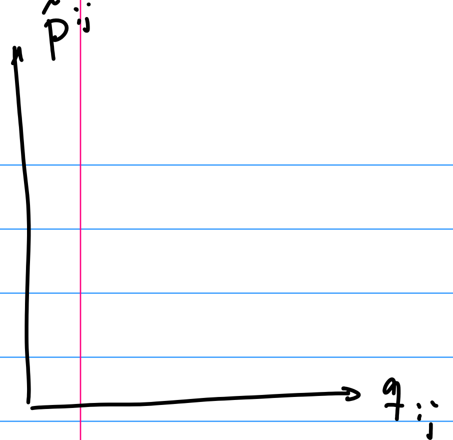


$$\dot{q}_{ij} = \frac{\delta H}{\delta \tilde{p}_{ij}} \qquad \dot{\tilde{p}}^{ij} = -\frac{\delta H}{\delta q_{ij}}$$

$k = 16\pi G$

$$H = \int_\Sigma N C + N_i C^i$$

$$C(x) = \frac{1}{k}\left[\frac{1}{\sqrt{\det q}}\left(\tilde{p}^{ij}\tilde{p}_{ij} - \frac{1}{2}p^2\right) - \sqrt{\det q}\ {}^3R\right]$$

$$C^i = -\frac{2}{k} D_i \tilde{p}^{ij}$$

<u>Def</u> Given phase space $(P, \{\ ,\ \})$, $\dim P = m$ (can be ∞)

and constraints C_A (function on P) $A = 1 \cdots n$ $n < m$ (can be ∞)

if $\{C_A, C_B\} = \sum_C f_{AB}{}^C C_C$ ←

Poisson bracket of constraints is a linear combination of constraints

we say the set of $C_{A=1..n}$ is of first class

otherwise they are called second class constraints.

constraint algebra

remark: $f_{AB}{}^C$ may be a function on phase space

↑ structure function

if $f_{AB}{}^C$ is constant function.

constraint algebra = Lie algebra

usually first class constraint ⟺ gauge transformations.

Hamiltonian & diffeo. constraints $C_A \equiv (C_i(x), C(x))$

$$C(\vec{N}) = \int_\Sigma d^3x\, N^i C_i \qquad C(N) = \int d^3x\, N C$$

$$\underline{\underline{C_A}} \equiv \left(\underline{C(\vec{N})} ,\ C(N) \right)$$

$$\left[\begin{aligned}
\{C(\vec{N}), C(\vec{N}')\} &= C(\mathcal{L}_{\vec{N}'} \vec{N}) \\
\{C(N), C(\vec{N}')\} &= C(\underline{\mathcal{L}_{\vec{N}'} N}) \\
\underline{\{C(N), C(N')\}} &= \underline{C\left((N\partial_a N' - N'\partial_a N)\, q^{ab} \right)}
\end{aligned}\right.$$

diffeomorphism

phase space dependent structure function.

$N^\mu = (N, N^i)$

$$\int_\Sigma d^3x \; N^\mu(x) C_\mu(x) \equiv N^A C_A \qquad A = (\mu, \vec{x})$$

DeWitt notation

$N^\mu = (N, 0, 0, 0)$ Hamiltonian constraint

$N^\mu = (0, \vec{N})$ Diffeo. constraint.

$$\{N^A C_A, M^B C_B\} = N^A M^B \{C_A, C_B\} = \boxed{N^A M^B f_{AB}{}^C} C_C$$

by def. of 1st class constraints

① $N^A = (0, \vec{N})$, $M^B = (0, \vec{M})$ $\qquad f_{AB}{}^C N^A M^B = (0, \mathcal{L}_{\vec{M}} \vec{N})$

② $N^A = (N, 0, 0, 0)$, $M^B = (0, \vec{M})$ $\qquad f_{AB}{}^C N^A M^B = (\mathcal{L}_{\vec{M}} N, 0, 0, 0)$

③ $N^A = (N, 0, 0, 0)$, $M^B = (M, 0, 0, 0)$ $\qquad \underline{f_{AB}}{}^C \underline{N}^A \underline{M}^B = \left(0, (N\partial_a M - M\partial_a N)\, \underline{q}^{ab} \right)$

$$f_{AB}{}^C = \underline{f_{AB}{}^C(q)}$$ phase space dep. structure function.

in GR, constraint algebra is NOT a Lie algebra,
gauge transformations generated by constraints can not form a Lie group.

(Lee & Wald, 1990)

$$\delta q = \{ q, C_A \} \qquad H = \int N C + N^i C_i$$

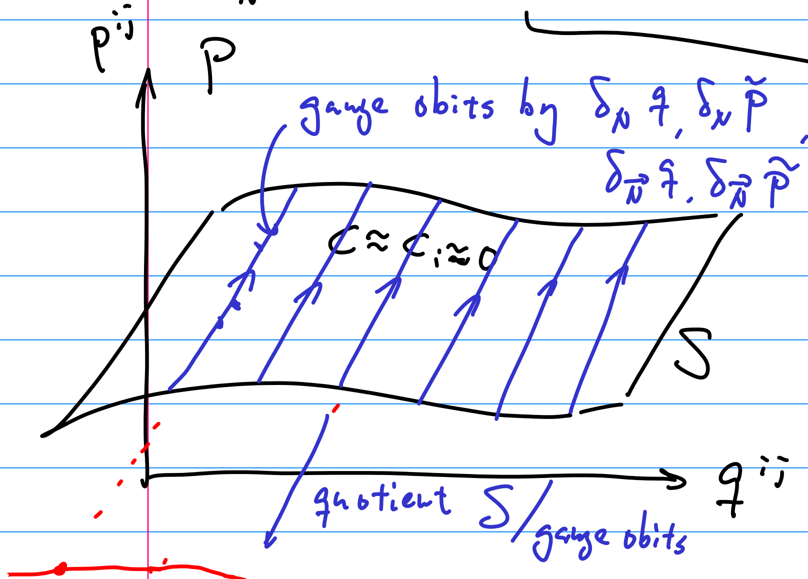
$$\delta_N H = \{ H, C(N) \} \approx 0 \quad \text{(on constraint surface)}$$

$$\delta_{\vec{N}} H = \{ H, C(\vec{N}) \} \approx 0$$

$\Rightarrow$ EOM is gauge inv.

$\dot{C}(N)$

$\dot{C}(\vec{N})$



lying on constraint surface

because $\delta C^A = \{ C^A, C^B \}$

$= f_{AB}{}^C C_C \approx 0$

(gauge transf. leaves S inv.)

quotient space (space of gauge obits)

space of physical DOFs.

reduced phase space P_{red}

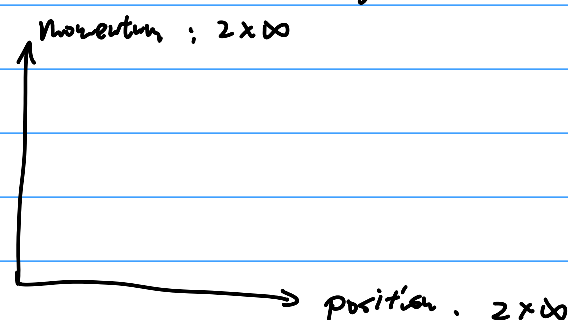
functions on P_{red} : constant along gauge obit. , gauge inv. observables.

$\tilde{p}^{ij}(\vec{x}), \; q_{ij}(\vec{x})$: $(6+6=12) \times \infty = \dim(P)$

$C^i(x) \quad C(x)$: $4 \times \infty$ constraints

$\dim(S) = 8 \times \infty$

gauge trasf. : $4 \times \infty \quad \rightarrow \dim(P_{\text{red}}) = 4 \times \infty$



DOFs of Einstein gravity $= 2 \times \infty$

(2 per space point)

recall linearized gravity : 2 polarizations of gravitational waves quantization $q \to \hat{q}$. $p \to \hat{p}$ $\mathcal{H} = \{\text{functions of } q\}$

functional diff eqn. (WDW eqn)

$$C(N) \to \hat{C}(N) \qquad \hat{C}(N)\Psi = 0$$
$$C(\vec{N}) \to \hat{C}(\vec{N}) \qquad \hat{C}(\vec{N})\Psi = 0$$

} quantum constraint eqns.

$$q_{ij}(x) \qquad \hat{p}^{ij}(x) = -i\hbar \frac{\delta}{\delta q_{ij}(x)}$$

Extend the ADM phase space :

$$\tilde{p}^{ab} = \sqrt{\det q}\,(K^{ab} - K q^{ab})$$

$$\to \begin{cases} q_{ab} = \delta_{ij} e^i_a e^j_b \\ K_{ab} = K_{ai} e^i_b \end{cases}$$

$e^j_a \equiv (e^j)_a$ orthonormal dual basis on Σ (cotriad)

e^a_i inverse of e^i_a

triad

we can use K_{ai}, e^i_a as dynamical variables

$$(K_{ai}, e^i_a) \longrightarrow (q_{ab}\ K_{ab}) \longrightarrow (\underline{q_{ab}\ \tilde{p}^{ab}})$$

① but there are additional gauge redundancy :

$$e^i_a \to O^i{}_j e^j_a \qquad O \in SO(3)$$

special rotations of basis

$$K_{ai} \to K_{ai} (O^{-1})^i{}_j \qquad \text{i.e. } \underline{O^i{}_j O^k{}_l \delta_{ik}} = \delta_{jl}$$

$\det O = 1$

② there is a constraint :

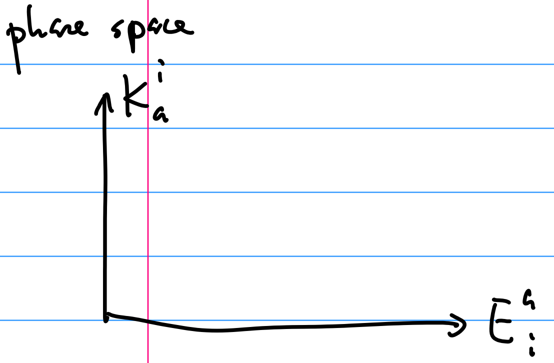
$$K_{ai} e^a_j = K_{ji} \to K_{[j i]} = 0$$

$$\Leftrightarrow K_{a[i} e^a_{j]} = 0 \qquad 3 \text{ constraints}$$

counting dimension of phase space : $\underset{(9+9)\times\infty}{(e^i_a\ K_{ai})} \xrightarrow[3\times\infty \text{ constraints}]{3\times\infty \text{ gauge transf}} \underset{(6+6)\times\infty}{(q_{ab}\ \tilde{p}^{ab})}$

canonical variable : $E^a_i = \sqrt{\det q}\, e^a_i$ densitized triad

(triad ADM) $K^i_a = K_{ai}$ extrinsic curvature



$$\{f, f'\} = \frac{k}{2}\int_\Sigma \left(\frac{\delta f}{\delta E^a_i(x)} \frac{\delta f'}{\delta K^i_a(x)} - \frac{\delta f}{\delta K^i_a(x)} \frac{\delta f'}{\delta E^a_i(x)} \right) d^3x$$

$$\{E^a_i(x), K^j_b(y)\} = \frac{k}{2}\, \delta^j_i\, \delta^a_b\, \delta^{(3)}(x, y)$$

$$\{E, E\} = \{K, K\} = 0$$

$$q_{ab} = \bar{E}^j_a \bar{E}^j_b\, |\det(E)| \qquad \tilde{P}^{ab} = 2\,|\det(E)|^{-1} E^a_k E^d_k K^j_{[d}\, \delta^b_{c]} E^c_j$$

inverse of E^a_j

$$\{\tilde{P}^{ab}(x), q_{cd}(y)\} = k\, \delta^a_{(c}\, \delta^b_{d)}\, \delta^{(3)}(x, y)$$

HW check this

Ref. Thiemann "Modern Canonical QGR"