$$\text{(diagram)} = -4g^2\, i n_f\, C(N)\, \delta^{ab} \int_0^1 dx \underbrace{\int \frac{d^d l_E}{(2\pi)^d}}_{\int d|l_E| \int d\Omega} \frac{-\overset{\frac{1}{2}}{\overset{\shortparallel}{\frac{2}{d}}} g^{\mu\nu} l_E^2 + g^{\mu\nu} l_E^2 - 2x(1-x) q^\mu q^\nu + g^{\mu\nu}(m^2 + x(1-x)q^2)}{(l_E^2+\Delta)^2}$$

- $$\int \frac{d^d l_E}{(2\pi)^d} \frac{1}{(l_E^2+\Delta)^2} = \int \frac{d\Omega_d}{(2\pi)^d} \int_0^\infty d l_E \frac{l_E^{d-1}}{(l_E^2+\Delta)^2} \qquad \int d\Omega_d = \frac{2\pi^{d/2}}{\Gamma(d/2)}$$

$$= \frac{2\pi^{d/2}}{\Gamma(d/2)} \frac{1}{(2\pi)^d} \underbrace{\frac{1}{2}\int_0^\infty d l_E^2 \frac{(l_E^2)^{\frac{d}{2}-1}}{(l_E^2+\Delta)^2}}_{\frac{1}{2}\left(\frac{1}{\Delta}\right)^{2-\frac{d}{2}} \int_0^1 dy\, y^{1-\frac{d}{2}} (1-y)^{\frac{d}{2}-1}} \qquad y = \frac{\Delta}{l_E^2+\Delta}$$

$$\int_0^1 dy\, y^{\alpha-1}(1-y)^{\beta-1} = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

$$= \frac{1}{(4\pi)^{d/2}} \frac{\Gamma(2-\frac{d}{2})}{\Gamma(2)} \left(\frac{1}{\Delta}\right)^{2-\frac{d}{2}}$$

$d = 4-\varepsilon$

log - div.

$\Gamma(2-\frac{d}{2})$ has pole at $d=4$

- $$\int \frac{d^d l_E}{(2\pi)^d} \frac{l_E^2}{(l_E^2+\Delta)^2} = \frac{d/2}{(4\pi)^{d/2}} \frac{\Gamma(1-\frac{d}{2})}{\Gamma(2)} \left(\frac{1}{\Delta}\right)^{1-\frac{d}{2}}$$

$\Gamma(1-\frac{d}{2})$ has pole at $d=2$

$\to \Lambda^2$-div.

$$\Gamma\left(\frac{\varepsilon}{2}\right) = \frac{2}{\varepsilon} - \gamma + O(\varepsilon)$$

$$\left(-\frac{2}{d}+1\right) g^{\mu\nu} \int \frac{d^d l_E}{(2\pi)^d} \frac{l_E^2}{(l_E^2+\Delta)^2} = \frac{-1}{(4\pi)^{d/2}} \frac{(1-\frac{d}{2})\Gamma(1-\frac{d}{2})}{\Gamma(2)} \left(\frac{1}{\Delta}\right)^{1-\frac{d}{2}} g^{\mu\nu}$$

$$(1-\tfrac{d}{2})\Gamma(1-\tfrac{d}{2}) = \Gamma(2-\tfrac{d}{2}) \qquad = \frac{1}{(4\pi)^{d/2}} \frac{\Gamma(2-\frac{d}{2})}{\Gamma(2)} \left(\frac{1}{\Delta}\right)^{2-\frac{d}{2}} (-\Delta g^{\mu\nu})$$

pole at $d=4 \to$ log - div.

$$\text{(diagram)} = -4 i g^2 n_f\, C(N)\, \delta^{ab} \int_0^1 dx \frac{1}{(4\pi)^{d/2}} \frac{\Gamma(2-\frac{d}{2})}{\Gamma(2)} \left(\frac{1}{\Delta}\right)^{2-d/2} \times$$

$$\left[-g^{\mu\nu}(\underline{m^2} - x(1-x)q^2) + g^{\mu\nu}(\underline{m^2} + x(1-x)q^2) - 2x(1-x) q^\mu q^\nu \right]$$

$$-\Delta = x(1-x)q^2 - m^2$$

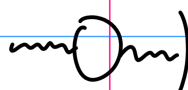
$$= i\left(g^{\mu\nu}q^2 - q^\mu q^\nu\right)\frac{-g^2}{(4\pi)^{d/2}} C(N)\, n_f\, \delta^{ab}\, \Gamma\left(2-\tfrac{d}{2}\right)\int_0^1 dx \frac{8(1-x)x}{(m^2 - x(1-x)q^2)^{2-\frac{d}{2}}}$$

$d = 4-\varepsilon$ $\quad \Delta^{2-\frac{d}{2}} = \Delta^{\frac{\varepsilon}{2}} = e^{\frac{\varepsilon}{2}\log\Delta} = 1 + \frac{\varepsilon}{2}\log\Delta + O(\varepsilon^2)$

$$= i\left(g^{\mu\nu}q^2 - q^\mu q^\nu\right)\frac{-g^2}{(4\pi)^2} C(N)\, n_f\, \delta^{ab}\left(\frac{2}{\varepsilon} - \gamma + O(\varepsilon)\right)$$

$$\cdot \int_0^1 dx\, 8x(1-x)\left(1 - \frac{\varepsilon}{2}\log\Delta + O(\varepsilon^2)\right)$$

$$= i\left(g^{\mu\nu}q^2 - q^\mu q^\nu\right)\underbrace{\frac{-g^2}{(4\pi)^2} C(N)\, n_f\, \delta^{ab}\int_0^1 dx\, 8x(1-x)\left(\frac{2}{\varepsilon} - \gamma - \log\Delta + O(\varepsilon)\right)}_{\Pi(q^2)}$$

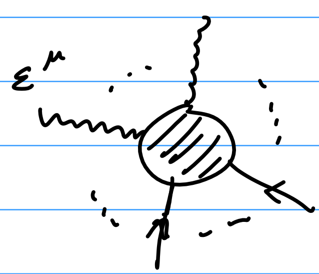
$\left(\right.$$\left.\right)^{\mu\nu} = \left(g^{\mu\nu}q^2 - q^\mu q^\nu\right)\cdot i\,\Pi(q^2)$ Lorentz covariant.

$q_\mu\left(\right.$$\left.\right)^{\mu\nu} = 0$ $\qquad \left(g^{\mu\nu}q^2 - q^\mu q^\nu\right)q_\mu = \left(q^\nu q^2 - q^2 q^\nu\right)$

example of Ward-Takahashi identity. $\qquad = 0$

(consequence of gauge inv.)

in general,



$= \underline{\varepsilon^\mu(q)}\cdot M_\mu(q \ldots)$

gauge transf. $A_\mu^a \to A_\mu^a + D_\mu \chi^a$

$\sim A_\mu^a + \partial_\mu \chi^a + O(g)$

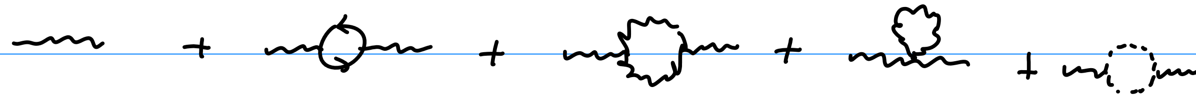
$A_\mu J^\mu$

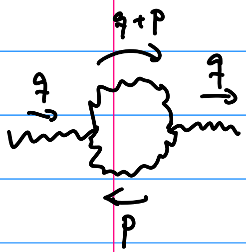
$\varepsilon_\mu^a(q) \sim \varepsilon_\mu^a(q) + q_\mu \chi^a(q)$

scattering amplitude has to be gauge inv.

$$\longrightarrow \varepsilon^\mu M_\mu \sim \varepsilon^\mu M_\mu + \overbrace{q^\mu M_\mu \chi} \qquad \forall\, \chi(q)$$

we must have $q^\mu M_\mu = 0$ ← Ward-Takahashi identity

$\langle TAA \rangle =$ 



$$= \frac{g^2}{2} \underbrace{f^{acd} f^{bcd}} \int \frac{d^4p}{(2\pi)^4} \frac{-i}{p^2} \frac{-i}{(p+q)^2} N^{\mu\nu}$$

symm factor

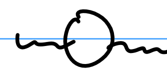
$$N^{\mu\nu} = \left[g^{\mu\rho}(q-p)^\sigma + g^{\rho\sigma}(2p+q)^\mu + g^{\sigma\mu}(-p-2q)^\rho \right]$$
$$\left[\delta^\nu_\rho (p-q)_\sigma + g_{\rho\sigma}(-2p-q)^\nu + \delta^\nu_\sigma (p+2q)_\rho \right]$$

HW derive this ↗

$G = SU(N)$

$$f^{acd} f^{bcd} = C_2(G)\, \delta^{ab}$$

$SU(N)$ $\quad C_2(G) = N$



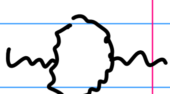
$[T^a, T^b] = i f^{abc} T^c$ $\quad$ fix normalization of T^a s.t. $tr_N(T^a T^b) = \frac{1}{2}\delta^{ab}$

$C(N) = \frac{1}{2}$ $\qquad C_2(N) = \frac{N^2-1}{2N}$

$\sum_{a=1}^{\dim G} T^a T^a = T^2 \propto 1$

$[T^2, T^a] = 0$

$T^2 = C_2(N)\, \mathbb{1}_{N\times N}$

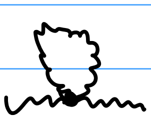


$$= \frac{i g^2}{(4\pi)^{d/2}} C_2(G)\, \delta^{ab} \int_0^1 dx \frac{1}{\Delta^{2-d/2}} x$$
$$\Big(\Gamma(1-\tfrac{d}{2})\, g^{\mu\nu} q^2 \left[\tfrac{3}{2}(d-1)\, x(1-x)\right]$$
$$+ \Gamma(2-\tfrac{d}{2})\, g^{\mu\nu} q^2 \left[\tfrac{1}{2}(2-x)^2 + \tfrac{1}{2}(1+x)^2\right]$$
$$- \Gamma(2-\tfrac{d}{2})\, q^\mu q^\nu \left[(1-\tfrac{d}{2})(1-2x)^2 + (1+x)(2-x)\right] \Big)$$

pole at $d=2$

quadratic div. break Ward identity

pole at $d=2$ cannot cancel



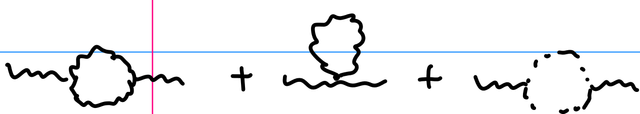
$$= \frac{i g^2}{(4\pi)^{d/2}} C_2(G)\, \delta^{ab} \int_0^1 dx \frac{1}{\Delta^{2-d/2}} \Big(-\Gamma(1-\tfrac{d}{2})\, g^{\mu\nu} q^2 \left[\tfrac{1}{2} d(d-1)\, x(1-x)\right]$$
$$- \Gamma(2-\tfrac{d}{2})\, g^{\mu\nu} q^2 \left[(d-1)(1-x)^2\right] \Big)$$

 $= \frac{i g^2}{(4\pi)^{d/2}} C_2(G)\, \delta^{ab} \int_0^1 dx\, \frac{1}{\Delta^{2-d/2}} \Big(-\Gamma(1-\frac{d}{2})\, g^{\mu\nu} q^2 [\frac{1}{2} x(1-x)]$

$+ \Gamma(2-\frac{d}{2})\, q^\mu q^\nu [x(1-x)] \Big)$

pole at $d=2$ $\quad \underline{\Gamma(1-\frac{d}{2})}\, q^2 g^{\mu\nu} x(1-x) \underbrace{\left[\frac{3}{2} d - \frac{3}{2} - \frac{1}{2} d^2 + \frac{d}{2} - \frac{1}{2} \right]}_{\underline{(1-\frac{d}{2})}(d-2)}$

$$(1-\frac{d}{2})\,\Gamma(1-\frac{d}{2}) = \Gamma(2-\frac{d}{2})$$



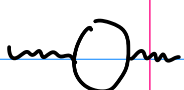
$= i(g^{\mu\nu} q^2 - q^\mu q^\nu)\, \delta^{ab} \frac{g^2}{(4\pi)^{d/2}} C_2(G)\, \Gamma(2-\frac{d}{2}) \int_0^1 dx\, \frac{1}{\Delta^{2-d/2}} \left[(1-\frac{d}{2})(1-2x)^2 + 2 \right]$

$\Delta = m^2 - x(1-x) q^2$

$\Delta^{2-\frac{d}{2}} = 1 + \frac{\varepsilon}{2} \log \Delta + O(\varepsilon^2)$

$\Gamma(2-\frac{d}{2}) = \frac{2}{\varepsilon} - \gamma + O(\varepsilon)$

$= i(g^{\mu\nu} q^2 - q^\mu q^\nu)\, \delta^{ab} \frac{g^2}{(4\pi)^{d/2}} C_2(G) \quad \left[\frac{5}{3} (\frac{2}{\varepsilon}) + \text{finite} \right]$

 $= i(g^{\mu\nu} q^2 - q^\mu q^\nu)\, \delta^{ab} \frac{g^2}{(4\pi)^{d/2}} C(N)\, n_f \left[(-\frac{4}{3})(\frac{2}{\varepsilon}) + \text{finite} \right]$

$(\text{total})^{\mu\nu} = i(g^{\mu\nu} q^2 - q^\mu q^\nu)\, \delta^{ab} \frac{g^2}{(4\pi)^{d/2}} \underline{\left(\frac{5}{3} C_2(G) - \frac{4}{3} n_f C(N) \right)} \left[\frac{2}{\varepsilon} + \text{finite} \right]$

$\Rightarrow \quad q_\mu (\text{total})^{\mu\nu} = 0 \quad$ ward identity. (gauge inv. or massless gluon)

if there was any Λ^2-div.

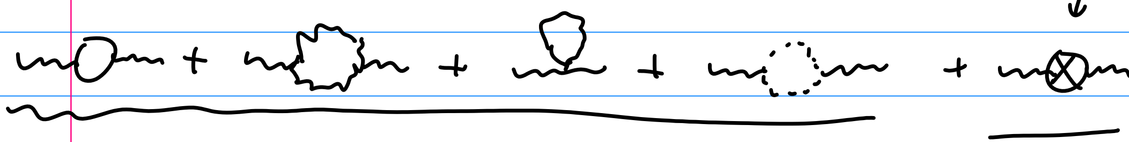
$\Lambda^2 \Sigma \qquad \Lambda^2 \Sigma \quad \Lambda^2 \Sigma$

$$\frac{1}{p^2} + \frac{1}{p^2} \Lambda^2 \Sigma \frac{1}{p^2} + \frac{1}{p^2} \Lambda^2 \Sigma \frac{1}{p^2} \Lambda^2 \Sigma \frac{1}{p^2} + \cdots = \frac{1}{p^2 + \Lambda^2 \Sigma}$$

gluon mass
break gauge inv.

perturbative renormalization

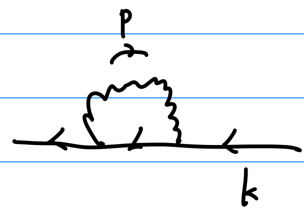
from counter term

 = finite

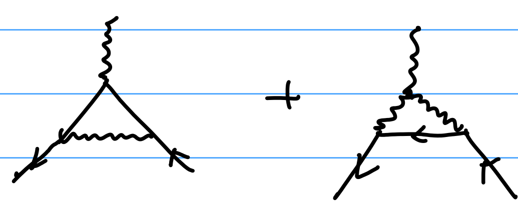
 $= -i\left(q^2 g^{\mu\nu} - q^\mu q^\nu\right)\delta^{ab}\,\delta_3$

$$\delta_3 \sim \frac{g^2}{(4\pi)^{d/2}}\left(\frac{5}{3}C_2(G) - \frac{4}{3}n_f C(N)\right)\left(\frac{2}{\varepsilon} + \text{finite}\right)$$

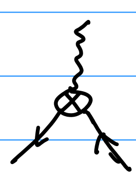
There are other loop diagrams

"quark self energy"  $= \frac{i g^2}{(4\pi)^{d/2}} C_2(N)\,\not{k}\int_0^1 dx\,(1-x)(d-2)\underbrace{\frac{\Gamma(2-\frac{d}{2})}{\Delta^{2-d/2}}}_{\frac{2}{\varepsilon}-\gamma-\log\Delta+O(\varepsilon)}$

 $= i\not{k}\,\delta_2$

"vertex correction"  $= ig\gamma^\mu T^a\left(\frac{g^2}{(4\pi)^{d/2}}\left[C_2(N)+C_2(G)\right]\right)$

$$\cdot\left(\frac{2}{\varepsilon} - \gamma - \log\Delta + O(\varepsilon)\right)$$

 $= ig\gamma^\mu T^a\,\delta_1$

Thm We only need finitely many local counter-term to make QCD or QED finite to all orders in g or e.

(see Ryder's book)