$$\int Dx(t)\, e^{i\int dt\, L} \qquad L = x(t)\,\hat{A}\,x(t)$$

$$\langle 0|T(\hat{x}(t_1)\,\hat{x}(t_2)|0\rangle = \frac{i}{2}A^{-1}(t_1,t_2)$$

$$\hat{A} = \frac{m}{2}\left(e^{i\theta}\frac{d^2}{dt^2} + \omega_0^2 e^{-i\theta}\right) \qquad A_{ij}A^{-1}_{jk} = \delta_{ik}$$

$$\hat{A}\; A^{-1}(t_1,t_2) = \delta(t_1-t_2) \qquad A^{-1}(t_1,t_2) = \frac{1}{2\pi}\int A^{-1}(\omega)\,e^{-i\omega(t_1-t_2)}\,d\omega$$

Green's function

$$\delta(t_1-t_2) = \frac{1}{2\pi}\int d\omega\; e^{-i\omega(t_1-t_2)}$$

$$\frac{1}{2\pi}\int d\omega\left(\underline{\frac{m}{2}\left(e^{i\theta}(-i\omega)^2 + \omega_0^2 e^{-i\theta}\right)A^{-1}(\omega)}\; e^{-i\omega(t_1-t_2)}\right)$$

$$= \frac{1}{2\pi}\int d\omega\; e^{-i\omega(t_1-t_2)}$$

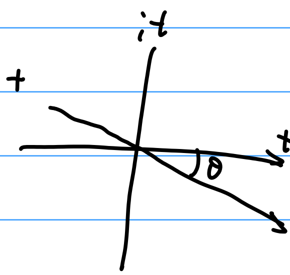
$$A^{-1}(\omega) = \frac{2m^{-1}}{\left(\omega_0^2 e^{-i\theta} - \omega^2 e^{i\theta}\right)} = \frac{-2m^{-1}}{\omega^2-\omega_0^2+i\varepsilon}$$

$i0^+$

$0<\theta\ll 1$

$$\omega_0^2(1-i\theta) - \omega^2(1+i\theta)$$

$$= \omega_0^2 - \omega^2 - \underbrace{i\theta(\omega_0^2+\omega^2)}_{i\varepsilon}$$



$$\langle 0|T(x(t_1)\,x(t_2))|0\rangle = \int \frac{-im^{-1}}{\omega^2-\omega_0^2+i\varepsilon}\, e^{-i\omega(t_1-t_2)}\,\frac{d\omega}{2\pi}$$

$$\langle 0|T(\phi(x_1)\,\phi(x_2))|0\rangle = \int \frac{d^4k}{(2\pi)^4}\, e^{ik(x_1-x_2)}\,\frac{1}{k^2-m^2+i\varepsilon}$$

finite tempreture (imaginary tile) correlation function

$$Z(\beta) = \sum_n e^{-\beta E_n} \qquad H|n\rangle = E_n|n\rangle$$

thermal correlation function : $\displaystyle\sum_n \langle n| T(\hat{O}_1(t_1) \cdots \hat{O}_n(t_n))|n\rangle \, e^{-\beta E_n}/Z.$

$\beta = \frac{1}{T} \qquad T \to 0 \quad \beta \to \infty \leftarrow$

thermal correlation func.
$\longrightarrow$ usual correlation func.

$\boxed{t \to i\tau}$

$e^{-itH} \to e^{\tau H} \qquad \hat{O}(t) \longrightarrow e^{\tau H}\hat{O}\, e^{-\tau H}$

$\beta > t_1 > t_2 > t_3 > \cdots t_n > 0$

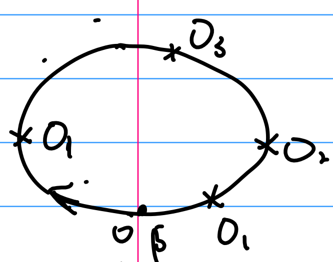
$$\to Z^{-1}\,\mathrm{Tr}\left(\hat{O}_1(t_1)\cdots\hat{O}_n(t_n)\, e^{-\beta H}\right)$$

$$= Z^{-1}\,\mathrm{Tr}\left(\hat{O}_1(\tau_1)\cdots\hat{O}_n(\tau_n)\, e^{-\beta H}\right) \equiv \langle O_1(\tau_1)\cdots O_n(\tau_n)\rangle_\beta$$

$$\to \quad = \frac{\mathrm{Tr}\left(e^{-H(\beta-\tau_1)}\,O_1\, e^{-H(\tau_1-\tau_2)}\cdots O_n\, e^{-H\tau_n}\right)}{\mathrm{Tr}\left(e^{-\beta H}\right)} \qquad \hat{O}_i = O_i[\hat{x}]$$

$$= \frac{\displaystyle\int_{\text{closed paths}} Dx(\tau)\; O_1[x(\tau_1)]\cdots O_n[x(\tau_n)]\; e^{-S_E[x(\tau)]}}{\displaystyle\int_{\text{closed paths.}} Dx(\tau)\; e^{-S_E[x(\tau)]}}$$

$x(0) = x(\beta)$



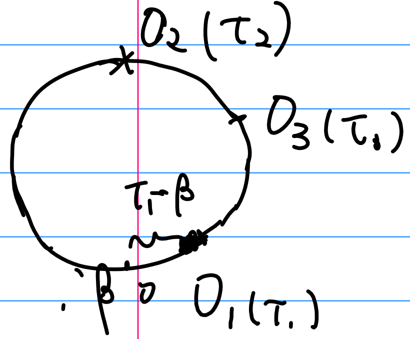
$\beta > \tau_1 > \tau_2 > \cdots > \tau_n > 0$

if $\tau_1 > \beta$, $\beta > \tau_2 > \cdots > \tau_n > 0$

$$\langle O_1(\tau_1)\cdots O_n(\tau_n)\rangle_\beta = \frac{\mathrm{Tr}\left(e^{H(\tau_1-\beta)}\, O_1\, e^{-H(\tau_1-\beta+\beta-\tau_2)}\cdots\right)}{\mathrm{Tr}\left(e^{-\beta H}\right)}$$

$$= \frac{\mathrm{Tr}\left(e^{H(\tau_1-\beta)}\, O_1\, e^{-H(\tau_1-\beta)}\, e^{H(\tau_2-\beta)}\cdots\right)}{\mathrm{Tr}\left(e^{-\beta H}\right)}$$

$$= \frac{\mathrm{Tr}\left(e^{-\beta H} e^{\tau_2 H} O_2 e^{(\tau_2-\tau_3)H} O_3 \cdots O_n e^{-H\tau_n} e^{H(\tau_1-\beta)} O_1 e^{-H(\tau_1-\beta)}\right)}{\mathrm{Tr}(e^{-\beta H})}$$



$$= \frac{\mathrm{Tr}\left(e^{-\beta H} O_2(\tau_2) O_3(\tau_3) \cdots O_1(\tau_1-\beta)\right)}{\mathrm{Tr}(e^{-\beta H})}$$

$$= \underline{\langle O_2(\tau_2) O_3(\tau_3) \cdots O_1(\tau_1-\beta) \rangle_\beta}$$

- Thermal correlation function is periodic in β.

Correlation function is translational inv.

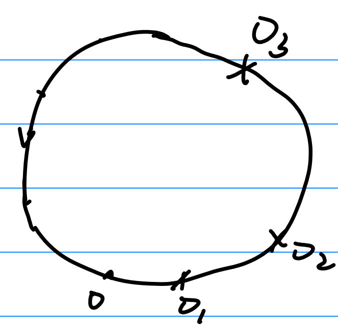
$$\tau_1 \to \tau_1 + a \quad \cdots \quad \tau_n \to \tau_n + a$$

$$\langle O_1(\tau_1+a) O_2(\tau_2+a) \cdots O_n(\tau_n+a) \rangle_\beta \qquad e^{-H\tau_n} \underline{\underline{e^{-Ha}}}$$

$$= \frac{\mathrm{tr}\left(e^{-H(\beta-\tau_1-a)} O_1 e^{H(\tau_1-\tau_2)} O_2 \cdots O_n e^{-H(\tau_n+a)}\right)}{\mathrm{tr}(e^{-\beta H})}$$

$$e^{-H(\beta-\tau_1)} \underline{\underline{e^{Ha}}}$$

$$= \langle O_1(\tau_1) O_2(\tau_2) \cdots O_n(\tau_n) \rangle$$

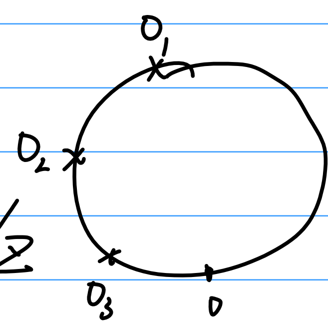


$$\langle O_1(\tau_1) O_2(\tau_2) \rangle_\beta \equiv \mathcal{G}^\beta(\tau_1-\tau_2)$$

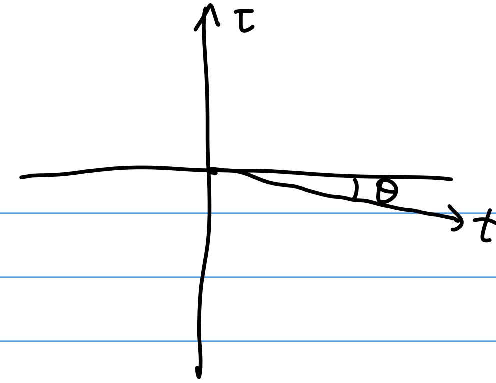
$$\|$$

$$iG(t) = \underline{\underline{\mathcal{G}(i\tau + \mathrm{sgn}(t)\, 0^+)}}$$

$$iG^\beta(t) = \sum_n \langle n | T(O_1(t_1) O_2(t_2)) | n \rangle e^{-\beta E_n}/Z$$



$$iG^\beta(t) = \mathcal{G}^\beta\left(i\tau + \underline{\underline{\mathrm{sgn}(t)\,0^+}}\right)$$



[proof: section 2.2.6 in Wen's book]

Example: Hawking radiation

schwarzschild

near horizon limit

$$ds^2 = -\left(1-\frac{2m}{r}\right)dt^2 + \left(1-\frac{2m}{r}\right)^{-1}dr^2 + r^2 d\Omega_2^2$$

$$\boxed{\left(\frac{\partial}{\partial t}\right)^a \quad \text{killing}}$$

$r = r_H = 2m$ horizon

$$\frac{\xi^2}{8m} = r - r_H \qquad \text{consider } \xi \to 0$$

$$dr = d\left(\frac{\xi^2}{8m} + 2m\right) = \frac{\xi}{4m}\,d\xi$$

$$\left(1-\frac{2m}{r}\right) = \left(1-\frac{2m}{\frac{\xi^2}{8m}+2m}\right) = \frac{\xi^2}{16m^2}\left(1-\frac{\xi^2}{16m^2}\right) + O(\xi^6)$$

$$r^2 = \left(\frac{\xi^2}{8m} + 2m\right)^2 = 4m^2 + O(\xi^2)$$

$$ds^2 = -\frac{\xi^2}{16m^2}dt^2 + \frac{16m^2}{\xi^2}\frac{\xi^2}{16m^2}d\xi^2 + 4m^2 d\Omega_2^2$$

$$= -\frac{\xi^2}{16m^2}dt^2 + d\xi^2 + 4m^2 d\Omega_2^2$$

$$= \underbrace{-\kappa^2 \rho^2 dt^2 + d\rho^2}_{\text{Rindler}} + 4m^2 \underbrace{d\Omega_2^2}_{S^2} \qquad \kappa = \frac{1}{4m}$$

Rindler $\times$ S^2 ↑ surface gravity

$t \to i\tau$

$$ds^2 \to \underbrace{\kappa^2 \rho^2 d\tau^2 + d\rho^2}_{\text{plane in polar coordinate.}} + 4m^2 d\Omega_2^2$$

proper angle coordinate $\quad \kappa\tau \equiv \theta \sim \theta + 2\pi$ $\qquad$ Hawking temperature

$$\kappa\tau \sim \kappa\tau + 2\pi$$

$$\underbrace{\tau \sim \tau + \frac{2\pi}{\kappa}} \qquad \boxed{\beta = \frac{2\pi}{\kappa}}$$

$\left(\frac{\partial}{\partial\tau}\right)^a$ is killing

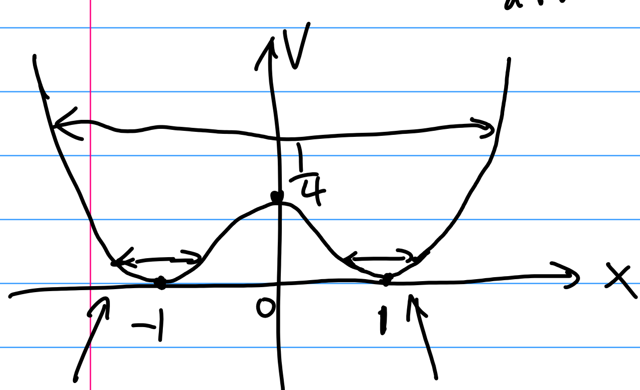
any quantum system. $\quad G(\tau_1 - \tau_2) = \langle O_1(\tau_1)\, O_2(\tau_2) \rangle$

$$\to \langle O_1(\tau_1)\, O_2(\tau_2) \rangle_\beta = \langle O_2(\tau_2)\, O_2(\tau_1 - \beta) \rangle_\beta$$

$\tau_1 > \beta$

Instanton in QM & quantum tunneling (kink)

$$L(\dot{x}, x) = \frac{1}{2} m \left(\frac{dx}{dt}\right)^2 - V(x) \qquad V(x) = \frac{1}{4}(x^2 - 1)^2$$



$x_0 = 1$

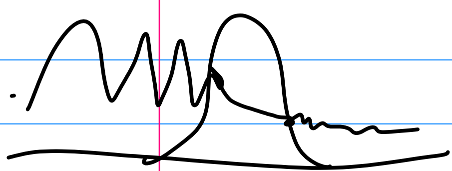
minima of $V(x)$: $x_0, -x_0$

low energy, like harmonic oscillator

low energy : $E < \frac{1}{4}$

high energy : $E > \frac{1}{4}$

quantum effect : low energy, you still have quantum tunneling from one well to the other



transition amplitude

$$\langle x_b | e^{-iHt} | x_a \rangle \equiv i A(x_b, x_a, t)$$

$\uparrow$ $\sim x_0$ $\qquad$ $\uparrow$ x_0

$$\hat{H} = \frac{\hat{p}^2}{2m} + V(x)$$

$$\underline{iA(x_b, x_a, t)} = \int \mathcal{D}x(t)\, e^{\frac{i}{\hbar}\int dt\, L(\dot{x}, x)}$$

semiclassical approximation : $= \sum_{\substack{\text{semiclassical} \\ \text{paths} \\ x_c(t)}} \dfrac{e^{\frac{i}{\hbar}\int dt\, L(\dot{x}_c, x_c)}}{\sqrt{\mathrm{Det}(S'')}} \left(1 + O(\hbar)\right)$

but classical paths cannot tunnel

we analytic continuous $t \to -i\tau$

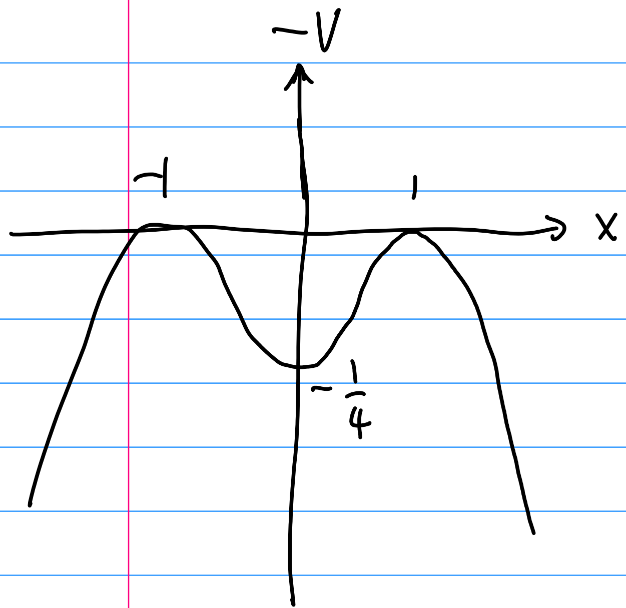
$$A(x_b, x_a, \tau) = \langle x_b | e^{-H\tau} | x_a \rangle = iA(x_b, x_a, -i\tau)$$

$$\parallel$$

$$\int \mathcal{D}x(\tau)\, e^{-\frac{1}{\hbar} S_E[x(\tau)]}$$

$$S_E = \int_0^{\tau} d\tau' \left(\frac{1}{2} m \dot{x}^2 + V(x) \right) \qquad L = K - V$$

Lagrangian with $-V(x)$ as the potential $\quad V = \frac{1}{4}(x^2-1)^2$



any positive E can tunnel from -1 to 1 even classically

Euclidean path integral is good for studying quantum tunneling effect.

$$A(x_b, x_a, \tau) = \sum_{\substack{\text{classical} \\ \text{paths} \\ x_c(\tau)}} \frac{e^{-\frac{1}{\hbar} S_E[x_c(\tau)]}}{\sqrt{\text{Det}(S'')}} \left(1 + O(\hbar)\right)$$