Exercises 3

Hidden Markov Models

Tim works underground in a secret bunker. He does not get to see outside or talk to anyone. But he does see if Sally, who is allowed out, brings her umbrella to work each day.

Tim thinks that if its raining today, its 80% likely to be so tomorrow. If it is sunny today, it is 75% likely to be so tomorrow. He also knows Sally from before this job, and knows she doesn’t like getting rained on. He believes that 99% of the time that it is raining, she would have an umbrella, and that 50% of the time it is not raining she would still have one.

Create the tranmission and emission matrices of a hidden markov model representing Tim’s beliefs, where the state are {raining, not raining}.

- Transmission: $\begin{bmatrix} .8 & .2 \\ .25 & .75 \end{bmatrix}$
- Emission: $\begin{bmatrix} .99 & .01 \\ .5 & .5 \end{bmatrix}$

Hidden Markov Models

Tim starts work Sunday night. He knows that it was raining on Sunday. Today is Friday, and he has seen the sequence <-U,-U,U,-U,U>. What is the probability it will rain tomorrow (Saturday)?

What is the probability it rained on Wednesday?

What is the most likely sequence of weather states given the sequence of observations?

Forward Algorithm To Friday

- Sunday$= \begin{bmatrix} 1 & 0 \end{bmatrix}$
- Monday$\propto \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} .8 & .2 \\ .25 & .75 \end{bmatrix} \begin{bmatrix} .01 & 0 \\ 0 & .5 \end{bmatrix} \rightarrow$Monday$\approx \begin{bmatrix} .07 & .93 \end{bmatrix}$
- Tuesday$\propto \begin{bmatrix} .07 & .93 \end{bmatrix} \begin{bmatrix} .8 & .2 \\ .25 & .75 \end{bmatrix} \begin{bmatrix} .01 & 0 \\ 0 & .5 \end{bmatrix} \rightarrow$Tuesday$\approx \begin{bmatrix} .01 & .99 \end{bmatrix}$
- Wed. $\propto \begin{bmatrix} .01 & .99 \end{bmatrix} \begin{bmatrix} .8 & .2 \\ .25 & .75 \end{bmatrix} \begin{bmatrix} .99 & 0 \\ 0 & .5 \end{bmatrix} \rightarrow$Wed. $\approx \begin{bmatrix} .40 & .60 \end{bmatrix}$
- Thur. $\propto \begin{bmatrix} .40 & .60 \end{bmatrix} \begin{bmatrix} .8 & .2 \\ .25 & .75 \end{bmatrix} \begin{bmatrix} .01 & 0 \\ 0 & .5 \end{bmatrix} \rightarrow$Thur. $\approx \begin{bmatrix} .02 & .98 \end{bmatrix}$
- Fri. $\propto \begin{bmatrix} .02 & .98 \end{bmatrix} \begin{bmatrix} .8 & .2 \\ .25 & .75 \end{bmatrix} \begin{bmatrix} .99 & 0 \\ 0 & .5 \end{bmatrix} \rightarrow$Fri. $\approx \begin{bmatrix} .41 & .59 \end{bmatrix}$

Probability of rain on Saturday

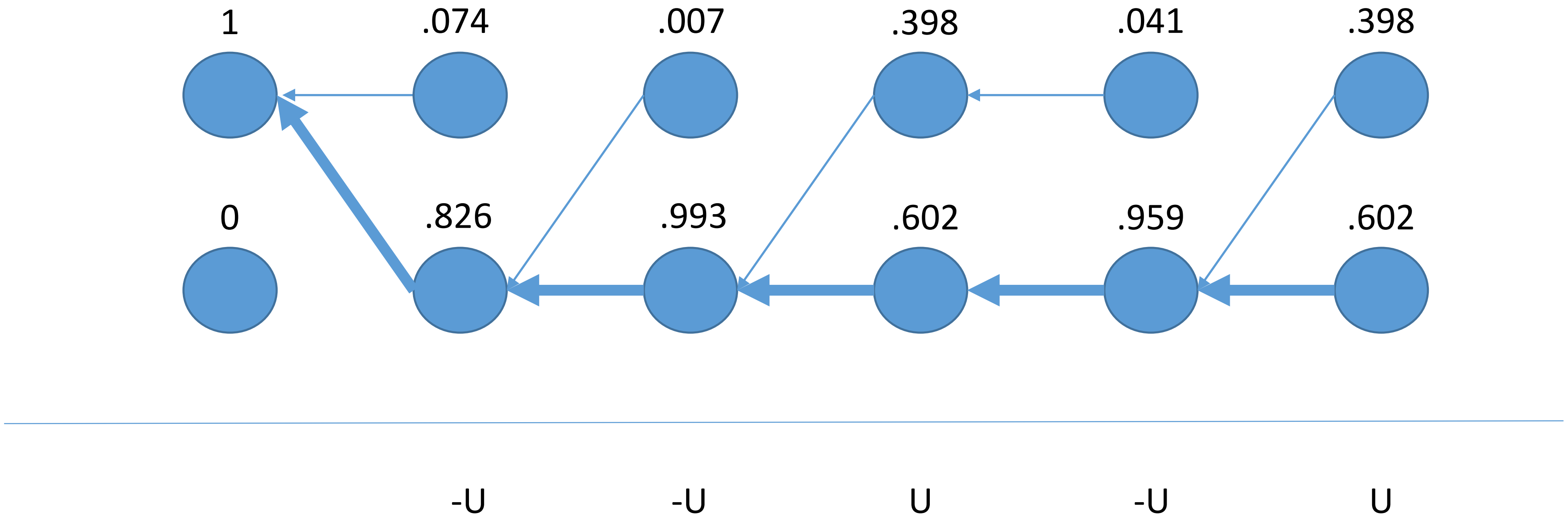
Sat.$\approx [.41 \quad .59]\begin{bmatrix} .8 & .2 \\ .25 & .75 \end{bmatrix} = [.48 \quad .52]$

Backward Algorithm To Wednesday

- Friday$= [1 \quad 1]$
- Thursday $= [1 * .8 * .99 + 1 * .2 * .5 \quad 1 * .25 * .99 + 1 * .75 * .5] \approx [.892 \quad .6225]$
- Wed.$\approx [.892 * .8 * .01 + .6225 * .2 * .5 \quad .892 * .25 * .01 + .6225 * .75 * .5] \approx [.069 \quad .236]$

So the smoothed probability for Wednesday is:

Wed. $\propto [.40 \quad .60]^T [.069 \quad .236] \rightarrow$ Wed. $\approx [.163 \quad .837]$



The most probable sequence of states is given by the large arrows.
I normalize at each step to avoid having to deal with really small numbers. When computing a better alternative is to use log values.
Rounding errors are accumulating.

Restricted Boltzmann Machines, Markov Random Fields and Gibbs Sampling

- You have a RBM that has been trained to model the patterns found in a dataset of three binary variables. The weights are as in the diagram.
- You have an incomplete datum, d=<1,?,1>, and you want to find the probability distribution over the missing value given your RBM model using Gibbs Sampling.
- Describe the complete process for using Gibbs Sampling in this case.
- Find the first sample (ignoring the original completely random value assignment), presuming that the most probable value is always the one sampled.

Weight	Value
$w_{1,1}$	2.7
$w_{1,2}$	1.6
$w_{1,3}$	-1.1
$w_{2,1}$	14.5
$w_{2,2}$	2.4
$w_{2,3}$	1
$w_{3,1}$	0.01
$w_{3,2}$	-7.6
$w_{3,3}$	-2.3

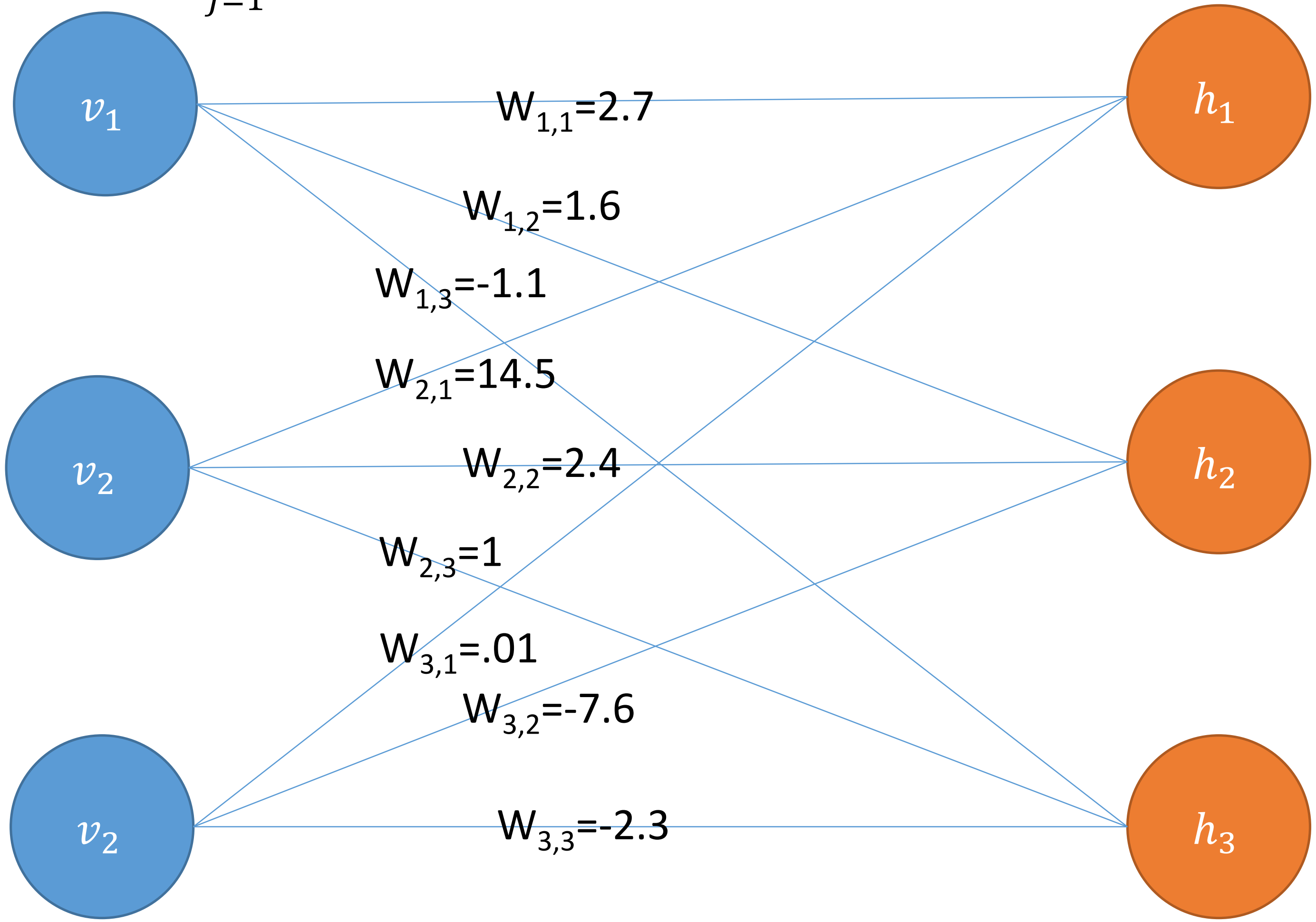
Process for using Gibbs Sampling

1. Assign all unknown variables a random value. Let these values be sample 0.
2. Generate a new sample i+1 from sample i by iterating through the unknown variables. For each calculate the its probability distribution given the values currently assigned to its neighbors. Then assign it a new value drawn from this distribution.
3. Generate as many samples as desired. Discard the first *m* samples (the burn), and thin the remainder if desired.
4. Count the number of times our unknown second variable takes the value 0 and 1. Normalize these counts for our probability distribution estimation.

I forgot to state clearly how many hidden variables there are in the RBM, but from the weights we see that there are three.

$$P(v_i = 1) = \sigma(\sum_{j=1}^{3} h_j\, w_{i,j})$$

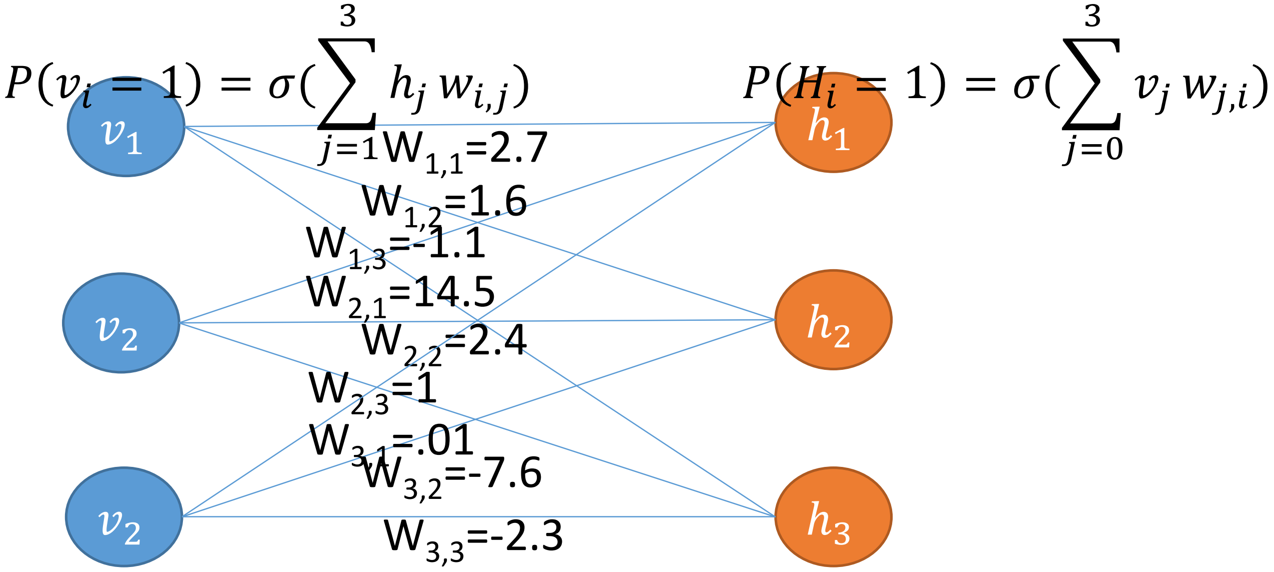
$$P(H_i = 1) = \sigma(\sum_{j=0}^{3} v_j\, w_{j,i})$$



- For sample 0 we assign random values to all unknown variables:

  <**1**,0,**1**,0,1,1> (bold are known variables)

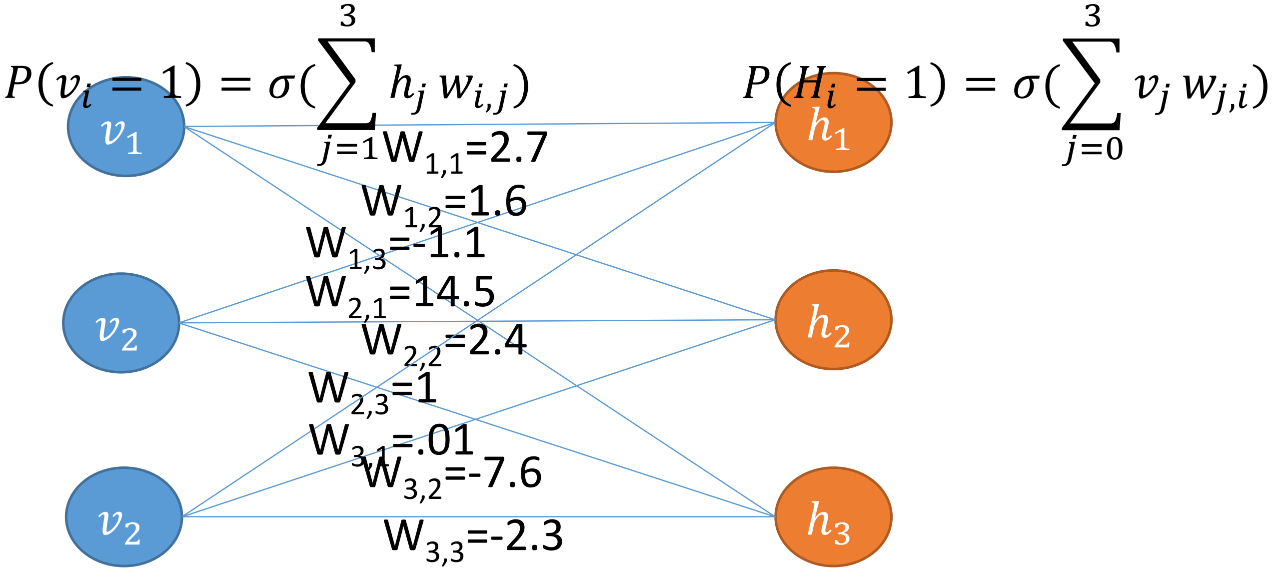
- To generate sample 1, we generate new values for each unknown variable from its probability distribution, given the current assigned values of its neighbors.



- For the second visible variable, its distribution is:

$$P(v_2 = 1) = \sigma\left(\sum_{j=1}^{3} h_j w_{i,j}\right) = \sigma(3.4) = .97$$

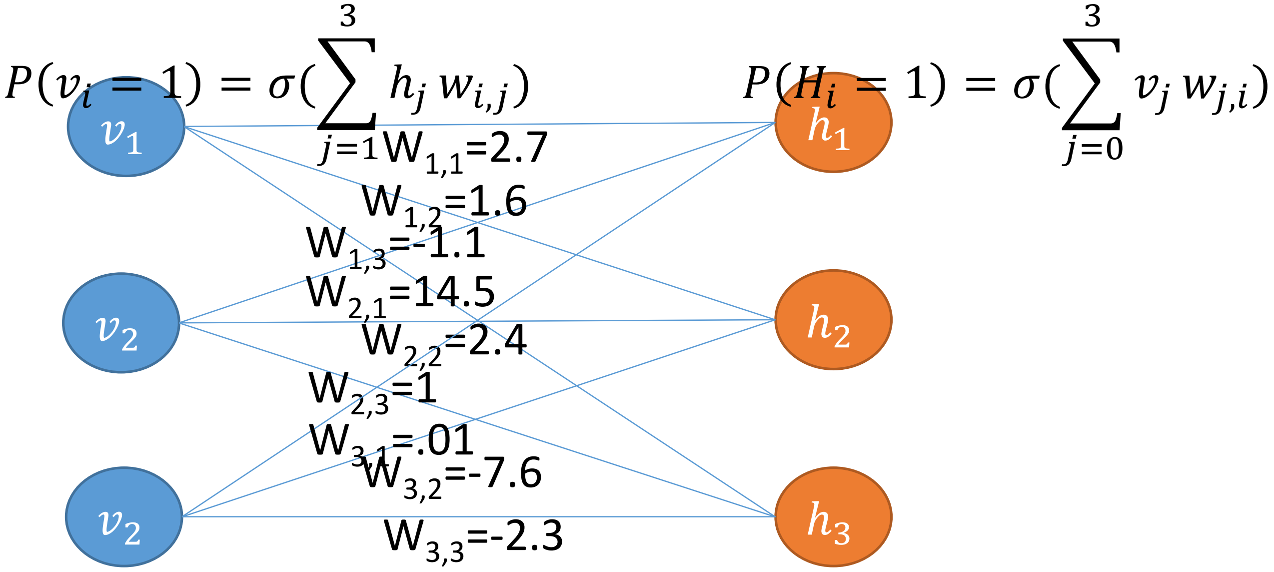
So we assign it value 1, and the currently assigned values are: <**1**,1,**1**,0,1,1>



- For the first hidden variable, its distribution is:

$$P(h_1 = 1) = \sigma\left(\sum_{j=1}^{3} v_j w_{j,i}\right) = \sigma(17.21) = 1$$

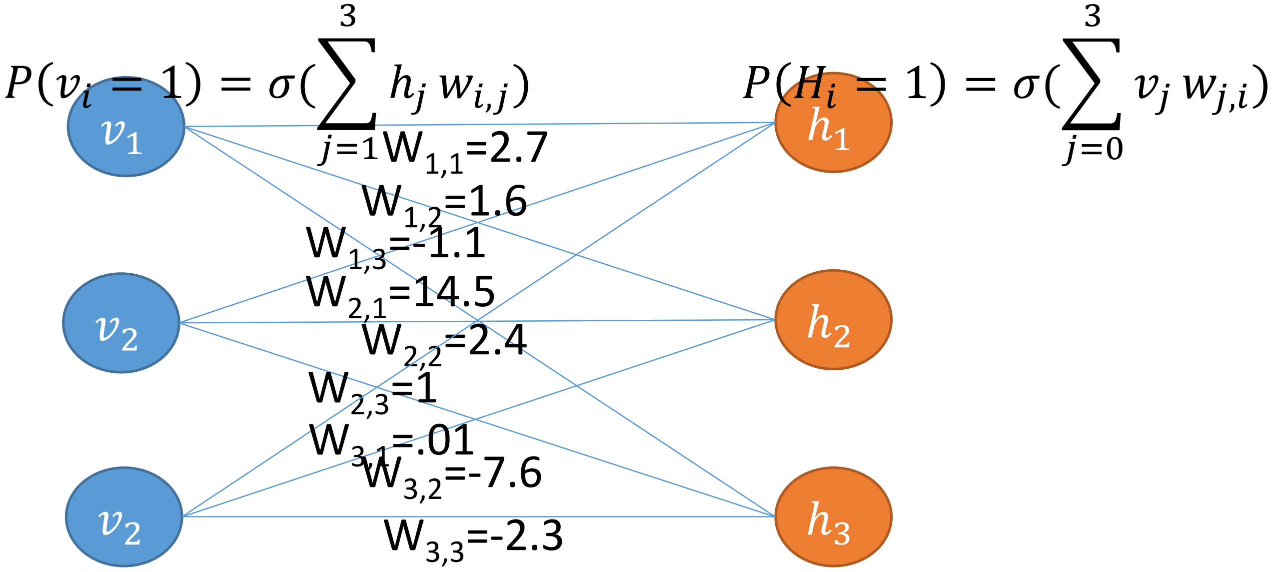
So we assign it value 1, and the currently assigned values are: <**1**,1,**1**,1,1,1>



- For the second hidden variable, its distribution is:

$$P(h_2 = 1) = \sigma\left(\sum_{j=1}^{3} v_j w_{j,i}\right) = \sigma(-3.6) = .03$$

So we assign it value 0, and the currently assigned values are: <**1**,1,**1**,1,0,1>



- For the third hidden variable, its distribution is:

$$P(h_2 = 1) = \sigma\left(\sum_{j=1}^{3} v_j\, w_{j,i}\right) = \sigma(-2.4) = .08$$

So we assign it value 0, and the currently assigned values are: <**1**,1,**1**,1,0,0>

And we are done, so our first sample is: <**1**,1,**1**,1,0,0>