exam of GR class : on May 6 at 10 am

no lecture & group meeting on this Friday May 1st

Next Lecture & groupmeeting will be on May 8th.

Next lecture : May 8th at 11am

Thm Wigner D-matrix is representation matrix of $SU(2)$ acting on V_j (spin j rep. of $SU(2)$)

$$\pi^j_{mn}(h) = D^j_{mn}(\alpha,\beta,\gamma) = \langle j m | e^{-i\alpha \hat{J}_3} e^{-i\beta \hat{J}_2} e^{-i\gamma \hat{J}_3} | j n \rangle, \quad |j m\rangle \in V_j$$

elements of representation matrices form a ortlogonal basis of $L^2(G)$

G is any compact Lie group. e.g. $SU(2)$

$$\int d\mu_H(h)\, \pi^j_{mn}(h)\, \pi^{j'}_{m'n'}(h) = \frac{1}{2j+1} \delta_{jj'} \delta_{mm'} \delta_{nn'}$$

Ortlogonal basis in $\mathcal{H}^0_\gamma$: $T_{\vec{j},\vec{m},\vec{n}}(\vec{h}) \equiv \prod_{e\in E(\gamma)} \pi^{j_e}_{m_e n_e}(h_e)$

Gauge transformation : $f(\{h_e\}_{e\in E(\gamma)}) \to f(\{g_{s(e)} h_e g_{t(e)}^{-1}\}_{e\in E(\gamma)})$

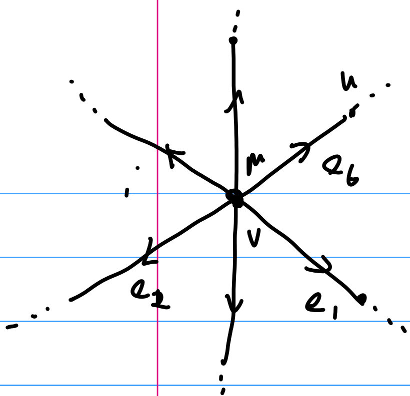
gauge tranformations at vertices. $g_v \in SU(2)$

we look for gauge inv wave functions

and $\mathcal{H}_\gamma \subset \mathcal{H}^0_\gamma$, $\mathcal{H}_\gamma$ is spanned by gauge inv. wave functions

gives the quantization of $P_\gamma \simeq T^* SU(2)^{|E(\gamma)|} / SU(2)^{|V(\gamma)|}$

How to construct gauge inv. wave functions



$e_1 \cdots e_6$ at v

$$T_{\vec{j}\vec{m}\vec{n}} = \prod_{e=1}^{6} \pi^{j_e}_{m_e n_e}(h_e) \cdots$$

$$\xrightarrow{\text{gauge transf.}} \prod_{e=1}^{6} \pi^{j_e}_{m_e n_e}(g_v h_e g^{-1}_{t(e)}) \cdots$$

π^{j}_{mn} is matrix element of representation matrix

$$\pi^{j}_{mn}(h) = \langle j m | \hat{h} | j n \rangle \leftarrow$$

$$\pi^{j}_{mn}(h h') = \sum_{k=-j}^{j} \pi^{j}_{mk}(h)\, \pi^{j}_{kn}(h') \quad \text{def. of rep.}$$

$$\pi^{j}_{mn}(h h') = \langle j m | \hat{h}\, \hat{h}' | j n \rangle = \sum_{k} \langle j m | \hat{h} | j k \rangle \langle j k | \hat{h}' | j n \rangle$$

$$T_{\vec{j}\vec{m}\vec{n}}(\vec{h}) \to \prod_{e=1}^{6} \pi^{j_e}_{m_e k_e}(g_v)\, \pi^{j_e}_{k_e l_e}(h_e)\, \pi^{j_e}_{l_e n_e}(g^{-1}_{t(e)}) \cdots$$

consider a tensor $C^{m_1 \cdots m_6}_{j_1 \cdots j_6}$, $C_{j_1 \cdots j_6} \in V_{j_1} \otimes \cdots \otimes V_{j_6}$

$$C_{j_1 \cdots j_6} = \sum_{\vec{m}} C^{m_1 \cdots m_6}_{j_1 \cdots j_6} |j_1 m_1\rangle \otimes \cdots \otimes |j_6 m_6\rangle$$

consider linear combination

$$\sum_{m_1 \cdots m_6} C^{m_1 \cdots m_6}_{j_1 \cdots j_6} \prod_{e=1}^{6} \pi^{j_e}_{m_e n_e}(h_e) \cdots$$

$$\xrightarrow[\text{transf.}]{\text{gauge}} \sum_{m_1 \cdots m_6} C^{m_1 \cdots m_6}_{j_1 \cdots j_6} \prod_{e=1}^{6} \pi^{j_e}_{m_e k_e}(g_v)\, \pi^{j_e}_{k_e l_e}(h_e)\, \pi^{j_e}_{l_e n_e}(g^{-1}_{t(e)}) \cdots$$

gauge inv. at the vertex v. if

$$\sum_{m_1 \cdots m_6} C^{m_1 \cdots m_6}_{j_1 \cdots j_6} \prod_{e=1}^{6} \pi^{j_e}_{m_e k_e}(g_v) = C^{k_1 \cdots k_6}_{j_1 \cdots j_6} \qquad (1)$$

i.e. $C_{j_1 \cdots j_6}$ is an invariant tensor in $V_{j_1} \otimes \cdots \otimes V_{j_6}$

$\forall g \in SU(2)$ $\pi^{j}(g) \circlearrowright V_j$ $\qquad V_{j_1} \otimes \cdots \otimes V_{j_6}$

tensor product rep. $\longrightarrow$ $\pi^{j_1}(g) \otimes \pi^{j_2}(g) \otimes \cdots \otimes \pi^{j_6}(g)$ $\quad \forall g \in SU(2)$

<u>Lemma</u> on each V_{j_e} $\exists$ $\hat{\vec{J}}_e = (\hat{J}^1_e \ \hat{J}^2_e \ \hat{J}^3_e)$

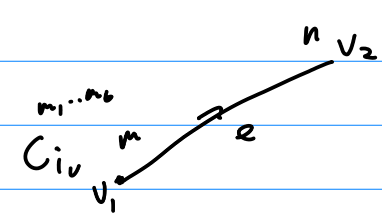
① $\iff$ $\underline{\sum_{e=1}^{6} \hat{\vec{J}}_e \, | C_{j_1 \cdots j_6} \rangle = 0}$ ② zero total angular momentum state

$| C_{j_1 \cdots j_6} \rangle \in \underline{V_{j_1} \otimes \cdots \otimes V_{j_6}}$

<u>HW</u> prove this.

$| C_{j_1 \cdots j_6} \rangle$ satisfying ① or ② is called an intertwiner, and span $C^{n_1 \cdots n_6}$

$\mathrm{Inv}\,(V_{j_1} \otimes \cdots \otimes V_{j_6}) \subset V_{j_1} \otimes \cdots \otimes V_{j_6}$
(space of intertwiners)



basis of gauge inv. wave functions

$$T_{\gamma, \vec{j}, \vec{i}}(\vec{h}) := \prod_{e \in E(\gamma)} \pi^{j_e}(h_e) \prod_{v \in V(\gamma)} C_{\vec{j}, \underline{\underline{i_v}}}$$

contract the the indices at all vertices

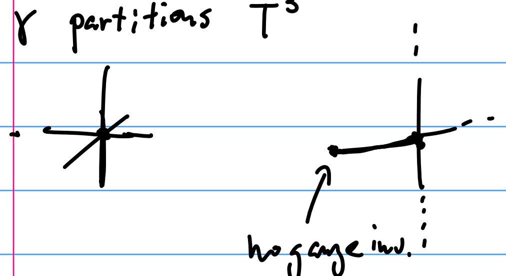
$\vec{i} = (i_v)_{v \in V(\gamma)}$

intertwiner basis label

assume γ has no dangling edge $\quad \vec{j} = (j_e)_{e \in E(\gamma)}$

$$\langle C_{\vec{j}\, i_v} | C_{\vec{j}, i_v'} \rangle_{\mathrm{Inv}(V_{j_1} \cdots V_{j_6})} = \delta_{i_v i_v'}$$

eg γ partitions T^3



$$|n\rangle = |\psi_n\rangle$$

$T_{\gamma, \vec{j}, \vec{i}}$ orthogonal basis in $\mathcal{H}_\gamma$ hilbert space of gauge inv. wave functions.

called spin-network basis.

HW prove $\langle T_{\gamma,\vec{j},\vec{i}} | T_{\gamma,\vec{j}',\vec{i}'} \rangle_{\mathcal{H}_\gamma} = \prod_{e\in E(\gamma)} \frac{1}{2j_e+1} \delta_{\vec{j}\vec{j}'} \delta_{\vec{i},\vec{i}'}$

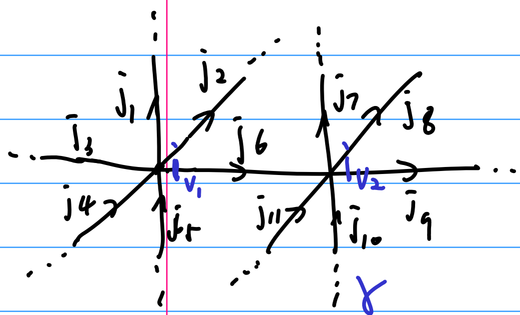
$$\langle f | f' \rangle_{\mathcal{H}^0_\gamma} = \int \prod_e d\mu_H(h_e) \overline{f(\vec{h})} f'(\vec{h})$$

spin-network : $(\gamma, \vec{j}, \vec{i})$ labels the basis in $\mathcal{H}_\gamma$ ← Hilbert space of LQG

↑ oriented graph whose edges e are colored by spins j_e

(lattice)

vertices v are colored by intertwiners i_v



(oriented graph = a set of oriented edges and vertices s.t. vertices are end points of edges)

Holonomy & flux operators $\underbrace{\hat{h}(e), \hat{p}^j(e)}_{\text{not gauge inv. operators}}$ acting on $\mathcal{H}^0_\gamma$

$$\hat{h}_{AB}(e) f(\vec{h}) := h_{AB}(e) f(\vec{h})$$

$$\hat{p}^j(e) f(\vec{h}) := i\frac{\ell_p^2}{2} \hat{R}^j_e f(\vec{h}) = i\frac{\ell_p^2}{2} \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} f(\cdots e^{\varepsilon\tau^j} h(e) \cdots)$$

$\ell_p^2 = \hbar\kappa$

$\kappa = 16\pi G_N$

↑ right-inv. vector field on $SU(2)$ acting only on $h(e)$

$$= i\frac{\ell_p^2}{2} \frac{\partial f}{\partial h(e)_{AB}} \left(\tau^j h(e)\right)_{AB} \qquad \tau^j = -i\sigma^j$$

on $SU(2)$ $\hat{R}^j f(h) = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} f(\underbrace{e^{\varepsilon\tau^j} h}_{\text{left translation.}})$ right inv vector field is generator of left translation.

$h \in SU(2)$

HW $[\hat{h}_{AB}(e)\ \hat{h}_{CD}(e')] = 0$

$$[\hat{p}^j(e)\ \hat{h}_{AB}(e')] = \underline{i\ell_p^2} \delta_{ee'} \left(\frac{\tau^j}{2} h(e')\right)_{AB}$$

quantum holonomy-flux algebra

$$[\hat{p}^j(e), \hat{p}^k(e')] = -i\, l_p^2\, \delta_{ee'}\, \varepsilon^{jkl}\, \hat{p}^l(e)$$

- they quantize the poisson algebra between $h(e)$ & $p^j(e)$ by $[\ ,\ ] = i\hbar \widehat{\{\ \ \}}$
- $e = e'$ analog of $[\hat{J}^j, \hat{J}^k] = i\,\varepsilon^{jkl}\,\hat{J}^l$

$\hat{p}^j \sim \hat{J}^j$, $f(h) \sim$ spin states.

here $[\hat{R}^j_e, \hat{R}^k_e] = -2\,\varepsilon^{jkl}\,\hat{R}^l$ $\quad -i\hat{R}^j/2 \equiv \hat{J}^j$

relates to $[\tau^j, \tau^k] = 2\,\varepsilon^{jkl}\,\tau^l$