1. Symmetry of QM
2. Theory of Angular momentum.
3. Scattering theory.

Symmetry of transformation of space $\mathbb{R}^3$

wave function $\psi(\vec{r})$ $\quad \vec{r} \in \mathbb{R}^3$.

consider linear transf. $Q: \mathbb{R}^3 \to \mathbb{R}^3$

$$Q\vec{r} = \vec{r}'$$

$\vec{r}$: 3d column vector

Q : 3×3 matrix

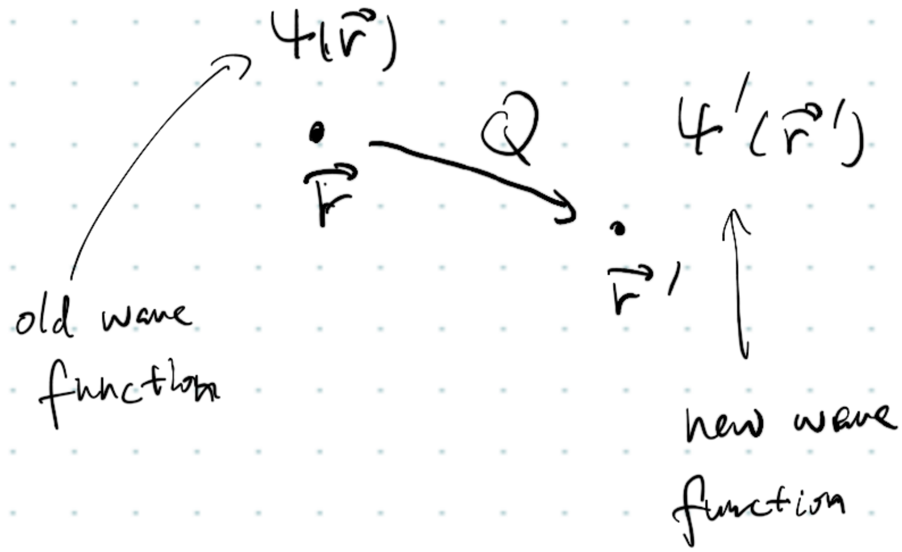
Q is a symmetry of the space, if it preserves the inner product

$$\vec{r}_1 \cdot \vec{r}_2 = (Q\vec{r}_1)\cdot(Q\vec{r}_2)$$

in particular, Q preserves the distance in $\mathbb{R}^3$ ✓

$$\vec{r}\cdot\vec{r} = (Q\vec{r})\cdot(Q\vec{r})$$

Q induces transformation of $\psi(\vec{r})$



s.t. $\psi'(\vec{r}\,') = \psi(\vec{r})$

$$\underbrace{\psi'(Q\vec{r})}_{Q\vec{r}\,=\,\vec{r}\,'} = \psi(\underbrace{\vec{r}}_{=\,Q^{-1}\vec{r}\,'})$$

$$\boxed{\begin{aligned}\psi'(\vec{r}) &= \psi(Q^{-1}\vec{r}) \\ &\equiv \hat{D}(Q)\psi(\vec{r})\end{aligned}}$$

$$Q : \mathbb{R}^3 \to \mathbb{R}^3$$

$$\hat{D}(Q) : \mathcal{H} \to \mathcal{H} \qquad \mathcal{H} = L^2(\mathbb{R}^3, d^3x)$$

$$|\psi\rangle \to |\psi'\rangle \qquad \begin{aligned}\psi'(\vec{r}) &= D(Q)\psi(\vec{r}) \\ &= \psi(Q^{-1}\vec{r})\end{aligned}$$

properties of $D(Q)$: linear operator on $\mathcal{H}$.

- unitarity : $\langle\psi_1|\psi_2\rangle = \int d^3\vec{r}\ \overline{\psi_1(\vec{r})}\ \psi_2(\vec{r})$

change of variable $\vec{r} \to Q^{-1}\vec{r}$

$$= \int d^3(Q^{-1}\vec{r})\ \overline{\psi_1(Q^{-1}\vec{r})}\ \psi_2(Q^{-1}\vec{r})$$

<u>Lemma</u> $\det Q = \pm 1$

Pf: $\vec{r}\cdot\vec{r}' = \underbrace{\delta_{ij} r^i r'^j}$

$\qquad \parallel$

$Q\vec{r}\cdot Q\vec{r}' = \underbrace{\delta_{ij} Q^i{}_k r^k Q^j{}_l r'^l}$

$\Rightarrow \delta_{ij} Q^i{}_k Q^j{}_l = \delta_{kl}$

$\Updownarrow$

$Q^T Q = 1_{3\times 3}$

$$\Rightarrow Q^T = Q^{-1}, \quad (\det Q)^2 = 1$$

$$\det Q = \pm 1$$

$$d^3(Q^{-1}\vec{r}) = d^3\vec{r}\,|\det Q^{-1}| = d^3\vec{r}$$

$$\langle \psi_1 | \psi_2 \rangle = \int d^3\vec{r}\; \overline{\psi_1(Q^{-1}\vec{r})}\; \psi_2(Q^{-1}\vec{r})$$

$$= \langle D(Q)\psi_1 | D(Q)\psi_2 \rangle$$

$\Rightarrow$ $D(Q)$ is unitary operator.

- Consider 2 transformations $Q_1, Q_2 : \mathbb{R}^3 \to \mathbb{R}^3$

$$Q = Q_1 \circ Q_2 \qquad Q\vec{r} = Q_1 \circ Q_2 \vec{r}$$

$$D(Q_1)\, D(Q_2)\, \psi(\vec{r}) = D(Q_1)\, \psi(Q_2^{-1}\vec{r})$$

$$= \psi\left(Q_2^{-1}(Q_1^{-1}\vec{r})\right)$$

$$= \psi(Q_2^{-1} Q_1^{-1}\vec{r}) = \psi\left((Q_1 Q_2)^{-1}\vec{r}\right)$$

$$= D(Q_1 Q_2)\, \psi(\vec{r})$$

$$\boxed{D(Q_1)\, D(Q_2) = D(Q_1 Q_2)}$$

if we view D to be a map from symmetry transf. on $\mathbb{R}^3$ to linear operators on $\mathcal{H}$.

D respects the product of transf.

- $D(Q)\, D(Q^{-1}) = D(Q Q^{-1}) = D(1_{3\times 3})$

$$D(1)\, \psi(\vec{r}) = \psi(1\vec{r}) = \psi(\vec{r})$$

$$D(1) = 1_{\mathcal{H}}$$

$$D(Q)D(Q^{-1}) = 1_{\mathcal{H}}$$

$$\boxed{D(Q)^{-1} = D(Q^{-1})}$$ D respects the inverse.

Def A group is a set G together with a binary operation

$a \cdot b \in G \quad \forall a, b \in G$

called group multiplication

s.t. (1) $(a \cdot b) \cdot c = a \cdot (b \cdot c) \quad \forall a, b, c \in G$

(associativity)

(2) $\exists ! \, e \in G$, s.t. $e \cdot a = a \quad \forall a \in G$

(identity)

(3) $\forall a \in G$, $\exists \, a^{-1} \in G$ s.t. $a \cdot a^{-1} = a^{-1} \cdot a = e$

(inverse)

Example (1) $\mathbb{R}^1 \setminus \{0\}$ with multiplication

(2) set of symmetry transf. Q's on $\mathbb{R}^3$

multiplication ; composition of Q_1, Q_2

- Q ; 3×3 matrix $\rightarrow$ associativity
- identity $\mathbb{1}_{3\times 3}$
- inverse $Q^T Q = \mathbb{1}$ $\quad Q^{-1} = Q^T$

the set of all 3×3 matrices with $Q^{-1} = Q^T$ is called $O(3)$ group.

(orthogonal group on $\mathbb{R}^3$)