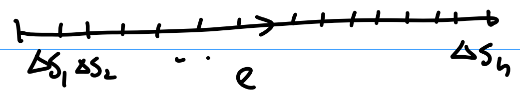
Smooth fields $\qquad\qquad\qquad\qquad$ lattice variables

$$\left(A_a^i(x),\ E_i^a(x)\right) \xrightarrow[\text{cubic lattice}]{} \left(h(e),\ \underline{\underline{p^i(e)}}\right)_{e\in E(\gamma)}$$

holonomy $\quad h(e) = P e^{\int_e A} \qquad A = A_a^i \frac{\tau^i}{2} \dot{e}^a \qquad \vec{\tau} = -i\vec{\sigma}$

① $$h(e) = 1_{2\times 2} + \sum_{n=1}^{\infty} \int_0^1 ds_1 \int_{s_1}^1 ds_2 \cdots \int_{s_{n-1}}^1 ds_n\ A(s_1)\cdots A(s_n)$$

② $$h(e) = \lim_{\substack{n\to\infty \\ (\Delta s_i \to 0)}} \left[1+\int_{\Delta s_1} A(s)\,ds\right]\left[1+\int_{\Delta s_2} A(s)\,ds\right]\cdots\left[1+\int_{\Delta s_n} A(s)\,ds\right]$$

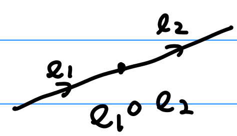


$A(s) \in su(2)$

$h(e) \in SU(2)$

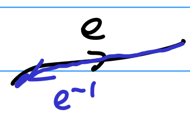
Thm: $h(e)$ satisfies following properties

(1) give e_1, e_2



$$\underset{\substack{\uparrow \\ \text{product of curves}}}{h(e_1 \circ e_2)} = \underset{\substack{\uparrow \\ \text{product in group.}}}{h(e_1)\,h(e_2)}$$

(2)



$$\underset{\substack{\uparrow \\ \text{inverse of curve}}}{h(e^{-1})} = \underset{\substack{\uparrow \\ \text{inverse of group.}}}{h(e)^{-1}}$$

} h is a homomorphism from space of curves to gauge group

(3) $h(e)$ is a solution of $\quad \frac{d}{ds} h(s) = h(s)\,A(s) \qquad A = A_a^i \dot{e}^a \frac{\tau_i}{2}$

parallel transportation eqn

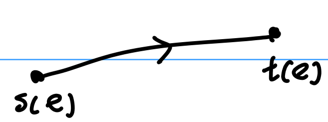
initial condition $h(0) = 1_{2\times 2}, \quad h(e) = h(s=1)$

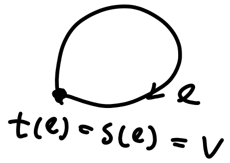
(4) gauge transformation, $\quad \underline{A_a(x) \to g(x)\,A_a(x)\,g(x)^{-1} - (\partial_a g(x))\,g(x)^{-1}}$

$A_a = A_a^i \frac{\tau_i}{2}. \qquad g(x)$: $SU(2)$-valued function on Σ

$$\Rightarrow \underline{h(e)} \to g(s(e))\,h(e)\,g(t(e))^{-1}$$

$s(e)$: source of $e \qquad$ target of e





$h(e) \to g(v)\, h(e)\, g(v)^{-1}$

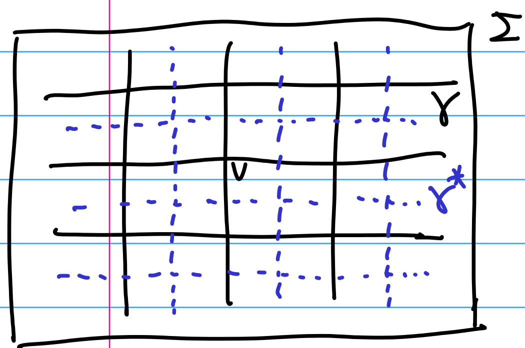
$\mathrm{tr}\, h(e) \to \mathrm{tr}\, h(e)$ SU(2) gauge inv.

Wilson loop variable.

HW prove (1) – (4)

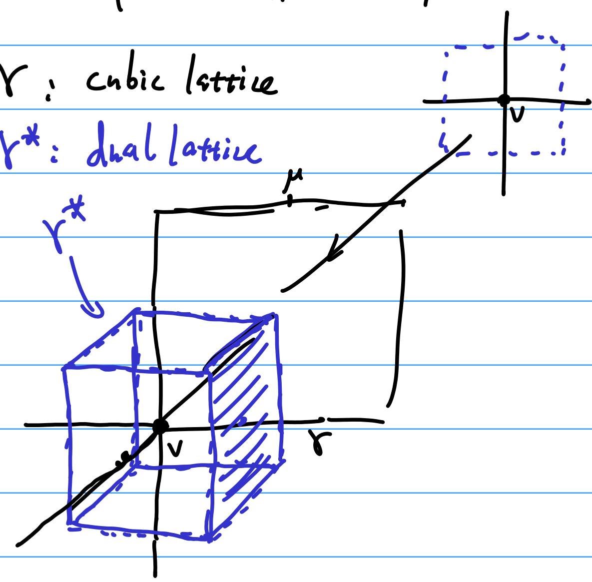
hint : (1), (2), (3) ; use ② (4) ; use (3)

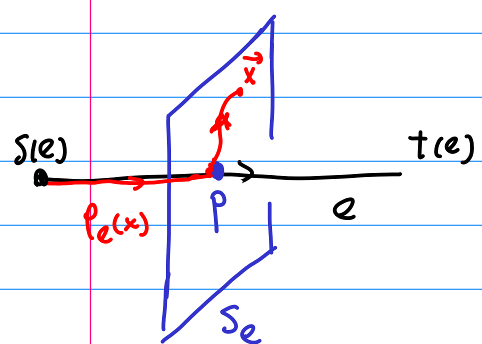
Gauge covariant flux : $P^j(e)$ function of both A_a^i and E_i^a



γ : cubic lattice

γ^* : dual lattice





Given one edge e and dual face S_e :

We choose an arbitrary path system on S_e, i.e. $\forall \vec{x} \in S_e$, $\exists$ a path connecting $\vec{x}$ to $p = S_e \cap e$

$\rho_e(\vec{x})$ is a curve from $s(e)$ firstly traveling to p then following this path to $\vec{x}$

$$\rho_e(\vec{x}) \longrightarrow h(\rho_e(\vec{x})) \qquad E^c = E^c_k \tau^k$$

Gauge covariant flux

$$P^j(e) := -\frac{1}{2\beta}\, \mathrm{tr}\left[\tau^j \int_{S_e} \varepsilon_{abc}\, dx^a \wedge dx^b\, h(\rho_e(\vec{x}))\, \underline{E^c_k(\vec{x})\tau^k}\, h(\rho_e(\vec{x}))^{-1} \right]$$

$$= -\frac{1}{2\beta}\, \mathrm{tr}\left[\tau^j \int_{S_e} h(\rho_e(\vec{x}))\, {*E}(\vec{x})\, h(\rho_e(\vec{x}))^{-1} \right]$$

$$({*E})_{ab} = \varepsilon_{abc} E^c$$

$[p^j] = L^2 \qquad [h] = 1$

dimensionless $p^j(e) := P^j(e)/a^2$ $\qquad$ a is a length unit, e.g. 1mm

$[a] = L$

Thm (1) $p^i(e)$ is gauge covariant : $p(e) \equiv p^i(e) \frac{\tau^j}{2} \in su(2)$

gauge trasf. $\qquad p(e) \longrightarrow g(s(e))\, p(e)\, g(s(e))^{-1}$

$\begin{pmatrix} A \to gAg^{-1} - \partial g\, g^{-1} \\ E \to g E g^{-1} \end{pmatrix}$ $\qquad$ i.e. $p(e)$ trasf. as a vector in the adjoint rep. of $SU(2)$

or namely $p(e) \in su(2)$

hint: $E^b_k \tau^k \to g(\vec{x})\, E^b_k(x) \tau^k\, g(\vec{x})^{-1}$

$h(\rho_e(\vec{x})) \to g(s(e))\, h(\rho_e(\vec{x}))\, g(\vec{x})^{-1}$

(2) if the coordinate length of e and coordinate area of S_e are μ and μ^2 in $\vec{x}$-coordinate

as $\mu \to 0$ continuum limit $\qquad p^j(e) = \frac{2\mu^2}{\beta a^2} E^a_j(s(e)) + O(\mu^3)$

(3) $p^j(e^{-1}) = \frac{1}{2} \mathrm{tr}\left[\tau^j h(e)\, p^i(e) \tau^i\, h(e)^{-1}\right]$

HW prove (1), (2), (3) $\qquad SU(2) \times \mathbb{R}^3 \simeq T^* SU(2)$

Now

smooth fields		Lattice fields
$(A^i_a(x),\ E^a_i(x))$	$\longrightarrow$	$(h(e),\ p^j(e))_{e \in E(\gamma)} \in (SU(2) \times \mathbb{R}^3)^{\lvert E(\gamma)\rvert}$
smooth gauge trasf. $g(x) \in SU(2)$ $\forall x \in \Sigma$	$\longrightarrow$	$(g(v))_{v \in V(\gamma)} \in SU(2)^{\lvert V(\gamma)\rvert}$

$\lvert E(\gamma)\rvert$: number of edges

$\lvert V(\gamma)\rvert$: number of vertices.

background indep. discretization

Holonomy - Flux algebra : Poisson brackets of $h(e)$, $p^j(e)$

<u>Thm</u>

$$\{ h_{AB}(e), h_{CD}(e') \} = 0 \qquad A,B,C,D = 1,2 \tag{1}$$

$$\{ p^j(e), h_{AB}(e') \} = \kappa\, \delta_{ee'} \left(\frac{\tau^j}{2} h(e) \right)_{AB} \tag{2}$$

$$\{ p^j(e), p^k(e') \} = -\kappa\, \delta_{ee'}\, \varepsilon^{jkl}\, p^l(e) \tag{3}$$

remarks:
- they are derived from $\{A, E\}$ of smooth field.
- h, p^j are "conjugate variables"
- (3) $\sim$ $\underline{[\hat{J}^i, \hat{J}^j] = i\hbar\, \varepsilon^{ijk} \hat{J}^k}$

<u>proof</u> (1) trivial from $\{ A_a^i(x), A_b^j(y) \} = 0$