Before 1987-88 : QG with ADM formalism (p^{ab}, q_{ab})

wave func $f[q]$ $\quad p^{ab} \to \hat{p}^{ab}(x) = -i\hbar \frac{\delta}{\delta q^{ab}(x)}$

$$C = \frac{1}{\kappa}\left[\frac{1}{\sqrt{\det q}}\left(p^{ab}p_{ab} - \frac{1}{2}p^2\right) - \sqrt{\det q}\ {}^3R(q)\right]$$

$$\to \hat{C}\left(q_{ab}(x) \to \hat{q}_{ab}(x),\ p^{ab}(x) \to -i\hbar \frac{\delta}{\delta q^{ab}(x)}\right)$$

$\hat{C}\,\Psi[q] = 0$ functional diff.

Wheeler-DeWitt eqn

1988 → 1996 : Ashtekar variable (E^a_i, A^i_a)

wave func. $f(h(e))$

$$C = \frac{1}{\kappa} F_{ab}\left(\frac{\epsilon_{ijk} E^a_j E^b_k}{\sqrt{\det q}}\right) + \cdots$$

$E^a_j(x) \to -i\frac{\beta l_p^2}{2}\frac{\delta}{\delta A^j_a(x)}$ $\quad C \to \hat{C}$

How to get $\frac{1}{\sqrt{\det q}}$ without divergence

1996-1996 Thiemann's Hamiltonian constraint.

$$C_0(x) = \frac{\sim 2}{\kappa}\,\mathrm{tr}\left(F_{ab}\frac{[E^a, E^b]}{\sqrt{\det q}}\right)$$

commutator of su(2)

$E^a = E^a_j \tau_j/2$ $\quad \vec{\tau} = -i\vec{\sigma}$

<u>Lemma</u> $\{A_c(x), V(R)\} = \sim \frac{\kappa\beta}{8}\,\mathrm{sgn}(\det e)\,\frac{[E^a, E^b]}{\sqrt{\det q}}\,\epsilon_{abc}$

$x \in R$

$E^a_j = \sqrt{\det q}\, e^a_j$ $\quad e^a_j$ right/left hand

$\mathrm{sgn}(\det e) = \pm 1$

<u>Proof</u>: $V(R) = \int_R d^3x \sqrt{\det q(x)}$

$$\frac{\delta V(R)}{\delta E^c_i(x)} = \int_R d^3x' \frac{1}{2\sqrt{\det q(x')}} \frac{\delta}{\delta E^c_i(x)} \underbrace{\det q(x')}_{\left(\det E^a_i(x)\right)\, \mathrm{sgn}(\det e)}$$

$$\det E = \frac{1}{6} \varepsilon_{abd}\, \varepsilon^{jkl} E^a_j E^b_k E^c_l$$

$$= \int_R d^3x' \frac{1}{2\sqrt{\det q(x')}} \mathrm{sgn}(\det e) \frac{1}{2} \varepsilon_{abd} \varepsilon^{jkl} \underbrace{\frac{\delta E^a_j(x')}{\delta E^c_i(x)}}_{\delta^a_c \delta^i_j \delta^{(3)}(x', x)} E^b_k E^c_l(x')$$

$$= \frac{\mathrm{sgn}(\det e)}{4\sqrt{\det q}} \varepsilon_{cbd} \varepsilon^{ikl} E^b_k E^c_l$$

$$\{A_c(x), V(R)\} = \frac{\tau_i}{2} \{A^i_c(x), V(R)\} = \frac{\tau_j}{2} \int \underbrace{\{A^i_c(x), E^a_j(x')\}}_{-\frac{\kappa\beta}{2} \delta^i_j \delta^a_c \delta^{(3)}(x, x')} \frac{\delta V(R)}{\delta E^a_j(x')} d^3x'$$

$$= -\frac{\kappa\beta}{2} \frac{\tau_i}{2} \frac{\delta V(R)}{\delta E^c_i(x)}$$

$$= -\frac{\kappa\beta}{2} \frac{\tau_i}{2} \mathrm{sgn}(\det e) \frac{1}{4\sqrt{\det q}} \varepsilon_{cbd} \underbrace{\varepsilon^{ikl} E^b_k E^c_l}$$

$$= -\frac{\kappa\beta}{2} \mathrm{sgn}(\det e) \frac{1}{4} \varepsilon_{cbd} \frac{[E^b, E^c]}{\sqrt{\det q}} \qquad \blacksquare$$

$$\frac{[E^a\ E^b]}{\sqrt{\det q}} = -\frac{4}{\kappa\beta} \{A_c, V(R)\} \varepsilon^{abc} \underline{\underline{\mathrm{sgn}(\det e)}}$$

$$\underline{\underline{C_0}} = -\frac{2}{\kappa} \mathrm{tr}\left(F_{ab} \frac{[E^a, E^b]}{\sqrt{\det q}}\right)$$

$$\mathrm{sgn}(\det e)\, C_0 = -\frac{2}{\kappa} \mathrm{tr}\left(F_{ab} \underbrace{\mathrm{sgn}(\det e) \frac{[E^a, E^b]}{\sqrt{\det q}}}\right)$$

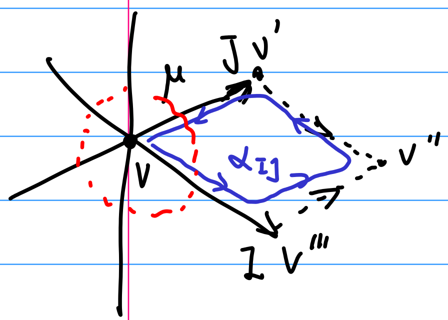
this is what we are going to quantize

$$\underbrace{\mathrm{sgn}(\det e)\, C_o(x)} = \frac{8}{\kappa^2 \beta}\, \varepsilon^{abc}\, \mathrm{tr}\left(\underline{\underline{F_{ab}^{(x)}}} \{ \underbrace{A_c^{(x)}, V(R)}_{x \in R} \} \right)$$

$$F_{ab}^{i} = \partial_a A_b^i - \partial_b A_a^i + \varepsilon_{ijk} A_a^j A_b^k$$

$SU(2)$ curvature

discretization $\quad \mathrm{sgn}(\det e)\, C_o(x) \longrightarrow C_{o,v}$



α_{IJ} : minimal loop

$$\frac{1}{2}\int dx^a \wedge dx^b\, F_{ab} = \int dx^I dx^J\, F_{IJ} \quad (\text{no sum } I, J)$$

$$h(\alpha_{IJ}) = 1 + \int_{\underbrace{S_{IJ}}_{O(\mu^2)}} F^i \frac{\tau^i}{2} + O(\mu^3)$$

$\partial S_{IJ} = \alpha_{IJ}$

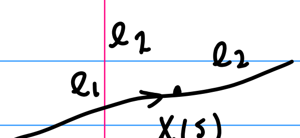
plaquette S_{IJ}

nonabelian stokes Thm

$U(1)$ loop holonomy : $\underline{e^{\oint_{\partial S} A} = e^{\int_S dA} = e^{\int_S F} = 1 + \int_S F + O(\mu^3)}$

$$V(R) \to V_v \qquad \{ \underline{A_I(x)}\ V_v \} \sim \underline{h(e_I) \{ h(e_I)^{-1}, V_v \}}$$

$$\underline{h(e_I) \{ h(e_I)^{-1}, V_v \}} = h(e_I) \int d^3x\, \frac{\delta h(e_I)^{-1}}{\delta A_a^i(x)} \underline{\underline{\{ A_a^i(x), V_v \}}}$$



$$= -\int d^3x \int ds\, \dot{e}_I^a \delta^{(3)}(\vec{x}, \vec{x}(s))\, h(e_I)\, \underbrace{h(e_2)^{-1} \frac{\tau_i}{2} h(e_1)^{-1}}_{\{A_a^i(x)\ V_v\}}$$

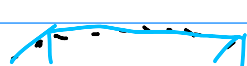
$$= -\int_{e_I} ds\, \dot{e}_I^a\, h(e_I)\, h(e_2)^{-1} \{ A_a(\vec{x}(s)), V_v \}\, h(e_1)^{-1}$$

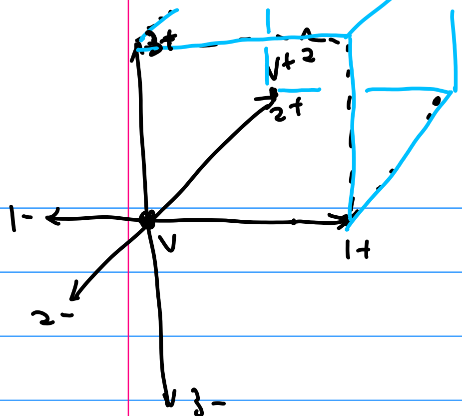
$$h(e) = 1 + O(\mu) \qquad \int ds \sim O(\mu)$$

$$= -\int_{e_I} ds\, \dot{e}_I^a \{ A_a(\vec{x}(s)), V_v \} + O(\mu^2)$$

let $\dot{e}_I^a = \left(\frac{\partial}{\partial x^I}\right)^a$

$$\underline{= -\int dx^I \{ A_I(\vec{x}), V_v \}} + O(\mu^2)$$





$$\int_{\square_{+++}} dx^1 dx^2 dx^3 \ \mathrm{sgn}(\det e)\, C_0$$

$$= \frac{8}{\kappa^2 \beta} \int dx^1 dx^2 dx^3 \ \varepsilon^{IJK}\, \underbrace{\mathrm{tr}\left(F_{IJ} \{ A_K , V_v \} \right)}_{= \mathrm{tr}\left((1+F_{IJ}) \{A_K , V_v\} \right)}$$

$$= \frac{8}{\kappa\beta} \Big[\int dx^1 dx^2 dx^3 \ \mathrm{tr}\left(F_{12} \{ A_3 , V_v \} \right) - \int dx^1 dx^2 dx^3 \ \mathrm{tr}\left(F_{13} \{ A_2 , V_v \} \right) + \cdots \Big]$$

$x \in [0, \mu]$

$$F(x) = F(0) + \underbrace{x\, \partial_x F}_{O(\mu)} + \underbrace{\tfrac{1}{2} x^2 \partial_x^2 F}_{O(\mu^2)} \cdots$$

$$= -\frac{8}{\kappa^2 \beta} \sum_{I,J,K=1}^{3} \varepsilon^{IJK}\, \mathrm{tr}\left(h(\alpha_{IJ})\, h(e_K) \{ h(e_K)^{-1} , V_v \} \right)$$

there are 8 cubes at v. $(1+, 2+, 3+)$ right hand coordinate

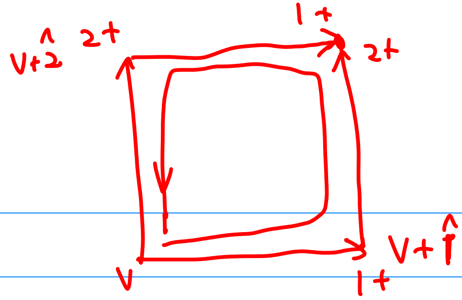
$(1-, 2+, 3+)$ left hand coordinate add over minus sgn

average $\frac{1}{8} \sum_{8\text{ cubes}} \cdots$

$$C_{0,v} := -\frac{1}{\kappa^2 \beta} \sum_{s_1, s_2, s_3 = \pm 1} \ \sum_{I_1 I_2 I_3 = 1}^{3} s_1 s_2 s_3\, \varepsilon^{I_1 I_2 I_3}$$

$$\mathrm{tr}\left[h\left(\alpha_{I_1 s_1, I_2 s_2}(v)\right) h(e_{v, I_3 s_3}) \{ h(e_{v I_3 s_3})^{-1} , V_v \} \right]$$

$$h\left(\alpha_{I_1 s_1, I_2 s_2}(v)\right) = h(e_{v, I_1 s_1})\, h(e_{v+s_1 \hat{I}_1, I_2 s_2})\, h(e_{v+s_2 \hat{I}_2, I_1 s_1})^{-1}\, h(e_{v, I_2 s_2})^{-1}$$



quantization: $h \to \hat{h}$, $V_v \to \hat{V}_v$, $\{\ \} \to \dfrac{[\ ]}{i\hbar}$

$$\hat{C}_{0,v} := -\underbrace{\frac{1}{i\hbar\kappa^2\beta}}_{-\frac{1}{i\ell_p^2\kappa\beta}} \sum_{\substack{s_1,s_2,s_3\\=\pm1}} \sum_{\substack{I_1,I_2,I_3\\=1}}^{3} s_1 s_2 s_3\, \epsilon^{I_1 I_2 I_3}\, \mathrm{tr}\left(\hat{h}(\alpha_{I_1 s_1 I_2 s_2}(v))\, \hat{h}(e_{v I_3 s_3}) \left[\hat{h}(e_{v I_3 s_3})^{-1}, \hat{V}_v\right]\right)$$

$\ell_p^2 = \hbar\kappa$

<u>HW</u> check $\hat{C}_{0,v}$ is gauge inv.

i.e. $\hat{C}_{0,v} \circlearrowright \mathcal{H}_\gamma$