$$\langle 0| T\hat\phi(x_1)\cdots\hat\phi(x_n)|0\rangle = \frac{1}{Z[0]}\frac{\delta}{i\delta J(x_1)}\cdots\frac{\delta}{i\delta J(x_n)} Z[J]\Big|_{J\to 0}$$

$$\underline{Z[J] = \int D\phi\, e^{iS_0[\phi(x)] + i\int d^4x\,\phi(x)J(x)}}$$

$$S[\phi] = -\frac{1}{2}\int d^4x\left(\partial_\mu\phi\,\partial^\mu\phi - m^2\phi^2\right) \qquad \underline{\underline{t\to te^{-i\theta}}}$$

$$\langle 0| T\hat\phi(x_1)\hat\phi(x_2)|0\rangle = \int\frac{d^4k}{(2\pi)^4}\frac{i}{k^2-m^2+i\varepsilon}e^{-ik(x_1-x_2)}$$

self-interacting scalar field

$$p^2\to m^2 \qquad \theta(E^2+|\vec{k}|^2+m^2)$$

$$S[\phi] = \int d^4x\left[\frac{1}{2}\phi\underline{\underline{(-\partial^2-m^2+i\varepsilon)}}\phi\right] + V(\phi)$$

$$\partial^2 = \partial_\mu\partial^\mu \sim -k^2 \qquad V(\phi) = \underline{-\frac{\lambda}{4!}\phi^4}$$

coupling const.

mass term

$$V(\phi) = 1 + \cancel{\phi} + \phi^2 + \cancel{\phi^3} + \phi^4 + \cdots$$

killed by symmetry

$\phi\to-\phi$

$$\langle 0| T\phi(x_1)\cdots\phi(x_n)|0\rangle = \frac{\int D\phi\,\phi_1(x_1)\cdots\phi_n(x_n)\,e^{iS}}{\int D\phi\, e^{iS}}$$

$Z[J=0] = \langle 0|0\rangle$

$$\underline{Z[J] = \int D\phi\, e^{i\int\left[\frac{1}{2}\phi(-\partial^2-m^2+i\varepsilon)\phi - \underline{\underline{\frac{\lambda}{4!}}}\phi^4 + J\phi\right]d^4x}}$$

small λ

$$= \int D\phi\, e^{i\int d^4x\left[\frac{1}{2}\phi(-\partial^2-m^2+i\varepsilon)\phi + J\phi\right]}\left(1 - \frac{i\lambda}{4!}\int d^4x\,\phi^4 - \frac{\lambda^2}{2(4!)^2}\left[\int d^4x\,\phi^4\right]^2 + O(\lambda^3)\right)$$

$$Z[J=0] = \langle 0| e^{-i\int d^4x\frac{\lambda}{4}\hat\phi^4}|0\rangle_{\text{free}}$$

$$Z_0[J] = \int D\phi\, e^{i\int d^4x\left[\frac{1}{2}\phi(-\partial^2-m^2+i\varepsilon)\phi + J\phi\right]}$$

$$Z[J] = e^{-i\int d^4x \frac{\lambda}{4!}\left(\frac{\delta}{i\delta J(x)}\right)^4} Z_0[J]$$

$$= \left[1 - i\frac{\lambda}{4!}\int d^4x \left(\frac{\delta}{i\delta J(x)}\right)^4 + O(\lambda^2)\right] Z_0[J]$$

$$Z_0[J] = [\cdots]\, e^{-\frac{i}{2}\int d^4x\, d^4y\, J(x)\Delta(x-y)J(y)}$$

$\langle 0|T\phi(x)\phi(y)|0\rangle_{\text{free}} = i\Delta(x-y)$

$$\Delta(x-y) = \int \frac{d^4k}{(2\pi)^4}\frac{1}{k^2-m^2+i\varepsilon} e^{-ik(x-y)}$$

$\Delta(x-y) = \Delta_{xy}$

$$\phi(x) \to \phi_x \qquad J(x)\to J_x \qquad \int \phi(x)J(x)\,d^4x = \phi_x J_x$$

2-pt function:

$$\langle 0|T\phi_x\phi_y|0\rangle = \frac{1}{Z[0]}\frac{\delta}{i\delta J_x}\frac{\delta}{i\delta J_y}\left[1 - \frac{i\lambda}{4!}\int\left(\frac{\delta}{i\delta J_z}\right)^4 d^4z\right]\underbrace{e^{-\frac{i}{2}J_x\Delta_{xy}J_y}}_{Z_0[J]}$$

$$\underline{Z[J] = \left[1 - \frac{i\lambda}{4}\int d^4z\left(\frac{\delta}{i\delta J_z}\right)^4\right] e^{-\frac{i}{2}J_x\Delta_{xy}J_y}} \qquad Z[0] = Z[J\to 0]$$

$\Delta_{xy} = \Delta_{yx}$

4-pt function

$\Delta(0) = \Delta(z-z)$
$= \Delta_{zz}$

$$\langle 0|T\phi_{x_1}\cdots\phi_{x_4}|0\rangle = \frac{1}{Z[0]}\frac{\delta}{i\delta J_{x_1}}\cdots\frac{\delta}{i\delta J_{x_4}} Z[J]\Big|_{J\to 0}$$

$$\frac{\delta}{i\delta J_z} e^{-\frac{i}{2}J_x\Delta_{xy}J_y} = e^{-\frac{i}{2}J_x\Delta_{xy}J_y}(-\Delta_{zx}J_x) \qquad \frac{\delta J_x}{\delta J_z} = \delta_{xz}$$

$$\left(\frac{\delta}{i\delta J_z}\right)^2 e^{-\frac{i}{2}J_x\Delta_{xy}J_y} = [\overset{\text{divergent}}{i\Delta(0)} + (\Delta_{zx}J_x)^2]\, e^{-\frac{i}{2}J_x\Delta_{xy}J_y}$$

$$\left(\frac{\delta}{i\delta J_z}\right)^3 e^{-\frac{i}{2}J_x\Delta_{xy}J_x} = 2(\Delta_{zx}J_x)\frac{\Delta(0)}{i} e^{-\frac{i}{2}J_x\Delta_{xy}J_y}$$

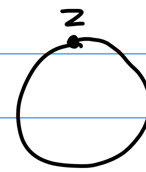
$$+ [i\Delta(0) + (\Delta_{zx}J_x)^2](-\Delta_{zx}J_x)\, e^{-\frac{i}{2}J_x\Delta_{xy}J_y}$$

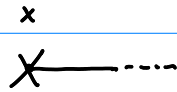
$$= [-3i(\Delta_{zx}J_x)\Delta(0) - (\Delta_{zx}J_x)^3]\, e^{-\frac{i}{2}J_x\Delta_{xy}J_y}$$

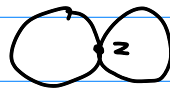
$$\left(\frac{\delta}{i\delta J_z}\right)^4 e^{-\frac{i}{2} J_x \Delta_{xy} J_y}$$

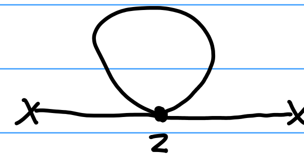
$$= \left[-3\Delta(0)^2 + 6i\,\Delta(0)\left(\Delta_{zx} J_x\right)^2 + \left(\Delta_{zx} J_x\right)^4\right] e^{-\frac{i}{2} J_x \Delta_{xy} J_y}$$

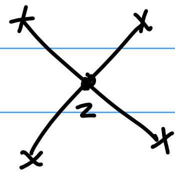
$\Delta_{xy} =$ 

$\Delta(0) \simeq \Delta_{zz} =$ 

$J_x =$ 

$\Delta(0)^2 =$ 

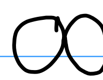
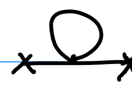
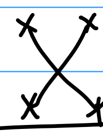
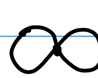
$\Delta(0)\left(\Delta_{zx} J_x\right)^2 = \Delta(0)\left(\Delta_{zx} J_x\right)\left(\Delta_{zy} J_y\right) =$ 

$\left(\Delta_{zx} J_x\right)^4 =$ 

$$\frac{\delta}{\delta J_w} \Delta_{zx} J_x \Delta_{zy} J_y$$

$$= 2\Delta_{zw} \Delta_{zy} J_y$$

$$\frac{Z[J]}{Z[0]} = \frac{\left[1 - \frac{i\lambda}{4}\int d^4z \left(\frac{\delta}{i\delta J_z}\right)^4\right] e^{-\frac{i}{2} J_x \Delta_{xy} J_y}}{\left[1 - \frac{i\lambda}{4}\int d^4z \left(\frac{\delta}{i\delta J_z}\right)^4\right] e^{-\frac{i}{2} J_x \Delta_{xy} J_y}\Big|_{J\to 0}}$$

$= \dfrac{\left[1 - \frac{i\lambda}{4}\int d^4z \left(-3\right.\right.}{\left[1 - \frac{i\lambda}{4}\int d^4z \left(-3\right.\right.}$  $+\,6i$  $+$  $\left.\left.\right)\right] e^{-\frac{i}{2} J_x \Delta_{xy} J_y}$ ... denominator: $\left[1 - \frac{i\lambda}{4}\int d^4z \left(-3\right.\right.$  $\left.\left.\right)\right]$

$\uparrow$

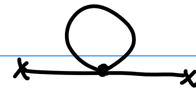
divergent vaccum diagram

but it will cancel the divergent numerator @ $O(\lambda)$

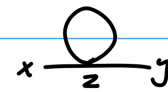
generalize to all order in λ

2-pt function : $\langle 0| T\phi(x)\phi(y)|0\rangle = \dfrac{\delta}{i\delta J_x}\dfrac{\delta}{i\delta J_y}\dfrac{Z[J]}{Z[0]}\Bigg|_{J\to 0}$

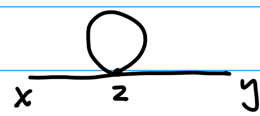
2 contributions : (1) $\dfrac{\delta}{\delta J}\dfrac{\delta}{\delta J}\, e^{-\frac{i}{2}J\Delta J}$ $\qquad O(1)$

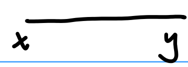
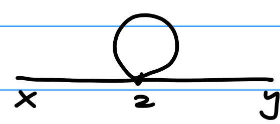
(2) $\dfrac{\delta}{\delta J}\dfrac{\delta}{\delta J}$  $\qquad O(\lambda)$

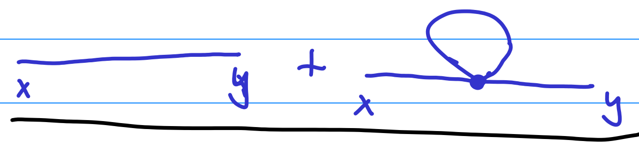
(1) $$\frac{\left[1-\frac{i\lambda}{4!}\int d^4z\,(-3\,\infty)\right] i\Delta(x-y)}{\left[1-\frac{i\lambda}{4!}\int d^4z\,(-3\,\infty)\right]} = i\Delta(x-y)$$

(2) $$\frac{1}{\underline{\left[1-\frac{i\lambda}{4!}\int d^4z\,(-3\,\infty)\right]}}\left[\underbrace{-(6i)\times 2\left(-\frac{i\lambda}{4!}\right)}_{-\frac{\lambda}{2}}\int d^4z\right.$$  $]$

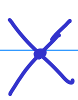
$\dfrac{2\,\cancel{6}\times\cancel{2}}{4\times\cancel{3}\times\cancel{2}}$

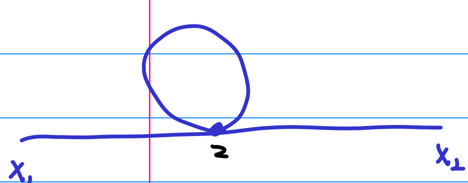
$= -\dfrac{\lambda}{2}\int d^4z$  $\left(1+O(\lambda)\right)$

$\langle 0|T\phi(x)\phi(y)|0\rangle = i$  $-\dfrac{\lambda}{2}\int d^4z$  $+O(\lambda^2)$

$=$  $+ O(\lambda^2)$

Feynman rule for $\lambda\phi^4$ (position space)

-  $= i\Delta(x_1-x_2)$
-  $= -i\lambda\int d^4x$
- symmetry factor : here, $\frac{1}{2}$

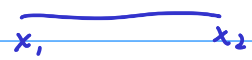
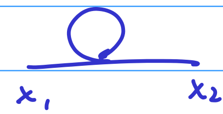
 $$= \frac{-i\lambda}{2}\int d^4z\,\big(i\underset{\Delta(z-z)}{\Delta(0)}\big)\big(i\,\Delta(x_1-z)\,i\,\Delta(x_2-z)\big)$$

$$= -\frac{\lambda}{2}\int d^4z\,\Delta(0)\,\underline{\Delta(x_1-z)}\,\Delta(x_2-z)$$

$$= -\frac{\lambda\Delta(0)}{2}\int d^4z \underbrace{\int\frac{d^4p}{(2\pi)^4}\frac{e^{-ip(x_1-z)}}{p^2-m^2+i\varepsilon}}\int\frac{d^4q}{(2\pi)^4}\frac{e^{-iq(x_2-z)}}{q^2-m^2+i\varepsilon}$$

$$= -\frac{\lambda\Delta(0)}{2}\int\frac{d^4p}{(2\pi)^4}\frac{d^4q}{(2\pi)^4}\frac{e^{-ipx_1-iqx_2}}{(p^2-m^2+i\varepsilon)(q^2-m^2+i\varepsilon)}(2\pi)^4\delta(p+q)$$

$$= -\frac{\lambda\Delta(0)}{2}\int\frac{d^4p}{(2\pi)^4}\frac{e^{-ip(x_1-x_2)}}{(p^2-m^2+i\varepsilon)^2}$$

$\langle 0|T\phi(x_1)\phi(x_2)|0\rangle =$  $+$  $+ O(\lambda^2)$

$$= i\int\frac{d^4p}{(2\pi)^4}\underbrace{\frac{e^{-ip(x_1-x_2)}}{p^2-m^2+i\varepsilon}}_{\text{pole at } m^2}\underbrace{\left[1+\frac{i\lambda\Delta(0)}{2}\frac{1}{p^2-m^2+i\varepsilon}+O(\lambda^2)\right]}_{\left[1-\frac{i\lambda\Delta(0)}{2}\frac{1}{p^2-m^2+i\varepsilon}\right]^{-1}}$$

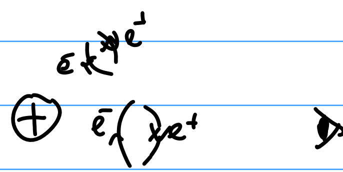
$$= i\int\frac{d^4p}{(2\pi)^4}\frac{e^{-ip(x_1-x_2)}}{\underbrace{p^2-m^2-\frac{i}{2}\lambda\Delta(0)}+i\varepsilon}$$

bare mass (divergent)

pole at $m^2+\delta m^2 = \underbrace{m^2+\frac{i}{2}\lambda\Delta(0)}_{m^2_{phy.}}$

physical mass is the pole of propagator.

physical mass



free Lagrangian

quantum correction (divergent)

$$\int\phi(\partial^2+m^2)\phi$$

finite

physical mass. = bare mass

mass renormalization (pre-baby version)

interacting Lagrangian

$O(\lambda)$

$$\int\phi_0(\partial^2+m_0^2)\phi_0+\frac{\lambda_0}{4!}\phi_0^2$$

$m_{phys}^2 = m_0^2+\delta m$

$m=m_0$ bare mass $\neq$ physical mass

$$\begin{cases} \text{mass renormalization} & m^2 \to m^2 + \delta m^2 & \text{2-pt function} \\ \text{charge renormalization} & \lambda \to \lambda + \delta\lambda & \text{4-pt function} \\ \text{wave function renormalization} & \phi \to \phi + \delta\phi & \end{cases}$$

<u>HW</u>: repeat all perturbative computation today

derive mass renormalization.