perturbative renormalization of QCD

$$\mathcal{L}_0 = \frac{1}{4}\left(F^a_{\mu\nu}\right)_0^2 + \bar\psi_0\left(i\gamma^\mu D^0_\mu - m_0\right)\psi_0 + \bar c_0\, \partial^\mu D^0_\mu c_0$$

$$F^a_{\mu\nu 0} = \partial_\mu A_{\nu 0} - \partial_\nu A_{\mu 0} + g_0 f^{abc} A^b_{\mu 0} A^c_{\nu 0}$$

$$D^0_\mu \psi_0 = \partial_\mu \psi_0 + i g_0 A^a_{\mu 0} T_a \psi_0$$

$$A^a_\mu = \frac{1}{Z_3^{1/2}} A^a_{\mu 0} \qquad \psi = \frac{1}{Z_2^{1/2}}\psi_0 \qquad c = \frac{1}{(Z_2^c)^{1/2}} c_0$$

(1) $$-\frac{1}{4}\left(\partial_\mu A^a_{\nu 0} - \partial_\nu A^a_{\mu 0}\right)^2 = -\frac{Z_3}{4}\left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu\right)$$

$$= -\frac{1}{4}\left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu\right) + \frac{1}{4}\underbrace{(Z_3 - 1)}_{\delta_3}\left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu\right)$$

in $\mathcal{L}_{ren}$ in $\mathcal{L}_{c.t.}$ $\delta_3 = Z_3 - 1$

(2) $$\bar\psi_0(i\partial\!\!\!/ - m_0)\psi_0 = Z_2\bar\psi(i\partial\!\!\!/ - m_0)\psi - m\bar\psi\psi + m\bar\psi\psi$$

$$= \bar\psi(i\partial\!\!\!/ - m)\psi + \bar\psi\left(i(Z_2 - 1)\partial\!\!\!/ - Z_2 m_0 + m\right)\psi$$

$$= \bar\psi(i\partial\!\!\!/ - m)\psi + \bar\psi\left(i\delta_2\partial\!\!\!/ - \delta_m\right)\psi$$

in $\mathcal{L}_{ren}$ in $\mathcal{L}_{c.t.}$

$\delta_2 = Z_2 - 1$

$\delta_m = Z_2 m_0 - m$

(3) $$g_0 A^a_{\mu 0}\bar\psi_0\gamma^\mu T^a\psi_0 = Z_3^{1/2} Z_2 g_0 A^a_\mu\bar\psi\gamma^\mu T^a\psi - g A^a_\mu\bar\psi\gamma^\mu T^a\psi + g A^a_\mu\bar\psi\gamma^\mu T^a\psi$$

$$= g A^a_\mu\bar\psi\gamma^\mu T^a\psi + g\underbrace{\left(Z_3^{1/2} Z_2\frac{g_0}{g} - 1\right)}_{\delta_1} A^a_\mu\bar\psi\gamma^\mu T^a\psi$$

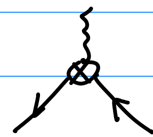
in $\mathcal{L}_{ren}$ in $\mathcal{L}_{c.t.}$ $\delta_1 = Z_3^{1/2} Z_2\frac{g_0}{g} - 1$

.... similarly we can work out other terms. (see P&S P532)

$$\mathcal{L}_0 = \mathcal{L}_{ren} + \mathcal{L}_{c.t.}$$

(1) $-\frac{1}{4}\delta_3\left(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a\right)^2$  $-i\left(q^2 g^{\mu\nu} - q^\mu q^\nu\right)\delta^{ab}\delta_3$

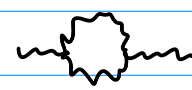
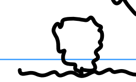
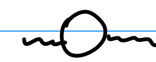
(2) $\bar{\psi}\left(i\delta_2 \not{\partial}\right)\psi$  $i \not{k}\, \delta_2$

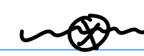
(3) $g\,\delta_1 A_\mu^a \bar{\psi}\gamma^\mu T^a \psi$  $i g \gamma^\mu T^a \delta_1$

to determine renormalization constants $\delta_1\ \delta_2\ \delta_3$, we need renormalization conditions.
(P&S sec. 12.2)

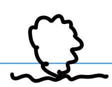
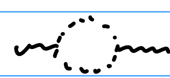
condition (1) $\langle 0|T A_\mu^a(q) A_\nu^a(-q)|0\rangle = \dfrac{-i\delta^{ab}}{q^2+i\varepsilon} g_{\mu\nu}$ at $q^2 = -M^2$

we introduce an energy scale

$\langle 0|T A_\mu^a(q) A_\nu^a(-q)|0\rangle =$  $+$  $+$  $+$ 

$+$  $+$  $+$ higher loops

should cancel at $q^2 = -M^2$

 $+$  $+$  $+$ 

$$= i\left(g^{\mu\nu}q^2 - q^\mu q^\nu\right)\delta^{ab}\frac{g^2}{(4\pi)^{d/2}}\Bigg(C_2(G)\int_0^1 dx \frac{\Gamma(2-\frac{d}{2})}{\Delta^{2-d/2}}\left[(1-\tfrac{d}{2})(1-2x)^2+2\right]$$
$$+ C(N)\, n_f \int_0^1 dx \frac{\Gamma(2-\frac{d}{2})}{\Delta_m^{2-d/2}}\left[8x(x-1)\right]\Bigg)$$

$\Delta_m = m^2 - x(1-x)q^2$, $\Delta = -x(1-x)q^2$

 $= -i\left(q^2 g^{\mu\nu} - q^\mu q^\nu\right)\delta^{ab}\delta_3$

condition (1) $\Rightarrow$
$$\delta_3 = \frac{g^2}{(4\pi)^{d/2}}\Bigg(C_2(G)\int_0^1 dx \frac{\Gamma(2-\frac{d}{2})}{\Delta^{2-d/2}}\left[(1-\tfrac{d}{2})(1-2x)^2+2\right]$$
$$+ C(N)\, n_f \int_0^1 dx \frac{\Gamma(2-\frac{d}{2})}{\Delta_m^{2-d/2}}\left[8x(x-1)\right]\Bigg)_{q^2=-M^2}$$

$\varepsilon = 4-d$

$$\left.\frac{\Gamma(2-\frac{d}{2})}{\Delta_m^{2-d/2}}\right|_{q^2=-M^2} = \frac{2}{\varepsilon} - \gamma - \log\Delta_m \Big|_{q^2=-M^2} + O(\varepsilon)$$

$$= \underbrace{\frac{2}{\varepsilon} - \gamma - \log M^2}_{\simeq \frac{\Gamma(2-\frac{d}{2})}{(M^2)^{2-\frac{d}{2}}}} + \log\left(\frac{m^2}{M^2} + x(1-x)\right)$$

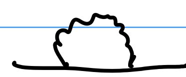
① $M \to UV$ $\quad \frac{m^2}{M^2} \sim 0$

② $M \to m$ @ IR $\quad \frac{m^2}{M^2} \sim 1$

$$\delta_3 = \frac{g^2}{(4\pi)^{d/2}} \frac{\Gamma(2-\frac{d}{2})}{(M^2)^{2-\frac{d}{2}}} \left[\frac{5}{3} C_2(G) - \frac{4}{3} n_f C(N)\right] + \text{finite}$$

Condition (2) quark propagator

$$\langle 0 | T \psi_i(k) \bar{\psi}_j(-k) | 0 \rangle = \frac{i}{\not{k} - m} \delta_{ij} \quad \text{at } k^2 = -M^2$$

$\langle 0 | \psi\bar{\psi} | 0 \rangle =$  $+$  $+$ 

$i\not{k}\delta_2$

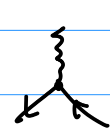
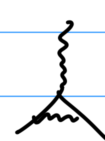
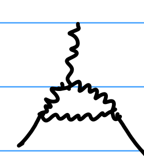
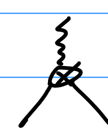
$$\frac{i g^2}{(4\pi)^{d/2}} C_2(N) \not{k} \int_0^1 dx\, (1-x)(d-2) \frac{\Gamma(2-\frac{d}{2})}{\Delta^{2-d/2}}$$

$$\Delta = -x(1-x)k^2$$

$$\Rightarrow \delta_2 = -\frac{ig^2}{(4\pi)^{d/2}} C_2(N) \int_0^1 dx\, (1-x)(d-2) \left.\frac{\Gamma(2-\frac{d}{2})}{\Delta^{2-d/2}}\right|_{k^2=-M^2}$$

$$= -\frac{g^2}{(4\pi)^{d/2}} C_2(N) \frac{\Gamma(2-\frac{d}{2})}{(M^2)^{2-\frac{d}{2}}} + \text{finite}$$

Condition (3) gluon - quark vertex

 $+$  $+$  $+$  $+ \cdots$

$ig T^a \gamma^\mu \delta_1$

$$= i g \gamma^\mu T^a$$

$$\Rightarrow \delta_1 = -\frac{g^2}{(4\pi)^2}\frac{\Gamma(2-\frac{d}{2})}{(M^2)^{2-\frac{d}{2}}}\left[C_2(N) + C_2(G)\right] + \text{finite}$$

$$C_2(G) \sim f^{abc} f^{dbc} \longrightarrow 0 \quad \text{for QED}$$

QED: $\delta_1 = \delta_2$ (consequence of gauge inv.)

<u>β-function</u> $\quad M\frac{\partial}{\partial M} g(M) = \beta(M)$

$$\delta_1 = Z_3^{\frac{1}{2}} Z_2 \frac{g_0}{g} - 1 \qquad \delta_2 = Z_2 - 1 \qquad \delta_3 = Z_3 - 1$$

$$\delta_1 - 1 = (1+\delta_3)^{\frac{1}{2}}(1+\delta_2)\frac{g_0}{g} \quad \Rightarrow \quad g = g_0 \frac{(1+\delta_3)^{\frac{1}{2}}(1+\delta_2)}{1+\delta_1}$$

$$\frac{\partial}{\partial M} g = g_0\left[\frac{1}{2}(1+\delta_3)^{-\frac{1}{2}}\left(\frac{\partial}{\partial M}\delta_3\right)\frac{1+\delta_2}{1+\delta_1} + \frac{(1+\delta_3)^{\frac{1}{2}}}{1+\delta_1}\frac{\partial}{\partial M}\delta_2 - \frac{(1+\delta_3)^{\frac{1}{2}}(1+\delta_2)}{(1+\delta_1)^2}\frac{\partial}{\partial M}\delta_1\right]$$

$$= \underbrace{g_0\frac{(1+\delta_3)^{\frac{1}{2}}(1+\delta_2)}{1+\delta_1}}_{g}\left[\frac{1}{2}\frac{1}{1+\delta_3}\partial_M\delta_3 + \frac{1}{1+\delta_2}\partial_M\delta_2 - \frac{1}{1+\delta_1}\partial_M\delta_1\right]$$

$$\delta_3, \delta_2, \delta_1 \sim O(g^2)$$

$$\beta(g) = M\frac{\partial}{\partial M} g = g\, M\frac{\partial}{\partial M}\underbrace{\left(-\delta_1 + \delta_2 + \frac{1}{2}\delta_3\right)}_{O(g^2)} + O(g^4)$$

$$M\frac{\partial}{\partial M}\frac{\Gamma(2-\frac{d}{2})}{M^{4-d}} = -\frac{2}{M^{4-d}}\underbrace{(2-\frac{d}{2})\Gamma(2-\frac{d}{2})}_{\Gamma(3-\frac{d}{2})} \xrightarrow{d=4} -2$$

$$\beta(g)_{QCD} = -\frac{g^3}{(4\pi)^2}\left[\frac{11}{3}C_2(G) - \frac{4}{3}n_f C(N)\right] \qquad C(N) = \frac{1}{2}$$

$$\beta(g)_{QED} = \frac{e^3}{12\pi^2} > 0 \qquad g = e \quad C(N) = 1 \quad n_f = 1$$

$$C_2(G) \sim f^{abc} f^{dbc} \to 0 \quad \text{QED}$$

QED $\quad \mu \frac{\partial}{\partial \mu} g = \beta(g) > 0 \qquad \mu \nearrow, \; g \nearrow$ $\quad$ perturbative QED is defined at IR

QCD : $\quad G = SU(3) \qquad N = 3$ $\qquad \psi^{I=1,2}_{i=1,2,3}$ $\quad I$ ← isospin, i quark

$$n_f = 2$$

$$\begin{array}{cccc} & \text{I} & \text{II} & \text{III} \\ SU(2)\left\{\begin{array}{c} \\ \\ \end{array}\right. & \begin{array}{c} u \\ d \end{array} & \begin{array}{c} c \\ s \end{array} & \begin{array}{c} t \\ b \end{array} \end{array}$$

$$\underbrace{\qquad\qquad\qquad}_{SU(3)}$$

6 flavor of quark.

$$C_2(G) = 3 \qquad C(N) = \frac{1}{2}$$

$$\frac{11}{3} C_2(G) - \frac{4}{3} n_f C(N) = 11 - \frac{4}{3} > 0$$

$$\beta(g)_{QCD} < 0 \implies \mu \nearrow \quad g \downarrow$$

UV

IR

Asymptotical freedom of QCD at UV.

- QCD is weakly coupled at UV., at UV you have free quark and gluon.
- QCD is strongly coupled at IR. no free quark & gluon

$\implies$ quark confinement