$\sqrt{p^i p^i}$

$R \subset \Sigma$, $\hat{V}(R) = \sum_{v \in R} \hat{V}_v$, $\hat{V}_v = \sqrt{|\hat{Q}_v|} = (\hat{Q}_v^2)^{1/4}$

$$\hat{Q}_v = \beta^3 \varepsilon_{ijk} \frac{\hat{p}^i(e_{v1+}) - \hat{p}^i(e_{v1-})}{4} \frac{\hat{p}^j(e_{v2+}) - \hat{p}^j(e_{v2-})}{4} \frac{\hat{p}^k(e_{v3+}) - \hat{p}^k(e_{v3-})}{4}$$

properties of $\hat{V}$: • $\hat{V}$ is self-adj. because $\hat{Q}_v$ is self-adj. $\quad$ $\hat{V}$ is gauge inv.

$$\hat{Q}|q\rangle = q|q\rangle \qquad \hat{V}|q\rangle = \sqrt{|q|}\,|q\rangle \qquad \hat{V} \circlearrowleft \mathcal{H}_\gamma$$

$\hat{p}^i(e)$ is self-adj.

- $\hat{V}(R)\, T_{r\vec{j}\vec{i}} = \sum_v \lambda_v(\vec{j},\vec{i})\, T_{r\vec{j}\vec{i}}$ $\qquad \lambda_v = \sqrt{q_v}$

  dep. on both spin & intertwiner

- λ_v is discrete spectrum, but is complicated

  $\lambda_v(\vec{j},\vec{i})$ can be computed numerically, but no simple analytic formula for eigenvalues & eigenvectors

- minimum of λ_v is $\lambda_v = 0$ when $\vec{j} = 0$ (vaccum state "no geometry")

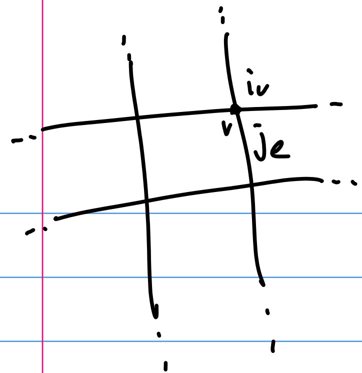
Summary: in LQG, both $\hat{A}_r$ and $\hat{V}$ are self-adj. and have discrete spectrum:

→ quantum geometry is fundamentally discrete, gaps in area & volume

$$\Delta A_r \sim \ell_p^2 \qquad A_r \sim j(j+1)\,\ell_p^2\,\gamma \qquad \Delta j = \tfrac{1}{2}$$
$$\Delta V \sim \ell_p^3$$

zoom out to macroscopic → ℓ_p negligible → smooth geometry

- quantum geometry = spin-network states $T_{r,\vec{j},\vec{i}}$



↪ every $e \to j_e \in \mathbb{Z}/2$ = quanta of area $Ar(S_e) \sim \sqrt{j_e(j_e+1)}$

every $v \to i_v$ = quanta of volume

quantum geometries are only excited on edges & vertices

" " are distributional & nonsmooth

Kinematics = quantum geometry on Σ

dynamics = time evolution in spacetime $\qquad H = \int \left(\Lambda^i G_j + NC + N^a C_a\right)$

Quantization of constraints

classical constraints $C_I = 0 \rightsquigarrow \hat{C}_I \to \hat{C}_I \Psi = 0$

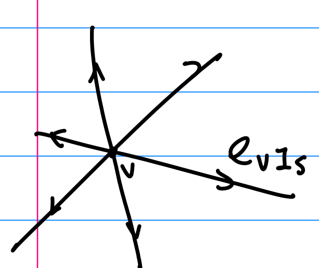
↑ solve quantum constraints

schematically, solution space of Ψ

= physical Hilbert space

quantizing Gauss constraint. classically $G_j(x) = D_a E^a_j(x) = \partial_a E^a_j + \varepsilon_{jkl} A^k_a E^a_l(x)$

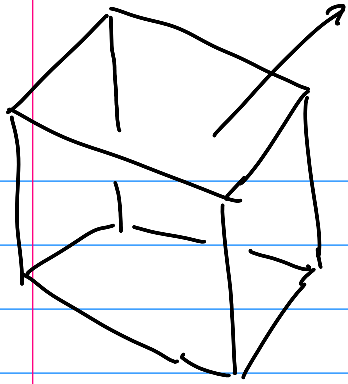
discrete Gauss constraint. $G^i(v) = \sum_{s=\pm} \sum_{I=1}^{3} p^i(e_{v,Is}) = 0$ (closure condition)



HW verify $G^i(v) = \mu^3 \frac{2}{\beta} D_a E^a_i(v) + O(\mu^4)$

use $p^i(e_{vI-}) = \frac{1}{2} Tr\left[\tau^i h(e_{v-\hat{I}\,I+})^{-1} p^j(e_{vI+}) \tau^j h(e_{v-\hat{I},I+})\right]$

Geometrical interpretation: geometrical cube of Euclidean geometry



$\vec{P}_{\alpha=1\ldots 6}$

oriented area

$|\vec{P}_\alpha|$ = area of αth face

$\dfrac{\vec{P}_\alpha}{|\vec{P}_\alpha|}$ = unit normal of face α

closure condition $\sum_{\alpha=1}^{6} \vec{P}_\alpha = 0$, 6 faces form a closed surface

<u>Minkowski Thm</u>: Give n vectors $\vec{P}_{\alpha=1\ldots n} \in \mathbb{R}^3$ satisfying $\sum_{\alpha=1}^{n} \vec{P}_\alpha = 0$

$\to$ $\exists$ unique polyhedron in $\mathbb{E}^3$ of n faces

s.t. $\vec{P}_\alpha$ is the oriented area of the α-th face

LQG closure condition $\sum_{I,s} P^i(e_{vIs}) = 0 \iff$ a geometrical $\underbrace{\text{cube}}_{\text{in } \mathbb{E}^3}$ centered at v.

s.t. $\vec{P}_\alpha = \vec{P}(e_{vIs})$

quantum Gauss constraint (quantum cube)

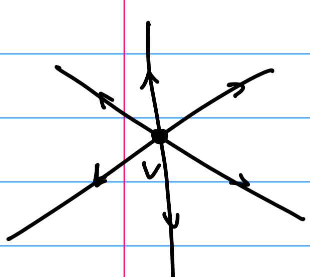
$$\sum_{\alpha=1}^{6} \hat{\vec{P}}_\alpha \psi = 0 \qquad \text{i.e.} \quad \sum_{s=\pm1} \sum_{I=1}^{3} \hat{P}^j(e_{vIs}) \, |\Psi\rangle = 0$$

$\vec{J}_\alpha$

$\hat{P}^j \sim \hat{R}^j \longleftrightarrow \hat{J}^j$

$$\to \sum_{s=\pm1} \sum_{I=1}^{3} \hat{R}^j_{e_{vIs}} \psi(\vec{h}) = 0$$

$\searrow$ zero total angular momentum state



$$\to \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \sum_{e \text{ at } v} \psi(e^{\varepsilon \tau_j} h_e \cdots) = 0$$

gauge transf. at v

solution: gauge inv. wave functions.

states satisfying quantum Gauss constraint span $\mathcal{H}_\gamma$:

Hilbert space of SU(2) gauge inv. states

$\mathrm{Tr}\,\vec{j}\,\vec{i}$

Quantum Diffeo. & Hamiltonian constraints

$$\left.\begin{array}{lll} C_a \rightsquigarrow \hat{C}_a : & \hat{C}_a \Psi = 0 \\ C \rightsquigarrow \hat{C} : & \hat{C} \Psi = 0 \end{array}\right\} \text{Quantum Constraint eqns}$$

Hamiltonian contraints: (Thiemann 1996)

$$C(x) = \underbrace{\frac{1}{\kappa} F^j_{ab} \frac{\varepsilon_{jkl} E^a_k E^b_l}{\sqrt{\det q}}(x)}_{\text{Euclidean term } C_0(x)} - \underbrace{\frac{1}{\kappa}(\beta^2+1)\, \varepsilon_{jmn} K^m_a K^n_b \frac{\varepsilon_{jkl} E^a_k E^b_l}{\sqrt{\det q}}(x)}_{\text{Lorentz term } C_L(x)}$$

$$C(x) \longrightarrow \underset{\substack{\uparrow \\ \text{discretization}}}{C_v} \longrightarrow \underset{\substack{\uparrow \\ \text{operator}}}{\hat{C}_v}$$

(1) Euclidean term $C_0(x) = -\frac{2}{\kappa} \mathrm{tr}\left(F_{ab} \frac{[E^a, E^b]}{\sqrt{\det q}} \right)$

$$F_{ab} = F^j_{ab} \frac{\tau_j}{2} \qquad \tau_j = -i\,\sigma_j$$

$$E^a = E^a_j \frac{\tau_j}{2}$$

$$[E^a, E^b] = E^a_j E^b_k \underbrace{\left[\frac{\tau_j}{2}, \frac{\tau_k}{2}\right]}_{\varepsilon^{jkl}\frac{\tau_l}{2}}$$

$$\mathrm{tr}\left[\frac{\tau_j}{2}\frac{\tau_k}{2}\right] = -\frac{1}{2}\delta_{jk}$$

$\frac{1}{\sqrt{\det q}} \rightsquigarrow \frac{1}{V_v}$?

No, since diverge if $V_v = 0$

<u>Lemma</u> $\{A_c(x), V(R)\} = \sim \frac{k\beta}{8} \operatorname{sgn}(\det e) \frac{[E^a \ E^b]}{\sqrt{\det q}} \varepsilon^{abc}$

$\forall$ region R, $x \in R$

(Thiemann's trick)

proof: