boson no spin $T_0^2 = 1$ $\quad T_0 \psi(t,\vec{r}) = \psi^*(-t,\vec{r})$

fermion spin $\frac{1}{2}$ $T^2 = -1$ $\quad T\begin{pmatrix}\psi_1(t,\vec{r})\\ \psi_2(t,\vec{r})\end{pmatrix} = e^{i\varphi}\sigma_y\begin{pmatrix}\psi_1^*(-t)\\ \psi_2^*(-t)\end{pmatrix}$

zero spin $\hat{H}\psi_{nl} = E_n\psi_{nl}$ $\quad l = 1..d(n)$

general sol of Schrödingen eqn $\quad \psi(\vec{r},t) = \sum_n a_n \psi_{nl}(\vec{r}) e^{-\frac{i}{\hbar}E_n t}$

($\hat{H}$ is real, not explicitly dep. on t) $\quad \hat{T}_0\psi(\vec{r},t) = \sum_n a_n^* \psi_{nl}^*(\vec{r}) e^{-\frac{i}{\hbar}E_n t}$

effectively $\hat{T}_0 : \psi_{nl}(\vec{r}) \to \psi_{nl}^*(\vec{r})$

if $\hat{H}$ is real, $\quad \hat{H}\psi_{nl} = E_n\psi_{nl}$ $\quad \psi_{nl}$ is eigenstate

$\Rightarrow \hat{H}\psi_{nl}^* = E_n\psi_{nl}^*$ $\quad \psi_{nl}^*$ is eigenstate

ψ_{nl} basis in $\mathcal{H}^{(n)}$, ψ_{nl}^* is basis in $\mathcal{H}^{(n)*}$

whether $\mathcal{H}^{(n)}$ and $\mathcal{H}^{(n)*}$ <u>the same</u> ?

(1) $\mathcal{H}^{(n)} \simeq \mathcal{H}^{(n)*}$ equivalent rep of symm. group

(2) $\mathcal{H}^{(n)} \not\simeq \mathcal{H}^{(n)*}$ not equivalent ....

$$\text{eigenspace} = \mathcal{H}^{(n)} \oplus \mathcal{H}^{(n)*}$$

<u>real rep, pseudo real rep, and complex rep</u>

Given a symm. group G, unitary irrep $D: g \mapsto D(g) \circlearrowright \mathcal{H}^{(n)}$

Def (complex conjugate irrep) $D(g)$ is unitary on $\mathcal{H}^{(n)}$, ψ_{ni} orthonormal basis in $\mathcal{H}^{(n)}$

$$D(g)\psi_{ni}(\vec{r}) = \sum_j \psi_{nj}(\vec{r}) D_{ji}(g)$$

$D_{ji}(g)$ unitary matrix

complex conjugate : $D(g)^* \psi_{ni}^*(\vec{r}) = \sum_j \psi_{nj}(\vec{r})^* D_{ji}(g)^*$

$$\psi_{ni}^*(\vec{r}) = T_0 \psi_{ni}(\vec{r})$$

$D_{ij}(g)$ unitary irrep matrix $\xrightarrow{T_0}$ $D_{ij}^*(g)$ unitary irrep matrix

(irrep D) irrep D^* : <u>complex conjugate irrep</u>

relation between D and D^*

(1) if $\exists$ unitary transf. $U : \mathcal{H}^{(n)} \to \mathcal{H}^{(n)}$ s.t

$$U D(g) U^{-1} = \underset{\substack{\uparrow \\ \text{real matrix}}}{D_0(g)} \qquad \forall g \in G$$

$\Rightarrow$ <u>D is equivalent to D^*</u>, since $U^* D(g)^* U^{*-1} = D_0(g)$

$\Rightarrow$ <u>$D^*(g)$</u> $= U^{*-1} D_0(g) U^* =$ <u>$U^{*-1} U D(g) U^{-1} U^*$</u>

$U^{*-1}U$ is unitary

we say $D \simeq D^*$ is a real rep.

(2) D is equivalent to D^*, but they are not equivalent to
$\underbrace{\text{is}}_{\text{unitarily}}$ real rep.

$$\nexists\, U \text{ s.t. } D(g) = U \underset{\substack{\uparrow \\ \text{real matrix}}}{D_0(g)} U^{-1} \quad \forall g \in G$$

We say, $D \simeq D^*$ is pseudo-real

(3) D is inequivalent to D^*.

We say, D and D^* are complex reps

distinguish (3) from (1) and (2), check character

if $D \simeq D^*$, then $\chi(g) = \chi^*(g) \quad \forall g \in G$

if $\chi(g) \neq \chi^*(g)$, then D & D^* are complex reps.

distinguish between (1) & (2)

$$D^* \simeq D \rightarrow D^*(g) = \underset{\substack{\uparrow \\ \text{unitary matrix}}}{Z} D(g) Z^{-1} \quad \forall g \in G$$

Complex conjugate $D(g) = Z^* D^*(g) Z^{*-1}$

$$\Rightarrow D^*(g) = Z Z^* D^*(g) (Z Z^*)^{-1}$$

i.e. $[Z Z^*, D^*(g)] = 0 \quad \forall g \in G$

by Schur's lemma, when D^* is irrep, $ZZ^* = c\mathbb{1}$
$c \in \mathbb{C}$

Z is unitary, $Z^* = (Z^T)^{-1}$

$$\Rightarrow \quad Z(Z^T)^{-1} = c\mathbb{1} \Rightarrow \underline{Z = cZ^T}$$

$$\xrightarrow[\text{transpose}]{} Z^T = cZ$$

$$\Rightarrow \quad Z = c^2 Z \quad \Rightarrow \quad c^2 = 1, \quad c = \pm 1$$

$$D^* \simeq D \quad \Rightarrow \quad ZZ^* = \pm\mathbb{1}$$

$\underline{\text{Thm}}$: $ZZ^* = \mathbb{1}$ iff D is real

$ZZ^* = -\mathbb{1}$ iff D is pseudo-real

Pf if D is real $\rightarrow$ $D^*(g) = (U^*)^{-1} D_0(g)\, U^*$

$$= (U^*)^{-1} U D(g) U^{-1} U^*$$

on the other hand $D^*(g) = Z D(g) Z^{-1}$

$$\Rightarrow \quad (U^*)^{-1} U = bZ \qquad b \in \mathbb{C}$$

Complex conjugate $U^{-1}U^* = b^* Z^*$

$$\underbrace{U^{-1}U^*(U^*)^{-1}U}_{=1} = b^* b\, Z^* Z = |b|^2 Z^* Z$$

$$\underline{Z^* Z = |b|^{-2}\mathbb{1}}$$

$\underline{ZZ^* = c\mathbb{1}}$ $\Rightarrow$ $|b|^2 c = 1$

$c = 1$ or -1 $\qquad$ $c = 1$ $\quad |b|^2 > 0$

conversely if $c = 1$ $\quad (ZZ^* = \mathbb{1})$

Z unitary, $Z = e^{iA}$ $\quad A$ hermitian

$Z^* = Z^{-1} \Leftrightarrow Z^T = Z$, $\quad Z$ symmetric

$Z^* = (Z^T)^{-1}$

def. $U = Z^{\frac{1}{2}} = e^{iA/2}$ unitary

$U^2 = Z$

$$\underline{U^* Z = e^{-iA/2} e^{iA} = e^{\frac{1}{2}iA} = U}$$

$$\forall g \in G, \ [U D(g) U^{-1}]^* = U^* D(g)^* U^{*-1}$$

$$= U^* Z D(g) Z^{-1} U^{*-1}$$

$$= U D(g) U^{-1} = D_0(g) \text{ real}$$

$\Rightarrow D$ is real rep.

$ZZ^* = \mathbb{1} \Leftrightarrow D$ is real rep

$\Rightarrow ZZ^* = -\mathbb{1} \Leftrightarrow D$ is psudo-real rep. $\qquad \square$

Distinguish $\underbrace{\text{real \& pseudo-real}}$ reps by characters

$$\underline{D^*(g) = Z\,D(g)\,Z^{-1}}$$

orthogonality $\displaystyle\sum_g D^*_{\alpha\delta}(g)\, D_{\beta\gamma}(g) = \delta_{\alpha\beta}\,\delta_{\gamma\delta}\,\frac{h}{\dim(D)=d}$

$$\shortparallel$$

$$\sum_{\alpha,\delta} Z^{-1}_{\tau\alpha} \times \sum_g \sum_{\sigma,\rho} Z_{\alpha\sigma} D_{\sigma\rho}(g) Z^{-1}_{\rho\delta} D_{\beta\gamma}(g) \times Z_{\delta\chi}$$

$$\Rightarrow \sum_g D_{\tau\chi}(g)\, D_{\beta\gamma}(g) = Z_{\gamma\chi}\, Z^{-1}_{\tau\beta}\,\frac{h}{d}$$

let $\chi=\beta$. $\sum_\beta$, $\displaystyle\sum_g\sum_\beta D_{\tau\beta}(g)\, D_{\beta\gamma}(g)$

$$\shortparallel$$

$$\sum_g D_{\tau\gamma}(g^2) = (Z\,Z^*)_{\tau\gamma}\,\frac{h}{d}$$

$$= \pm\,\delta_{\tau\gamma}\,\frac{h}{d}$$

let $\tau=\gamma$, $\sum_\tau$,

$$\boxed{\sum_g \chi(g^2) = \pm h}$$

for complex rep. $D \neq D^*$ $\Rightarrow$ $\displaystyle\sum_g \left[D^*_{\alpha\beta}(g)\right]^* D_{\gamma\delta}(g) = 0$

$$\shortparallel$$

$$\sum_g D_{\alpha\beta}(g) D_{\gamma\delta}(g)$$

$$\Rightarrow \sum_g \chi(g^2) = 0$$

summary : $\frac{1}{h} \sum_{g \in G} \chi(g^2) = \begin{cases} 1 & \text{real rep} \\ -1 & \text{psudo-real rep} \\ 0 & \text{complex.} \end{cases}$

Extra-degeneracy of $\hat{H}$ due to time-reversal inv.

$\hat{H} \psi_{nl} = E_n \psi_{nl} \quad l = 1 \cdots d$

$\psi_{nl} \in \mathcal{H}^{(n)}$ unitary irrep. of symmetry group G

$l = 1 \cdots d \qquad \dim \mathcal{H}^{(n)} = d$

find degeneracy at E_n, deg. at E_n is either d or $2d$ (spin-zero)

if $\hat{H}$ is real and not explicitly dep on t

$\hat{H} \psi_{nl} = E_n \psi_{nl}$

$\hat{H} \psi_{nl}^* = E_n \psi_{nl}^* \qquad \psi_{nl}^*(\vec{r}) = \hat{T}_0 \psi_{nl}(\vec{r})$

- if $\forall \psi_{nl}^*$, $\psi_{nl}^* \in \mathcal{H}^{(n)}$ spanned by ψ_{nl}

then $\mathcal{H}^{(n)}$ is the final eigenspace i.e. $\mathcal{H}^{(n)} = \mathcal{H}^{(n)*}$

degeneracy at E_n is d

- otherwise, eigenspace $= \mathcal{H}^{(n)} \oplus \mathcal{H}^{(n)*}$

  degeneracy at $E_n = 2d$ (extra degeneracy)

<u>spin-zero</u>

$$\hat{H}\psi_i = E\psi_i \qquad \psi_i \in \underline{\mathcal{H}^{(n)}}$$

$$D(g)\psi_i = \sum_j \psi_j D_{ji}(g) \quad \forall g \in G$$

$\hat{H}$ is real $\quad \hat{H}\psi_i^* = E\psi_i^* \qquad \psi_i^* \in \mathcal{H}^{(n)*}$

$$D^*(g)\psi_i^* = \sum_j \psi_j^* D_{ji}(g)^* \quad \forall g \in G$$

$\mathcal{H}^{(n)}$ carries irrep D of G

$\mathcal{H}^{(n)*}$ carries complex conjugate irrep D^* of G

$\mathcal{H}^{(n)} \simeq \mathcal{H}^{(n)*}$ or not relates to $D \simeq D^*$ or not

firstly if $D \not\simeq D^*$ complex irrep, $\mathcal{H}^{(n)} \perp \mathcal{H}^{(n)*}$ by orthogonality theorem.

$\Rightarrow$ extra degeneracy $2d$

let's look at case (1) real rep and (2) psudo-real rep.

$\exists$ unitary Z, $D^*(g) = Z D(g) Z^{-1}$, $ZZ^* = \begin{cases} 1 & \text{real} \\ -1 & \text{psudo-real} \end{cases}$

<u>Lemma</u> if $\mathcal{H}^{(n)} = \mathcal{H}^{(n)*}$, then D is of case (1) i.e. real rep.

pf. $\underline{\underline{\mathcal{H}^{(n)} = \mathcal{H}^{(n)*}}}$, $\hat{H}\psi_i = E\psi_i$

$\hat{H}\psi_i^* = E\psi_i^*$

both $\{\psi_i\}$, $\{\psi_i^*\}$ are both orthonormal basis

then $\exists$ unitary U s.t. $\psi_i = \sum_k \psi_k^* U_{ki}$

$\psi_i^* = \sum_k \psi_k U_{ki}^*$

$\Rightarrow \psi_i = \sum_{kl} \psi_l U_{lk}^* U_{ki}$ i.e. $UU^* = \mathbb{1}$

$D(g)\psi_i = \sum_j \psi_j D_{ji}(g)$

$\Vert \qquad\qquad\qquad \Vert$

$D(g)\sum_k \psi_k^* U_{ki} \qquad \sum_j\sum_k \psi_k^* U_{kj} D_{ji}(g)$

$\times U^{-1}_{il}, \sum_i$

$D(g)\psi_l^* = \sum_k \psi_k^* (UD(g)U^{-1})_{kl}$

$\uparrow$

same as $D^*(g)\psi_l^*$

Compare to $D^*(g)\psi_l^* = \sum_j \psi_j^* D_{jl}^*(g)$

$D^*(g) = U D(g) U^{-1}$ $\Big\} \Rightarrow D$ is real

$$U U^* = 1 \qquad J$$

$\square$

$$U = Z$$

Lemma if D is real, then $\mathcal{H}^{(n)} = \mathcal{H}^{(n)*}$

pf D is real, $\underline{Z D Z^{-1} = D^*}$ & $Z Z^* = 1$

$$D^*(g) \psi_i^* = \sum_j \psi_j D^*_{ji}(g)$$

$$= \sum_j \psi_j \sum_{k,l} Z_{jk} D_{kl}(g) Z^{-1}_{li}$$

$\times Z_{im}$

and $\sum_i$

$$\Rightarrow D^*(g) \left(\sum_i \psi_i^* Z_{im} \right) = \sum_k \left(\sum_j \psi_j^* Z_{jk} \right) D_{km}(g)$$

$$D(g) \psi_m = \sum_k \psi_k D_{km}(g)$$

$$\Rightarrow \sum_i \psi_i^* Z_{im} = \psi_m$$

$$\underbrace{\psi_i^* \in \mathcal{H}^{(n)*} \qquad \psi_i \in \mathcal{H}^{(n)}}$$

they are linear dep. by $\sum_i \psi_i^* Z_{im} = \psi_m$

$$\Rightarrow \mathcal{H}^{(n)} = \mathcal{H}^{(n)*}$$

above 2 lemmas $\Rightarrow$ $\mathcal{H}^{(n)} = \mathcal{H}^{(n)*}$ iff D is real

(no extra degeneracy) d

then if D is pseudo-real then $\mathcal{H}^{(n)} \neq \mathcal{H}^{(n)*}$

$\Rightarrow$ extra degeneracy $2d$

if D is $\begin{cases} \text{real} & \text{degeneracy} = d = \dim(D) \\ \text{pseudo-real} & \text{degeneracy} = 2d \\ \text{complex} & \text{degeneracy} = 2d \end{cases}$

Examples (1) 1d free particle $\hat{H} = \frac{1}{2m}\hat{p}^2$

$$\hat{p} = -i\hbar\frac{\partial}{\partial x}$$

symmetry: transl. inv. $Q(\lambda)x = x + \lambda$ $\lambda \in \mathbb{R}$

$$D(\lambda) = e^{-\frac{i}{\hbar}\lambda\hat{p}}$$

$$D(\lambda)\psi(x) = \psi(x+\lambda)$$

$G = \mathbb{R} = \{\lambda\}$ group multiplication: $+$: $\lambda_1 + \lambda_2$

irrep of $\mathbb{R}$: $D^{(k)}(\lambda) = e^{ik\lambda}$

$\mathcal{H}^{(k)} = \mathbb{C}$ $\dim(D^{(k)}) = 1$

all irrep of G are 1-dim

$$D^{(k)}(\lambda_1) D^{(k)}(\lambda_2) = e^{ik(\lambda_1+\lambda_2)} = D^{(k)}(\lambda_1+\lambda_2)$$

Complex conjugate of $D^{(k)}$: $D^{(k)}(\lambda)^* = e^{-ik\lambda}$

$\neq e^{ik\lambda}$

$D^{(k)} \neq D^{(k)*}$ cannot transf between them by unitary on $\mathbb{C}$

Complex irrep.

energy level degeneracy $= 2 \dim(D^{(k)})$

$= 2$

eigenstates of H : $\hat{H} \psi_k = \frac{\hbar^2 k^2}{2m} \psi_k$

$\psi_k = \underset{\substack{\cap \\ D^{(k)}}}{e^{ikx}}$, $\psi_{-k} = \underset{\substack{\cap \\ D^{(k)*}}}{e^{-ikx}}$

indeed degeneracy $= 2$

$$U e^{ik\lambda} U^{-1} = e^{-ikx}$$

$$\underset{\substack{\| \\ e^{ik\lambda}}}{=} \quad \underset{\substack{\circlearrowleft \\ \mathbb{C}}}{}$$

$$e^{ik\lambda} \neq e^{-ik\lambda} \Rightarrow D^{(k)} \neq D^{(k)*}$$

(2) central potential : $\hat{H} \psi_{nlm}(\vec{r}) = E_{nl} \psi_{nlm}$

$l = 0, 1, 2 \cdots$

$m = -l, -l+1, \dots, l$

symmetry group $G = SO(3)$

eigenspace $\mathcal{H}^{(n,l)}$ relates to irrep of $SO(3)$

$\forall$ Rotation $Q(\alpha, \beta, \gamma) \in SO(3)$

$\uparrow$ Euler angles

$D^{(l)}$

irrep of $SO(3)$, $\mathcal{H}^{(l)}$ is spanned by $Y_{lm}(\theta, \varphi)$

is labelled by l,

$m = -l, \cdots l$

$(l = 0, 1, 2, 3 \dots)$

$\dim(\mathcal{H}^{(l)}) = 2l+1$

$$\langle l m | D^{(l)}(\alpha, \beta, \gamma) | l m' \rangle = \int_{S^2} d\theta \, d\varphi \sin\theta \, Y^*_{lm} \hat{D}^l \, Y_{lm'}$$

$$= D^{(l)}_{mm'}(\alpha, \beta, \gamma) = e^{-im'\alpha} \, d^l_{mm'}(\beta) \, e^{-im\gamma}$$

$\uparrow$ Wigner D-function

$\uparrow$ Wigner d-func

$d(0) = 1$

$$\chi^l(\alpha) = \operatorname{tr} D^{(l)}(\alpha, \beta, \gamma) = \operatorname{tr} D^{(l)}(\alpha, 0, 0)$$

$$= \sum_{m=-l}^{l} e^{-im\alpha} = \frac{\sin\left((l+\frac{1}{2})\alpha\right)}{\sin \alpha/2} \quad \text{real}$$

$\chi^{l*} = \chi^l$. D^l is not complex basis in $\mathcal{H}^{(l)}$: $Y_{lm}(\theta,\varphi)$, $Y^*_{lm}(\theta,\varphi) = Y_{l,-m}(\theta,\varphi)$

$$Y^*_{lm} \in \mathcal{H}^{(l)}$$

$$\Rightarrow \mathcal{H}^{(l)*} = \mathcal{H}^{(l)}$$

$\Rightarrow D^{(l)}$. $l = 0, 1 \ldots$ are all real rep. , no extra degenercy

$\mathcal{H}^{(n,l)}$ is the eigenspace , deg. $= 2l+1$

single spin-$\frac{1}{2}$ particle : $T^2 = -1$

Lemma $\langle \psi | T\psi \rangle = 0$

pf. Let $\varphi = T\psi$ $\forall \psi \in \mathcal{H}^{(0)} \otimes \mathbb{C}^2$

$$\langle \psi | \varphi \rangle = \langle \psi | T\psi \rangle = \langle T\psi | T^2 \psi \rangle^*$$

T is antiunitary

$$= -\langle T\psi | \psi \rangle^* = -\langle \psi | \varphi \rangle$$

$$\Rightarrow \langle \psi | \varphi \rangle = 0$$

ψ, $T\psi$ are orthogonal. $\square$

$\hat{H} \psi_i = E \psi_i$ $i = 1 \ldots d$ $\psi_i \in \mathcal{H}^{(d)}$ carry irrep $D^{(d)}$ of G

$\hat{H}(T\psi_i) = E(T\psi_i)$ $\varphi_i \equiv T\psi_i$

(1) $\langle \psi_i | \varphi_i \rangle = 0$

(2) $\langle \psi_i | \psi_j \rangle = \delta_{ij}$

(3) $\langle \varphi_i | \varphi_j \rangle = \delta_{ij}$

<u>Thm</u> (Kramer's Thm) single spin-$\frac{1}{2}$ particle, $\forall$ energy level degeneracy d' is always even and

$$d \leq d' \leq 2d$$

Pf we have (1), (2), & (3) whether $\{\psi_i, \varphi_i\}$ form a complete basis

but it's possible $\langle \psi_i | \varphi_j \rangle \neq 0 \quad i \neq j$

if $|\varphi_k\rangle = \sum_l |\psi_l\rangle c_l$ the φ_k should be removed from the set of basis

but the $|\psi_k\rangle$ should be removed as well

since $T^2 = -1$

$$T|\varphi_k\rangle = \sum_l T\left(|\psi_l\rangle c_l\right)$$

$$T|\varphi_k\rangle = T^2|\psi_k\rangle = -|\psi_k\rangle$$

$$\sum_l T\left(|\psi_l\rangle c_l\right) = \sum_l c_l^* |\varphi_l\rangle$$

$$\Rightarrow |\psi_k\rangle = -\sum_l c_l^* |\varphi_l\rangle$$

$$\psi_1 \cdots \widehat{\psi_k} \cdots \psi_d \; \varphi_1 \cdots \widehat{\varphi_k} \cdots \varphi_d$$

removed in pair.

find degeneracy $d' = 2d - 2n$ # of removed pairs

d' is even

$$d \le d' \le 2d$$

$\square$

Theory of angular momentum

spatial rotation & SO(3) group

finite rotation : Q 3×3 matrix s.t. $Q\vec{r}_1 \cdot Q\vec{r}_2 = \vec{r}_1 \cdot \vec{r}_2$

$$\Rightarrow Q^T Q = 1 \qquad Q \in O(3)$$

$$\det(Q^T Q) = 1$$

$$\det(Q)^2 = 1$$

$$\det Q = \pm 1$$

$SO(3)$: $Q^T Q = 1$ & $\det Q = 1$