holonomy & flux operators acting $\mathcal{H}_\gamma^0$ (non-gauge inv. wave func.)

$$\hat{h}_{AB}(e)\, f(h) = h_{AB}(e)\, f(h) \qquad A,B = 1,2$$

$$\hat{P}^j(e)\, f(h) = i\frac{l_p^2}{2}\, \hat{R}^j_e\, f(h) = i\frac{l^2}{2} \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} f(\cdots e^{\varepsilon\tau^j} h(e) \cdots)$$

$$= i\frac{l_p^2}{2} \frac{\partial f}{\partial h(e)_{AB}} (\tau^j h(e))_{AB}$$

quantum holonomy-flux algebra

$$[\hat{h}(e)\ \hat{h}(e')] = 0$$

$$[\hat{P}^j(e)\ \hat{h}(e')] = i l_p^2\, \delta_{ee'} \left(\frac{\tau_j}{2} h(e)\right)$$

$$[\hat{P}^j(e)\ \hat{P}^k(e')] = -i l_p^2\, \delta_{ee'}\, \varepsilon^{jkl}\, \hat{P}^l(e)$$

$$[\ ,\ ] = i\hbar \widehat{\{\ \ \}}$$

$l_p^2 = \hbar k$

$k = 16\pi G_N$

$\tau_j = -i\sigma_j$

$$\boxed{-\frac{i}{2}\hat{\vec{R}} = \hat{\vec{J}}}$$

Geometrical operators :

classically : gravitational field = curved spacetime geometry

quantum : operators of gravitational field = operators of geometry

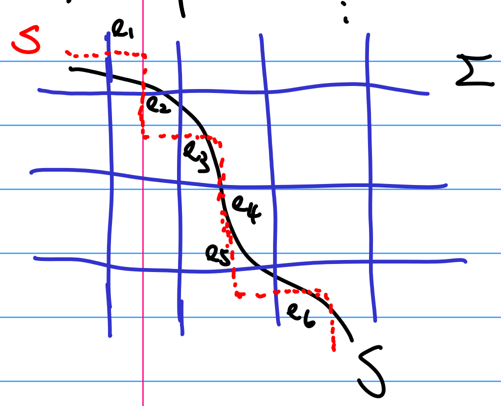
Area operator & volume operator.

(Quantum Riemanian geometry)

$M = \Sigma \times R$

Σ: space, R: time

Area operator :



$$S \simeq \sum_{i=1}^{n} S_{e_i} \qquad e_i \cap S \neq \phi$$

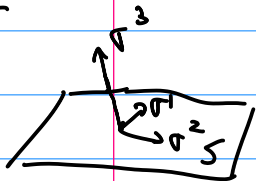
classically $\quad Ar(S) = \int_S d^2\sigma \sqrt{\det h}$

$d^2\sigma$: coordinate on S; $\det h$: induced metric

$\dim(S) = 2$

$$Ar(S) = \int_S d^2\sigma \sqrt{n_a E^a_j n_b E^b_j(\sigma)} \qquad n_a = (d\sigma^3)_a$$

Σ



$S = \{\sigma^3 = 0\}$

$$E^a_j = \sqrt{\det q}\; \underbrace{e^a_j}_{\text{triad}} = \underbrace{sgn(e)}_{=+1/-1 \text{ for right/left hand } e^a_j} \frac{1}{2} \underbrace{\epsilon^{abc}\epsilon_{jkl}}_{\epsilon\text{-symbol e.g. } \epsilon_{123}=1} \underbrace{e^k_b e^l_c}_{\text{co-triad}}$$

$n_a E^a_j n_b E^b_j(\sigma)$

$$= \frac{1}{4} n_a \epsilon^{abc} \underline{\epsilon_{jkl}} e^k_b e^l_c n_d \epsilon^{def} \underline{\epsilon_{jmn}} e^m_e e^n_f = \frac{1}{2} n_a n_d \epsilon^{abc}\epsilon^{def} \delta_{k[m}\delta_{l|n]} e^k_b e^l_c e^m_e e^n_f$$

$$= \frac{1}{2} n_a n_d \epsilon^{abc}\epsilon^{def} e^k_b e^l_c e^k_e e^l_f$$

$$= \frac{1}{2} \epsilon^{abc}\epsilon^{def} n_a n_d q_{be} q_{cf} \qquad n_a = (d\sigma^3)_a$$

$$= \frac{1}{2} \epsilon^{3IJ}\epsilon^{3KL} q_{IK} q_{JL} = \det(h) \qquad n_a = \delta^3_a$$

$$\underline{\underline{q_{IJ}}} = q_{ab} \frac{\partial x^a}{\partial \sigma^I} \frac{\partial x^b}{\partial \sigma^J} = h_{IJ}$$

$$Ar(S) \simeq \sum_{i=1}^{n} A_r(S_{e_i}) \qquad A_r(S_{e_i}) = \int_{S_{e_i}} d^3\sigma \sqrt{n_a E^a_j n_b E^b_j}$$

because S_{e_i} are all small.

$$Ar(S_{e_i}) = \sqrt{\underbrace{\int_{S_{e_i}} d^2\sigma\, n_a E^a_j}_{} \underbrace{\int_{S_{e_i}} d^2\sigma\, n_b E^b_j}_{}} + \underbrace{O(\mu^3)}_{\text{Coordinate Length of edges}}$$

$$\text{IS}$$

$$\sqrt{n_a E^a_j n_b E^b_j \left[\int_{S_{e_i}} d^2\sigma\right]^2} = \int d^2\sigma \sqrt{n_a E^a_j n_b E^b_j}$$

$n_a = (d\sigma^3)_a$

$$\int_{S_{e_i}} d^2\sigma\, n_a E^a_j = \int_{S_{e_i}} d^2\sigma\, E^3_j = \frac{1}{2} \int_{S_{e_i}} dx^a \wedge dx^b \epsilon_{abc} E^c_j$$

$$= -\frac{1}{4}\, tr\left[\tau^j \int_{S_{e_i}} dx^a \wedge dx^b\, \varepsilon_{abc}\, E^c_k\, \tau^k\right]$$

$h(p_e(x)) = 1 + O(\mu)$, $\quad h(p_e(x))^{-1}$

$$= \frac{\beta}{2} p^j(e_i) + O(\mu^3)$$

Lattice discretization of $Ar(S)$

$$\hat{Ar}(S) \equiv \sum_{i=1}^{n} \frac{\beta}{2} \sqrt{\underbrace{\hat{p}^i(e_i)\,\hat{p}^i(e_i)}} = \frac{\beta l_p^2}{4} \sum_{i=1}^{n} \sqrt{-\hat{R}^j_{e_i}\hat{R}^j_{e_i}}$$

$$\boxed{-\frac{i}{2}\hat{\vec{R}} = \hat{\vec{J}}}$$

$$= \frac{\beta l_p^2}{2} \sum_{i=1}^{n} \sqrt{\hat{\vec{J}}_{e_i} \cdot \hat{\vec{J}}_{e_i}} \qquad \hat{\vec{J}}\cdot\hat{\vec{J}} = \hat{J}^2 = j(j+1)$$

$$\hat{Ar}(S)\, T_{\gamma,\vec{j},\vec{i}} = \left(\sum_{\substack{e \\ e\cap S \neq \phi}} \frac{\beta l_p^2}{2} \sqrt{j_e(j_e+1)}\right) T_{\gamma,\vec{j},\vec{i}}$$

$\hat{Ar}(S)$ is $SU(2)$ gauge inv. $\qquad \frac{\beta l_p^2}{2} = 8\pi G \hbar \beta$

$l_p^2 = \hbar k = 16\pi G \hbar$

$\hat{Ar}(S)$ has discrete spectrum $\left(j_e = 0, \frac{1}{2}, 1, \cdots\right)$

$\longrightarrow$ <u>quantum Riemanian geometry is fundamentally discrete.</u>

gap of eigenvalues: $\Delta Ar(S) \sim l_p^2$

	operators		eigenstates
QM	$\hat{\vec{J}}$	$\hat{J}_3$	$\vert j\, m\rangle$
LQG	$-\frac{i}{2}\hat{\vec{R}}_e$, $-\frac{i}{2}\hat{\vec{L}}_e$	$\hat{R}^3$, $\hat{L}^3$	$\pi^{j_e}_{m_e n_e}(h)$

$$[R^i\ L^j] = 0$$

on $SU(2)$, $h \in SU(2)$

$$\hat{R}^i f(h) = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} f(e^{\varepsilon\tau^i} h(e)) \quad \leftarrow \text{right inv. vector field}$$

$$\hat{L}^i f(h) = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} f(h(e)\, e^{\varepsilon\tau^i}) \quad \leftarrow \text{Left inv. vector field}$$

$$\pi^j_{mn}(h) \longrightarrow \pi^j_{mn}(e^{\varepsilon_1 \tau^k} h\, e^{\varepsilon_2 \tau^l})$$

ε_1 for left translation
ε_2 for right translation

$$= \pi^j_{ma}(e^{\varepsilon_1 \tau^k})\, \pi^j_{ab}(h)\, \underline{\pi^j_{bn}(e^{\varepsilon_2 \tau^l})}$$

$$\pi^j_{ma}(e^{\varepsilon_1 \tau^k}) = \langle j\, m|\ \pi^j(e^{\varepsilon_1 \tau^k})\ |j\, a\rangle \qquad |j\, n\rangle \in V_j$$

$$= \langle j\, m|\ e^{i\varepsilon_1 \hat{J}^k}\ |j\, a\rangle$$

$\hat{R}^k$: $\left.\dfrac{d}{d\varepsilon_1}\right|_{\varepsilon_1\to 0} \longrightarrow i\langle j\, m|\ \hat{J}^k\ |j\, a\rangle$ acting on π^j_{ab}

$\vec{J}$ operator acting on the index a

$\hat{L}^l$: $\left.\dfrac{d}{d\varepsilon_2}\right|_{\varepsilon_1\to 0} \longrightarrow i\langle j\, b|\ J^l\ |j\, n\rangle$ acting on π^j_{ab}

$\vec{J}$ operator acting on the index b

π^j_{mn} : vectors in V_j $\to$ vectors in V_j

rep. matrix at spin j

acting on V_j $\qquad (\pi^j)^m{}_n\, v^n = w^m$

π^j_{mn} is a tensor $\quad \pi^j_{mn} \in V_j \otimes V_j^*$

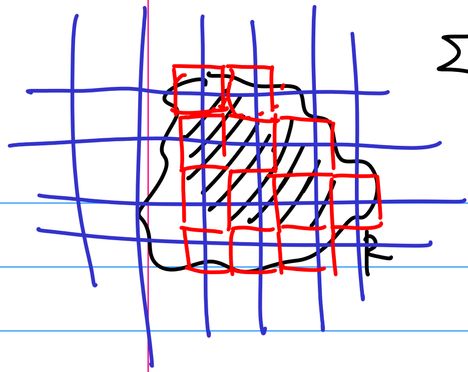
for index m $\quad \vec{R}\sim\vec{J} \quad \vec{L}\sim\vec{J} \quad$ for index n

$$\Rightarrow [\vec{R}.\vec{L}] = 0$$

$$-\hat{\vec{R}}\cdot\hat{\vec{R}} \sim \hat{J}^2 \text{ acting on } V_j = j(j+1)$$

$$-\hat{\vec{L}}\cdot\hat{\vec{L}} \sim \hat{J}^2 \text{ acting on } V_j^* = j(j+1)$$

Volume operator



Σ

$$V(R) \simeq \sum_{v \in R} V_v$$

V_v is the volume at vertex v

$\dim(R) = 3$

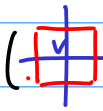
$$E^a_i = \sqrt{\det q}\, e^a_i$$

$$V(R) = \int_R d^3x \sqrt{\det q} = \int_R d^3x \sqrt{|\det(E^a_i)|}$$

$$= \int_R d^3x \sqrt{\left| \frac{1}{6} \varepsilon_{abc} \varepsilon^{ijk} E^a_i E^b_j E^c_k \right|}$$

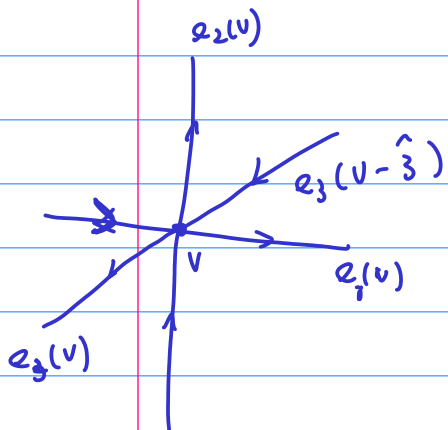
ε - symbols

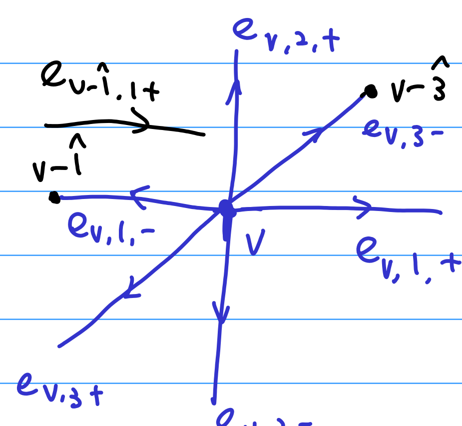
$$= \sum_v V_v \qquad V_v = V(\;\cdot\;)$$



$$V_v = \sqrt{|Q_v|}$$

$$Q_v = \beta^3 \varepsilon_{ijk} \left(\frac{p^i(e_{v,1+}) - p^i(e_{v,1-})}{4}\right)\left(\frac{p^j(e_{v,2+}) - p^j(e_{v,2-})}{4}\right)\left(\frac{p^k(e_{v,3+}) - p^k(e_{v,3-})}{4}\right)$$





$$e_{vI+} = e_I(v)$$

$$e_{vI-} = e_I(v-\hat{I})^{-1}$$

$$X^j_I(v) = p^j(e_{v,I+}) - p^j(e_{v,I,-})$$

$$Q_v = \beta^3 \varepsilon_{ijk} \frac{X^i_1(v)}{4} \frac{X^j_2(v)}{4} \frac{X^k_3(v)}{4} \qquad V_v = \sqrt{|Q_v|}$$

$$V(R) = \sum_v \sqrt{|Q_v|} + O(\mu^4)$$

HW

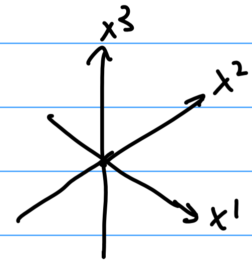
- prove $p^i(\underline{e_{vI-}}) = \frac{1}{2}\mathrm{Tr}\left[\tau^i h(e_{v-\hat{I},I+})^{-1} p^j(e_{v-\underline{\hat{I},I+}})\tau^j h(e_{v-\hat{I},I+})\right]$

- prove continuum limit of volume

$$V_v = \mu^3\sqrt{|\det(E^a_i)|} + O(\mu^4)$$

~~hint~~: prove firstly $p^j(e_{v,Is}) = \frac{\mu^2 s}{\beta} E^I_j(v) + O(\mu^3)$

$I=1,2,3,\ S=\pm 1$



x^I

$$\hat{V}(R) = \sum_v \hat{V}_v$$

$$\hat{V}_v = \sqrt{|\hat{Q}_v|} = (\hat{Q}_v^2)^{\frac{1}{4}}$$

$$\hat{Q}_v = \beta^3 \varepsilon_{ijk}\frac{\hat{X}^i_1(v)}{4}\frac{\hat{X}^j_2(v)}{4}\frac{\hat{X}^k_3(v)}{4}$$

$$\hat{X}^j_I(v) = \hat{p}^j(e_{vI+}) - \hat{p}^j(e_{vI-})$$