

MODULE - 01

Fundamentals of logic.

Introduction:- Logic \rightarrow Method of Reasoning
logic is expressed in a symbolic language is
called Mathematical logic.

(*) PROPOSITIONS :-

A proposition is a statement (declaration)
which in a given context, can be said to be either
true or false, but not both.

Ex:- 1. Bangalore is in Karnataka

2. 9 is divisible by 3.

3. $x^y = y^x$, for this we cannot decisively
say whether it is true (or not), unless we
know that what is x and y (whether it is integer

(or Real no) so it is not a proposition.

\rightarrow propositions are usually represented by small
letters such as p, q, r, t, s, \dots

\rightarrow The truth (or the falsity of a proposition is
called its truth value.

\rightarrow If a proposition is true it is denoted by the
truth value 1

→ If a proposition is false it is denoted by the truth value '0'.

(*) Logical connectives :-

The words like 'not', 'and', 'if then', if and only if, such words are called as logical connectives.

(*) Compound proposition :-

The new proposition is obtained by combining the two given propositions using the logical connectives are called as compound proposition.

(*) Simple proposition :-

proposition which do not contains any logical connectives are called simple propositions.

(*) Negation (\sim)

A proposition is obtained by inserting the word 'not' in an appropriate place in the negation of the given proposition.

The negation of a proposition 'p' is denoted by $\sim p$.

e.g.: - If P: 3 is a prime number - 1

$\sim p$: 3 is not a prime number - 0

Truth table:

P	$\sim p$
1	0
0	1

(2) P : 8 is divisible by 3 - 0

$\sim P$: 8 is not divisible by 3 - 1

Truth table

P	$\sim P$
0	1
1	0

(*) conjunction (\wedge)

A compound proposition is obtained by inserting the word 'and' between two given propositions is called conjunction of the given proposition.

The conjunction of p and q_1 is denoted by $p \wedge q_1$.

The conjunction of p and q_1 is true only when both p and q_1 are true, in all other cases it is false.

Truth table:

p	q_1	$p \wedge q_1$
0	0	0
0	1	0
1	0	0
1	1	1

(*) Disjunction (\vee)

A compound proposition is obtained by inserting the word 'or' between two given proposition is called disjunction of the given proposition.

The disjunction of p and q_1 is denoted by $p \vee q_1$.

Truth table:

(*) The disjunction of p and q_1

p	q_1	$p \vee q_1$
0	0	0
0	1	1
1	0	1
1	1	1

(*) The disjunction $p \vee q_1$ is false only when only both p & q_1 are false otherwise it is true

(*) Exclusive disjunction ($\underline{\vee}$)

The exclusive disjunction of two propositions p and q_1 is denoted by $p \underline{\vee} q_1$ (Read it as either p or q_1)

The exclusive disjunction is true only when either p is true ($\underline{\vee}$ q_1 is true but not both)

Truth table:

p	q_1	$p \underline{\vee} q_1$
0	0	0
0	1	1
1	0	1
1	1	0

(*) conditional (\rightarrow)

A compound proposition is obtained by inserting the word 'if then' in an appropriate place is called the conditional.

The conditional of p and q_1 is denoted by $p \rightarrow q_1$.
(read as if p then q_1).

The conditional $p \rightarrow q_1$ is false only when p is true and q_1 is false.

Truth table:

P	q_1	$p \rightarrow q_1$
0	0	1
0	1	1
1	0	0
1	1	1

(*) Bi-conditional:- (\leftrightarrow)

A compound proposition is obtained by inserting the word 'if and only if' in an appropriate place is called as bi-conditional of the given proposition.

It is denoted by $p \leftrightarrow q_1$.

The bi-conditional $p \leftrightarrow q_1$ is true only when both p and q_1 have the same truth values otherwise it is false.

The biconditional of p and q_1 is denoted by

$$p \leftrightarrow q_1 = (p \rightarrow q_1) \wedge (q_1 \rightarrow p)$$

Truth table :

p	q_1	$p \leftrightarrow q_1$
0	0	1
0	1	0
1	0	0
1	1	1

combined truth table :-

p	q_1	$\sim p$	$\sim q_1$	$p \wedge q_1$	$p \vee q_1$	$\underline{p \vee q_1}$	$p \rightarrow q_1$	$p \leftrightarrow q_1$
0	0	1	1	0	0	0	1	1
0	1	1	0	0	1	1	1	0
1	0	0	1	0	1	1	0	0
1	1	0	0	1	1	0	1	1

$q_1 \rightarrow p$
1
0
1
1

PROBLEMS:

1. Let p : A circle is a conic

q : $\sqrt{5}$ is a real number.

r : Exponential series is convergent.

Express the following compound propositions in words

(i) $p \wedge (\sim q)$ (ii) $(\sim p) \vee q$ (iii) $p \vee (\sim q)$ (iv) $q \rightarrow (\sim p)$

(v) $p \rightarrow (q \vee r)$ (vi) $\sim p \leftrightarrow q$

SOLN:- (i) $p \wedge (\sim q)$:

A circle is a conic and $\sqrt{5}$ is not a real no.

(ii) $(\sim p) \vee q$:

A circle is not a conic or $\sqrt{5}$ is a real number.

(iii) $p \vee (\sim q)$:

Either a circle is a conic or $\sqrt{5}$ is a real number.
(but not both)

(iv) $q \rightarrow (\sim p)$:

If $\sqrt{5}$ is a real number, then a circle is not a conic.

(v) $p \rightarrow (q \vee r)$:

If a circle is a conic then either $\sqrt{5}$ is a real no
or the exponential series is convergent (but not both).

(vi) $\sim p \leftrightarrow q$

If a circle is not a conic then $\sqrt{5}$ is a real number
and if $\sqrt{5}$ is a real number then a circle is not a conic.

(a) Construct the truth tables for the following compound propositions:

$$(i) P \wedge (\sim q_1) \quad (ii) (\sim p) \vee q_1 \quad (iii) p \rightarrow (\sim q_1) \quad (iv) (\sim p) \vee (\sim q_1)$$

:-

P	q_1	$\sim p$	$\sim q_1$	$P \wedge (\sim q_1)$	$(\sim p) \vee q_1$	$p \rightarrow (\sim q_1)$	$(\sim p) \vee (\sim q_1)$
0	0	1	1	0	1	1	0
0	1	1	0	0	1	1	1
1	0	0	1	1	0	1	1
1	1	0	0	0	1	0	0

(3) Let p, q_1 and r be the propositions having truth values 0, 0 and 1 respectively. Find the truth values of the following compound propositions.

$$(i) (P \vee q_1) \vee r$$

$$(ii) (P \wedge q_1) \rightarrow r$$

$$(iii) P \rightarrow [q_1 \rightarrow (\sim r)]$$

$$\text{Soln: } (i) (P \vee q_1) \vee r = (0 \vee 0) \vee 1 \\ = 0 \vee 1 \\ = 1$$

$$(ii) (P \wedge q_1) \rightarrow r = (0 \wedge 0) \rightarrow 1 \\ = 0 \rightarrow 1 \\ = 1$$

$$(iii) P \rightarrow [q_1 \rightarrow (\sim r)] = 0 \rightarrow [0 \rightarrow (0)] \\ = 0 \rightarrow 1 \\ = 1$$

(4) construct the truth table for the following compound propositions.

(i) $(p \wedge q) \rightarrow (\sim r)$

(ii) $q \vee (\sim r \rightarrow p)$

Soln:-

p	q	r	$p \wedge q$	$\sim r$	$(p \wedge q) \rightarrow (\sim r)$	$\sim r \rightarrow p$	$q \vee (\sim r \rightarrow p)$
0	0	0	0	1	1	0	0
0	0	1	0	0	1	1	0
0	1	0	0	1	1	0	0
0	1	1	0	0	1	1	1
1	0	0	0	1	1	1	0
1	0	1	0	0	1	1	0
1	1	0	1	1	1	1	1
1	1	1	1	0	0	1	1

(*) Tautology:

A compound proposition which is true for all possible truth values of its components is called a tautology.

(*) Contradiction:

A compound proposition which is false for all possible truth values of its components is called a contradiction.

(*) Contingency:

A compound proposition that can be true or false (depending upon the truth values of its components) is called a contingency.

Note:- Contingency is a compound proposition which is neither a tautology nor a contradiction.

PROBLEMS:

(1) prove that for any proposition P , $P \vee \sim P$ is tautology and $P \wedge \sim P$ is a contradiction.

-	P	$\sim P$	$P \vee \sim P$	$P \wedge \sim P$
	0	1	1	0
	1	0	1	0

Hence $P \vee \sim P$ is a tautology and $P \wedge \sim P$ is a contradiction.

(2) show that, for any propositions p and q
 the compd. proposition $p \rightarrow (p \vee q)$ is a tautology
 and the compd. proposition $p \wedge (\sim p \wedge q)$ is a contradiction

p	q	$p \vee q$	$p \rightarrow (p \vee q)$	$\sim p$	$\sim p \wedge q$	$p \wedge (\sim p \wedge q)$
0	0	0	1	1	0	0
0	1	1	1	1	0	0
1	0	1	1	0	0	0
1	1	1	1	0	0	0

Hence, $p \rightarrow (p \vee q)$ is true for all possible truth value
 hence it is tautology

$p \wedge (\sim p \wedge q)$ is false for all possible truth values

hence it is contradiction.

(3) prove that for any propositions p, q_1, γ then
 the compound proposition $\{p \rightarrow (q_1 \rightarrow \gamma)\} \rightarrow \{(p \rightarrow q_1) \rightarrow (p \rightarrow \gamma)\}$
 is a tautology.

p	q_1	γ	$q_1 \rightarrow \gamma$	$p \rightarrow (q_1 \rightarrow \gamma)$	$p \rightarrow q_1$	$p \rightarrow \gamma$	$(p \rightarrow q_1) \rightarrow (p \rightarrow \gamma)$	$(p \rightarrow (q_1 \rightarrow \gamma)) \rightarrow ((p \rightarrow q_1) \rightarrow (p \rightarrow \gamma))$
0	0	0	1	1	1	1	1	1
0	0	1	1	1	1	1	1	1
0	1	0	0	1	1	1	1	1
0	1	1	1	1	1	1	1	1
1	0	0	1	1	0	0	1	1
1	0	1	1	1	0	1	1	1
1	1	0	0	1	0	0	1	1
1	1	1	1	1	1	1	1	1

(4) prove that for any proposition p, q_1, γ the compound proposition $[(p \vee q_1) \wedge \{(p \rightarrow \gamma) \wedge (q_1 \rightarrow \gamma)\}] \rightarrow \gamma$ is a tautology.

$[(p \vee q_1) \wedge \{(p \rightarrow \gamma) \wedge (q_1 \rightarrow \gamma)\}] \rightarrow \gamma$ is a tautology.

P	q_1	γ	$p \vee q_1$	$p \rightarrow \gamma$	$q_1 \rightarrow \gamma$	$(p \rightarrow \gamma) \wedge (q_1 \rightarrow \gamma)$	$[(p \vee q_1) \wedge \{(p \rightarrow \gamma) \wedge (q_1 \rightarrow \gamma)\}]$	$\rightarrow \gamma$
0	0	0	0	1	1	1	0	1
0	0	1	0	1	0	0	0	1
0	1	0	1	0	1	0	0	1
0	1	1	1	0	1	0	0	1
1	0	0	1	0	1	0	0	1
1	0	1	1	1	1	1	1	1
1	1	0	1	0	0	0	0	1
1	1	1	1	1	1	1	1	1

Hence, $[(p \vee q_1) \wedge \{(p \rightarrow \gamma) \wedge (q_1 \rightarrow \gamma)\}] \rightarrow \gamma$ is a tautology.

- (5) prove that (i) $(p \vee q_1) \vee (p \leftarrow q_1)$ is a tautology
(ii) $(p \vee q_1) \wedge (p \leftarrow q_1)$ is a contradiction
(iii) $(p \vee q_1) \wedge (p \rightarrow q_1)$ is a contingency.

:-

P	q_1	$p \vee q_1$	$p \leftarrow q_1$	$p \rightarrow q_1$	(i)	(ii)	(iii)
0	0	0	1	1	1	0	0
0	1	1	0	1	1	0	1
1	0	1	0	0	1	0	0
1	1	0	1	1	1	0	0

Logical Equivalence (\leftrightarrow)

TWO propositions u and v are said to be logically equivalent whenever u and v have the same truth value.

We write, $u \leftrightarrow v$. Here the symbol \leftrightarrow stands for "logically equivalent to".

Note:- 1. When the propositions u and v are not logically equivalent, we write $u \not\leftrightarrow v$.

2. Whenever u and v are logically equivalent then $u \leftrightarrow v$ is always tautology.

problems:-

(1) prove that for any three propositions p, q, r
 $[(p \vee q) \rightarrow r] \leftrightarrow [(p \rightarrow r) \wedge (q \rightarrow r)]$.

Soln:-

P	q	r	$p \vee q$	$(p \vee q) \rightarrow r$	$p \rightarrow r$	$q \rightarrow r$	$(p \rightarrow r) \wedge (q \rightarrow r)$
0	0	0	0	1	1	1	1
0	0	1	0	1	1	1	1
0	1	0	1	0	1	0	0
0	1	1	1	1	1	1	1
1	0	0	1	0	0	1	0
1	0	1	1	1	1	1	1
1	1	0	1	0	0	0	0
1	1	1	1	1	1	1	1

If is logically equivalent.

(Q) Examine whether compound proposition
 $[(p \vee q_1) \rightarrow r] \Leftrightarrow [\sim r \rightarrow \sim(p \vee q_1)]$

Soln:-

P	q ₁	r	p ∨ q ₁	(p ∨ q ₁) → r	~r	~(p ∨ q ₁)	~r → ~(p ∨ q ₁)
0	0	0	0	1	1	1	1
0	0	1	0	1	0	1	1
0	1	0	1	0	1	0	0
0	1	1	1	1	0	0	1
1	0	0	1	0	1	0	0
1	0	1	1	1	0	0	1
1	1	0	1	0	1	0	0
1	1	1	1	1	0	0	1

Hence, It is logically equivalent.

* The laws of logic

To

1. prove that, for any three propositions p, q, r

$$[p \rightarrow (q \wedge r)] \Leftrightarrow [(p \rightarrow q) \wedge (p \rightarrow r)]$$

2. show that the compound propositions $p \wedge (\sim q \vee r)$
 and $p \vee (q \wedge \sim r)$ are logically equivalent.

(*) The Laws of Logic

In the following laws, T_0 denotes a tautology and F_0 denotes a contradiction.

1. Law of Double negation.

$$(\sim\sim p) \Leftrightarrow p$$

2. Idempotent law

$$(i) (p \vee p) \Leftrightarrow p \quad (ii) (p \wedge p) \Leftrightarrow p$$

3. Identity law

$$(i) (p \vee F_0) \Leftrightarrow p \quad (ii) (p \wedge T_0) \Leftrightarrow p$$

4. Inverse law

$$(i) (p \vee \sim p) \Leftrightarrow T_0 \quad (ii) (p \wedge \sim p) \Leftrightarrow F_0$$

5. Domination laws.

$$(i) (p \vee T_0) \Leftrightarrow T_0 \quad (ii) (p \wedge F_0) \Leftrightarrow F_0$$

6. Commutative Law

$$(i) (p \vee q_1) \Leftrightarrow (q_1 \vee p) \quad (ii) (p \wedge q_1) \Leftrightarrow (q_1 \wedge p)$$

7. Absorption law

$$(i) [p \vee (p \wedge q_1)] \Leftrightarrow p$$
$$(ii) [p \wedge (p \vee q_1)] \Leftrightarrow p$$

~~(8)~~ De-Morgan's Law

$$(i) \sim(p \vee q) \Leftrightarrow \sim p \wedge \sim q$$

$$(ii) \sim(p \wedge q) \Leftrightarrow \sim p \vee \sim q$$

~~(9)~~ Associative Law

$$(i) p \vee (q \vee r) \Leftrightarrow (p \vee q) \vee r$$

$$(ii) p \wedge (q \wedge r) \Leftrightarrow (p \wedge q) \wedge r$$

~~(10)~~ Distributive Law

$$(i) p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$$

$$(ii) p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$$

~~(11)~~ Law for the conditional

$$p \rightarrow q \Leftrightarrow \sim p \vee q$$

~~(*) 8(a) proof.~~

p	q	$\sim p$	$\sim q$	$p \vee q$	$\sim(p \vee q)$	$\sim p \wedge \sim q$
0	0	1	1	1	0	1
0	1	1	0	1	0	0
1	0	0	1	1	0	0
1	1	0	0	1	0	0

~~(*) 8(i) proof.~~

p	q	$p \vee q$	$\sim(p \vee q)$	$\sim p$	$\sim q$	$\sim p \wedge \sim q$
0	0	0	1	1	1	1
0	1	1	0	1	0	0
1	0	1	0	0	1	0
1	1	1	0	0	0	0

Hence proof

PROBLEMS:

(i) prove the following logical equivalence without using truth table.

$$(i) p \vee [p \wedge (p \vee q)] \Leftrightarrow p.$$

Soln:- $p \vee [p \wedge (p \vee q)] \Leftrightarrow p \vee p$, by Absorption law
 $\Leftrightarrow p$, by idempotent law.

$$(ii) [p \vee q \vee (\sim p \wedge \sim q \wedge r)] \Leftrightarrow (p \vee q \vee r)$$

Soln:- $[p \vee q \vee (\sim p \wedge \sim q \wedge r)]$
 $\Leftrightarrow [p \vee q \vee \sim (\sim p \wedge \sim q \wedge r)]$, by De-morgan's law
 $\Leftrightarrow [(p \vee q) \vee (\sim (\sim p \wedge \sim q) \wedge r)]$
 $\Leftrightarrow [(p \vee q) \vee \sim (\sim p \wedge \sim q)] \wedge [(p \vee q) \vee r]$ (\because by distributive property)
 $\Leftrightarrow T_0 \wedge [p \vee q \vee r]$ (\because by inverse law & associativity)
 $\Leftrightarrow (p \vee q \vee r)_{//}$ (\because by identity).

$$(iii) [(\sim p \vee \sim q) \rightarrow (p \wedge q \wedge r)] \Leftrightarrow p \wedge q$$

Soln:- consider,

$$[(\sim p \vee \sim q) \rightarrow (p \wedge q \wedge r)]$$
 $\Leftrightarrow [\sim (\sim p \wedge \sim q) \rightarrow (p \wedge q \wedge r)]$ (\because N.I.T)
 $\quad (p \wedge q) \vee (\sim p \wedge \sim q)$
 $\quad P \rightarrow Q \Leftrightarrow \sim P \vee Q$

$$\Leftrightarrow (p \wedge q) \vee ((p \wedge q) \wedge \sim q)$$

WIKT Absorption law

$$\Leftrightarrow p \wedge q //$$

$$p \vee (p \wedge q) \Leftrightarrow p.$$

(Q) prove the following logical equivalences:

$$(i) [(p \vee q) \wedge (p \vee \sim q)] \vee q \Leftrightarrow p \vee q$$

$$\text{soln: } [(p \vee q) \wedge (p \vee \sim q)] \vee q.$$

$$\Leftrightarrow [p \vee (q \wedge \sim q)] \vee q. \quad \begin{array}{l} \text{WIKT distributive law} \\ p \vee (q_1 \wedge q_2) \Leftrightarrow (p \vee q_1) \wedge (p \vee q_2) \end{array}$$

$$\Leftrightarrow [p \vee F_0] \vee q$$

$$\Leftrightarrow p \vee q //$$

$$(ii) (p \rightarrow q) \wedge [\sim q \wedge (\sim q \vee \sim q)] \Leftrightarrow \sim(q \vee p)$$

$$\text{soln: } (p \rightarrow q) \wedge [\sim q \wedge (\sim q \vee \sim q)]$$

$$\Leftrightarrow (p \rightarrow q) \wedge [\sim q \wedge (\sim q \vee \sim q)] \quad \text{by commutative}$$

$$\Leftrightarrow (p \rightarrow q) \wedge [\sim q] \quad \therefore \text{by Absorption law.}$$

$$\Leftrightarrow (\sim p \vee q) \wedge \sim q,$$

$$\Leftrightarrow (\sim p \wedge \sim q) \vee (q \wedge \sim q)$$

$$\Leftrightarrow \sim(p \vee q) \vee F_0$$

$$\Leftrightarrow \sim(p \vee q)$$

$$\Leftrightarrow \sim(q \vee p) //$$

$$(iii) p \rightarrow (q_1 \rightarrow \gamma) \Leftrightarrow (p \wedge q_1) \rightarrow \gamma$$

$$\begin{aligned} :- \quad p \rightarrow (q_1 \rightarrow \gamma) &\Leftrightarrow p \rightarrow (\neg q_1 \vee \gamma) \\ &\Leftrightarrow \neg p \vee (\neg q_1 \vee \gamma) \\ &\Leftrightarrow (\neg p \vee \neg q_1) \vee \gamma \\ &\Leftrightarrow \neg(p \wedge q_1) \vee \gamma \\ &\Leftrightarrow (p \wedge q_1) \rightarrow \gamma // \end{aligned}$$

$$(iv) [\neg p \wedge (\neg q_1 \wedge \gamma)] \vee (q_1 \wedge \gamma) \vee (p \wedge \gamma) \Leftrightarrow \gamma$$

$$\begin{aligned} \text{soln:- } [\neg p \wedge (\neg q_1 \wedge \gamma)] \vee (q_1 \wedge \gamma) \vee (p \wedge \gamma) \\ &\Leftrightarrow [(\neg p \wedge \neg q_1) \wedge \gamma] \vee (q_1 \wedge \neg \gamma) \vee (p \wedge \gamma) \\ &\Leftrightarrow [\neg(p \vee q_1) \wedge \gamma] \vee (q_1 \wedge \neg \gamma) \vee (p \wedge \gamma) \\ &\Leftrightarrow [\neg(p \vee q_1) \wedge \gamma] \vee [(q_1 \vee p) \wedge \gamma] \\ &\Leftrightarrow [\neg(p \vee q_1) \vee (q_1 \vee p)] \wedge \gamma \\ &\Leftrightarrow [\neg(p \vee q_1) \vee (p \vee q_1)] \wedge \gamma \\ &\Leftrightarrow [T_0] \wedge \gamma \\ &\Leftrightarrow \gamma // \end{aligned}$$

$$(v). \neg\{\{(p \vee q_1) \wedge \gamma\} \rightarrow \neg q_1\} \Leftrightarrow \neg\{\neg\{[(p \vee q_1) \wedge \gamma] \vee \neg q_1\}\}$$

$$\Leftrightarrow \neg q_1 \wedge \gamma$$

$$\begin{aligned} \text{soln:- } \neg\{\{(p \vee q_1) \wedge \gamma\} \rightarrow \neg q_1\} &\quad \text{by demorgan's law} \\ &\Leftrightarrow \neg\{\neg\{[(p \vee q_1) \wedge \gamma] \vee \neg q_1\}\} \quad (\text{by double negation}) \\ &\Leftrightarrow \neg\{[(p \vee q_1) \wedge \gamma] \wedge \neg \neg q_1\} \\ &\Leftrightarrow \neg\{[(p \vee q_1) \wedge \gamma] \wedge q_1\} \end{aligned}$$

$$\Leftrightarrow [P \vee (Q \wedge R)] \wedge Q$$

$$\Leftrightarrow [(P \vee Q) \wedge \tau] \wedge Q$$

$$\Leftrightarrow [(P \vee Q) \wedge (\tau \wedge Q)] \quad (\text{Associative law})$$

$$\Leftrightarrow (P \vee Q) \wedge (Q \wedge \tau) \quad \text{commutative law.}$$

$$\Leftrightarrow [(P \vee Q) \wedge Q] \wedge \tau \quad \text{Absorption law.}$$

$$\Leftrightarrow Q \wedge \tau //$$

(3) prove that

$$[(P \vee Q) \wedge \sim \{\sim P \wedge (\sim Q \vee \sim R)\}] \vee (\sim P \wedge \sim Q) \vee (\sim P \wedge \sim R)$$

tautology

Soln:- Let ω denote the given proposition,

$$\omega = [(P \vee Q) \wedge \sim \{\sim P \wedge (\sim Q \vee \sim R)\}] \vee (\sim P \wedge \sim Q) \vee (\sim P \wedge \sim R)$$

$$\omega = u \vee v \vee w$$

$$\text{Now } u \Leftrightarrow [(P \vee Q) \wedge \sim \{\sim P \wedge (\sim Q \vee \sim R)\}]$$

$$\Leftrightarrow [(P \vee Q) \wedge \sim \{\sim P \wedge \sim(Q \wedge R)\}] \quad \text{by demorgan's}$$

$$\Leftrightarrow [(P \vee Q) \wedge \sim \{\sim(P \vee (Q \wedge R))\}] \quad \text{Again demorgan's}$$

$$\Leftrightarrow [(P \vee Q) \wedge (P \vee (Q \wedge R))] \quad \text{by double negation}$$

$$\Leftrightarrow P \vee \{Q \wedge (Q \wedge R)\} \quad \text{by}$$

$$\Leftrightarrow P \vee \{(Q \wedge Q) \wedge R\}$$

$$\Leftrightarrow P \vee (Q \wedge R)$$

$$w \Leftrightarrow \sim(p \vee q) \vee \sim(p \vee r)$$

$$\Leftrightarrow \sim\{(p \vee q) \wedge (p \vee r)\} \quad \text{by demorgan's law.}$$

$$\Leftrightarrow \sim\{p \vee (q \wedge r)\}$$

$$\text{Now } (1) \Rightarrow w \Leftrightarrow p \vee \sim$$

$$\Leftrightarrow \{p \vee (q \wedge r)\} \vee \{\sim(p \vee (q \wedge r))\}$$

$$\Leftrightarrow T_0 //$$

Thus the given compound proposition is a tautology.

(*) Simplify the following compound proposition using the laws of logic.

$$(a) (p \vee q) \wedge [\sim\{(\sim p \wedge q)\}]$$

$$\Leftrightarrow (p \vee q) \wedge [\sim \sim p \vee \sim q] \quad \text{by demorgan's law}$$

$$\Leftrightarrow (p \vee q) \wedge [p \vee \sim q]$$

$$\Leftrightarrow p \vee (q \wedge \sim q) \quad (\text{by distributive law})$$

$$\Leftrightarrow p \vee F_0 \quad (\text{by inverse law})$$

$$\Leftrightarrow P // \quad (\text{by identity law})$$

$$(b) \sim[\sim\{(p \vee q) \wedge r\} \vee \sim q]$$

$$\Leftrightarrow \sim[\cancel{\sim\{(p \vee q) \wedge r\}}] \wedge \cancel{\sim q}$$

$$\sim \sim [(p \vee q) \wedge r] \wedge \sim q$$

demorgan's law

$$\Leftrightarrow [(p \vee q) \wedge r] \wedge \sim q$$

$$\Leftrightarrow (p \vee q) \wedge (\neg r \wedge q) \quad \text{distributive law.}$$

$$\Leftrightarrow (p \vee q) \wedge (q \wedge \neg r) \quad \text{by commutative law}$$

$$\Leftrightarrow [(p \vee q) \wedge q] \wedge \neg r \quad \text{Associative law}$$

$$\Leftrightarrow q \wedge \neg r // \quad \text{by Absorption law.}$$

(c) Let x be a specified no. write down the negation of the following conditional

"If x is an integer, then x is a rational no".

Soln:- It is of the form $p \rightarrow q$

where, p : x is an integer

q : x is a rational number.

$$\sim(p \rightarrow q) \Leftrightarrow \sim(\sim p \vee q)$$

$$\Leftrightarrow \sim \sim p \wedge \sim q \quad \text{demorgan's law}$$

$$\Leftrightarrow p \wedge \sim q$$

i.e " x is an integer and x is not a rational no".

(d) Let x be a specified no write down the negation of the following proposition:

"If x is not a real no, then it is not

soln:- " If x is not a real no, then it is not a rational no and not an irrational no".

Soln:- p : x is a real number

q : x is a rational number

r : x is a irrational no

The given statement is of the form,

$$\sim p \rightarrow (\sim q \wedge \sim r)$$

$$\text{Now, } \sim [\sim p \rightarrow (\sim q \wedge \sim r)]$$

$$\Leftrightarrow \sim [\sim p \rightarrow \{\sim (q \vee r)\}]$$

$$\Leftrightarrow \sim [\sim \sim p \vee \sim (q \vee r)]$$

$$\Leftrightarrow \sim (p \vee \sim (q \vee r)) \text{ demorgan's law}$$

$$\Leftrightarrow \sim p \wedge \sim \sim (q \vee r)$$

$$\Leftrightarrow \sim p \wedge (q \vee r)$$

$\therefore x$ is not a real no and it is a rational no or irrational no.

(*) Duality

Suppose ' u ' is a compound proposition and its duality is obtained by replacing

- (i) each \wedge and \vee by \wedge and \wedge respectively
- (ii) each T_0 & F_0 by F_0 & T_0 respectively

And it is denoted by u^d .

$$\text{ex:- } u: p \wedge (q \vee \neg r) \vee (s \wedge T_0)$$

$$u^d: p \vee (q \wedge \neg r) \wedge (s \vee F_0)$$

(*) Converse, Inverse and contrapositive:

consider a conditional $p \rightarrow q$ then

- (1) $q \vee \neg p$ is called the converse of $p \rightarrow q$
- (2) $\neg p \rightarrow \neg q$ is called the inverse of $p \rightarrow q$
- (3) $\neg q \rightarrow \neg p$ is called the contrapositive of $p \rightarrow q$.

(*) RULE OF INFERENCE:

consider a set of propositions $P_1, P_2, P_3, \dots, P_n$ and a proposition Q . Then a compound proposition of the form $(P_1 \wedge P_2 \wedge P_3 \wedge \dots \wedge P_n) \rightarrow Q$ is called a argument. Here P_1, P_2, \dots, P_n are called the premises of the argument and Q is called a conclusion of the argument.

If it is represented in the form of tabular form, i.e

$$\begin{array}{c} P_1 \\ P_2 \\ P_3 \\ \vdots \\ \vdots \\ P_m \\ \hline \therefore Q \end{array}$$

\Rightarrow The preceding argument is said to be valid if whenever each of the premises P_1, P_2, \dots, P_m is true, then the conclusion Q is likewise true.

In other words, the argument

$$(P_1 \wedge P_2 \wedge P_3 \wedge \dots \wedge P_m) \rightarrow Q \text{ is valid when}$$

$$(P_1 \wedge P_2 \wedge P_3 \wedge \dots \wedge P_m) \Rightarrow Q$$

For finding the validity of Argument use the Rules of Logic and these rules are called the Rules of Inference.

(1) Rule of conjunctive simplification.

For any two propositions p and q , if $p \wedge q$ is true then p is true

$$\text{i.e } p \wedge q \Rightarrow p$$

2) Rule of Disjunctive Amplification

for any two propositions p and q_V , if p is true
then $p \vee q_V$ is true i.e

$$p \Rightarrow p \vee q_V$$

3) Rule of Syllogism:

for any three propositions, p, q_V, τ
if $p \rightarrow q_V$ is true and $q_V \rightarrow \tau$ is true then $p \rightarrow \tau$
is true

$$\text{i.e } \{(p \rightarrow q_V) \wedge (q_V \rightarrow \tau)\} \Rightarrow p \rightarrow \tau$$

In tabular form

$$\begin{array}{c} p \rightarrow q_V \\ q_V \rightarrow \tau \\ \hline \therefore p \rightarrow \tau \end{array}$$

4) Modus Ponens

This rule states that if p is true and $p \rightarrow q_V$
is true, then q_V is true

$$\text{i.e } \{p \wedge (p \rightarrow q_V)\} \Rightarrow q_V$$

In tabular form

$$\begin{array}{c} p \\ p \rightarrow q_V \\ \hline \therefore q_V \end{array}$$

5) Modus Tollens :

This rule states that if $p \rightarrow q$ is true and q_V is false, then p is false

i.e. $\{(p \rightarrow q) \wedge \sim q\} \Rightarrow \sim p$

In tabular form,

$$\begin{array}{c} p \rightarrow q \\ \sim q \\ \hline \therefore \sim p \end{array}$$

6) Rule of Disjunctive Syllogism :

This rule states that if $p \vee q_V$ is true and p is false, then q_V is true

i.e. $[(p \vee q_V) \wedge \sim p] \Rightarrow q_V$

In tabular form,

$$\begin{array}{c} p \vee q_V \\ \sim p \\ \hline \therefore q_V \end{array}$$

PROBLEMS: Test whether the following are valid argument.

(1) If Sachin hits a century, then he gets a free car.

Sachin hits a century

∴ Sachin gets a free car.

Soln:- Let p : Sachin hits a century
 q : Sachin gets a free car.

The given argument is of the form

$$\frac{p \rightarrow q}{\therefore q}$$

$$[(p \rightarrow q) \wedge p] \Rightarrow q$$

$$\Leftrightarrow [p \wedge (p \rightarrow q)] \Rightarrow q \quad (\text{By Modus ponens})$$

This is a valid argument.

(2) If Sachin hits a century, he gets a free car

Sachin does not get a free car

∴ Sachin has not hit a century

Soln:- Let p : Sachin hits a century
 q : Sachin gets a free car

The given argument is of the form

$$\begin{array}{c} p \rightarrow q \\ \hline \therefore \neg p \\ \hline \neg q \end{array}$$

$$[(p \rightarrow q) \wedge \neg q] \Rightarrow \neg p \quad (\text{by Modus Tollens})$$

This is a valid argument.

(3) If Sachin hits a century, he gets a free car.

Sachin gets a free car

∴ Sachin has hit a century.

Soln:- Let p : Sachin hits a century

q : Sachin gets a free car.

The given argument is

$$\begin{array}{c} p \rightarrow q \\ \hline \neg p \\ \hline q \end{array}$$

$$[(p \rightarrow q) \wedge q] \Rightarrow p$$

$$\Leftrightarrow [(\neg p \vee q) \wedge q] \cancel{\Rightarrow} p$$

$$\Leftrightarrow q \quad (\text{absorption law})$$

∴ The given argument is not a valid.

(A) If I drive to work, then I will arrive tired
I am not tired (when I arrive at work)
 \therefore I do not drive to work.

Soln:- Let p : I drive to work
 q : I arrive tired.

The given argument is of the form

$$\begin{array}{c} p \rightarrow q \\ \hline \neg q \\ \therefore \neg p \end{array}$$

$$[(p \rightarrow q) \wedge \neg q] \Rightarrow \neg p \quad [\text{by Modus Tollens}]$$

It is a valid argument.

(5) I will become famous (\Rightarrow) I will not become a Musician

I will become a musician

\therefore I will become famous.

Soln:- Let p : I will become famous
 q : I will become a musician

The given argument is

$$\begin{array}{c} p \vee \neg q \\ \hline q \\ \therefore p \end{array}$$

$$[(p \vee \neg q) \wedge q] \Rightarrow p$$

$$\Leftarrow (q \rightarrow p) \wedge q$$

$$\Leftarrow q \wedge (q \rightarrow p) \quad \cancel{\text{Modus ponens rule}}$$

$$\Rightarrow p$$

WKT by

Modus ponens rule

$$p \wedge (p \rightarrow q) \Rightarrow q$$

Hence it is a valid argument.

(6) If I study, then I do not fail in the examination

If I do not fail in the examination, my father gifts a two-wheeler to me

\therefore if I study ~~then~~ my father gifts a two-wheeler to me.

Soln:- Let p : I study

q : I do not fail in the examination

r : My father gifts a two-wheeler to me.

The given argument reads $p \rightarrow q$

$$\frac{q \rightarrow r}{\therefore p \rightarrow r}$$

$$[(p \rightarrow q) \wedge (q \rightarrow r)] \Rightarrow p \rightarrow r \quad (\text{by Rule of Syllogism})$$

It is a valid argument.

(7) If Ravi goes out with friends, he will not study
If Ravi does not study, his father becomes angry
His father is not angry

∴ Ravi has not gone out with friends.

Soln:- Ref p: Ravi goes out with friends

q: Ravi does not study

r: His father becomes angry

The given argument is

$$\begin{array}{c} p \rightarrow q \\ q \rightarrow r \\ \hline \therefore \neg p \end{array}$$

$$[(p \rightarrow q) \wedge (q \rightarrow r) \wedge \neg r] \Rightarrow \neg p$$

$$\text{Now, } [(\underline{p \rightarrow q}) \wedge (\underline{q \rightarrow r}) \wedge \neg r] \quad (\text{Rule of syllogism})$$

$$\Rightarrow [(p \rightarrow r) \wedge \neg r] \quad (\text{by Rule of Modus tollens})$$

$$\Rightarrow \neg p$$

(8) If I study, I will not fail in the examination

If I do not watch TV in the evenings, I will study

I failed in the examination

∴ I must have watched TV in the evenings.

Soln:- Let p : I study

q : I fail in the examination

r : I watch TV in the evenings.

The given argument is

$$p \rightarrow \sim q,$$

$$\sim r \rightarrow p$$

$$\frac{q}{\therefore r}$$

$$[(p \rightarrow \sim q) \wedge (\sim r \rightarrow p) \wedge q] \Rightarrow r$$

$$\Rightarrow [(\sim r \rightarrow p) \wedge (p \rightarrow \sim q) \wedge q] \quad \text{Rule of syllogism}$$

$$\Rightarrow [(\sim r \rightarrow \sim q) \wedge q] \quad \text{Modus tollens} \\ [(p \rightarrow q) \wedge \sim q \Rightarrow \sim p]$$

$$\Rightarrow \sim(\sim r)$$

$$\Rightarrow r$$

It's a valid argument.

(q) I will get grade A in this course or I will not graduate

If do not graduate, I will join the army

\therefore I will not join the army.

Soln:- Let p : I get grade A in this course

q : I do not graduate

γ : I join the army

Then the given argument is

$$P \vee Q$$

$$Q \rightarrow \gamma$$

$$\frac{P}{\therefore \sim \gamma}$$

$$[(P \vee Q) \wedge (Q \rightarrow \gamma) \wedge P]$$

$$\Leftrightarrow [(\sim \sim P \vee Q) \wedge (Q \rightarrow \gamma) \wedge P]$$

Law for conditional

$$\Leftrightarrow [(\sim P \rightarrow Q) \wedge (Q \rightarrow \gamma) \wedge P]$$

Rule of syllogism.

$$\Leftrightarrow [(\sim P \rightarrow \gamma) \wedge P] \quad \text{there is no rule}$$

$$\Leftrightarrow [(\sim \sim P \vee \gamma) \wedge P]$$

$$\Leftrightarrow [(P \vee \gamma) \wedge P] \quad \text{by Absorption law}$$

$$\Rightarrow \gamma //$$

It is not a valid.

(10) If I have talent and work hard, then I will become successful in life

If I become successful in life, then I will be happy

\therefore If I will not be happy, then I did not work hard
 \therefore If I will not be happy, then I do not have talent.

Soln:- Let P : I have talent

Q : I will work hard

γ : I will become successful in life

8: I will be happy

The given argument is $(p \wedge q) \rightarrow r$
 $r \rightarrow s$
 $\therefore \sim s \rightarrow (\sim q \vee \sim p)$

$$[((p \wedge q) \rightarrow r) \wedge (r \rightarrow s)] \Rightarrow \sim s \rightarrow (\sim q \vee \sim p)$$

Now, $((p \wedge q) \rightarrow r) \wedge (r \rightarrow s)$ Rule of Syllogism

$$\Rightarrow (p \wedge q) \rightarrow s$$

$$\Rightarrow \sim s \rightarrow \sim(p \wedge q) \quad (\text{demorgan's})$$

$$\Rightarrow \sim s \rightarrow \sim p \vee \sim q$$

$$\Rightarrow \sim s \rightarrow \sim q \vee \sim p$$

It is valid argument.

(ii) If Ravi studies, then he will pass in Discrete Maths paper

If Ravi does not play cricket, then he will study
Ravi failed in Discrete Maths paper

\therefore Ravi played cricket

Soln:- p: Ravi studies

q: Ravi will pass in Discrete Maths paper

r: Ravi play cricket.

Given argument, $p \rightarrow q$

$$\sim r \rightarrow p$$

$$\frac{\sim q}{\therefore r}$$

i.e. $\{ (p \rightarrow q_1) \wedge (\neg r \rightarrow p) \wedge \neg q_1 \} \Rightarrow r$

Now, $\{ (p \rightarrow q_1) \wedge (\neg r \rightarrow p) \wedge \neg q_1 \}$

$$\Leftrightarrow \{ (\neg r \rightarrow p) \wedge (p \rightarrow q_1) \wedge \neg q_1 \} \quad \text{commutative law}$$

$$\Rightarrow \{ (\neg r \rightarrow q_1) \wedge \neg q_1 \} \quad (\text{syllogism})$$

$$\Rightarrow \neg(\neg r) \quad (\text{by Modus tollens})$$

$$\Rightarrow r$$

It is a valid argument.

(2) P

$$P \rightarrow \sim q_V$$

$$\begin{array}{c} \sim q_V \rightarrow \sim \gamma \\ \hline \therefore \sim \gamma \end{array}$$

soln: $P \wedge (\underline{P \rightarrow \sim q_V}) \wedge (\underline{\sim q_V \rightarrow \sim \gamma})$

$$\Rightarrow P \wedge (P \rightarrow \sim \gamma) \quad (\text{syllogism})$$

$$\Rightarrow \sim \gamma // \quad \text{by Modus ponens}$$

(3) $P \rightarrow \gamma$

$$q_V \rightarrow \gamma$$

$$\underline{(P \vee q_V) \rightarrow \gamma}$$

soln:- $(P \rightarrow \gamma) \wedge (q_V \rightarrow \gamma) \Rightarrow (P \vee q_V) \rightarrow \gamma$

$$\Leftrightarrow (\sim P \vee \gamma) \wedge (\sim q_V \vee \gamma) \quad \text{distributive law}$$

$$\Leftrightarrow (\sim P \wedge \sim q_V) \vee \gamma$$

$$\Leftrightarrow \sim (P \vee q_V) \vee \gamma \quad \text{de Morgan's law}$$

$$\Rightarrow (P \vee q_V) \rightarrow \gamma // \quad \text{conditional law.}$$

(4)

$$P \rightarrow q_V$$

$$\gamma \rightarrow \beta$$

$$P \vee \gamma$$

$$\underline{\therefore q_V \vee \beta}$$

soln:- $(P \rightarrow q_V) \wedge (\gamma \rightarrow \beta) \wedge (P \vee \gamma)$

$$\Leftrightarrow (P \rightarrow q_V) \wedge (\gamma \rightarrow \beta) \wedge (\sim \sim P \vee \gamma)$$

$$\Leftrightarrow (P \rightarrow q_V) \wedge (\gamma \rightarrow \beta) \wedge (\sim P \rightarrow \gamma)$$

$$\Leftrightarrow (p \rightarrow q) \wedge (\neg p \rightarrow r) \wedge (r \rightarrow s)$$

$$\Leftrightarrow (p \rightarrow q) \wedge (\neg p \rightarrow s) \quad \begin{array}{l} \text{(using contrapositive} \\ p \rightarrow q \Leftrightarrow \neg q \rightarrow \neg p \end{array}$$

$$\Leftrightarrow (\neg q \rightarrow \neg p) \wedge (\neg p \rightarrow s) \quad \text{(syllogism)}$$

$$\Leftrightarrow (\neg q \rightarrow s)$$

$$\Rightarrow q \vee s //$$

If is valid

(5) $p \rightarrow q$

$$r \rightarrow s$$

$$\neg q \vee \neg s$$

$$\therefore \neg(p \wedge r)$$

Soln:- $(p \rightarrow q) \wedge (r \rightarrow s) \wedge (\underline{\neg q \vee \neg s})$ law for cond'n

$$\Leftrightarrow (p \rightarrow q) \wedge (r \rightarrow s) \wedge (q \rightarrow \neg s)$$

by commutative

$$\Leftrightarrow \underline{(p \rightarrow q) \wedge (q \rightarrow \neg s)} \wedge (r \rightarrow s) \quad \text{law}$$

$$\Rightarrow (p \rightarrow \neg s) \wedge (\underline{r \rightarrow s}) \quad \text{by contrapositive}$$

$$\Rightarrow (p \rightarrow \neg s) \wedge (\neg s \rightarrow \neg r)$$

(by syllogism)

$$\Rightarrow (p \rightarrow \neg r)$$

$$\Rightarrow \neg p \vee \neg r$$

$$\Rightarrow \neg(p \wedge r) //$$

If is a valid argument.

$$(6) \quad p \rightarrow \tau$$

$$\sim p \rightarrow q_V$$

$$\frac{q_V \rightarrow s}{\therefore \sim \tau \rightarrow s}$$

Soln:- $(p \rightarrow \tau) \wedge (\sim p \rightarrow q_V) \wedge (q_V \rightarrow s)$ Syllogism

$$\Rightarrow \underline{(p \rightarrow \tau) \wedge (\sim p \rightarrow s)} \quad \text{by contrapositive}$$

$$\Rightarrow (\sim \tau \rightarrow \sim p) \wedge (\sim p \rightarrow s)$$

$$\Rightarrow \sim \tau \rightarrow s // \quad \text{by Syllogism}$$

If is valid.

$$(7) \quad (\sim p \vee \sim q_V) \rightarrow (\tau \wedge s)$$

$$\tau \rightarrow t$$

$$\sim t$$

$$\therefore p$$

Soln:- $((\sim p \vee \sim q_V) \rightarrow (\tau \wedge s)) \wedge \underline{(\tau \rightarrow t) \wedge \sim t}$ by MT

$$\Rightarrow [(\sim p \vee \sim q_V) \rightarrow (\tau \wedge s)] \wedge \sim \tau$$

$$\Rightarrow [(\sim p \vee \sim q_V) \rightarrow (\tau \wedge s)] \wedge (\sim \tau \vee \sim s) \quad (\text{by disjunction})$$

$$\Leftrightarrow [(\sim p \vee \sim q_V) \rightarrow (\tau \wedge s)] \wedge \sim(\tau \wedge s) \quad \text{demorgan}$$

$$\Rightarrow \sim(\sim p \vee \sim q_V) \quad \text{by MIT}$$

$$\Rightarrow \sim(\sim(p \wedge q_V))$$

$$\Rightarrow p \wedge q_V \Rightarrow p \quad (\text{by conjunction})$$

$$(8) p \rightarrow (q \rightarrow r)$$

$$\sim q \rightarrow \sim p$$

$$\frac{p}{\therefore r}$$

Soln:- $\underbrace{(p \rightarrow (q \rightarrow r)) \wedge (\sim q \rightarrow \sim p)}_{\rightarrow (p \rightarrow (q \rightarrow r)) \wedge p} \wedge p$ by commutative

$$\Rightarrow \underbrace{(p \rightarrow (q \rightarrow r)) \wedge p}_{\Rightarrow (p \rightarrow (q \rightarrow r)) \wedge p} \wedge (\sim q \rightarrow \sim p) \text{ by Modus ponens.}$$

$$\Rightarrow (q \rightarrow r) \wedge \underbrace{(\sim q \rightarrow \sim p)}_{\Rightarrow (q \rightarrow r) \wedge (\sim q \rightarrow \sim p)} \text{ by contrapositive}$$

$$\Rightarrow (q \rightarrow r) \wedge (p \rightarrow q) \text{ by commutative law}$$

$$\Rightarrow (p \rightarrow q) \wedge (q \rightarrow r) \text{ by Syllogism}$$

$$\Rightarrow p \rightarrow r$$

$$\Rightarrow r \quad (\because p \text{ is true then } r \text{ is also true})$$

$$(9) \sim p \leftrightarrow q$$

$$q \rightarrow r$$

$$\sim r$$

$$\therefore p$$

Soln:- $\underbrace{(\sim p \leftrightarrow q)}_{\Rightarrow (\sim p \rightarrow q) \wedge (q \rightarrow \sim p)} \wedge (q \rightarrow r) \wedge \sim r$ (WKT: $p \leftrightarrow q \Leftrightarrow (p \rightarrow q) \wedge (q \rightarrow p)$)

$$\Rightarrow (\sim p \rightarrow q) \wedge (q \rightarrow \sim p) \wedge \underbrace{(q \rightarrow r) \wedge \sim r}_{\Rightarrow (\sim p \rightarrow q) \wedge (q \rightarrow \sim p) \wedge \sim q} \text{ by MT}$$

$$\Rightarrow (\sim p \rightarrow q) \wedge (q \rightarrow \sim p) \wedge \underbrace{\sim q}_{\Rightarrow (\sim p \rightarrow q) \wedge (\sim q \rightarrow \sim p) \wedge (q \rightarrow \sim p)} \text{ commutative law}$$

$$\Rightarrow \underbrace{[(\sim p \rightarrow q) \wedge \sim q]}_{\Rightarrow \sim (\sim p) \wedge (q \rightarrow \sim p)} \wedge (q \rightarrow \sim p) \text{ by M.T}$$

$$\Rightarrow \sim (\sim p) \wedge (q \rightarrow \sim p)$$

$$\Rightarrow p \wedge (q \rightarrow \sim p) \Rightarrow (q \rightarrow \sim p) \wedge p \Rightarrow p$$

If it is valid.

$$(70) \quad p \rightarrow (q \rightarrow r)$$

$$p \vee \sim q$$

$$q \vee$$

$$\therefore s \rightarrow r$$

$$\text{SOLN: } (p \rightarrow (q \rightarrow r)) \wedge (\underline{p \vee \sim q}) \wedge q / \text{ commutative law}$$

$$\Rightarrow (p \rightarrow (q \rightarrow r)) \wedge (\underline{\sim q \vee p}) \wedge q / \text{ conditional law}$$

$$\Rightarrow (\underline{p \rightarrow (q \rightarrow r)}) \wedge (\underline{s \rightarrow p}) \wedge q / \text{ commutative law}$$

$$\Rightarrow (\underline{s \rightarrow p}) \wedge (\underline{p \rightarrow (q \rightarrow r)}) \wedge q / \text{ syllogism}$$

$$\Rightarrow (\underline{s \rightarrow (q \rightarrow r)}) \wedge q / \text{ law for cond'n}$$

$$\Leftrightarrow (\sim s \vee (q \rightarrow r)) \wedge q / \text{ distributive law}$$

$$\Leftrightarrow (\sim s \wedge q) \vee (\underline{(q \rightarrow r) \wedge q}) / \text{ by M.P}$$

$$\Leftrightarrow (\underline{\sim s \wedge q}) \vee r / \text{ by conjunctive simplification}$$

$$\Leftrightarrow \sim s \vee r$$

$$\Leftrightarrow s \rightarrow r //$$

It is valid.

$$(11) \ p \rightarrow (q \vee \neg r)$$

$$r \rightarrow s$$

$$\neg(q \vee \neg s)$$

$$\therefore \neg p$$

$$\text{SOLN:- } [p \rightarrow (q \vee \neg r)] \wedge (r \rightarrow s) \wedge \underline{\neg(q \vee \neg s)} \text{ demorgan's law}$$

$$\Leftrightarrow [p \rightarrow (q \vee \neg r)] \wedge (r \rightarrow s) \wedge \underline{\neg q \vee \neg \neg s} \text{ by conditional law}$$

$$\Leftrightarrow [p \rightarrow (q \vee \neg r)] \wedge (r \rightarrow s) \wedge \underline{q \rightarrow \neg s} \text{ by contrapositive}$$

$$\Leftrightarrow [p \rightarrow (q \vee \neg r)] \wedge \underline{r \rightarrow s \wedge s \rightarrow \neg q} \text{ syllogism}$$

$$\Leftrightarrow [p \rightarrow (q \vee \neg r)] \wedge \underline{r \rightarrow \neg q} \text{ by conditional law}$$

$$\Leftrightarrow [p \rightarrow (q \vee \neg r)] \wedge \underline{\neg r \vee \neg q} \text{ demorgan's law}$$

$$\Leftrightarrow [p \rightarrow (q \vee \neg r)] \wedge \underline{\neg r \vee \neg q} \text{ commutative law}$$

$$\Leftrightarrow \underline{[p \rightarrow (q \vee \neg r)] \wedge \neg(r \vee \neg q)} \text{ by MT}$$

$$\Rightarrow \neg p //$$

It is valid.

(12)

$$\neg(p \vee q) \rightarrow r$$

$$r \rightarrow (s \vee t)$$

$$\neg s \wedge \neg t$$

$$\frac{\neg u \rightarrow \neg t}{\therefore p}$$

$$\underline{\text{Soln:-}} \quad \frac{[(\sim p \vee q) \rightarrow r] \wedge [r \rightarrow (s \vee t)] \wedge [\sim s \wedge \sim u]}{\sim u \rightarrow r} \quad \begin{array}{l} \text{Syllogism} \\ \text{Associative} \end{array}$$

$$\Rightarrow [(\sim p \vee q) \rightarrow (s \vee t)] \wedge \sim s \wedge [\sim u \wedge (\sim u \rightarrow r)] \quad \text{by M.P}$$

$$\Rightarrow [(\sim p \vee q) \rightarrow (s \vee t)] \wedge (\sim s \wedge \sim t) \quad \text{demorgan's law}$$

$$\Rightarrow [(\sim p \vee q) \rightarrow (s \vee t)] \wedge \sim(s \vee t) \quad \text{by M.T}$$

$$\Rightarrow \sim(\sim p \vee q) \quad \text{demorgan's}$$

$$\Rightarrow \sim \sim p \wedge \sim q$$

$$\Rightarrow p \wedge \sim q \quad \text{by conjunctive simplification.}$$

$$\Rightarrow p //$$

If is valid.

$$\begin{array}{c} \text{Q3)} \\ \frac{p \vee q}{\sim p \vee r} \end{array}$$

$$\frac{\sim r}{\therefore q}$$

$$\underline{\text{Soln:-}} \quad (p \vee q) \wedge \underline{(\sim p \vee r)} \wedge (\sim r) \quad \text{conditional law}$$

$$\Rightarrow (p \vee q) \wedge \underline{(p \rightarrow r) \wedge \sim r} \quad \text{by M.T}$$

$$\Rightarrow \underline{(\sim p \vee q) \wedge \sim p}$$

$$\Rightarrow (\sim p \rightarrow q) \wedge \sim p \quad \text{by M.P}$$

$\Rightarrow q //$ valid argument.

$$(14) \quad p \rightarrow r$$

$$r \rightarrow s$$

$$t \vee \sim s$$

$$\sim t \vee u$$

$$\sim u$$

$$\therefore \sim p$$

$$\underline{\text{Soln:-}} \quad \underbrace{(p \rightarrow r) \wedge (r \rightarrow s) \wedge (t \vee \sim s) \wedge (\sim t \vee u)}_{\text{syllogism}} \wedge \underbrace{(\sim t \vee u)}_{\substack{\text{commutative} \\ \text{and conditional}}} \wedge \underbrace{\sim u}_{\text{conditional}}$$

$$\Rightarrow \underbrace{(p \rightarrow s) \wedge (s \rightarrow t) \wedge (t \rightarrow u)}_{\text{syllogism}} \wedge \sim u$$

$$\Rightarrow \underbrace{(p \rightarrow s) \wedge (s \rightarrow u)}_{\text{syllogism}} \wedge \sim u$$

$$\Rightarrow (p \rightarrow u) \wedge \sim u \quad \text{by MT}$$

$$\Rightarrow \sim p // \quad \text{It is valid}$$

$$(15) \quad p \rightarrow q$$

$$q \rightarrow (\sim s \wedge s)$$

$$\sim s \nrightarrow (\sim t \vee u)$$

$$p \wedge t$$

$$\therefore u$$

$$\underline{\text{Soln:-}} \quad \underbrace{(p \rightarrow q) \wedge (q \rightarrow (\sim s \wedge s)) \wedge (\sim s \vee (\sim t \vee u)) \wedge (p \wedge t)}_{\text{syllogism}} \wedge (\sim s \wedge s)$$

$$\Rightarrow [p \rightarrow (\sim s \wedge s)] \wedge [\sim s \vee (\sim t \vee u)] \wedge (p \wedge t)$$

$$\Rightarrow [p \rightarrow (\gamma \wedge s)] \wedge [\underbrace{(\neg \gamma \vee \neg t) \vee u}_{\text{demorgan's}}] \wedge (p \wedge t)$$

commutative

$$\Rightarrow [p \rightarrow (\gamma \wedge s)] \wedge (p \wedge t) \wedge [\underbrace{\neg(\gamma \wedge t) \vee u}_{\text{conditional}}]$$

$$\Rightarrow [p \rightarrow (\gamma \wedge s)] \wedge (p \wedge t) \wedge [(\gamma \wedge t) \rightarrow u]$$

$$\Rightarrow \underbrace{(p \rightarrow (\gamma \wedge s)) \wedge p}_{\text{by M.P.}} \wedge t \wedge [(\gamma \wedge t) \rightarrow u]$$

$$\Rightarrow \underbrace{(\gamma \wedge s) \wedge t}_{\text{commutative}} \wedge [(\gamma \wedge t) \rightarrow u]$$

$$\Rightarrow \underbrace{(\gamma \wedge t) \wedge s}_{\text{commutative}} \wedge [(\gamma \wedge t) \rightarrow u]$$

$$\Rightarrow \underbrace{(\gamma \wedge t) \wedge [f \wedge t \rightarrow u]}_{\text{by M.P.}} \wedge s$$

$$\Rightarrow u \wedge s \quad \text{by conjunctive \underline{simplic.}}$$

$$\Rightarrow u$$

Tay

$$(1) \quad p \qquad (2) \quad c \vee d$$

$$p \rightarrow q$$

$$(c \vee d) \rightarrow \neg h$$

$$s \vee \neg r$$

$$\neg h \rightarrow (A \wedge \neg B)$$

$$\neg r \rightarrow \neg q /$$

$$(A \wedge \neg B) \rightarrow R \vee s$$

$$\therefore s$$

$$\therefore R \vee s.$$

OPEN STATEMENTS ; QUANTIFIERS STATEMENTS

(*) Open statements

The declarative sentences like $x+2=3$, $x=\sqrt{2}$, $x > 0$, x divisible by 2, in these sentences 'x' is not defined, such statements are called as open statement and 'x' is called the free variables.

- This open statement can be converted into propositions by giving the values to 'x' from universal set (U)
- Open statements are denoted by $p(x)$, $q(x)$, $r(x)$ and so on.

Consider $x+3=6$ and the set of real number R. Now, this sentence becomes a proposition if x is replaced by any element of R.

For example :-
 if x is replaced by 3 \rightarrow true proposition
 and if x is replaced by 5, it becomes a false proposition. Here we say R is a universe (or universe of discourse).

Example (1) : Suppose the universal set consists of all integers, consider the following open statements
 $p(x) : x \leq 3$, $q(x) : x+1$ is odd, $r(x) : x > 0$
 Write down the truth values of the following,

(i) $p(2)$

:- We've $p(x) : x \leq 3$

$x = 2, \Rightarrow p(2) : 2 \leq 3$ It is true $\rightarrow 1$

(ii) $\sim q_V(4)$

:- We've $q_V(x) : x + 1$ is odd

$\Rightarrow q_V(4) : 4 + 1$ is odd

$\sim q_V(4)$ is false $\rightarrow 0$

(iii) $p(-1) \wedge q_V(1)$

:- $p(-1) : -1 \leq 3 \rightarrow \text{true} \rightarrow 1$

$q_V(1) : 1 + 1 = 2$ is not odd $\rightarrow \text{false} = 0$.

$\therefore p(-1) \wedge q_V(1) [1 \wedge 0]$ is false.

(iv) $\sim p(3) \vee r(0)$

:- $p(3) : 3 \leq 3$ is true $= 1$

$\Rightarrow \sim p(3)$ is false $= 0$

$r(0) : 0 > 0$ is false $= 0$

$\therefore \sim p(3) \wedge r(0) = 0 \wedge 0 = 0$
 $\Rightarrow \text{false}$

(v) $p(2) \wedge [q_V(0) \vee \sim r(2)]$

:- $p(2) : 2 \leq 3$ true $= 1$

$q_V(0) : 0 + 1 = 1$ is odd, true $= 1$

$\sim r(2) : 2 > 0$

$\Rightarrow \sim r(2) : 2 < 0$ is false $= 0$

$$\therefore p(\alpha) \wedge [q_1(0) \vee \neg r(\alpha)]$$

$$1 \wedge [1 \vee 0] \Rightarrow 1 \wedge 1 \\ = 1$$

$$\Rightarrow p(\alpha) \wedge [q_1(0) \vee \neg r(\alpha)] - \text{True} - \boxed{1}$$

Quantifiers

consider the statements

- (i) All integers are divisible by 2
- (ii) Some integers are divisible by 3
- (iii) Every integer is real
- (iv) There exists an integer which is prime

In these propositions, the words 'all', "every", "Some", "there exists" are associated with the idea of a quantity. Such words are called Quantifiers.

\Rightarrow The symbol \forall has been used to denote the phrases 'for all', "for every", "for any", "for each" and these are called universal quantifiers.

\Rightarrow The symbol \exists has been used to denote the phrases 'there exists', "for some", "for at least", these are called as existential quantifiers.

\Rightarrow A proposition involving universal or existential

quantifier is called a quantifier statement

⇒ Thus a quantified statement is a proposition of the form " $\forall x \in S, p(x)$ " or " $\exists x \in S, p(x)$ " where $p(x)$ is an open statement and S is the universe for x in $p(x)$.

"The variable present in a quantified statement is called a bound variable".

Example:- For the universe of all integers, let

$$p(x) : x > 0$$

$$q(x) : x \text{ is even}$$

$$r(x) : x \text{ is a perfect square}$$

$$(4, 9, 16, \dots)$$

$$s(x) : x \text{ is divisible by 3.}$$

$$t(x) : x \text{ is divisible by 7.}$$

Write down the following quantified statements in symbolic form.

(i) At least one integer is even.

$$\text{Soln:- } \exists x \in S, q(x)$$

(ii) There exists a positive integer that is even

$$\text{Soln:- } \exists x \in S, (p(x) \wedge q(x))$$

(iii) Some even integers is either even or odd.

$$\text{Soln:- } \exists$$

(iii) Some even integers are divisible by 3.

Soln:- $\exists x \in S, [q(x) \wedge s(x)]$

(iv) Every integer is either even (or) odd

:- $\forall x \in S, [q(x) \vee \sim q(x)]$

(v) If x is even and a perfect square, then x is not divisible by 3.

:- $\forall x \in S, [(q_1(x) \wedge r(x)) \rightarrow \sim s(x)]$

(vi) If x is odd (or) is not divisible by 7, then x is divisible by 3.

:- $\forall x \in S, \{[\sim q_1(x) \vee \sim r(x)] \rightarrow s(x)\}$.

(*) Truth value of a Quantifier Statement

Rule 1:- $\forall x \in S, p(x)$ is true, then it should be true for all values of x , suppose if it is false for at least one value, i.e. $p(a)$ is false where $a \in S$ - this element 'a' is counter example, then $\forall x \in S, p(x)$ is false.

Rule 2:- For proving $\exists x \in S, p(x)$ is true, it is enough to show that $p(a)$ is true where $a \in S$.

(*) Truth value of a Quantified statement

Rule 1: The statement " $\forall x \in S, p(x)$ " is true only when $p(x)$ is true for each $x \in S$.

Rule 2: The statement " $\exists x \in S, p(x)$ " is false^{only when $p(x)$ is false} for every $x \in S$.

Accordingly, a proposition of the form " $\forall x \in S, p(x)$ ", is false, it is enough to exhibit one element 'a' of S such that $p(a)$ is false. This element 'a' is called counter example.

Similarly a proposition of the form " $\exists x \in S, p(x)$ " is true, it is enough to exhibit one element 'a' of S s.t $p(a)$ is true.

(*) Two Rules of Inference:

- (*) If an open statement $p(x)$ is known to be true for all x in a universe S and if $a \in S$, then $p(a)$ is true (Rule of universal Specification)
- (*) If an open statement $p(x)$ is proved to be true for any x chosen from a set S , then the quantified statement $\forall x \in S, p(x)$ is true (Rule of universal Generalization).

Problems

(1) Consider the open statement $p(x) : x > 0$;
 $q(x) : x \text{ is even}$; $r(x) : x \text{ is a perfect square}$
 $s(x) : x \text{ is divisible by } 3$; $t(x) : x \text{ is divisible by } 7$
 Express each of the following symbolic statements in words.

- (i) $\forall x, [r(x) \rightarrow p(x)]$
- (ii) $\exists x, [s(x) \wedge q(x)]$
- (iii) $\forall x, [\sim r(x)]$
- (iv) $\forall x, [r(x) \vee t(x)]$

Soln:- (i) For all integer x , if x is a perfect square

then $x > 0$

(ii) For some integer x , x is divisible by 3 and
 x is odd.

(iii) For all integer x , x is not a perfect square

(iv) For all integer x , x is a perfect square \wedge
 x is divisible by 7 \rightarrow False (Eg:- $x = 8$)

(2) Consider the following open statements with the
 set of all real no as the universe.

$p(x) : |x| > 3$, $q(x) : x > 3$, Find the truth

values of the statement $\forall x, [p(x) \rightarrow q(x)]$

Soln:- Given $p(x) : |x| > 3$

For $x = -4$, i.e $p(-4) : |-4| > 3$

$4 > 3$ is true - 1

And $q(x) : x > 3$, For $x = -4$, $q(-4) : -4 > 3$ - false - 0

For $x = -4$, $p(x) : 1$ and $q_V(x) : 0$

$$\therefore p(x) \rightarrow q_V(x) = 1 \rightarrow 0 \\ = 0$$

\Rightarrow false

{ We've got this result by taking $x = -4$ counter example}

(3) consider the following open statements with the set of all real numbers as the universe

$$p(x) : x \geq 0 ; q_V(x) : x^2 \geq 0 ; r(x) : x^2 - 3x - 4 = 0$$

$$s(x) : x^2 - 3 > 0$$

Determine the truth values of the following statements.

$$(i) \exists x, p(x) \wedge q_V(x) \quad (ii) \forall x, p(x) \rightarrow q_V(x)$$

$$(iii) \forall x, q_V(x) \rightarrow s(x) \quad (iv) \forall x, r(x) \vee s(x)$$

$$(v) \exists x, p(x) \wedge r(x) \quad (vi) \forall x, r(x) \rightarrow p(x).$$

Soln:-

(i) there exist a real number ' x ' for which both $p(x)$ and $q_V(x)$ are true ; for instance $x = 1$

$$\therefore p(x) \wedge q_V(x) = 1 \wedge 1 = 1 \rightarrow \text{true}$$

(ii) For all x , $q_V(x)$ is true hence $p(x) \rightarrow q_V(x)$ is true.

(iii) For every real no x , $s(x)$ is false for $x = 1$ and $q_V(x)$ is true

thus. $q_V(x) \rightarrow s(x)$ is false for $x = 1$

i.e The statement $q_1(x) \rightarrow s(x)$ is not always true

$\therefore \forall x, q_1(x) \rightarrow s(x)$ is false

(iv) $\tau(x): x^2 - 3x - 4 = 0$

$$(x-4)(x+1)$$

i.e $\tau(x)$ is true only for $x=4$ and $x=-1$

but for all real no x , $\tau(x)$ is not true i.e it is false and $s(x)$ is false for $x=1$

Hence $\tau(x) \wedge s(x)$ is false (not always true)

(v) $\exists x, p(x) \wedge \tau(x)$

for $x=4$, both $p(x)$ and $\tau(x)$ is true

hence, $p(x) \wedge \tau(x) = 1 \wedge 1 = 1$ is true.

(vi) $\forall x, \tau(x) \rightarrow p(x)$

$\tau(x)$ is true for $x=-1$ but $p(x)$ is false for $x=-1$

thus. $\tau(x) \rightarrow p(x)$ is not always true

$\therefore \forall x, \tau(x) \rightarrow p(x)$ is false

- (4). Let $p(x) = x^2 - 7x + 10 = 0$; $q_1(x) : x^2 - 2x - 3 = 0$,
- $\tau(x) : x < 0$. Determine the truth (or falsity of the following statements where the universe U contains only the integers 2 and 5. If a statement is false, provide a counter example (or explanation).
- (i) $\forall x, p(x) \rightarrow \neg \tau(x)$ (ii) $\forall x, q_1(x) \rightarrow \tau(x)$

$$(iii) \exists x, q_1(x) \rightarrow r(x) \quad (iv) \exists x, p(x) \rightarrow r(x)$$

Soln: Let $U = \{2, 5\}$

$$p(x) : x^2 - 7x + 10 = 0$$

$$\forall x : x^2 - 2x - 3 = 0$$

$$p(x) : (x-5)(x-2) = 0$$

$$q_1(x) : (x-3)(x+1) = 0$$

$$p(x) : x=5, x=2$$

$$q_1(x) : x=3, x=-1$$

$$(i) \forall x, p(x) \rightarrow \sim r(x)$$

$p(x)$ is true for all $x \in U$ and $\sim r(x)$ is true

for all $x \in U$

$\therefore \forall x, p(x) \rightarrow \sim r(x)$ is true.

$$(ii) \forall x, q_1(x) \rightarrow r(x)$$

$q_1(x)$ is false for all $x \in U$ and $r(x)$ is false

for all $x \in U$

$\therefore \forall x, q_1(x) \rightarrow r(x)$ is true.

$$(iii) \exists x, q_1(x) \rightarrow r(x)$$

$q_1(x)$ and $r(x)$ is false at $x=2$

$\therefore \exists x, q_1(x) \rightarrow r(x)$ is true

$$(iv) \exists x, p(x) \rightarrow r(x)$$

$p(x)$ is true for $x=2$ but $r(x)$ is false

for $x=2$

$\therefore \exists x, p(x) \rightarrow r(x)$ is false.

(*) Logical Equivalence :

Two quantified statements are said to be logically equivalent whenever they have the same truth values in all possible situations.

$$(i) \forall x, [p(x) \wedge q_1(x)] \Leftrightarrow (\forall x, p(x)) \wedge (\forall x, q_1(x))$$

$$(ii) \exists x, [p(x) \vee q_1(x)] \Leftrightarrow (\exists x, p(x)) \vee (\exists x, q_1(x))$$

$$(iii) \exists x, [p(x) \rightarrow q_1(x)] \Leftrightarrow \exists x, [\sim p(x) \vee q_1(x)]$$

(*) Rule for Negation of a Quantified Statement

$$(1) \sim \{\forall x, p(x)\} \Leftrightarrow \exists x, \{\sim p(x)\}$$

$$(2) \sim \{\exists x, p(x)\} \Leftrightarrow \forall x, \{\sim p(x)\}$$

PROBLEMS

(1) Consider the open statements $p(x)$; $x > 0$,
 $q_1(x)$; x is even, $r(x)$; x is perfect square,
 $s(x)$; x is divisible by 3, $t(x)$; x is divisible by 7
Express each of the following symbolic statements in words and indicate its truth table,

$$(i) \forall x, [r(x) \rightarrow p(x)]$$

Soln. For any integer x , if x is perfect square
then $x > 0$ — false (take $x=0$)

(Q) Negate and simplify each of the following

(i) $\exists x, [p(x) \vee q_1(x)]$

Soln:- $\sim \{ \exists x, [p(x) \vee q_1(x)] \}$

$$\Leftrightarrow \forall \exists x, p \ A x, [\sim (p(x) \vee q_1(x))]$$

$$\Leftrightarrow \forall x, [\sim p(x) \wedge \sim q_1(x)] //$$

(ii) $\forall x, [p(x) \wedge \sim q_1(x)]$

Soln:- $\sim \{ \forall x, [p(x) \wedge \sim q_1(x)] \}$

$$\Leftrightarrow \exists x, [\sim (p(x) \wedge \sim q_1(x))]$$

$$\Leftrightarrow \exists x, [\sim p(x) \vee q_1(x)] //$$

(iii) $\forall x, [p(x) \rightarrow q_1(x)]$

Soln:- $\sim \{ \forall x, [p(x) \rightarrow q_1(x)] \}$

$$\Leftrightarrow \exists x, [\sim (p(x) \rightarrow q_1(x))] \text{ Law for cond'n.}$$

$$\Leftrightarrow \exists x, [\sim (\sim p(x) \vee q_1(x))] \text{ demorgan's}$$

$$\Leftrightarrow \exists x, [p(x) \wedge \sim q_1(x)]$$

(iv) $\exists x, [\{p(x) \vee q_1(x)\} \rightarrow r(x)]$

Soln:- $\sim \{ \exists x, [\{p(x) \vee q_1(x)\} \rightarrow r(x)] \}$

$$\Leftrightarrow \forall x, \{\sim [(p(x) \vee q_1(x)) \rightarrow r(x)] \}$$

$$\Leftrightarrow \forall x, \{\sim [\sim (p(x) \vee q_1(x)) \vee r(x)] \}$$

$$\Leftrightarrow \forall x, [(p(x) \vee q_1(x)) \wedge \neg r(x)] //$$

(3) Write down the following proposition in symbolic form and find its negation

"All integers are rational numbers and some rational numbers are not integers".

$\therefore p(x)$: x is a rational number

$q_1(x)$: x is an integer

\mathbb{Z} = set of all integers, \mathbb{Q} = set of all rational no

Symbolic form,

$$[\forall x \in \mathbb{Z}, p(x)] \wedge [\exists x \in \mathbb{Q}, \neg q_1(x)]$$

$$\Leftrightarrow \neg\{\forall x \in \mathbb{Z}, p(x)\} \wedge \{\exists x \in \mathbb{Q}, \neg q_1(x)\}$$

$$\Leftrightarrow \neg[\forall x \in \mathbb{Z}, p(x)] \vee \neg[\exists x \in \mathbb{Q}, \neg q_1(x)]$$

$$\Leftrightarrow [\exists x \in \mathbb{Z}, \neg p(x)] \vee [\forall x \in \mathbb{Q}, q_1(x)] //$$

Some integers are not rational no (or) all rational numbers are integers.

(4) Let the set \mathbb{Z} of all integers be the universe, consider the statements

$p(x)$: $8x+1=5$ and $q_1(x)$: $x^2=9$

obtain the negation of the quantified statements $\exists x \in \mathbb{Z}, [p(x) \wedge q_1(x)]$ and express it in words.

Soln: The negation of the given statement is

$$\neg \{ \exists x \in \mathbb{Z}, [p(x) \wedge q_1(x)] \}$$

$$\Leftrightarrow \forall x \in \mathbb{Z}, \neg [p(x) \wedge q_1(x)]$$

$$\Leftrightarrow \forall x \in \mathbb{Z}, \neg p(x) \vee \neg q_1(x).$$

In words, for all integers x , $2x+1 \neq 5$ (or) $x^2 \neq 9$.

(5) Write down the following proposition in symbolic form and find its negation.

"If all triangles are right angled, then no triangle is equiangular".

Soln: Let S be the set of all triangle

$p(x)$: x is right angled

$q(x)$: x is equiangular.

In symbolic form,

$$[\forall x \in S, p(x)] \rightarrow [\forall x \in S, \neg q(x)]$$

Now, its negation,

$$\neg \{ [\forall x \in S, p(x)] \rightarrow [\forall x \in S, \neg q(x)] \}$$

$$\Leftrightarrow \neg \{ \neg [\forall x \in S, p(x)] \vee [\forall x \in S, \neg q(x)] \}$$

$$\Leftrightarrow [\forall x \in S, \neg p(x)] \wedge \neg [\forall x \in S, \neg q(x)]$$

$$\Leftrightarrow [\forall x \in S, p(x)] \wedge [\exists x \in S, q(x)]$$

In words, for all triangle are right angled and some triangles are equiangular.

(6) Write down the negation of each of the following statements.

- (i) For all integers 'n', if n is not divisible by 2 then n is odd
- (ii) If k, m, n are any integers where $(k-m)$ and $(m-n)$ are odd, then $(k-n)$ is even.
- (iii) For all real numbers x , if $|x-3| < 7$, then $-4 < x < 10$.
- (iv) If x is a real number where $x^2 \geq 16$, then $x < -4$ or $x \geq 4$.

Soln:- 'Z' be the set of all integers and 'R' be the set of all real no

(i) $p(x)$: n is divisible by 2

$q(x)$: n is odd

Symbolic form, $\forall x \in Z, \sim p(x) \rightarrow q(x)$

Its negation, $\sim [\forall x \in Z, \sim p(x) \rightarrow q(x)]$

$\Leftrightarrow \exists x \in Z, \sim (\sim p(x) \rightarrow q(x))$

$\Leftrightarrow \exists x \in Z, \sim [\sim \sim p(x) \vee q(x)]$

$\Leftrightarrow \exists x \in Z, \sim p(x) \wedge \sim q(x)$

"For some integers n , n is not divisible by 2 and n is not odd".

(ii) $p(x)$: $(k-m)$ is odd ; $\sigma(x)$: $(k-n)$ is even

$q_1(x)$: $(m-n)$ is odd

Symbolic form; $\forall k, m, n \in \mathbb{Z}, (p(x) \wedge q_1(x)) \rightarrow \sigma(x)$

\therefore The negation of the statement is,

$$\neg\{\forall k, m, n \in \mathbb{Z}, (p(x) \wedge q_1(x)) \rightarrow \sigma(x)\}$$

$$\Leftrightarrow \exists k, m, n \in \mathbb{Z}, \neg[(p(x) \wedge q_1(x)) \rightarrow \sigma(x)]$$

$$\Leftrightarrow \exists k, m, n \in \mathbb{Z}, \neg[\neg(p(x) \wedge q_1(x)) \vee \sigma(x)]$$

$$\Leftrightarrow \exists k, m, n \in \mathbb{Z}, (p(x) \wedge q_1(x)) \wedge \neg \sigma(x)$$

In words,

"For some integers k, m, n , $(k-m)$ and $(m-n)$ are odd and $(k-n)$ is not even".

(iii) Here, \mathbb{R} .

$p(x)$: $|x-3| < 7$ & $q_1(x)$: $-4 < x < 10$ (i.e $x \in (-4, 10)$)

Symbolic form, $\forall x \in \mathbb{R}, p(x) \rightarrow q_1(x)$

The negation is

$$\neg\{\forall x \in \mathbb{R}, p(x) \rightarrow q_1(x)\}$$

$$\Leftrightarrow \exists x \in \mathbb{R}, \neg(p(x) \rightarrow q_1(x))$$

$$\Leftrightarrow \exists x \in \mathbb{R}, \neg(\neg p(x) \vee q_1(x))$$

$$\Leftrightarrow \exists x \in \mathbb{R}, p(x) \wedge \neg q_1(x)$$

In words, For some real no., $|x-3| < 7$ and $x \notin (-4, 10)$.

(iv) $p(x): x^2 > 16$, $q(x): x < -4$, $r(x): x > 4$

Symbolic form, $\forall x \in \mathbb{R}, p(x) \rightarrow (q(x) \vee r(x))$

Its negation is

$$\neg \{\forall x \in \mathbb{R}, [p(x) \rightarrow (q(x) \vee r(x))] \}$$

$$\Leftrightarrow \exists x \in \mathbb{R}, \neg [p(x) \rightarrow (q(x) \vee r(x))]$$

$$\Leftrightarrow \exists x \in \mathbb{R}, \neg [\neg p(x) \vee (q(x) \vee r(x))]$$

$$\Leftrightarrow \exists x \in \mathbb{R}, p(x) \wedge \neg (q(x) \vee r(x))$$

$$\Leftrightarrow \exists x \in \mathbb{R}, p(x) \wedge \neg q(x) \wedge \neg r(x)$$

In words,

For some real no., $x^2 > 16$ and $x \geq -4$ and

$x \leq 4$.

(* Logical Implication Involving Quantifiers)

A quantified statement P is said to logically imply a quantified statement Q if Q is true whenever P is true.

Then we write $P \Rightarrow Q$

An argument $P_1 \wedge P_2 \wedge P_3 \wedge \dots \wedge P_n \rightarrow Q$ is valid argument if Q is true whenever each of P_1, P_2, \dots, P_n is true (M)

$$P_1 \wedge P_2 \wedge P_3 \wedge \dots \wedge P_n \Rightarrow Q$$

PROBLEMS

(1) P.T the following argument is valid

All men are mortal

Sachin is a man

∴ Sachin is mortal

:- Let S denote the set of all men

$p(x)$: x is mortal

a : Sachin

The given argument, $\forall x \in S, p(x)$

$a \in S$

∴ $p(a)$

$p(x)$ is true and $a \in S$, means $p(a)$ is true

∴ This is valid by universal specification.

(2) Find whether the following is a valid argument
for which the universe is the set of all students

No Engineering student is bad in studies

Anil is not bad in studies

∴ Anil is an engineering student.

Soln:- S : set of all students.

$p(x)$: x is an engineering student

$q(x)$: x is bad in studies

a : Anil.

Argument,

$$\forall x, [p(x) \rightarrow \neg q_1(x)]$$

$$\frac{\neg q_1(a)}{\therefore p(a)}$$

$$[p(a) \rightarrow \neg q_1(a)] \wedge \neg q_1(a)$$

$$\Leftrightarrow [p(a) \rightarrow \neg q_1(a)] \wedge \neg q_1(a) \quad (\text{by universal specification})$$

$\not\vdash p(a)$ (\because It is not M.T (or M.P)).

Because no rule is applicable

\therefore It is an invalid statement.

~~(3)~~ Find whether the following variable is valid.

No engineering students of Ist or IInd Sem studies logic

Anil is an engineering student who studies logic

\therefore Anil is not in second semester.

Soln:- S : set of all engineering students.

$p(x)$: x is in Ist Sem

$q(x)$: x is in IInd Sem

$r(x)$: x studies logic

a : Anil

$$\begin{array}{c} \therefore \forall x \in S, (p(x) \vee q_1(x)) \rightarrow \sim r(x) \\ \hline \therefore \sim q_1(a) \end{array}$$

$$\{(p(x) \vee q_1(x)) \rightarrow \sim r(x)\} \wedge r(a)$$

$$\Rightarrow \{(p(a) \vee q_1(a)) \rightarrow \sim r(a)\} \wedge r(a) \quad [\text{WKT by MT}]$$

$$\Rightarrow \sim (p(a) \vee q_1(a)) \quad (p \rightarrow \sim q) \wedge q \Leftrightarrow \sim p$$

demorgan's law

$$\Rightarrow \sim p(a) \wedge \sim q_1(a) \quad \text{by conjunction elimination}$$

$$\Rightarrow \sim q_1(a) \wedge \sim p(a) \quad \text{commutative}$$

$$\Rightarrow \sim q_1(a) \quad (\text{by conjunction Simplification})$$

It is valid.

~~(4)~~ p.t the following argument is not valid

All squares have four sides

The quadrilateral ABCD has four sides

\therefore ABCD is a square

Soln:- S : set of all quadrilaterals

p(x) : x is a square

q_1(x) : x has 4 sides.

a : ABCD

$\forall x \in S, p(x) \rightarrow q_1(x)$

$$\underline{q_1(a)}$$

$$\therefore p(a)$$

$$[p(x) \rightarrow q_1(x)] \wedge q_1(a)$$

$$\Rightarrow [p(a) \rightarrow q_1(a)] \wedge q_1(a)$$

[neither M.P nor M.T]

$$\not\rightarrow p(a)$$

\therefore If is valid.

(5) Over the universe of all quadrilaterals in plane geometry. Verify the validity of the argument "Since every square is a rectangle and every rectangle is a parallelogram, it follows that every square is a parallelogram."

Soln:- S: set of all ~~parallel~~ quadrilaterals

$p(x)$: x is a square

$q(x)$: x is a rectangle

$r(x)$: x is a parallelogram.

$\forall x \in S, p(x) \rightarrow q(x)$

$$\underline{q(x) \rightarrow r(x)}$$

$$\therefore p(x) \rightarrow r(x)$$

Now, $[p(x) \rightarrow q(x)] \wedge [q(x) \rightarrow r(x)]$

$$\Rightarrow [p(a) \rightarrow q_1(a)] \wedge [q_1(a) \rightarrow r(a)]$$

$$\Rightarrow p(a) \rightarrow r(a) \quad [\because \text{Syllogism}]$$

\therefore It is valid.

(6) prove the following argument is valid

$$\forall x, [p(x) \rightarrow \{q_1(x) \wedge r(x)\}]$$

$$\forall x, [p(x) \wedge s(x)]$$

$$\therefore \forall x, [r(x) \wedge s(x)]$$

$$\text{soln: } [p(x) \rightarrow \{q_1(x) \wedge r(x)\}] \wedge [p(x) \wedge s(x)]$$

$$\Rightarrow [p(x) \rightarrow \{q_1(x) \wedge r(x)\}] \wedge [p(x) \wedge s(x)] \quad \xrightarrow{\text{M.P}}$$

$$\Rightarrow \{q_1(x) \wedge r(x)\} \wedge s(x)$$

(by conjunctive Simplification)

$$\Rightarrow r(x) \wedge s(x)$$

$$\Rightarrow r(x) \wedge s(x)$$

It is valid.

(7) prove the following argument is valid

$$\forall x, [p(x) \vee q_1(x)]$$

$$\forall x, [\{\sim p(x) \wedge q_1(x)\} \rightarrow r(x)]$$

$$\therefore \forall x, [\sim r(x) \rightarrow p(x)]$$

$$\text{soln: } [p(x) \vee q_1(x)] \wedge [\{\sim p(x) \wedge q_1(x)\} \rightarrow r(x)]$$

$$\Rightarrow [p(a) \vee q_1(a)] \wedge [\underbrace{\{\sim p(a) \wedge q_1(a)\}}_{\text{law for condn.}} \rightarrow r(a)]$$

$$\Rightarrow [p(a) \vee q_1(a)] \wedge [\sim(\sim p(a) \wedge q_1(a)) \vee r(a)]$$

demorgan's

$$\Rightarrow \underline{[p(a) \vee q_1(a)]} \wedge [p(a) \vee \sim q_1(a)] \vee r(a)$$

distributive

$$\Rightarrow [p(a) \vee (q_1(a) \wedge \sim q_1(a))] \vee r(a)$$

$$\Rightarrow p(a) \vee F_0 \vee r(a)$$

$$\Rightarrow p(a) \vee r(a)$$

$$\Rightarrow r(a) \vee p(a)$$

$$\Rightarrow \sim \sim r(a) \vee p(a)$$

$$\Rightarrow \sim r(a) \rightarrow p(a)$$

$$\Rightarrow \sim r(a) \rightarrow p(a) //$$

If is valid.

(8) prove that the following argument is valid

$$A x, [p(x) \vee q_1(x)]$$

$$\exists x, \sim p(x)$$

$$A x, [\sim q_1(x) \vee r(x)]$$

$$A x, [s(x) \rightarrow \sim r(x)]$$

$$\therefore \exists x, \sim s(x)$$

$$\text{Soln: } [p(x) \vee q_1(x)] \wedge \sim p(x) \wedge [\sim q_1(x) \vee r(x)] \wedge [s(x) \rightarrow \sim r(x)]$$

$$\Rightarrow \underline{[p(x) \vee q_1(x)]} \wedge \sim p(x) \wedge [\sim q_1(x) \vee r(x)] \wedge [s(x) \rightarrow \sim r(x)]$$

by disjunctive syllogism

- $$\Rightarrow \underline{q_1(a) \wedge [\sim q_1(a) \vee r(a)] \wedge [s(a) \rightarrow \sim r(a)]} \quad (\text{by disjunctive syllogism})$$
- $$\Rightarrow r(a) \wedge [s(a) \rightarrow \sim r(a)] \quad ((P \vee Q) \wedge \sim Q = P)$$
- $$\Rightarrow r(a) \wedge [\sim s(a) \vee \sim r(a)] \quad \text{by disjunctive syllogism}$$
- $$\Rightarrow \sim s(a)$$
- $$\Rightarrow \sim s(a)$$
- \therefore It is valid.

(9) Find whether the following argument is valid.

If a triangle has 2 equal sides, then it is isosceles

If a triangle is isosceles, then it has 2 equal angles

The triangle $\triangle ABC$ does not have 2 equal angles.

\therefore $\triangle ABC$ does not have 2 equal sides.

Soln:- S: set of all triangles.

$p(x)$: x has 2 equal sides

$q_1(x)$: x is isosceles.

$r(x)$: x has 2 equal angles.

a: $\triangle ABC$ (counter example)

Now, $\forall x \in S, p(x) \rightarrow q_1(x)$

$q_1(x) \rightarrow r(x)$

$\sim r(a)$

$\sim p(a)$

$$\therefore [p(x) \rightarrow q_1(x)] \wedge [q_1(x) \rightarrow r(x)] \wedge \sim r(a)$$

$$\Rightarrow [\underbrace{p(a) \rightarrow q_1(a)}_{\text{Syllogism}}] \wedge [q_1(a) \rightarrow r(a)] \wedge \sim r(a)$$

$$\Rightarrow [p(a) \rightarrow r(a)] \wedge \sim r(a) \quad \text{Modus ponens}$$

$$\Rightarrow \sim p(a)$$

\therefore It is valid.

(10) Determine if the argument is valid (or) not.

All people concerned about the environment, recycle their plastic containers. B is not concerned about the environment. Therefore, B does not recycle his plastic containers.

Soln:- S: Set of all people.

$p(x)$: x is concerned about the environment

$q(x)$: x recycle their containers.

a : B

$$\forall x \in S, \quad p(x) \rightarrow q(x)$$

$$\frac{\sim p(a)}{\therefore \sim q(a)}$$

$$\therefore [p(x) \rightarrow q(x)] \wedge \sim p(a) \Rightarrow [p(a) \rightarrow q(a)] \wedge \sim p(a)$$

$\not\Rightarrow \sim q(a)$ [both MP and MT are invalid.]

\therefore Argument is invalid.

(*) Open statements with more than one variable

consider the following statements $x - 2y \geq 0$ and $x - y + z = 0$, these are the open statements which contains more than one free variables. These becomes propositions if each variables is replaced by an element in a universal set.

open statements containing two variables are denoted by $p(x, y)$, $q(x, y)$ --- so on

open statements containing 3 variables is denoted by $p(x, y, z)$, $q(x, y, z)$ --- so on

ex: (1) let $p(x, y) : x^2 \geq y$, $q(x, y) : (x+2) < y$. where x and y are the set of all real numbers. Determine the truth values of the following statements.

soln:- (i) $p(2, 4)$ (ii) $p(-3, 8)$ (iii) $p(-3, 8) \wedge q(1, 3)$

(iv) $p\left(\frac{1}{2}, \frac{1}{3}\right) \vee \neg q(-2, -3)$

soln:- (i) $p\left(\frac{x}{y}, 4\right) : (2)^2 \geq 4$
 $4 \geq 4 \rightarrow \text{true} - 1$

(ii) $p(-3, 8) : (-3)^2 \geq 8$
 $\Rightarrow 9 \geq 8 \rightarrow \text{true} - 1$

(iii) $p(-3, 8) : -3^2 \geq 8 = 9 \geq 8 - \text{true} - 1$

$$q_V(1,3) : (1+2) < 3$$

$3 < 3 \rightarrow \text{false} - 0$

$$\therefore p(-3,8) \wedge q_V(1,3) = 1 \wedge 0 = 0 \rightarrow \text{false}$$

$$(iv) p\left(\frac{1}{2}, \frac{1}{3}\right) : \left(\frac{1}{2}\right)^2 \geq \frac{1}{3}$$

$$\Rightarrow \frac{1}{4} \geq \frac{1}{3} \rightarrow \text{False} - 0$$

$$q_V(-2, -3) : (-2+2) < -3$$

$$\Rightarrow 0 < -3 \Rightarrow \text{false} - 0$$

$$\therefore p\left(\frac{1}{2}, \frac{1}{3}\right) \vee \neg q_V(-2, -3)$$

$$\Rightarrow 0 \vee 1$$

$$\Rightarrow 1 - \text{true.}$$

(*) Quantified Statement with more than one variable

When an open statement contains more than one free variable, quantification may be applied to each of the variables.

Properties

- (i) $\forall x, \forall y p(x,y) \Leftrightarrow \forall y, \forall x p(x,y) \Leftrightarrow \forall x, \forall y p(x,y)$
 - (ii) $\exists x, \exists y p(x,y) \Leftrightarrow \exists y, \exists x p(x,y) \Leftrightarrow \exists x, \exists y p(x,y)$
 - (iii) $\forall x, \exists y p(x,y) \not\Leftrightarrow \exists y, \forall x, p(x,y)$
 - (iv) $\forall y, \exists x p(x,y) \not\Leftrightarrow \exists x, \forall y p(x,y)$
- similarly for 3 variables.

problems

(1) Let x and y denote the integers. Consider the statement

$p(x, y)$: $x+y$ is even

Write down the following statements in words

(i) $\forall x, \exists y, p(x, y)$ (ii) $\exists x, \forall y, p(x, y)$

(iii) $\forall x, \forall y, p(x, y)$

Soln:- (i) For all integers x , there exist an integer y , $x+y$ is even

(ii) There exists an integer x such that $x+y$ is even for every integer y .

(iii) For all integers x and y , $x+y$ is even.

(2) Find the Negation for the following statements.

(i) $\forall x, \exists y, \{ \{ p(x, y) \wedge q(x, y) \} \rightarrow r(x, y) \}$

Soln:- $\sim \{ \forall x, \exists y, \{ \{ p(x, y) \wedge q(x, y) \} \rightarrow r(x, y) \} \}$

$\Rightarrow \exists x, \sim \{ \exists y, \{ (p(x, y) \wedge q(x, y)) \rightarrow r(x, y) \} \}$

$\Rightarrow \exists x, \forall y, \sim \{ (p(x, y) \wedge q(x, y)) \rightarrow r(x, y) \}$

$\Rightarrow \exists x, \forall y, \sim [\sim (p(x, y) \wedge q(x, y)) \vee r(x, y)]$

$\Rightarrow \exists x, \forall y, [p(x, y) \wedge q(x, y)] \wedge \sim r(x, y) //$

$$(ii) \forall x, \forall y [(x < y) \rightarrow \exists z, (x < z < y)]$$

soln:- $\sim \{\forall x, \forall y [(x < y) \rightarrow \exists z, (x < z < y)]\}$

$$\Rightarrow \exists x, \exists y \sim [(x < y) \rightarrow \exists z, (x < z < y)]$$

$$\Rightarrow \exists x, \exists y, (x < y) \wedge \sim \exists z, (x < z < y)$$

$$\Rightarrow \exists x, \exists y, (x < y) \wedge \forall z (x > z \geq y)$$

$$(iii) \forall x, \forall y [|x| = |y| \rightarrow (y = \pm x)]$$

soln:- $\sim \{\forall x, \forall y [|x| = |y| \rightarrow (y = \pm x)]\}$

$$\Rightarrow \exists x, \exists y \sim [|x| = |y| \rightarrow (y = \pm x)]$$

$$\Rightarrow \exists x, \exists y \sim [\sim (|x| = |y|) \vee (y = \pm x)]$$

$$\Rightarrow \exists x, \exists y (|x| = |y|) \wedge (y \neq \pm x) //$$

$$(iv) [\forall x, \forall y, ((x < 0) \wedge (y > 0))] \rightarrow [\exists z, (xz > y)]$$

soln:-

$$\sim \{\forall x, \forall y, ((x < 0) \wedge (y > 0)) \rightarrow [\exists z, (xz > y)]\}$$

$$\Rightarrow \sim \{\sim (\forall x, \forall y, [(x < 0) \wedge (y > 0)]) \vee \exists z, (xz > y)\}$$

$$\Rightarrow \sim \forall x \forall y, [(x < 0) \wedge (y > 0)] \wedge \sim \exists z, (xz > y)$$

$$\Rightarrow \forall x, \forall y, [(x < 0) \wedge (y > 0)] \wedge \forall z, (xz \leq y) //$$

Methods of proof and Disproof

The propositions that commonly appear in mathematical discussions are conditionals of the form $p \rightarrow q$, where p & q are simple or compd proposition & which may involves quantifiers also.

The process of establishing that the conditional is true by using the rules or laws of logic and other known facts constitutes a proof of the conditional.

The process of establishing that a proposition is false constitutes a disproof.

(1) Direct proof

The direct proof of a conditional $p \rightarrow q$ has the following steps.

(1) Hypothesis : First assume that p is true

(2) Analysis : Starting with the hypothesis and employing the rules or laws of logic and other known facts, infer that q is true

(3) Conclusion : $p \rightarrow q$ is true.

problems

(1) Give a direct proof of the statement:

"The square of an odd integer is an odd integer".

Soln:- Is of the form,

"If n is an odd integer, then n^2 is an odd integer"

Now, p : n is an odd integer

q : n^2 is an odd integer.

Hypothesis:- Assume that n is an odd integer

i.e. $n = 2k + 1$ for some k

Analysis: consequently $n^2 = (2k + 1)^2$

$$n^2 = 4k^2 + 1 + 4k, \text{ which is}$$

not divisible by 2.

$\therefore n^2$ is an odd integer.

Conclusion: $p \rightarrow q$ is true.

~~(2)~~ p.t, for all integers k and l , if k & l are both odd, then $k+l$ is even and kl is odd.

Soln:- p : k & l are both odd

q : $k+l$ is even and kl is odd

Hypo: Assume that k & l are odd

i.e. $k = 2m+1$, $l = 2n+1$, $m, n \in \mathbb{Z}$

Analy : consequently ,

$$k+l = (2m+1) + (2n+1)$$

$k+l = 2m+2n+2$, which is divisible by 2

$\therefore k+l$ is even

$$kl = (2m+1)(2n+1)$$

$= 4mn+2m+2n+1$, which is not divisible by 2

$\therefore kl$ is odd

conclu : $p \rightarrow q_1$ is true

(*) Indirect proof

WKT $p \rightarrow q_1 \Leftrightarrow \sim q_1 \rightarrow \sim p$ [contrapositive statement]

First show that $\sim q_1 \rightarrow \sim p$ is true by direct proof , this shows that $p \rightarrow q_1$ is true indirectly , this method of proving a conditional is called indirect proof.

problems

(1) Let n be an integer, P.T "if n^2 is odd then n is odd".

Sol:- p : n^2 is odd

q : n is odd

s.t : $\sim q \rightarrow \sim p$ is true

Hypothesis: Assume that $\neg q$ is true
i.e. m is even
 $m = 2k$.

Analysis: $m^2 = (2k)^2$
 $\Rightarrow m^2 = 4k^2$, divisible by 2
 $\therefore m^2$ is even

Hence $\neg p$ is true

Conclusion: $\neg q \rightarrow \neg p$ is true

Hence $p \rightarrow q$ is true.

(2) "The product of two even integers is an even integer".

Soln: If m and n are even integers then mn is an even integer.

p : m & n are even integers

q : mn is an even integer.

S.T: $\neg q \rightarrow \neg p$ is true.

Hypo: Assume that $\neg q$ is true
i.e. mn is an odd integer

Analys: mn is not divisible by 2

$\Rightarrow m$ is not divisible by 2

n is not divisible by 2

$\therefore m$ is odd & n is odd $\Rightarrow m$ & n are odd integers

$\Rightarrow \neg p$ is true

conclu: $\neg q \rightarrow \neg p$ is true,
Hence $p \rightarrow q$ is true

(3) For all integers k and l , if $k+l$ is even then
 k & l are both even (or) both odd

Soln:- p : $k+l$ is even
 q : k & l are both even (or) both odd.

S.T: $\neg q \rightarrow \neg p$ is true

Hypoth: Assume that $\neg q$ is true
i.e k & l are both not even (or) both not odd
(that means one is even & other one is odd)
i.e $k = 2m$, $l = 2n+1$

Analysis: $k+l = 2m+2n+1$, is not divisible by 2

$\therefore k+l$ is odd

$\Rightarrow \neg p$ is true

conclu: $\neg q \rightarrow \neg p$ is true

Hence $p \rightarrow q$ is true

(4) provide an indirect proof of the following statement. "For all positive real numbers say if the product xy exceeds 25 then $x > 5$ or $y > 5$ ".

Soln:- p : $xy > 25$

s.t. $\neg q \rightarrow \neg p$ is true

q : $x > 5$ or $y > 5$

Hypo: Assume $\neg q$ is true

i.e. $x \leq 5$ and $y \leq 5$

Analysis: $xy \leq 5 \times 5$

$xy \leq 25$

$\Rightarrow \neg p$ is true

Conclusion: $\neg q \rightarrow \neg p$ is true

Hence $p \rightarrow q$ is true.

(*) proof by contradiction

Hypothesis: Assume that $p \rightarrow q$ is false

i.e. p is true and q is false.

Analysis: Take q is false, use any rules or laws

of known facts to s.t. p is false.

Hence our assumption is wrong.

Conclusion: Hence $p \rightarrow q$ is true.

problems:

- (1) provide a proof by contradiction of the following statement.

"For every integer n , if n^2 is odd, then n is odd".

Sol: - $p: n^2$ is odd

$q_1: n$ is odd

S.T: $p \rightarrow q_1$ is true by contradiction

Hypothesis: Assume that $p \rightarrow q_1$ is false

i.e p is true and q_1 is false

Analysis: take q_1 is false

i.e n is even

$$n = 2k$$

consequently, $n^2 = 4k^2$, divisible by 2

$\Rightarrow n^2$ is even

Hence p is false

Hence our assumption is wrong

Conclusion: $p \rightarrow q_1$ is true.

(Q) "If n^2 is an even integer then n is an even integer".

Sol: - $p: n^2$ is an even integer

$q_1: n$ is an even integer.

S.T: $p \rightarrow q_1$ is true by contradiction

Hypothesis: Assume that $p \rightarrow q_1$ is false

i.e p is true and q_1 is false

Analysis: Take q_V is false, means n is an odd integer

i.e $n = 2k + 1$

consequently, $n^2 = (2k+1)^2$

$$n^2 = 4k^2 + 1 + 4k, \text{ not divisible by } 2$$

$\therefore n^2$ is an odd integer

$\Rightarrow p$ is false

Hence our assumption is wrong

Conclusion: $p \rightarrow q_V$ is true.

(3) P.T for all real numbers x and y ,

"if $x+y \geq 100$, then $x \geq 50$ or $y \geq 50$

Soln:- $p: x+y \geq 100$

$q_V: x \geq 50$ or $y \geq 50$

Hypo: Assume that $p \rightarrow q_V$ is false

i.e p is true and q_V is false.

Analysis: Take q_V is false, means $x < 50$ and $y < 50$

consequently, $x+y < 50+50$

$$x+y < 100$$

$\Rightarrow p$ is false, hence our assumption is wrong.

Conclusion: $p \rightarrow q_V$ is true

(A) P.T, "if 'm' is an even integer then 'm+7' is an odd integer".

Soln:- p: m is an even integer

q_v: m+7 is an odd integer

Hypo: Assume that $p \rightarrow q_v$ is false

i.e p is true and q_v is false.

Analy: take q_v is false

m+7 is an even integer

$$\Rightarrow m+7 = 2k$$

$m = 2k+7$, which is not divisible by 2

$\therefore m$ is an odd integer

$\Rightarrow p$ is false, our assumption is wrong

Conclusion: $p \rightarrow q_v$ is false.

(B) Give a (i) direct proof (ii) indirect proof

(iii) proof by contradiction, for the following statement

"If n is an odd integer, then n+9 is an even integer".

Soln:- p: n is an odd integer

q_v: n+9 is an even integer

(i) direct proof:

S.T $p \rightarrow q_v$ is true

Hypothesis: Assume that p is true, i.e. n is odd

$$n = 2k + 1$$

Analysis: $n+q = 2k+1+q$

$$n+q = 2k+1+q \text{, divisible by } 2$$

$\therefore n+q$ is even integer

$\therefore q_1$ is true

conclu: $p \rightarrow q_1$ is true.

(ii) indirect proof:

s.t.: $\sim q_1 \rightarrow \sim p$ is true

Hypo: Assume that $\sim q_1$ is true

i.e. $n+q$ is an odd integer

Analysis: $n+q = 2k+1$

$$n = 2k+1-q = 2k-8 \text{, is divisible by } 2.$$

$\therefore n$ is even integer

i.e. $\sim p$ is true

conclu: $\sim q_1 \rightarrow \sim p$ is true

Hence $p \rightarrow q_1$ is true

(iii) proof by contradiction:

Hypothesis: Assume $p \rightarrow q_1$ is false

i.e. p is true and q_1 is false

Analysis: Take q_1 is false

$n+q$ is an odd integer

$$\therefore m+q = 2k+1$$

$$m = 2k+1 - q$$

$m = 2k-8$, is divisible by 2

$\therefore m$ is an even integer

i.e. p is false

Hence our assumption is wrong.

Conclusion: $p \rightarrow q$ is true.

(*) Disproof by contradiction

Hypothesis: Assume that $p \rightarrow q$ is true

i.e. p is true and q_V is true

Analysis: Take p is true by using rules, laws or any other known facts to s.t. q_V is false.

Hence our assumption is wrong.

Conclusion: $p \rightarrow q_V$ is false.

Disproof the Statement:

(1) "If m is an even integer then $m+7$ is an even integer".

Soln: p: m is an even integer

q_V : $m+7$ is an even integer.

Hypothesis: Assume that $p \rightarrow q_V$ is true

i.e. p is true and q_V is true

Analysis: Take p is true i.e m is an even integer

$$m = 2K$$

conseq, $m+7 = 2K+7$, is not divisible by 2

$\therefore m+7$ is an odd integers

$\therefore q$ is false

Hence our assumption is wrong

conclus: $p \rightarrow q$ is false.

(*) Disproof by counter example:

Disprove the proposition.

(1) The product of any two odd integers is a perfect square.

Soln:- take $m=3$ and $n=5$

$$mn = (3)(5) = 15$$

is not a perfect square

Hence the proposition is disproved.