(Goal of procedure)
physical Hilbert space
$\downarrow$

Quantum constraint anomaly.

$$\left.\begin{array}{l} C \rightsquigarrow \hat{C}\Psi = 0 \\ 2e^a_j C_a = C_j \rightsquigarrow \hat{C}_j \Psi = 0 \end{array}\right\}$$ look for solutions Ψ which span $\mathcal{H}_{phys}$

Naive proposal of solving constraints at quantum level

problem 1: eqns are hard to solve: both $\hat{C}$, $\hat{C}_j$ are nonpolynomial operators dep. on $\hat{V}_v = (\hat{Q}^2_v)^{1/4}$, one needs to diagonalized $\hat{Q}^2_v$ to compute e.g. matrix element of $\hat{V}_v$, but $\hat{Q}^2$ is hard to be diagonalized analytically. so it's hard to compute matrix elements of $\hat{C}$, $\hat{C}_j$. solving quantum constraint eqn. is even harder.

problem 2: quantum anomaly

classically C, C_j are 1st class constraints

$$C_I = \{C(x), C_j(x)\} \qquad I = (j, x)$$

$$\{C_I, C_J\} = f_{IJ}{}^k(q)\, C_k \leftarrow$$

Suppose our quantization gives $\underline{[\hat{C}_I, \hat{C}_J] = \hat{f}_{IJ}{}^k \hat{C}_k}$

$\forall$ solution Ψ, $C_I\Psi = 0 \ \forall I \Rightarrow [C_I, C_J]\Psi = 0$ firstly

$$\overset{\shortparallel}{\hat{C}_I \underline{\underline{\hat{C}_J\Psi}} - \hat{C}_J \underline{\underline{\hat{C}_I\Psi}}}$$

secondly $\hat{f}_{IJ}{}^k \underline{\underline{\hat{C}_k\Psi}} = 0$

all consistent.

however, our quantization actually gives

$$[\hat{C}_I, \hat{C}_J] = \hat{f}_{IJ}{}^k \hat{C}_k + \mu \hat{O}_1{}^{IJ} + \hbar \hat{O}_2{}^{IJ}$$

discretization corrections | quantum corrections

$\Rightarrow$ given $\hat{C}_I \Psi = 0$ $\Rightarrow$ left: $[\hat{C}_I, \hat{C}_J]\Psi = 0$

right: $\left(\hat{f}_{IJ}{}^{k} \hat{C}_k + \mu \hat{O}_1 + \hbar \hat{O}_2\right)\Psi$

$= \left(\mu \hat{O}_1 + \hbar \hat{O}_2\right)\Psi \neq 0$

$\Rightarrow$ inconsistency unless we impose in addition

$$\underbrace{\hat{O}_1 \Psi = \hat{O}_2 \Psi = 0}_{\text{hard to solve \& no classical analog}}$$

too many quantum constraints. $\rightarrow$ solution space $\mathcal{H}_{phys}$ is too small.

Constraint anomaly is problematic.

In general, QFT: global symm. classical $\{Q_I, Q_J\} = f_{IJ}{}^{k} Q_k$

constant (Lie algebra)

symm charge. e.g. H, $\vec{P}$, $\vec{J}$

quantum. $[\hat{Q}_I, \hat{Q}_J] = f_{IJ}{}^{k} \hat{Q}_k + \underline{\underline{\hbar \hat{O}}}$

quantum correction (quantum anomaly)

because in QM $|\psi\rangle \sim e^{i\theta}|\psi\rangle$

$$e^{[\hat{Q}_I, \hat{Q}_J]}|\Psi\rangle = \underline{\underline{e^{i\theta}}}\, e^{f_{IJ}{}^{k}\hat{Q}_k}|\Psi\rangle$$

a symmetry broken by quantum effect is called a quantum anomaly and is fine (e.g. axial anomaly breaking scaling inv. in QFT) but for gauge symm, $Q_I = C_I$ constraint.

gauge anomaly or constraint anomaly is NOT fine

because $[\hat{C}_I, \hat{C}_J] = f_{IJ}{}^k \hat{C}_k + \underline{\hbar \hat{O}}$

$\rightarrow$ additional constraints $\hat{O}\Psi = 0$ $\rightarrow$ $\mathcal{H}_{phys}$ is too small

in our case $[\hat{C}_I, \hat{C}_J] = \hat{f}_{IJ}{}^k \hat{C}_k + \underbrace{\mu \hat{O}_1}_{\text{discretization anomaly}} + \underbrace{\hbar \hat{O}_2}_{\text{quantum anomaly}}$

so far there is no better proposal to quantize C, C_j.

How to accept anomaly but make $\mathcal{H}_{phy}$ not small.

idea: weakly imposing quantum constraints

strongly imposing $\hat{C}_I$: $\hat{C}_I \Psi = 0$ $\quad \Psi \in \mathcal{H}$

solutions Ψ span $\mathcal{H}_{phys}$

2 ways of imposing $\hat{C}_I$

(1) Gupta-Bleuler formalism : look for subspace $\mathcal{H}^{(w)}_{phys} \subset \mathcal{H}$

s.t. $\forall \Psi, \Phi \subset \mathcal{H}^{(w)}_{phys}$

$$\langle \Psi | \underbrace{\hat{C}_I | \Phi \rangle}_{\text{not necessarily zero but orthogonal to } \Psi \in \mathcal{H}^{(w)}_{phys}} = 0$$

(used in e.g. covariant quantization of strings)

(2) Master constraint formalism (Thiemann 2003)

1. define a classical Master constraint

$$M = \int d^3x \, \frac{C^2 + q^{ab} C_a C_b}{\sqrt{\det q}}$$

$M = \underline{\underline{K_{IJ}}} C_I C_J$

$\boxed{\int d^3x \sqrt{\det q} \, f}$

$$= \int d^3x \, \frac{C^2 + \frac{1}{4} C_j C_j}{\sqrt{\det q}}$$

is $\overbrace{\text{nondeg.}}^{\text{positive}}$ quadratic form of C, C_j

$M = 0 \iff C = C_j = 0$

classical constraint algebra: $\{M, M\} = 0$ trivial

(we have only a single constraint)

2. quantum Master constraint

$$\hat{M} = \sum_v \left[\widehat{\frac{C_v}{(\det q)^{1/4}}}^{\dagger} \widehat{\frac{C_v}{(\det q)^{1/4}}} + \frac{1}{4} \widehat{\frac{C_{jv}}{(\det q)^{1/4}}}^{\dagger} \widehat{\frac{C_{jv}}{(\det q)^{1/4}}} \right]$$

$\widehat{\frac{C_v}{(\det q)^{1/4}}} = C_v \leftarrow$ replacing $h[h^{-1} V_v]$ by $2h[h^{-1}, \widehat{\sqrt{V_v}}]$

$\uparrow$ $(\hat{Q}_v^2)^{1/8}$

$$2h\{h^{-1}, \sqrt{V_v}\} = \frac{1}{\sqrt{V_v}} h\{h^{-1} V_v\}$$

$\uparrow$ $(\det q)^{1/4}$

$\widehat{\frac{C_{jv}}{(\det q)^{1/4}}}$ is similar

hermitian: $\langle \psi, O\phi \rangle = \langle O\psi, \phi \rangle$

self adj: $O = O^\dagger$

$\langle \psi, \hat{O}\phi \rangle = \langle \hat{O}^\dagger \psi, \phi \rangle$

both $\hat{C}_v$, $\hat{C}_{j,v}$ are non hermitian

$\Rightarrow$ $\hat{M}$ is hermitian and positive-semidefinite

$$\langle \psi | \hat{C}^\dagger \hat{C} | \psi \rangle = \langle \hat{C}\psi | \hat{C}\psi \rangle \geq 0$$

$\longrightarrow$ admit a self-adjoint extension (Friedrich extension)

QM: $\hat{H} = \frac{\hat{p}^2}{2m} + V(x)$

therefor $\hat{M}$ can be viewed as self-adj. operator.

$\hat{M}$ can be diagonalized in principle

$[\hat{M}, \hat{M}] = 0$ trivially no anomaly

quantum constraint eqn. $\hat{M}\Psi = 0$, solutions span $\mathcal{H}_{phys}$.

eigenspace of $\hat{M}$ with zero eigenvalue

$\mathcal{H}_{phys}$ is well-defined

$\boxed{\hat{M}\Psi = 0}$ is "weaker" than $\hat{C}_v \Psi = \hat{C}_{j,v}\Psi = 0$

$\hat{M} \sim \sum_I \hat{C}_I \hat{C}_I$ $\qquad$ $\hat{C}_I \Psi = 0 \Rightarrow \sum_I \hat{C}_I \hat{C}_I \Psi = 0$

but not the inverse at the quantum level

$\hat{M}$ may be a resolution of quantum constraint anomaly.

How to resolve both problem 1 & 2

$\rightarrow$ reduced phase space quantization (solve constraints classically then quantize)

Giesel - Thiemann 2007, useful recently