Theorem of orthonormal basis

$D_i : G \to L(\mathcal{H})$

$D_j : G \to L(\mathcal{H})$

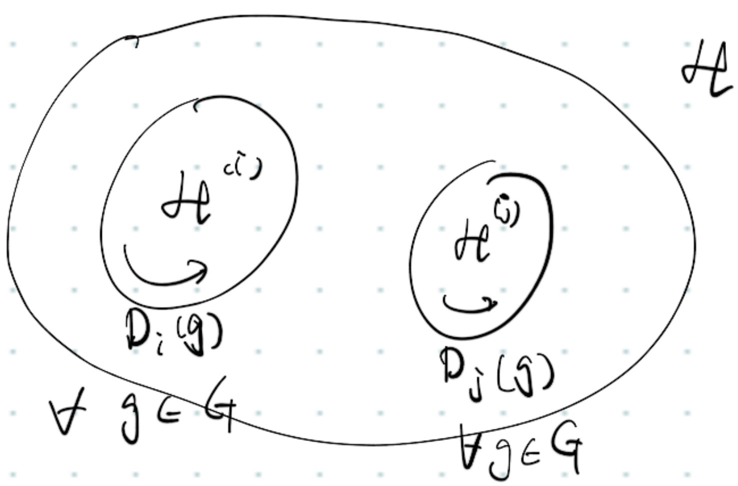
D_i, D_j are unitary irrep

if D_i, D_j are inequivalent

$\Rightarrow \exists \ \mathcal{H}^{(i)} \subset \mathcal{H} \quad D_i : G \to L(\mathcal{H}^{(i)})$

$\quad \mathcal{H}^{(j)} \subset \mathcal{H} \quad D_j : G \to L(\mathcal{H}^{(j)})$

and $\underline{\underline{\mathcal{H}^{(i)} \perp \mathcal{H}^{(j)}}}$



inequivalent unitary irreps are carried by mutually orthogonal subspaces in $\mathcal{H}$

!

<u>Theorem</u> (orthogonality of matrix elements)

Let's assume G to be a finite group, i.e., G has only a finite number of elements

$$\left.\begin{aligned} D^{(i)} &: G \to L(\mathcal{H}^{(i)}) \\ D^{(j)} &: G \to L(\mathcal{H}^{(j)}) \end{aligned}\right\} \text{ unitary irrep.}$$

$f^{(i)}_{\alpha=1\cdots\dim(\mathcal{H}^{(i)})}$ orthogonal basis in $\mathcal{H}^{(i)}$

$f^{(j)}_{\beta=1\cdots\dim(\mathcal{H}^{j})}$ orthogonal basis in $\mathcal{H}^{(j)}$

$$\langle f^{(i)}_\alpha | f^{(j)}_\beta \rangle = \delta^{ij}\,\delta_{\alpha\beta}\, r$$

constant.

matrix elements of rep.

functions on G (rep. functions)

$$\begin{cases} D^{(i)}_{\alpha\beta}(g) = \langle f^{(i)}_\alpha | D^{(i)}(g) | f^{(i)}_\beta \rangle \\ D^{(j)}_{\alpha\beta}(g) = \langle f^{(j)}_\alpha | D^{(j)}(g) | f^{(j)}_\beta \rangle \end{cases}$$

orthogonality of rep. functions

$$\frac{\sum_{g\in G} D^{(i)}_{\alpha\delta}(g)^* D^{(j)}_{\beta\gamma}(g)}{}$$

$$= \delta_{ij}\,\delta_{\alpha\beta}\,\delta_{\delta\gamma}\,\frac{h}{\dim(\mathcal{H}^{(i)})}$$

rank of the group

Lie group (infinite group)

$$\sum_{g\in G} \rightarrow \int_G dg$$

<u>Character</u> : group G , rep D : $G \rightarrow L(\mathcal{H})$

$$g \mapsto D(g) \in L(\mathcal{H})$$

character : $\chi(g) = \mathrm{tr}(D(g))$

$$= \sum_{\alpha} D_{\alpha\alpha}(g)$$

properties : (1) conjugacy class : group conjugate :

$|$ $\quad g \mapsto hgh^{-1} \quad \forall, g, h \in G$

obit of group conjugate

$$g \to \text{obit of } hgh^{-1}$$

$$= \{hgh^{-1} \mid \forall h \in G\}$$

$$\chi(g) \to \chi(hgh^{-1}) = tr(D(hgh^{-1}))$$

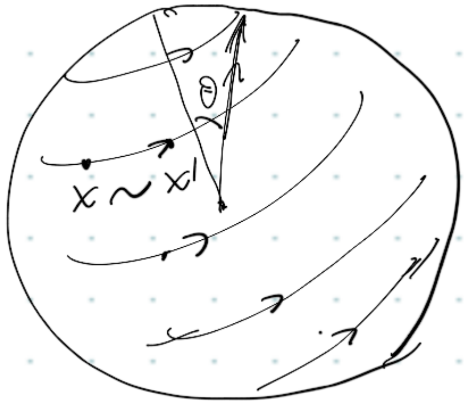
$$= tr(D(h) D(g) D(h^{-1}))$$

$$= tr(\underline{D(h)} D(g) \underline{D(h)^{-1}})$$

$$= tr(D(g)) = \chi(g)$$

i.e. $\chi(g)$ inv. under conjugate

χ is function of conjugacy classes in G



S^2

an orbit = a circle with const. θ

= one class

$\{\text{classes}\} = \{\text{orbits}\}$

$= [0, \pi]$

$f(\theta, \varphi)$

$\underline{\underline{\cos\theta}}$ is a function of $\theta \in [0, \pi]$

is a function of classes (orbits)