# Group-Sparse Matrix Factorization for Transfer Learning of Word Embeddings

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## Abstract

Sparse regression has recently been applied to enable transfer learning from very limited data. We study an extension of this approach to unsupervised learning—in particular, learning word embeddings from unstructured text corpora using low-rank matrix factorization. Intuitively, when transferring word embeddings to a new domain, we expect that the embeddings change for only a small number of words—e.g., the ones with novel meanings in that domain. We propose a novel group-sparse penalty that exploits this sparsity to perform transfer learning when there is very little text data available in the target domain—e.g., a single article of text. We prove generalization bounds for our algorithm. Furthermore, we empirically evaluate its effectiveness, both in terms of prediction accuracy in downstream tasks as well as the interpretability of the results.

## 1. Introduction

While machine learning algorithms have proven to be tremendously effective at solving supervised and unsupervised problems, achieving good performance typically requires large training datasets. Yet, in many domains, there is very little data available for training. Thus, there has been a great deal of interest in transfer learning, where the goal is to leverage knowledge in a data-rich source domain to improve performance in a data-poor target domain.

A surprisingly effective transfer learning strategy is to simply fine-tune a model trained on data from the source domain (which we call the *proxy data*) on data from the target domain (which we call the *gold data*). For instance, this strategy has been used in transferring image classification models (Esteva et al., 2017), healthcare decision-making (Bastani, 2020), and word embeddings (Dingwall & Potts, 2018). Intuitively, stochastic gradient descent has regularization properties similar to $\ell_2$ regularization (Ali et al., 2020), so this strategy can be interpreted as regularizing the parameters towards those of the proxy model—i.e., adopting a loss function of the form

$$\widetilde{\ell}(\theta; X_g, \widehat{\theta}_p) = \ell(\theta; X_g) + \|\theta - \widehat{\theta}_p\|_2^2,$$

where $X_g$ is the gold training data, $\ell(\theta; X_g)$ is the unregularized loss, and $\widehat{\theta}_p$ are the parameters estimated using the proxy training data.

With this viewpoint, a natural strategy is to leverage alternative regularization strategies towards the source domain, instead of the $\ell_2$ norm. Recent work has investigated using the $\ell_1$ norm in the setting of (generalized) linear regression (Bastani, 2020)—i.e.,

$$\widetilde{\ell}(\theta; X_g, Y_g, \widehat{\theta}_p) = \|Y_g - X_g\theta\|_2^2 + \lambda \cdot \|\theta - \widehat{\theta}_p\|_1.$$

Intuitively, $\ell_1$ regularization enables efficient learning in domains with very little data (Tibshirani, 1996; Chen et al., 1995; Candes & Tao, 2007; Bickel et al., 2009). The key assumption for this approach to work is that the values of the true parameters $\theta_g$ (for the target domain) must differ from $\theta_p$ (for the source domain) in only a few components—i.e., $\theta_g - \theta_p$ is sparse. If this assumption holds, then they prove that their strategy can learn from $\mathcal{O}(s \log d)$ samples instead of $\mathcal{O}(d)$ samples, where $d$ is the dimension of $\theta_g$ and $s = \|\theta_g - \theta_p\|_0$ is the sparsity. A natural question is whether these techniques can be leveraged beyond the setting of generalized linear regression.

In this paper, we apply this approach to matrix factorization, which underlies one of the most basic unsupervised learning algorithms—namely, learning word embeddings from large-scale unlabeled text corpora such as Wikipedia (Pennington et al., 2014). While more sophisticated techniques have been developed (Devlin et al., 2018), approaches based on generalizations of matrix factorization remain competitive and widely used, and also tend to be more interpretable since we can visualize vector embeddings of individual words.

The key question is identifying a notion of sparsity that we can leverage in this setting. Intuitively, we expect that only a small number of words in the target domain may change

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meaning compared to the source domain. For instance, in computer science, the term "object" (as in "object oriented") has a very different meaning than the usual English definition. Thus, we might expect very few word embeddings to change from the source domain to the target domain. More formally, let $U_p$ denote the proxy word embedding matrix, whose $i^{\text{th}}$ row $U_p^i$ is the true word embedding of word $i \in [d] = \{1, ..., d\}$ based on the proxy data; analogously, $U_g$ denotes the gold word embedding matrix. Then, we expect that the word embeddings for most words are equal in both domains—i.e., $U_g^i = U_p^i$ for most $i \in [d]$.

Based on this intuition, we formulate an objective that encodes a *group-sparse* penalty (Friedman et al., 2010; Simon et al., 2013), where each row is a group. Intuitively, a group sparse penalty partitions the parameters into groups, and encodes that only a small number of groups contain non-zero parameters; it does so by encoding an $\ell_1$ norm over the $\ell_2$ norm of each group. We propose a two-stage estimator that uses this penalty to solve the transfer learning problem. The first stage estimates the proxy word embeddings using only proxy data. Then, the second stage estimates the word embeddings of the gold data using $\ell_{2,1}$ regularization to impose group sparsity compared to the proxy word embeddings:

$$\|U_g - \widehat{U}_p\|_{2,1} = \sum_{i=1}^{d} \|U_g^i - \widehat{U}_p^i\|_2,$$

where $\widehat{U}_p$ is the estimated proxy word embedding matrix.

We prove sample complexity bounds for our estimator, demonstrating how it can substantially improve the quality of the word embeddings for the gold data. In particular, assuming that most word vectors are preserved between the source and the target domains, we show that our estimator requires exponentially less gold data to achieve the same accuracy compared to using the gold data alone. Our proof relies on a tail inequality for the group lasso (Lounici et al., 2011), combined with an error bound for low-rank matrix problems (Ge et al., 2017).

While our main results are for word embeddings trained using matrix factorization, we show that our approach also applies to nonlinear extensions of matrix factorization. In particular, we show how our group sparse penalty term can be leveraged in conjunction with the GloVe word embedding objective (Pennington et al., 2014).

We have evaluated our approach to learn word embeddings for Wikipedia articles in domains such as finance, math, computing etc. In particular, we demonstrate that our approach identifies words with novel meanings in this domain at a high rate. These results demonstrate the interpretability of word embeddings learned using our algorithm. Finally, we demonstrate the efficacy of our approach in a downstream task where the goal is to predict clinical trial eligibility based on unstructured clinical statements regarding inclusion or exclusion criteria.

**Related work.** Typically, transfer learning refers to learning with a large amount of data in the source domain, and a small amount of data in the target domain. Broadly speaking, the two domains must be connected in some way: they can either have the same covariate distribution $p(x)$ but different label distributions $p(y \mid x)$ and $q(y \mid x)$ (called *label shift*), or vice versa (called *covariate shift*). Approaches targeting the latter setting are typically referred to as *domain adaptation*.

Recently, Bastani (2020) has applied sparse regression to handle label shift, when the shift is sparse—i.e., $y = x^T \beta_p + \epsilon$ vs. $y = x^T \beta_g + \epsilon$, where $\|\beta_p - \beta_g\|_0$ is much smaller than the dimension of $\beta_p, \beta_g$; here $p$ refers to *proxy data* from the source domain, and $g$ refers to *gold data* from the target domain. When the parameters $\beta_g$ are sparse (i.e., $s = \|\beta_g\|_0$ is small), existing theory shows that the sample complexity of estimating $\beta_g$ is $\mathcal{O}(s \log d)$ instead of $\mathcal{O}(d)$, where $d$ is the dimension of $\beta_g$ (Bühlmann & Van De Geer, 2011). Their key theoretical result is that the sample complexity of estimating $\beta_g$ scales as $s$ even though $\beta_g$ itself may not be sparse. Instead, *relative sparsity* between $\beta_g$ and $\beta_p$ is sufficient to enable efficient transfer learning in high dimensions. A key limitation of their work is that they are limited to the supervised learning setting. Our motivation is to study the sample complexity of transfer learning in settings beyond supervised learning.

Given multiple proxy datasets as well as their "disparities" from the source domain, Crammer et al. (2008) study which proxy sources to use in supervised learning to minimize generalization error. Zhang et al. (2013) propose importance reweighting and sample transformation to correct the data distribution shift, and Ganin & Lempitsky (2015) add a domain classifier into their deep feed-forward neural network framework to fine-tune the source model. More relatedly, there has been work proving generalization bounds for unsupervised domain adaptation (Ben-David et al., 2007; 2010); unlike our work, they assume a large number of unlabeled examples from the target domain.

For word embedding models, a standard approach is to fine-tune the pre-trained word embeddings end-to-end. A closely related approach is to add an $\ell_2$ penalty to the objective to regularize the word embeddings towards the existing ones Dingwall & Potts (2018); Yang et al. (2017). However, these approaches do not provide theoretical guarantees on their performance. In contrast, we prove theoretical bounds on the performance of our estimator under sparsity assumptions motivated by the domain distinction of word embeddings. We show that in the very low-data regime (e.g., a single article), using $\ell_1$ regularization outperforms $\ell_2$ regularization.

Another approach is to train *contextual embeddings* that capture different meanings of the same word based on their context (Devlin et al., 2018). Assuming the training corpus contains some data that covers the target domain, then one can automatically tailor word embeddings based on the given context. However, this such techniques lack the interpretability of traditional word embeddings, since we cannot examine or visualize the embedding of a single word in isolation. Also, they may not work when the training corpus altogether omits content from the target domain.

We build on approaches to word embeddings based on low-rank matrix factorization. Given a few observations $X_i$ about a matrix $\Theta \in \mathbb{R}^{d_1 \times d_2}$ with rank $r$, the goal is to compute a low-rank estimate $\widehat{\Theta}$. Recent work has provided an algorithm based on nuclear norm regularization, and proves a bound $\|\widehat{\Theta} - \Theta\|_F = \mathcal{O}(\sqrt{d/n})$ on the estimation error (Negahban & Wainwright, 2011). A more practical algorithm is the Burer-Monteiro approach (Burer & Monteiro, 2003), which replaces $\Theta$ explicitly with a low-rank representation $UV^T$, with $U \in \mathbb{R}^{d_1 \times r}$ and $V \in \mathbb{R}^{d_2 \times r}$, and minimizes the objective in terms of $U$ and $V$. This approach is nonconvex but is simpler to implement and computationally efficient. Ge et al. (2017) show that the local minima of this nonconvex problem are also global minima under the restricted isometry property (RIP).

A simple way to construct word embeddings is to take $\Theta$ to encode the relationships between words (e.g., the co-occurrence matrix, in which $\Theta_{ij}$ counts how many times words $i$ and $j$ occur together in a fixed-length window), run low-rank matrix factorization to compute $UV^T \approx \Theta$, and then choose the $i^{\text{th}}$ row of $U$ to be the embedding of word $i$. Levy & Goldberg (2014) shows that skip-gram with negative sampling implicitly factorizes a word-context matrix, described by pointwise mutual information (PMI) matrix. GloVe (Pennington et al., 2014), which can be thought of as a nonlinear version of this approach, was a state-of-the-art technique until recently. We show how our approach can be extended to GloVe, although our theoretical guarantees only hold for the linear setting. Recently, contextual embeddings have been shown to outperform GloVe. However, they assign vectors to sequences of words, not to individual words, making them less widely applicable as well as less interpretable.

## 2. Problem Formulation

In this section, we formalize the problem of group sparse transfer learning for word embeddings. We begin by giving background on matrix factorization, and then formalize the transfer learning problem along with our assumptions on the group sparse structure of the word embeddings.

**Notation.** For a matrix $\Theta \in \mathbb{R}^{d_1 \times d_2}$ of rank $r$, we denote its singular values by $\sigma_1(\Theta) \geq \sigma_2(\Theta) \geq \cdots \geq \sigma_r(\Theta) > 0$, its Frobenius norm by $\|\Theta\|_F = \sqrt{\sum_{j=1}^r \sigma_j^2(\Theta)}$, its operator norm by $\|\Theta\| = \sigma_1(\Theta)$, its vector $\ell_\infty$ norm by $|\Theta|_\infty = \max_{i,j} |\Theta_{ij}|$, its vector $\ell_1$ norm by $|\Theta|_1 = \sum_{i,j} |\Theta_{ij}|$, its $j^{\text{th}}$ row by $\Theta^j$, and

$$\|\Theta\|_{2,1} = \sum_{j=1}^{d_1} \|\Theta^j\|$$

to denote its matrix $\ell_{2,1}$ norm. Given $\Theta, \Theta' \in \mathbb{R}^{d_1 \times d_2}$, we denote the matrix dot product by $\langle \Theta, \Theta' \rangle = \sum_{i=1}^{d_1} \sum_{j=1}^{d_2} \Theta_{ij} \Theta'_{ij}$. We let $[k] = \{1, 2, \cdots, k\}$.

**Matrix sensing.** In our formalism, we consider the general setting of matrix sensing—i.e., given noisy observations $X_i$ of linear projections $\Theta^* = U^* U^{*T}$, recover the underlying matrix $U^*$. In the case of word embeddings, $X_i$ are simply noisy observations of entries of $\Theta^*$; we give details below.

Formally, consider an unknown matrix $U^* \in \mathbb{R}^{d \times r}$, and let $\Theta^* = U^* U^{*T}$; note that $\Theta^* \in \mathbb{R}^{d \times d}$ is symmetric and has rank $r$. The goal is to estimate $U^*$ given observations $A_i \in \mathbb{R}^{d \times d}$ and $X_i \in \mathbb{R}$, for $i \in [n]$, where

$$X_i = \langle A_i, \Theta^* \rangle + \epsilon_i \qquad (1)$$

and $\epsilon_1, \ldots, \epsilon_n$ are i.i.d. $\sigma$-subgaussian random variables. For instance, in the application to word embeddings, the $A_1, \ldots A_{d^2}$ are the basis matrices—i.e., $A_{i+j \cdot d}$ equals 1 in position $(i, j)$ and equals 0 elsewhere.

In this formulation, we can only compute $U^*$ up to orthogonal change-of-basis since $\Theta^*$ is preserved under this transformation—i.e., if $\widetilde{U}^* = U^* R$ for an orthogonal matrix $R \in \mathbb{R}^{r \times r}$, then we have $\widetilde{U}^* \widetilde{U}^{*T} = U^* R R^T U^{*T} = U^* U^{*T} = \Theta^*$. Thus, the goal is to compute $\widehat{U}$ such that $\widehat{U} \approx U^* R$ for some orthogonal matrix $R$.

To simplify notation, we define the linear operator $\mathcal{A} : \mathbb{R}^{d \times d} \to \mathbb{R}^n$, where $\mathcal{A}(\Theta)_i = \langle A_i, \Theta \rangle$. Then, (1) becomes

$$X = \mathcal{A}(\Theta^*) + \epsilon,$$

where $X = \begin{bmatrix} X_1 & \ldots & X_n \end{bmatrix}^T$ and $\epsilon = \begin{bmatrix} \epsilon_1 & \ldots & \epsilon_n \end{bmatrix}^T$.

Now, given an estimator $\widehat{U}$, we measure the estimation error using the $\ell_{2,1}$ norm. We use this norm instead of the more typical Frobenius norm since it is more naturally compatible with the group-sparse structure. It is analogous to the fact that the $\ell_1$ error of $\ell_1$ regularized linear regression is more natural to bound; bounding the $\ell_2$ norm requires additional regularity assumptions. We refer the reader to Chapter 6 of Bühlmann & Van De Geer (2011) for a discussion. In addition, since we can only identify $U^*$ up to orthogonal change-of-basis, we consider the error in a specific direction as in Ge et al. (2017).

**Definition 1.** *Given $\widehat{U}, U^* \in \mathbb{R}^{d\times r}$, the error of $\widehat{U}$ is*

$$\ell(\widehat{U}, U^*) = \|\widehat{U} - R_{(\widehat{U},U^*)} U^*\|_{2,1},$$

*where $R_{(\widehat{U},U^*)} = \arg\min_{R:R^TR=RR^T=I} \|\widehat{U} - U^*R\|_F$.*

This definition of error is invariant under rotation.

**Transfer learning.** Consider unknown parameters $U_p^* \in \mathbb{R}^{d\times r}$ for the source domain, and unknown parameters $U_g^* \in \mathbb{R}^{d\times r}$ for the target domain. Our goal is to use data from the source domain to help estimate $U_g^*$. In particular, we assume given *proxy observations* $\mathcal{A}_p : \mathbb{R}^{d\times d} \to \mathbb{R}^{n_p}$ and $X_p \in \mathbb{R}^{n_p}$ from the source domain, along with *gold observations* $\mathcal{A}_g : \mathbb{R}^{d\times d} \to \mathbb{R}^{n_g}$ and $X_g \in \mathbb{R}^{n_g}$ from the target domain, where

$$X_p = \mathcal{A}_p(\Theta_p^*) + \epsilon_p$$
$$X_g = \mathcal{A}_g(\Theta_g^*) + \epsilon_g,$$

and $\epsilon_p \in \mathbb{R}^{n_p}$ and $\epsilon_g \in \mathbb{R}^{n_g}$ are vectors of i.i.d. $\sigma_p$ and $\sigma_g$-subgaussian random variables, respectively.

We are interested in the setting $(n_g/\sigma_g) \ll (n_p/\sigma_p)$. Intuitively, this condition says that either we have many more proxy observations than gold observations (i.e., $n_g \ll n_p$), or that the proxy observations are much lower variance than the gold observations (i.e., $\sigma_p \ll \sigma_g$). The latter case sometimes arises in low-data settings due to the observation structure; for instance, as we describe below, this is the case for our application to word embeddings.

**Group sparse structure.** To leverage the proxy observations to help estimate the gold parameters $U_g^*$, we need to assume some relationship between the two. Letting

$$\Delta_U^* = U_g^* - U_p^*,$$

we assume that $\Delta_U^*$ has a row-sparse structure—i.e., most of its rows are 0. More precisely, letting

$$J = \left\{ j \in [d] \;\middle|\; \|\Delta_U^{*j}\| \neq 0 \right\},$$

the *group sparsity* of $\Delta_U^*$ is $s = |J|$. Then, an accurate estimate of $\Delta_U^*$ can help to recover $U_g^*$, since estimating $\Delta_U^*$ requires less data to recover due to its sparse structure.

Importantly, note that the row-sparse structure of $\Delta_U^*$ is preserved under *simultaneous* orthogonal transformations of $U_g^*$ and $U_p^*$—i.e., if $\widetilde{U}_p^* = U_p^*R$ and $\widetilde{U}_g^* = U_g^*R$ for an orthogonal matrix $R$, then $\widetilde{\Delta}_U^* = (U_g^* - U_p^*)R = \Delta_U^*R$ has the same row sparsity as $\Delta^*$.

**Application to word embeddings.** To apply matrix factorization to compute word embeddings, we begin by constructing the *word co-occurrence* matrix $\widehat{\Theta} \in \mathbb{R}^{d\times d}$, where $\widehat{\Theta}_{ij}$ counts the number of times the two words indexed by $i$ and $j$ appear together (e.g., in some fixed-length window of text); here, $d$ is the total number of the words. In addition, we normalize $\widehat{\Theta}$ (i.e., divide by the total count $\sum_{i,j} \widehat{\Theta}_{ij}$).

Intuitively, we think of $\widehat{\Theta}$ as an empirical estimate of $\Theta^*$, and take the observations $X_i$ to be the entries of $\widehat{\Theta}$. In particular, let $A_1, ..., A_{d^2} \in \mathbb{R}^{d\times d}$ such that $A_{i+j\cdot d} = \mathbb{1}(i = j)$ Then, we take

$$X_i = \langle A_i, \widehat{\Theta}\rangle,$$

in which case $\epsilon_i = \langle A_i, \widehat{\Theta}\rangle - \Theta^*$ is the error. This error is bounded (since $\widehat{\Theta}$ is normalized) and zero mean (by definition), so it is subgaussian. Thus, we can use matrix factorization on $X_i$'s and $A_i$'s to compute $\widehat{U}$ such that $\Theta^* \approx \widehat{U}\widehat{U}^T$. Finally, we take $\widehat{U}^i$ to be the word vector for word $i \in [d]$.

As discussed above, in this setting, the number of observations $n$ scales the subgaussian parameter of $\epsilon_i$ rather than the number of observations, which is always $d^2$. In particular, as more observations become available, the variance of our estimate $\widehat{\Theta}$ of $\Theta^*$ becomes smaller.

## 3. Naïve Estimators

We begin by describing two naïve strategies for estimating $U_g^*$: one based on only using the gold data, and one based on only using the proxy data. Our proposed estimator (described in Section 4) builds on these ones.

### 3.1. Gold Estimator

First, we consider estimating $U_g^*$ using only the gold data:

$$\widehat{U}_g = \arg\min_{U_g} \frac{1}{n_g}\|X_g - \mathcal{A}_g(U_gU_g^T)\|^2. \tag{2}$$

Now, we analyze the sample complexity of $\widehat{U}_g$ under standard regularity assumptions. In particular, we assume restricted well-conditionedness (RWC) (Li et al., 2019).

**Definition 2.** *A linear operator $\mathcal{A}$ satisfies the $r$-RWC condition if*

$$\alpha\|Z\|_F^2 \leq \frac{1}{n}\|\mathcal{A}(Z)\|^2 \leq \beta\|Z\|_F^2,$$

*with $3\alpha > 2\beta$ and for any $Z \in \mathbb{R}^{d\times d}$ with $\operatorname{rank}(Z) \leq r$.*

This property is a generalizations of the restricted isometry property (RIP), which is a common assumption in matrix sensing problems. Under the RIP condition, common low-rank matrix problems have no spurious local minima—i.e. all local minima are also global minima (Bhojanapalli et al., 2016; Ge et al., 2017). However, the RIP condition is very restrictive as it requires all the eigenvalues of the Hessian matrix to be within a small range of 1. The RWC condition applies to more general situations and also guarantees the statistical consistency for all local minima (Li et al., 2019).

**Theorem 1.** *Assume $\mathcal{A}_g$ satisfies $2r$-RWC. Then, we have*

$$\ell(\widehat{U}_g, U_g^*) \le C_g\sqrt{\frac{\sigma_g^2(2rd(2d+1)\log(36\sqrt{2}) + d\log(\frac{2}{\delta}))}{n_g}} = \mathcal{O}\left(\sqrt{\frac{\sigma_g^2(d^2 + d\log(\frac{1}{\delta}))}{n_g}}\right)$$

*with probability at least $1-\delta$.*

Here, $C_g$ is only a function of $\alpha$, $\beta$, and $\sigma_r(U_g^*)$. We give a full statement of this theorem in Appendix B, and a proof in Appendix C.

### 3.2. Proxy Estimator

Next, we consider a strategy that estimates $U_g^*$ by estimating $U_p^*$ and ignoring the bias term $\Delta_U^*$:

$$\widehat{U}_p = \underset{U_p}{\arg\min}\frac{1}{n_p}\|X_p - \mathcal{A}_p(U_pU_p^T)\|^2. \tag{3}$$

We have the following result:

**Theorem 2.** *Assume $\mathcal{A}_p$ satisfies $2r$-RWC. Then, we have*

$$\ell(\widehat{U}_p, U_g^*) = \mathcal{O}\left(\|\Delta_U^*\|_{2,1} + \omega + \sqrt{\frac{\sigma_p^2(d^2 + d\log(\frac{1}{\delta}))}{n_p}}\right),$$

*with probability at least $1-\delta$, where*

$$\omega = \|U_p^*(R_{(\widehat{U}_p,U_p^*)} - R_{(\widehat{U}_p,U_g^*)})\|_{2,1}.$$

Since $U_p^*$ may not be aligned with $U_g^*$, the estimation error using $\widehat{U}_p$ as an estimator of $U_g^*$ includes a term $\omega$ accounting for the difference between $U_p^*$ and $U_g^*$. When $R_{(\widehat{U}_p,U_p^*)} = R_{(\widehat{U}_p,U_g^*)}$, we have $\omega = 0$. In this case, the error decomposes into the bias term $\|\Delta_U^*\|_{2,1}$ plus the error of $\widehat{U}_p$ compared to $U_p^*$. We give a full statement in Appendix B, and a proof in Appendix D.

## 4. Group Sparse Transfer Learning

In this section, we describe our proposed estimator that combines the gold and proxy data. Then, we state the quadratic compatibility condition, which extends the standard compatibility condition from the sparse regression literature (Lounici et al., 2011) to the matrix factorization setting, and prove sample complexity bounds assuming this condition holds. Finally, we describe how our group-sparse penalty term can be leveraged in conjunction with the GloVe objective.

### 4.1. Estimation Procedure

We define our proposed joint estimator for gold task as through the following two steps:

$$\begin{aligned}\widehat{U}_p &= \underset{U_p}{\arg\min}\frac{1}{n_p}\|X_p - \mathcal{A}_p(U_pU_p^T)\|^2\\ \widehat{U}_g &= \underset{g(U_g)\le 2L}{\arg\min}\frac{1}{n_g}\|X_g - \mathcal{A}_g(U_gU_g^T)\|^2 + \lambda\|U_g - \widehat{U}_p\|_{2,1}\end{aligned} \tag{4}$$

Since our problem is nonconvex, we follow Loh & Wainwright (2015) and define a compact search region for $U_g$: $g(U_g) = \|U_g - \widehat{U}_p\|_{2,1} \le 2L$. $L$ is a tuning parameter that should be chosen carefully to make $U_g$ feasible—specifically, $\|U_g^* - U_p^*\|_{2,1} \le L$.

### 4.2. Quadratic Compatibility Condition

We make the following key assumption, which generalizes the compatibility condition required in the traditional sparse regression setting:

**Definition 3.** *The quadratic compatibility condition is*

$$\frac{s}{n_g}\|\mathcal{A}_g(\Delta U_g^{*T} + U_g^*\Delta^T + \Delta\Delta^T)\|^2 \ge \kappa(\sum_{j\in J}\|\Delta^j\|)^2$$

*for any $\Delta \in \mathbb{R}^{d\times r}$ that satisfies*

$$\sum_{j\in J^c}\|\Delta^j\| \le 7\sum_{j\in J}\|\Delta^j\|.$$

Compared to the standard compatibility condition in the group sparse setting (Lounici et al., 2011), we have an extra quadratic term $\Delta\Delta^T$ since we are considering the nonconvex matrix factorization setting. We give a detailed discussion of this condition in Appendix A.

### 4.3. Main Result

Our main result characterizes the estimation error of our joint estimator $\widehat{U}_g$.

**Theorem 3.** *Let $\lambda = \lambda_0$ (where $\lambda_0$ is defined in Appendix B). Assuming $\mathcal{A}_p$ satisfies $(2r)$-RWC, $\mathcal{A}_g$ satisfies the quadratic compatibility condition, and*

$$\lambda_0 \le \frac{L\sigma_r(U_p^*)(3\alpha - 2\beta)}{16},$$

*then, we have*

$$\ell(\widehat{U}_g, U_g^*) = \mathcal{O}\left(\sqrt{\frac{\sigma_g^2 s^2\log(\frac{d}{\delta})}{n_g}} + \sqrt{\frac{\sigma_p^2 s^2(d^2 + d\log(\frac{1}{\delta}))}{n_p}}\right),$$

*with probability at least $1-\delta$.*

*Table 1.* Error bound for naïve estimators and joint estimator.

| **Estimator** | Joint | Gold | Proxy |
|---|---|---|---|
| **Error Bound** | $\mathcal{O}\left(\sqrt{\frac{s^2 \log d}{n_g}} + \sqrt{\frac{s^2 d^2}{n_p}}\right)$ | $\mathcal{O}\left(\sqrt{\frac{d^2}{n_g}}\right)$ | $\mathcal{O}\left(\\|\Delta_U^*\\|_{2,1} + \omega + \sqrt{\frac{d^2}{n_p}}\right)$ |

We give the expression of $\lambda_0$ and a full statement of the theorem in Appendix B, and a proof in Appendix E. We note that the required condition on $\lambda_0$ in Theorem 3 is easily satisfied in our "gold-scarce and proxy-rich" setting—for instance $n_g \gg \log(d)$ and $n_p \gg d^2$.

We summarize the estimation error bounds we derived for the three different estimators in Table 1. In the regime of interest—we have access to lots of proxy data ($n_p \gg d^2$) but limited gold data ($n_g \ll d^2$)—the upper bound of our joint estimator is much smaller in contrast to the typical proxy and gold estimators. In particular, taking $n_p \to \infty$, our bound scales as $\sqrt{s^2 \log d / n_g}$ whereas the gold bound scales as $\sqrt{d^2/n_g}$, which is an improvement of $s/d$. Alternatively, the proxy bound scales as at least $\|\Delta_U^*\|_{2,1}$, which does not go to zero with $n_g$; in contrast, our bound does.

### 4.4. Application to GloVe

The original GloVe method solves the following optimization problem (Pennington et al., 2014):

$$\min_{U^i, V^j, b_i, c_j} \sum_{i,j \in [d]} f(X_{ij})(\log(X_{ij}) - (U^i V^{jT} + b_i + c_j))^2,$$

where $X_{ij}$ is the total number of co-occurrences of word $i$ and $j$, $\{U^i\}$ and $\{V^j\}$ are the two sets of word embeddings, and $d$ is the vocabulary size; $f(X_{ij})$ is a weighting function that is non-decreasing in co-occurrence; $b_i, c_j \in \mathbb{R}$ are bias terms. In practice, GloVe takes the sum of the two sets of embeddings as the final embeddings, i.e. $U^i + V^i$ is the word vector for word $i$. To leverage our approach, we add a group lasso penalty to this objective:

$$\min_{U^i, V^j, b_i, c_j} \sum_{i,j \in [d]} f(X_{ij})(\log(X_{ij}) - (U^i V^{jT} + b_i + c_j))^2 + \lambda \sum_{i \in [d]} \|(U^i + V^i) - \widehat{U}_p^i\|, \tag{5}$$

where $\widehat{U}_p$ is the pre-trained GloVe embedding matrix.

## 5. Experiments

We evaluate our joint estimator on both synthetic and real data. On the synthetic data, we compare the error of our estimator with the ground truth parameters. Then, we consider two real datasets; in this case, we leverage our penalty in conjunction with the GloVe objective. First, we apply it to Wikipedia articles from specific domains (e.g., math), and evaluate whether it can identify words with novel meanings in that domain; this experiment demonstrates the interpretability of our approach. Second, we evaluate the downstream prediction accuracy of our word embeddings on a clinical trial eligibility data.

### 5.1. Experiments on Synthetic Data

**Data.** We focus on the low-data setting; in particular, we let $n_g = 50$, $n_p = 5,000$, and $d = 20$. We consider the exact low-rank case with $r = 5$. The observation matrices $A_{p,i}$'s (and $A_{g,i}$'s) are independent Gaussian random matrices whose entries are i.i.d. $N(0,1)$. We generate $\Theta_p^*$ by choosing $U_p^*$ with i.i.d. $N(0,1)$ elements. To construct the gold data, we set the row sparsity of $\Delta_U^*$ to 10% ($s = 2$). Then, we randomly pick $s$ rows and set the value of each entry to 1. We take both noise terms to be $\epsilon_{p,i}, \epsilon_{g,i} \sim N(0,1)$.

**Setup.** We compute the gold, proxy, and joint estimators by solving optimization problems (2), (3), and (4), respectively. To construct the joint estimator, we also need to pick a proper value for the hyperparameter $\lambda$. We use 5-fold cross validation to tune $\lambda$ and we keep 20% of the gold data as the cross validation set. As all the final estimates of $U_g$ might suffer an orthogonal change-of-basis, we instead report the Frobenius norm of the estimation error of $\Theta_g$. We average this error over 50 random trials.

**Results.** Figure 1 shows the Frobenius error of the naïve estimators (i.e., the gold and proxy estimators from Section 3 and our joint estimator, with a 95% confidence interval. Our joint estimator significantly outperforms the other two estimators—in particular, the Frobenius error of our joint estimator is only around 7% of the proxy estimator and 3% of the gold estimator.

### 5.2. Experiments on Wikipedia

One advantage of our method is that it is more interpretable—in particular, we show that it can be used to identify the domain words (i.e., words that have a special meaning in a certain domain). We apply our method to GloVe and evaluate its performance on single domain Wikipedia articles in terms of the accuracy at identifying domain words. We compare our joint estimator with Mittens (Dingwall & Potts, 2018) as well as with randomly selecting the words.

**Data.** We manually curated 37 Wikipedia articles from the

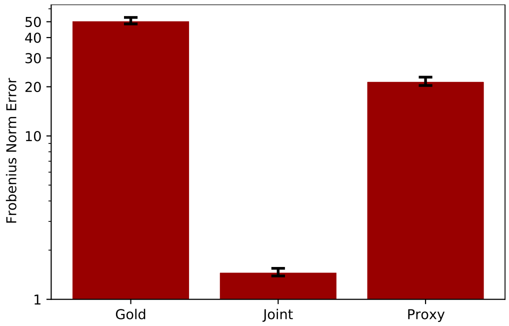

*Figure 1.* Frobenius norm estimation error of $\Theta_g$ over 50 trials. A 95% confidence interval is provided for each estimator. Y-axis is in log scale.

following four domains: finance, math, computer science, and politics. The articles selected all have a domain-specific word in their title—e.g., "put" in the article "put option" (in finance), "closed" in "closed set" (in math), "object" in computing, and "left" in "left wing politics" (in politics). All the Wikipedia text data were downloaded from the English Wikipedia database dumps[1] in January 2020. We preprocess the text by splitting and tokenizing sentences, removing short sentences that contain less than 20 characters or 5 tokens, and removing stopping words.

We download the pretrained word embeddings from GloVe's official website[2]. We take those trained using the 2014 Wikipedia dump and Gigaword 5, which contains around 6 billion tokens and 400K vocabulary words.

**Setup.** We solve the optimization problem (5) to construct our joint estimator for each single article. We take the pretrained GloVe word embedding as described above. Similar to GloVe, we create the co-occurrence matrix using a symmetric context window of length 5. We choose the dimension of the word embedding to be 100 and use the default weighting function of GloVe. The Mittens word embeddings are obtained solving a similar problem as (5), but with the Frobenius norm penalty—i.e.,

$$\sum_{i\in[d]} \|(U^i + V^i) - \widehat{U}_p^i\|^2.$$

We fix $\lambda = 0.05$ for both approaches; we found our results to be robust to this choice. Then, to identify domain words, we score each word $i$ by the $\ell_2$ distance between its new embedding (i.e., our joint estimator or Mittens) and its pretrained embedding; a higher score indicates more likely to be a domain word.

To compare the domain word identification accuracy, we set select the top 10% of words according to this score—i.e., treat all words in the top 10% as positives. We define a word to be a domain word if any of its definitions on Wiktionary is labeled with key words from that domain—i.e., "finance" or "business" for finance, "math", "geometry", "algebra", or "group theory" for math, "computing" or "programming" for computer science, and "politics" for politics. Finally, we compute the $F_1$-score of the selected domain words across articles in each domain. To calculate the $F_1$-score of random selection, we compute the precision and recall in close form; in particular, the precision is the fraction of domain words among all vocabulary words, and the recall is the top fraction we set (i.e., 10%).

*Table 2.* Weighted F1-score of domain word identification across four domains, where we select the top 10% of words.

| Domain | Joint | Mittens | Random |
|---|---|---|---|
| Finance | **0.2280** | 0.1912 | 0.1379 |
| Math | **0.2546** | 0.2171 | 0.1544 |
| Computing | **0.2613** | 0.1952 | 0.1436 |
| Politics | **0.1852** | 0.1543 | 0.0634 |

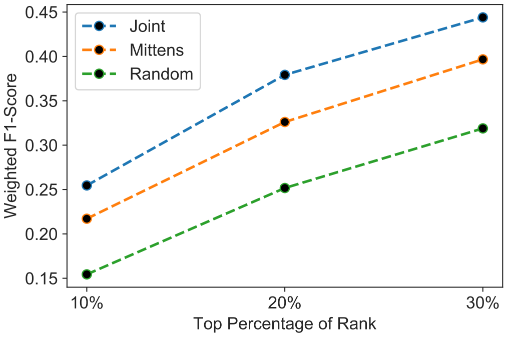

*Figure 2.* Weighted F1-score versus top percentage of the rank set for the threshold in the math domain.

**Results.** Table 2 shows the $F_1$-score of each approach for each domain. Our method consistently outperforms Mittens and random selection across all domains. While these results show that Mittens also learns domain-specific word embeddings, our estimator does so more effectively, most likely since our sparsity assumption fits text data well.

Next, we evaluate how our result varies with the value of selection threshold; in particular, we consider 10%, 20%, and 30%. Figure 2 shows the weighted $F_1$-score versus the top percentage set for the threshold in the math domain. Our approach consistently outperforms the baselines. Finally, Table 3 shows the top 10 words ranked by our approach and by Mittens for one article in each domain. As can be seen, our approach is effective at identifying domain words.

[1] https://dumps.wikimedia.org/enwiki/latest/

[2] https://nlp.stanford.edu/projects/glove/

*Table 3.* Top 10 words in the rank sorted by absolute change of word embedding from source to target domain. We pick one article from each domain, with the domain words labeled in bold. The threshold is set to top 10% of the rank.

| **Short** | | **Prime Number** | | **Cloud Computing** | | **Conservatism** | |
|---|---|---|---|---|---|---|---|
| Joint | Mittens | Joint | Mittens | Joint | Mittens | Joint | Mittens |
| **short** | **short** | **prime** | **prime** | **cloud** | **cloud** | **party** | **party** |
| **shares** | percent | **formula** | still | **data** | **private** | **conservative** | **conservative** |
| price | due | **numbers** | **formula** | **computing** | large | social | second |
| **stock** | public | **number** | de | **service** | information | conservatism | social |
| **security** | customers | **primes** | **numbers** | **services** | devices | government | research |
| selling | prices | **theorem** | **number** | **applications** | **applications** | **liberal** | svp |
| **securities** | high | **natural** | great | **private** | security | **conservatives** | government |
| **position** | hard | integers | side | users | work | political | de |
| may | **shares** | **theory** | way | use | **engine** | **right** | also |
| **margin** | price | **product** | algorithm | **software** | allows | economic | church |

### 5.3. Clinical Trial Eligibility

Another measure of the quality of the word embeddings is the prediction accuracy in downstream tasks. To this end, we consider a clinical trial eligibility prediction task, where the words are from the medical domain. We apply our approach in conjunction with GloVe, and compare it with the two naïve estimators as well as Mittens.

**Data.** The inclusion standard for cancer clinical trial is restrictive, and the protocols of inclusion or exclusion are only described in text. Bustos & Pertusa (2018) use deep neural networks to classify short clinical statements into inclusion or exclusion criteria, which aims to help determine the eligibility of patients for cancer clinical trials. In this experiment, we analyze a 1-million subsample of the original 6-million clinical trial eligibility data used in Bustos & Pertusa (2018), which has been made publicly available by the authors on Kaggle[3]. The data provides a label (eligible or not eligible), and a corresponding short free-text statement that describes the eligibility criterion and the study intervention and condition.

**Setup.** We study the low-data setting; in particular, we restrict to 50 observations; we use a balanced sample as in Bustos & Pertusa (2018). As before, we solve (5) for our joint estimator, and solve the same objective with the Frobenius norm penalty for Mittens. As in many semi-supervised studies, we train word embeddings using all text from both the training and test sets. In this task, we set $\lambda = 0.0001$ and the word embedding dimension to 100.

To predict eligibility, we use logistic regression with $\ell_2$ penalty. We split the 50 observations into 20% for testing and 80% for training and cross-validation. We use 5-fold cross-validation to tune the hyperparameter in regularized logistic regression. Since it is computationally expensive to feed all embeddings into the model, we instead take the average of embeddings of all words in an observation as the features to feed to the logistic regression model.

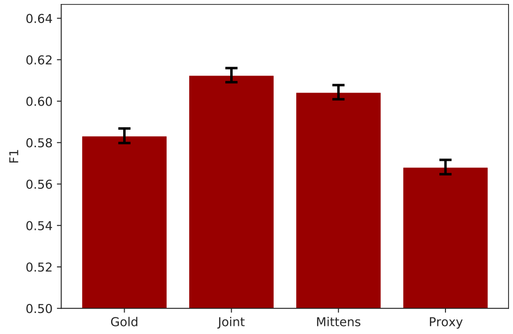

*Figure 3.* F1-score of prediction over 10K trials. A 95% confidence interval is provided for each estimator.

**Results.** We compare the $F_1$-score of our joint estimator with the two naïve estimators and Mittens. Figure 3 shows the average of $F_1$-score over 10K trials and a 95% confidence interval. As can be seen, our joint estimator outperforms all baselines on the downstream prediction task.

## 6. Conclusion

We have proposed a novel estimator for transfering word embeddings to new domains. We cast the problem as a low-rank matrix factorization problem with a group-sparse penalty, regularizing the learned embeddings towards existing domain-agnostic embeddings such as GloVe. Under a sparsity assumption and standard regularity conditions, our estimator provably requires exponentially less data to achieve the same error compared to the gold and proxy estimators. Our experiments demonstrate the effectiveness of our approach in the low-data regime on synthetic data, a domain word identification task on single Wikipedia articles, and a downstream clinical trial eligibility prediction task.

[3] https://www.kaggle.com/auriml/eligibilityforcancerclinicaltrials

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## A. Discussion of Quadratic Compatibility Condition

In this section, we show a connection between our quadratic compatibility condition (3) and the traditional compatibility condition commonly used for linear models. Consider the following convex optimization problem,

$$\min \frac{1}{n}\|X - \mathcal{A}(\Theta)\|^2 + \lambda\|\Theta\|_{2,1},$$

assuming $\Theta$ has a row-sparse structure. The compatibility condition of such a problem has a form of

$$\frac{s}{n}\|\mathcal{A}(\Delta)\|^2 \geq (\sum_{j\in J}\|\Delta^j\|)^2, \tag{6}$$

for any $\Delta$ in a constrained set, usually $\{\Delta \mid \sum_{j\in J^c}\|\Delta^j\| \leq C\sum_{j\in J}\|\Delta^j\|, C > 1\}$. The biggest distinction of our condition is that we put a quadratic form of $\Delta$ in the linear operator $\mathcal{A}$ instead of a linear form as above. We show that we can derive our quadratic compatibility condition from the compatibility condition (6) if we narrow down our search region to a small neighbourhood of $\Delta_U^*$.

Define a new operator $\bar{\mathcal{A}}_g : \mathbb{R}^{d\times d} \to \mathbb{R}^{n_g}$ with $\bar{\mathcal{A}}_g(\Theta)_i = \langle A_{g,i} + A_{g,i}^T, \Theta\rangle$. We have

$$\begin{aligned}
&\mathcal{A}_g(\Delta U_g^{*T} + U_g^*\Delta^T + \Delta\Delta^T)_i \\
=&\langle A_{g,i}, \Delta U_g^{*T} + U_g^*\Delta^T + \Delta\Delta^T\rangle \\
=&\langle A_{g,i} + A_{g,i}^T, \Delta U_g^{*T} + \frac{1}{2}\Delta\Delta^T\rangle \\
=&\bar{\mathcal{A}}_g(\Delta U_g^{*T} + \frac{1}{2}\Delta\Delta^T)_i.
\end{aligned}$$

Assume the following compatibility condition holds

$$\frac{s}{n_g}\|\bar{\mathcal{A}}_g(\nabla)\|^2 \geq \kappa_1(\sum_{j\in J}\|\nabla^j\|)^2,$$

for $\nabla = \Delta U_g^{*T} + \frac{1}{2}\Delta\Delta^T$ with $\sum_{j\in J^c}\|\Delta^j\| \leq 7\sum_{j\in J}\|\Delta^j\|$. To derive our quadratic compatibility condition, we only need

$$\sum_{j\in J}\|(\Delta U_g^{*T} + \frac{1}{2}\Delta\Delta^T)^j\| \geq \kappa_2\sum_{j\in J}\|\Delta^j\|. \tag{7}$$

Define a projection onto $U_g^*$ as $P_g = U_g^*(U_g^{*T}U_g^*)^{-1}U_g^{*T}$. Let $M_g = I - P_g$. Then

$$\begin{aligned}
\Delta U_g^{*T} + \frac{1}{2}\Delta\Delta^T &= \Delta U_g^{*T} + \frac{1}{2}\Delta\Delta^T P_g + \frac{1}{2}\Delta\Delta^T M_g \\
&= \Delta(I + \frac{1}{2}\Delta^T U_g^*(U_g^{*T}U_g^*)^{-1})U_g^{*T} + \frac{1}{2}\Delta\Delta^T M_g.
\end{aligned}$$

By the orthogonality of $M_g$ and $P_g$,

$$\begin{aligned}
&\|(\Delta U_g^{*T} + \frac{1}{2}\Delta\Delta^T)^j\|^2 \\
=&\|\Delta^j(I + \frac{1}{2}\Delta^T U_g^*(U_g^{*T}U_g^*)^{-1})U_g^{*T} + \frac{1}{2}\Delta^j\Delta^T M_g\|^2 \\
=&\|\Delta^j(I + \frac{1}{2}\Delta^T U_g^*(U_g^{*T}U_g^*)^{-1})U_g^{*T}\|^2 + \|\frac{1}{2}\Delta^j\Delta^T M_g\|^2 \\
\geq&\|\Delta^j(I + \frac{1}{2}\Delta^T U_g^*(U_g^{*T}U_g^*)^{-1})U_g^{*T}\|^2
\end{aligned}$$

If $I + \frac{1}{2}\Delta^T U_g^*(U_g^{*T}U_g^*)^{-1}$ have only non-zero eigenvalues for any $\Delta$, then we can guarantee a lower bound like (7). One sufficient condition is to have $\sigma_1(\Delta) \leq 2(1-\mu)\sigma_r(U_g^*)$ for some $1 > \mu > 0$. In this case,

$$\sigma_{\min}(\frac{1}{2}\Delta^T U_g^*(U_g^{*T}U_g^*)^{-1}) \geq -(1-\mu),$$

which guarantees

$$\sigma_{\min}(I + \frac{1}{2}\Delta^T U_g^* (U_g^{*T} U_g^*)^{-1}) \geq \mu > 0.$$

Therefore, we have

$$\frac{s}{n_g} \|\mathcal{A}_g(\Delta U_g^{*T} + U_g^* \Delta^T + \Delta\Delta^T)\|^2 \geq \kappa(\sum_{j\in J} \|\Delta^j\|)^2,$$

with $\kappa = \kappa_1 \kappa_2^2$ and $\kappa_2 = \sigma_r(U_g^*)\mu$. We derive our quadratic compatibility condition from a common compatibility condition on a more constrained set

$$\{\Delta | \sum_{j\in J^c} \|\Delta^j\| \leq 7 \sum_{j\in J} \|\Delta^j\|, \|\Delta\| \leq 2(1-\mu)\sigma_r(U_g^*)\}.$$

If one goes through our main proof in Appendix E, one can find we require

$$\|\widehat{U}_g - U_g^*\| \leq 2(1-\mu)\sigma_r(U_g^*)$$

in order to use the above common compatibility condition. This means we need to further search within a small neighbourhood of $U_g^*$ when solving the optimization problem (4) besides $g(U_g) \leq 2L$.

## B. Statement of Results

**Gold estimator.** First, we have the following result for the gold estimator:

**Theorem 4.** *Assume $\mathcal{A}_g$ is $2r$-RWC. Then, we have*

$$\ell(\widehat{U}_g, U_g^*) \leq \frac{16c\sqrt{d}}{(3\alpha - 2\beta)\sigma_r(U_g^*)}$$

*with probability at least*

$$1 - 2(36\sqrt{2})^{2r(2d+1)} \exp\left(-\frac{c^2 n_g}{8\beta\sigma_g^2}\right).$$

We give a proof in Appendix C.

**Proxy estimator.** Next, we have the following result for the proxy estimator:

**Theorem 5.** *Assume $\mathcal{A}_p$ is $2r$-RWC. Then, we have*

$$\ell(\widehat{U}_p, U_g^*) \leq \|\Delta_U^*\| + \omega + \frac{16c\sqrt{d}}{(3\alpha - 2\beta)\sigma_r(U_p^*)}$$

*with probability at least*

$$1 - 2(36\sqrt{2})^{2r(2d+1)} \exp\left(-\frac{c^2 n_p}{8\beta\sigma_p^2}\right),$$

*where*

$$\omega = \|U_p^*(R_{(\widehat{U}_p, U_p^*)} - R_{(\widehat{U}_p, U_g^*)})\|_{2,1}.$$

We give a proof in Appendix D.

**Joint estimator.** Finally, we have the following result for the joint estimator:

**Theorem 6.** *Assume $\mathcal{A}_p$ satisfies $2r$-RWC and $\mathcal{A}_g$ satisfies the quadratic compatibility condition (3). Let $A_{g,i}^{lk}$ represent the $(l,k)$ entry of matrix $A_{g,i}$, $A_g^{lk} = \begin{bmatrix} A_{g,1}^{lk} & \dots & A_{g,n_g}^{lk} \end{bmatrix}^T$. Define $\Psi_j, \Phi_j \in \mathbb{R}^{r\times r}$ to be*

$$\Psi_j = U_g^{*T} \frac{A_g^{jT} A_g^j}{n_g} U_g^*, \quad \Phi_j = U_g^{*T} \frac{(A_g^{Tj})^T A_g^{Tj}}{n_g} U_g^*,$$

*where $A_g^j, A_g^{Tj} \in \mathbb{R}^{n_g \times d}$ are matrices that stacks up the $j^{th}$ rows of $A_{g,i}$, $i \in [n_g]$ and $A_{g,i}^T$, $i \in [n_g]$ respectively, i.e.*

$$A_g^j = \begin{bmatrix} A_{g,1}^j \\ A_{g,2}^j \\ \vdots \\ A_{g,n_g}^j \end{bmatrix}, \quad A_g^{Tj} = \begin{bmatrix} A_{g,1}^{Tj} \\ A_{g,2}^{Tj} \\ \vdots \\ A_{g,n_g}^{Tj} \end{bmatrix}.$$

*Then, our two-step joint estimator satisfies*

$$\|\widehat{\Delta}_U - \widetilde{\Delta}_U\|_{2,1} \geq 8\lambda(\frac{2s}{\kappa} + \frac{16}{\sigma_r(U_p^*)(3\alpha - 2\beta)})$$

*with probability lower than*

$$\begin{aligned} & 2(36\sqrt{2})^{2r(2d+1)} \exp(-\frac{L^2\sigma_r^2(U_p^*)(3\alpha-2\beta)^2 n_p}{2048\beta\sigma_p^2 d}) \\ & + 2d^2 \exp\left(-\frac{\lambda^2 n_g}{2048L^2\sigma_g^2(\max_{l,k}\|A_g^{lk}\|^2/n_g)}\right) \\ & + d\max_{j\in[d]} \exp\left(-(\sqrt{\frac{\frac{\lambda^2 n_g}{256\sigma_g^2} - (\mathrm{tr}(\Psi_j) - \frac{\|\Psi_j\|_F^2}{2\|\Psi_j\|})}{2\|\Psi_j\|}} - \frac{\|\Psi_j\|_F}{2\|\Psi_j\|})^2\right) \\ & + d\max_{j\in[d]} \exp\left(-(\sqrt{\frac{\frac{\lambda^2 n_g}{256\sigma_g^2} - (\mathrm{tr}(\Phi_j) - \frac{\|\Phi_j\|_F^2}{2\|\Phi_j\|})}{2\|\Phi_j\|}} - \frac{\|\Phi_j\|_F}{2\|\Phi_j\|})^2\right) \\ & + 2(36\sqrt{2})^{2r(2d+1)} \exp(-\frac{\lambda^2 n_p}{8\beta\sigma_p^2 d}). \end{aligned}$$

The proof is given in Appendix E.

## C. Proofs of Theorem 1 & 4

To formally state the result, we define the following concept similar to Chi et al. (2019):

**Definition 4.** *A linear operator $\mathcal{A} : \mathbb{R}^{d\times d} \to \mathbb{R}^n$ satisfies the $\beta$-smoothness condition in a set $\mathcal{Z}$ if for any $Z \in \mathcal{Z}$*

$$\frac{1}{n}\|\mathcal{A}(Z)\|^2 \leq \beta\|Z\|_F^2.$$

Before discussing the estimation error bound of the gold estimator, we first introduce a lemma that helps with our proof.

**Lemma 1.** *Let $\mathcal{Z} \subset \mathbb{R}^{d\times d}$ be the subspace of matrices with rank at most $r$. The operator $\mathcal{A}$ is $\beta$-smooth in $\mathcal{Z}$. $\epsilon$ is $\sigma$-subgaussian. Then, we have*

$$P(\sup_{Z\in\mathcal{Z}} |\frac{1}{n}\sum_{i=1}^n \epsilon_i\langle A_i, Z\rangle| \leq c\|Z\|_F) \geq 1 - 2(36\sqrt{2})^{r(2d+1)} \exp\left(-\frac{c^2 n}{8\beta\sigma^2}\right).$$

*Proof.* Without loss of generality, consider $\mathcal{Z} = \{Z \in \mathbb{R}^{d\times d} | \operatorname{rank}(Z) \leq r, \|Z\|_F = 1\}$. Define $\mathcal{N}$ to be a $\frac{1}{4\sqrt{2}}$-net of $\mathcal{Z}$. Lemma 3 gives the covering number for the set $\mathcal{Z}$:

$$|\mathcal{N}| \leq (36\sqrt{2})^{r(2d+1)}.$$

For any $Z \in \mathcal{Z}$, there exists $Z' \in \mathcal{N}$ with $\|Z - Z'\|_F \leq \frac{1}{4\sqrt{2}}$, such that

$$|\sum_{i=1}^n \epsilon_i\langle A_i, Z\rangle| \leq |\sum_{i=1}^n \epsilon_i\langle A_i, Z'\rangle| + |\sum_{i=1}^n \epsilon_i\langle A_i, Z - Z'\rangle|. \tag{8}$$

Set $\Delta_Z = Z - Z'$ and note that $\mathrm{rank}(\Delta_Z) \leq 2r$. We decompose $\Delta_Z$ into two matrices, $\Delta_Z = \Delta_{Z,1} + \Delta_{Z,2}$, that satisfy $\mathrm{rank}(\Delta_{Z,j}) \leq r$ for $j = 1, 2$ and $\langle \Delta_{Z,1}, \Delta_{Z,2} \rangle = 0$ (e.g. through SVD). As $\|\Delta_{Z,1}\|_F + \|\Delta_{Z,2}\|_F \leq \sqrt{2}\|\Delta_Z\|_F$, we have $\|\Delta_{Z,j}\|_F \leq \frac{1}{4}$, $j = 1, 2$. Combined with inequality (8), we have

$$|\sum_{i=1}^n \epsilon_i \langle A_i, Z\rangle| \leq \sup_{Z' \in \mathcal{N}} |\sum_{i=1}^n \epsilon_i \langle A_i, Z'\rangle| + \frac{1}{2} \sup_{Z \in \mathcal{Z}} |\sum_{i=1}^n \epsilon_i \langle A_i, Z\rangle|.$$

Since the above holds for any $Z \in \mathcal{Z}$, we have

$$\sup_{Z \in \mathcal{Z}} |\sum_{i=1}^n \epsilon_i \langle A_i, Z\rangle| \leq 2 \sup_{Z' \in \mathcal{N}} |\sum_{i=1}^n \epsilon_i \langle A_i, Z'\rangle|.$$

Then it follows from the union bound that

$$\begin{aligned} P(\sup_{Z \in \mathcal{Z}} |\frac{1}{n}\sum_{i=1}^n \epsilon_i \langle A_i, Z\rangle| \geq c) &\leq P(\sup_{Z \in \mathcal{N}} |\frac{1}{n}\sum_{i=1}^n \epsilon_i \langle A_i, Z\rangle| \geq \frac{c}{2}) \\ &\leq |\mathcal{N}| \max_{Z \in \mathcal{N}} P(|\frac{1}{n}\sum_{i=1}^n \epsilon_i \langle A_i, Z\rangle| \geq \frac{c}{2}) \\ &\leq 2|\mathcal{N}| \exp\left(-\frac{c^2 n}{8\beta\sigma^2}\right) \\ &= 2(36\sqrt{2})^{r(2d+1)} \exp\left(-\frac{c^2 n}{8\beta\sigma^2}\right). \end{aligned}$$

The last inequality uses $\beta$-smoothness of $\mathcal{A}$ and a tail inequality of $\sigma$-subgaussian random variables. □

*Proof.* (Theorem 4) The proof mainly follows Theorem 8 and Theorem 31 of Ge et al. (2017) but we also provides here for completeness. As in Ge et al. (2017), we use the notation $U : \mathcal{H} : V$ to denote the inner product $\langle U, \mathcal{H}(V)\rangle$ for $U, V \in \mathbb{R}^{d_1 \times d_2}$. The linear operator $\mathcal{H}$ can be viewed as a $d_1 d_2 \times d_1 d_2$ matrix. In our problem (2), we define

$$\Theta : \mathcal{H} : \Theta = \frac{1}{n_g}\|\mathcal{A}_g(\Theta)\|^2$$

for any $\Theta \in \mathbb{R}^{d \times d}$. We can rewrite problem (2) as

$$\min_{U_g} f(U_g),$$

where

$$\begin{aligned} f(U_g) &= \frac{1}{n_g}\|X_g - \mathcal{A}_g(U_g U_g^T)\|^2 \\ &= (U_g U_g^T - \Theta_g^*) : \mathcal{H} : (U_g U_g^T - \Theta_g^*) + Q(U_g), \end{aligned}$$

and

$$Q(U_g) = -\frac{2}{n_g}\sum_{i \in [n_g]} \langle A_{g,i}, \Theta_g - \Theta_g^*\rangle \epsilon_{g,i} + \frac{1}{n_g}\sum_{i \in [n_g]} \epsilon_{g,i}^2.$$

Define $\Delta = \widehat{U}_g - U_g^* R_{(\widehat{U}_g, U_g^*)}$. By Lemma 7 from Ge et al. (2017), we have for the Hessian $\nabla^2 f(\widehat{U}_g)$ with $\nabla f(\widehat{U}_g) = 0$

$$\Delta : \nabla^2 f(\widehat{U}_g) : \Delta = \Delta\Delta^T : \mathcal{H} : \Delta\Delta^T - 3(\widehat{\Theta}_g - \Theta_g^*) : \mathcal{H} : (\widehat{\Theta}_g - \Theta_g^*) + \Delta : \nabla^2 Q(\widehat{U}_g) : \Delta - 4\langle \nabla Q(\widehat{U}_g), \Delta\rangle.$$

Using Lemma 4 and the $2r$-RWC assumption, we have

$$\Delta : \nabla^2 f(\widehat{U}_g) : \Delta \leq -(3\alpha - 2\beta)\|\widehat{\Theta}_g - \Theta_g^*\|_F^2 + \Delta : \nabla^2 Q(\widehat{U}_g) : \Delta - 4\langle \nabla Q(\widehat{U}_g), \Delta\rangle.$$

We then bound the part related to function $Q$:

$$\Delta : \nabla^2 Q(\widehat{U}_g) : \Delta - 4\langle \nabla Q(\widehat{U}_g), \Delta\rangle = \frac{4}{n_g}\sum_{i\in[n_g]}\langle A_{g,i}, \widehat{\Theta}_g - \Theta_g^*\rangle \epsilon_{g,i} + \frac{4}{n_g}\sum_{i\in[n_g]}\langle A_{g,i}, \widehat{U}_g\Delta^T - \Delta\widehat{U}_g^T\rangle \epsilon_{g,i}$$

Define $\mathcal{Z} = \{Z \in \mathbb{R}^{d\times d} \mid \operatorname{rank}(Z) \le 2r\}$. Under the event

$$\mathcal{E}_g = \left\{ \sup_{Z\in\mathcal{Z}} |\frac{1}{n_g}\sum_{i=1}^{n_g}\epsilon_{g,i}\langle A_{g,i}, Z\rangle| \le c\|Z\|_F \right\},$$

we have

$$\frac{4}{n_g}\sum_{i\in[n_g]}\langle A_{g,i}, \widehat{\Theta}_g - \Theta_g^*\rangle \epsilon_{g,i} \le 4c\|\widehat{\Theta}_g - \Theta_g^*\|_F$$
$$\frac{4}{n_g}\sum_{i\in[n_g]}\langle A_{g,i}, \widehat{U}_g\Delta^T - \Delta\widehat{U}_g^T\rangle \epsilon_{g,i} \le 4(1+\sqrt{2})c\|\widehat{\Theta}_g - \Theta_g^*\|_F,$$

where the second inequality uses Lemma 4. Therefore, we have

$$\Delta : \nabla^2 f(\widehat{U}_g) : \Delta \le -(3\alpha - 2\beta)\|\widehat{\Theta}_g - \Theta_g^*\|_F^2 + (8+4\sqrt{2})c\|\widehat{\Theta}_g - \Theta_g^*\|_F.$$

Since $\widehat{U}_g$ is a local minimum, we must have

$$-(3\alpha - 2\beta)\|\widehat{\Theta}_g - \Theta_g^*\|_F^2 + (8+4\sqrt{2})c\|\widehat{\Theta}_g - \Theta_g^*\|_F \ge 0,$$

that is, $\widehat{\Theta}_g$ should satisfy

$$\|\widehat{\Theta}_g - \Theta_g^*\|_F \le \frac{(8+4\sqrt{2})c}{3\alpha - 2\beta}.$$

Using Lemma 4 again gives

$$\|\widehat{U}_g - U_g^* R_{(\widehat{U}_g, U_g^*)}\|_F \le \frac{1}{\sqrt{2(\sqrt{2}-1)\sigma_r(\Theta_g^*)}}\|\widehat{\Theta}_g - \Theta_g^*\|_F \le \frac{16c}{(3\alpha-2\beta)\sigma_r(U_g^*)}.$$

Further by Cauchy-Schwarz, we have

$$\|\widehat{U}_g - U_g^* R_{(\widehat{U}_g, U_g^*)}\|_{2,1} \le \frac{16c\sqrt{d}}{(3\alpha-2\beta)\sigma_r(U_g^*)}.$$

By Lemma 1,

$$P(\mathcal{E}_g) \ge 1 - 2(36\sqrt{2})^{2r(2d+1)}\exp\left(-\frac{c^2 n_g}{8\beta\sigma_g^2}\right).$$

□

*Proof.* (Theorem 1) Theorem 1 follows from Theorem 4 by taking

$$c = \sqrt{\frac{8\beta\sigma_g^2(2r(2d+1)\log(36\sqrt{2}) + \log(\frac{2}{\delta}))}{n_g}}.$$

□

## D. Proofs of Theorem 2 & 5

*Proof.* (Theorem 5) Same as the proof of Theorem 4, we get

$$\|\widehat{U}_p - U_p^* R_{(\widehat{U}_p, U_p^*)}\|_{2,1} \leq \frac{16c\sqrt{d}}{(3\alpha - 2\beta)\sigma_r(U_p^*)}.$$

under the event

$$\mathcal{E}_p = \left\{ \sup_{Z \in \mathcal{Z}} |\frac{1}{n_p} \sum_{i=1}^{n_p} \epsilon_{p,i} \langle A_{p,i}, Z \rangle| \leq c \|Z\|_F \right\}.$$

To measure the estimation error of $\widehat{U}_p$ for $U_g^*$, we need to align $\widehat{U}_p$ with $U_g^*$. The estimation error of using proxy estimator for gold data is

$$\begin{aligned}\|\widehat{U}_p - U_g^* R_{(\widehat{U}_p, U_g^*)}\|_{2,1} =& \|\widehat{U}_p - U_p^* R_{(\widehat{U}_p, U_p^*)} + U_p^* R_{(\widehat{U}_p, U_p^*)} - (U_p^* + \Delta_U^*) R_{(\widehat{U}_p, U_g^*)}\|_{2,1} \\ \leq& \|\widehat{U}_p - U_p^* R_{(\widehat{U}_p, U_p^*)}\|_{2,1} + \|U_p^*(R_{(\widehat{U}_p, U_p^*)} - R_{(\widehat{U}_p, U_g^*)})\|_{2,1} + \|\Delta_U^*\|_{2,1}.\end{aligned}$$

Therefore, we have

$$\|\widehat{U}_p - U_g^* R_{(\widehat{U}_p, U_g^*)}\|_{2,1} \leq \|\Delta_U^*\| + \|U_p^*(R_{(\widehat{U}_p, U_p^*)} - R_{(\widehat{U}_p, U_g^*)})\|_{2,1} + \frac{16c\sqrt{d}}{(3\alpha - 2\beta)\sigma_r(U_p^*)}.$$

By Lemma 1,

$$P(\mathcal{E}_p) \geq 1 - 2(36\sqrt{2})^{2r(2d+1)} \exp\left(-\frac{c^2 n_p}{8\beta\sigma_p^2}\right).$$

□

*Proof.* (Theorem 2) Similarly, Theorem 2 follows from Theorem 5 by taking

$$\omega = \|U_p^*(R_{(\widehat{U}_p, U_p^*)} - R_{(\widehat{U}_p, U_g^*)})\|_{2,1},$$

and

$$c = \sqrt{\frac{8\beta\sigma_p^2(2r(2d+1)\log(36\sqrt{2}) + \log(\frac{2}{\delta}))}{n_p}}.$$

□

## E. Proofs of Theorem 3 & 6

*Proof.* (Theorem 6) The problem (4) is equivalent to the following problem:

$$\widehat{\Delta}_U = \operatorname*{arg\,min}_{\|\Delta_U\|_{2,1} \leq 2L} \frac{1}{n_g} \|X_g - \mathcal{A}_g((\widehat{U}_p + \Delta_U)(\widehat{U}_p + \Delta_U)^T)\|^2 + \lambda \|\Delta_U\|_{2,1},$$

with the joint estimator $\widehat{U}_g = \widehat{\Delta}_U + \widehat{U}_p$.

Note that the row sparsity is immune to rotations, that is, for any orthogonal matrix $R$, $\Delta_U^* R$ is still row sparse. After our first step of finding the proxy estimator, we align $\widehat{U}_p$ with $U_p^*$ in the direction of $R_{(\widehat{U}_p, U_p^*)}$. By our definition,

$$U_g^* R_{(\widehat{U}_p, U_p^*)} = U_p^* R_{(\widehat{U}_p, U_p^*)} + \Delta_U^* R_{(\widehat{U}_p, U_p^*)}.$$

Through our previous analyses, $\widehat{U}_p$ is close to $U_p^* R_{(\widehat{U}_p, U_p^*)}$ with a high probability. Therefore, in our second step, we aim to find an estimator $\widehat{\Delta}_U$ for $\Delta_U^* R_{(\widehat{U}_p, U_p^*)}$ through $\ell_{2,1}$ penalty. For simplicity, we use $U_g^*$, $U_p^*$ and $\Delta_U^*$ to represent $U_g^* R_{(\widehat{U}_p, U_p^*)}$, $U_p^* R_{(\widehat{U}_p, U_p^*)}$ and $\Delta_U^* R_{(\widehat{U}_p, U_p^*)}$ respectively in the following analyses, which are aligned in the direction of $R_{(\widehat{U}_p, U_p^*)}$. Define the first step estimation error $\nu = \widehat{U}_p - U_p^*$ and $\widetilde{\Delta}_U = \Delta_U^* - \nu$. Thus, $U_g^* = U_p^* + \Delta_U^* = \widehat{U}_p + \widetilde{\Delta}_U$. Since $\widehat{U}_p$ carries the estimation error from the first step, the parameter we actually want to recover is $\widetilde{\Delta}_U$, which is approximately row sparse.

We introduce a few definitions that will be used in the proof. Define the adjoint of an operator $\mathcal{A} : \mathbb{R}^{d\times d} \to \mathbb{R}^n$ to be $\mathcal{A}^* : \mathbb{R}^n \to \mathbb{R}^{d\times d}$, with $\mathcal{A}^*(\epsilon) = \sum_{i=1}^n \epsilon_i A_i$. Let $\sigma_{\max}(\mathcal{A}^*\mathcal{A})$ be the maximum eigenvalue of $\mathcal{A}^*\mathcal{A}$, with

$$\sigma_{\max}(\mathcal{A}^*\mathcal{A}) = \sup_{\|R\|_F=1} \langle R, \mathcal{A}^*(\mathcal{A}(R))\rangle.$$

As we search within $\|\Delta_U\|_{2,1} \le 2L$ and $\|\Delta_U^*\|_{2,1} \le L$, we require the following event to hold

$$\mathcal{I} = \{\|\nu\|_{2,1} \le L\}$$

for $\widetilde{\Delta}_U$ to be feasible. Using a similar analysis to Theorem 4, we can show the event $\mathcal{I}$ takes place with a high probability:

$$P(\mathcal{I}) \ge 1 - 2(36\sqrt{2})^{2r(2d+1)} \exp\left(-\frac{L^2\sigma_r^2(U_p^*)(3\alpha-2\beta)^2 n_p}{2048\beta\sigma_p^2 d}\right)$$

Under the event $\mathcal{I}$, the global optimality of $\widehat{\Delta}_U$ implies

$$\frac{1}{n_g}\|X_g - \mathcal{A}_g((\widehat{U}_p + \widehat{\Delta}_U)(\widehat{U}_p + \widehat{\Delta}_U)^T)\|^2 + \lambda\|\widehat{\Delta}_U\|_{2,1} \le \frac{1}{n_g}\|X_g - \mathcal{A}_g((\widehat{U}_p + \widetilde{\Delta}_U)(\widehat{U}_p + \widetilde{\Delta}_U)^T)\|^2 + \lambda\|\widetilde{\Delta}_U\|_{2,1}.$$

Plugging in $X_g = \mathcal{A}_g((\widehat{U}_p + \widetilde{\Delta}_U)(\widehat{U}_p + \widetilde{\Delta}_U)^T) + \epsilon_g$ yields

$$\begin{aligned}&\frac{1}{n_g}\|\mathcal{A}_g((\widehat{U}_p + \widehat{\Delta}_U)(\widehat{U}_p + \widehat{\Delta}_U)^T - (\widehat{U}_p + \widetilde{\Delta}_U)(\widehat{U}_p + \widetilde{\Delta}_U)^T)\|^2 + \lambda\|\widehat{\Delta}_U\|_{2,1}\\ &\qquad \le \frac{2}{n_g}\langle \epsilon_g, \mathcal{A}_g((\widehat{U}_p + \widehat{\Delta}_U)(\widehat{U}_p + \widehat{\Delta}_U)^T - (\widehat{U}_p + \widetilde{\Delta}_U)(\widehat{U}_p + \widetilde{\Delta}_U)^T)\rangle + \lambda\|\widetilde{\Delta}_U\|_{2,1}.\end{aligned}$$

Rearranging the RHS with $U_g^* = \widehat{U}_p + \widetilde{\Delta}_U$, we get

$$\begin{aligned}&\frac{1}{n_g}\|\mathcal{A}_g((\widehat{U}_p + \widehat{\Delta}_U)(\widehat{U}_p + \widehat{\Delta}_U)^T - (\widehat{U}_p + \widetilde{\Delta}_U)(\widehat{U}_p + \widetilde{\Delta}_U)^T)\|^2 + \lambda\|\widehat{\Delta}_U\|_{2,1}\\ &\qquad \le \frac{2}{n_g}\langle \epsilon_g, \mathcal{A}_g((\widehat{\Delta}_U - \widetilde{\Delta}_U)U_g^{*T} + U_g^*(\widehat{\Delta}_U - \widetilde{\Delta}_U)^T + (\widehat{\Delta}_U - \widetilde{\Delta}_U)(\widehat{\Delta}_U - \widetilde{\Delta}_U)^T)\rangle + \lambda\|\widetilde{\Delta}_U\|_{2,1}\end{aligned} \tag{9}$$

The first part of the first term on the RHS of inequality (9) has

$$\begin{aligned}&\langle \epsilon_g, \mathcal{A}_g((\widehat{\Delta}_U - \widetilde{\Delta}_U)U_g^{*T} + U_g^*(\widehat{\Delta}_U - \widetilde{\Delta}_U)^T)\rangle\\ =&\langle \mathcal{A}_g^*(\epsilon_g), (\widehat{\Delta}_U - \widetilde{\Delta}_U)U_g^{*T} + U_g^*(\widehat{\Delta}_U - \widetilde{\Delta}_U)^T\rangle\\ =&\langle \mathcal{A}_g^*(\epsilon_g)U_g^*, \widehat{\Delta}_U - \widetilde{\Delta}_U\rangle + \langle \mathcal{A}_g^*(\epsilon_g)^T U_g^*, \widehat{\Delta}_U - \widetilde{\Delta}_U\rangle\\ \le&(\max_{j\in[d]}\|(\mathcal{A}_g^*(\epsilon_g)^j U_g^*\| + \max_{j\in[d]}\|\mathcal{A}_g^*(\epsilon_g)^{Tj}U_g^*\|)\|\widehat{\Delta}_U - \widetilde{\Delta}_U\|_{2,1}.\end{aligned}$$

Correspondingly, the second part of the first term on the RHS of inequality (9) has

$$\begin{aligned}&\langle \epsilon_g, \mathcal{A}_g((\widehat{\Delta}_U - \widetilde{\Delta}_U)(\widehat{\Delta}_U - \widetilde{\Delta}_U)^T)\rangle\\ =&\langle \mathcal{A}_g^*(\epsilon_g), (\widehat{\Delta}_U - \widetilde{\Delta}_U)(\widehat{\Delta}_U - \widetilde{\Delta}_U)^T\rangle\\ \le&|\mathcal{A}_g^*(\epsilon_g)|_\infty|(\widehat{\Delta}_U - \widetilde{\Delta}_U)(\widehat{\Delta}_U - \widetilde{\Delta}_U)^T|_1\\ \le&|\mathcal{A}_g^*(\epsilon_g)|_\infty\|\widehat{\Delta}_U - \widetilde{\Delta}_U\|_{2,1}^2.\end{aligned} \tag{10}$$

Consider the following two events

$$\mathcal{G}_1 = \left\{\frac{2}{n_g}\max_{j\in[d]}\|\mathcal{A}_g^*(\epsilon_g)^j U_g^*\| \le \frac{\lambda}{8}\right\},$$

$$\mathcal{G}_2 = \left\{ \frac{2}{n_g} \max_{j \in [d]} \|\mathcal{A}_g^*(\epsilon_g)^{Tj} U_g^*\| \leq \frac{\lambda}{8} \right\},$$

and

$$\mathcal{F} = \left\{ \frac{2}{n_g} |\mathcal{A}_g^*(\epsilon_g)|_\infty \leq \frac{\lambda}{16L} \right\}.$$

**Lemma 2.** *The events $\mathcal{G}_1$, $\mathcal{G}_2$ and $\mathcal{F}$ satisfy the following concentration inequalities*

$$P(\mathcal{G}_1^c) \leq d \max_{j \in [d]} \exp\left( -(\sqrt{\frac{\frac{\lambda^2 n_g}{256\sigma_g^2} - (\operatorname{tr}(\Psi_j) - \frac{\|\Psi_j\|_F^2}{2\|\Psi_j\|})}{2\|\Psi_j\|}} - \frac{\|\Psi_j\|_F}{2\|\Psi_j\|})^2 \right),$$

$$P(\mathcal{G}_2^c) \leq d \max_{j \in [d]} \exp\left( -(\sqrt{\frac{\frac{\lambda^2 n_g^2}{256\sigma_g^2} - (\operatorname{tr}(\Phi_j) - \frac{\|\Phi_j\|_F^2}{2\|\Phi_j\|})}{2\|\Phi_j\|}} - \frac{\|\Phi_j\|_F}{2\|\Phi_j\|})^2 \right),$$

*and*

$$P(\mathcal{F}^c) \leq 2d^2 \exp(-\frac{\lambda^2 n_g^2}{128 L^2 \sigma_g^2 \|A_{lk}\|^2}).$$

The proof is given after the proof of this theorem.

Under the events $\mathcal{G}_1$, $\mathcal{G}_2$ and $\mathcal{F}$, we derive from inequality (9) that

$$\begin{aligned}
&\frac{1}{n_g}\|\mathcal{A}_g((\widehat{U}_p + \widehat{\Delta}_U)(\widehat{U}_p + \widehat{\Delta}_U)^T - (\widehat{U}_p + \widetilde{\Delta}_U)(\widehat{U}_p + \widetilde{\Delta}_U)^T)\|^2 + \lambda\|\widehat{\Delta}_U\|_{2,1} \\
\leq& \frac{\lambda}{4}\|\widehat{\Delta}_U - \widetilde{\Delta}_U\|_{2,1} + \frac{\lambda}{16L}\|\widehat{\Delta}_U - \widetilde{\Delta}_U\|_{2,1}^2 + \lambda\|\widetilde{\Delta}_U\|_{2,1} \\
\leq& \frac{\lambda}{2}\|\widehat{\Delta}_U - \widetilde{\Delta}_U\|_{2,1} + \lambda\|\widetilde{\Delta}_U\|_{2,1}.
\end{aligned}$$

The second inequality uses the definition of the search region $\|\Delta_U\|_{2,1} \leq 2L$, the definition of event $\mathcal{I}$, and the feasibility of $\Delta_U^*$ that $\|\Delta_U^*\|_{2,1} \leq L$. We can further arrange the inequality to get

$$\begin{aligned}
\frac{1}{n_g}\|\mathcal{A}_g((\widehat{U}_p + \widehat{\Delta}_U)(\widehat{U}_p + \widehat{\Delta}_U)^T - (\widehat{U}_p + \widetilde{\Delta}_U)(\widehat{U}_p + \widetilde{\Delta}_U)^T)\|^2 + \frac{\lambda}{2}\sum_{j \in J^c}\|(\widehat{\Delta}_U - \widetilde{\Delta}_U)^j\| \\
\leq \frac{3\lambda}{2}\sum_{j \in J}\|(\widehat{\Delta}_U - \widetilde{\Delta}_U)^j\| + 2\lambda\sum_{j \in J^c}\|\nu^j\|. \quad (11)
\end{aligned}$$

Now we consider the following two cases respectively:

(i). $\sum_{j \in J^c}\|\nu^j\| \leq \sum_{j \in J}\|(\widehat{\Delta}_U - \widetilde{\Delta}_U)^j\|$,

(ii). $\sum_{j \in J^c}\|\nu^j\| > \sum_{j \in J}\|(\widehat{\Delta}_U - \widetilde{\Delta}_U)^j\|$.

Under Case (i), we derive from the inequality (11) that

$$\frac{1}{n_g}\|\mathcal{A}_g((\widehat{U}_p + \widehat{\Delta}_U)(\widehat{U}_p + \widehat{\Delta}_U)^T - (\widehat{U}_p + \widetilde{\Delta}_U)(\widehat{U}_p + \widetilde{\Delta}_U)^T)\|^2 + \frac{\lambda}{2}\sum_{j \in J^c}\|(\widehat{\Delta}_U - \widetilde{\Delta}_U)^j\| \leq \frac{7\lambda}{2}\sum_{j \in J}\|(\widehat{\Delta}_U - \widetilde{\Delta}_U)^j\|.$$

Thus, we have $\sum_{j\in J^c} \|(\widehat{\Delta}_U - \widetilde{\Delta}_U)^j\| \leq 7\sum_{j\in J} \|(\widehat{\Delta}_U - \widetilde{\Delta}_U)^j\|$ and the quadratic compatibility condition of $\mathcal{A}_g$ is satisfied. Further write the above as

$$\begin{aligned}\frac{1}{n_g}\|\mathcal{A}_g((\widehat{U}_p + \widehat{\Delta}_U)(\widehat{U}_p + \widehat{\Delta}_U)^T - (\widehat{U}_p + \widetilde{\Delta}_U)(\widehat{U}_p + \widetilde{\Delta}_U)^T)\|^2 + \frac{\lambda}{2}\|\widehat{\Delta}_U - \widetilde{\Delta}_U\|_{2,1} &\leq 4\lambda\sum_{j\in J}\|(\widehat{\Delta}_U - \widetilde{\Delta}_U)^j\| \\ &\leq \frac{8\lambda^2 s}{\kappa} + \frac{\kappa}{2s}(\sum_{j\in J}\|(\widehat{\Delta}_U - \widetilde{\Delta}_U)^j\|)^2.\end{aligned}$$

where the second inequality uses $2ab \leq a^2 + b^2$. Now we use the quadratic compatibility condition on the RHS, and

$$\frac{1}{2n_g}\|\mathcal{A}_g((\widehat{U}_p + \widehat{\Delta}_U)(\widehat{U}_p + \widehat{\Delta}_U)^T - (\widehat{U}_p + \widetilde{\Delta}_U)(\widehat{U}_p + \widetilde{\Delta}_U)^T)\|^2 + \frac{\lambda}{2}\|\widehat{\Delta}_U - \widetilde{\Delta}_U\|_{2,1} \leq \frac{8\lambda^2 s}{\kappa}$$

Under Case (ii), the inequality (11) gives

$$\frac{1}{n_g}\|\mathcal{A}((\widehat{U}_p + \widehat{\Delta}_U)(\widehat{U}_p + \widehat{\Delta}_U)^T - (\widehat{U}_p + \widetilde{\Delta}_U)(\widehat{U}_p + \widetilde{\Delta}_U)^T)\|^2 + \frac{\lambda}{2}\|\widehat{\Delta}_U - \widetilde{\Delta}_U\|_{2,1} \leq 4\lambda\sum_{j\in J^c}\|\nu^j\|.$$

Therefore, under any circumstances, we have

$$\|\widehat{\Delta}_U - \widetilde{\Delta}_U\|_{2,1} \leq 8(\frac{2\lambda s}{\kappa} + \sum_{j\in J^c}\|\nu^j\|) \leq 8(\frac{2\lambda s}{\kappa} + \|\nu\|_{2,1}).$$

Consider the event

$$\mathcal{J} = \left\{\|\nu\|_{2,1} \leq \frac{16\sqrt{d}c}{(3\alpha - 2\beta)\sigma_r(U_p^*)}\right\}.$$

Using a similar analysis to Theorem 4 as our analysis on event $\mathcal{I}$, we have

$$P(\mathcal{J}) \geq 1 - 2(36\sqrt{2})^{2r(2d+1)}\exp(-\frac{c^2 n_p}{8\beta\sigma_p^2}).$$

Therefore, on the event $\mathcal{J}$, taking $c = \frac{\lambda}{\sqrt{d}}$, the estimation error is bounded by

$$\|\widehat{\Delta}_U - \widetilde{\Delta}_U\|_{2,1} \leq 8(\frac{2\lambda s}{\kappa} + \frac{16\lambda}{(3\alpha - 2\beta)\sigma_r(U_p^*)}).$$

Combining all the above, we have the following concentration inequality

$$\begin{aligned}P\left(\|\widehat{\Delta}_U - \widetilde{\Delta}_U\|_{2,1} \geq 8(\frac{2\lambda s}{\kappa} + \frac{16\lambda}{(3\alpha - 2\beta)\sigma_r(U_p^*)})\right) \leq & P(\mathcal{I}^c) + P(\mathcal{G}_1^c) + P(\mathcal{G}_2^c) + P(\mathcal{F}^c) + P(\mathcal{J}^c) \\ \leq & 2(36\sqrt{2})^{2r(2d+1)}\exp(-\frac{L^2\sigma_r^2(U_p^*)(3\alpha - 2\beta)^2 n_p}{2048\beta\sigma_p^2 d}) \\ & + 2d^2\exp\left(-\frac{\lambda^2 n_g}{2048L^2\sigma_g^2(\max_{l,k}\|A_g^{lk}\|^2/n_g)}\right) \\ & + d\max_{j\in[d]}\exp\left(-(\sqrt{\frac{\frac{\lambda^2 n_g}{256\sigma_g^2} - (\mathrm{tr}(\Psi_j) - \frac{\|\Psi_j\|_F^2}{2\|\Psi_j\|})}{2\|\Psi_j\|}} - \frac{\|\Psi_j\|_F}{2\|\Psi_j\|})^2\right) \\ & + d\max_{j\in[d]}\exp\left(-(\sqrt{\frac{\frac{\lambda^2 n_g}{256\sigma_g^2} - (\mathrm{tr}(\Phi_j) - \frac{\|\Phi_j\|_F^2}{2\|\Phi_j\|})}{2\|\Phi_j\|}} - \frac{\|\Phi_j\|_F}{2\|\Phi_j\|})^2\right) \\ & + 2(36\sqrt{2})^{2r(2d+1)}\exp(-\frac{\lambda^2 n_p}{8\beta\sigma_p^2 d}).\end{aligned}$$

□

*Proof.* (Lemma 2) Consider the event $\mathcal{F}$ first. With $\epsilon_g$ being $\sigma_g$-subgaussian,

$$\begin{aligned} P(\mathcal{F}^c) &= P(\frac{2}{n_g}|\mathcal{A}_g^*(\epsilon_g)|_\infty \geq \frac{\lambda}{16L}) \\ &\leq d^2 \max_{l,k\in[d]} P(\frac{2}{n_g}|\sum_{i=1}^{n_g} A_{g,i}^{lk}\epsilon_{g,i}| \geq \frac{\lambda}{16L}) \\ &\leq 2d^2 \exp\left(-\frac{\lambda^2 n_g}{2048L^2\sigma_g^2(\max_{l,k}\|A_g^{lk}\|^2/n_g)}\right), \end{aligned}$$

In the last inequality, we use the fact that $\epsilon_g$ is $\sigma_g$-subgaussian in the final inequality.

Next, we look at the event $\mathcal{G}_1$.

$$\begin{aligned} P(\mathcal{G}_1^c) &= P(\frac{2}{n_g}\max_{j\in[d]}\|\mathcal{A}_g^*(\epsilon_g)^j U_g^*\| \geq \frac{\lambda}{8}) \\ &\leq d\max_{j\in[d]} P(\frac{2}{n_g}\|\mathcal{A}_g^*(\epsilon_g)^j U_g^*\| \geq \frac{\lambda}{8}). \end{aligned}$$

For a given $j$, observe that

$$\begin{aligned} \frac{4}{n_g^2}\|\mathcal{A}_g^*(\epsilon_g)^j U_g^*\|^2 &= \frac{4}{n_g^2}\|\sum_{i=1}^{n_g} A_{g,i}^j U_g^*\epsilon_{g,i}\|^2 \\ &= \frac{4}{n_g}\epsilon_g^T\frac{A_g^j U_g^* U_g^{*T} A_g^{jT}}{n_g}\epsilon_g, \end{aligned}$$

Note that $\Psi_j$ has the same positive eigenvalues as $\frac{A_g^j U_g^* U_g^{*T} A_g^{jT}}{n_g}$. Different from Lounici et al. (2011), we assume subgaussian random noises instead of Gaussian noises. Therefore, instead, we have from Lemma 5

$$P(\frac{4}{n_g}\epsilon_g^T\frac{A_g^j U_g^* U_g^{*T} A_g^{jT}}{n_g}\epsilon_g \geq \frac{\lambda^2}{64}) \leq \exp\left(-(\sqrt{\frac{\frac{\lambda^2 n_g}{256\sigma_g^2} - (\mathrm{tr}(\Psi_j) - \frac{\|\Psi_j\|_F^2}{2\|\Psi_j\|})}{2\|\Psi_j\|}} - \frac{\|\Psi_j\|_F}{2\|\Psi_j\|})^2\right).$$

Combining the results above, we can derive that

$$P(\mathcal{G}_1^c) \leq d\max_{j\in[d]}\exp\left(-(\sqrt{\frac{\frac{\lambda^2 n_g}{256\sigma_g^2} - (\mathrm{tr}(\Psi_j) - \frac{\|\Psi_j\|_F^2}{2\|\Psi_j\|})}{2\|\Psi_j\|}} - \frac{\|\Psi_j\|_F}{2\|\Psi_j\|})^2\right).$$

Similarly for event $\mathcal{G}_2$, we have

$$P(\mathcal{G}_2^c) \leq d\max_{j\in[d]}\exp\left(-(\sqrt{\frac{\frac{\lambda^2 n_g^2}{256\sigma_g^2} - (\mathrm{tr}(\Phi_j) - \frac{\|\Phi_j\|_F^2}{2\|\Phi_j\|})}{2\|\Phi_j\|}} - \frac{\|\Phi_j\|_F}{2\|\Phi_j\|})^2\right).$$

□

*Proof.* (Theorem 3) Theorem 3 follows from Theorem 6.

Assume $\frac{L\sigma_r(U_p^*)(3\alpha-2\beta)}{16} \geq \lambda$. On this event, the first term on the RHS is smaller than the last term on the RHS. In order to make each term on the RHS to be smaller than $\frac{\delta}{5}$, we require $\lambda \geq \lambda_0$ with

$$\lambda_0 = \max\left\{ \sqrt{\frac{2048L^2\sigma_g^2(\max_{l,k}\|A_g^{lk}\|^2/n_g)}{n_g}\log(\frac{10d^2}{\delta})}, \right.$$
$$\max_{j\in[d]}\sqrt{\frac{256\sigma_g^2}{n_g}(\mathrm{tr}(\Psi_j)+2\|\Psi_j\|_F\sqrt{\log(\frac{5d}{\delta})}+2\|\Psi_j\|\log(\frac{5d}{\delta}))},$$
$$\max_{j\in[d]}\sqrt{\frac{256\sigma_g^2}{n_g}(\mathrm{tr}(\Phi_j)+2\|\Phi_j\|_F\sqrt{\log(\frac{5d}{\delta})}+2\|\Phi_j\|\log(\frac{5d}{\delta}))},$$
$$\left. \sqrt{\frac{8\beta\sigma_p^2 d}{n_p}(2r(2d+1)\log(36\sqrt{2})+\log(\frac{10}{\delta}))}\right\}.$$

Note that

$$\frac{1}{n_g}\|A_g^{lk}\|^2 = \langle E_{lk}, \frac{1}{n_g}\mathcal{A}_g^*(\mathcal{A}_g(E_{lk}))\rangle \leq \sigma_{\max}(\frac{\mathcal{A}_g^*\mathcal{A}_g}{n_g}),$$

where $E_{lk}\in\mathbb{R}^{d\times d}$ is a matrix whose $(l,k)$ entry is 1 and otherwise 0. On the other hand,

$$\|\Psi_j\| = \max_{\|x\|=1, x\in\mathbb{R}^r} x^T U_g^{*T}\frac{A_g^{jT}A_g^j}{n_g}U_g^* x$$
$$= \max_{\|x\|=1} x^T U_g^{*T}\frac{A_g^{jT}A_g^j}{n_g}U_g^* x.$$

If we define a matrix $E_j(x)$ whose $j^{\text{th}}$ row is $x^T$ and otherwise 0, then

$$\|\Psi_j\| = \max_{\|x\|=1}\frac{1}{n_g}\langle E_j(U_g^* x), A_g^*(A_g(E_j(U_g^* x)))\rangle.$$

As $\|x\|=1$, we have

$$\|E_j(U_g^* x)\|_F = \|U_g^* x\| \leq \sigma_1(U_g^*).$$

Therefore, we have

$$\|\Psi_j\| \leq \max_{\|R\|_F\leq\sigma_1(U_g^*)}\frac{1}{n_g}\langle R, A_g^*(A_g(R)\rangle$$
$$\leq \sigma_{\max}(\frac{\mathcal{A}_g^*\mathcal{A}_g}{n_g})\sigma_1^2(U_g^*).$$

With a similar analysis, we have

$$\|\Phi_j\| \leq \sigma_{\max}(\frac{\mathcal{A}_g^*\mathcal{A}_g}{n_g})\sigma_1^2(U_g^*).$$

Given the above results, we can bound the trace and Frobenius norm of $\Psi_j$ and $\Phi_j$ proportional to their rank:

$$\mathrm{tr}(\Psi_j) \leq r\|\Psi_j\| \leq r\sigma_{\max}(\frac{\mathcal{A}_g^*\mathcal{A}_g}{n_g})\sigma_1^2(U_g^*)$$
$$\|\Psi_j\|_F \leq \sqrt{r}\|\Psi_j\| \leq \sqrt{r}\sigma_{\max}(\frac{\mathcal{A}_g^*\mathcal{A}_g}{n_g})\sigma_1^2(U_g^*)$$
$$\mathrm{tr}(\Phi_j) \leq r\sigma_{\max}(\frac{\mathcal{A}_g^*\mathcal{A}_g}{n_g})\sigma_1^2(U_g^*)$$
$$\|\Phi_j\|_F \leq \sqrt{r}\sigma_{\max}(\frac{\mathcal{A}_g^*\mathcal{A}_g}{n_g})\sigma_1^2(U_g^*).$$

Combining all the above results, we can instead set $\lambda_0$ as:

$$\lambda_0 = \max \Bigg\{ \sqrt{\frac{2048 L^2 \sigma_g^2 \sigma_{\max}(\frac{\mathcal{A}_g^* \mathcal{A}_g}{n_g})}{n_g} \log(\frac{10d^2}{\delta})},$$

$$\sqrt{\frac{256 \sigma_g^2 \sigma_{\max}(\frac{\mathcal{A}_g^* \mathcal{A}_g}{n_g}) \sigma_1^2(U_g^*)}{n_g} (r + 2\sqrt{r \log(\frac{5d}{\delta})} + 2\log(\frac{5d}{\delta}))},$$

$$\sqrt{\frac{8\beta \sigma_p^2 d}{n_p} (2r(2d+1)\log(36\sqrt{2}) + \log(\frac{10}{\delta}))} \Bigg\}.$$

Therefore, with $\lambda \geq \lambda_0$ and with $n_p$, $n_g$ and $d$ such that $\lambda \leq \frac{L\sigma_r(U_p^*)(3\alpha - 2\beta)}{16}$,

$$\|\widehat{\Delta}_U - \widetilde{\Delta}_U\|_{2,1} = \mathcal{O}\left( \sqrt{\frac{\sigma_g^2 s^2 \log(\frac{d}{\delta})}{n_g}} + \sqrt{\frac{\sigma_p^2 s^2 (d^2 + d\log(\frac{1}{\delta}))}{n_p}} \right),$$

with probability greater than $1 - \delta$. Moreover,

$$\ell(\widehat{U}_g, U_g^*) \leq \|\widehat{\Delta}_U - \widetilde{\Delta}_U\|_{2,1} = \mathcal{O}\left( \sqrt{\frac{\sigma_g^2 s^2 \log(\frac{d}{\delta})}{n_g}} + \sqrt{\frac{\sigma_p^2 s^2 (d^2 + d\log(\frac{1}{\delta}))}{n_p}} \right),$$

with probability at least $1 - \delta$. □

## F. Useful Lemmas

**Lemma 3.** *(Covering Number for Low-rank Matrices) Let $\mathcal{Z} = \{Z \in \mathbb{R}^{d_1 \times d_2} | \operatorname{rank}(Z) \leq r, \|Z\|_F = 1\}$. Then there exists an $\epsilon$-net $\mathcal{N} \subseteq \mathcal{Z}$ with respect to the Frobenius norm obeying*

$$|\mathcal{N}| \leq (9/\epsilon)^{(d_1+d_2+1)r}.$$

*Proof.* See Lemma 3.1 of Candes & Plan (2011). □

**Lemma 4.** *Let $\Delta = \widehat{U} - U^* R_{(\widehat{U}, U^*)}$, $\Theta^* = U^* U^{*T}$ and $\widehat{\Theta} = \widehat{U}\widehat{U}^T$, where $R_{(\widehat{U}, U^*)}$ is defined in Definition 1. Then,*

$$\|\Delta\Delta^T\|_F^2 \leq 2\|\widehat{\Theta} - \Theta^*\|_F^2$$
$$\sigma_r(\Theta^*)\|\Delta\|_F^2 \leq \frac{1}{2(\sqrt{2}-1)} \|\widehat{\Theta} - \Theta^*\|_F^2.$$

*Proof.* See Lemma 6 of Ge et al. (2017). □

**Lemma 5.** *(Concentration Inequality for Quadratic Subgaussian) Let $x \in \mathbb{R}^n$ be a zero-mean $\sigma$-subgaussian random vector that*

$$E[\exp(a^T x)] \leq \exp(\|a\|^2 \sigma^2 / 2)$$

*for any $a \in \mathbb{R}^n$. $A \in \mathbb{R}^{m \times n}$ and $\Sigma = A^T A$. Then, for any $t > 0$,*

$$P(\|Ax\|^2 > \sigma^2(\operatorname{tr}(\Sigma) + 2\|\Sigma\|_F \sqrt{t} + 2\|\Sigma\| t)) \leq \exp(-t).$$

*Proof.* See Theorem 1 of Hsu et al. (2012). □