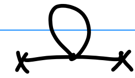
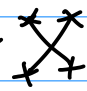
4 pt function in $\lambda\phi^4$

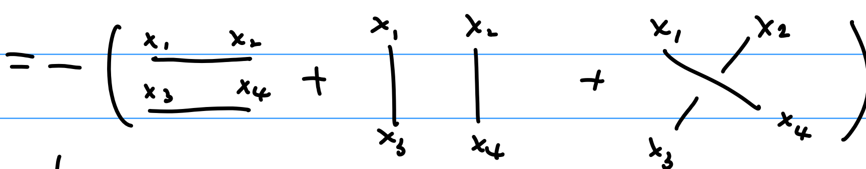
$$\tau(x_1\cdots x_4) = \langle 0|T\phi(x_1)\cdots\phi(x_4)|0\rangle = \frac{\delta^4}{\delta J_{x_1}\delta J_{x_2}\delta J_{x_3}\delta J_{x_4}}\frac{Z[J]}{Z[J]}\Big|_{J\to 0}$$

$$\frac{Z[J]}{Z[0]} = \frac{\left[1-\frac{i\lambda}{4!}\int d^4z\left(-3\,\infty + 6i\right.\right.}{1-\frac{i\lambda}{4!}\int d^4z\,(-3\,\infty)}$$

$+$

$$\left.\left.\right)\right]e^{-\frac{i}{2}J_x\Delta_{xy}J_y}$$

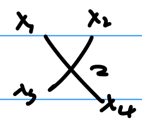
- all $4\times\frac{\delta}{\delta J}$ acting on $e^{-\frac{i}{2}J_x\Delta_{xy}J_y}$

$$\frac{\delta}{\delta J}\frac{\delta}{\delta J}e^{-\frac{i}{2}J\Delta J} = \frac{\delta}{\delta J}e^{-\frac{i}{2}J\Delta J}\Delta J = e^{-\frac{i}{2}J\Delta J}\Delta$$

$$O(\lambda^0):\ -\left[\Delta_{x_1x_2}\Delta_{x_3x_4} + \Delta_{x_1x_3}\Delta_{x_2x_4} + \Delta_{x_1x_4}\Delta_{x_2x_3}\right]$$

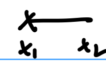


HW derive this



$= \Delta_{x_1z}\Delta_{x_2z}\Delta_{x_3z}\Delta_{x_4z}$

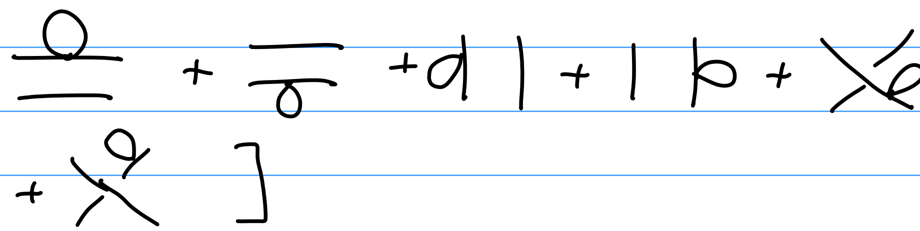
x_1 —— x_2 $= \Delta_{x_1x_2}$



$= \Delta_{x_1x_2}J_{x_1}$

- $2\times\frac{\delta}{\delta J}$ acting on $e^{-\frac{i}{2}J\Delta J}$

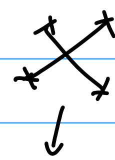
$O(\lambda)$: $\frac{i\lambda}{2}\Big[$



$\Big]$

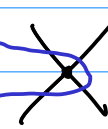
HW: derive this

- $0\times\frac{\delta}{\delta J}$ acting on $e^{-\frac{i}{2}J\Delta J}$

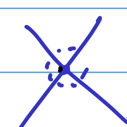


$$O(\lambda):\ \frac{-i\lambda}{4!}\frac{\delta^4}{\delta J_{x_1}\cdots\delta J_{x_4}}\int d^4z\,\left[\Delta_{zx}J_x\right]^4\ e^{-\frac{i}{2}J_x\Delta_{xy}J_y}$$

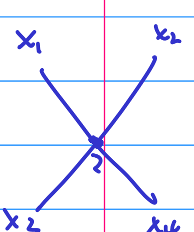
$$= -i\lambda\int d^4z\ \Delta_{x_1z}\Delta_{x_2z}\Delta_{x_3z}\Delta_{x_4z} = \int d^4z\,-i\lambda$$



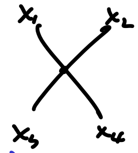
x_1 —— x_2 $= i\Delta_{x_1x_2}$

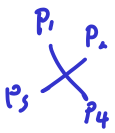


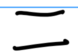
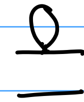
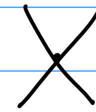
$= -i\lambda\int d^4z\ldots$



$$= -i\lambda\int d^4z\,(i\Delta_{x_1z})(i\Delta_{x_2z})(i\Delta_{x_3z})(i\Delta_{x_4z})$$

$$= -i\lambda \int d^4z\, \Delta_{x_1 z}\Delta_{x_2 z}\Delta_{x_3 z}\Delta_{x_4 z} = -i\lambda \int d^4 z$$ 

 $= -i\lambda \frac{i}{p_1^2-m^2+i\epsilon}\frac{i}{p_2^2-m^2+i\epsilon}\cdots(2\pi)^4\delta(\Sigma p)$ S: $-i\lambda(2\pi)^4\delta(\Sigma p)$

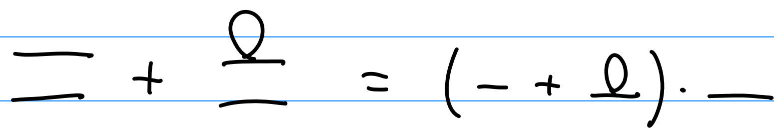
$\tau(x_1 \cdots x_4) = -\big(3 \times$  $\big) - \frac{i\lambda}{2}\big(6\times$  $\big) - i\lambda$ 

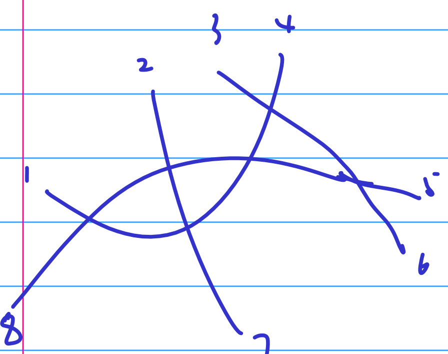
disconnected diagram

factorized into 2 pt functions

Connected diagram

(more interesting)

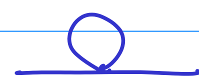




$\sim \lambda^6$

$$\Delta_{xy} = \int \frac{d^4q}{(2\pi)^4}\frac{1}{q^2-m^2} e^{iq(x-y)}$$

divergence in $\lambda\phi^4$

2 pt function = $O(\lambda^0)$ + $O(\lambda^1)$ 

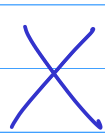
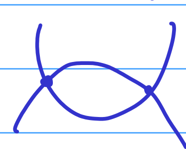
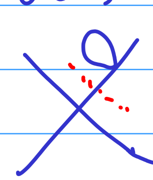
$\frac{\lambda}{2}\int \frac{d^4q}{(2\pi)^4}\frac{1}{q^2-m^2}$

$\times$ external Δ's

$\Delta(0)$ divergent

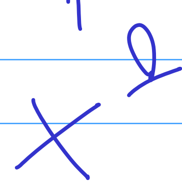
self energy

- $q\to\infty$: quadratic UV div.
- $m\to 0$ $q\to 0$ IR div.

4pt function = $O(\lambda)$  + $O(\lambda^2)$  + $O(\lambda^2)$ 

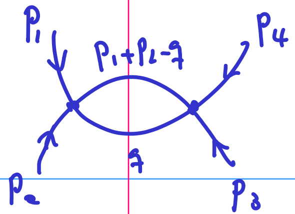
1-particle irreducible (1-PI) diagram

(more interesting)



not 1-PI

we only focus on UV div.



$$\int T(x_1 \cdots x_4)\, e^{ip_1x_1+\cdots+ip_4x_4}\, d^4x_1 \cdots d^4x_4$$

in momentum rep.

$$= (2\pi)^4 \delta^{(4)}(p_1+p_2+p_3+p_4)\, \frac{\lambda^2}{2}\int_{-\infty}^{\infty} \frac{d^4q}{(2\pi)^4} \frac{1}{(q^2-m^2)\left[(p_1+p_2-q)^2-m^2\right]}$$

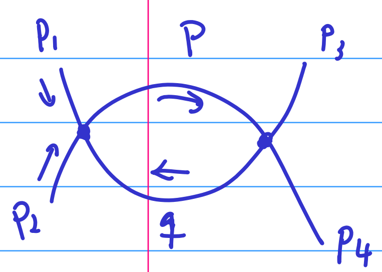
$\times$ external propagators

$q \to \infty \quad \int d^4q \frac{1}{q^4}$

Logarithmic div.

Hw derive this

- $\delta^{(4)}(\Sigma p)$: momentum conservation (from transl. inv. of 4-pt func.)



- loop diagram : free internal momentum not constrainted by momentum conservation

- LSZ reduction : n-pt function $\xrightarrow{}$ S-matrix

removing external propagator

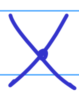
$p_1+p_2 = p-q \quad \to \quad p = p_1+p_2-q$

$p_3+p_4 = p-q \quad \to \quad p_1+p_2 = p_3+p_4$

$S = 1 + iM$

$\sigma \sim \int |M|^2 ds$

Momentum space Feynman rule

- $= \dfrac{i}{p^2-m^2+i\varepsilon}$
-  $= -i\lambda\mu^{4-d}$
- overall $\delta^{(4)}(\Sigma p) \times (2\pi)^4$
- n-loops : n fold $\int \dfrac{d^4q}{(2\pi)^4}$ ← d
- symmetry factor.

$f(x) \qquad x \to \infty \quad . \quad f(x) \to \infty$

$x \to \Lambda \qquad f(\Lambda)$

Regularizing the divergence.

$$\int^{\Lambda} d^4 q \ (\dots)$$ UV cut-off

break Lorentz inv.

dimensional regularization $\displaystyle\int d^4 q \frac{1}{q^4} \to \int d^d q \frac{1}{q^4}$ $\quad d = 4 - \varepsilon$

$\varepsilon > 0$

Convergent if $\varepsilon > 0$

div. as $\varepsilon \to 0$

Compute the integral for $d = 4 - \varepsilon$

analytic continue to $\varepsilon \to 0$

e^{iS}

Scalar Lagrangian in d-dim

$$\int d^d x \, \mathcal{L} = \int d^d x \left[\frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{m^2}{2} \phi^2 - \frac{\lambda \mu^{4-d}}{4!} \phi^4 \right]$$

$\hbar = c = 1$

$[x] = -1$ $\quad [\phi] = \frac{d-2}{2}$

$[\partial] = 1$ $\quad [\mu] = 1$

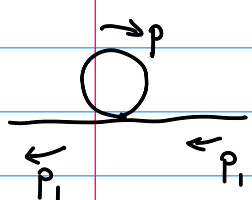
$[m] = 1$ $\quad [\lambda] = 0$

μ : arbitrary mass parameter

λ : dimensionless

$d = 4 - \varepsilon$ $\quad d \in \mathbb{C}$

$d^d p = d|p| \, d\Omega^{d-1}$ $\quad \dfrac{1}{-|p|^2 - m^2}$ $\quad a^{\frac{\varepsilon}{2}} = e^{\frac{\varepsilon}{2} \ln a} = 1 + \frac{\varepsilon}{2} \ln a$



$$= \frac{1}{2} \lambda \mu^{4-d} \int \frac{d^d p}{(2\pi)^d} \frac{1}{p^2 - m^2} = \dots = - \frac{i\lambda}{32\pi^2} m^2 \left(\frac{4\pi \mu^2}{-m^2} \right)^{2 - d/2} \Gamma\left(1 - \frac{d}{2}\right)$$

- $\vec{p} = (|p|, \theta_1 \cdots \theta_{d-1})$
- $p^0 \to i p^0_E \Rightarrow p^2 \to -p_E^2$

HW derive this

(Ryder's book or others)

Γ - function : simple poles at $x = 0, -1, -2 \dots$

$\Gamma(x)$

$$\Gamma(-n + \varepsilon) = \frac{(-1)^n}{n!} \left[\frac{1}{\varepsilon} + \psi_1(n+1) + O(\varepsilon) \right]$$

$n = 0, 1, 2 \dots$

$1 + \frac{1}{2} + \dots + \frac{1}{n} - \gamma$

0.577

Euler-Mascheroni const.

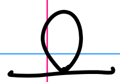
$d = 4 - \varepsilon$

$$\Gamma\left(1 - \frac{d}{2}\right) = \Gamma\left(-1 + \frac{\varepsilon}{2}\right) = -\frac{2}{\varepsilon} - 1 + \gamma + O(\varepsilon)$$

pole @ -1

quadratic div. integral

$d = 4$



$$= \frac{-i\lambda m^2}{32\pi^2}\left[1 + \frac{\varepsilon}{2}\ln\left(\frac{4\pi\mu^2}{-m^2}\right) + O(\varepsilon^2)\right]\left[-\frac{2}{\varepsilon} - 1 + \gamma + O(\varepsilon)\right]$$

$$= \frac{i\lambda m^2}{16\pi^2\varepsilon} + \text{finite}$$

div. if $\varepsilon \to 0$

dep. on μ^2