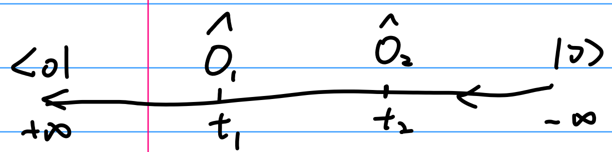
$$\langle 0|T(\hat O_1(t_1)\hat O_2(t_2))|0\rangle = \begin{cases} \langle 0|\hat O_1(t_1)\hat O_2(t_2)|0\rangle & t_1>t_2 \\ \langle 0|\hat O_2(t_2)\hat O_1(t_1)|0\rangle & t_2>t_1\end{cases}$$

$t_1>t_2$

$$iG(t_1,t_2) = \langle 0|\hat O_1(t_1)\hat O_2(t_2)|0\rangle = \frac{\langle 0|U(\infty,t_1)\hat O_1 U(t_1,t_2)\hat O_2 U(t_2,-\infty)|0\rangle}{\langle 0|U(\infty,-\infty)|0\rangle} \leftarrow$$

$$U(t_1,t_2) = e^{-i\hat H(t_1-t_2)}$$

$H|0\rangle = E_0|0\rangle$



$$\langle x_b|U(t_b,t_a)|x_a\rangle = \int Dx\, e^{iS}$$

$$iG^\theta(t_1,t_2) := \frac{\langle\psi|U^\theta(\infty,t_1)O_1U^\theta(t_1,t_2)O_2U^\theta(t_2,-\infty)|\psi'\rangle}{\langle\psi|U^\theta(\infty,-\infty)|\psi'\rangle}$$

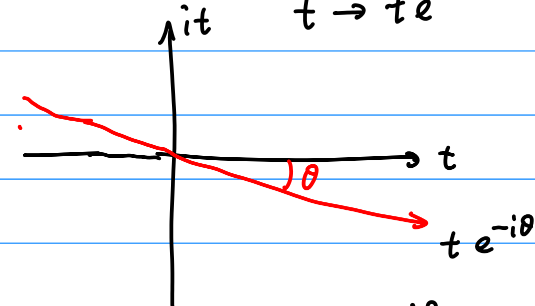
$|\psi\rangle = \sum_E a_E|E\rangle$

$|\psi'\rangle = \sum_E a'_E|E\rangle$

$$U^\theta(t) = e^{-iH(te^{-i\theta})}$$

$t\to te^{-i\theta}$ $\quad \theta>0$

requirement $a_0\neq 0 \quad a_0'\neq 0$



$T\to\infty$

$$\frac{\sum_{EE'} a_E^* a'_E \langle E|e^{-iE(T-t_1)e^{-i\theta}}\hat O_1 e^{-i\hat H(t_1-t_2)e^{-i\theta}}\hat O_2 e^{-iE'(t_1+T)e^{-i\theta}}|E'\rangle}{\sum_{EE'} a_E^* a'_E\, e^{-iE(T+T)e^{-i\theta}}\langle E|E'\rangle}$$

$$e^{-iETe^{-i\theta}} = e^{-iET(\cos\theta - i\sin\theta)} = e^{-iET\cos\theta}e^{-ET\sin\theta} \quad T\to\infty$$

$0<\theta\ll 1$

$\sum_{EE'} \longrightarrow$ ground state

$$iG^\theta(t_1,t_2) = \frac{\cancel{a_0^*a_0'}\langle 0|e^{-iH(T-t_1)e^{-i\theta}}O_1e^{-iH(t_1-t_2)e^{-i\theta}}O_2e^{-iH(t_1+T)e^{-i\theta}}|0\rangle}{\cancel{a_0^*a_0'}\langle 0|e^{-iH(T+T)e^{-i\theta}}|0\rangle}$$

$$= \frac{\langle 0|U^\theta(\infty,t_1)O_1U^\theta(t_1,t_2)O_2U^\theta(t_1,-\infty)|0\rangle}{\langle 0|U^\theta(\infty,-\infty)|0\rangle}$$

$$\lim_{\theta\to\infty} iG^\theta(t_1,t_2) = G(t_1,t_2) = \underline{\langle 0|\hat{O}_1(t_1)\hat{O}_2(t_2)|0\rangle}$$

we $|\psi\rangle = |x\rangle$, $|\psi'\rangle = |x'\rangle$ arbitrary $|x\rangle$ $|x'\rangle$

$$\underline{iG^\theta(t_1,t_2)} = \frac{\langle x|U^\theta(\infty,t_1)O_1 U^\theta(t_1,t_2)O_2 U^\theta(t_2,-\infty)|x'\rangle}{\langle x|U^\theta(\infty,-\infty)|x'\rangle}$$

$$= \frac{\int dx(t_1)\,dx'(t_1)\,dx(t_2)\,dx'(t_2)\langle x|U^\theta(\infty,t_1)|x(t_1)\rangle\langle x(t_1)|O_1|x'(t_1)\rangle\langle x'(t_1)|U^\theta(t_1,t_2)|x(t_2)\rangle\langle x(t_2)|O_2|x'(t_2)\rangle\langle x'(t_2)|U^\theta(t_2,-\infty)|x'\rangle}{\langle x|U^\theta(\infty,-\infty)|x'\rangle}$$

$$\underline{\langle x|U^\theta(t_1,t_2)|x'\rangle = \int Dx(t)\, e^{i\int L(\dot{x},x)dt}}$$

$\hat{O}_1$, $\hat{O}_2$ are operator depending on $\hat{x}$ only $\hat{O}_1 = O_1(\hat{x})$ $\hat{O}_2 = O_2(\hat{x})$

$$\langle x(t_1)|\hat{O}_1|x'(t_1)\rangle = O_1(x(t_1))\,\delta(x(t_1)-x'(t_1))$$

$$\langle x(t_2)|\hat{O}_2|x'(t_2)\rangle = O_2(x(t_2))\,\underline{\delta(x(t_2)-x'(t_2))}$$

$$= \frac{\int dx(t_1)\,dx(t_2)\langle x|U^\theta(\infty,t_1)|x(t_1)\rangle O_1(x(t_1))\langle x(t_1)|U^\theta(t_1,t_2)|x(t_2)\rangle O_2(x(t_2))\underline{\langle x(t_2)|U^\theta(t_2,-\infty)|x'\rangle}}{\langle x|U^\theta(\infty,-\infty)|x'\rangle}$$

$$= \frac{\int Dx(t)\; O_1(x(t_1))\,O_2(x(t_2))\; e^{i\int_{-\infty}^{+\infty}L(\dot{x},x)dt}}{\int Dx(t)\; e^{i\int_{-\infty}^{+\infty}L(\dot{x},x)dt}}$$

path integral formula for $\underline{\langle 0|\hat{O}_1(t_1)\hat{O}_2(t_2)|0\rangle}$ when $\theta\to 0$ when $t_1>t_2$

$x(t\to-\infty) = x'$ $\quad x(t\to\infty) = x$

$t_1>t_2$ $\quad = \langle 0|T(\hat{O}_1(t_1)\hat{O}_2(t_2))|0\rangle$

How about $t_2>t_1$: the path integral formula for correlation function is automatically time-ordered

$$\frac{\int Dx\, O_1(t_1) O_2(t_2)\, e^{i\int L\,dt}}{\int Dx\, e^{i\int L\,dt}} = \frac{\int Dx\, O_2(t_2) O_1(t_1)\, e^{i\int L\,dt}}{\int Dx\, e^{i\int L\,dt}}$$

$t_2 > t_1$

$$= \frac{\langle x | \hat{U}^\theta(\infty, t_2)\, \hat{O}_2(t_2)\, \hat{U}^\theta(t_2, t_1)\, \hat{O}_1(t_1)\, \hat{U}^\theta(t_1, -\infty) | x' \rangle}{\langle x | U^\theta(\infty, -\infty) | x' \rangle}$$

$$= \langle 0 | \hat{O}_2(t_2)\, \hat{O}_1(t_1) | 0 \rangle \quad = \langle 0 | T(\hat{O}_1(t_1)\, \hat{O}_2(t_2)) | 0 \rangle$$

$$\boxed{\begin{aligned} \langle 0 | T(\hat{O}_1(t_1)\, \hat{O}_2(t_2)) | 0 \rangle &= \frac{\int Dx\, O_1(t_1) O_2(t_2)\, e^{i\int L\,dt}}{\int Dx\, e^{i\int L\,dt}} \\ \langle 0 | T(\hat{O}_1(t_1) \cdots \hat{O}_n(t_n)) | 0 \rangle &= \frac{\int Dx\, O_1(t_1) \cdots O_n(t_n)\, e^{i\int \underline{L}\,dt}}{\int Dx\, e^{i\int L\,dt}} \end{aligned}}$$

Example : Harmonic Oscillator $\quad H = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 x^2$

$$\int dt\, L = \int dt \left(\frac{m}{2} \underline{\frac{dx}{dt}\frac{dx}{dt}} - \frac{1}{2} m\omega^2 x^2 \right)$$

$$= \int_{-\infty}^{+\infty} dt \left[-\frac{m}{2} x \frac{d^2}{dt^2} x - \frac{1}{2} m\omega^2 x^2 + \cancel{\frac{m}{2} \frac{d}{dt}\left(x \frac{dx}{dt} \right)} \right] \quad 0$$

$$= -\frac{m}{2} \int_{-\infty}^{+\infty} dt \left(x(t) \left[\frac{d^2}{dt^2} + \omega^2 \right] x(t) \right)$$

$t \to t e^{-i\theta}$

$$\longrightarrow -\frac{m}{2} \int_{-\infty}^{+\infty} dt\, e^{-i\theta} \left(x(t) \left[e^{2i\theta} \frac{d^2}{dt^2} + \omega^2 \right] x(t) \right)$$

$$= -\frac{m}{2} \int_{-\infty}^{+\infty} dt \left(x(t) \left[e^{i\theta} \frac{d^2}{dt^2} + e^{-i\theta} \omega^2 \right] x(t) \right)$$

$$\hat{A} \equiv \left[e^{i\theta} \frac{d^2}{dt^2} + e^{-i\theta} \omega^2 \right] \left(\frac{m}{2} \right)$$

$$\langle 0 | T\left(\hat{x}(t_1)\, \hat{x}(t_2) \right) | 0 \rangle = \frac{\int Dx(t)\; x(t_1)\, x(t_2)\, e^{-i \int_{-\infty}^{+\infty} dt\; x(t) \hat{A} x(t)}}{\int Dx(t)\; e^{-i \int_{-\infty}^{+\infty} dt\; x(t) \hat{A} x(t)}}$$

$$\underline{\int \prod_{i=1}^{N} dx_i \;\; x_{i_1}\, x_{i_2}\; e^{-i \sum_{ij} x_i A_{ij} x_j}} \qquad \vec{x} = (x_1 \dots x_N) \;\to x(t)$$

generating functional $\quad Z(\vec{J}) = \int \prod_{i=1}^{N} dx_i \; e^{-i \sum_{ij} \underline{x_i A_{ij} x_j} + i \sum_i x_i J_i}$

$\vec{J} = (J_1 \cdots J_N)$ $\qquad A_{ij} = A_{ji}$, A invertible

$\to J(t)$ $\qquad J_i$: source

$$Z(J) = \int Dx(t)\; e^{-i \int dt\; x(t) \hat{A} x(t) + i \int dt\; x(t) J(t)}$$

$$S = \sum_{ij} x_i A_{ij} x_j - \sum_i x_i J_i$$

$$\delta S = 0 \;\Rightarrow\; 2 \sum_j A_{ij} x_j - J_i = 0$$

$$\Rightarrow \vec{x} = \frac{1}{2} A^{-1} \vec{J}$$

$$S = x^T A x - x^T J = \frac{1}{4} J^T \cancel{A^{-1T} A} A^{-1} J - \frac{1}{2} J^T A^{-1T} J$$

$$= -\frac{1}{4} J^T A^{-1} J$$

$$Z(J) = N \sqrt{\frac{1}{\operatorname{Det} A}}\, e^{-\frac{i}{4} \sum_{ij} J_i A^{-1}_{ij} J_j} = N \sqrt{\frac{1}{\operatorname{Det} A}}\; e^{-\frac{i}{4} \int dt_1 dt_2\, J(t_1) A^{-1}(t_1, t_2) J(t_2)}$$

$$\Rightarrow \int Dx\; x(t_1)\, x(t_2)\; e^{-i \int dt\; x \hat{A} x} = \frac{\delta}{\delta i J(t_1)} \frac{\delta}{\delta i J(t_2)} \int Dx\; e^{-i \int dt\; x \hat{A} x + i \int J x\, dt} \Bigg|_{J \to 0}$$

$$\to \quad = N \sqrt{\frac{1}{\operatorname{Det} A}}\; \frac{\delta}{\delta i J(t_1)} \frac{\delta}{\delta i J(t_2)}\; e^{-\frac{i}{4} \int dt_1 dt_2\; J(t_1) A^{-1}(t_1, t_2) J(t_2)} \Bigg|_{J=0}$$

$$\langle 0| T(x(t_1)\, x(t_2))|0\rangle = \frac{1}{Z[0]} \frac{\delta}{i\delta J(t_1)} \frac{\delta}{i\delta J(t_2)} Z[J] \Big|_{J=0}$$

$$\langle 0| T(x(t_1)\cdots x(t_n))|0\rangle = \frac{1}{Z[0]} \frac{\delta}{i\delta J(t_1)} \cdots \frac{\delta}{i\delta J(t_n)} Z[J] \Big|_{J=0}$$

$$\frac{\delta}{\delta J_i} e^{-\frac{i}{4} J_m A^{-1}_{mn} J_n} = e^{-\frac{i}{4} J_m A^{-1}_{mn} J_n} \left(-\frac{i}{2} A^{-1}_{in} J_n\right)$$

$$\frac{\delta}{\delta J_j} \frac{\delta}{\delta J_i} e^{-\frac{i}{4} J_m A^{-1}_{mn} J_n} = \frac{\delta}{\delta J_j} \left[e^{-\frac{i}{4} J_m A^{-1}_{mn} J_n} \left(-\frac{i}{2} A^{-1}_{in} J_n\right)\right] \Big|_{J\to 0}$$

$$= e^{-\frac{i}{4} J^T A^{-1} J} \left(-\frac{i}{2} A^{-1}_{jn} J_n\right)\left(-\frac{i}{2} A^{-1}_{im} J_m\right) \Big|_{J\to 0}$$

$$+ e^{-\frac{i}{4} J^T A^{-1} J} \left(-\frac{i}{2} A^{-1}_{ij}\right) \Big|_{J\to 0}$$

$$= -\frac{i}{2} A^{-1}_{ij}$$

$$\langle 0| T(\hat{x}(t_1)\, \hat{x}(t_2))|0\rangle = \frac{-N\sqrt{\frac{1}{\mathrm{Det} A}} \left(-\frac{i}{2}\right) A^{-1}(t_1, t_2)}{N\sqrt{\frac{1}{\mathrm{Det} A}}} = \frac{i}{2} A^{-1}(t_1, t_2)$$

$$\hat{A} = \frac{m}{2}\left(e^{i\theta}\left(\frac{d}{dt}\right)^2 + \omega^2 e^{-i\theta}\right)$$

$$A A^{-1} = 1$$

$$A_{ij} A^{-1}_{jk} = \delta_{ik}$$

$$\hat{A}\; A^{-1}(t_1\, t_2) = \delta(t_1 - t_2)$$

$A^{-1}(t_1\, t_2)$ is Green's function for $\hat{A}$

$$A^{-1}(t_1\, t_2)$$