boson no spin $T_0^2 = 1$ $T_0 \psi(t,\vec{r}) = \psi^*(-t,\vec{r})$

fermion spin $\frac{1}{2}$ $T^2 = -1$ $T\begin{pmatrix}\psi_1(t,\vec{r})\\ \psi_2(t,\vec{r})\end{pmatrix} = e^{i\varphi}\sigma_y\begin{pmatrix}\psi_1^*(-t)\\ \psi_2^*(-t)\end{pmatrix}$

zero spin $\hat{H}\psi_{nl} = E_n \psi_{nl}$ $l = 1..d(n)$

general sol of Schrödingen eqn $\psi(\vec{r},t) = \sum_n a_n \psi_{nl}(\vec{r}) e^{-\frac{i}{\hbar}E_n t}$

($\hat{H}$ is real, not explicitly dep. on t) $\hat{T}_0 \psi(\vec{r},t) = \sum_n a_n^* \psi_{nl}^*(\vec{r}) e^{-\frac{i}{\hbar}E_n t}$

effectively $\hat{T}_0 : \psi_{nl}(\vec{r}) \to \psi_{nl}^*(\vec{r})$

if $\hat{H}$ is real, $\hat{H}\psi_{nl} = E_n \psi_{nl}$ ψ_{nl} is eigenstate

$\Rightarrow \hat{H}\psi_{nl}^* = E_n \psi_{nl}^*$ ψ_{nl}^* is eigenstate

ψ_{nl} basis in $\mathcal{H}^{(n)}$, ψ_{nl}^* is basis in $\mathcal{H}^{(n)*}$

whether $\mathcal{H}^{(n)}$ and $\mathcal{H}^{(n)*}$ the same ?

(1) $\mathcal{H}^{(n)} \simeq \mathcal{H}^{(n)*}$ equivalent rep of symm. group

(2) $\mathcal{H}^{(n)} \not\simeq \mathcal{H}^{(n)*}$ not equivalent ....

eigenspace $= \mathcal{H}^{(n)} \oplus \mathcal{H}^{(n)*}$

real rep, psudo real rep, and complex rep

Given a symm. group G, unitary irrep $D: g \mapsto D(g) \circlearrowright \mathcal{H}^{(n)}$

Def $D(g)$ is unitary on $\mathcal{H}^{(n)}$, ψ_{nl} orthonormal basis in $\mathcal{H}^{(n)}$

(complex conjugate irrep)

$$D(g)\,\psi_{nl}(\vec{r}) = \sum_j \psi_{nj}(\vec{r})\, D_{ji}(g)$$

$D_{ji}(g)$ unitary matrix

complex conjugate : $D(g)^*\,\psi^*_{ni}(\vec{r}) = \sum_j \psi_{nj}(\vec{r})^*\, D_{ji}(g)^*$

$$\psi^*_{ni}(\vec{r}) = T_0\,\psi_{ni}(\vec{r})$$

$D_{ij}(g)$ unitary irrep matrix $\xrightarrow{T_0}$ $D^*_{ij}(g)$ unitary irrep matrix

(irrep D) irrep D^* : <u>complex conjugate irrep</u>

relation between D and D^*

(1) if $\exists$ unitary transf. $U : \mathcal{H}^{(n)} \to \mathcal{H}^{(n)}$ s.t

$$U\,D(g)\,U^{-1} = D_0(g) \qquad \forall g \in G$$

real matrix

$\Rightarrow$ <u>D is equivalent to D^*</u>, since $U^* D(g)^* U^{*-1} = D_0(g)$

$\Rightarrow$ $\underline{D^*(g)} = U^{*-1} D_0(g)\, U^* = \underline{U^{*-1} U\, D(g)\, U^{-1} U^*}$

$U^{*-1} U$ is unitary

we say $D \simeq D^*$ is a real rep.

(2) D is unitarily equivalent to D^*, but they are not equivalent to real rep.

$$\nexists\, U \text{ s.t. } D(g) = U D_0(g) U^{-1} \quad \forall g \in G$$

D_0 : real matrix

We say, $D \simeq D^*$ is pseudo-real

(3) D is inequivalent to D^*.

We say, D and D^* are complex reps

distinguish (3) from (1) and (2), check character

if $D \simeq D^*$, then $\chi(g) = \chi^*(g) \quad \forall g \in G$

if $\chi(g) \neq \chi^*(g)$, then D & D^* are complex reps.

distinguish between (1) & (2)

$$D^* \simeq D \rightarrow D^*(g) = Z D(g) Z^{-1} \quad \forall g \in G$$

Z : unitary matrix

Complex conjugate $D(g) = Z^* D^*(g) Z^{*-1}$

$$\Rightarrow D^*(g) = Z Z^* D^*(g) (Z Z^*)^{-1}$$

i.e. $[Z Z^*, D^*(g)] = 0 \quad \forall g \in G$

by Schur's lemma, when D^* is irrep, $ZZ^* = c\,\mathbb{1}$
$c \in \mathbb{C}$

Z is unitary, $Z^* = (Z^T)^{-1}$

$$\Rightarrow \quad Z(Z^T)^{-1} = c\,\mathbb{1} \Rightarrow \underline{Z = c Z^T}$$

$$\xrightarrow[\text{transpose}]{} Z^T = cZ$$

$$\Rightarrow \quad Z = c^2 Z \Rightarrow c^2 = 1, \quad c = \pm 1$$

$$D^* \simeq D \Rightarrow Z Z^* = \pm \mathbb{1}$$

$\underline{\text{Thm}}$: $ZZ^* = \mathbb{1}$ iff D is real

$ZZ^* = -\mathbb{1}$ iff D is pseudo-real

Pf if D is real $\rightarrow$ $D^*(g) = (U^*)^{-1} D_0(g)\, U^*$

$$= (U^*)^{-1} U D(g) U^{-1} U^*$$

on the other hand $D^*(g) = Z D(g) Z^{-1}$

$$\Rightarrow (U^*)^{-1} U = b Z \qquad b \in \mathbb{C}$$

Complex conjugate $U^{-1} U^* = b^* Z^*$

$$U^{-1}U^*(U^*)^{-1}U = b^* b Z^* Z = |b|^2 Z^* Z$$

$$\begin{array}{c} \| \\ 1 \end{array} \qquad \underline{Z^* Z = |b|^{-2}\,\mathbb{1}}$$

$$\underline{Z Z^* = c\mathbb{1}} \quad \Rightarrow \quad |b|^2 c = 1$$

$$c = 1 \text{ or } -1 \qquad c = 1 \quad |b|^2 > 0$$

conversely if $c = 1$ $\quad (Z Z^* = \mathbb{1})$

Z unitary, $Z = e^{iA}$ $\quad A$ hermitian

$$Z^* = Z^{-1} \Leftrightarrow Z^T = Z, \quad Z \text{ symmetric}$$

$$Z^* = (Z^T)^{-1}$$

def. $U = Z^{\frac{1}{2}} = e^{iA/2}$ unitary

$$U^2 = Z$$

$$\underline{U^* Z = e^{-iA/2} e^{iA} = e^{\frac{1}{2} iA} = U}$$

$$\forall g \in G, \ [U D(g) U^{-1}]^* = U^* D(g)^* U^{*-1}$$

$$= U^* Z D(g) Z^{-1} U^{*-1}$$

$$= U D(g) U^{-1} = D_0(g) \text{ real}$$

$\Rightarrow D$ is real rep.

$Z Z^* = \mathbb{1} \Leftrightarrow D$ is real rep

$\Rightarrow Z Z^* = -\mathbb{1} \Leftrightarrow D$ is psudo-real rep. $\square$

Distinguish $\underbrace{\text{real \& pseudo-real}}$ reps by characters

$$\underline{D^*(g) = Z\, D(g)\, Z^{-1}}$$

orthogonality $\displaystyle\sum_g D^*_{\alpha\delta}(g)\, D_{\beta\gamma}(g) = \delta_{\alpha\beta}\,\delta_{\gamma\delta}\,\frac{h}{\dim(D)=d}$

$$\parallel$$

$$\sum_{\alpha,\delta} Z^{-1}_{\tau\alpha} \times \sum_g \sum_{\sigma,\rho} Z_{\alpha\sigma} D_{\sigma\rho}(g)\, Z^{-1}_{\rho\delta}\, D_{\beta\gamma}(g) \times Z_{\delta\chi}$$

$$\Rightarrow \sum_g D_{\tau\chi}(g)\, D_{\beta\gamma}(g) = Z_{\gamma\chi}\, Z^{-1}_{\tau\beta}\,\frac{h}{d}$$

let $\chi = \beta$. $\sum_\beta$, $\displaystyle\sum_g \sum_\beta D_{\tau\beta}(g)\, D_{\beta\gamma}(g)$

$$\parallel$$

$$\sum_g D_{\tau\gamma}(g^2) = \left(Z\, Z^*\right)_{\tau\gamma}\frac{h}{d}$$

$$= \pm\,\delta_{\tau\gamma}\,\frac{h}{d}$$

let $\tau = \gamma$, $\sum_\tau$,

$$\boxed{\sum_g \chi(g^2) = \pm h}$$

for complex rep. $D \neq D^*$ $\Rightarrow$ $\displaystyle\sum_g \left[D^*_{\alpha\beta}(g)\right]^* D_{\gamma\delta}(g) = 0$

$$\parallel$$

$$\sum_g D_{\alpha\beta}(g) D_{\gamma\delta}(g)$$

$$\Rightarrow \sum_g \chi(g^2) = 0$$

summary : $\frac{1}{h} \sum_{g \in G} \chi(g^2) = \begin{cases} 1 & \text{real rep} \\ -1 & \text{psudo-real rep} \\ 0 & \text{complex.} \end{cases}$

Extra-degeneracy of $\hat{H}$ due to time-reversal inv.

$\hat{H} \psi_{nl} = E_n \psi_{nl} \quad l = 1 \dots d$

$\psi_{nl} \in \mathcal{H}^{(n)}$ unitary irrep. of symmetry group G

$l = 1 \cdots d \qquad \dim \mathcal{H}^{(n)} = d$

find degeneracy at E_n , deg. at E_n is either d or $2d$ (spin-zero)

if $\hat{H}$ is real and not explicitly dep on t

$\hat{H} \psi_{nl} = E_n \psi_{nl}$

$\hat{H} \psi_{nl}^* = E_n \psi_{nl}^* \qquad \psi^*_{nl}(\vec{r}) = \hat{T}_0 \psi_{nl}(\vec{r})$

- if $\forall \psi_{nl}^*$, $\psi^*_{nl} \in \mathcal{H}^{(n)}$ spanned by ψ_{nl}

then $\mathcal{H}^{(n)}$ is the final eigenspace i.e. $\mathcal{H}^{(n)} = \mathcal{H}^{(n)*}$

degeneracy at E_n is d

- otherwise, eigenspace $= \mathcal{H}^{(n)} \oplus \mathcal{H}^{(n)*}$

  degeneracy at $E_n = 2d$ (extra degeneracy)