# Theorem of orthonormal basis

$D_i : G \to L(\mathcal{H})$

$D_j : G \to L(\mathcal{H})$

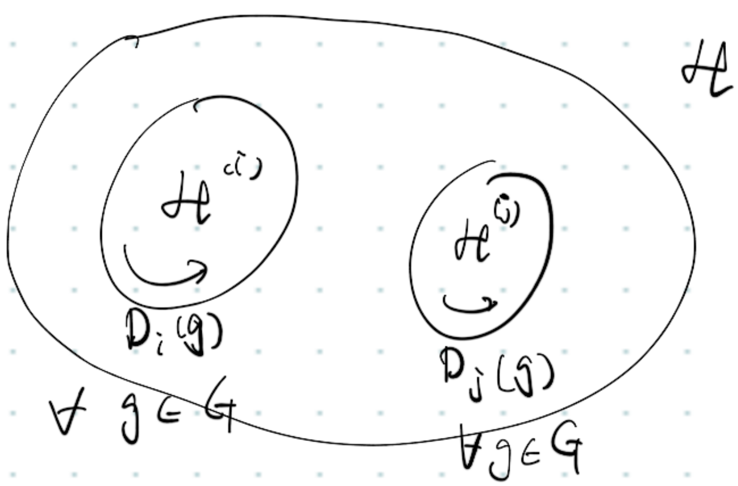
$D_i, D_j$ are unitary irrep

if $D_i, D_j$ are inequivalent

$\Rightarrow \exists \; \mathcal{H}^{(i)} \subset \mathcal{H} \quad D_i : G \to L(\mathcal{H}^{(i)})$

$\qquad \mathcal{H}^{(j)} \subset \mathcal{H} \quad D_j : G \to L(\mathcal{H}^{(j)})$

and $\underline{\underline{\mathcal{H}^{(i)} \perp \mathcal{H}^{(j)}}}$




inequivalent unitary irreps are carried by mutually orthogonal subspaces in $\mathcal{H}$

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Theorem (orthogonality of matrix elements)

Let's assume $G$ to be a finite group, ie, $G$ has only a finite number of elements

$$\left.\begin{array}{l} D^{(i)} : G \to L(\mathcal{H}^{(i)}) \\ D^{(j)} : G \to L(\mathcal{H}^{(j)}) \end{array}\right\} \text{unitary irrep.}$$

$f^{(i)}_{\alpha = 1 \cdots \dim(\mathcal{H}^{(i)})}$ orthogonal basis in $\mathcal{H}^{(i)}$

$f^{(j)}_{\beta = 1 \cdots \dim(\mathcal{H}^{j})}$ orthogonal basis in $\mathcal{H}^{(j)}$

$$\langle f^{(i)}_\alpha | f^{(j)}_\beta \rangle = \delta^{ij} \delta_{\alpha\beta} \, r$$

$r$: constant.

matrix elements of rep.

functions on $G$ (rep. functions)
$$\left\{\begin{array}{l} D^{(i)}_{\alpha\beta}(g) = \langle f^{(i)}_\alpha | D^{(i)}(g) | f^{(i)}_\beta \rangle \\ D^{(j)}_{\alpha\beta}(g) = \langle f^{(j)}_\alpha | D^{(j)}(g) | f^{(j)}_\beta \rangle \end{array}\right.$$

orthogonality of rep. functions

$$\sum_{g \in G} D^{(i)}_{\alpha\delta}(g)^* D^{(j)}_{\beta\gamma}(g)$$

$$= \delta_{ij}\delta_{\alpha\beta}\delta_{\delta\gamma} \frac{h}{\dim(\mathcal{H}^{(i)})}$$

rank of the group

Lie group (infinite group)

$$\sum_{g \in G} \to \int_G dg$$

<u>Character</u>: group $G$, rep $D$: $G \to L(\mathcal{H})$

$$g \mapsto D(g) \in L(\mathcal{H})$$

character: $\chi(g) = \mathrm{tr}(D(g))$

$$= \sum_{\alpha} D_{\alpha\alpha}(g)$$

properties: (1) conjugacy class: group conjugate:

$$| \qquad g \mapsto hgh^{-1} \quad \forall\, g, h \in G$$

obit of group conjugate

$$g \to \text{obit of } hgh^{-1}$$

$$= \{ hgh^{-1} \mid \forall h \in G \}$$

$$\chi(g) \to \chi(hgh^{-1}) = tr(D(hgh^{-1}))$$

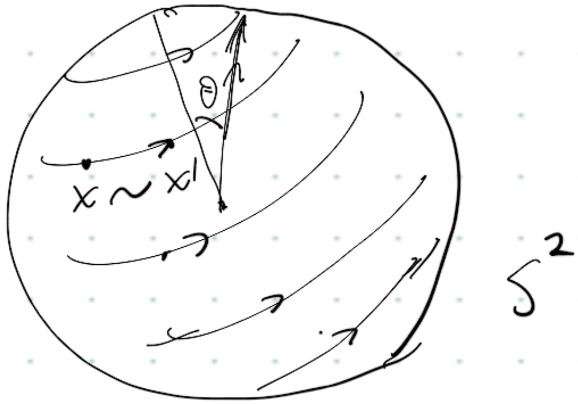
$$= tr(D(h)D(g)D(h^{-1}))$$

$$= tr(\underline{D(h)}\, D(g)\, \underline{D(h)^{-1}})$$

$$= tr(D(g)) = \chi(g)$$

i.e. $\chi(g)$ inv. under conjugate

$\chi$ is function of conjugacy classes in $G$




an obit = a circle with const. $\theta$

= one class

{classes} = {obits}

$= [0, \pi]$

$f(\theta, \varphi)$

$\underline{\underline{\cos\theta}}$ is a function of $\theta \in [0,\pi]$

is a function of classes (obits)

2) unitary reps $D'$ is equivalent to $D$

$\forall g \in G$, $D'(g) = U D(g) U^{-1}$ $\quad U: \mathcal{H} \to \mathcal{H}$ unitary

$$\chi'(g) = \operatorname{tr} D'(g) = \operatorname{tr}\left(U D(g) U^{-1}\right)$$

$$= \operatorname{tr} D(g) = \chi(g)$$

equivalent reps have the same charater.

3) reducible rep. $D = D_1 \oplus D_2 \oplus \cdots$

$\forall g \in G \quad \chi(g) = \mathrm{tr}\, D(g) = \sum_i \mathrm{tr}\, D_i(g) = \sum_i \chi_i(g)$

$$D(g) = \begin{pmatrix} \boxed{D_1(g)} & & 0 \\ & \boxed{D_2(g)} & \\ 0 & & \ddots \end{pmatrix}$$

charater of reducible rep = sum of characters of irreps.

<u>Theorem (orthogonality of $\chi$)</u>

(1) $G$ finite group, $D^{(i)}$, $D^{(j)}$ irrep, then

$$\sum_{g \in G} \underline{\chi^{(i)}(g)^* \chi^{(j)}(g)} = h\,\delta^{ij}$$

rank of the group.

$$= \sum_{C \in \left(\substack{\text{conjugacy} \\ \text{classes}}\right)} h_C\, \chi^{(i)}(C)^* \chi^{(j)}(C)$$

$h_C$ = # of elements in conjugacy class $C$.

(2) $D^{(i)}$ irrep of $G$ (finite group)

$i \in I \leftarrow$ set of all irreps of $G$.

$$\sum_{i \in I} \chi^{(i)}(C_l)^* \chi^{(i)}(C_m) = \frac{h}{h_l} \delta_{lm}$$

$l$th conjugacy class

$h_l$ : # of elements in $C_l$

<u>symmetries of schrödinger eqn</u>

Let $G$ be the symmetry group of the system

i.e. $\forall g \in G$ . $D(g)$ : unitary transf. on $\mathcal{H}$

($D$ : unitary rep $G$, carried by $\mathcal{H}$)

s.t. $\underline{D(g) \hat{H} D(g)^{-1} = \hat{H}}$ $\quad \forall g \in G$

eigen-eqn of $\hat{H}$ $\qquad \hat{H} |\psi_{ni}\rangle = E_n |\psi_{ni}\rangle$

$i = 1 \cdots d(n)$

$d(n)$ : degeneracy of energy level $E_n$

$\{\psi_{ni}\}_{i=1\cdots d(n)}$ span the $\underbrace{\text{eigen space}}_{\mathcal{H}^{(n)}}$ of $\hat{H}$ at the eigenvalue $E_n$

$$|\psi_{ni}\rangle \to D(g)|\psi_{ni}\rangle \equiv |\psi'_{ni}\rangle \quad \forall g \in G$$

$$\underline{\hat{H} D(g)|\psi_{ni}\rangle} = D(g)\hat{H}|\psi_{ni}\rangle = \underline{E_n D(g)|\psi_{ni}\rangle}$$

$D(g)|\psi_{ni}\rangle$ belongs to the eigenspace $\mathcal{H}^{(n)}$

($D(g)$ leaves $\mathcal{H}^{(n)}$ inv) $\forall g \in G$

$\{|\psi_{ni}\rangle\}_{i=1\cdots d(n)}$ basis in $\mathcal{H}^{(n)}$

$$D(g)|\psi_{ni}\rangle = \sum_j |\psi_{nj}\rangle D^{(n)}_{ji}(g)$$

rep matrix.

Any eigenspace $\mathcal{H}^{(n)}$ of $\hat{H}$ is a rep space of the

Symmetry group

$$\mathcal{H} = \bigoplus_{n=1}^{\infty} \mathcal{H}^{(n)}$$

$$D = \bigoplus_{n=1}^{\infty} D^{(n)}$$

$(\mathcal{H}, D)$ is a reducible rep of $G$.

<u>Example</u> (1) central potential $\hat{H} = \dfrac{\hat{p}^2}{2m} + V(r)$

rotational inv.

$$\boxed{G = SO(3)}$$

group of 3d rotations.

on the hand, $\hat{H}\,\psi_{nlm}(r,\theta,\varphi) = E_{nl}\,\psi_{nlm}(r,\theta,\varphi)$

$$\psi_{nlm}(r,\theta,\varphi) = R_{nl}(r)\,Y_{lm}(\theta,\varphi)$$

spherical harmonics

$l = 0, 1, 2 \ldots$

$m = -l, -l+1 \ldots l$

eigenspace $\underline{\underline{\mathcal{H}^{(n,l)}}}$ is spanned by $\{Y_{lm}\}_{m=-l..l}$

$$\dim \mathcal{H}^{(n,l)} = 2l+1$$

$\underline{\underline{\mathcal{H}^{(n,l)}}}$ are irrep. of $G = SO(3)$

all irreps of $SO(3)$ are $(\mathcal{H}_l, D_l)$

s.t. $\dim \mathcal{H}_l = 2l+1$

$\{|l,m\rangle\}_{m=-l,\dots,l}$ spans $\mathcal{H}_l$ carrier space of irrep of $SO(3)$

(2) hydrogen atom : $\hat{H} = \dfrac{\hat{p}^2}{2m} - \dfrac{e^2}{r}$

$$\hat{H}\psi_{nlm}(r,\theta,\varphi) = E_n \psi_{nlm}(r,\theta,\varphi)$$

$$\psi_{nlm}(r,\theta,\varphi) = R_{nl}(r)\, \overline{Y_{lm}(\theta,\varphi)}$$

eigen space $\mathcal{H}^{(n)}$ is spanned by $\{R_{nl}(r) Y_{lm}(\theta,\varphi)\}$

$l = 0, 1 \cdots n-1$

$m = -l \cdots l$

$$\dim \mathcal{H}^{(n)} = \sum_{l=0}^{n-1} (2l+1) = n^2$$

$$\mathcal{H}^{(n)} = \bigoplus_{l=0}^{n-1} \mathcal{H}^{(n,l)}$$

$\dim(\mathcal{H}^{(n,l)}) = 2l+1$

reducible rep of SO(3)

irrep of SO(3)