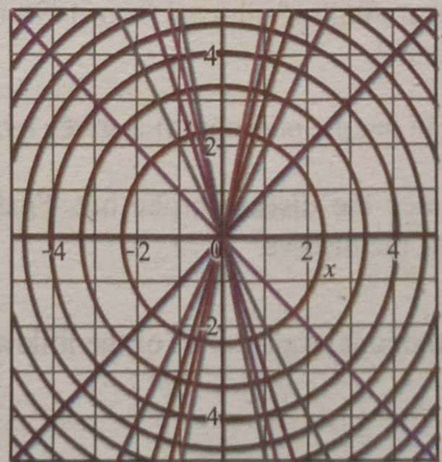
$$\ln y=\ln x+\ln C \Rightarrow y=Cx \quad \text{(iii)}$$

Thus, the family of curves G represents a family of homogeneous straight lines that pass through the origin. This is the result, we would expect, since the radii of a circle are the homogeneous lines ($y=Cx$, C is any real number) perpendicular to the lines tangent to a circle.

10.4.2 MAPLE graphic commands to view the graphs of given family of curves and its orthogonal trajectories

The general solution $y(x)=sqrt\left[-x^2+c_1\right]$ of the above problem in example 12 is the first family of curves. This can also be written as $x^2+y^2=c$. The orthogonal trajectories of a first family of curves is the second family of curves represented by $y=Cx$.
This equation can be viewed through command on line by typing MAPLE commands as,

```
> ?conturplot
> with(plots):
> F := contourplot(x^2 + y^2, x=-5..5, y=-5..5)
                                   F := PLOT(...)                    (1)
> G := plot({seq(c·x, c=-5..5)}, x=-5..5, y=-5..5) :
> display({F, G});
```

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>
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Exercise 10.2

1. Find general solution of the following differential equations:

 a. $\left(\dfrac{dy}{dx}\right)^2 = 1 - y^2$ b. $e^x \dfrac{dy}{dx} + y^2 = 0$

 c. $\sqrt{1-x^2}\,dy = \sqrt{1-y^2}\,dx$ d. $\operatorname{cosec}^2 x dy + \sec y dx = 0$

2. Reduce the following differential equations in separable form and then solve:

 a. $y' = (y+x)^2$ b. $y' = \tan(x+y) - 1$

 c. $\dfrac{dy}{dx} = \dfrac{y}{x} + \dfrac{y^2}{x^2}$ d. $\dfrac{dy}{dx} = \dfrac{y}{x} + \dfrac{x^2}{y^2}$

3. Solve the following homogeneous differential equations:

 a. $\dfrac{dy}{dx} = \dfrac{x+y}{x-y}$ b. $\dfrac{dy}{dx} = \dfrac{xy - y^2}{x^2}$

 c. $\dfrac{dy}{dx} = \dfrac{x^2 + 3y^2}{2xy}$ d. $\dfrac{dy}{dx} = \dfrac{xy + y^2}{x^2 + xy + y^2}$

4. Reduce the following differential equations in the standard form of homogeneous form and then solve:

 a. $x\dfrac{dy}{dx} = y + \sqrt{x^2 + y^2},\ y(4) = 3$ b. $(x^4 + y^4)dx = 2x^3 y dy,\ y(1) = 0$

5. The slope of a family of curves at a point $P(x, y)$ is $\dfrac{y-1}{1-x}$. Determine the equation of the curve that passes through the point P(4, –3).

6. Find the solution curve of the differential equation $xyy' = 3y^2 + x^2$ which passes through the point P(–1, 2).

7. Find the real portion from the solution curves of the differential equation $xe^{\frac{y}{x}}dx + ydx = xdy$ which passes through the point P(1, 0).

8. A particle moves along the *x-axis* so that its velocity at any point is equal to half its abscissa minus three times the time. At a time $t = 2, x = -4$, determine the motion of a particle along the *x-axis*.

9. The rate of consumption of oil (billions of barrels) is given by $\dfrac{dx}{dt} = 1.2e^{0.04t}$, Where t=0 correspond to 1990. Find the total amount of oil used from 1990 to year 1995. At this rate, how much oil will be used in 8(t =8) years?

10. The rate of infection of a disease (in people per month) is given by: $\dfrac{dI}{dt} = \dfrac{100t}{t^2+1}$

 Where t is the time in months since the disease broke out. Find the total number of infected people over the first four months of the disease.

10. Determine the equations of the orthogonal trajectories of the following families of curves.

 a. $y = cx^3$ b. $xy = c$ c. $y = cxe^x$ d. $y^2 = x^2 + C$

Review Exercise 10

Choose the correct option.

i. The order of equation $3\frac{dy}{dx}+2y=5$ is:

(a). 1 (b). 2 (c). 3 (d). 4

ii. The degree of the equation $\left(\frac{d^2y}{dx^2}\right)^2-\frac{dy}{dx}+y=0$ is:

(a). 1 (b). 2 (c). 3 (d). 4

iii. The solution that depends on an arbitrary constant quantity is called:

(a). exact solution (b). particular
(c). general solution (d). none of these

iv. The solution of $\frac{dy}{dx}=\sin(x)$ at $y(0)=1$ is:

(a). $y=\sin(x)+2$ (b). $y=-\sin(x)+2$
(c). $y=\cos(x)+2$ (d). $y=-\cos(x)+2$

v. The solution of $y\frac{dy}{dx}=\cos(x)$ at $y(0)=-1$ is:

(a). $y=-\sqrt{2\sin(x)+1}$ (b). $y=\sqrt{2\sin(x)+1}$
(c). $y=-\sqrt{2\cos(x)+1}$ (d). $y=\sqrt{2\cos x+1}$

vi. The differential equation for the orthogonal trajectory of the family of curve $x^2+c^2=c$ is:

(a). $\frac{dy}{dx}=cx$ (b). $y=cx$ (c). $\frac{dy}{dx}=ct$ (d). $cy=\frac{dy}{d\theta}$

vii. The Orthogonal trajectories for the family of $y^2=4ax$:

(a). $y^2=k-2x^2$ (b). $x^2=k-2y^2$ (c). $x^2=k+2y^2$ (d). $y^2=2x^2-k$

Summary

- A differential equation is an equation that involves the derivatives of an unknown function (dependent variable) of one or more variables (independent variables).
- The order of a differential equation is the order of the highest-order derivative which appears in the equation.
- The degree of a differential equation is the power of the highest-order derivative which appears in the equation.
- A solution of an ordinary differential in one dependent variable y on an interval I is a function $y(x)$ which, when substituted for the dependent variable y and its derivatives $y', y'', \ldots$, reduces the differential equation to an identity in the independent variable x over interval I.
- A differential equation is **linear** in a set of one or more of its dependent variables if and only if each term of the equation which contains a variable of the set or any of their derivatives is of the first degree in those variables and their derivatives.
- A function $f(x, y)$ is homogeneous function of degree $\boldsymbol{n}$ in variables x and y if and only if for all values of the variables x, and y and for every positive value of λ, the identity is true:

$$f(\lambda x, \lambda y) = \lambda^n f(x, y), \quad n = 1, 2, 3, \ldots$$

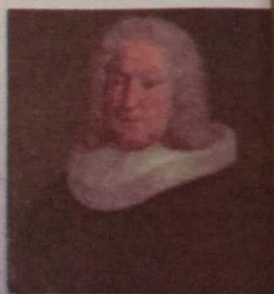
- The differential equation $$\frac{dy}{dx} = \frac{f(x,y)}{g(x,y)}$$

is called a homogenous differential equation, if it defines a homogenous function of degree zero.

History

Jacob Bernoulli (1654)-(1705)

Jacob Bernoulli and Johann Bernoulli we two Swiss mathematicians. They were first interpreters of Leibniz's version of differential equations. Both brothers orgue about Newton's theories and maintained that the Newton's theory of fluxions was plagiarized from the Leibniz's original work. Both brothers worked hard using differential calculus to disprove Newton's principle. The could not accept the theory which Newton had given, that earth and other plants rotate around the sun in elliptical orbits. The first book on differential calculus was written on the bases of Bernoulli brothers ideas by Gabriele Manfred in 1707. Most of the publications were made on differential equations and partial differential equations in 18th century.



Johann Bernoulli (1667)-(1748)

Unit 11

PARTIAL DIFFERENTIATION

By the end of this unit, the students will be able to:

11.1 Differentiation of function of two variables

i. Define a function of two variables.
ii. Define partial derivative.
iii. Find partial derivatives of a function of two variables.

11.2 Euler's Theorem

i. Define a homogeneous function of degree n.
ii. State and prove Euler's theorem on homogeneous functions.
iii. Verify Euler's theorem for homogeneous functions of different degrees (simple cases).
iv. Use MAPLE command diff to find partial derivatives.

Introduction

The goal of this unit is to extend the methods of single variable differential calculus to functions of two variables. In many practical situations, the value of one quantity may depend on the values of two or more others. For example, the amount of water in a reservoir may depend on the amount of rainfall and on the amount of water consumed by local residents. The current in an electrical circuit may vary with the electromotive force, the capacitance, the resistance, and the impedance in the circuit. The flow of blood from an artery into a small capillary may depend on the diameter of the capillary and the pressure in both the artery and the capillary. The output of a factory may depend on the amount of capital invested in the plant and on the size of the labor force. We will analyze such situations using functions of several variables.

In many problems involving functions of several variables, the goal is to find the derivative of the function with respect to one of its variables when all the others are held constant. In this unit, we need to develop the concept and shall see how it can be used to find slopes and rates of change in case of two variables function.

11.1 Differentiation of the function of two variables

In the real world, physical quantities often depend on two or more variables. For example, we might be concerned with the temperature on a metal plate at various points at time t. The locations of temperature on the plate are given as ordered pairs (x, y), so that the temperature T can be considered as a function of two location variables x and y, as well as a time variable t. The notation of a function of single variable, we might extend this as $T(x, y, t)$. We begin our study of function of two variables by examining this notation and a few other basic concepts.

For illustration, if a company produces x items at a cost of 10 rupees per item, then the total cost $C(x)$ of producing x items is given by: $C(x) = 10x$

The cost is a function of one independent variable, the number of items produced. If the company wants to produce two products, with x of one product at a cost of rupees 10 each, and y of another product at a cost of rupees 15 each, then the total cost to the firm is a function of two independent variables x and y:

$C(x, y) = 10x + 15y$

When $x = 5$ and $y = 12$, the total cost is written with $C(5,12) = 10(5) + 15(12) = 230$ rupees.

11.1.1 Function of two variables

"A function $z = f(x, y)$ is a function of two variables x and y, if for each given pair (x, y), we determine a single value of z." *Where, x, y and z are real variables.*

The real numbers x and y are **independent variables**; z is the dependent variable. The set of all ordered pairs of real numbers (x, y) such that $f(x, y)$ is a real number, is **the domain** of f and the set of all values of $f(x, y)$ is the **range**.

Example 1 How to show that $z = f(x,y) = \sqrt{1-x+y}$ is a function of two independent variables x and y? Find also the domain and range of a given function.

Solution For this function, we need to show the transformation of two independent variables (inputs) x and y is just *a* single dependent variable z. In respect of any two real values of independent variables x and y, say, $x = 2$ and $y = 1$, the function $z = f(2,1) = \sqrt{1-2+1} = 0$ gives response of just one real value of z which is $z = 0$.

The function $z = f(x,y) = \sqrt{1-x+y}$ is therefore declared a function of two independent variables x and y.

> **Remember**
> Functions of more than one independent variable are called **multivariate functions.**

The domain of $f(x, y)$ is the set of all ordered (x, y) for which $\sqrt{1-x+y}$ is defined. We must have $1-x+y \geq 0$ or $y \geq x-1$, in order for the square root to be defined.

In a function $z = f(x,y) = \sqrt{1-x+y}$, we see that $z = f(x, y)$ must be nonnegative and the range of $f(x, y)$ is all $z \geq 0$

11.1.2 Partial derivative

To give clear concept to partial derivative, the problem related to our real-life situations is considered as:

Suppose, a small firm makes only two products, radios and audiocassette recorders. The profit of the firm from these two products is given by: $P(x,y) = 40x^2 - 10xy + 5y^2 - 80,$ (i)

Where x is the number of units of radios sold and y is the number of units of recorders sold. How changes in x will (radios) or y (recorders) affects P (profit)?

Suppose that sales of radios have been steady at 10 units; only the sales of recorders vary. The management would like to find the rate (marginal profit/ derivative of the profit function) at which the y number of recorders sold.

If x is fixed at 10 units, then this information reduces the profit two variables function to a **new single variable function** that can be found from equation (i) by putting $x = 10$:

$$P(10, y) = 40(10)^2 - 10(10)y + 5y^2 - 80 = 3920 - 100y + 5y^2$$

The function $P(10, y)$ shows the profit from the sales of y recorders, assuming that x is fixed at 10 units. The rate, at which the y number of recorders sold, is the ordinary derivative of $P(10, y)$ with respect to y: $\frac{d}{dy}P(10, y) = -100 + 10y$ (ii)

This represents the per unit profit from y number of audiocassette recorders.

The notation of $\frac{d}{dy}P(10, y)$ is usually stands for **ordinary derivative,** when the function is a single variable function. **In our case,** the profit function (i) is a function of two variables; its rate with respect to y should be a **partial derivative**. For partial derivative with respect to y, the ordinary derivative in equation (ii) is replaced by $\frac{\partial}{\partial y}P(10, y)$ to obtain

$$\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}P(10, y) = P_y = -100 + 10y,$$ prime notation "/" is not allowed.

Informally,

- the partial derivative $\frac{\partial}{\partial x}f(x, y)$ with respect to x is the derivative of $f(x,y)$ obtained by keeping x as a variable and y as a **constant quantity**.

- the partial derivative $\frac{\partial}{\partial y} f(x, y)$ with respect to y is the derivative of $f(x,y)$ obtained by keeping y as a variable and x as a **constant quantity**.

11.1.3 Partial derivatives of a function of two variables

If $z = f(x, y)$ is a function of two variables, then the first partial derivatives of $z = f(x, y)$ with respect to x and y are the functions f_x and f_y respectively, defined by,

$$\frac{\partial f}{\partial x} = f_x(x, y) = f_x = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x, y)}{\Delta x} \qquad \text{(i)}$$

$$\frac{\partial f}{\partial y} = f_y(x, y) = f_y = \lim_{\Delta y \to 0} \frac{f(x, y+\Delta y) - f(x, y)}{\Delta y} \qquad \text{(ii)}$$

Provided that limits exist.

Example 2 If function is $f(x, y) = x^3 y + xy^2$. Find partial derivatives f_x and f_y.

Solution The partial derivatives of $f(x, y)$ w.r.t. x and y are the following:

$$\frac{\partial f}{\partial x} = f_x = \frac{\partial}{\partial x}\left(x^3 y + xy^2\right) = 3x^2 y + y^2, \quad y \text{ is constant}$$

$$\frac{\partial f}{\partial y} = f_y = \frac{\partial}{\partial y}\left(x^3 y + xy^2\right) = x^3 + 2xy, \quad x \text{ is constant}$$

Example 3 The function is $z = x^2 \sin(3y + x^3)$. Find z_x and z_y at a point $\left(\frac{\pi}{3}, 0\right)$.

Solution The partial derivative of $z(x, y)$ w.r.t. x is:

$$z_x = \frac{\partial}{\partial x}\left[x^2 \sin(3x + y^3)\right], \quad y \text{ is constant}$$

$$= 2x\sin(3x + y^3) + x^2 \cos(3x + y^3)\frac{\partial}{\partial x}\left(3x + y^3\right) = 2x\sin(3x + y^3) + x^2 \cos(3x + y^3)(3 + 0)$$

$$= 2x\sin(3x + y^3) + 3x^2 \cos(3x + y^3)$$

$$\left[z_x\right]_{\left(\frac{\pi}{3}, 0\right)} = 2\frac{\pi}{3}\sin\left(\frac{3\pi}{3}\right) + 3\frac{\pi^2}{9}\cos\left(3\frac{\pi}{3}\right) = \frac{2\pi}{3}\sin\pi + \frac{3\pi^2}{9}\cos\pi = -\frac{\pi^2}{3}$$

The partial derivative of $z(x, y)$ w.r.t. y is:

$$z_y = \frac{\partial}{\partial y}\left[x^2 \sin(3x + y^3)\right], \quad x \text{ is constant}$$

$$= x^2 \frac{\partial}{\partial y}\left[\sin(3x + y^3)\right] = x^2 \cos(3x + y^3)\frac{\partial}{\partial y}\left(3x + y^3\right) = x^2 \cos(3x + y^3)(0 + 3y^2)$$

$$= 3x^2 y^2 \cos(3x + y^3)$$

$$\left[z_y\right]_{\left(\frac{\pi}{3}, 0\right)} = 3\left(\frac{\pi^2}{9}\right)(0)^2 \cos\left(\frac{3\pi}{3}\right) = 0$$

Example 4 Suppose that the temperature of the water at the point on a river where a nuclear power plant discharges its hot waste water is approximated by $T(x, y) = 2x + 5y + xy - 40$ (i)

Where x represents the temperature of the river water in degree Celsius before it reaches the power plant and y is the number of megawatts (in hundreds) of electricity being produced by the plant.

(a). Find and interpret $T_x(9,5)$. *(b).* Find and interpret $T_y(9,5)$.

Solution

a. The partial derivative of (i) w.r.t. x is the rate of change in T with respect to x: $T_x = 2 + y$, y is constant

This rate with $x = 9$ and $y = 5$ is $[T_x]_{(9,5)} = 2 + y = 2 + 5 = 7$ the approximate change in temperature resulting from a one degree increase in input water, if the input electricity y remains constant at 500 megawatts.

b. The partial derivative of (i) w.r.t. y is the rate of change in T with respect to y: $T_y = 5 + x$, x is constant

This rate with $x = 9$ and $y = 5$ is $[T_y]_{(9,5)} = 5 + x = 5 + 9 = 14$ the approximate change in temperature resulting from a one megawatt increase in production of electricity if the input water temperature x remains constant at 9°C.

Exercise 11.1

1. If $f(x,y) = x^2y + xy^2$ and t is any real number, then find out the following:

   a. $f(t, 0)$ b. $f(-1, 0)$ c. $f(0, -1)$

   d. $f(t, t)$ e. $f(t, t^2)$ f. $f(1-t, t)$

2. The function is $f(x,y,z) = x^2ye^{2x} + (x+y-z)^2$. Find the function value at the following points:

   a. $f(0,0,0)$ b. $f(1,-1,1)$ c. $f(-1,1,-1)$

   d. $\dfrac{\partial}{\partial x} f(x,x,x)$ e. $\dfrac{\partial}{\partial y} f(1,y,1)$ f. $\dfrac{\partial}{\partial z} f(1,1,z^2)$

3. Find the partial derivatives f_x and f_y of each of the following functions:

   a. $f(x,y) = \sin(x^2)\cos y$ b. $f(x,y) = \sqrt{3x^2 + y^4}$

   c. $f(x,y) = xy^3 \tan^{-1} y$ d. $f(x,y) = x^3 + x^2y + xy^2 + y^3$

   e. $f(x,y) = \sin^{-1} xy$ f. $f(x,y) = x^2e^{x+y}\cos y$

4. The production function z for the United States was once estimated as:

   $z = f(x,y) = x^{0.7}y^{0.3}$

   Where x stands for the amount of labor and y stands for the amount of capital. Find the marginal productivity of labor $\left(\dfrac{\partial z}{\partial x}\right)$ and of capital $\left(\dfrac{\partial z}{\partial y}\right)$.

5. A similar production function for Canada is:

   $z = f(x,y) = x^{0.4}y^{0.6}$

   Where x stands for the amount of labor and y stands for the amount of capital. Find the marginal productivity of labor $\left(\dfrac{\partial z}{\partial x}\right)$ and of capital $\left(\dfrac{\partial z}{\partial y}\right)$.

6. If $f(x,y) = x^2y + xy^2$, then find f_x and f_y by using definition of partial derivatives.

History

Leonhard Euler was a Swiss Mathematician, Physicist, Astronomer and Engineer. He made the important and influential contributions in many branches of Mathematics, such as calculus, graph theory, topology and analytic number theory. He also made significant contribution in mechanics, fluid dynamics, optics and music theory, He was the first person who introduced $f(x)$ to denoted the function f applied to the argument 'X'. In 1735 he introduced a theorem known by his name Euler theorem.

Leonhard Euler (1707-1783)

11.2 Euler's Theorem

The specialty of Euler's theorem is to verify the degree of a homogeneous function. The homogeneous function is a function $z = f(x, y)$ not altered if the real numbers x and y of a function $z = f(x, y)$ are stretched or squeezed by any real scalar quantity λ.

11.2.1 Homogeneous function of degree n

"A function $f(x, y)$ is said to be a homogeneous function of degree n if, for all values of λ and some constant values of n, we have $f(\lambda x, \lambda y) = \lambda^n f(x, y)$ " (i)

For example,

a. $f(x,y) = 3x + 4y$ is a homogeneous function of degree 1.

Since, $f(\lambda x, \lambda y) = 3(\lambda x) + 4(\lambda y) = \lambda(3x) + \lambda(4y) = \lambda(3x, 4y) \Rightarrow f(\lambda x, \lambda y) = \lambda f(x, y)$

b. $f(x,y) = 3x^2 + 4y^2$ is a homogeneous function of degree 2.

Since, $f(\lambda x, \lambda y) = 3(\lambda x)^2 + 4(\lambda y)^2$

$$= \lambda^2(3x^2, 4y^2)$$

$$\Rightarrow f(\lambda x, \lambda y) = \lambda^2 f(x, y)$$

Remember

A homogeneous function can also define another way.
"A function (x, y) is said to be homogeneous function of degree n if it is expressed in the form
$$f(x,y) = x^n . f\left(\frac{y}{x}\right)"$$

In general, A function $f(x_1, x_2, x_3, \dots, x_n)$ is said to be homogeneous of degree n if, for all values of λ and some constant values of n, we have $f(\lambda x_1, \lambda x_2, \lambda x_3, \dots, \lambda x_n) = \lambda^n f(x_1, x_2, x_3, \dots, x_n)$.

Example 5 Show that the function $f(x, y) = 2xy + y^2$ is a homogeneous function of degree 2.

Solution The function $f(x, y) = 2xy + y^2$ is a homogeneous function of degree 2, if the identity (i) is true:

$$f(\lambda x, \lambda y) = 2(\lambda x)(\lambda y) + (\lambda y)^2 = 2\lambda^2 xy + \lambda^2 y^2 = \lambda^2(2xy + y^2) = \lambda^2 f(x, y),\ n = 2$$

Thus, the given function is a homogeneous function of degree 2.

11.2.2 Verification of Euler's theorem for homogeneous functions of different degrees

Statement: If $z = f(x, y)$ is continuously differentiable and defines a homogeneous function of degree n, then

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nz \qquad \text{(i)}$$

Proof: If $z = x^n f\left(\frac{y}{x}\right)$, then, its partial derivatives with respect x and y are the following:

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x} x^n f\left(\frac{y}{x}\right)$$

$$\frac{\partial z}{\partial x} = nx^{n-1} f\left(\frac{y}{x}\right) + x^n f'\left(\frac{y}{x}\right)\left(\frac{-y}{x^2}\right) = nx^{n-1} f\left(\frac{y}{x}\right) - yx^{n-2} f'\left(\frac{y}{x}\right) \qquad \text{(ii)}$$

$$\frac{\partial z}{\partial y} = x^n f'\left(\frac{y}{x}\right)\left(\frac{1}{x}\right) = x^{n-1} f'\left(\frac{y}{x}\right) \qquad \text{(iii)}$$

The addition of the products of (ii) by x and y (iii) by y to obtain the Euler's method of order n:

$$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=x\left[nx^{n-1}f\left(\frac{y}{x}\right)-yx^{n-2}f'\left(\frac{y}{x}\right)\right]+y\left[x^{n-1}f'\left(\frac{y}{x}\right)\right]=nx^{n}f\left(\frac{y}{x}\right)-yx^{n-1}f'\left(\frac{y}{x}\right)+yx^{n-1}f'\left(\frac{y}{x}\right)$$

$$=nx^{n}f\left(\frac{y}{x}\right)=nz \qquad \text{(iv)}$$

Example 6 Use Euler's theorem to verify that the function $z=f(x,y)=ax^2+2bxy+cy^2$ is a homogeneous function of degree 2.

Solution The homogeneous function and its derivatives

$$z=f(x,y)=ax^2+2bxy+cy^2,\ \frac{\partial z}{\partial x}=2ax+2by,\ \frac{\partial z}{\partial y}=2bx+2cy$$ are used in Euler's result (iv) to

confirm the degree of homogeneous function:

$$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=x(2ax+2by)+y(2bx+2cy)\ =2ax^2+2bxy+2bxy+2cy^2=2(ax^2+2bxy+cy^2)=2z,\ n=2$$

The Euler's procedure confirmed the second degree of homogeneous function.

Example 7 If $u=\tan^{-1}\dfrac{x^3+y^3}{x-y}$, then, show that

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\sin 2u$$

Solution The function $u=(x, y)$ is not a homogeneous function, however, it can be reduced to homogeneous form by introducing a new variable z:

$$z=\tan u=\frac{x^3+y^3}{x-y}=\frac{x^3\left(1+\left(\frac{y}{x}\right)^3\right)}{x\left(1-\left(\frac{y}{x}\right)\right)}=x^2f\left(\frac{y}{x}\right)$$

The given function is a homogeneous of degree 2. The Euler's theorem in this situation is:

$$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=2z,\quad n=2 \qquad \text{(i)}$$

The partial derivatives of $z=\tan u$ w.r.t. x and y

$$\frac{\partial z}{\partial x}=\sec^2 u\frac{\partial u}{\partial x},\quad \frac{\partial z}{\partial y}=\sec^2 u\frac{\partial u}{\partial y}$$

are substituting in (i) to obtain the required result:

$$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=2z$$

$$\sec^2 u\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)=2z,\ z=\tan u$$

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=2z\cos^2 u=2\frac{\sin u}{\cos u}\cos^2 u=2\sin u\cos u=\sin 2u$$

11.2.3 MAPLE command "*diff*" to find partial derivatives

Look at the following example the use of MAPLE command "diff" is illustrated in this examples.

Example 8 Use MAPLE command "diff" to find the partial derivation of

(a). $f(x,y)=x^3+y^3+3xy^2+4x^2y$ w.r.t, variables x and y.

(b). $f(x,y)=y\sin x+x\cos y+x^2$ w.r.t, variables x and y.

Solution

a. **Command:**

> $diff(x^3+y^3+3\cdot x\cdot y^2+4\cdot x^2y,x);$

$$3x^2+3y^2+8xy$$

> $diff(x^3+y^3+3\cdot x\cdot y^2+4\cdot x^2y,y);$ (2)

$$3y^2+6xy+4x^2$$

b.

> $diff(y\sin(x)+x\cdot\cos(y)+x^2,x);$ (1)

$$y\cos(x)+\cos(y)+2x$$

> $diff(y\sin(x)+x\cdot\cos(y)+x^2,y);$ (2)

$$\sin(x)-x\sin(y)$$

Using Palettes: Use cursor button to select expression in which you are interested. In this problem, the expression is partial derivative palette. Click-partial derivative palette, insert the given function, then press "ENTER" key to obtain the partial derivatives of a given function:

$$\frac{\partial}{\partial x}(x^3+y^3+3\cdot x\cdot y^2+4x^2\cdot y)$$

$$3x^2+3y^2+8xy$$

$$\frac{\partial}{\partial y}(x^3+y^3+3\cdot x\cdot y^2+4x^2\cdot y)$$

$$3y^2+6xy+4x^2$$

Exercise 11.2

Are the following functions homogeneous?

a. $u=f(x,y)=\sin^{-1}\dfrac{x+y}{\sqrt{x}+\sqrt{y}}$?

b. $z=f(x,y)=\dfrac{x+y}{\sqrt{x}+\sqrt{y}}$?

c. $z=f(x,y)=x^3e^{\frac{y}{x}}-3y^2\sqrt{x^2+y^2}$

d. $z=f(x,y)=(x^2+3y^2)^{\frac{1}{3}}$

Verify Euler's theorem for the following homogeneous functions:

a. $z=f(x,y)=ax^2+2hxy+by^2$

b. $z=f(x,y)=(x^2+xy+y^2)^{-1}$

If $u=f\left(\dfrac{y}{x}\right)$, then show that $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}=0.$

If $z=xyf\left(\dfrac{x}{y}\right)$, then show that $x\dfrac{\partial z}{\partial x}+y\dfrac{\partial z}{\partial y}=2z.$

a. If $u=\tan^{-1}\dfrac{x^2+y^2}{x+y}$, then show that $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}=\sin(u).\cos(u)$

b. If $u = \sin^{-1}\dfrac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}$, then show that $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}=0.$

6. Use MAPLE command "*diff*" to find the partial derivation of

a. $f(x,y)=\dfrac{x^3+y^3}{x^2+y^2}, w\cdot r\cdot t' y'$ b. $-\dfrac{\sin^2 x}{y}+x\cos(y) w\cdot r\cdot t' y'$

c. $\dfrac{1}{y}\tan(x)-x\cot^2(y) w\cdot r\cdot t' x'$ d. $\left(\dfrac{4t^3+3s^2}{s-t}\right) w\cdot r\cdot t' s'$

Review Exercise 11

1. **Choose the correct option.**

i. In the function $z=f(x,y)$, x and y are:
(a). alphanumeric variables (b). dependent variables
(c). independent variables (d). dependent constants

ii. If $f(x,y)=\sqrt{x^2+y^2-1}$ then $f(1,5)$ is:
(a). 25 (b). 5 (c). 125 (d). $3\sqrt{3}$

iii. If $f(x,y)=x^2+y^2$ then f_x or $\dfrac{\partial f}{\partial x}$ is:
(a). $2y$ (b). $2x$ (c). $2x+2y$ (d). $2x+y^2$

iv. $diff\left(\dfrac{1}{x+y}, y\right)$; is:
(a). $\dfrac{1}{x^2}$ (b). $-\dfrac{1}{y^2}$ (c). $-\dfrac{1}{(x+y)^2}$ (d). $\dfrac{1}{(x+y)^2}$

v. The Euler's Theorem stated that:
(a). $\dfrac{\partial z}{\partial x}+\dfrac{\partial z}{\partial y}=nz$ (b). $x\dfrac{\partial z}{\partial x}+\dfrac{\partial z}{\partial y}=nz$ (c). $x\dfrac{\partial z}{\partial x}+y\dfrac{\partial z}{\partial y}=nz$ (d). $-x\dfrac{\partial z}{\partial x}+\dfrac{\partial z}{\partial y}=nz$

vi. The functions of more than one independent variables are called:
(a). monomial functions (b). polynomial functions
(c). multivariate function (d). none of these

vii. If $f(x,y,z)=\sqrt{x^3+y^3-z}$ then $f(1, 0, 1)$ is:
(a). 1 (b). -1 (c). 0 (d). 2

viii. If $f(x,y)=\cos y\cdot e^{3x}$ then f_x is:
(a). $3\cos y\cdot e^{3x}-\sin y$ (b). $3\cos ye^{3x}$
(c). $3e^{3x}\cos y$ (d). $8\cos ye^{3x}$

ix. If $f(x,y)=3xe^y+x^3y^2$ then f_y is:
(a). $3xe^y+2x^3y$ (b). $3x^ye+2y3x^2$
(c). $3xe^y-2x^3y$ (d). $3x^2y^2+3e^y$

Summary

- A function $z = f(x, y)$ is a **function of two variables x and y**, if a unique of z is obtained from each ordered pair of real numbers (x, y). The real numbers x and y are **independent variables**; z is the **dependent variable**. The set of all ordered pairs of real numbers (x, y) such that $f(x, y)$ is a real number, is **the domain** of f and the set of all values of $f(x, y)$ is the **range**.
- A polynomial function in x and y is the sum of functions of the form $f(x,y) = Cx^m y^n$
- If $z = f(x,y)$ is a function of two variables, then the first partial derivatives of $z = f(x, y)$ with respect to x and y are the functions f_x and f_y respectively, defined by,

$$f_x(x,y) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x}, \quad f_y(x,y) = \lim_{\Delta x \to 0} \frac{f(x, y+\Delta y) - f(x,y)}{\Delta y}$$

- A function $f(x, y)$ is a **homogeneous** function of degree $\boldsymbol{n}$ in variables x and y, if for all values of the variables and for every positive value of λ, for which the identity is true:

$$f(\lambda x, \lambda y) = \lambda^n f(x,y)$$

- The specialty of **Euler's theorem** is to verify the degree of a homogeneous function. The homogeneous function is a function $z = f(x, y)$ not altered if the real numbers x and y of a function $z = f(x, y)$ are stretched or squeezed by any real scalar quantity λ.

History

Carl Neumann was German mathematician. He shidied physics from his father and later became a mathematician. His father was professor at Konigsberg university. In 1875 be introduce the new standard notation (d-hat) during a lecture on the mechanical theory of heat. The symbol was popularized by his name as Neumann notation or δ (Greek delta). He worked on the drichlet principle and can be considered one of the initiators of the thory of integral equations. The Neumann series. That is analogous to the geometric series

$$\frac{1}{1-x} = a + x + x^2 + x^3 + \ldots$$

Carl Neumann
(1832)-(1925)

is named after him.

Carl Neumann also founded a mathematical research journal mathematische arnalen. The Neumann boundary condition for different types of ordinary differential equations and partial differential equation is also named after him. He also developed an interest in thermodynamics via the overlap of heat and electricity.

Unit 12

INTRODUCTION TO NUMERICAL METHODS

By the end of this unit, the students will be able to:

12.1 Numerical solution of non-linear equations

i. Describe importance of numerical methods.

ii. Explain the basic principles of solving a non-linear equation in one variable.

iii. Calculate real roots of a non-linear equation in one variable by
- Bisection method,
- Regula-falsi method,
- Newton-Raphson method.

iv. Use MAPLE command *fsolve* to find numerical solution of an equation and demonstrate through examples.

12.2 Numerical quadrature

i. Define numerical quadrature. Use
- Trapezoidal rule,
- Simpson's rule,

to compute the approximate value of definite integrals without error terms.

ii. Use MAPLE command **trapezoid** for trapezoidal rule and **simpson** for Simpson's rule and demonstrate through examples.

Introduction

Scientists, economists, engineers, and other researchers study relationships between quantities. For example, an engineer may need to know how the illumination from a light source on an object is related to the distance between the object and the source; a biologist may wish to investigate how the population of a bacterial colony varies with time in the presence of a toxin; an economist may wish to determine the relationship between demand for a certain commodity and its market price. The mathematical study of such relationships involves the concept of non-linear equations. For example, the value of the assets of a certain company at time t years is modeled by a non-linear equation $f(t)=100,000-75,000e^{-2t}$ where t is measured in years. The standing rule for solving non-linear equations algebraic is quadratic formula. In this case, it is not valid to obtain the actual number of t (years) at which the asset function $f(t)$ is going to be zero. Now we are in position to obtain the approximate number of year's t that can be found by using some numerical procedures. In this unit, we will learn the numerical procedures recommended are the bracketing methods and iterative methods.

12.1 Numerical Solution of Non-linear Equations

Numerical analysis is the theory of constructive methods in mathematical analysis. Constructive methods in their turn mean a procedure that permits us to obtain the solution of a mathematical problem with an arbitrary precision in a finite number of steps that can be prepared rationally.

12.1.1 Importance of numerical methods

Numerical analysis is both a Science and an Art. As a Science, it is concerned with the process by which a mathematical problem can be solved arithmetically. As an Art, numerical analysis is concerned with choosing that procedure which is best suited to the solution of a particular problem.

Students learning numerical solution of non-linear equations should have the following objectives in view. First, he should obtain an intuitive and working understanding of some numerical methods for the basic problems of numerical analysis. Second, he should gain some appreciation of the concept of error and of the need to analyze and predict it. Third, he should develop some experience in the implementation of numerical method by using **computer software**.

12.1.2 Basic principles of solving non-linear equations in one variable

*"If $f(x)$ is any **continuous function** of a single variable x, then any number r for which $f(r)=0$ is called a root of $f(x)=0$. Also we say that r is a zero of the function $f(x)$."*

If $f(x)$ is any algebraic function (non-linear equation), then the actual roots of $f(x)$ can be found by direct rules, such as quadratic formula, common factors procedure and synthetic division.

For example, the quadratic equation $x^2+5x+6=0$ has two **actual** (or exact) roots $r_1=-2$ and $r_2=-3$ obtained by common factors procedure/quadratic formula:

$$f(x)=x^2+5x+6=0$$
$$=x^2+2x+3x+6=0$$
$$=(x+2)(x+3)=0 \Rightarrow x+2=0,\ x+3=0 \Rightarrow x=-2=r_1,\ x=-3=r_2$$

On the other hand, the actual roots of no algebraic equation $x^{-2}+5x+6=0$ are not possible by applying quadratic formula. The only way is to find out the **approximate roots** that can be found by using some **numerical procedures**. The numerical procedures are the bracketing methods and iterative methods.

12.1.3 Real roots of non-linear equation in one variable

For approximate roots of a non-linear equations, the numerical procedures **Bisection method** and **regula-falsi method** are the bracketing methods that depend on two initial approximations that must be in the shape of a closed interval $[a, b]$.

The non-linear curve $f(x)$ has the function values $f(a)$ and $f(b)$ in the interval $[a, b]$ that must be opposite in signs for showing its continuity. Once the interval has been found, no matter how large, the **iteration** will be preceded until an approximate root is obtained. The fundamental principle in computer science is the **iteration.** As the name suggests, it means that a process of bisection or regula-falsi method is repeated until an answer is achieved.

(a) Bisection Method

If $y = f(x)$ is continuous function in the interval $[a, b]$, then, it will cross the x-axis at a point $(r,0)$ whose x-coordinate $x = r$ will keep as actual root that lies somewhere in the interval $[a, b]$. This is shown in the Figure 12.1.

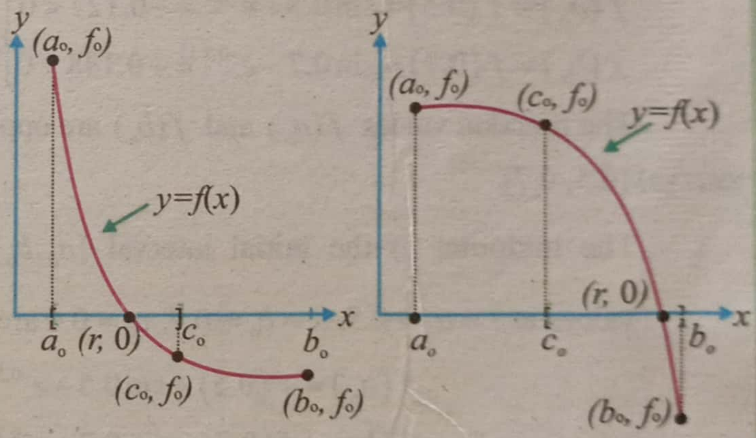


Figure 12.1

The bisection method systematically moves the endpoints of the interval $[a,b]$ closer and closer together till it reaches an interval of small width that brackets the root r. The decision step for this process of interval halving is to choose the midpoint $c=\dfrac{(a+b)}{2}$ and then analyze the three possibilities that might arise:

1. If the function values $f(a)$ and $f(c)$ at $x=a$ and $x=c$ have opposite signs, then the approximate root lies in interval $[a, c]$ and discard b.
2. If the function values $f(c)$ and $f(b)$ at $x=c$ and $x=b$ have opposite signs, then the approximate root lies in interval $[c, b]$ and discard a.
3. If the function value at $x=c$ is $f(c)=0$, then **c is our approximate root to actual root r.**

If either of cases 1 or 2 occurs, we have an interval half as wide as the original interval that contains the root, and we are "squeezing down on it" see Figure 12.1. To continue the process, relabel the new smaller interval and repeat the sequence of nested intervals and their midpoints.

The procedure in detail is as under:

The given interval $[a_0, b_0]$ is the initial interval at which the function $f(x)$ must be opposite in signs. At this stage, the initial interval brackets the actual root r whose midpoint is $c_0 = \dfrac{(a_0 + b_0)}{2}$.
It develops the first iterate c_o (through properties 1 or 2) to actual root r in the interval $[a_0, b_0]$.

The next interval $[a_1, b_1]$ is the first interval which brackets the actual root r and c_1 is its midpoint $c_1 = \dfrac{(a_1 + b_1)}{2}$. It develops the second iterate c_1 (through properties 1 or 2) to actual root r in the interval $[a_1, b_1]$.

Similarly, the interval $[a_n, b_n]$ is the nth interval which brackets the actual root r and c_n is its midpoint $c_n = \dfrac{(a_n + b_n)}{2}$. It develops the $(n-1)$-th iterate c_{n-1} to actual root r in the n-th interval $[a_n, b_n]$.

This completes the n times iteration of the bisection method and the midpoint c_{n-1} is taken as the desired approximation to the actual root $r \approx c_{n-1}$ of $y = f(x)$.

Remember

To find approximate root of an equation $f(x) = 0$ in the interval $[a, b]$. Proceed with the method only if $f(x)$ is continuous and $f(a)$ and $f(b)$ have opposite in signs.

Example 1 Perform two iterations of the bisection method to approximate the actual root r of the non-linear equation $f(x) = \sin x - e^{-x}$ (x is in radians) in the interval [0.5, 0.7].

Solution Reset the given interval to obtain the initial interval $[a_0, b_0] = [0.5, 0.7]$ and compute the function $f(x) = \sin x - e^{-x}$ values at $x = a_0 = 0.5$ and $x = b_o = 0.7$ to obtain:

$$\left.\begin{aligned} f(a_o) &= f(0.5) = \sin 0.5 - e^{-0.5} = -0.127 < 0 \\ f(b_o) &= f(0.7) = \sin 0.7 - e^{-0.7} = +0.148 > 0 \end{aligned}\right\} \textit{opposite in signs}$$

The function values $f(a_o)$ and $f(b_o)$ are opposite in signs, so the actual root r of $f(x)$ lies in the interval [0.5, 0.7].

i. The midpoint of the initial interval $[a_0, b_0]$ is $c_o = \dfrac{a_o + b_o}{2} = \dfrac{0.5 + 0.7}{2} = 0.6$, and the function values at $x = a_0 = 0.5$, $x = b_0 = 0.7$, $c_0 = 0.6$ are the following:

$$\left.\begin{aligned} f(a_o) &= f(0.5) = \sin 0.5 - e^{-0.5} = -0.127 < 0 \\ f(b_o) &= f(0.7) = \sin 0.7 - e^{-0.7} = +0.148 > 0 \\ f(c_o) &= f(0.6) = \sin 0.6 - e^{-0.6} = +0.016 > 0 \end{aligned}\right\}$$

The function values $f(a_o)$ and $f(c_o)$ are opposite in signs, so the approximation to the actual root r of $f(x)$ lies in the interval [0.5, 0.6] and discard $b_0 = 0.7$. The initial interval is reset to obtain the first interval $[a_1, b_1] = [0.5, 0.6]$.

ii. The midpoint of the first interval is $c_1 = \dfrac{a_1 + b_1}{2} = \dfrac{0.5 + 0.6}{2} = 0.55$, and the function values at $x = a_1 = 0.5$, $x = b_1 = 0.6$, and $x = c_1 = 0.55$ are the following:

$$\left.\begin{aligned} f(a_1) &= f(0.5) = \sin 0.5 - e^{-0.5} = -0.127 < 0 \\ f(c_1) &= f(0.55) = \sin 0.55 - e^{-0.55} = -0.054 < 0 \\ f(b_1) &= f(0.6) = \sin 0.6 - e^{-0.6} = +0.016 > 0 \end{aligned}\right\}$$

The function values $f(b_1)$ and $f(c_1)$ are opposite in signs, so the approximation to actual root r of $f(x)$ lies in the interval [0.55,0.6] and discard $a_1 = 0.5$. The first interval is reset to obtain the second interval

$$[a_2, b_2] = [0.55, 0.6].$$

After second iteration of the bisection method, the midpoint $c_1 = 0.55$ is declared approximate root to actual root $r \approx c_1$. The approximation value of a function $f(x) = \sin x - e^{-x}$ at approximate root $r \approx 0.55$ is $f(0.055) = -0.054$.

(b) Regula-Falsi method

The next bracketing method is the method of regula-falsi method. It was developed because the bisection method converges at a fairly slow speed. As before, we assume that $f(a)$ and $f(b)$ have opposite signs. The bisection method always used the midpoint of the interval as the next iterate, but in regula-falsi method, the next iterate is anywhere in the interval $[a, b]$ represented by the point of intersection $(c, 0)$ of the straight line formed by the points $(a, f(a))$, and $(b, f(b))$ and the x-axis

$$\left.\begin{aligned} \frac{y-f(a)}{x-a} &= \frac{f(b)-f(a)}{b-a} \\ \frac{0-f(a)}{c-a} &= \frac{f(b)-f(a)}{b-a} \\ c &= a - \frac{(a-b)f(a)}{f(a)-f(b)} \end{aligned}\right\}, \ at\ (x, y) = (c, 0) \qquad \text{(i)}$$

and then analyze the three possibilities that might arise :

i. If the function values $f(a)$ and $f(c)$ at $x = a$ and $x = c$ have opposite signs, then the root lies in the interval $[a, c]$ and discard b.

ii. If the function values $f(c)$ and $f(b)$ at $x = c$ and $x = b$ have opposite signs, then the root lies in the interval $[c, b]$ and discard a.

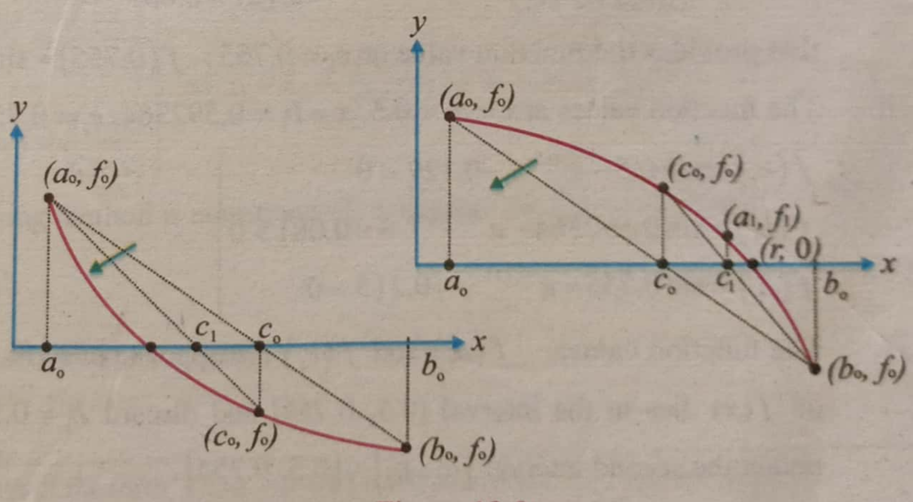


Figure 12.2

iii. If the function value at $x = c$ is $f(c) = 0$, then **c is our approximate root**. This is shown in the Figure 12.2.

Example 2 Perform two iterations of the regula-falsi method to approximate the actual root r of the non-linear equation $f(x) = \sin x - e^{-x}$ (x is in radians) in the interval [0.5, 0.7].

> **Remember**
> To find approximate root of an equation $f(x) = 0$ in the interval $[a, b]$. Proceed with the method only if $f(x)$ is continuous and $f(a)$ and $f(b)$ have opposite in signs.

Solution Reset the given interval to obtain the initial interval $[a_0, b_0] = [0.5, 0.7]$ and compute the function $f(x) = \sin x - e^{-x}$ values at $x = a_0 = 0.5$ and $x = b_o = 0.7$ to obtain:

$$\left.\begin{array}{l} f(a_o) = f(0.5) = \sin 0.5 - e^{-0.5} = -0.127 < 0 \\ f(b_o) = f(0.7) = \sin 0.7 - e^{-0.7} = +0.148 > 0 \end{array}\right\} \text{opposite in signs}$$

The function values $f(a_o)$ and $f(b_o)$ are opposite in signs, so the actual root r of $f(x)$ lies in the interval [0.5,0.7]. Equation (i) is used to obtain

$$c_o = a_o - \frac{(a_o - b_o) f(a_o)}{f(a_o) - f(b_o)} = 0.5 - \frac{(0.5 - 0.7)(-0.127)}{-0.127 - 0.148} = 0.592364$$

That provides the function value at $c_o = 0.592364$: $f(c_o) = \sin(0.592364) - e^{-0.592364} = +0.081$

i. The function values at $x = a_0 = 0.5$, $x = b_0 = 0.7$, $c_0 = 0.592364$ are the following:

$$\left.\begin{array}{l} f(a_o) = \sin 0.5 - e^{-0.5} = -0.127 < 0 \\ f(b_o) = \sin 0.7 - e^{-0.7} = +0.148 > 0 \\ f(c_o) = \sin 0.592364 - e^{-0.592364} = +0.081 > 0 \end{array}\right\}$$

The function values $f(a_o)$ and $f(c_o)$ are opposite in signs, so the approximation to actual root r of $f(x)$ lies in the interval [0.5, 0.592364] and discard $b_0 = 0.7$. The initial interval is reset to obtain the first interval $[a_1, b_1] = [0.5, 0.592364]$.

Equation (i) is used to obtain

$$c_1 = a_1 - \frac{(a_1 - b_1) f(a_1)}{f(a_1) - f(b_1)} = 0.5 - \frac{(0.5 - 0.592364)(-0.127)}{-0.127 + 0.081} = 0.755$$

that provides the function value at $c_1 = 0.755$: $f(0.755) = \sin(0.755) - e^{-0.755} = 0.215$

ii. The function values at $x = a_1 = 0.5$, $x = b_1 = 0.592364$, $c_1 = 0.755$ are the following:

$$\left.\begin{array}{l} f(a_1) = \sin 0.5 - e^{-0.5} = -0.127 < 0 \\ f(b_1) = \sin 0.592364 - e^{-0.592364} = +0.081 > 0 \\ f(c_1) = \sin 0.755 - e^{-0.755} = +0.215 > 0 \end{array}\right\}$$

The function values $f(a_1)$ and $f(c_1)$ are opposite in signs, so the approximation to actual root r of $f(x)$ lies in the interval [0.5, 0.755] and discard $b_1 = 0.592364$. The first interval is reset to obtain the second interval $[a_2, b_2] = [0.5, 0.755]$.

After second iteration of the regula-falsi, the point $c_1 = 0.755$ is declared as approximation to actual root $r \approx c_1$. The approximate value of a function $f(x) = sinx - e^{-x}$ at approximate root $r \approx c_1 = 0.755$ is $f(0.755) = 0.215$.

(c) Newton-Raphson-method

Another numerical procedure is the Newton-Raphson method under the umbrella of **iterative methods**. The Newton-Raphson method is one-point iterative **method** which requires **one** previous approximation in contrast of bracketing methods which require **two** in computing the successive approximation.

The Newton-Raphson method uses the **slopes of the tangent lines to the graph of a function $f(x)$** to approximate roots of the equation $f(x) = 0$.

If $f(x)$ and $f'(x)$ are continuous near actual roots, then this extra information regarding the nature of $f(x)$ can be used to develop a sequence of iterates $\{x_n\}$ that will converge faster to actual root than either the bisection and regula-falsi methods.

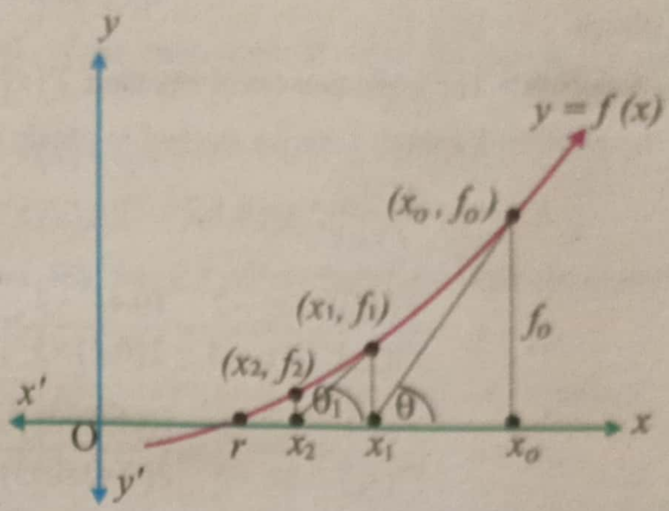


Figure 12.3

If r is any actual root of an equation of the form $f(x) = 0$, and x_0 is an initial approximation to the actual r, then the tangent line on a curve $f(x)$ at a point (x_0, f_0) crosses the x-axis at a point $(x_1, 0)$. The point $(x_1, 0)$ is the point of intersection of the tangent line and the x-axis, and the x-coordinate x_1 of the point of intersection $(x_1, 0)$ is our first approximation to the actual root r of an equation $f(x) = 0$. This is shown in the Figure 12.3.

The slope of the tangent line on a curve $y = f(x)$ at a point (x_0, f_0) is used to obtain the first approximation (iterate) x_1 :

$$\left.\begin{aligned} \tan\theta &= \frac{f(x_o)}{x_o - x_1} \\ f'(x_o) &= \frac{f(x_o)}{x_o - x_1} \end{aligned}\right\} \Rightarrow x_o - x_1 = \frac{f(x_o)}{f'(x_o)} \Rightarrow x_1 = x_o - \frac{f(x_o)}{f'(x_o)} \qquad \text{(i)}$$

Similarly, the slope of the tangent line on a curve $y = f(x)$ at a point (x_1, f_1) is used to obtain the second iterate x_2 :

$$\left.\begin{aligned} \tan\theta_1 &= \frac{f(x_1)}{x_1 - x_2} \\ f'(x_1) &= \frac{f(x_1)}{x_1 - x_2} \end{aligned}\right\} \Rightarrow x_1 - x_2 = \frac{f(x_1)}{f'(x_1)} \Rightarrow x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

This procedure of slope finding method is continued till it reaches the $(i+1)$-th iterate:

$$x_{i+1} = x_1 - \frac{f(x_i)}{f'(x_i)}, \; i = 0, 1, 2, \ldots \qquad \text{(ii)}$$

which is called the **Newton-Raphson method.**

Remember

To find approximate root of an equation of the form $f(x) = 0$ with initial iterate x_o, the iteration method

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}, \; i = 0,1,2,3,\ldots, \; i = 1, 2, 3,\ldots$$

develops a sequence of successive iterates $\{x_n\}$ that will converge faster to actual root r than either the bisection and regula-falsi methods.

Example 3 Use Newton-Raphson iterative method to approximate the actual root $r = 0.438447$ of the non-linear equation $f(x) = x^2 - 5x + 2$ with initial start $x_o = 0.4$ that must be accurate to six decimal places.

Solution The given non-linear equation $f(x) = x^2 - 5x + 2$ and its derivative $f'(x) = 2x + 5$ are used in the Newton-Raphson iterative method to obtain the successive iterates

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}, \quad i = 0, 1, 2$$

$$x_1 = x_o - \frac{f(x_o)}{f'(x_o)} = \frac{x_o^2 - 2}{2x_o - 5} = \frac{(0.4)^2 - 2}{2(0.4) - 5} = 0.438095, \quad i = 0$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = \frac{x_1^2 - 2}{2x_1 - 5} = \frac{(0.438095)^2 - 2}{2(0.438095) - 5} = 0.438447, \quad i = 1$$

that converge the actual root $r = 0.438447$.

The second iterate $x_2 = 0.438447$ agrees to six decimal accuracy of actual root $r = 0.438447$. We achieved in just two iterates of the Newton-Raphson method the six decimal accuracy.

12.1.4 MAPLE command "fsolve" to find numerical solution of an equation and demonstrate through examples

The use of MAPLE command "fsolve" to find the approximate solution of given function is illustrated in the following example.

Example 4 Use maple command "fsolve" to solve.

(a). Linear equation $x^2 - 5x + 6 = 0$ with initial start $x_0 = 1.8$.

(b). Nonlinear equation $x^2 - 5x + 6 = 0$ with initial start $x_0 = 0.5$.

Solution The command below will show you full detail of the approximate root of linear and non-linear equations on line by typing without initial start:

a.

> ?fsolve
> fsolve$(x^2 - 5 \cdot x + 6)$

2.000000000, 3.000000000

The quadratic function $f(x)$ is also a second degree polynomial. The numerical solution through polynomial is:

> Polynomial $:= x^2 - 5 \cdot x + 6$

Polynomial $:= x^2 - 5x + 6$

> fsolve(Polynomial)

2.000000000, 3.000000000

b.

> fsolve$(\sin(x) - \exp(-x))$

0.5885327440

Context Menu:

> $\sin(x) - \exp(-x)$
> fsolve($\sin(x) - \exp(-x)$)

0.5885327440

This result is obtained through right-click on the last end of the expression by selecting "Solve < Numerically Solve" on the context menu.

Exercise 12.1

1. Find an interval $a \le x \le b$ at which $f(a)$ and $f(b)$ have opposite signs for the following functions:

   a. $f(x) = e^x - 2 - x$ b. $f(x) = \cos x + 1 - x$ x is in radians

   c. $f(x) = \ln(x) - 5 + x$ d. $f(x) = x^2 - 10x + 23$

2. Compute four iterates of the bisection method for the following functions with indicated interval $[a_0, b_0]$:

   a. $f(x) = e^x - 2 - x,$ $[1, 1.8]$ b. $f(x) = \cos x + 1 - x,$ $[0.8, 1.6]$ x is in radians

   c. $f(x) = \ln(x) - 5 + x$ $[3.2, 4]$ d. $f(x) = x^2 - 10x + 23$ $[3.2, 4]$

3. Compute four iterates of the regula-falsi method for the following functions with indicated interval $[a_0, b_0]$:

   a. $f(x) = e^x - x,$ $[-2.4, -1.6]$ b. $f(x) = \cos x + 1 - x,$ $[0.8, 1.6]$ x is in radians

   c. $f(x) = \ln(x) - 5 + x,$ $[-2.4, -1.6]$ d. $f(x) = x^2 - 10x + 23,$ $[-2.4, -1.6]$

4. What will happen if the bisection method is used with the function $f(x) = \dfrac{1}{(x-2)}$ for the following intervals:

   a. $[3, 7]$ b. $[1, 7]$

5. Find iterate x_3 of Newton-Raphson iterative method for the following functions with initial start x_o:

   a. $f(x) = x^3 - 3,$ $x_o = 1$ b. $f(x) = \sin(x), x_o = 1$

   c. $f(x) = x^3 + 2x - 1,$ $x_o = 0$ d. $f(x) = \sin x,$ $x_0 = -2$

6. Use Newton-Raphson iterative method to approximate the actual r of the following non-linear equations with indicated interval:

   a. $f(x) = x^3 + 3x - 1 = 0$ on $(0,1)$ b. $f(x) = x^3 + 2x^2 - x + 1 = 0$ on $(-3, -2)$

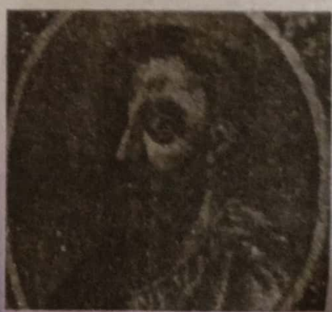
   c. $\sqrt[3]{x-3} = x + 1$ on $[-3, -2]$

   Continue the process until two consecutive iterates will agree to three decimal places.

7. Use MAPLE command '*f*solve' to solve $3x^2 + 4x - 3 = 0$ with initial start at $x_0 = 0.5$

History

Qusta Ibn Luqa was a Syrian mathematician, astronomer and philosopher. He contribute in many fields of science, medicine, astronomy. His translation on the difference between the spirit and the soul was one of the few works not attributed to Aristotle that was included in a list of books to be read or lectured on. He was the first person to write the double false position in 10th century. He justified the technique by a formal, Euclidean-style geometric proof, within the tradition of Muslim mathematics. Double false position was known as Hisab-al-Khata'ayn. It was used for centuries to solve practical problems such as commercial and recreational problems.



Qusta Ibn Luqa
(820-912)

12.2 Numerical Quadrature

Numerical integration is a primary tool used by engineers and scientists to obtain approximate solutions for definite integrals that cannot be solved analytically. For example, the integral

$$I = \int_0^1 e^{x^2} dx \qquad \text{(i)}$$

has no **actual** solution. This means that there is no any integral formula that could be used directly to obtain the actual solution. The only way is to find out the approximate solution that can be found by using some numerical procedures, such, as numerical integration.
We now approach the subject of numerical integration. The goal is to approximate the definite integral of $f(x)$

$$I = \int_a^b f(x)dx \qquad \text{(ii)}$$

over the interval [a, b] by evaluating $f(x)$ at a finite number of equally spaced grid points:

$$\left.\begin{array}{ll} x: & a = x_o \quad x_1 \;....\; x_{n-1} \quad x_n = b \\ f(x): & f_o \quad f_1 \;....\; f_{n-1} \quad f_n \end{array}\right\}$$

12.2.1 Numerical quadrature

"If a set of points in the interval [a, b] is $a = x_o < x_1 < x_2 < ... < x_{n-1} < x_n = b$, then an expression of the form $Q[f(x)] = \sum_{j=0}^{n} w_j f(x_j) = w_o f(x_0) + w_1 f(x_1) + ... + w_n f(x_n)$

with the property $$\int_a^b f(x)dx = Q[f(x)] + E[f(x)] \qquad \text{(iii)}$$

is called a numerical integration or ***quadrature formula.****"*

The term E [f (x)] is called the truncation error for integration. The values $\{x_j\}_{j=0}^{n}$ are called the **quadrature nodes** and $\{w_j\}_{j=0}^{n}$ are called the **weights**.

Depend on the given numerical procedure, the grid points $\{x_j\}$ are chosen in various ways. For trapezoidal rule and Simpson's rule, the grid points are chosen to be **equally spaced**. Before discussion of trapezoidal rule and Simpson's rule it must be familiar about approximation by rectangles and approximate area by rectangles.

Approximate by rectangles

If $f(x) = 0$ is a function over the interval [a, b], then the definite integral (ii) represents the actual area under the graph of $f(x)$ on the interval [a, b]. This is shown in the Figure 12.4.

For approximate area, the function $f(x)$ must be known at equally spaced grid points in the interval[a, b], each of width $\Delta x = (b-a)/n$:

$$\left.\begin{array}{ll} x: a = x_o & x_1 \quad x_2 ...... x_{n-1} \quad x_n = b \\ f(x): & f_o \quad f_1 \quad f_2 ...... f_{n-1} \quad f_n \end{array}\right\}, \; x_k = x_o + k\Delta x, \; k = 0, 1, 2, ..., n$$

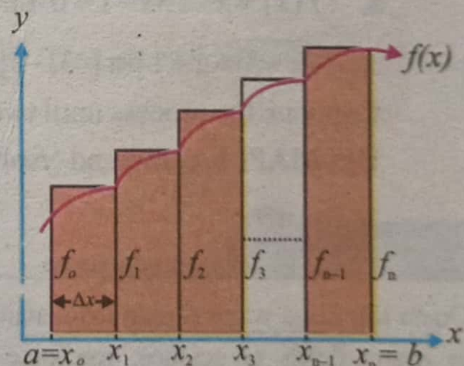


Figure 12.4

In light of above equally spaced grid points, the actual area (ii) under a curve $f(x)$ over the interval [a, b] is rearranged as under:

$$I = \int_a^b f(x)dx = \int_{x_0}^{x_1} f(x)dx + \int_{x_1}^{x_2} f(x)dx + ... + \int_{x_{n-1}}^{x_n} f(x)dx \qquad \text{(iv)}$$

Approximate Area by Rectangles

Consider the initial subinterval $[x_0, x_1]$. Let x_1 denote the right endpoint of the initial subinterval, and the base of the rectangle is of course the initial subinterval and its height is $f(x_0)$. The area of the rectangle in the initial subinterval is therefore $f(x_0)\Delta x$.

If $\int_{x_0}^{x_1} f(x)dx$ is the actual area in the initial subinterval, then the approximate area $f(x_0)\Delta x$ is of course the area of the rectangle lies in the initial subinterval $[x_0, x_1]$. Thus, the sum of the areas of all n rectangles is giving approximate area under the curve $f(x)$ to actual area represented by definite integral (iv).

$$I = \int_a^b f(x)dx \approx R_n = f(x_0)\Delta x_0 + f(x_1)\Delta x + f(x_2)\Delta x + ... + f(x_n)\Delta x \qquad \text{(v)}$$

This approximation improves as the number of rectangles increases, and we can estimate the integral to any desired degree of accuracy by taking n large enough. However, because fairly large values of n are usually required to achieve reasonable accuracy, approximation by rectangles is rarely used in practice.

i. **The Trapezoidal rule**

The accuracy of the approximation can be improved if trapezoids are used instead of rectangles. Figure 12.5 shows the area approximated by n trapezoids instead of n rectangles.

If $\int_{x_0}^{x_1} f(x)dx$ is the actual area in the initial subinterval, then the approximate area $\left[\frac{f(x_0)+f(x_1)}{2}\right]\Delta x$ is of course the area of the trapezoid lies in the initial subinterval $[x_0, x_1]$. Thus, the sum of the areas of all n trapezoids is giving approximate area under the curve $f(x)$ to actual area represented by definite integral (iv):

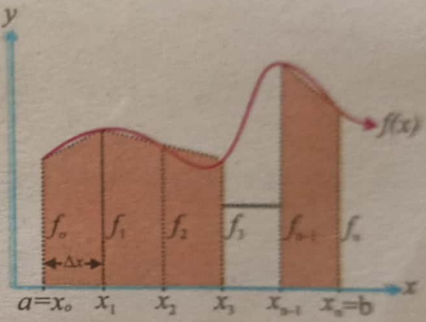


Figure 12.5

$$\int_a^b f(x)dx \approx T_n$$

$$= \frac{1}{2}[f(x_0)+f(x_1)]\Delta x + \frac{1}{2}[f(x_1)+f(x_2)]\Delta x + ... + \frac{1}{2}[f(x_{n-1})+f(x_n)]\Delta x$$

$$= \frac{1}{2}[f_0 + 2f_1 + 2f_2 + ... + 2f_{n-1} + f_n]\Delta x \qquad \text{(vi)}$$

In general, If $f(x)$ is continuous on $[a, b]$, then the trapezoidal rule is

$$I = \int_a^b f(x) \approx T_n = \frac{\Delta x}{2}[f_0 + 2f_1 + 2f_2 + ... + 2f_{n-1} + f_n],$$

The n^{th} subinterval is $x_n = x_o + n\Delta x \Rightarrow b = a + n\Delta x$ that gives $\Delta x = \frac{(b-a)}{n}$. Moreover, the larger the value of n, the better the approximation.

Example 5 Approximate the definite integral $I=\int_{-1}^{2} x^2 dx$

for $n=4$ subintervals and then compare your approximate answer with the actual value of the definite integral that must be accurate to 5 decimal places.

Solution The given interval is $[a, b]=[-1, 2]$ and the width for $n=4$ subintervals is

$$\Delta x=\frac{b-a}{n}=\frac{2-(-1)}{4}=\frac{3}{4}=0.75$$

The trapezoidal rule (vi) is used for $\Delta x=0.75$ and $n=4$ to obtain:

$$\int_{-1}^{2} x^2 dx \approx T_4=\frac{1}{2}[f_0+2f_1+2f_2+2f_3+f_4](0.75) \qquad \text{(i)}$$

The function values f_0, f_1, f_2, f_3, f_4 at grid points x_0, x_1, x_2, x_3, x_4 are

$x_o=a=-1,$ $\qquad f_0=f(-1)=(-1)^2=1$

$x_1=a+1\Delta x=-1+\frac{3}{4}=-\frac{1}{4},$ $\qquad f_1=\left(-\frac{1}{4}\right)^2=\frac{1}{16}0.0625$

$x_2=a+2\Delta x=-1+2\frac{3}{4}=\frac{1}{2},$ $\qquad f_2=\left(\frac{1}{2}\right)^2=\frac{1}{4}=0.25$

$x_3=a+3\Delta x=-1+3\frac{3}{4}=\frac{5}{4},$ $\qquad f_3=\left(\frac{5}{4}\right)^2=\frac{25}{16}=0=1.5625$

$x_4=a+4\Delta \dot{x}=-1+4\frac{3}{4}=2,$ $\qquad f_4=(2)^2=4$

used in (i) to obtain: $\int_{-1}^{2} x^2 dx \approx T_4=\frac{1}{2}[f_0+2f_1+2f_2+2f_3+f_4]\Delta x$

$$=\frac{1}{2}[1+2(0.0625)+2(0.25)+2(1.5625)+4](0.75)=3.28125$$

The exact value of the definite integral is: $I=\int_{-1}^{2} x^2 dx=\left|\frac{x^3}{3}\right|_{-1}^{2}=\frac{8}{3}+\frac{1}{3}=\frac{9}{3}=3$

The trapezoidal approximation T_4 developed an error, which we denote by E_4:

$E_4=$ *Actual* $-$ *Approximation* $=I-T_4=3-3.28125=-0.28125$

The negative sign indicates that the trapezoidal formula overestimated the true value of the definite integral.

In terms of numerical quadrature notation, $Q[f(x)]=T_4=3.28125$ and $E_4[(x)]=-0.28125$ with nodes $x_o=-1,\ x_1=\frac{-1}{4},\ x_2=\frac{1}{2},\ x_3=\frac{5}{4},\ x_4=2.$

ii. The Simpson's rule

The accuracy of the approximation can be improved if instead of trapezoidal strips, the parabolic strips are used in three consecutive equally spaced grid points. The name given to the procedure in which the approximating strip has a parabolic arc is the **Simpson's rule.**

Mathematically, the parabolic arc is represented by a second degree polynomial $p(x)=Ax^2+Bx+C.$

Simpson's rule approximates the actual area in an interval $[a, b]$ by parabolic arc if $f(x)$ is replaced by second degree polynomial $p(x)$

$$\int_a^b f(x)dx \approx \int_a^b p(x)dx = \int_a^b (Ax^2+Bx+C)dx = \left(\frac{b-a}{6}\right)\left[p(a)+4p\left(\frac{a+b}{2}\right)+p(b)\right] \quad \text{(i)}$$

that requires three consecutive grid points in an even interval $[a, b]$. The definite integral of the second degree polynomial in equation (i) is simplified by a rule called **prismoidal rule**. This rule is valid for a polynomial $p(x)$ of degree **less than or equal to 3.**

For Simpson's rule, the function $f(x)$ is known at equally even spaced grid points in the interval $[a, b]$:

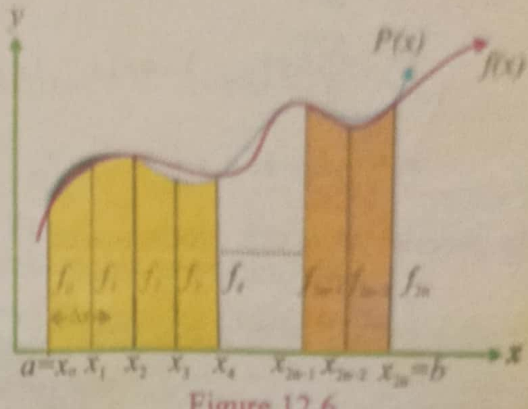


Figure 12.6

$$\left.\begin{matrix} x: a=x_0 & x_1 & x_2 \dots x_{2n-2} & x_{2n-1} & x_{2n}=b \\ f(x): \quad f_0 & f_1 & f_2 \dots f_{2n-2} & f_{2n-1} & f_{2n} \end{matrix}\right\}, \text{ with}$$

$$x_{2n} = x_o + 2n\Delta x, \; n = 0, 1, 2, \dots.$$

$$b = a + 2n\Delta x, \; x_{2n} = b, \; x_0 = a \Rightarrow \quad \Delta x = \frac{b-a}{2n} \quad \text{(ii)}$$

In light of even spaced grid points, the definite integral (iv) is rearranged as under:

$$I = \int_a^b f(x)dx = \int_{x_0}^{x_2} f(x)dx + \int_{x_2}^{x_4} f(x)dx + \dots\dots + \int_{x_{2n-2}}^{x_{2n}} f(x)dx \quad \text{(iii)}$$

If $\int_{x_0}^{x_2} f(x)dx$ is the actual area in the initial even subinterval $[x_0, x_2]$, then the parabolic arc in the subinterval $[x_0, x_2]$ represents the approximate area:

$$\int_{x_0}^{x_2} f(x)dx \approx \int_{x_0}^{x_2} p(x)dx, \; a = x_0, \; b = x_2,$$

$$= \left(\frac{x_2 - x_0}{6}\right)\left[p(x_0) + 4p\left(\frac{x_0+x_2}{2}\right) + p(x_2)\right] \quad \text{(iv)}$$

If the width of one subinterval is $\Delta x = \frac{(b-a)}{n}$, then the width of two consecutive subintervals is $x_2 - x_0 = 2\Delta x$. The subintervals are of equal width that gives $\frac{x_2 + x_0}{2} = x_1$. Using these in equation (iv) to obtain

$$\int_{x_0}^{x_2} f(x)dx \approx \int_{x_0}^{x_2} p(x)dx = \frac{2\Delta x}{6}\left[p(x_0) + 4p\left(\frac{x_0+x_2}{2}\right) + p(x_2)\right] = \frac{\Delta x}{3}[p(x_0) + 4p(x_1) + p(x_2)] \quad \text{(v)}$$

Since the polynomial passes through the three consecutive grid points on the curve, the best approximations would be the function itself: that is $p(x_0) = f(x_0)$, and $p(x_1) = f(x_1)$, and $p(x_2) = f(x_2)$. These are substituted in equation (v) to obtain:

$$\int_{x_0}^{x_2} f(x)dx \approx \int_{x_0}^{x_2} p(x)dx = \frac{\Delta x}{3}[p(x_0) + 4p(x_1) + p(x_2)] = \frac{\Delta x}{3}[f(x_0) + 4f(x_1) + f(x_2)] \quad \text{(vi)}$$

Thus, the sum S_{2n} of the areas of all n parabolic strips is giving the approximate area under the curve $f(x)$ to actual area represented by definite integral (iii):

$$S_{2n}=\frac{\Delta x}{3}\left[f(x_0)+4f(x_1)+2f(x_2)\right]+\frac{\Delta x}{3}\left[f(x_2)+4f(x_3)+f(x_4)\right]$$
$$+....+\frac{\Delta x}{3}\left[f(x_{2n-2})+4f(x_{2n-1})+f(x_{2n})\right]$$
$$=\frac{\Delta x}{3}\left[f_0+4(f_1+f_3+.....+f_{2n-1})+2(f_2+f_4+.....+f_{2n-2})+f_{2n}\right] \quad \text{(vii)}$$

In general: If $f(x)$ is continuous on $[a, b]$, then the Simpson's rule is

$$I=\int_a^b f(x)\approx S_{2n}=\frac{\Delta x}{3}\left[f_0+4(f_1+f_3+....+f_{2n-1})+2(f_2+f_4+....+f_{2n-2})+f_{2n}\right]$$

The even n[th] subinterval is $x_{2n}=x_0+2n\Delta x \Rightarrow b=a+2n\Delta x$ that gives $\Delta x=\frac{((b-a)}{2n}$.

Moreover, the larger the value for n, the better the approximation.

Example 6 Approximate the definite integral $I=\int_{-1}^{2} x^2dx$ for $n=2$ subintervals and then compare your approximate answer with the actual value of the definite integral that must be accurate to 5 decimal places.

Solution The given interval is $[a, b]=[-1, 2]$ and the width for $n=2$ subintervals is

$$\Delta x=\frac{b-a}{2n}=\frac{2-(-1)}{4}=\frac{3}{4}=0.75.$$

The Simpson's rule (vii) is used for $\Delta x=0.75$ and $n=2$ to obtain:

$$\int_{-1}^{2} x^2dx \approx S_4=\frac{0.75}{3}\left[f_0+4(f_1+f_3)+2(f_2)+f_4\right] \quad \text{(i)}$$

The function values f_0, f_1, f_2, f_3, f_4 at grid points x_0, x_1, x_2, x_3, x_4 are

$$x_o=a=-1, \qquad f_0=f(-1)=(-1)^2=1$$

$$x_1=a+1\Delta x=-1+\frac{3}{4}=-\frac{1}{4}, \qquad f_1=\left(-\frac{1}{4}\right)^2=\frac{1}{16}0.0625$$

$$x_2=a+2\Delta x=-1+2\frac{3}{4}=\frac{1}{2}, \qquad f_2=\left(\frac{1}{2}\right)^2=\frac{1}{4}=0.25$$

$$x_3=a+3\Delta x=-1+3\frac{3}{4}=\frac{5}{4}, \qquad f_3=\left(\frac{5}{4}\right)^2=\frac{25}{16}=0=1.5625$$

$$x_4=a+4\Delta x=-1+4\frac{3}{4}=2, \qquad f_4=(2)^2=4$$

These function values are used in equation (i) to obtain:

$$\int_{-1}^{2} x^2dx \approx S_4=\frac{0.75}{4}[f_0+4(f_1+f_3)+2f_2+f_4]$$

$$= \frac{0.75}{3}[1+4(0.0625+1.5625)+2(0.25)+4]$$

$$= 0.25(1+6.5+0.50+4) = 3$$

The exact value of the integral is:

$$I = \int_{-1}^{2} x^2 dx = \left|\frac{x^3}{3}\right|_{-1}^{2} = \frac{8}{3}+\frac{1}{3} = 3$$

The error term is therefore:

$$E_2 = I - S_4 = 3 - 3 = 0$$

This develops the idea that Simpson's rule is much more accurate than trapezoidal rule.

12.2.2 MAPLE commands "trapezoid" for trapezoidal rule and "Simpson" for Simpson's rule and demonstrate through examples

The use of MAPLE command "trapezoid" and "Simpson" is illustrated in the following example.

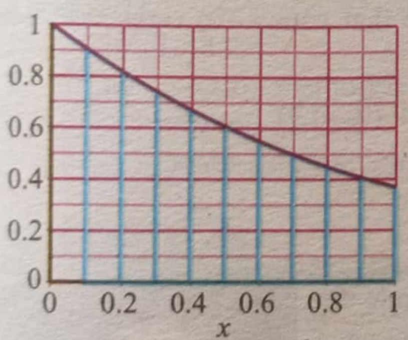
Example 7 Use of MAPLE command to find the approximate area $\int_0^1 e^{(-x)} dx$ in the interval [0, 1] by

(a). Trapezoidal rule. (b). Simpson rule

Solution

a. The command below will show you full detail about Trapezoidal rule on line by typing:

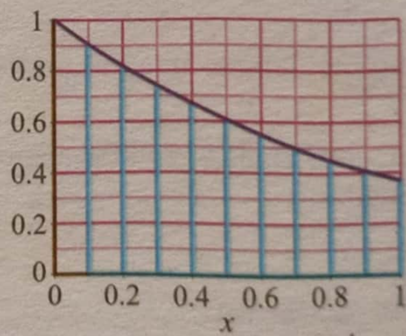
```
?trapezoid
with(Student[Calculus1]) :
ApproximateInt(exp(-x), x=0..1,
method=trapezoid, output=plot);
```



An approximation of $\int_0^1 f(x)\,dx$ using trapezoid rule, where $f(x) = e^{-x}$ and the partition is uniform. The approximate value of the integral is 0.6326472382. Number of subintervals used: 10.

b. The command below will show you full detail about Simpson rule on line by typing:

```
?simpson
with(Student[Calculus1]) :
ApproximateInt(exp(-x), x=0..1,
method=simpson, output=plot);
```



An approximation of $\int_0^1 f(x)\,dx$ using Simpson's rule, where $f(x) = e^{-x}$ and the partition is uniform. The approximate value of the integral is 0.6321205808. Number of subintervals used: 10.

Exercise 12.2

1. Use the trapezoidal rule to approximate the value of each definite integral. Round the answer to the nearest hundredth and compare your results with the exact value of the definite integral:

   a. $I=\int_1^3 x^2 dx,\ n=4$
   b. $I=\int_0^1 \left(\frac{x^2}{2}+1\right)dx,\ n=4$
   c. $I=\int_1^3 \frac{dx}{x},\ n=6$
   d. $I=\int_0^2 \sqrt{1+x^2}\,dx,\ n=6$

2. Use Simpson's rule to approximate the value of each definite integral. Round the answer to the nearest hundredth and compare your results with the exact value of the definite integral:

   a. $I=\int_2^4 x^2 dx,\ n=3$
   b. $I=\int_2^3 \left(\frac{x^2}{3}-1\right)dx,\ n=4$
   c. $I=\int_1^3 \frac{dx}{x},\ n=3$
   d. $I=\int_0^1 e^{2x}dx,\ n=4$

3. A quarter circle of radius 1 has the equation $y=\sqrt{1-x^2}$ for $0\le x\le 1$, which means that: $\int_0^1 \sqrt{1-x^2}\,dx=\frac{\pi}{4},\ n=4$

   Approximate the definite integral on the left by trapezoidal rule that equals the right side when $\pi=3.1$.

4. A quarter circle of radius 1 has the equation $y=\sqrt{1-x^2}$ for $0\le x\le 1$, which means that $\int_0^1 \sqrt{1-x^2}\,dx=\frac{\pi}{4},\ n=4$

   Approximate the definite integral on the left by Simpson's rule that equals the right side when $\pi=3.1$.

5. Use MAPLE commands to find the approximate area in the interval [0, 2] by

   a. Trapezoidal rule
   b. Simpson's rule

History

Thomas Simpson was a British mathematician and creator of Simpson's rule for approximate definite integrals. This rule was known and used earlier by Bonoventura Cavalieri in 1639 and later by James Gregory. The long popularity of Simpson's textbook invites this association with his name therefore many readers would have learnt it from them. A challenge proposed by 'Pierre de Fermat' to find a point 'D' such that the sum of the distances to three given points A, B and C is least. This challenge was popularized in Italy Sampson treats this challenge in this first part of Doctrine and application of Fluxions (1750) by the description of circular arc at which the edges of the triangle ABC subtend and angle of $\frac{\pi}{3}$. In second part of the book he extends this geometrical method. Sampson's several textbooks contains most of the optimization problems treated by simple techniques.

Thomas Simpson (1710-1761)

Review Exercise 12

Choose the correct option.

1.

i. When $f(x)=0$, the roots of the equation $f(x)=6x^2+5x-6$ are:
(a). $\left\{\frac{2}{3},-\frac{2}{3}\right\}$ (b). $\left\{\frac{2}{3},-\frac{3}{2}\right\}$ (c). $\left\{\frac{3}{2},-\frac{2}{3}\right\}$ (d). $\left\{-\frac{3}{2},-\frac{2}{3}\right\}$

ii. When $f(x)=0$, then the value of x in the equation $f(x)=x^2+6x-5$ is:
(a). $x\approx-3.7187...$ (b). $x\approx0.37187...$ (c). $x\approx-0.37187...$ (d). $x\approx4.7847...$

iii. The Newton Raphson method is also known as:
(a). tangent method (b). second method
(c). diameter method (d). chord method

iv. The iterative formula for newton Raphson method $x_3=$
(a). $x_1-\dfrac{f(x_2)}{f'(x_2)}$ (b). $x_2-\dfrac{f(x_3)}{f'(x_3)}$ (c). $x_2-\dfrac{f(x_2)}{f'(x_2)}$ (d). $x_0+\dfrac{f(x_2)}{f'(x_2)}$

v. If the function values $f(a)$ and $f(c)$ at $x=a$ and $x=c$ have the opposite signs, then the roots lies in the interval:
(a). $[a, b]$ (b). $[a, c]$ (c). (a, b) (d). (a, c)

vi. The Newton Raphson method fails at:
(a). stationary point (b). floating point
(c). continuous point (d). non-of these

vii. The positive roots for $3t-\cos t-1$ by using the Regula Falsi method and correct up to 3 decimal places are:
(a). 0.507 (b). 0.670 (c). 0.570 (d). 0.607

viii. The MAPLE command to solve equation $x^2-5x+3=0$ is:
(a). solve $(x^2-5x+3)=0$ (b). *f*solve $(x^2-5x+3=0)$
(c). Simplon $(x^2-5x+3=0)$ (d). Psolve $(x^2-5x+3=0)$

ix. The roots of the equation $x-\sin x-\frac{1}{2}=0$ by using the bisection method between $1<x<2$:
(a). 1.88 (b). 1.48 (c). 1.49 (d). 1.897

x. The bisection method is also known as:
(a). Newton method (b). Binary Chopping method
(c). Quaternary method (d). none of these

Project

i. Create a quadratic function and find its roots by using Newton-Raphson method. Also plot its graph by using maple commands.
ii. Write a function and find its integral limit −1 to 2. Use trapezoidal rule for approximate value.

Summary

- If $f(x)$ is continuous and $f(a)$ and $f(b)$ have opposite in signs, then the bisection method will be used to approximate the actual root r of the non-linear equation $f(x)=0$ in the interval $[a, b]$. In bisection method, the approximate root of r will always be the midpoint of the interval $[a, b]$.
- If $f(x)$ is continuous and $f(a)$ and $f(b)$ have opposite in signs, then the regula-falsi method will be used to approximate the actual root **r** of the non-linear equation $f(x)=0$ in the interval $[a, b]$. In Regula-Falsi method, the approximate root of r will not be the midpoint of the interval $[a, b]$ that should be anywhere in the interval $[a, b]$.
- If $f(x)$ and its derivatives are continuous functions and x_0 is the initial iterate, then the Newton-Raphson method $x_{i+1}=x_i-\dfrac{f(x_i)}{f'(x_i)}$, $i=0,1,2,3,\ldots\ldots$, i = 1, 2, 3,.......

  produces a sequence of successive iterates $\{x_n\}$ that will converge faster to r than either the bisection and regula-falsi methods.
- If $f(x)$ is continuous on $[a, b]$, then the triangle rule

$$R_n=f(x_1)\Delta x+f(x_2)\Delta x+\ldots.+f(x_n)\Delta x$$

  is used to approximate the definite integral $\int_a^b f(x)dx$.
- If $f(x)$ continuous on $[a, b]$, then the trapezoidal rule is

$$T_n=\frac{\Delta x}{2}[f_0+2f_1+2f_2+\ldots\ldots+2f_{n-1}+f_n],\ \Delta x=\frac{(b-a)}{n}$$

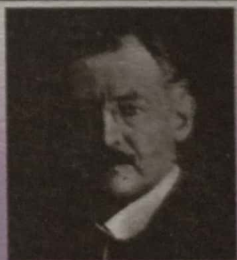
  is used to approximate the definite integral $\int_a^b f(x)dx$.
- If $f(x)$ is continuous on $[a, b]$, then the Simpson's rule

$$S_{2n}=\frac{\Delta x}{3}\left[f_0+4(f_1+f_3+\ldots.+f_{2n-1})+2(f_2+f_4+\ldots.+f_{2n-2})+f_{2n}\right]$$

  with $\Delta x=\dfrac{(b-a)}{2n}$ is used to approximate the definite integral $\int_a^b f(x)dx$.

History

Carl Runge was a German mathematician and physicist. He was co-developer of the runge – kutta method in the field of numerical analysis. He received his Ph.D degree in mathematics at Berlin in 1880. In 1886 he became the professor in Hanover, Germany. He also studied spectral lines of various elements and was very interested in the application of this work to astronomical spectrosphy. Runge reached to his retirement in 1923 but he continued to run his institute until his successor Gustave Herglotz, arrived in 1925.



Carl David Telme Runge
(1710-1761)

Answers

Unit-2

Exercise 2.1

1. a. Domain: $(-\infty,\infty)$ or $-\infty < x < \infty$
   Range: $(-\infty,5)$

   b. Domain: $(-\infty,\infty)$ or $-\infty < x < \infty$
   Range: $(-\infty,\infty)$ or $-\infty < y < \infty$

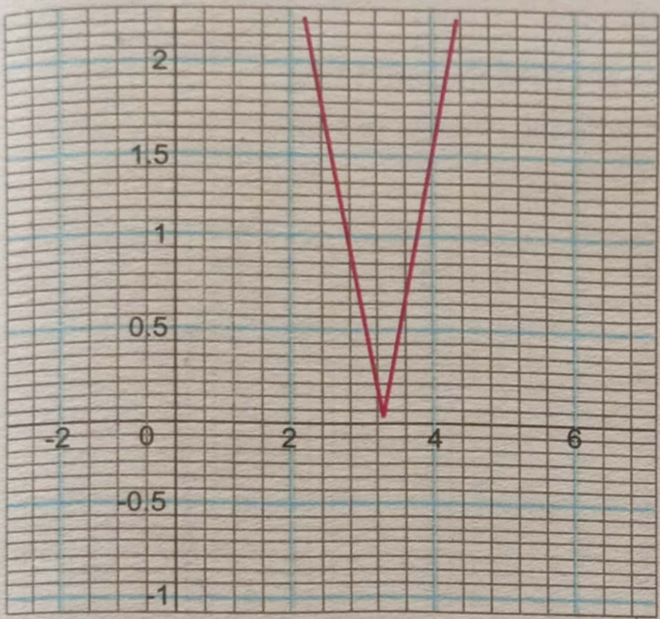
   c. Domain: $(-\infty,-2) \cup (-2, \infty)$
   Range: $(-\infty,0) \cup (0, \infty)$

   d. Domain: $(-\infty, \infty)$
   Range: $(-\infty,\infty)$

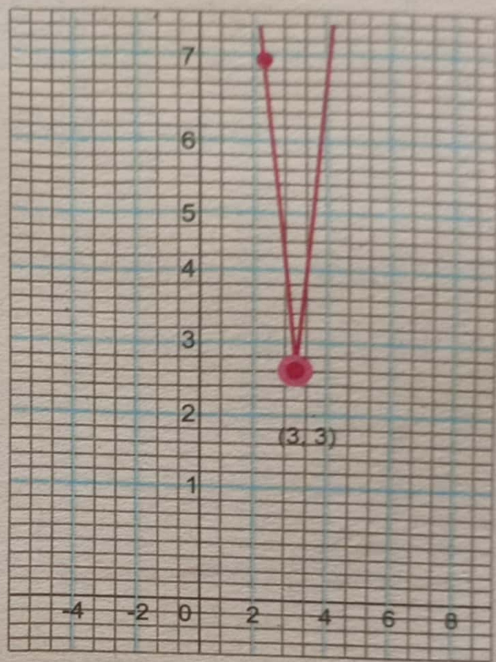
   e. Domain: $(-\infty, \infty)$
   Range: $[0, \infty)$

   f. Domain: $(-\infty, \infty)$
   Range: $[0,\infty)$

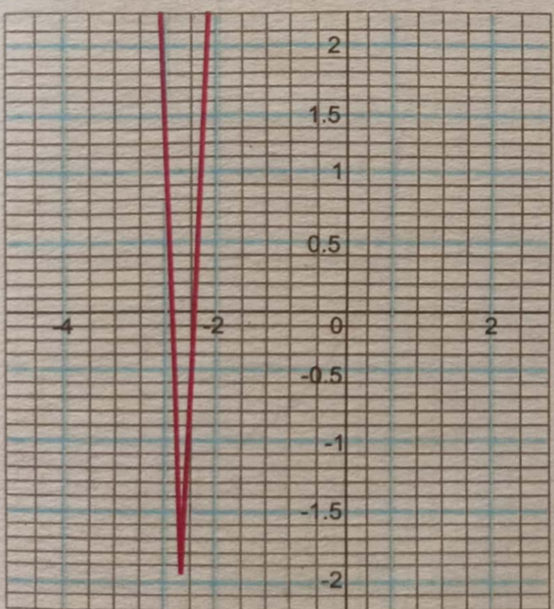
2. a.



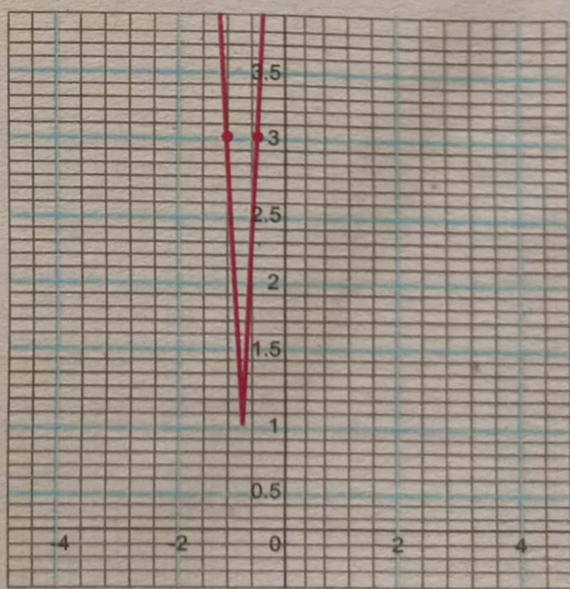
b.



c.



d.



3. a. $f[g(x)] = 4x^2 + 1, g[f(x)] = 2x^2 + 2$ b. $f[g(x)] = \sin(1-x^2), g[f(x)] = 1 - \sin^2 x = \cos^2 x$
c. $f[g(x)] = x, g[f(x)] = x$ d. $f[g(x)] = Sin(2x+3), g[f(x)] = 2Sinx + 3$

4. a. $h^{-1}(x) = f^{-1}(g(x)) = x - 1$ b. $h^{-1}(x) = f^{-1}(g(x) = \frac{x-7}{4}$

$k^{-1}(x) = g^{-1}f(x) = x - 1$ $k^{-1}(x) = g^{-1}(f(x)) = \frac{x-14}{4}$

c. $h^{-1}(x) = f^{-1}[g(x)] = x + 3$ d. $h^{-1}(x) = f^{-1}[g(x)] = x$ self-inverse

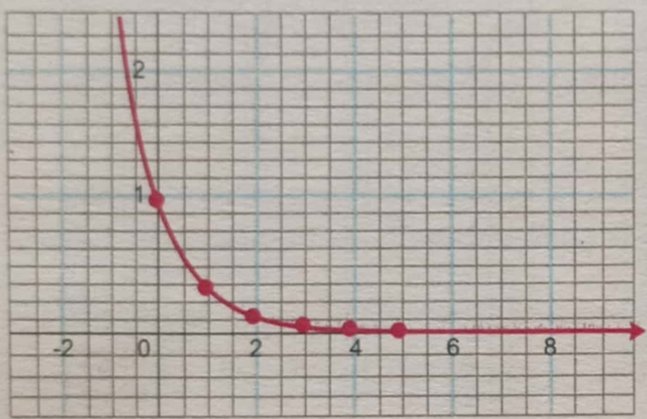
$k^{-1}(x) = g^{-1}[f(x)] = \frac{2x-3}{2}$ $k^{-1}(x) = g^{-1}[f(x)] = x$ self-inverse

Exercise 2.2

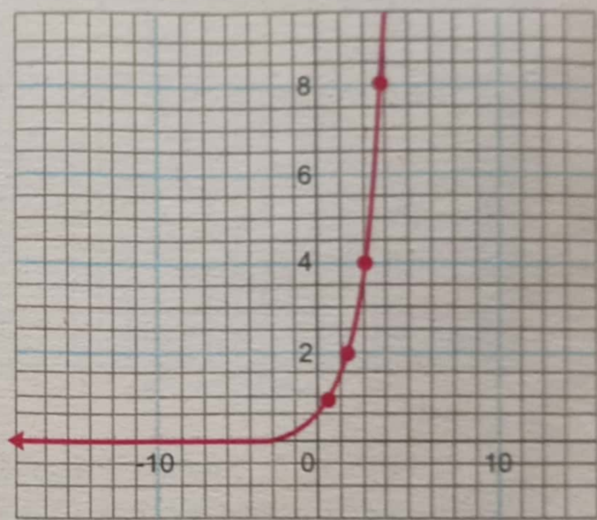
1. a. Trigonometric b. Algebraic c. Inverse trigonometric
d. Logarithmic e. Implicit f. Hyperbolic g. Explicit

2. a. 1 in gram 100 b. 0.7089 in gram 70.9g
c. 0.5025 in gram 50.3g d. 0.11648 in gram 11.6g
e. The amount of radium in the box decreases with the passage of time.

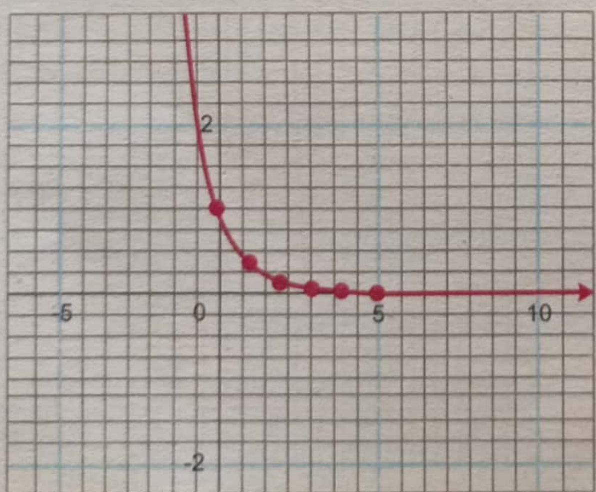
3. a.



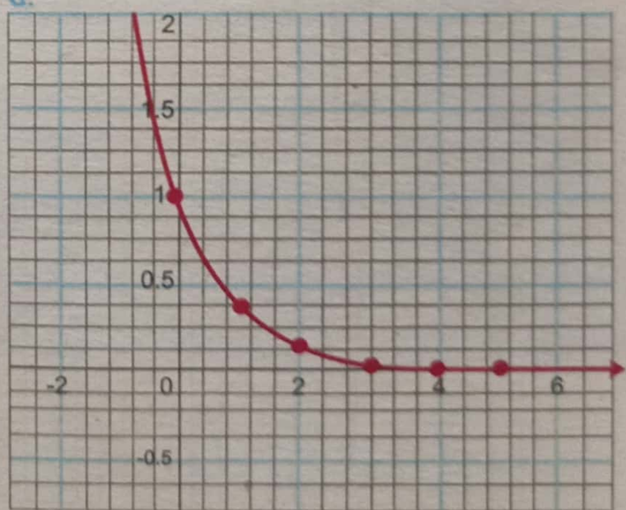
b.



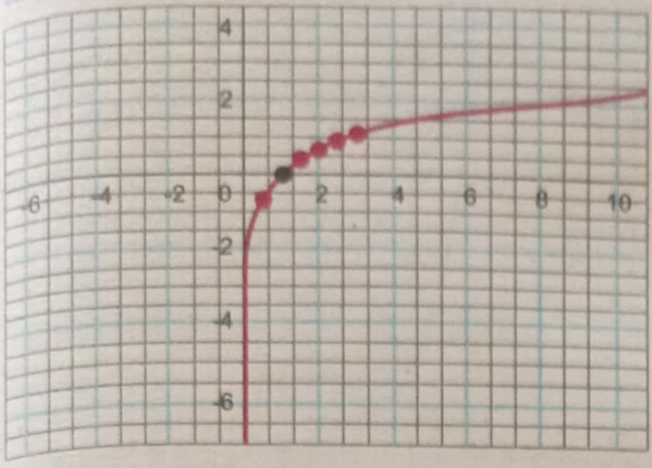
c.



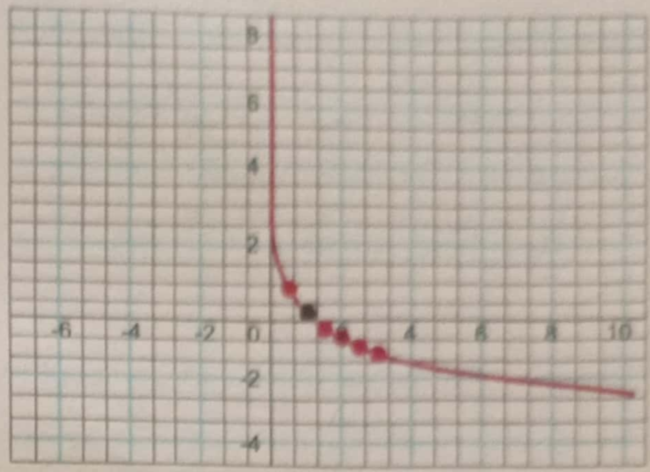
d.



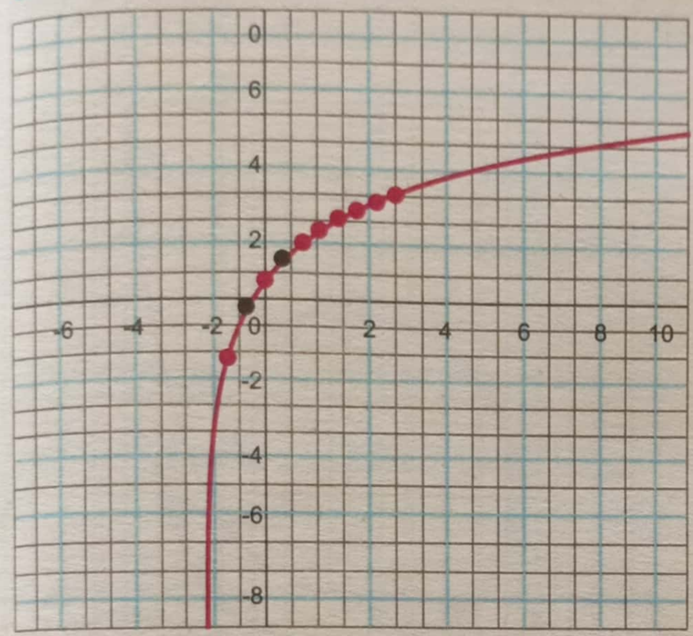
4. a.


4
2
-6
-4
-2
0
2
4
6
8
10
-2
-4
-6

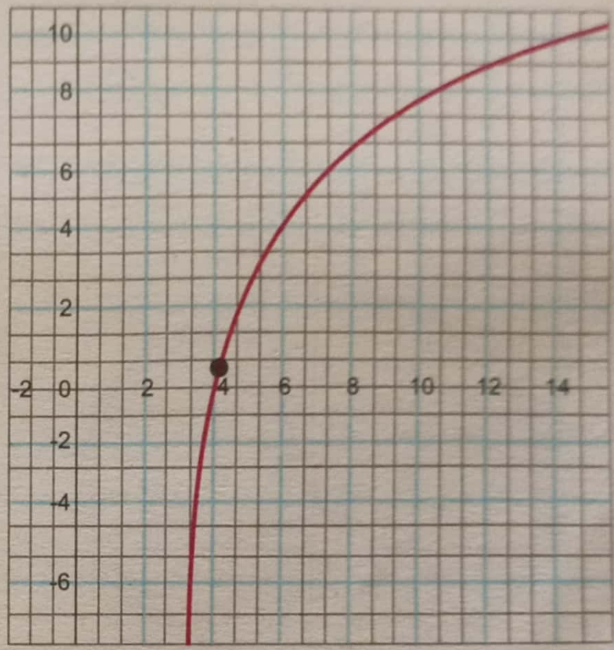
b.


8
6
4
2
-6
-4
-2
0
4
6
8
10
-2
-4

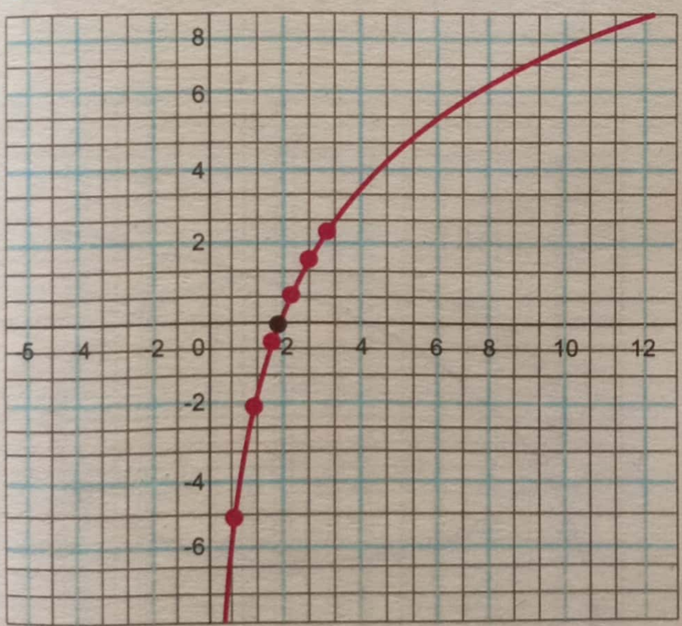
c.


0
6
4
2
-6
-4
-2
0
2
4
6
8
10
-2
-4
-6
-8

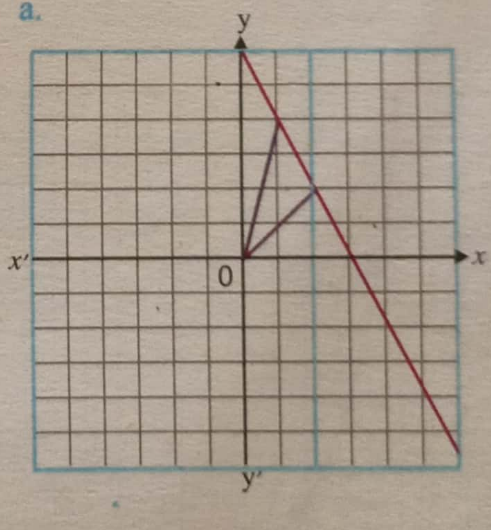
d.


10
8
6
4
2
-2
0
2
4
6
8
10
12
14
-2
-4
-6

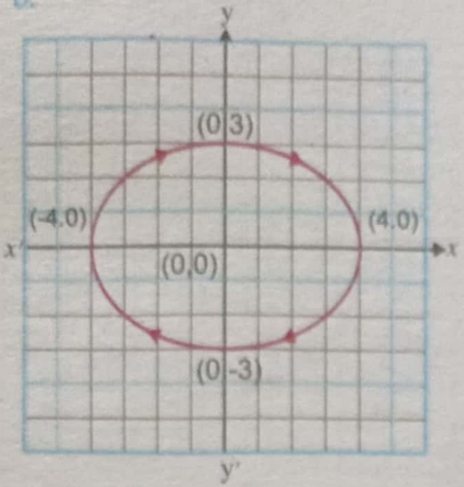
e.


8
6
4
2
-5
-4
-2
0
2
4
6
8
10
12
-2
-4
-6

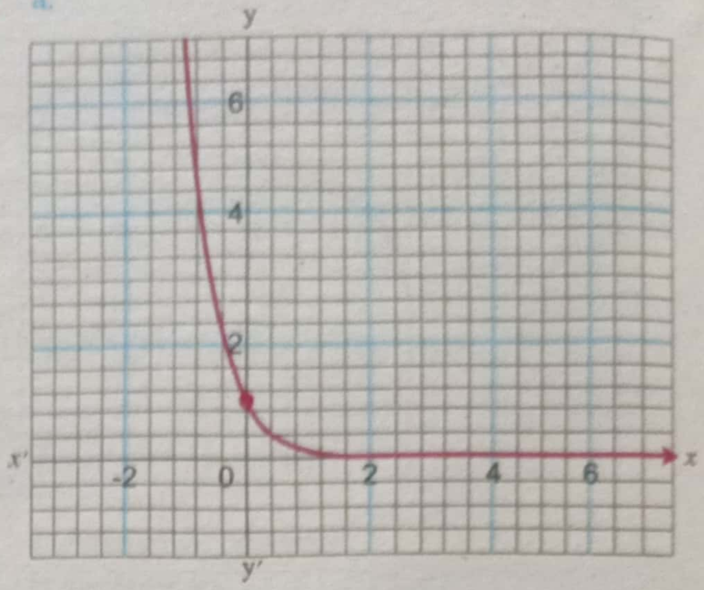
5. a.


y
x'
0
x
y'

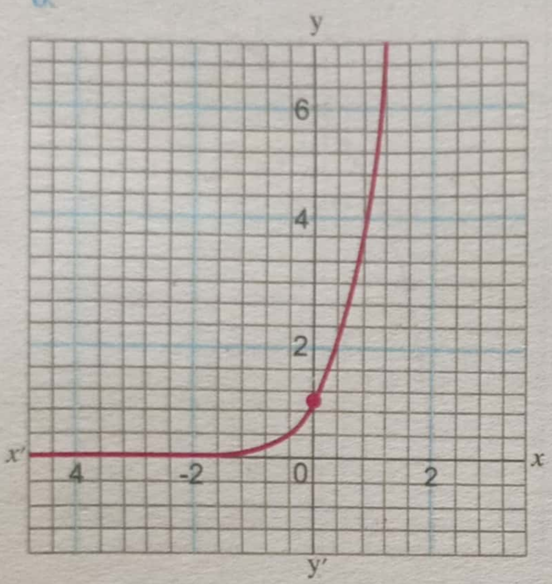
b.


y
(0,3)
(-4,0)
(4,0)
x'
x
(0,0)
(0,-3)
y'

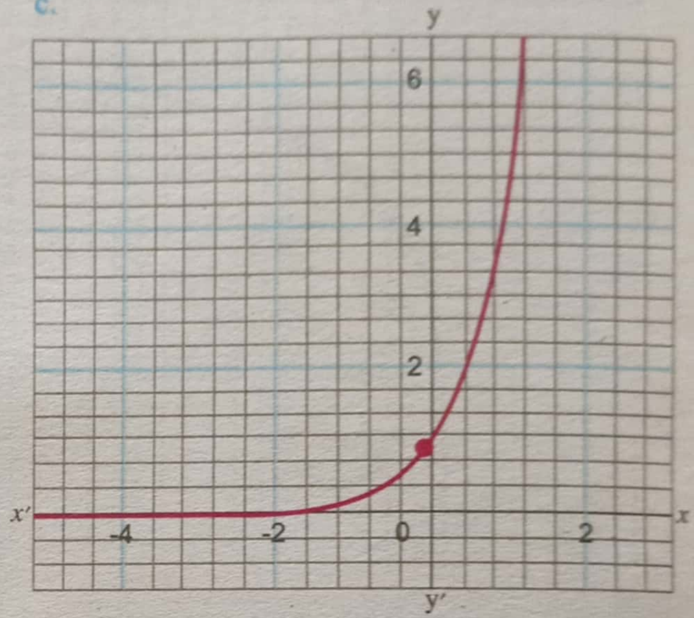
6. a.


y
6
4
2
x'
-2
0
2
4
6
x
y'

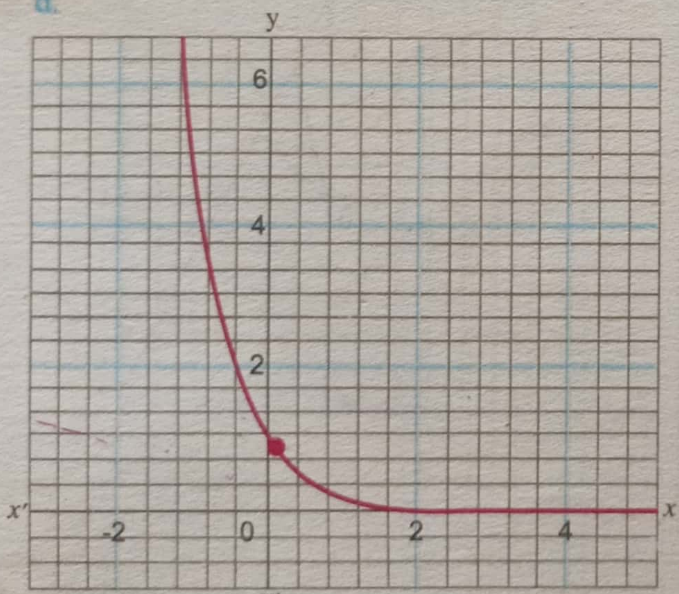
b.


y
6
4
2
x'
4
-2
0
2
x
y'

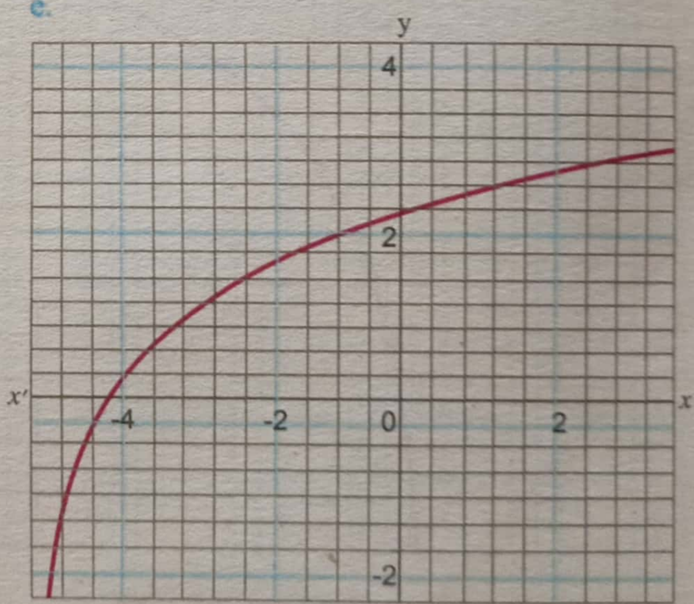
c.


y
6
4
2
x'
-4
-2
0
2
x
y'

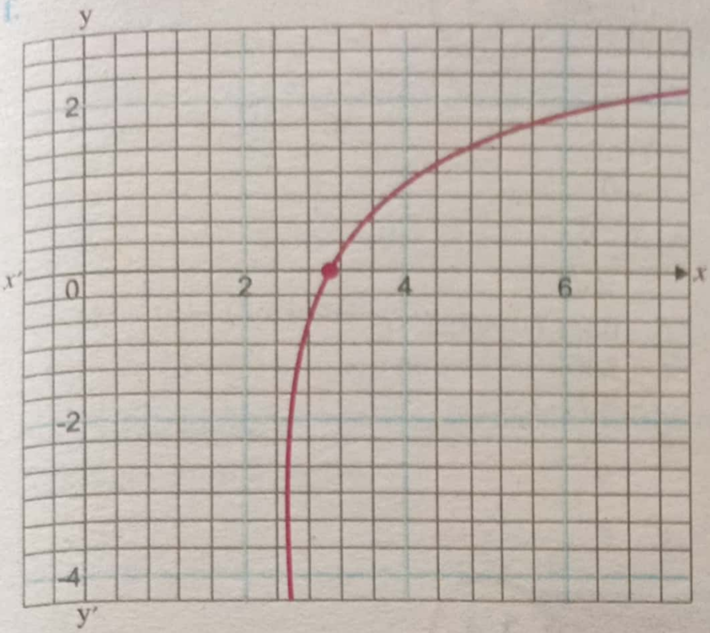
d.


y
6
4
2
x'
-2
0
2
4
x
y'

e.

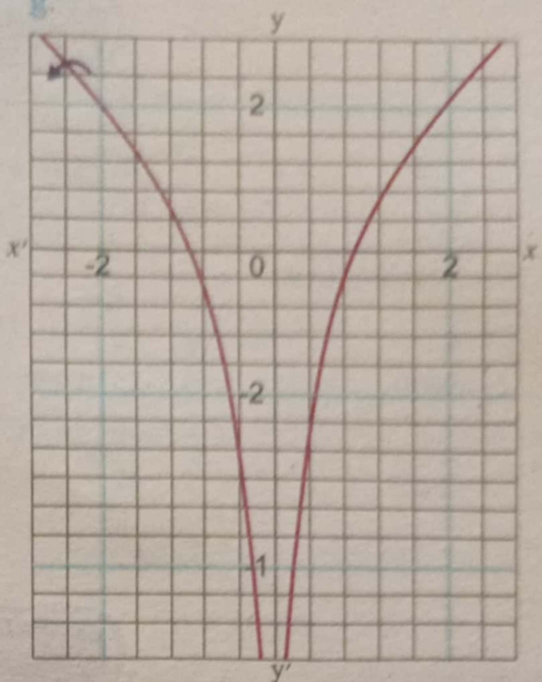

y
4
2
x'
-4
-2
0
2
x
-2
y'

f.



y
2
x'
0
2
4
6
x
-2
-4
y'

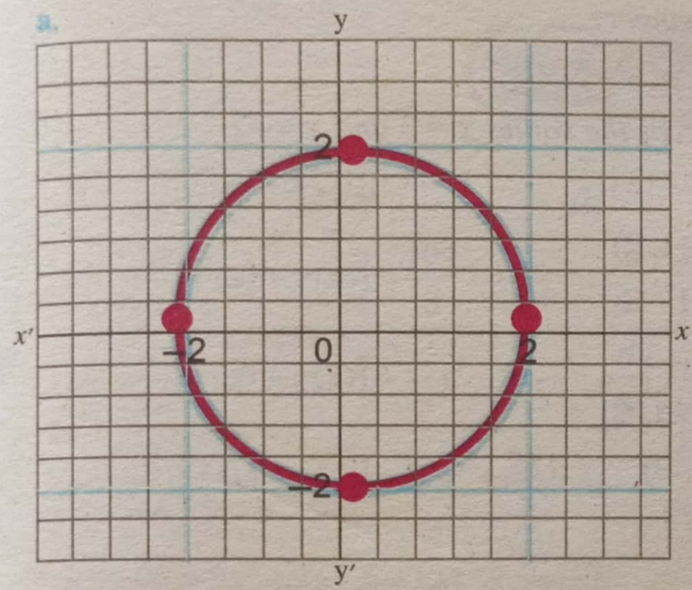
g.



y
2
x'
-2
0
2
x
-2
-4
y'

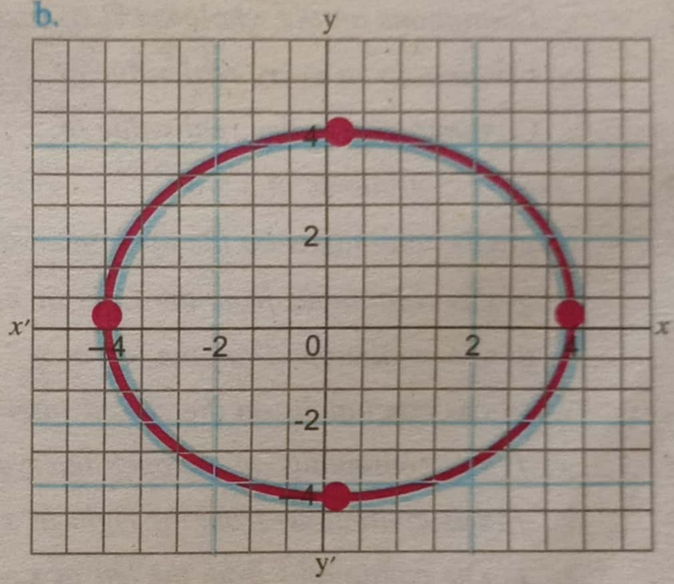
7\.

a.



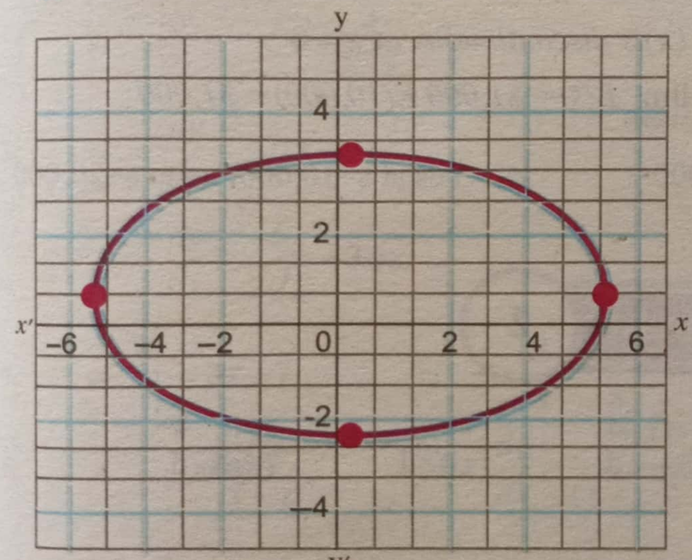
y
2
x'
-2
0
2
x
-2
y'

b.



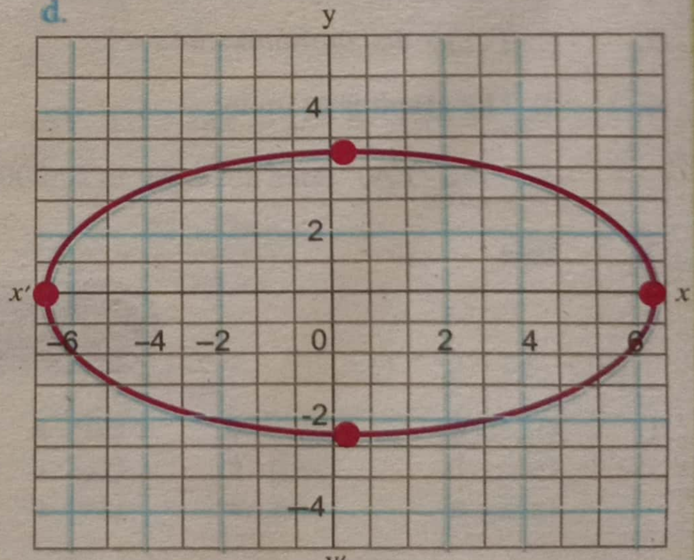
y
4
2
x'
-4
-2
0
2
4
x
-2
-4
y'

c.



y
4
2
x'
-6
-4
-2
0
2
4
6
x
-2
-4
y'

d.



y
4
2
x'
-6
-4
-2
0
2
4
6
x
-2
-4
y'

Exercise 2.3

1. a. $-\frac{11}{4}$ b. $\frac{9}{4}$ c. $\frac{1}{2}$ d. -1 e. 1
   f. 1 g. 0 h. 0
2. a. does not exist b. $-\frac{1}{9}$ c. $\frac{1}{2\sqrt{5}}$
3. a. 5 b. -7 c. does not exist
4. a. 5900 b. 5514.29 c. 5200
6. a. $\frac{1}{2\sqrt{x}}$ b. $\frac{n}{m}a^{n-m}$ c. $\frac{p^2}{q^2}$ d. does not exist
   e. e^{-1} *or* $\frac{1}{e}$ f. $e^{1/2}$ g. e^{-1}

Exercise 2.4

1. a. *Continuous on the whole set* R *of real numbers*
   b. *Continuous at every real value of* x, *except* $x=5$
   c. *Continuous at every real value of* x , *except the values* $x=3$ *and* $x=-2$
2. $f(x)$ *is continuous at* $x=0$
3. a. $g(x)$*is continuous on the open interval* $(-1, 2)$
   b. $g(x)$*is continuous from the right at* $x=-1$
   c. $g(x)$*is continuous from the left at* $x=2$
   d. $g(x)$*is continuous on the closed interval*$[-1, 2]$
4. a. $f(x)$*is continuous on the open interval*$(0, 3)$
   b. $f(x)$*is continuous from the right at* $x=0$
   c. $f(x)$*is continuous from the left at* $x=3$
   d. $f(x)$*is continuous on the closed interval*$[0, 3]$
5. a. $f(x)$*is discontinuous at* $x=1$ b. $f(x)$*is discontinuous at* $x=0$
6. a. The graph is as under: b. $\lim_{s\to 10{,}000} E(s)=\$1{,}000$ $E(10{,}000)=\$1{,}000$
   c. $\lim_{s\to 20{,}000} E(s)$ does not exist; E(20,000)=\$2,000. d. Yes at s=10,000, no at s=20,000
7. true

Review Exercise 2

1. (i). a (ii). d (iii). a (iv). d (v). c (vi). a
   (vii). b (viii). c (ix). a (x). d (xi). d (xii). d
   (xiii). a (xiv). a (xv). c

Unit-3

Exercise 3.1

1. a. 5 b. $\frac{1}{3}$ c. 21 d. 0.25
2. a. 2 b. $\Delta x = 0.1$ c. 4.1π d. $\frac{1}{7}$
3. a. $\Delta t = 0.1$; 14.4 units/second b. $\Delta t = 0.01$; 15.84 units/second
4. $\Delta t = 2$; 25; the average rate of inflation is Rs.25 per year.
5. $\Delta t = 30$; 1170; the profit is increasing at the rate of Rs.1170 per acre, when the number of acres changes from 20 to 50.
6. a. 3 b. $\frac{5}{2\sqrt{5x+6}}$ c. $2x$ d. $-2x$ e. $32x-7$ f. $\frac{-7}{x^2}$
7. a. 3 b. $y=3x-19$ c. $f'(1)=-5, f'(5)=3$, yes
8. a. 1, $y=x+9$ b. 1, $y=x-16$ c. -6, $y=-6x-10$ d. 7, $y=7x-6$

Exercise 3.2

1. a. $\frac{d}{dx}[f(x)+g(x)]=12x+5$, $\frac{d}{dx}[f(x)-g(x)]=-12x+1$

   b. $\frac{d}{dx}[f(x)+g(x)]=\frac{104}{3}x+\frac{1}{2}$, $\frac{d}{dx}[f(x)-g(x)]=\frac{100}{3}x-\frac{1}{2}$

   c. $\frac{d}{dx}[f(x)+g(x)]=12x^2-8x$, $\frac{d}{dx}[f(x)-g(x)]=-6x^2+8x$

   d. $\frac{d}{dx}[f(x)+g(x)]=12x^2+\frac{6x}{5}-7$, $\frac{d}{dx}[f(x)-g(x)]=12x^2-\frac{6}{5}x-3$

2. a. $\frac{dy}{dx}=9x^2+2x-6=12x^2+\frac{6x}{5}-7$ b. $\frac{dy}{dx}=18x^5-16x^3+6x^2-8$

   c. $\frac{dy}{dx}=-\sqrt{x}-\frac{3}{2\sqrt{x}}-2$ d. $\frac{dy}{dx}=\frac{15}{\sqrt{x}}-12$

3. a. $\frac{dy}{dx}=-7(x-4)^{-2}$ b. $\frac{dy}{dx}=\frac{4x^4-48x^3-2x+6}{(4x^3+1)^2}$

   c. $\frac{dy}{dx}=\frac{5}{2\sqrt{x}}+3\sqrt{x}$ d. $f'(p)=\frac{24^2-28p+9}{(3p+2)^2}$

4. a. $y=3x-7$ b. $y=\frac{3}{4}x-\frac{1}{8}$ c. $y=-\frac{1}{16}x+\frac{13}{40}$ d. $y=-2x+9$

5. a. $y=\frac{Px}{x-p}$ b. $\frac{dy}{dx}=\frac{-P^2}{(x-p)^2}$

Exercise 3.3

1. a. $20(x^3-4x+2)^4(3x^2-4)$ b. $\frac{11}{5}(4x-x^3)^{10}(4-3x^2)$

c. $-2t(1-3t^2)^{-\frac{2}{3}}$ d. $\frac{-21}{(3t+1)^8}$

2. a. $f'(x)=(2x-5)^2(40x-67)$ b. $f'(x)=\frac{(x^2-2x-8)}{(x-1)^2}$

c. $f'(x)=\frac{-12(2x-5)^3}{(x-4)^5}$ d. $f'(x)=\frac{4x^2+11}{(2x^2+11)^{\frac{1}{2}}}$

3. a. $y''=\frac{3}{2}t+2$ b. $y''=\frac{12}{a}t^2$ c. $y''=\frac{b(t^2-1)}{2at}$ d. $y''=\frac{[2t-t^4]}{[1-2t^3]}$

4. 4222.8 dollars/hour

5. a. $-\frac{x}{y}$ b. $-\frac{y^2+4xy}{2x^2+6xy}$ c. $-\frac{(x+y)^2}{(x+y)^2+1}$ d. $-\frac{y^2}{x^2}$

6. a. $-\frac{2}{3}x(x^2+1)^{-\frac{4}{3}}$ b. $\frac{x^2+4x}{(x+2)^2}$ c. $\frac{1}{(5-x)^2}$ d. $\frac{-3}{(x-1)^2}$

7. a. $\frac{du}{dv}=-a^2v/b^2u$ b. $\frac{dv}{du}=-b^2u/a^2v$

8. $y'=9/7$

9. 25 miles per hour.

Exercise 3.4

1. a. $2Cosx(2x)$ b. $-3\cos ec^2(3x)$ c. $3\left[\sec^2(3x)-\sin(3x)\right]$

d. $-2\cot(x).\cos ec^2(x)$ e. $\frac{1}{2\sqrt{x}}\sec^2\sqrt{x}$ f. $3\sin^2(x).\cos(x)$

2. a. $-3x^2\cos ec^2(3x)+2x\cot(3x)$ b. $2(\sin 2x+\cot 3x)(2\cos 2x-3\cos ec^2 3x)$

c. $-2\cos ec2x.\cot 2x$ d. $4(x+3)\sec^2(x+3)^2$

e. $\frac{\sqrt{x}.\cos\sqrt{x}.\sec^2 x+\tan x.\sin\sqrt{x}}{2\sqrt{\tan x}.\cos^2\sqrt{x}}$ f. $\frac{2\sec^2 2x+3\cot 3x.(1+\tan 2x)}{\cos ec 3x}$

3. a. $\frac{dy}{dx}=\frac{-1}{\sqrt{a^2-x^2}}$ b. $\frac{dy}{dx}=\frac{p}{p^2+x^2}$ c. $\frac{dy}{dx}=\frac{a}{x^2+a^2}$

d. $\frac{dy}{dx}=\frac{-1}{1+x^2}$ e. $\frac{dy}{dt}=\frac{-1}{(t+3)\sqrt{t^2+6t+8}}$ f. $\frac{dy}{dx}=\frac{-1}{x(x^2+1)}+\frac{1}{x^2}\tan^{-1}\left(\frac{x+1}{x-1}\right)$

4. a. $P'(t)=\frac{\pi}{26}\sin\left(\frac{\pi}{26}\right)$ b. $P'(8)=9.94$ dollars/ week, $P'(26)=0\ dollars/week$

$P'(50)=-2.89\ dollars/week$

5. a. $V'(t) = \frac{7\pi}{40}\sin\left(\frac{\pi t}{2}\right)$

b. $V'(3) = -0.5498 \approx -0.55$ cubic units/ sec,

$V'(3) = 0$ *cubic units / sec* $\quad V'(3) = 0.5498 \approx 0.55$ *cubic units / sec*

Exercise 3.5

1. a. $2e^{2x}$ b. e^{3x} c. $\frac{5}{4}x.e^{x^2+1}$ d. $2^x.\ln(2)$ e. $256.4^x.\ln 4$

f. $\frac{1}{(x+1)}$ g. $\frac{2}{x\ln a}$ h. $\frac{3}{x+2}$

2. a. $-20x^4 11^{-4x^5+3}.\ln(11)$ b. $\frac{1}{2\sqrt{x}}.e^{\sqrt{x}-5}$ c. $(5x^4 - x^3)e^{\frac{1}{x}}$

d. $\frac{2(e^{2x}+2)}{(e^{-2x}+1)^2}$ e. $\frac{m.e^{mx}+me^{-mx}}{e^{mx}-e^{-mx}}$ f. $\frac{a.e^{ax}+a.e^{-ax}}{e^{ax}+e^{-ax}} - \frac{(e^{ax}-e^{-ax})(a.e^{ax}-a.e^{-ax})}{(e^{ax}+e^{-ax})^2}$

3. a. $\frac{x^2}{2}(1+3\ln x)$ b. $\frac{x(1+4\ln x)}{2\sqrt{\ln x}}$ c. $\frac{2x}{1-x^4}$ d. $\frac{-1}{\sqrt{x^2+1}}$

e. $-2e^{-2x}[\sin(2x)+\cos(2x)]$ f. $x(x.\cos x+2)e^{\sin x}$

4. a. $2.\cosh(2x)$ b. $3.\sinh(3x)$ c. $\frac{-\sin x}{\sqrt{1+\cos^2 x}}$

d. $\frac{2}{4-x^2}$ e. $-2\,\mathrm{cosec}\,(2x)$ f. $\cosh^{-1}(x)$

5. 2.27 mm of mercury/yr; 0.81 mm of mercury/yr; 0.41 mm of mercury/yr.

6. $A'(t) \approx 10{,}000(\ln 2)2^{2t}$; $A'(1) = 27{,}726$ bacteria/hr

$A'(5) = 7{,}097{,}827$ bacteria/hr

Review Exercise 3

1. (i) b (ii) d (iii) b (iv) c (v) c

(vi) d (vii) d (viii) c (ix) c (x) d

Unit-4

Exercise 4.1

1. a. $f''(x) = 18x$ b. $f'''(x) = -\frac{6}{x^4}$ c. $s''(t) = \frac{-25}{4}(5t+7)^{\frac{-3}{2}}$ d. $\frac{4}{(x-1)^3}$

2. a. $y'' = -\frac{b^4}{a^2y^3}$ b. $y'' = -\frac{r^2}{y^3}$ c. $y'' = 0$ d. $\frac{(e^y-e^x)(e^{x+y}-1)}{(e^y+1)^3}$

3. a. $y'' = \frac{9}{32t}$ b. $y'' = \frac{4}{3a^2}$ c. $y'' = \frac{-b}{a^2\sin^3 2t}$

d. $\frac{d^2y}{dx^2} = \frac{(1+t^3)[6a(3a-6at^3)(1-7t^3+t^6)+54a^2t^3(2-t^3)^2}{(3a-6at^3)^3}$

4. a. $24(3x^2+4)^4+864(x^3+4x-5)(3x^2+4)^2.x+1296(x^3+4x-5)^2.x^2+288(x^3+4x-5)^3.(3x^2+4)$

b. $16(1+\tan^2 x)^2.\tan x+24\tan^3 x(1+\tan^2 x)$ c. $\frac{729}{2}e^{6x}.\ln|e|^3$ d. $24x.\ln(x^3)+52x$

5. a. $\sqrt{\sec(2x)}\tan(2x)^3+7\sqrt{\sec(2x)}\tan(2x)(2+2\tan(2x))^2$

b. $-\cos(\sin(x))\cos(x)^3+3\sin(\sin(x))\cos(x)\sin(x)-\cos(\sin(x))\cos(x)$

6. a. $1663200x3^7(3x+4)^4$ b. $-2^{11}(181440)(2x-4)^{-10}$

c. $(3)^6 4\cdot\cos(3x+8+3\pi)$ d. $(5)^{12}\cdot 7\,e^{5x+4}$

Exercise 4.2

1. a. $y=\frac{(x+5)}{4}$ b. $y=-2x$ c. $y=\frac{x}{e}$ d. $y=\frac{1}{2}$

2. a. $y=\frac{-1}{2e}x+\left(\frac{1+2e^2}{2e}\right)$ b. $b=\frac{-1}{12}x+1$ c. $-x+\frac{\pi}{2}$ d. $y=-x+1$

3. a. $y=\frac{2}{3}x+\frac{13}{13}$ b. $y=(1+\pi)x+\pi$ c. $x=1$ d. $x=1$

5. a. $1-x+x^2-x^3+......$ b. $x^2-\frac{1}{3}x^4+\frac{2}{45}x^6-......$

c. $1+\frac{x^2}{2!}+\frac{x^4}{4!}+\frac{x^6}{6!}+......$ d. $-4x-8x^2-\frac{64}{3}x^3-......$

6. b. 2.7183 7. a. $\frac{\pi}{4}$ b. $\tan^{-1}\sqrt[3]{16}$ 8. $x=\frac{4+\sqrt{91}}{15}$ *and* $\frac{4-\sqrt{91}}{15}$

Exercise 4.3

1. a. i. critical point $(0,-2)$ ii. Increasing on $(-\infty,-2)\cup(0,+\infty)$; decreasing on $(-2,0)$

iii. $(0, 1)$ is relative minimum; $(-2, 5)$ is relative maximum

b. i. critical point $\left(\frac{5}{3},-25\right)$ ii. increasing on $(-\infty,-25)\cup\left(\frac{5}{3},\infty\right)$ decreasing on $\left(-25,\frac{5}{3}\right)$

iii. relative minimum $(-25,0)$ relative maximum $\left(\frac{5}{3},-9481\right)$

2. a. $x=-3, 4$ b. $x=-3, 5$ c. $x=0, 1$ d. $x=0, 1$

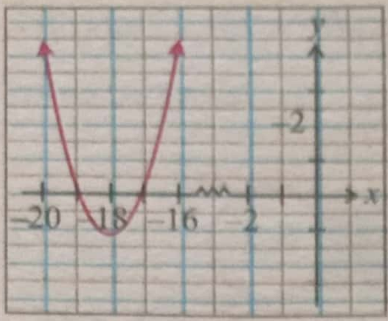
3. a. Relative minimum at $x=1$ b. relative minimum at $x=1$

c. neither maximum nor minimum d. Relative minimum at $x=4$

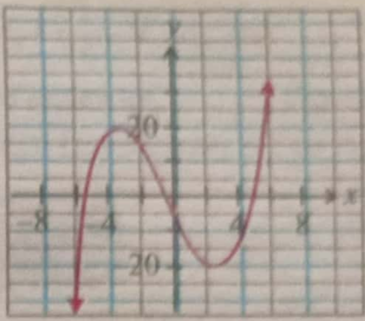
4. a. Critical point: $(-18,-1)$ is relative minimum; increasing on $(-18,+\infty)$; decreasing on $(-\infty,-18)$; concave up on $(-\infty,+\infty)$.

b. Critical points: $(-3, 20)$ is relative maximum; $(3,-16)$ is relative minimum; increasing on $(-\infty,-3)U(3,+\infty)$; decreasing on $(-3,3)$; inflection point: $(0, 2)$; concave up on $(0,+\infty)$; concave down on $(-\infty,0)$.

(a)



(b)



a. Relative maximum of 1 at $x=0$, relative minimum of -3 at $x=2$
b. Relative maximum of 2 at $x=-3$, relative minimum of -2 at $x=-1$
a. Relative maximum at $x=2$, relative minimum at $x=4$, neither at $x=1,-5$
b. Relative maximum at $x=-3$; relative minimum at $x=\frac{1}{2}$; neither at $x=1$
a. The expenditure on advertising that leads to maximum profit is $x=6$.
b. The maximum profit at $x=6$ is 512 hundred dollars.
The number of hundred thousands of tires is $x=1{,}000{,}000$ tires; the number of hundred thousands of tires x that developed the maximum profit is P(10)=\$700=\$700,000.
a. The drug concentration is increasing in the interval (0, 3) and is decreasing in the interval $(3,\infty)$.
b. The maximum drug concentration time is $x=3$.
c. The maximum drug concentration at a time $x=3$ is K (3) = 0.22% = 0.0022.
d. The concentration is maximum at $x=3$ as $k(x)$ changes from increasing function to decreasing function. Maximum concentration $k(x)=0.22\%$

Hint: $C(x)=[G(x)][32]2.25]=\frac{1}{48}\left(\frac{300}{x}+2x\right)(32)(2.25)=1.5\left(\frac{300}{x}+2x\right)$

$C'(x)=1.5\left(\frac{-300}{x^2}+2\right)=0 \Rightarrow x=\pm\sqrt{150}$

Minimum cost at $x=\sqrt{150}=12.2$ is C (12. 2) = \$7.50.

Review Exercise 4

(i) b	(ii) c	(iii) d	(iv) a	(v) a
(vi) b	(vii) c	(viii) d	(ix) c	(x) b

Unit-5

Exercise 5.1

a. $t\neq 0,\ t>0$

b. $t\neq 2,\ t\geq 0.$

c. $t\neq(2n-1)\frac{\pi}{2},\ n=0,\pm1,\pm2,\ldots$

d. $t\neq n\pi,\ n=0,\pm1,\pm2,\ldots$

a. $(7t-3)\hat{i}-10\hat{j}+\left(\frac{2t^2-3}{t}\right)\hat{k}$

b. $(2t+4)\hat{i}-15\hat{j}+(3t^2+\frac{1}{t})\hat{k}$

c. $(1-t)\sin t$

d. $-t^2e^t\hat{i}+t^2\sin t\hat{j}+\left(2te^t+5\sin t\right)\hat{k}$

3. a. $3\hat{i}+e^2\hat{j}$ b. $3\hat{i}-\frac{1}{3}\hat{j}+2\hat{k}$ c. $\hat{i}+e\hat{k}$ d. $\hat{i}+\ln(4)\hat{j}$

4. a. All values of t. b. All values of t except t = 0 c. All values of t except $t \neq 0, t \neq -1$
d. All values of $t \in$ positive real number R^+

Exercise 5.2

1. a. $F'(t)=\hat{i}+2t\hat{j}+(1+3t^2)\hat{k}$ b. $F'(s)=(1+4s)\hat{i}+(2s-1)\hat{j}+2s\hat{k}$
c. $-\sin\theta\hat{i}+\cos\theta\hat{j}-3\sin\theta\hat{k}$

2. a. $2\hat{i}+18t\hat{j}-16\hat{k}$ b. $2\hat{i}-2\hat{j}+36s^2\hat{k}$ c. $-\frac{1}{x^2}\hat{i}-2\hat{k}$
d. $2[\cos^2\theta-\sin^2\theta]\hat{i}+2[\cos^2\theta-\sin^2\theta]\,\hat{j}$

3. a. $-2x-9x^2$ b. $\frac{4x}{\sqrt{4x^2+1}}$

4. a. $v(t)=\hat{i}+2t\hat{j}+2\hat{k};\ v(1)=\hat{i}+2\hat{j}+2\hat{k}, |v(1)|=3$
in the direction of $\frac{1}{3}\hat{i}+\frac{2}{3}\hat{j}+\frac{2}{3}\hat{k},\ A(t)=2\hat{j};\ A(1)=2\hat{j}$,

b. $v(t)=-\sin t\hat{i}+\cos t\hat{j}+3\hat{k},\ v\left(\frac{\pi}{4}\right)=-\frac{\sqrt{2}}{2}\hat{i}+\frac{\sqrt{2}}{2}\hat{j}+3\hat{k}$
$\left|v\left(\frac{\pi}{4}\right)\right|=\sqrt{10}$ *in the direction of* $-\frac{1}{2\sqrt{5}}\hat{i}+\frac{1}{2\sqrt{5}}\hat{j}+\frac{3}{\sqrt{10}}\hat{k}$
$A(t)=-\cos t\hat{i}-\sin t\,\hat{j},\ A\left(\frac{\pi}{2}\right)=-\frac{\sqrt{2}}{2}\hat{i}-\frac{\sqrt{2}}{2}\hat{j}$

c. $v(t)=e^t\hat{i}-e^{-t}\hat{j}+2e^{2t}\hat{k},\ v(\ln 2)=2\hat{i}-\frac{1}{2}\hat{j}+8\hat{k}$
$|v(\ln 2)|=\frac{\sqrt{273}}{2}$ *in the direction of* $\frac{4}{\sqrt{273}}\hat{i}-\frac{1}{\sqrt{273}}\hat{j}+\frac{16}{\sqrt{273}}\hat{k}$
$A(t)=e^t\hat{i}+e^{-t}\hat{j}+4e^{2t}\hat{k};\ A(\ln 2)=2\hat{i}+\frac{1}{2}\hat{j}+16\hat{k}$

Review Exercise 5

1. (i) a (ii) b (iii) c (iv) d (v) a
(vi) c (vii) b (viii) a (ix) d (x) b

Unit-6

Exercise 6.1

1. a. $\frac{x^4}{4}+C$ b. $(a+15)x$ or $ax+15x$ c. $\frac{1}{5}\ln|x|+c$
d. x^3+2x^2-5x+c e. $\frac{t^4}{4}+\frac{3}{4}t^5+C$ f. $\frac{1}{2}x^{-2}+C$

g. $\frac{e^{5x}}{5}+C$ h. $5\tan^{-1}x$ i. $\frac{2}{7}t^{\frac{7}{2}}$

a. $\frac{1}{27}(3x+4)^9+C$ b. $\frac{(x^3-4)^2}{2}+C$ c. $\frac{(x^3+7x)^9}{9}+C$

d. $-\frac{1}{3}(x^3-7x)^{-3}+C$ e. $2\left(\sqrt{3t^2-7}\right)$ f. $2\left(r^{\frac{1}{3}}+4\right)^{\frac{3}{2}}$

g. $\frac{2}{3}x^{\frac{3}{2}}+\frac{6}{5}(x)^{\frac{5}{2}}+C$ h. $\frac{-1}{2\left(x^2+2x+2\right)}+C$

a. $e^{6t}+C$ b. $\frac{1}{10}e^{(5x^2+1)}$ c. $\frac{1}{3}e^{(x^3-6x+4)}$ d. $\frac{8^{(7-3x^2)}}{\ln 8}+C$

$y=x^4+2x^3-3$ 5. $y=\frac{\left(2x^2-1\right)^3}{12}-406.42$

$-e^{-0.01t}$ 7. a. $w=0.0005h^3$ b. 171.5 ponds 8. 19,400

Exercise 6.2

a. $\frac{\sin^5 x}{5}+C$ b. $\frac{2}{5}(\sin x)^{\frac{5}{2}}$ c. $-\frac{(\ln(\sin x))^2}{2}+C$

d. $-\cos e^x+C$ e. $2\ln\left|\sin\sqrt{x}\right|+C$ f. $-\ln\left|\sin x+\cos x\right|+C$

a. $\frac{1}{4}tan^{-1}\left(\frac{x}{4}\right)+C$ b. $-\tan^{-1}(\cos x)+C$ c. $\frac{1}{2}\sec^{-1}\left(\frac{e^x}{2}\right)+C$

d. $\ln\left|x^2+4x+5\right|+\tan^{-1}(x+2)+C$ e. $\sin^{-1}\left(\frac{x+1}{\sqrt{5}}\right)-(5-(x+1)^2)^{\frac{1}{2}}+C$

a. $x^2\frac{e^{2x}}{2}-x\frac{e^{2x}}{2}+\frac{e^{2x}}{8}+C$ b. $-x\sin x+\cos x+C$ c. $-\frac{1}{2}e^x\cos x+\frac{1}{2}e^x\sin x$

d. $\frac{1}{2}e^x\sin x-\frac{1}{2}e^x\cos x+C$ e. $x\sin^{-1}x+\sqrt{1-x^2}+C$ f. $\frac{1}{3}\sin(e^{3t})+C$

Exercise 6.3

a. $\frac{1}{3}\ln\left|\frac{x-3}{x}\right|+C$ b. $3x-\ln x+C$ c. $3\ln x+\ln(x+1)+\frac{1}{x}+\frac{2}{x+1}+C$

d. $\frac{1}{3}\ln(x-1)-\frac{1}{6}\ln(x^2+x+1)-\frac{1}{\sqrt{3}}\tan^{-1}\left(\sqrt{\frac{4}{3}}\left(x+\frac{1}{2}\right)\right)+C$

e. $\left[\frac{x^2}{2}+x+2\ln(x-1)-2\ln(x)\right]$ f. $\frac{1}{2}\ln\left|\frac{x-1}{x+1}\right|+C$

g. $\ln\left|\frac{(2x+1)^{\frac{5}{6}}}{(x-1)^{\frac{4}{3}}}\right|+C$ h. $x+\ln\left|\frac{(x-5)^3}{(x+3)}\right|+C$

i. $\frac{1}{2}\tan^{-1}x+\ln\left[\frac{(x^2+1)^{\frac{1}{4}}}{(x+1)^{\frac{1}{2}}}\right]+C$ j. $\frac{3}{2}\tan^{-1}x+\frac{1}{4}\sin(2\tan^{-1}x)+C$

2. $1.89ml$ fourth hour 3. 45000 4. 109.86 feet

Exercise 6.4

1. a. 1705 b. 34816 c. −2 d. 14 e. 15625 f. 0 g. $-2+\sqrt{2}$ h. 0
2. a. 4.357 b. 0.458
3. a. $\frac{23}{3}$ b. $\frac{29}{6}$ c. 8 d. $\frac{68}{6}$
4. a. $A=4.01$ square units b. $A=2.953$ square units
   c. $A=3.83$ square units d. $\frac{8}{3}$ square units
5. a. L_1 day = 414.44 barrels b. L_2 day = 191.6 barrels
   c. As is apparent from the amount of oil leaked on 1st and 2nd day, the number of barrel of oil leakage per day decreases with the passage of time.
6. a. $\frac{x^3}{3}+\frac{x^2}{2}-x+C$ b. $-\frac{e^{2x}\cos(x)}{5}-\frac{2e^{2x}\sin(x)}{5}+C$

Review Exercise 6

1. i. (b) ii. (c) iii. (c) iv. (c) v. (a)
   vi. (b) vii. (b) viii. (c) ix. (a) x. (d)

Unit-7

Exercise 7.1

2. a. (−1, 6) b. (0, 0) c. $\left(\frac{1}{20},\frac{1}{28}\right)$
3. a. $\left(\frac{11}{6},\frac{17}{6}\right)$ b. (−24, −2) c. $\left(-\frac{32}{9},\frac{41}{9}\right)$
4. a. 3:4 b. 14:11
5. a. $\left(\frac{7}{3},\frac{7}{3}\right)$ b. $\left(\frac{10}{3},\frac{10}{3}\right)$ c. $\left(1,\frac{4}{3}\right)$

Exercise 7.2

1. a. $x-y+5=0$ b. $3x+y-6=0$ c. $3x+4y+12=0$
2. a. $\frac{1}{4}$ b. $-\frac{4}{7}$ c. 2 d. $-\frac{1}{6}$
3. a. x–intercept, $a=-3$ and y–intercept, $b=6$
   b. x–intercept, $a=3$ and y–intercept, $b=9$
   c. x–intercept, $a=-2$ and y–intercept, $b=2$

4. a. perpendicular b. neither perpendicular nor parallel c. parallel d. parallel
5. a. $3x-y=0$ b. $5x-y-5=0$ c. $x-y=0$
6. a. $4x-y-2=0$ b. $x+2y+5=0$ c. $3x+y+4=0$ d. $5x-y-43=0$
7. a. $2x-y+2=0$ b. $8x-y+4=0$ c. $x-2y-8=0$
8. a. $y=\frac{7}{10}x+\frac{13}{7}$ b. $\frac{x-\frac{13}{7}}{\cos 35^0}=\frac{y+\frac{13}{10}}{\sin 35^0}$ c. *Normal form* $\theta=45^{\circ}$, $P=1$, $x\cos\theta+y\sin\theta=1$

Exercise 7.3

1. a. Below the line, on the opposite sides of the line
   b. Above the line, on the some side of the line
2. a. on the opposite sides b. on the same sides
3. a. $d=6$ b. $d=\frac{6}{13}$ c. $d=1$
4. a. $45°$ b. $\tan^{-1}\left(\frac{1}{7}\right), 180-\tan^{-1}\left(\frac{1}{7}\right)$
5. a. $90°$ b. $\tan^{-1}\left(\frac{-7}{6}\right)$
6. a. $45°$ b. $\tan^{-1}(13)$
7. a. $\tan^{-1}\left(\frac{3}{19}\right)$ b. $180-\tan^{-1}(a)$
8. a. $119x+102y-125=0$ b. $23x+23y-11=0$ c. $3x+4y-5a=0$ d. $x-3y=0$

Exercise 7.4

1. a. $(1, 2)$ b. $(-16, 12)$
2. a. concurrent; $(3, 1)$ b. concurrent; $(-1,2)$ c. not concurrent
4. a. 6 b. 15 c. $\frac{5}{2}$
5. a. 0 b. 0 c. yes
6. a. $3x-5y=0$ and $x+y=0$ b. $x-y=0$ and $4x-5y=0$
8. a. $2x^2+7xy+3y^2=0$ b. $x^2-2\tan\theta\, xy-y^2=0$ c. $ay^2-2hxy+bx^2=0$

Review Exercise 7

1. i. (b) ii. (c) iii. (a) iv. (a) v. (b)
   vi. (a) vii. (d) viii. (a) ix. (b) x. (d)

Unit -8

Exercise 8.1

1. a. $x^2+y^2=16$ b. $x^2+y^2-6x-4y+12=0$ c. $x^2+y^2+8x+6y+9=0$
   d. $x^2+y^2+2ax+2by-2ab=0$

2. a. $x^2+y^2=25$ b. $x^2+y^2=36$ c. $x^2+y^2-12x+12y=0$
d. $x^2+y^2+18x+12y+117=317$ e. $(x+5)^2+(y-4)^2=16$
f. $(x-5)^2+(y-3)^2=25$

3. a. $center=c(4,3), r=4$ b. $center=c\left(-2,\frac{3}{2}\right), r=2\sqrt{2}$ c. $center=c(-2,3), r=0$
d. $center=c\left(\frac{1}{2},4\right), r=\sqrt{\frac{-7}{4}}$

4. a. $x^2+y^2-2x-21=0$ b. $x^2+y^2+4x+6y-72=0$ c. $x^2+y^2-22x-4y+25=0$

5. a. $x^2+y^2-2x+2y-48=0$ b. $x^2+y^2-6x-8y+15=0$

6. a. $5x^2+5y^2+12x-15y=0$ b. $x^2+y^2-3x+y=0$

7. a. $16x^2+16y^2+128x-56y+49=0$ b. $x^2+y^2-8x-4=0$
c. $x^2+y^2+6x-10y+9=0$

8. a. $x^2+y^2=25$ b. $x^2+y^2=20$

Exercise 8.2

1. a. equation of tangent : $1x+2y=55$, equation of normal : $2x-y=0$
b. equation of tangent: $3x+2y+13=0$ equation of normal $2x+3y=0$
c. equation of tangent: $x+y-5=0$ equation of normal: $y-x=-3$

2. a. equation of tangent: $18x+18\sqrt{3}y=13$, equation of normal: $18\sqrt{3}x+18y=0$,
b. equation of tangent : $x+y-2\sqrt{2}=0$, equation of normal : $x-y=0$
c. equation of tangent : $\sqrt{3}x+y-2=0$, equation of normal : $x-\sqrt{3}y=0$

3. a. $n=\pm a\sqrt{m^2+l^2}$ b. $n=\pm 3\sqrt{2}$ c. $n=\pm 18\sqrt{2}$

4. a. $c=\pm a\sqrt{1+m^2}$ b. $c=\pm 3\sqrt{2}$ c. $c=\pm 9a\sqrt{2}$

5. a. $g^2m^2+f^{2\ 2}-2\ ng-2gf\ m-n^2+2fnm-m^2c-c\ ^2=0$

6. a. $n=7,-43$ b. $n=4.52,-13.52$ c. Does not touch the circle $C=\frac{-152\pm\sqrt{1472}i}{32}$

10. a. $y=\frac{-1}{2}x+\sqrt{5}$ b.

12. a. b. $y=\frac{x}{\sqrt{3}}\pm\frac{2}{3}$ c. $y=\sqrt{3}x\pm\frac{\sqrt{13}}{2}$

Review Exercise 8

1. i. (d) ii. (d) iii. (c) iv. (a) v. (a)
vi. (d) vii. (b) viii. (d) ix. (c) x. (b)

Unit -9

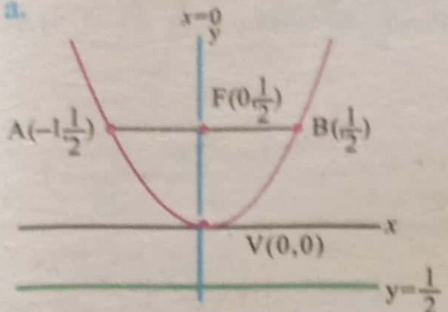
Exercise 9.1

1. a. Vertex (0,0), focus $\left(0,\frac{1}{2}\right)$, latus rectum $A\left(-1,\frac{1}{2}\right), B\left(1,\frac{1}{2}\right)$ axis of symmetry $x=0$ (y-axis)

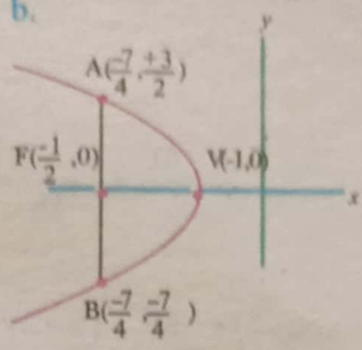
b. Vertex $(-0,0)$, focus $\left(\frac{-7}{4},0\right)$, latus rectum $A\left(\frac{-7}{4},\frac{+3}{2}\right), B\left(\frac{-7}{4},\frac{-3}{2}\right)$ axis of symmetry $y=0$

c. Vertex $(0,3)$, focus $\left(\frac{1}{4},3\right)$, latus rectum $A\left(\frac{1}{4},\frac{5}{2}\right), B\left(\frac{1}{4},\frac{7}{2}\right)$ axis of symmetry $y=3$

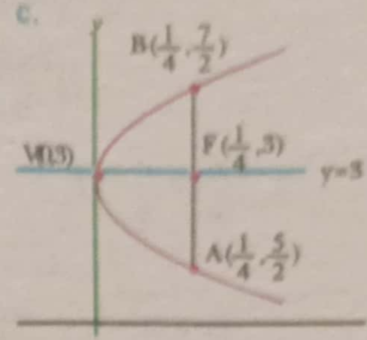
a.



b.



c.



2. a. $(x-2)^2=8(y+1)$ b. $y^2=4x$

3. a. $x^2=12y$ b. $y^2=16x$ c. $y^2=12x$ d. $x^2=-48y$

e. $(x-5)^2=\frac{64}{3}(y-1)$ f. $(y+7)^2=64(x-3)$

4. $3x+y+5=0$ 5. $(y+2)^2=6\left(x-\frac{5}{2}\right)$ 6. a. $P_1(1,3)$ and $P_2(4,6)$

b. $x=1\pm\sqrt{3}i$ which is imaginary so the line does not intersect the parabola.

7. a. $c=\frac{9}{4}$ b. $c=-\frac{1}{6}$

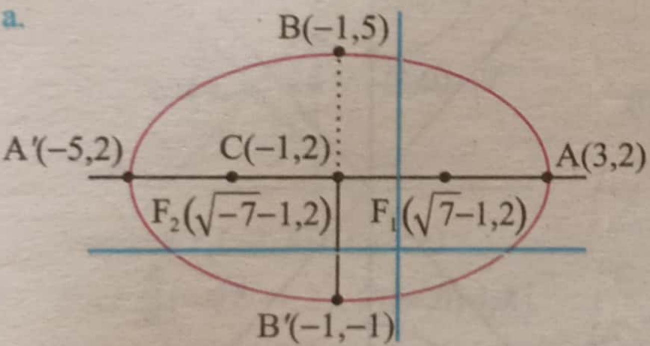
8. a. equation of tangent is $x-y+3=0$, equation of normal is $x+y-9=0$

b. equation of tangent is $4x+3y-1=0$, equation of normal is $8x+9y-1=0$

9. a. $4x+4y+1=0$ b. $y=\sqrt{3}x+\frac{1}{4\sqrt{3}}$ 10. $x^2=-12(y+3)$

Exercise 9.2

1. a.



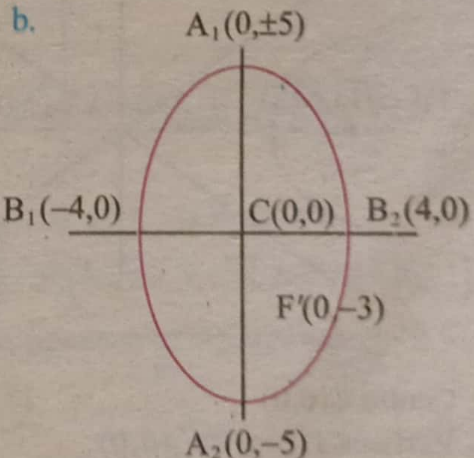
Centre C(0,0)

Foci $F(\pm\sqrt{5},0)$

End points of major axis $A(\pm3,0)$

End Points of minor axis $B(0,\pm2)$

b.



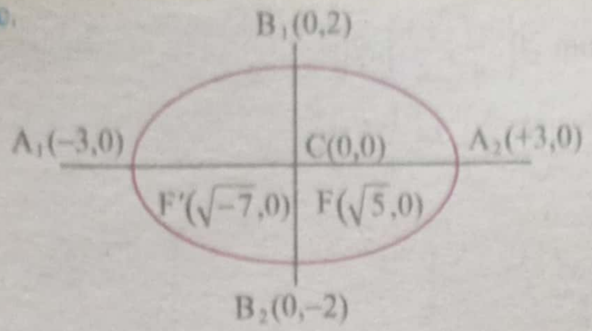
Centre C(0,0)

Foci $F(0,\pm3)$

End points of major axis $A(0,\pm5)$

End Points of minor axis $B(\pm2,0)$

c.



Centre: The center is $C(h,k)=C(-1,2)$
Foci: $F_1(\sqrt{7},-1,2), F_2(-\sqrt{7},-1,2)$
End points of major axis $A(3,2)A'(-5,2)$
End Points of minor axis $B(-1,5)B'(-1,-1)$

2. a. $\frac{x^2}{36}+\frac{y^2}{9}=1$ b. $\frac{(x-6)^2}{49}+\frac{(y+2)^2}{16}=1$ c. $\frac{(x-2)^2}{9}+\frac{y^2}{25}=1$

3. a. $\frac{(x+3)^2}{4}+\frac{(y-2)^2}{1}=1$ b. $\frac{(x-8)^2}{16}+\frac{(y-2)^2}{4}=1$ c. $\frac{\left(x-\frac{23}{16}\right)^2}{\left(\frac{73}{16}\right)^2}+\frac{(y-3)^2}{(3)^5}=1$

d. $\frac{x^2}{4}+\frac{(y-5)^2}{9}=1$

4. a. $e=\frac{3}{5}$ b. $\frac{x^2}{25}+\frac{y^2}{16}=1$ c. $e=\frac{\sqrt{3}}{2}$

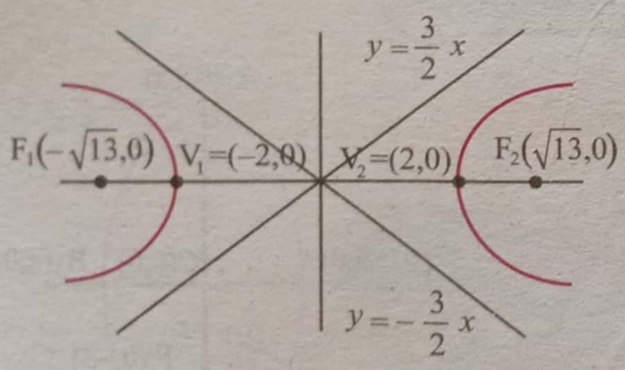
5. a. $c=\pm\sqrt{5}$ b. $c=\pm 4$ c. $c=\pm 6$

6. a. equation of tangent is $\frac{x}{4}+\frac{2y}{9}=1$ equation of normal $8x-9x+10=0$
 b. equation of tangent is $6x+21y-14=0$ equation of normal $7x-2y-9=0$
 c. equation of tangent is $x+4y-4=0$ equation of normal $4x-y-3=0$

7. a. $8x-9y=10=0$ b. $6x+21y-144=0$

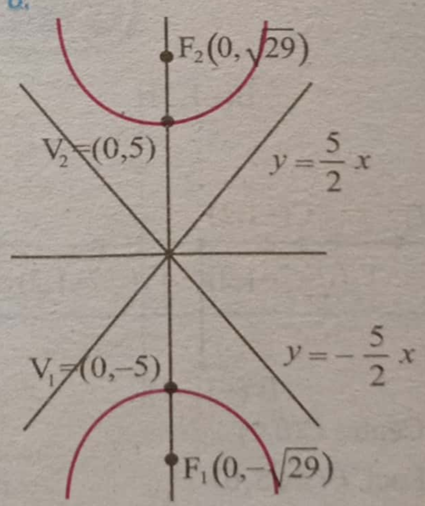
Exercise 9.3

1. a.



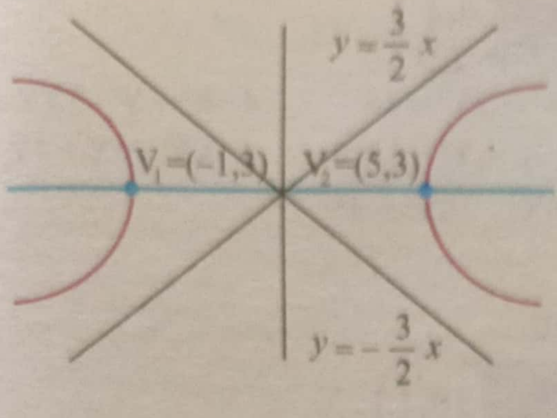
Centre $C(0,0)$
Vertices $V_1(-a,0)V_2(a,0)$
Foci $F_1(-\sqrt{13},0)\,F_2(\sqrt{13},0)$
Equation of asymptotes $y=\pm\frac{b}{a}x$

b.



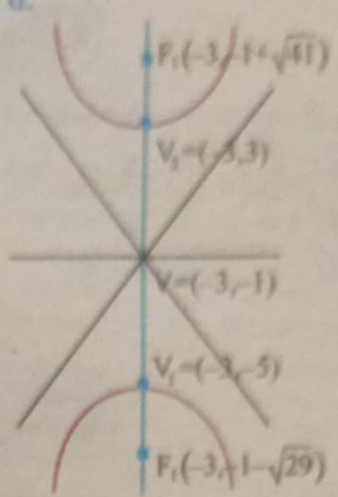
Centre $C(0,0)$
Vertices $V_1(0,-a)V_2(0,a)$
Foci $F_1(0,-c)F_2(0,c)$
Equation of asymptotes $y=\pm\frac{a}{b}x$

c.



Centre C(2,3)
Vertices $V_1(-1,3) V_2(5,3)$
Foci $F_1(-3,3)$ $F_2(7,3)$
Equation of asymptotes $y-k=\pm\frac{b}{a}(x-h)$

d.



Centre C(−3,−1)
Vertices $V_1(-3,-5) V_2(-3,3)$
Foci $F_1\left(-3,-1-\sqrt{41}\right)$ $F_2\left(-3,-1+\sqrt{41}\right)$
Equation of asymptotes $y+1=\pm\frac{4}{5}(x+3)$

2. a. $\frac{x^2}{16}-\frac{y^2}{4}=1$ b. $\frac{y^2}{25}-\frac{x^2}{16}=1$ c. $\frac{(x-4)^2}{9}-\frac{4(y-2)^2}{9}=1$ d. $\frac{(y+2)^2}{9}-\frac{(x+5)^2}{16}=1$

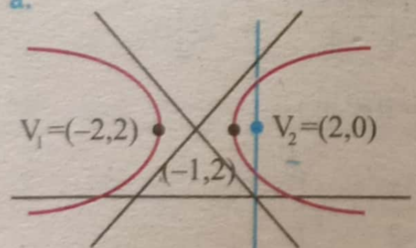
3. a. $\frac{y^2}{4}-\frac{x^2}{5}=1$ b. $\frac{x^2}{25}-\frac{y^2}{24}=1$ c. $9x^2-y^2=9$

d. $16x^2-9y^2=144$ e. ______

4. a. $\frac{x^2}{16}-\frac{y^2}{9}=1$ b. $\frac{y^2}{25}-\frac{x^2}{144}=1$

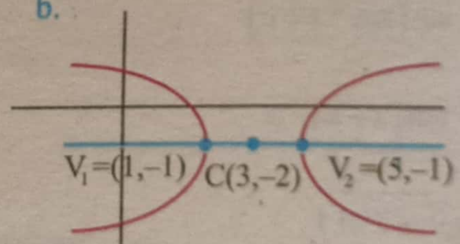
5. a. $\frac{(x+1)^2}{9}-\frac{(x+2)^2}{3}=1$ b. $\frac{(y-2)^2}{1}-\frac{(x+3)^2}{3}=1$

6. a.



Vertices $V_1(-2,27), V_2(0,2)$
Foci $F_1\left(-\sqrt{2}-1,2\right), F_2\left(\sqrt{2}-1,2\right)$
The equation of asymptotes $y-2=\pm1(x+1)$

b.



Vertices $V_1(1,-1), V_2(5,-1)$
Foci $F_1\left(-\sqrt{2}+3,-1\right), F_2\left(\sqrt{2}+3,-1\right)$
The equation of asymptotes $y-k=\pm\frac{b}{a}(x-h)$

7. a. $p_1(4,1)$, $p_2(-1,-4)$ b. $p_1(0,1)$, $p_2(-\frac{8}{3},\frac{5}{3})$

8. a. $c=\pm2\sqrt{3}$ b. $c=\pm\sqrt{7}$

9. a. Equation of tangent is $\sqrt{13}x+2y+4=0$, Equation of normal is $4x-2\sqrt{13}y+13\sqrt{13}=0$

b. Equation of tangent is $y=\frac{3x}{5}+\frac{9}{5}$, Equation of normal is $y=\frac{5x}{3}+\frac{125}{9}$

Exercise 9.4

1. a. $2X - Y = 0$ b. $X^2 + Y^2 = 4$ c. $X^2 + 2Y^2 = 16$ d. $X^2 + Y - 3XY + 1 = 0$
2. a. $2XY + a^2 = 0$ b. $X^2 = -4pY$ c. $3X^2 - Y^2 - 1 = 0$ d. $XY + 6X + 4Y - 1 = 0$
3. a. $3X^2 + 2Y^2 + 5$ b. $-X^2 + 4Y^2 = 1$ c. $6X^2 - 19Y^2 = 25$
4. a. $\theta = 36.87°$ b. $\theta = 30°$

Unit -10

Exercise 10.1

1. a. order = degree = 1, b. order = 2, degree = 1, c. order = 3, degree = 1, d. order = 2, degree = 1,
2. a. y is a solution, since the substitution of y and its derivative y' reduce the differential equation into an identity. b. y is a solution. c. y is a solution
   d. y is not a solution, since the substitution of y and its derivative y' does not reduce the differential equation into an identity.
3. a. $xy = 2$ b. $y = x - x\ln x + 1$ c. $\sin(xy) = 0.7071$ d. $\frac{y^2}{x} = \frac{x^2}{2} + \frac{1}{2}$
4. a. $y = \sin x + 1$ b. $f = \frac{1}{3}x^3 + 1$ c. $y = -\frac{1}{x} + \frac{1}{2}$ d. $y = \frac{1}{x^2 - 8}$
5. 276 students

Exercise 10.2

1. a. $y = \pm\sin(x + c)$ b. $y = \frac{-1}{e^{-x} + c}$
   c. $y = \sin(\sin^{-1} x + c)$ d. $y = \sin^{-1}\left[\frac{\sin 2x}{4} - \frac{x}{2} + c\right]$
2. a. $y = \tan(x + c) - x$ b. $y = \sin-1(e^x c_1) - x$
   c. $y = \sqrt{ce^{-x2} + 1}$ d. $y = x\left(\frac{3x^2}{2} + c_2\right)^{\frac{1}{3}}$
3. a. $\tan^{-1}\left(\frac{y}{x}\right) - \frac{1}{2}\ln\left(1 + \frac{y^2}{x^2}\right) = \ln(x) + c$ b. $y = \frac{x}{\ell n(x) + c}$
   c. $y = \pm x\sqrt{-1 + cx}$ d. $-\frac{x^2}{2y^2} - \frac{x}{y} + \ln(y) = c$
4. a. $y = \frac{1}{4}x^2 - 1$ b. $\frac{x^2 . \ln(x)}{1 + \ln(x)}$
5. $y = \frac{x - 13}{x - 1}$

6. $\left(\frac{2y^2}{x^2} + 1\right)^{\frac{1}{4}} = 9^{\frac{1}{4}}x$

7. $\frac{y}{4 + t} = \frac{2t}{3}$

8. $\frac{dx}{dt}=\frac{x}{2}-3t,\ x(t)=6t+12+ce^{\frac{t}{2}},\ x(t)=6t+12-28e^{\left(\frac{t}{2}-1\right)}$

9. General consumption of oil is $x(t)=30e^{0.04t}+c$. At time $t=0$ (1990), the oil consumption was $x=0$, that gives $c=-30$. The actual oil consumption at time t is $x(t)=30e^{0.04t}-30$.

10. 110.825 items

11. a. $x^2+3y^2=k$ b. $y^2-x^2=k$
c. $\frac{y^2}{2}=-x+\ nk(x+1)$ d. $yx=k$

Review Exercise 10

1. i. (a) ii. (b) iii. (c) iv. (d) v. (a) vi. (b) vii. (a)

Unit-11

Exercise 11.1

1. a. 0 b. 0 c. 0 d. $2t^3$ e. t^4+t^5 f. $t-t^2$
2. a. 0 b. $1-e^2$ c. $e^{-2}+1$ d. $2x^3e^{2x}+3x^2e^{2x}+2x$
e. e^2+2y f. $4z(z^2-2)$
3. a. $f_x=2x\cos x^2\cos y,\ f_y=-\sin x^2\sin y$ b. $f_x=\frac{3x}{\sqrt{3x^2+y^4}},\ f_y=\frac{2y^3}{\sqrt{3x^2+y^4}}$
c. $f_x=y^3\tan^{-1}y,\ f_y=xy^2(\frac{y}{1+y^2}+3\tan^{-1}y)$ d. $f_x=3x^2+2xy+y^2,\ f_y=x^2+2xy+3y^2$
e. $f_x=\frac{y}{\sqrt{1-x^2y^2}},\ f_y=\frac{x}{\sqrt{1-x^2y^2}}$ f. $f_x=x(x+2)e^{x+y}\cos y=,\ f_y=x^2e^{x+y}(\cos y-\sin y)$
4. $f_x=0.7x^{-0.3}y^{0.3},\ f_y=0.3x^{0.7}y^{-0.7}$
5. $f_x=0.4yx^{-0.6}y^{0.6}, f_y=0.6x^{0.4}y^{-0.4}$
6. $f_x=2xy+y^2, f_y=x^2+2xy$

Exercise 11.2

1. a. Not homogeneous b. Homogeneous
c. Homogeneous, degree=3 d. Homogeneous

Review Exercise 11

1. i. (c) ii. (b) iii. (b) iv. (c) v. (c)
vi. (c) vii. (c) viii. (b) ix. (a)

Unit -12

Exercise 12.1

1. a. Hence the function takes values with opposite sign in the interval [1,2].
   b. Hence the function takes values with opposite sign in the interval [1,2].
   c. Hence the function takes values with opposite sign in the interval [1,4].
   d. Hence the function takes values with opposite sign in the interval [3,4].
2. a. $c_0 = 1.4, c_1 = 1.2, c_2 = 1.1, c_3 = 1.15$ b. $c_0 = 1.2, c_1 = 1.4, c_2 = 1.3, c_3 = 1.25$
   c. $c_0 = 3.6, c_1 = 3.8, c_2 = 3.7, c_3 = 3.65$ d. $c_0 = 3.6, c_1 = 3.4, c_2 = 3.5, c_3 = 3.55$
3. a. $c_0 = -1.8300782,\ c_1 = -1.8409252,\ c_2 = -1.8413854,\ c_3 = -1.8414048$
   b. $c_0 = 1.27,\ c_1 = 1.283,\ c_2 = 1.283412,\ c_3 = 1.283402$
   c. $c_0 = 3.6979549,\ c_1 = 3.6935108,\ c_2 = 3.6934424,\ c_3 = 3.6934414$
   d. As both $f(a_0)$ and $f(b_0)$ are positive, there is no root in the given interval.
4. a. The bisection method is not valid, since the functions have not opposite signs in the interval [3,7].
   b. The bisection method is valid, since the functions have opposite signs in the interval [1,7].
5. a. $x_3 = 1.443$ b. $x_3 = 0$ c. $x_3 = 0.453$ d. $x_3 = 0$
6. a. For 5 decimal accuracy, the forth iteration $x_4 = 0.324$
   b. For 5 decimal accuracy, the third iteration $x_3 = -2.5468$
   c. For 5 decimal accuracy, the third iteration is $x_3 = -2.79553$

Exercise 12.2

1. a. $T_4 = 8.75$ b. $T_4 = 1.17$ c. $T_6 = 1.11$ d. $T_6 = 3.26$
2. a. $S_6 = 18.70$ b. $S_8 = 1.11$ c. $S_6 = 1.10$ d. $S_8 = 3.19$

Review Exercise 12

1. i. (b) ii. (c) iii. (a) iv. (c) v. (b)
   vi. (a) vii. (d) viii. (b) ix. (c) x. (b)

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