

boson no spin

$$T_0^2 = 1$$

$$T_0 \psi(t, \vec{r}) = \psi^*(-t, \vec{r})$$

fermion spin $\frac{1}{2}$

$$T^2 = -1$$

$$T \begin{pmatrix} \psi_1(t, \vec{r}) \\ \psi_2(t, \vec{r}) \end{pmatrix} = e^{i\varphi} \sigma_y \begin{pmatrix} \psi_1^*(-t) \\ \psi_2^*(-t) \end{pmatrix}$$

zero spin

$$\hat{H} \psi_{nl} = E_n \psi_{nl} \quad l=1 \dots d(n)$$

general sol of Schrödinger eqn $\psi(\vec{r}, t) = \sum_n a_n \psi_{nl}(\vec{r}) e^{-\frac{i}{\hbar} E_n t}$
(\hat{H} is real, not explicitly dep. on t)

$$\hat{T}_0 \psi(\vec{r}, t) = \sum_n a_n^* \psi_{nl}^*(\vec{r}) e^{-\frac{i}{\hbar} E_n t}$$

effectively $\hat{T}_0 : \psi_{nl}(\vec{r}) \rightarrow \psi_{nl}^*(\vec{r})$

if \hat{H} is real, $\hat{H} \psi_{nl} = E_n \psi_{nl}$ ψ_{nl} is eigenstate

$\Rightarrow \hat{H} \psi_{nl}^* = E_n \psi_{nl}^*$ ψ_{nl}^* is eigenstate

ψ_{nl} basis in $\mathcal{H}^{(n)}$, ψ_{nl}^* is basis in $\mathcal{H}^{(n)*}$

whether $\mathcal{H}^{(n)}$ and $\mathcal{H}^{(n)*}$ the same?

(1) $\mathcal{H}^{(n)} \cong \mathcal{H}^{(n)*}$ equivalent rep of symm. group

(2) $\mathcal{H}^{(n)} \not\cong \mathcal{H}^{(n)*}$ not equivalent

$$\text{eigenspace} = \mathcal{H}^{(n)} \oplus \mathcal{H}^{(n)*}$$

real rep, pseudo real rep, and complex rep

Given a symm. group G , unitary irrep $D: g \mapsto D(g) \in \mathcal{H}^{(n)}$

Def $D(g)$ is unitary on $\mathcal{H}^{(n)}$, ψ_{hi} orthonormal basis in $\mathcal{H}^{(n)}$

(complex
conjugate
irrep)

$$D(g) \psi_{hi}(\vec{r}) = \sum_j \psi_{hj}(\vec{r}) D_{ji}(g)$$

$D_{ji}(g)$ unitary matrix

complex conjugate : $D(g)^* \psi_{hi}^*(\vec{r}) = \sum_j \psi_{hj}^*(\vec{r}) D_{ji}^*(g)$

$$\psi_{hi}^*(\vec{r}) = T_0 \psi_{hi}(\vec{r})$$

$D_{ij}(g)$ unitary irrep matrix
(irrep D)

$\xrightarrow{T_0} D_{ij}^*(g)$ unitary irrep matrix
irrep D^* :

Complex conjugate irrep

relation between D and D^*

(1) if \exists unitary transf. $U: \mathcal{H}^{(n)} \rightarrow \mathcal{H}^{(n)}$ s.t

$$U D(g) U^{-1} = D_0(g) \quad \forall g \in G$$

↑
real matrix

$$\Rightarrow \underline{D \text{ is equivalent to } D^*}, \text{ since } U^* D(g)^* U^{*-1} = D_0(g)$$

$$\Rightarrow \underline{D^*(g) = U^{*-1} D_0(g) U^*} = \underline{U^{*-1} U D(g) U^{-1} U^*}$$

$U^{*-1} U$ is unitary

we say $D \simeq D^*$ is a real rep.

(2) D is ^{unitarily} equivalent to D^* , but they are not equivalent to real rep.

$$\nexists U \text{ s.t. } D(g) = U D_0(g) U^{-1} \quad \forall g \in G$$

\uparrow
real matrix

We say, $D \simeq D^*$ is pseudo-real

(3) D is inequivalent to D^* .

We say, D and D^* are complex reps

distinguish (3) from (1) and (2), check character

$$\text{if } D \simeq D^*, \text{ then } \chi(g) = \chi^*(g) \quad \forall g \in G$$

if $\chi(g) \neq \chi^*(g)$, then D & D^* are complex reps.

distinguish between (1) & (2)

$$D^* \simeq D \rightarrow D^*(g) = Z D(g) Z^{-1} \quad \forall g \in G$$

\uparrow
unitary matrix

Complex conjugate $D(g) = Z^* D^*(g) Z^{*-1}$

$$\Rightarrow D^*(g) = Z Z^* D^*(g) (Z Z^*)^{-1}$$

$$\therefore [Z Z^*, D^*(g)] = 0 \quad \forall g \in G$$

by Schur's lemma, when D^* is irrep, $Z Z^* = c \mathbb{1}$
 $c \in \mathbb{C}$

Z is unitary, $Z^* = (Z^T)^{-1}$

$$\Rightarrow Z (Z^T)^{-1} = c \mathbb{1} \Rightarrow \underline{Z = c Z^T}$$

$$\xRightarrow{\text{transpose}} Z^T = c Z$$

$$\Rightarrow Z = c^2 Z \Rightarrow c^2 = 1, c = \pm 1$$

$$D^* \simeq D \Rightarrow Z Z^* = \pm \mathbb{1}$$

Thm: $Z Z^* = \mathbb{1}$ iff D is real

$Z Z^* = -\mathbb{1}$ iff D is pseudo-real

pf if D is real $\rightarrow D^*(g) = (U^*)^{-1} D_0(g) U^*$
 $= (U^*)^{-1} U D(g) U^{-1} U^*$

on the other hand $D^*(g) = Z D(g) Z^{-1}$

$$\Rightarrow (U^*)^{-1} U = b Z \quad b \in \mathbb{C}$$

Complex conjugate $U^{-1} U^* = b^* Z^*$

$$U^{-1} U^* (U^*)^{-1} U = b^* b Z^* Z = |b|^2 Z^* Z$$

$$\parallel$$
$$1$$

$$\underline{Z^* Z = |b|^{-2} \mathbb{1}}$$

$$\underline{Z Z^* = c \mathbb{1}}$$

$$\Rightarrow |b|^2 c = 1$$

$$c = 1 \text{ or } -1$$

$$c = 1 \quad |b|^2 > 0$$

conversely if $c = 1$ ($Z Z^* = \mathbb{1}$)

Z unitary, $Z = e^{iA}$ A hermitian

$$Z^* = Z^{-1} \Leftrightarrow Z^T = Z, \quad Z \text{ symmetric}$$

$$Z^* = (Z^T)^{-1}$$

$$\text{def. } U = Z^{\frac{1}{2}} = e^{iA/2} \quad \text{unitary}$$

$$U^2 = Z$$

$$\underline{U^* Z = e^{-iA/2} e^{iA} = e^{\frac{1}{2}iA} = U}$$

$$\begin{aligned} \forall g \in G, [U D(g) U^{-1}]^* &= U^* D(g)^* U^{*-1} \\ &= U^* Z D(g) Z^{-1} U^{*-1} \\ &= U D(g) U^{-1} = D_0(g) \text{ real} \end{aligned}$$

$\Rightarrow D$ is real rep.

$$Z Z^* = \mathbb{1} \Leftrightarrow D \text{ is real rep}$$

$$\Rightarrow Z Z^* = -\mathbb{1} \Leftrightarrow D \text{ is pseudo-real rep.} \quad \square$$

Distinguish real & pseudo-real reps by characters

$$\underline{D^*(g) = Z D(g) Z^{-1}}$$

orthogonality $\sum_g D_{\alpha\delta}^*(g) D_{\beta\gamma}(g) = \delta_{\alpha\beta} \delta_{\gamma\delta} \frac{h}{\dim(D)=d}$

$$\sum_{\alpha, \delta} \underline{Z_{\tau\alpha}^{-1}} \sum_g \sum_{\sigma, \rho} \underline{Z_{\alpha\sigma}} D_{\sigma\rho}(g) \underline{Z_{\rho\delta}^{-1}} D_{\beta\gamma}(g) \times \underline{Z_{\delta\chi}}$$

$$\Rightarrow \sum_g D_{\tau\chi}(g) D_{\beta\gamma}(g) = Z_{\gamma\chi} Z_{\tau\beta}^{-1} \frac{h}{d}$$

let $\chi = \rho$. $\sum_{\beta} \underline{\quad}, \sum_g \sum_{\beta} D_{\tau\beta}(g) D_{\beta\gamma}(g)$

$$\sum_g D_{\tau\gamma}(g^2) = (Z Z^*)_{\tau\gamma} \frac{h}{d}$$

$$= \underline{\pm} \delta_{\tau\gamma} \frac{h}{d}$$

let $\tau = \gamma$, $\sum_{\tau} \underline{\quad},$

$$\boxed{\sum_g \chi(g^2) = \pm h}$$

for complex rep. $D \neq D^* \Rightarrow \sum_g \underline{\quad} [D_{\alpha\beta}^*(g)]^* D_{\gamma\delta}(g) = 0$

$$\sum_j \text{Dop}(j) \text{Drs}(j)$$

$$\Rightarrow \sum_j \chi(j^2) = 0$$

Summary :

$$\frac{1}{h} \sum_{j \in G} \chi(j^2) = \begin{cases} 1 & \text{real rep} \\ -1 & \text{pseudo-real rep} \\ 0 & \text{complex.} \end{cases}$$

Extra-degeneracy of \hat{H} due to time-reversal inv.

$$\hat{H} \psi_{nl} = E_n \psi_{nl} \quad l = 1 \dots d$$

$$\psi_{nl} \in \mathcal{H}^{(n)} \quad \text{unitary irrep. of symmetry group } G$$

$$l = 1 \dots d \quad \dim \mathcal{H}^{(n)} = d$$

find degeneracy at E_n , deg. at E_n is either d or $2d$ (spin-zero)

if \hat{H} is real and not explicitly dep on t

$$\hat{H} \psi_{nl} = E_n \psi_{nl}$$

$$\hat{H} \psi_{nl}^* = E_n \psi_{nl}^*$$

$$\psi_{nl}^*(\vec{r}) = \hat{T}_0 \psi_{nl}(\vec{r})$$

• if $\forall \psi_{nl}^*$, $\psi_{nl}^* \in \mathcal{H}^{(n)}$ spanned by ψ_{nl}

then $\mathcal{H}^{(n)}$ is the final eigenspace i.e. $\mathcal{H}^{(n)} = \mathcal{H}^{(n)*}$
 degeneracy at E_n is d

• otherwise, eigenspace $= \mathcal{H}^{(n)} \oplus \mathcal{H}^{(n)*}$

degeneracy at $E_n = 2d$ (extra degeneracy)

Spin-zero

$$\hat{H} \psi_i = E \psi_i \quad \psi_i \in \underline{\mathcal{H}^{(n)}}$$

$$D(g) \psi_i = \sum_j \psi_j D_{ji}(g) \quad \forall g \in G$$

\hat{H} is real $\hat{H} \psi_i^* = E \psi_i^* \quad \psi_i^* \in \mathcal{H}^{(n)*}$

$$D^*(g) \psi_i^* = \sum_j \psi_j^* D_{ji}^*(g) \quad \forall g \in G$$

$\mathcal{H}^{(n)}$ carries irrep D of G

$\mathcal{H}^{(n)*}$ carries complex conjugate irrep D^* of G

$\mathcal{H}^{(n)} \simeq \mathcal{H}^{(n)*}$ or not relates to $D \simeq D^*$ or not

firstly if $D \not\simeq D^*$ complex irrep, $\mathcal{H}^{(n)} \perp \mathcal{H}^{(n)*}$ by
 orthogonality theorem.

\Rightarrow extra degeneracy $2d$

let's look at case (1) real rep and (2) pseudo-real rep.

$$\exists \text{ unitary } Z, D^*(g) = Z D(g) Z^{-1}, \quad Z Z^* = \begin{cases} 1 & \text{real} \\ -1 & \text{pseudo-real} \end{cases}$$

Lemma if $\mathcal{H}^{(n)} = \mathcal{H}^{(n)*}$, then D is of case (1) i.e. real rep.

pf. $\mathcal{H}^{(n)} = \mathcal{H}^{(n)*}$, $\hat{H} \psi_i = E \psi_i$
 $\hat{H} \psi_i^* = \bar{E} \psi_i^*$

both $\{\psi_i\}$, $\{\psi_i^*\}$ are both orthonormal basis

then \exists unitary U s.t. $\psi_i = \sum_k \psi_k^* U_{ki}$

$$\psi_i^* = \sum_k \psi_k U_{ki}^*$$

$$\Rightarrow \psi_i = \sum_{kl} \psi_k U_{lk}^* U_{ki} \quad \text{i.e. } U U^* = \mathbb{1}$$

$$D(g) \psi_i = \sum_j \psi_j D_{ji}(g)$$

$$\parallel \quad \sum_j \sum_k \psi_k^* U_{kj} D_{ji}(g)$$

$$\times U_{il}^{-1} \sum_i$$

$$D(g) \psi_i^* = \sum_k \psi_k^* (U D(g) U^{-1})_{ki}$$

\uparrow
same as $D^*(g) \psi_i^*$

compare to $D^*(g) \psi_i^* = \sum_j \psi_j^* D_{ji}^*(g)$

$$D^*(g) = U D(g) U^{-1} \quad \} = D \text{ is real}$$

$$U U^* = I$$

J

□

$$U = Z$$

Lemma if D is real, then $\mathcal{H}^{(n)} = \mathcal{H}^{(n)*}$

pf D is real, $Z D Z^{-1} = D^*$ & $Z Z^* = I$

$$\begin{aligned} D^*(g) \psi_i^* &= \sum_j \psi_j D_{ji}^*(g) \\ &= \sum_j \psi_j \sum_{k,l} Z_{jk} D_{kl}(g) Z_{li}^{-1} \end{aligned}$$

$$\Rightarrow D^*(g) \left(\sum_i \psi_i^* Z_{im} \right) = \sum_k \left(\sum_j \psi_j^* Z_{jk} \right) \underline{D_{km}(g)}$$

× Z_{im}
and \sum_i

$$D(g) \psi_m = \sum_k \psi_k D_{km}(g)$$

$$\Rightarrow \sum_i \psi_i^* Z_{im} = \psi_m$$

$$\psi_i^* \in \mathcal{H}^{(n)*} \quad \psi_i \in \mathcal{H}^{(n)}$$

they are linear dep. by $\sum_i \psi_i^* Z_{im} = \psi_m$

$$\Rightarrow \mathcal{H}^{(n)} = \mathcal{H}^{(n)*}$$

above 2 lemmas $\Rightarrow \mathcal{H}^{(h)} = \mathcal{H}^{(h)*}$ iff D is real
(no extra degeneracy) d

then if D is pseudo-real then $\mathcal{H}^{(h)} \neq \mathcal{H}^{(h)*}$
 \Rightarrow extra degeneracy $2d$

if D is $\begin{cases} \text{real} \\ \text{pseudo-real} \\ \text{complex} \end{cases}$ degeneracy $= d = \dim(D)$
degeneracy $= 2d$
degeneracy $= 2d$

Examples (1) 1d free particle $\hat{H} = \frac{1}{2m} \hat{p}^2$

$$\hat{p} = -i\hbar \frac{\partial}{\partial x}$$

symmetry: transl. inv. $Q(\lambda)x = x + \lambda \quad \lambda \in \mathbb{R}$

$$D(\lambda) = e^{-\frac{i}{\hbar} \lambda \hat{p}}$$

$$D(\lambda)\psi(x) = \psi(x + \lambda)$$

$G = \mathbb{R} = \{\lambda\}$ group multiplication: $+$; $\lambda_1 + \lambda_2$

irrep of \mathbb{R} : $D^{(k)}(\lambda) = e^{ik\lambda}$

$$\mathcal{H}^{(k)} = \mathbb{C} \quad \dim(D^{(k)}) = 1$$

all irrep of G are 1-dim

$$\underline{D^{(k)}(\lambda_1) D^{(k)}(\lambda_2) = e^{ik(\lambda_1 + \lambda_2)} = D^{(k)}(\lambda_1 + \lambda_2)}$$

Complex conjugate of $D^{(k)}$: $D^{(k)}(\lambda)^* = e^{-ik\lambda}$
 $\neq e^{ik\lambda}$ ↗

$$D^{(k)} \neq D^{(k)*}$$

Complex irrep.

cannot transform
between them
by unitary on \mathbb{C}

$$\text{energy level degeneracy} = 2 \dim(D^{(k)}) \\ = 2$$

eigenstates of H : $\hat{H} \psi_k = \frac{\hbar^2 k^2}{2m} \psi_k$

$$\psi_k = e^{ikx} \quad , \quad \psi_{-k} = e^{-ikx}$$

indeed degeneracy = 2

$$U e^{ikx} U^{-1} = e^{-ikx}$$

\Downarrow
 \mathbb{C}

$$e^{ikx} \neq e^{-ikx} \Rightarrow D^{(k)} \neq D^{(k)*}$$

(2) central potential : $\hat{H} \psi_{nlm}(\vec{r}) = E_{nl} \psi_{nlm}$
 $l = 0, 1, 2, \dots$

$$m = -l, -l+1, \dots, l$$

symmetry group $G = SO(3)$

eigenspace $\mathcal{H}^{(n,l)}$ relates to irrep of $SO(3)$

\forall Rotation $Q(\alpha, \beta, \gamma) \in SO(3)$

$\mathcal{H}^{(l)}$ is spanned by $Y_{lm}(\theta, \varphi)$
 is labelled by l , $m = -l, \dots, l$
 Euler angles

$$(l = 0, 1, 2, 3, \dots)$$

$$\dim(\mathcal{H}^{(l)}) = 2l+1$$

$$\langle l m | D^{(l)}(\alpha, \beta, \gamma) | l m' \rangle = \int_{S^2} d\theta d\varphi \sin\theta Y_{lm}^* \hat{D}^{(l)} Y_{lm'}$$

$$= D_{mm'}^{(l)}(\alpha, \beta, \gamma) = e^{-im'\alpha} d_{mm'}^{(l)}(\beta) e^{-im\gamma}$$

\uparrow
Wigner D-function

\uparrow
Wigner d-function

$$d(0) = 1$$

$$\chi^l(\alpha) = \text{tr } D^{(l)}(\alpha, \beta, \gamma) = \text{tr } D^{(l)}(\alpha, 0, 0)$$

$$= \sum_{m=-l}^l e^{-im\alpha} = \frac{\sin((l+\frac{1}{2})\alpha)}{\sin \alpha/2} \quad \text{real}$$

$$\chi^{l*} = \chi^l \quad D^l \text{ is not complex}$$

basis in $\mathcal{H}^{(l)}$: $Y_{lm}(\theta, \varphi)$, $Y_{lm}^*(\theta, \varphi) = Y_{l, -m}(\theta, \varphi)$

$$Y_{lm}^* \in \mathcal{H}^{(l)}$$

$$\Rightarrow \mathcal{H}^{(l)*} = \mathcal{H}^{(l)}$$

$\Rightarrow D^{(l)}$, $l=0, 1, \dots$ are all real rep., no extra degeneracy

$\mathcal{H}^{(h, l)}$ is the eigenspace, $\deg. = 2l+1$

single spin- $\frac{1}{2}$ particle: $T^2 = -1$

lemma $\langle \psi | T\psi \rangle = 0$

pf. Let $\varphi = T\psi \quad \forall \psi \in \mathcal{H}^{(0)} \otimes \mathbb{C}^2$

$$\langle \psi | \varphi \rangle = \langle \psi | T\psi \rangle = \langle T\psi | T^2\psi \rangle^*$$

↑
T is antiunitary

$$= -\langle T\psi | \psi \rangle^* = -\langle \psi | \varphi \rangle$$

$$\Rightarrow \langle \psi | \varphi \rangle = 0$$

$\psi, T\psi$ are orthogonal. □

$$\hat{H} \psi_i = E \psi_i \quad i = 1 \dots d \quad \psi_i \in \mathcal{H}^{(d)} \text{ carry irrep } D^{(d)} \text{ of } G$$

$$\hat{H}(T\psi_i) = E(T\psi_i) \quad \varphi_i \equiv T\psi_i$$

$$(1) \quad \langle \psi_i | \psi_i \rangle = 0$$

$$(2) \quad \langle \psi_i | \psi_j \rangle = \delta_{ij}$$

$$(3) \quad \langle \varphi_i | \varphi_j \rangle = \delta_{ij}$$

Thm (Kramer's Thm) single spin- $\frac{1}{2}$ particle, \forall energy level degeneracy d' is always even and

$$d \leq d' \leq 2d$$

pf we have (1), (2), & (3) whether $\{\psi_i, \varphi_i\}$ form a complete basis

but it is possible $\langle \psi_i | \varphi_j \rangle \neq 0 \quad i \neq j$

if $|\varphi_k\rangle = \sum_i |\psi_i\rangle c_i$ the φ_k should be removed from the set of basis

but the $|\psi_k\rangle$ should be removed as well

$$\text{since } T^2 = -1$$

$$T|\varphi_k\rangle = \sum_i T(|\psi_i\rangle c_i)$$

$$\parallel$$

$$T^2|\varphi_k\rangle$$

\parallel

$$-|\varphi_k\rangle$$

\parallel

$$\sum_i c_i^* |\varphi_i\rangle$$

$$\Rightarrow |\varphi_k\rangle = -\sum_i c_i^* |\varphi_i\rangle$$

$$\psi_1 \dots \widehat{\psi_k} \dots \psi_d \quad \psi_1 \dots \widehat{\psi_k} \dots \psi_d$$

\uparrow removed in pair.

find degeneracy $d' = 2d - 2n$ # of removed pairs

d' is even

$$d \leq d' \leq 2d$$

□

Theory of angular momentum

spatial rotation & SO(3) group

finite rotation: Q 3×3 matrix s.t. $Q\vec{r}_1 - Q\vec{r}_2 = \vec{r}_1 - \vec{r}_2$

$$\Rightarrow Q^T Q = I \quad Q \in O(3)$$

$$\det(Q^T Q) = 1$$

$$\det(Q)^2 = 1$$

$$\det Q = \pm 1$$

$$SO(3) : Q^T Q = I \text{ \& } \det Q = 1 \quad P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

HW Tutorial

problem 1.

$$\hat{H} = \hat{H}_0 + V_{\text{lattice}}(\vec{r})$$

↑
central force, $SO(3)$ symmetry

eigenspace of H_0 : $\mathcal{H}_{L=0,1,\dots}$
carries irrep of $SO(3)$

V_{lattice} breaks $SO(3)$ to O (cubic lattice group)

irrep of $SO(3)$ $\left\{ \begin{array}{l} \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \end{array} \right\}$ irreps of O
 $l=0,1,\dots$ $T_i=1,\dots,5$

$$D^L = \bigoplus_T D_T$$

↑ irrep of $SO(3)$ ↑ irreps of O

$$\forall g \in O, \chi_L(g) = \sum_T \chi_T(g)$$

character of O

$\chi_T(g)$	E	$8C_3$	$6C_2$	$2C_2$	$6C_4$
T_1	1	1	1	1	1
T_2	1	1	-1	1	-1
T_3	2	-1	0	2	0
T_4	3	0	-1	-1	1
T_5	3	0	1	-1	-1

character of $SO(3)$ evaluated at elements in O

$\chi_l(g)$	E	$8C_3$	$6C_2$	$2C_2$	$6C_4$
$l=0$	1	1	1	1	1
$l=1$	3	0	-1	-1	1
$l=2$	5	-1	1	1	-1
$l=3$	7	1	-1	-1	-1
$l=4$	9	0	1	1	1

* $l=0$: $D^{l=0} = T_1$ * $l=1$: $D^{l=1} = T_4$

* $l=2$: $D^{l=2} = T_3 \oplus T_5$ $\chi_{l=2}(g) = \chi_{T_3}(g) + \chi_{T_5}(g)$

check: E : $\chi_{l=2}(E) = 5$ $\forall g \in O$

$$\chi_{T_3}(E) + \chi_{T_5}(E) = 2 + 3 = 5$$

$$8C_3: \chi_2(8C_3) = -1$$

$$\chi_{T_3}(8C_3) + \chi_{T_5}(8C_3) = -1 + 0 = -1$$

$$D_l = \bigoplus_T D_T \quad \forall g \in O, \quad D_l(g) = \bigoplus_T D_T(g)$$

$$\left(\begin{array}{ccc} \boxed{D_{T_1}(g)} & & \\ & \boxed{D_{T_2}(g)} & \\ & & \boxed{D_{T_3}(g)} \\ & & & \ddots \end{array} \right) \begin{array}{c} \circ \\ \circ \\ \circ \end{array} \left(\begin{array}{c} - \\ - \\ - \\ - \end{array} \right) \quad \text{---} \begin{array}{c} \vdots \\ \vdots \\ \vdots \end{array}$$

$$\chi_l(g) = \text{tr } D_l(g) = \sum_T \text{tr } D_T(g) = \sum_T \chi_T(g)$$

$$6C_2: \chi_{l=2}(6C_2) = 1$$

$$\chi_{T_3}(6C_2) + \chi_{T_5}(6C_2) = 0 + 1 = 1$$

$$2C_2: \chi_{l=2}(2C_2) = 1$$

$$\chi_{T_3}(2C_2) + \chi_{T_5}(2C_2) = 2 + (-1) = 1$$

$$6C_4: \chi_{l=2}(6C_4) = -1$$

$$\chi_{T_3}(6C_4) + \chi_{T_5}(6C_4) = 0 + (-1) = -1$$

$$\bullet l=3: D_3 = T_2 \oplus T_4 \oplus T_5$$

$$E: \chi_3 = 7$$

$$\chi_{T_2} + \chi_{T_4} + \chi_{T_5} = 1 + 3 + 3 = 7$$

$$8C_3: \chi_3 = 1$$

$$\chi_{T_2} + \chi_{T_4} + \chi_{T_5} = 1 + 0 + 0 = 1$$

$$6C_2: \chi_3 = -1$$

$$\chi_{T_2} + \chi_{T_4} + \chi_{T_5} = -1 + (-1) + 1 = -1$$

$$3C_2: \chi_3 = -1$$

$$\chi_{T_2} + \chi_{T_4} + \chi_{T_5} = 1 + (-1) + (-1) = -1$$

$$6C_4: \chi_3 = -1$$

$$\chi_{T_2} + \chi_{T_4} + \chi_{T_5} = -1 + 1 + (-1) = -1$$

$$\bullet l = 4 \quad D_4 = T_1 \oplus T_3 \oplus T_4 \oplus T_5$$

$$E: \quad \chi_4 = 9$$

$$\chi_{T_1} + \chi_{T_3} + \chi_{T_4} + \chi_{T_5} = 1 + 2 + 3 + 3 = 9$$

$$8C_3: \quad \chi_4 = 0$$

$$\chi_{T_1} + \chi_{T_3} + \chi_{T_4} + \chi_{T_5} = 1 + (-1) + 0 + 0 = 0$$

$$6C_2: \quad \chi_4 = 1$$

$$\chi_{T_1} + \chi_{T_3} + \chi_{T_4} + \chi_{T_5} = 1 + 0 + (-1) + 1 = 1$$

$$3C_2: \quad \chi_4 = 1$$

$$\chi_{T_1} + \chi_{T_3} + \chi_{T_4} + \chi_{T_5} = 1 + 2 + (-1) + (-1) = 1$$

$$6C_4: \quad \chi_4 = 1$$

$$\chi_{T_1} + \chi_{T_3} + \chi_{T_4} + \chi_{T_5} = 1 + 0 + 1 + (-1) = 1$$

$$\vec{r} = \sum_{i=1}^3 r_i \vec{e}_i \quad \text{choose basis } \vec{e}_{i=1,2,3}$$

$$\vec{r}' = Q \vec{r} = \sum_{i=1}^3 r'_i \vec{e}_i$$

Rotation of basis

$$\vec{r}' = Q \vec{r} = \sum_{i=1}^3 r_i Q \vec{e}_i = \sum_{i=1}^3 r_i \vec{e}'_i$$

$$\vec{e}'_i = Q \vec{e}_i = \sum_j \vec{e}_j (\underbrace{\vec{e}_j \cdot Q \vec{e}_i}_{Q_{ji}}) = \sum_{j=1}^3 \vec{e}_j Q_{ji}$$

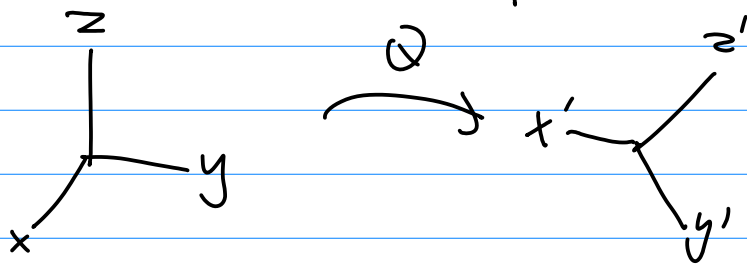
$$\vec{r}' = \sum_{i=1}^3 r'_i \vec{e}_i$$

$$\sum_{i=1}^3 r_i \sum_{j=1}^3 \vec{e}_j Q_{ji} = \sum_{j=1}^3 \left(\sum_{i=1}^3 Q_{ji} r_i \right) \vec{e}_j$$

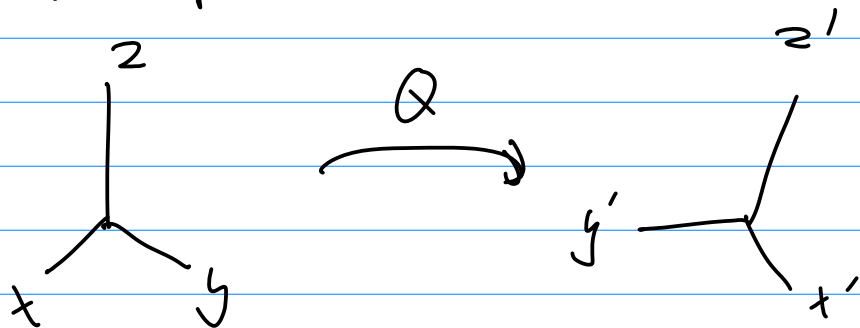
$$r'_j = \sum_{i=1}^3 Q_{ji} r_i$$

Parity $P: \vec{r} \rightarrow -\vec{r}$ $P_{ij} = -\delta_{ij}$ $\det P = -1$
 $P \notin SO(3)$ $P \in O(3)$

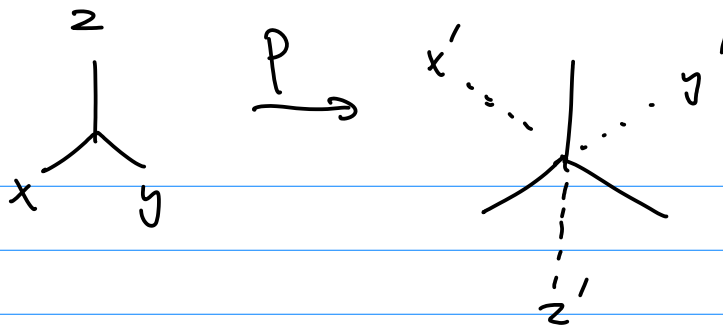
$Q \in SO(3)$ transf. right-hand frame to right-hand frame



$Q \notin SO(3)$, $Q \in O(3)$ transf. right-hand frame to left-hand frame



P:



Proof:

Let $\vec{z} = \vec{x} \times \vec{y}$ we should show that

$$Q\vec{x} \times Q\vec{y} = Q\vec{z} \quad \forall Q \in SO(3)$$

$$Q\vec{x} \times Q\vec{y} = -Q\vec{z} \quad \forall Q \notin SO(3)$$

$$Q \in O(3)$$

\forall vector $\vec{u} \in \mathbb{R}^3$

$$\underline{\underline{Q\vec{u}}} \cdot (Q\vec{x} \times Q\vec{y}) = \sum_{ijk} \varepsilon_{ijk} (\underline{\underline{Q\vec{u}}})_i (Q\vec{x})_j (Q\vec{y})_k$$

$$= \sum_{ijk} \varepsilon_{ijk} Q_{i\alpha} u_{\alpha} Q_{j\beta} x_{\beta} Q_{k\gamma} y_{\gamma}$$

$$= \sum_{\alpha\beta\gamma} \left(\underbrace{\sum_{ijk} \varepsilon_{ijk} Q_{i\alpha} Q_{j\beta} Q_{k\gamma}}_{\varepsilon_{\alpha\beta\gamma} \det Q} \right) u_{\alpha} x_{\beta} y_{\gamma}$$

$$= \det Q \sum_{\alpha\beta\gamma} \varepsilon_{\alpha\beta\gamma} u_{\alpha} x_{\beta} y_{\gamma}$$

$$= \det Q (\vec{u} \cdot (\vec{x} \times \vec{y}))$$

$$= \det Q (\underline{\underline{\vec{u}}} \cdot \vec{z}) = \det Q (\underline{\underline{Q\vec{u}}} \cdot Q\vec{z})$$

$$Q\vec{x} \times Q\vec{y} = \underbrace{\det Q}_{\pm 1} (Q\vec{z})$$

all rotations of rigid body are $SO(3)$

$\forall Q \in SO(3)$ can be composed by 2 types of simple rotations

$$Q(\hat{k}, \alpha), \quad Q(\hat{j}, \beta)$$

rotation around
z-axis

rotation around
y-axis

$\hat{i}, \hat{j}, \hat{k}$ basis
of x, y, z axis

$$Q(\hat{k}, \alpha) = \begin{pmatrix} \cos \alpha & -\sin \alpha & 0 \\ \sin \alpha & \cos \alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

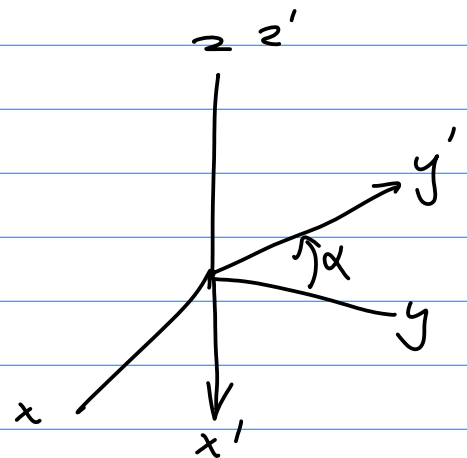
$$\vec{e}_i' = \sum_j \vec{e}_j Q_{ji}$$

$$\hat{i} \rightarrow \hat{i}' = \hat{i} \cos \alpha + \hat{j} \sin \alpha$$

$$\hat{j} \rightarrow \hat{j}' = \hat{i} (-\sin \alpha) + \hat{j} \cos \alpha$$

$$\hat{k} \rightarrow \hat{k}' = \hat{k}$$

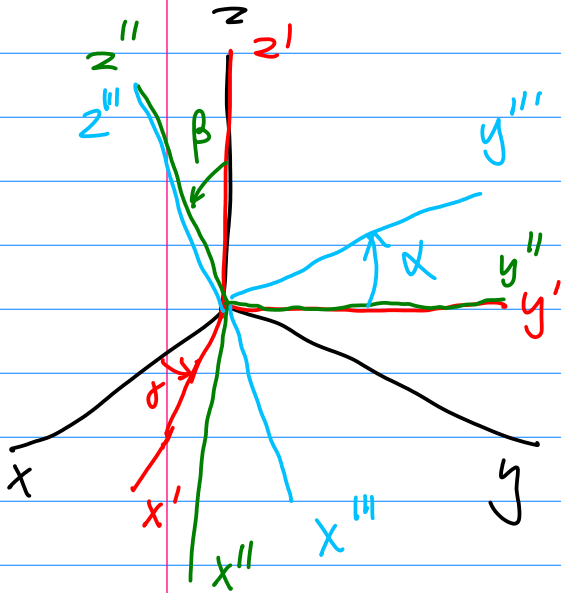
$$Q(\hat{j}, \beta) = \begin{pmatrix} \cos \beta & 0 & \sin \beta \\ 0 & 1 & 0 \\ -\sin \beta & 0 & \cos \beta \end{pmatrix}$$



Euler angles:
give a basis
 $\hat{i}, \hat{j}, \hat{k}$

- (1) rotation around z : γ
 - (2) rotation around y : β
 - (3) rotation around x : α
- } compose any rotations

$$\alpha \in [0, 2\pi), \beta \in [0, \pi], \gamma \in [0, 2\pi)$$



$$Q(\alpha, \beta, \gamma)$$

$$= Q(\hat{k}, \alpha) Q(\hat{j}, \beta) Q(\hat{i}, \gamma)$$

$$\vec{e}_i = \sum_j \vec{e}_j Q_{ji}(\alpha, \beta, \gamma)$$

$$Q(\alpha, \beta, \gamma) = \begin{pmatrix} \cos\alpha \cos\beta \cos\gamma - \sin\alpha \sin\gamma & -\cos\alpha \cos\beta \sin\gamma - \sin\alpha \cos\gamma & \sin\alpha \cos\beta \\ \sin\alpha \cos\beta \cos\gamma + \cos\alpha \sin\gamma & \sin\alpha \cos\beta \sin\gamma + \cos\alpha \cos\gamma & -\sin\alpha \sin\beta \\ -\sin\alpha \cos\beta & \cos\alpha \cos\beta & \sin\beta \end{pmatrix}$$

$$\forall Q(\alpha, \beta, \gamma) \in SO(3)$$

all group elements in $SO(3)$ can be parametrized by

3 Euler angles

$SO(3)$ is a 3-dim Lie group

$SO(3)$ and $SU(2)$

in QM, it's better to work with $SU(2)$, reps of $SU(2)$
since 1) $\{\text{irreps of } SU(2)\} \supset \{\text{irreps of } SO(3)\}$
2) we have spin.

$SU(2)$ group: special unitary transf. on \mathbb{C}^2

$$SU(2) \ni u = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix}, \quad \det(u) = \overbrace{aa^* + bb^*} = 1$$

$$u^\dagger = u^{-1}$$

$$a, b \in \mathbb{C}$$

$$2 \text{ complex} = 4 \text{ reals}$$

1 constraint.

3 real parameters

$\Rightarrow SU(2)$ 3 dim Lie group

$$aa^* + bb^* = 1 \quad \text{we write } a = \cos \eta e^{i\beta}$$

$$b = -\sin \eta e^{-i\beta}$$

$$u = \begin{pmatrix} \cos \eta e^{-i\beta} & -\sin \eta e^{-i\beta} \\ \sin \eta e^{i\beta} & \cos \eta e^{i\beta} \end{pmatrix} = u(\eta, \beta, \beta)$$

3 real parameters

$$\beta, \beta \in [0, 2\pi)$$

$$\eta \in [0, \frac{\pi}{2}]$$

homomorphism between $SU(2)$ and $SO(3)$

$$\begin{array}{ccc} \vec{r} \in \mathbb{R}^3 & \rightarrow & \vec{r} \cdot \vec{\sigma} = \begin{pmatrix} z & x-iy \\ x+iy & -z \end{pmatrix} \\ \parallel & & \parallel \\ (x, y, z) & & h \end{array}$$

$$\det(h) = x^2 + y^2 + z^2 = |\vec{r}|^2$$

$u \in SU(2)$, view h an operator on \mathbb{C}^2

$$h \rightarrow h' = \underline{u h u^{-1}} = \vec{r}' \cdot \vec{\sigma} = \begin{pmatrix} z' & x'-iy' \\ x'+iy' & -z' \end{pmatrix}$$

$$\det(h') = \det(u h u^{-1}) = \det(h)$$

$$\parallel \qquad \parallel$$

$$|\vec{r}'|^2 \qquad |\vec{r}|^2$$

This action by u from \vec{r} to \vec{r}' corresponds to a rotation $Q(u) \in SO(3)$, s.t. $Q(u) \vec{r} = \vec{r}'$

$$u (\vec{\sigma} \cdot \vec{r}) u^{-1} = \vec{\sigma} \cdot (Q(u) \vec{r})$$

we find a map from $SU(2)$ to $SO(3)$

homomorphism: $\forall u_1, u_2 \in SU(2)$

$$\begin{aligned} (u_1 u_2) (\vec{\sigma} \cdot \vec{r}) (u_1 u_2)^{-1} &= u_1 u_2 (\vec{\sigma} \cdot \vec{r}) u_2^{-1} u_1^{-1} \\ &= u_1 (\vec{\sigma} \cdot Q(u_2) \vec{r}) u_1^{-1} \\ &= \vec{\sigma} \cdot (Q(u_1) Q(u_2) \vec{r}) \end{aligned}$$

\forall rep D of $SO(3)$

$$SU(2) \xrightarrow[\text{hom}]{i} SO(3) \xrightarrow[\text{hom}]{D} L(\mathcal{H})$$

$D \circ i$ is a rep of $SU(2)$

$$u(\vec{r} \cdot \vec{r})u^{-1} = \vec{r} \cdot (Q(u)\vec{r})$$

$$\downarrow u \rightarrow -u \in SU(2)$$

$$(-u)(\vec{r} \cdot \vec{r})(-u)^{-1}$$

$$u = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix}$$

$$\underline{aa^* + bb^* = 1}$$

$$-u = \begin{pmatrix} -a & -b \\ b^* & -a^* \end{pmatrix}$$

$$\underline{aa^* + bb^* = 1}$$

$$SU(2) \quad SO(3)$$

$$\begin{array}{ccc} u & \searrow & \\ & Q(u) & \\ -u & \swarrow & \end{array}$$

2-to-1 homomorphism

" $SU(2)$ is double covering

Example (1) $u_1(\alpha) = \begin{pmatrix} e^{-i\alpha/2} & 0 \\ 0 & e^{i\alpha/2} \end{pmatrix}$ group of $SO(3)$ "

$$h = \vec{r} \cdot \vec{r} = \begin{pmatrix} z & x-iy \\ x+iy & -z \end{pmatrix}$$

$$h' = u_1(\alpha) h u_1(\alpha)^{-1} = \begin{pmatrix} z & (x-iy)e^{-i\alpha} \\ (x+iy)e^{i\alpha} & -z \end{pmatrix}$$

$$\vec{r} \cdot \vec{r}' = \begin{pmatrix} z' & x'-iy' \\ x'+iy' & -z' \end{pmatrix}$$

$$\Rightarrow \begin{cases} x' = x \cos \alpha - y \sin \alpha \\ y' = x \sin \alpha + y \cos \alpha \\ z' = z \end{cases}$$

$$\vec{r}' = \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \underbrace{\begin{pmatrix} \cos \alpha & -\sin \alpha & 0 \\ \sin \alpha & \cos \alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{Q(\hat{k}, \alpha)} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$Q(u, \alpha) = Q(\hat{k}, \alpha)$$

$$\pm u, (\alpha) \mapsto Q(\hat{k}, \alpha) \text{ rotation around } z\text{-axis}$$

$$(2) \quad u_2(\beta) = \begin{pmatrix} \cos \beta/2 & -\sin \beta/2 \\ \sin \beta/2 & \cos \beta/2 \end{pmatrix} \in SU(2)$$

$$\text{HW Show } \pm u_2(\beta) \mapsto Q(\hat{j}, \beta) \text{ rotation around } y\text{-axis}$$

$$u(\alpha, \beta, \gamma) = u_1(\alpha) u_2(\beta) u_1(\gamma)$$

↓

$$Q(\alpha, \beta, \gamma) = Q(\hat{k}, \alpha) Q(\hat{j}, \beta) Q(\hat{k}, \gamma) \quad \alpha, \beta, \gamma \text{ are Euler angles}$$

$$u(\alpha, \beta, \gamma) = \begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}} \cos \frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}} \sin \frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}} \sin \frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}} \cos \frac{\beta}{2} \end{pmatrix}$$

compare with 3.9.3

$$u = \begin{pmatrix} \cos \eta e^{-i\beta} & -\sin \eta e^{-i\beta} \\ \sin \eta e^{i\beta} & \cos \eta e^{i\beta} \end{pmatrix}$$

$$\alpha = \beta + \gamma, \quad \beta = 2\eta, \quad \gamma = \beta - \gamma$$

$$\beta, \gamma \in [0, 2\pi), \quad \eta \in [0, \frac{\pi}{2}]$$

$$\Rightarrow \alpha \in [0, 4\pi), \quad \beta \in [0, \pi], \quad \gamma \in [0, 2\pi)$$

↑
relate to 2-to-1 homomorphism $SU(2) \rightarrow SO(3)$

$$\left(\text{Vol}(SU(2)) = 2 \text{Vol}(SO(3)) \right)$$

Reps of $SU(2)$

fundamental rep of $SU(2)$ (defining rep) on \mathbb{C}^2

$$v = \begin{pmatrix} \beta \\ \eta \end{pmatrix} \in \mathbb{C}^2$$

(spin Hilbert space)

$$v' = \begin{pmatrix} \beta' \\ \eta' \end{pmatrix} = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix} \begin{pmatrix} \beta \\ \eta \end{pmatrix} = \begin{pmatrix} a\beta + b\eta \\ -b^*\beta + a^*\eta \end{pmatrix}$$

$$\beta \rightarrow \beta' = a\beta + b\eta$$

$$\eta \rightarrow \eta' = -b^*\beta + a^*\eta$$

homogeneous polynomials of \mathfrak{z}, η
basis

degree - 0 $1 \Leftrightarrow \mathfrak{z}^0 \eta^0$

degree - 1 $\mathfrak{z}, \eta \Leftrightarrow \mathfrak{z}^{\frac{1}{2}+\frac{1}{2}} \eta^{\frac{1}{2}-\frac{1}{2}}, \mathfrak{z}^{\frac{1}{2}-\frac{1}{2}} \eta^{\frac{1}{2}+\frac{1}{2}}$

degree - 2 $\mathfrak{z}^2, \mathfrak{z}\eta, \eta^2 \Leftrightarrow \mathfrak{z}^{1+1} \eta^{1-1}, \mathfrak{z}^{1+0} \eta^{1+0}, \mathfrak{z}^{1-1} \eta^{1+1}$

\vdots

degree - $2j$ $\mathfrak{z}^{\bar{j}+\bar{j}} \eta^{\bar{j}-\bar{j}}, \mathfrak{z}^{\bar{j}+(\bar{j}-1)} \eta^{\bar{j}-(\bar{j}-1)} \dots$
 $\mathfrak{z}^{\bar{j}+m} \eta^{\bar{j}-m} \dots \mathfrak{z}^{\bar{j}-\bar{j}} \eta^{\bar{j}+\bar{j}}$

$$m = \bar{j}, \bar{j}-1, \bar{j}-2, \dots, -\bar{j}$$

$$\mathfrak{z}^{\bar{j}+\bar{j}} \eta^{\bar{j}-\bar{j}} \sim \underline{|\bar{j}, \bar{j}\rangle}$$

$$\mathfrak{z}^{\bar{j}+m} \eta^{\bar{j}-m} \sim |\bar{j}, m\rangle$$

$$\mathfrak{z}^{\bar{j}-\bar{j}} \eta^{\bar{j}+\bar{j}} \sim |\bar{j}, -\bar{j}\rangle$$

any degree - $2j$ homogeneous polynomial $\sum_{m=-\bar{j}}^{\bar{j}} a_m \underline{\mathfrak{z}^{\bar{j}+m} \eta^{\bar{j}-m}}$

$$\sim \sum_{m=-\bar{j}}^{\bar{j}} a_m |\bar{j}, m\rangle$$

span $2j$ -dim vector space \mathcal{H}_j rep of $SU(2)$

$$SU(2) \text{ action: } \sum_{m=-j}^j a_m z^{j+m} y^{j-m}$$

$$\rightarrow \sum_{m=-j}^j a_m \underbrace{(a z + b y)^{j+m}} \underbrace{(-b^* z + a^* y)^{j-m}}$$

still homogeneous polynomial of $2j$

$$\text{basis vector } f_m^j(z, y) = \frac{z^{j-m} y^{j+m}}{\sqrt{(j-m)! (j+m)!}} \quad \leftarrow$$

$$m = -j, -j+1, \dots, j$$

$$\{f_m^j\}_{m=-j}^j \text{ span } \mathcal{H}_j \text{ irrep. of } SU(2)$$

$$\forall u \in SU(2) \quad D_j(u) f_m^j(z, y) := f_m^j(u^{-1} \begin{pmatrix} z \\ y \end{pmatrix})$$

$$u^{-1} \begin{pmatrix} z \\ y \end{pmatrix} = u^+ \begin{pmatrix} z \\ y \end{pmatrix} = \begin{pmatrix} a^* & -b \\ b^* & a \end{pmatrix} \begin{pmatrix} z \\ y \end{pmatrix} = \begin{pmatrix} a^* z - b y \\ b^* z + a y \end{pmatrix}$$

$$D_j(u) f_m^j = \frac{(a^* z - b y)^{j-m} (b^* z + a y)^{j+m}}{\sqrt{(j-m)! (j+m)!}}$$

$$= \sum_{m'=-j}^j f_{m'}^j D_{m'm}^j(u)$$

\uparrow
rep. matrix Wigner D-matrix $(2j+1) \times (2j+1)$

derive $D_{m, m'}^j(u)$

$$(x+y)^n = \sum_{r=0}^n \frac{n!}{r!(n-r)!} x^r y^{n-r}$$

$$D_j(u) f_m^j = \sum_{n=0}^{\bar{j}-m} \sum_{m'=n-\bar{j}}^{n+m} (-1)^n \frac{\sqrt{(\bar{j}-m)!(\bar{j}+m)!(\bar{j}-m')!(\bar{j}+m')!}}{n! (\bar{j}-m-n)! (n+m-m')! (\bar{j}+m'-m)!} \\ (a^*)^{\bar{j}-m-n} b^n (b^*)^{n+m-m'} a^{\bar{j}+m'-m} f_{m'}^j$$

$$\sum_{n=0}^{\bar{j}-m} \sum_{m'=n-\bar{j}}^{n+m} (\dots) = \sum_{m'=-\bar{j}}^{\bar{j}} \sum_n (\dots)$$

over all integers s.t. for

all $(\dots)!$

non negative.

$$D_{m, m'}^j(a, b) = \sum_n (-1)^n \frac{\sqrt{(\bar{j}-m)!(\bar{j}+m)!(\bar{j}-m')!(\bar{j}+m')!}}{(\bar{j}+m'-n)! (\bar{j}-m-n)! n! (n+m-m')!}$$

$$(a^*)^{\bar{j}-m-n} a^{\bar{j}+m'-n} b^n (b^*)^{n+m-n'}$$

$$\bar{j} = 0, \frac{1}{2}, 1, \dots$$

$$m, m' = -\bar{j} \dots \bar{j}$$

properties of D-matrix

1) Unitary: Lemma: $D(u) D(u)^\dagger = I \quad \forall u \in SU(2)$

$$\text{i.e. } \sum_m D_{m_1 m}^j(u) D_{m_2 m}^j(u)^* = \delta_{m_1 m_2}$$

pf: let's look at $\sum_m f_m^j(z, \eta)^* f_m^j(z, \eta)$

$$= \sum_{m=-j}^j \frac{(z^* z)^{j-m} (\eta^* \eta)^{j+m}}{(j-m)! (j+m)!}$$

$$= \frac{(z^* z + \eta^* \eta)^{2j}}{(2j)!} \quad \leftarrow \text{SU(2) inv.}$$

$\forall u \in \text{SU}(2)$

$$u \begin{pmatrix} z \\ \eta \end{pmatrix} = \begin{pmatrix} z' \\ \eta' \end{pmatrix} \quad (z', \eta')^* \begin{pmatrix} z' \\ \eta' \end{pmatrix} = z'^* z' + \eta'^* \eta'$$

$$= (z, \eta) u^\dagger u \begin{pmatrix} z \\ \eta \end{pmatrix}$$

$$= (z, \eta) \begin{pmatrix} z \\ \eta \end{pmatrix} = z^* z + \eta^* \eta$$

$$\sum_m \left(D_j^i(u) f_m^j \right)^* \left(D_j^i(u) f_m^j \right) = \sum_m \underline{f_m^j}^* \underline{f_m^j}$$

$$\sum_{m'' m'} \underline{f_{m''}^j}^* \underline{f_{m'}^j} \left(\sum_m \underline{D_{m'' m}^j}^*(u) \underline{D_{m' m}^j}(u) \right)$$

if $\underline{f_{m''}^j}^* \underline{f_{m'}^j} \equiv b_{m'' m'}^j$ are linear indep

then $\sum_m \underline{D_{m' m}^j}(u) \underline{D_{m'' m}^j}(u)^* = \delta_{m'' m'}$

$$D D^\dagger = I$$

To prove they are linearly indep.

solve eqn $\sum_{m'', m'} C_{m'' m'} f_{m''}^{j*} f_{m'}^j = 0$

HW \rightarrow Show $C_{m'' m'} = 0 \quad \forall m', m'' = -j \dots j$

□

Lemma D_j is irreducible.

Pf. Recall Schur's lemma; There is no matrix commuting with all

$D_j(u)$ except $\lambda \mathbb{1}$, then D_j is irrep

assume $\exists M$ s.t. $M D_j^j(a, b) = D_j^j(a, b) M \quad \forall u = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix}$

choose $a = e^{i\alpha/2}, b = 0, \quad \alpha \in [0, 4\pi]$

$$D_{m' m}^j(e^{i\alpha/2}, 0) = \sum_{m''} \underset{\text{const.}}{\delta_{m' m''}} e^{i m'' \alpha}$$

$$\sum_{m'} M_{m m'} D_{m' m''}^j(e^{i\alpha/2}, 0)$$

$$= \sum_{m'} D_{m m'}^j(e^{i\alpha/2}, 0) M_{m' m''}$$

$$\Leftrightarrow M_{m m''} e^{i m'' \alpha} = M_{m m''} e^{i m \alpha} \Leftrightarrow M_{m m''} (e^{i m'' \alpha} - e^{i m \alpha}) = 0$$

$$\Rightarrow M_{m m''} = 0 \quad \text{if } m \neq m'' \quad \text{i.e.} \quad M_{m m''} = M_m \delta_{m m''}$$

for other u s.t. D_j^j has nonzero off-diagonals

$$M D_j^j = D_j^j M \Leftrightarrow M_m D_{m m''} = D_{m m''} M_{m''}$$

$$\Rightarrow M_m = M_{m''} \Rightarrow M \propto \mathbb{1}$$

$\rightarrow D_j^j$ is irreducible

□

we have classified all irreps of $SU(2)$

D-matrix in terms of Euler angles

$$u(\alpha, \beta, \gamma) = \begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}} \cos \frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}} \sin \frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}} \sin \frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}} \cos \frac{\beta}{2} \end{pmatrix}$$

$$a = e^{-i\frac{\alpha+\gamma}{2}} \cos \frac{\beta}{2}$$

$$b = -e^{-i\frac{\alpha-\gamma}{2}} \sin \frac{\beta}{2}$$

$$D_{mm'}^j(\alpha, \beta, \gamma) = D_{mm'}^j(a = e^{-i\frac{\alpha+\gamma}{2}} \cos \frac{\beta}{2}, b = -e^{-i\frac{\alpha-\gamma}{2}} \sin \frac{\beta}{2})$$

$$\alpha \in [0, 4\pi), \beta \in [0, \pi], \gamma \in [0, 2\pi)$$

$$\alpha \rightarrow \alpha + 2\pi \Rightarrow a \rightarrow -a \quad b \rightarrow -b$$

$$u = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix} \rightarrow u(-a, -b) = -u(a, b)$$

$$u(\vec{r} \cdot \vec{r}) u^{-1} = \vec{r} \cdot Q(u) \vec{r} \quad \begin{matrix} u \in SU(2) \\ Q(u) \in SO(3) \end{matrix}$$

$$\begin{matrix} u(a, b) & u(-a, -b) \\ \searrow & \swarrow \\ & \text{the same element in } SO(3) \end{matrix}$$

$$D_{mm'}^j(-a, -b) = (-1)^{2j} D_{mm'}^j(a, b)$$

if j is integer

$$\begin{matrix} & \nearrow u(\alpha, \beta, \gamma) & \searrow \\ SO(3) \text{ rotation} & & D_{mm'}^j(\alpha, \beta, \gamma) \\ Q(\alpha, \beta, \gamma) & \searrow -u(\alpha, \beta, \gamma) & \nearrow \\ \alpha \in [0, 2\pi) & \text{||} & u(\alpha+2\pi, \beta, \gamma) \end{matrix}$$

1-to-1 map between $Q(\alpha, \beta, \gamma)$ and $D^j(\alpha, \beta, \gamma)$

if j is half-integer

SO(3) rotation $Q(\alpha, \beta, \gamma) \rightarrow D_{mm'}^j(\alpha, \beta, \gamma)$ and $D_{mm'}^j(\alpha+4\pi, \beta, \gamma)$ } not the same

$\alpha \in [0, 2\pi)$

1-to-2 map between $Q(\alpha, \beta, \gamma)$ and $D^j(\alpha, \beta, \gamma)$

double valued rep of SO(3)

HW compute $D^{j=1/2}(\alpha, \beta, \gamma)$ and $D^{j=1}(\alpha, \beta, \gamma)$

$\forall u_{2 \times 2} \in SU(2)$ $u = u_0 \mathbb{1} + \sum_{i=1}^3 u_i \sigma_i = \begin{pmatrix} u_0 + u_3 & u_1 - i u_2 \\ u_1 + i u_2 & u_0 - u_3 \end{pmatrix}$

$\det(u) = u_0^2 - \sum_{i=1}^3 u_i^2 = 1$

$u^\dagger = u^{-1}$

$\Rightarrow u_0 \in \mathbb{R}, u_{i=1,2,3} \in i\mathbb{R}$

$= u_0^2 + \sum_{i=1}^3 v_i^2 = 1$

$u_{i=1,2,3} = i v_{i=1,2,3}$

$\begin{pmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \end{pmatrix}$

$u_0 = \cos \theta/2$

$\vec{u} = (u_1, u_2, u_3) = i \vec{n} \sin \theta/2$

$\vec{n} \cdot \vec{n} = 1$

$u = \cos \frac{\theta}{2} \mathbb{1} + i \vec{n} \cdot \vec{\sigma} \sin \frac{\theta}{2} = \underline{\underline{e^{i\theta \vec{n} \cdot \vec{\sigma}/2}}}$

$\Rightarrow \forall u \in SU(2) \exists$ an angle $\theta \in [0, 4\pi)$, and unit vector \vec{n} (axis)

s.t. $\underline{\underline{u = e^{i\theta \vec{n} \cdot \vec{\sigma}/2}}}$

conjugacy classes of $SU(2)$:

$\forall u_i \in SU(2) \quad u \rightarrow u_i u u_i^{-1} = u_i e^{i\theta \vec{n} \cdot \vec{\sigma}/2} u_i^{-1} = e^{i\theta u_i (\vec{n} \cdot \vec{\sigma}) u_i^{-1}/2}$

recall $u_i (\vec{n} \cdot \vec{\sigma}) u_i^{-1} = \vec{\sigma} \cdot (Q(u_i) \vec{n})$

$u_i u u_i^{-1} = e^{i\theta \vec{\sigma} \cdot (Q(u_i) \vec{n})/2}$

rotation around new axis $Q(u_i) \vec{n}$, w/ same angle

$Q(u_i) \vec{n} \quad \forall u_i \quad \text{cover all directions in } \mathbb{R}^3$

A conjugacy class \leftrightarrow an angle θ
 \uparrow
 contain all axes

character of $SU(2)$: $\chi_j(u) \leftarrow$ class function, $\chi_j(u) = \chi_j(\theta)$

to compute $\chi_j(\theta)$: choose axis to be \hat{k} (z-direction) : $Q(\hat{k}, \theta)$

$$D_{m'm}^j(\theta, 0, 0) = e^{-im'\theta} \delta_{m'm}$$

$$\chi_j(\theta) = \text{tr } D^j(\theta, 0, 0) = \sum_{m=-j}^j e^{-im\theta} = \frac{\sin[(j+\frac{1}{2})\theta]}{\sin \theta/2}$$

Thm Orthogonality of D and χ

integral on $SU(2)$ $\frac{1}{16\pi^2} \int_0^{4\pi} d\alpha \int_0^\pi d\beta \sin\beta \int_0^{2\pi} d\gamma \quad D_{mn}^j(\alpha, \beta, \gamma)^* D_{m'n'}^{j'}(\alpha, \beta, \gamma)$

$$= \delta_{jj'} \delta_{mm'} \delta_{nn'} \frac{1}{2j+1}$$

$$d\mu_H = \frac{1}{16\pi^2} d\alpha d\gamma d\beta \sin\beta \quad \text{Haar measure of } SU(2)$$

$$\int_0^{4\pi} d\theta \cdot (1 - \cos\theta) \chi^j(\theta) \chi^{j'}(\theta) = \pi \delta_{jj'}$$

$$D^j(\alpha, \beta, \gamma) = D^j(u_1(\alpha) u_2(\beta) u_3(\gamma))$$

$$= D^j(u_1(\alpha)) D^j(u_2(\beta)) D^j(u_3(\gamma))$$

$$D_{m'm}^j(\alpha, \beta, \gamma) = \sum_{n,n'} D_{m'n'}^j(u_1(\alpha)) D_{n'n}^j(u_2(\beta)) D_{nm}^j(u_3(\gamma))$$

$$u_1(\alpha) = \begin{pmatrix} e^{-i\alpha/2} & 0 \\ 0 & e^{i\alpha/2} \end{pmatrix} = e^{-i\frac{\alpha}{2} \sigma_3}$$

$$D_{mn}^j(u_1(\alpha)) = e^{-im\alpha} \delta_{mn}$$

$$D_{m'm}^j(\alpha, \beta, \gamma) = e^{-im'\alpha} d_{m'm}^j(\beta) e^{-im\gamma}$$

$$d_{m'm}^j(\beta) = D_{m'm}^j(u_2(\beta)) \quad \text{Wigner } d\text{-matrix}$$

SU(2) irreps and angular momentum

Recall group of unitary operators $D(\vec{n}, \varphi) = e^{-\frac{i}{\hbar} \varphi \vec{n} \cdot \vec{J}}$
 $\vec{J} = \vec{L} + \vec{S}$ total angular momentum

$D(\vec{n}, \varphi)$ is unitary rep of SU(2)

$$u(\vec{n}, \varphi) = e^{-i\varphi \vec{n} \cdot \vec{\sigma}/2} \xrightarrow{\text{homomorphism}} D(\vec{n}, \varphi) = e^{-\frac{i}{\hbar} \varphi \vec{n} \cdot \vec{J}}$$

\mathbb{C}^2 \mathcal{H}
 $\vec{J} = \frac{\hbar}{2} \vec{\sigma}$

$$u(\vec{n}, \varphi) = e^{-\frac{i}{\hbar} \varphi \vec{n} \cdot \vec{J}}$$

$$D(\vec{n}, \varphi) f_m^j = \sum_{m'} f_{m'}^j D_{m'm}^j(\vec{n}, \varphi) \quad f_m^j = \frac{3^{j-m} \eta^{j+m}}{(j-m)!(j+m)!}$$

$m = -j, -j+1, \dots, j$

(1) take $\vec{n} = \hat{k}$ (z-axis) $D_{m'm}^j(\hat{k}, \varphi) = \delta_{m'm} e^{-im\varphi}$ ← check by yourselves

$$D(\hat{k}, \varphi) = e^{-\frac{i}{\hbar} \varphi \hat{J}_z}$$

$$e^{-\frac{i}{\hbar} \varphi \hat{J}_z} f_m^j = e^{-im\varphi} f_m^j$$

$$\left. \frac{d}{d\varphi} \right|_{\varphi=0} -\frac{i}{\hbar} \hat{J}_z f_m^j = -im f_m^j \rightarrow \hat{J}_z f_m^j = \hbar m f_m^j$$

(2) take $\vec{n} = \hat{j}$ (y-axis) $D_{m'm}^j(0, \varphi, 0)$

check by yourselves

$$\frac{d}{d\varphi} D_{m'm}^j(0, \varphi, 0) \Big|_{\varphi=0}$$

↖ in terms of Euler angles

$$= -\frac{1}{2} \sqrt{(j-m)(j+m+1)} \delta_{m', m+1} + \frac{1}{2} \sqrt{(j+m)(j-m+1)} \delta_{m', m-1}$$

$$D(\hat{j}, \varphi) = e^{-\frac{i}{\hbar} \varphi \hat{J}_y}$$

$$\Rightarrow \hat{J}_y f_m^j = -\frac{i\hbar}{2} \sqrt{(j-m)(j+m+1)} f_{m+1}^j + \frac{i\hbar}{2} \sqrt{(j+m)(j-m+1)} f_{m-1}^j$$

(3) take $\hat{n} = \hat{i}$ along x-axis

$$\hat{J}_x f_m^j = \frac{\hbar}{2} \sqrt{(j-m)(j+m+1)} f_{m+1}^j + \frac{\hbar}{2} \sqrt{(j+m)(j-m+1)} f_{m-1}^j$$

$$(\hat{J}_x \pm i\hat{J}_y) f_m^j = \hbar \sqrt{(j \mp m)(j \pm m + 1)} f_{m \pm 1}^j$$

$$\hat{J}_\pm^2 f_m^j = \hbar^2 j(j+1) f_m^j$$

combine $\hat{J}_x^2 + \hat{J}_y^2 + \hat{J}_z^2 f_m^j = \hbar^2 j(j+1) f_m^j$, we know $\underline{f_m^j = |j m\rangle}$

$$D^j(\alpha, \beta, \gamma) |j m\rangle = \sum_{m'} |j m'\rangle D_{m'm}^j(\alpha, \beta, \gamma)$$

$$\langle j' m' | D^j(\alpha, \beta, \gamma) | j m \rangle = \delta_{jj'} D_{m'm}^j(\alpha, \beta, \gamma)$$

relate $|j m\rangle$ or f_m^j to spherical harmonics

$$\langle \vec{r} | \psi \rangle = \psi(\vec{r})$$

let $j = l$ integer position rep. (r, θ, φ)

$$\langle \theta, \varphi | l, m \rangle \equiv f_m^l(\theta, \varphi)$$

$$D(\vec{n}, r) f_m^l(\theta, \varphi) = \sum_{m'} f_{m'}^l(\theta, \varphi) D_{m'm}^l(\vec{n}, r)$$

(1) let $\vec{n} = \hat{k}$ $D_{m'm}^l(\hat{k}, r) = \delta_{m'm} e^{-imr}$

$$\text{Left} = D(\hat{k}, r) f_m^l(\theta, \varphi) = f_m^l(Q^{-1}(\hat{k}, r) \cdot (\theta, \varphi)) = f_m^l(\theta, \varphi - r)$$

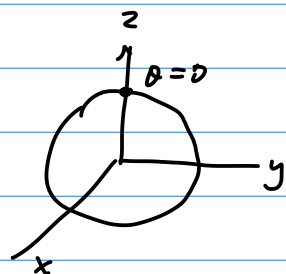
$$\Rightarrow f_m^l(\theta, \varphi - r) = e^{-imr} f_m^l(\theta, \varphi)$$

let $\theta = 0$ $f_m^l(0, \varphi - r) = e^{-imr} f_m^l(0, \varphi)$

$f_m^l(0, \varphi)$ is indep of φ .

$$f_m^l(0, \varphi) = e^{-imr} f_m^l(0, \varphi)$$

$$\Rightarrow f_m^l(0, \varphi) = \delta_{m0} f^{\leftarrow \text{const}} = f_m^l(0, 0)$$



(2) Consider rotation Q s.t. $Q(\theta, \varphi) = (0, 0)$ or $Q^{-1} \cdot (0, 0) = (\theta, \varphi)$

in term of Euler angle $Q = Q(0, -\theta, -\varphi)$

$$D(Q) f_m^L(0, 0) = f_m^L(Q^{-1} \cdot (0, 0)) = \underline{f_m^L(\theta, \varphi)}$$

$$\begin{aligned} D(Q) f_m^L(0, 0) &= \sum_{n'} f_{n'}^L(0, 0) D_{n'm}^L(0, -\theta, -\varphi) \\ &= f D_{0m}^L(0, -\theta, -\varphi) \end{aligned}$$

$$\Rightarrow f_m^L(\theta, \varphi) = f D_{0m}^L(0, -\theta, -\varphi) = f D_{m0}^L(\varphi, \theta, 0)^*$$

(3) determine f by normalization

$$\int f_m^{L*} f_m^L \sin\theta d\theta d\varphi = 1$$

$$= f^* f \int D_{m0}^L(\varphi, \theta, 0) D_{m0}^L(\varphi, \theta, 0)^* \sin\theta d\varphi d\theta$$

$$\begin{aligned} \sum_m (\dots) &\Rightarrow f^* f \int \underbrace{\sum_m D_{m0}^L(\varphi, \theta, 0) D_{m0}^L(\varphi, \theta, 0)^*}_{(DD^\dagger)_{00} = 1} \sin\theta d\varphi d\theta \\ &\quad (DD^\dagger)_{00} = 1 \end{aligned}$$

$$\Rightarrow f^* f (4\pi) = 2L+1 \quad f = \sqrt{\frac{2L+1}{4\pi}}$$

$$\langle \theta, \varphi | L, m \rangle = f_m^L(\theta, \varphi) = \sqrt{\frac{2L+1}{4\pi}} D_{m0}^L(\varphi, \theta, 0)^*$$

$$\sqrt{\frac{2L+1}{4\pi}} D_{m0}^L(\varphi, \theta, 0)^* = Y_{Lm}(\theta, \varphi)$$

Coupling 2 angular momenta

$$\underline{\hat{\vec{J}} = \hat{\vec{J}}_1 + \hat{\vec{J}}_2} \quad \text{acting on } \mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$$

$$\hat{\vec{J}} = \hat{\vec{J}}_1 \otimes 1 + 1 \otimes \hat{\vec{J}}_2$$

Example: (1) $\mathcal{H}_1 = L^2(\mathbb{R})$ wave functions

$\mathcal{H}_2 = \mathbb{C}^2$ spin d.o.f.

$$\hat{\vec{J}} = \hat{\vec{L}} + \hat{\vec{S}}$$

(2) $\hat{\vec{J}}_1$ angular momentum of system 1

\hat{J}_2 system 2

$\vec{J} = \vec{J}_1 + \vec{J}_2$
total angular momentum of
coupling systems 1 and 2.

in \mathcal{H}_1 basis $|j_1, m_1\rangle$

$$\hat{J}_1^2 |j_1, m_1\rangle = \hbar^2 j_1(j_1+1) |j_1, m_1\rangle$$
$$(\hat{J}_1)_z |j_1, m_1\rangle = \hbar m_1 |j_1, m_1\rangle$$

in \mathcal{H}_2 basis $|j_2, m_2\rangle$

$$\hat{J}_2^2 |j_2, m_2\rangle = \hbar^2 j_2(j_2+1) |j_2, m_2\rangle$$
$$(\hat{J}_2)_z |j_2, m_2\rangle = \hbar m_2 |j_2, m_2\rangle$$

$\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ $|j_1, m_1\rangle \otimes |j_2, m_2\rangle \equiv |j_1, j_2, m_1, m_2\rangle$

eigenbasis of $\hat{J}_{1,z}^2$ $(\hat{J}_{1,z})_z$

a different basis operator $(\hat{J}_1 + \hat{J}_2)^2 = \hat{J}^2$ $\hat{J}_z = (\hat{J}_1)_z + (\hat{J}_2)_z$

$$[\hat{J}^2, \vec{J}] = 0 \quad [\hat{J}_{1,z}^2, \hat{J}_z] = 0$$

$\hat{J}^2, \hat{J}_z, \hat{J}_1^2, \hat{J}_2^2$ commuting operators