Spontaneous symmetry breaking

symmetry in EOM, but the symmetry is broken by the solution.

example: $x^2 = 1$ solution $x = 1$ or $x = -1$

symm: $x \to -x$

this symm is broken at solution.

$x^2 = 0$ solution $x = 0$

$x^2 = \mu$

Def (spontaneous symm breaking)

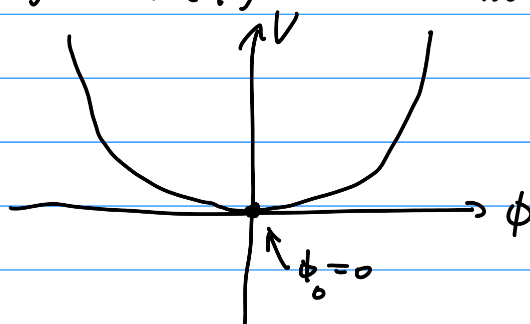
- Dynamics (Lagrangian, Hamiltonian, EOMs) invariant under a transf. group G
- vaccum (ground state, solution of EOMs) only invariant under a subgroup $H \subset G$

$\Rightarrow$ symm group G is spontaneously broken down to H.

real scalar field with potential $V(\phi) = \mu^2 \phi^2 + \lambda \phi^4$ $\lambda > 0$

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 - V(\phi) \quad \text{inv. under } \phi \to -\phi$$

- $\mu^2 > 0$

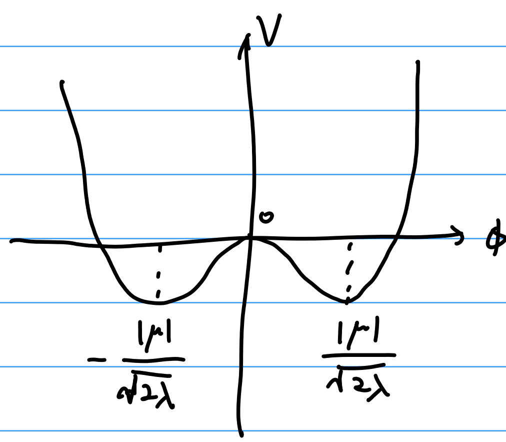


$$\partial^2\phi + \mu^2\phi + \lambda\phi^3 = 0$$

$$\partial^2\phi + \frac{\partial V}{\partial\phi}(\phi) = 0$$

ground state $\phi_0 = 0 \Rightarrow$ no spontaneous ~~symm~~

- $\mu^2 < 0$



2 ground state:

$$\phi_0 = \frac{|\mu|}{\sqrt{2\lambda}} \text{ or } -\frac{|\mu|}{\sqrt{2\lambda}}$$

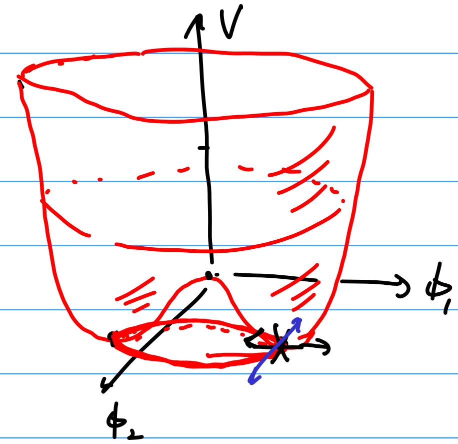
spontaneous ~~symm~~

2 real scalar fields (ϕ_1, ϕ_2) inv. under $\underbrace{SO(2)}_{U(1)}$ rotation

$SO(2) \times \mathbb{Z}_2$

$$\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \to \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$$

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu \phi_1 \partial^\mu \phi_1 + \partial_\mu \phi_2 \partial^\mu \phi_2\right) + \underbrace{\frac{\mu^2}{2}\left((\phi_1)^2 + (\phi_2)^2\right) - \frac{\lambda}{4}\left((\phi_1)^2 + (\phi_2)^2\right)^2}_{\mu^2 > 0\ ,\ \lambda > 0}$$



Mexican hat

minimum : $(\phi_1)^2 + (\phi_2)^2 = \frac{\mu^2}{\lambda} \equiv v^2$

∞ many ground states

$$\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \to \underbrace{\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}}_{\not\in SO(2)} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$$

$$\boxed{\begin{aligned} \phi_1 &\to -\phi_1 \\ \phi_2 &\to \phi_2 \end{aligned}}$$

Any one ground state . e.g. $(\phi_1, \phi_2) = (0, v)$

$\Rightarrow$ $SO(2)$ symm is spontaneously broken.

Non abelian symmetry : n complex scalar ϕ_i in the fund. rep. of $SU(n)$

$$\mathcal{L} = (\partial_\mu \phi)^\dagger (\partial^\mu \phi) + \mu^2 \phi^\dagger \phi - \lambda (\phi^\dagger \phi)^2 \qquad \mu > 0\ ,\ \lambda > 0$$

$$V = -\mu^2 \phi^\dagger \phi + \lambda (\phi^\dagger \phi)^2$$

minimum of $V(\phi)$: $\phi^\dagger \phi = \frac{\mu^2}{2\lambda} = v^2$, e.g. $\phi = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ v \end{pmatrix} \Big\} n-1$

$v = \frac{\mu}{\sqrt{2\lambda}} \neq 0$

$$\underbrace{\begin{array}{c} \overbrace{\quad}^{n-1} \\ n-1 \left\{ \begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix} \right. \end{array}}_{SU(n-1) \subset SU(n)} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ v \end{pmatrix}$$

spontaneously ~~symm~~

$SU(n) \to SU(n-1)$

Goldenstone boson (emergent massless scalars in broken direction)

- 2 scalar (ϕ_1, ϕ_2)

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu\phi_1\partial^\mu\phi_1 + \partial_\mu\phi_2\partial^\mu\phi_2\right) + \frac{\mu^2}{2}\left((\phi_1)^2 + (\phi_2)^2\right) - \frac{\lambda}{4}\left((\phi_1)^2+(\phi_2)^2\right)^2$$

this is an expansion of $\mathcal{L}$ at the ground state $\phi_1 = \phi_2 = 0$ $\quad \mu > 0$, $\lambda > 0$

$\uparrow$ unstable

not true ground state

QCD $\quad F^2 + \bar{\psi}\cancel{D}\psi + c\partial Dc$

$A = \psi = c = 0$

true ground state is $(\phi_1, \phi_2) = (v, 0)$

$\phi_1 = v + \chi_1$, $\quad \phi_2 = \chi_2$ $\qquad v = \frac{\mu}{\sqrt{\lambda}}$

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu\chi_1\partial^\mu\chi_1 + \partial_\mu\chi_2\partial^\mu\chi_2\right) - \frac{\mu^2}{2}(\chi_1)^2 + \frac{v^2}{4} + O(\chi_1, \chi_2)^2$$

χ_1 is fluctuation $\perp$ symm. broken direction

χ_2 is fluctuation $//$ symm broken direction

$\phi = \frac{\phi_1 + i\phi_2}{\sqrt{2}}$ $\qquad \phi^* = \frac{\phi_1 - i\phi_2}{\sqrt{2}}$ $\qquad \phi = \rho e^{i\theta}$

$$\mathcal{L} = (\partial_\mu\phi)^\dagger\partial_\mu\phi + \mu^2\phi^\dagger\phi - \lambda(\phi^\dagger\phi)^2$$

$\phi_0 = \frac{v}{\sqrt{2}} e^{i\theta_0}$

$\phi = e^{i\theta/v}\left(\frac{v+\chi}{\sqrt{2}}\right)$

$$= \frac{1}{2}\partial_\mu\chi\partial^\mu\chi - \frac{\mu^2}{2}\chi^2 + \frac{1}{2}\partial_\mu\theta\partial^\mu\theta + \frac{v^2}{4} + O(\theta, \chi)^2$$

θ is massless field (Goldstone boson)

ϕ_0 vaccum expectation value of ϕ at the ground state.

$\shortparallel$

$\langle\phi\rangle$ $\qquad$ Ref: P&S Sec. 11.3. quantum effective action

$\uparrow$

order parameter

1) quantum effective action

2) Ising model, mean field theory, Landau paradiam.

3) chiral anomaly.

4) electro-weak theory (Weinberg - Glashow - Salam model)

<u>Goldstone's theorem</u>: $V(\phi)$ inv. under a symm group G spontaneously broken to $H \subset G$ $\Rightarrow$ $\dim G - \dim H$ massless scalars associate to broken direction.

- Expand $V(\phi)$ around minimum $\phi_i = \underset{\substack{\uparrow \\ \text{VEV.}}}{v_i} + \chi_i$

$$V(\phi) = \underset{\substack{\uparrow \\ \text{set to zero}}}{V(v_i)} + \cancel{\frac{\partial V}{\partial \phi_i}}^{\,0} \chi_i + \sum_{ij} \frac{\partial^2 V}{\partial \phi_i \partial \phi_j}(v)\, \chi_i \chi_j + O(\chi^3)$$

- eigenvalue of $\frac{\partial^2 V}{\partial \phi_i \partial \phi_j}(v)$ $\longrightarrow$ $\underset{\substack{\uparrow \\ \text{all positive}}}{\text{mass}}$ of χ_i

- Group G $\Rightarrow$ infinitesimal transf. $\delta_\varepsilon \phi_i = \varepsilon^a \sum_j \rho(T^a)_{ij}\, \phi_j$

- invariance of V under G $\Rightarrow$ $\sum_{ij} \frac{\partial^2 V}{\partial \phi_i \partial \phi_j}(v)\, \underline{\delta_\varepsilon \phi_i}\big|_v\, \underline{\delta_\varepsilon \phi_j}\big|_v = 0$

- $G \supset H$ the ground state inv. under H.

$$\delta_\varepsilon \phi_i \big|_v = 0 \quad \forall g_\varepsilon \in H.$$

$$\sum_{i,j} \underset{\substack{\uparrow \\ \text{non zero.}}}{\frac{\partial^2 V}{\partial \phi_i \partial \phi_j}(v)}\, \underset{\uparrow}{\delta_\varepsilon \phi_i \big|_v}\, \underset{\uparrow}{\delta_\varepsilon \phi_j \big|_v} = 0$$

since they are zero

- G/H $\quad \delta_\varepsilon \phi_i \big|_v \neq 0$ $\quad \sum_{i,j} \frac{\partial^2 V}{\partial \phi_i \partial \phi_j}(v)\, \delta_\varepsilon \phi_i \big|_v\, \delta_\varepsilon \phi_j \big|_v = 0$

$$\Rightarrow \frac{\partial^2 V}{\partial \phi_i \partial \phi_j}(v) = 0$$

along these directions.

$\Rightarrow$ $\dim G - \dim H$ zero modes $\longrightarrow$ massless Goldstone scalars