boson no spin $T_0^2 = 1$ $T_0 \psi(t,\vec{r}) = \psi^*(-t,\vec{r})$

fermion spin $\frac{1}{2}$ $T^2 = -1$ $T \begin{pmatrix} \psi_1(t,\vec{r}) \\ \psi_2(t,\vec{r}) \end{pmatrix} = e^{i\varphi} \sigma_y \begin{pmatrix} \psi_1^*(-t) \\ \psi_2^*(-t) \end{pmatrix}$

zero spin $\hat{H} \psi_{nl} = E_n \psi_{nl}$ $l = 1 .. d(n)$

general sol of Schrödingen eqn $\psi(\vec{r},t) = \sum_n a_n \psi_{nl}(\vec{r}) e^{-\frac{i}{\hbar}E_n t}$

($\hat{H}$ is real, not explicitly dep. on t) $\hat{T}_0 \psi(\vec{r},t) = \sum_n a_n^* \psi_{nl}^*(\vec{r}) e^{-\frac{i}{\hbar}E_n t}$

effectively $\hat{T}_0 : \psi_{nl}(\vec{r}) \to \psi_{nl}^*(\vec{r})$

if $\hat{H}$ is real, $\hat{H} \psi_{nl} = E_n \psi_{nl}$ ψ_{nl} is eigenstate

$\Rightarrow \hat{H} \psi_{nl}^* = E_n \psi_{nl}^*$ ψ_{nl}^* is eigenstate

ψ_{nl} basis in $\mathcal{H}^{(n)}$, ψ_{nl}^* is basis in $\mathcal{H}^{(n)*}$

whether $\mathcal{H}^{(n)}$ and $\mathcal{H}^{(n)*}$ <u>the same</u> ?

(1) $\mathcal{H}^{(n)} \simeq \mathcal{H}^{(n)*}$ equivalent rep of symm. group

(2) $\mathcal{H}^{(n)} \not\simeq \mathcal{H}^{(n)*}$ not equivalent ....

$$\text{eigenspace} = \mathcal{H}^{(n)} \oplus \mathcal{H}^{(n)*}$$

<u>real rep, psudo real rep, and complex rep</u>

Given a symm. group G, unitary irrep $D: g \mapsto D(g) \circlearrowright \mathcal{H}^{(n)}$

Def $D(g)$ is unitary on $\mathcal{H}^{(n)}$, ψ_{nl} orthonormal basis in $\mathcal{H}^{(n)}$

(complex conjugate irrep)

$$D(g)\psi_{nl}(\vec{r}) = \sum_j \psi_{nj}(\vec{r}) D_{ji}(g)$$

$D_{ji}(g)$ unitary matrix

Complex conjugate : $D(g)^* \psi^*_{ni}(\vec{r}) = \sum_j \psi_{nj}(\vec{r})^* D_{ji}(g)^*$

$$\psi^*_{ni}(\vec{r}) = T_0 \psi_{ni}(\vec{r})$$

$D_{ij}(g)$ unitary irrep matrix $\xrightarrow{T_0}$ $D^*_{ij}(g)$ unitary irrep matrix

(irrep D) irrep D^* : <u>complex conjugate irrep</u>

relation between D and D^*

(1) if $\exists$ unitary transf. $U : \mathcal{H}^{(n)} \to \mathcal{H}^{(n)}$ s.t

$$U D(g) U^{-1} = D_0(g) \quad \forall g \in G$$

real matrix

$\Rightarrow$ <u>D is equivalent to D^*</u>, since $U^* D(g)^* U^{*-1} = D_0(g)$

$\Rightarrow$ $\underline{D^*(g)} = U^{*-1} D_0(g) U^* = \underline{U^{*-1} U D(g) U^{-1} U^*}$

$U^{*-1}U$ is unitary

we say $D \simeq D^*$ is a real rep.

(2) D is $\underbrace{\text{equivalent}}_{\text{unitarily}}$ to D^*, but they are not equivalent to real rep.

$$\not\exists\, U \text{ s.t. } D(g) = U \underset{\uparrow \text{ real matrix}}{D_0(g)} U^{-1} \quad \forall g \in G$$

We say, $D \simeq D^*$ is pseudo-real

(3) D is inequivalent to D^*.

We say, D and D^* are complex reps

distinguish (3) from (1) and (2), check character

if $D \simeq D^*$, then $\chi(g) = \chi^*(g) \quad \forall g \in G$

if $\chi(g) \neq \chi^*(g)$, then D & D^* are complex reps.

distinguish between (1) & (2)

$$D^* \simeq D \rightarrow D^*(g) = \underset{\uparrow \text{ unitary matrix}}{Z} D(g) Z^{-1} \quad \forall g \in G$$

Complex conjugate $D(g) = Z^* D^*(g) Z^{*-1}$

$$\Rightarrow D^*(g) = Z Z^* D^*(g) (Z Z^*)^{-1}$$

i.e. $[Z Z^*, D^*(g)] = 0 \quad \forall g \in G$

by Schur's lemma, when D^* is irrep, $ZZ^* = c\mathbb{1}$
$c \in \mathbb{C}$

Z is unitary, $Z^* = (Z^T)^{-1}$

$\Rightarrow \quad Z(Z^T)^{-1} = c\mathbb{1} \Rightarrow \underline{Z = cZ^T}$

$\xrightarrow[\text{transpose}]{} Z^T = cZ$

$\Rightarrow \quad Z = c^2 Z \quad \Rightarrow \quad c^2 = 1, \quad c = \pm 1$

$D^* \simeq D \quad \Rightarrow \quad ZZ^* = \pm\mathbb{1}$

<u>Thm</u>: $ZZ^* = \mathbb{1}$ iff D is real

$ZZ^* = -\mathbb{1}$ iff D is pseudo-real

<u>Pf</u> if D is real $\rightarrow$ $D^*(g) = (U^*)^{-1} D_0(g) U^*$

$= (U^*)^{-1} U D(g) U^{-1} U^*$

on the other hand $D^*(g) = Z D(g) Z^{-1}$

$\Rightarrow \quad (U^*)^{-1} U = bZ \quad b \in \mathbb{C}$

Complex conjugate $U^{-1} U^* = b^* Z^*$

$$U^{-1} U^* (U^*)^{-1} U = b^* b Z^* Z = |b|^2 Z^* Z$$

$$\begin{array}{c} \| \\ 1 \end{array} \qquad \underline{Z^* Z = |b|^{-2} \mathbb{1}}$$

$$\underline{Z Z^* = c\,\mathbb{1}} \quad \Rightarrow \quad |b|^2 c = 1$$

$$c = 1 \text{ or } -1 \qquad c = 1 \quad |b|^2 > 0$$

conversely if $c = 1$ $\quad (Z Z^* = \mathbb{1})$

$$Z \text{ unitary}, \quad Z = e^{iA} \quad A \text{ hermitian}$$

$$Z^* = Z^{-1} \Leftrightarrow Z^T = Z, \quad Z \text{ symmetric}$$

$$Z^* = (Z^T)^{-1}$$

$$\text{def. } U = Z^{\frac{1}{2}} = e^{iA/2} \quad \text{unitary}$$

$$U^2 = Z$$

$$\underline{U^* Z = e^{-iA/2} e^{iA} = e^{\frac{1}{2} iA} = U}$$

$$\forall g \in G, \ [U D(g) U^{-1}]^* = U^* D(g)^* U^{*-1}$$

$$= U^* Z D(g) Z^{-1} U^{*-1}$$

$$= U D(g) U^{-1} = D_0(g) \text{ real}$$

$$\Rightarrow D \text{ is real rep.}$$

$$Z Z^* = \mathbb{1} \Leftrightarrow D \text{ is real rep}$$

$$\Rightarrow Z Z^* = -\mathbb{1} \Leftrightarrow D \text{ is psudo-real rep.} \qquad \square$$

Distinguish $\underbrace{\text{real \& pseudo-real}}$ reps by characters

$$\underline{D^*(g) = Z\, D(g)\, Z^{-1}}$$

orthogonality $\sum_g D^*_{\alpha\delta}(g)\, D_{\beta\gamma}(g) = \delta_{\alpha\beta}\,\delta_{\gamma\delta}\,\dfrac{h}{\dim(D)=d}$

$$\|$$

$$\sum_{\alpha,\delta} \underline{Z_{\tau\alpha}^{-1}} \times \sum_g \sum_{\sigma,\rho} \underline{Z_{\alpha\sigma}}\, D_{\sigma\rho}(g)\, Z^{-1}_{\rho\delta}\, D_{\beta\gamma}(g) \times Z_{\delta\chi}$$

$$\Rightarrow \sum_g D_{\tau\chi}(g)\, D_{\beta\gamma}(g) = Z_{\gamma\chi}\, Z^{-1}_{\tau\beta}\, \frac{h}{d}$$

let $\chi = \beta$. $\sum_\beta$, $\sum_g \sum_\beta D_{\tau\beta}(g)\, D_{\beta\gamma}(g)$

$$\|$$

$$\sum_g D_{\tau\gamma}(g^2) = (Z\, Z^*)_{\tau\gamma}\, \frac{h}{d}$$

$$= \pm\, \delta_{\tau\gamma}\, \frac{h}{d}$$

let $\tau = \gamma$, $\sum_\tau$,

$$\boxed{\sum_g \chi(g^2) = \pm\, h}$$

for complex rep. $D \neq D^*$ $\Rightarrow \sum_g \left[D^*_{\alpha\beta}(g)\right]^* D_{\gamma\delta}(g) = 0$

$$\|$$

$$\sum_g D_{\alpha\beta}(g) D_{\gamma\delta}(g)$$

$$\Rightarrow \sum_g \chi(g^2) = 0$$

summary : $$\frac{1}{h} \sum_{g \in G} \chi(g^2) = \begin{cases} 1 & \text{real rep} \\ -1 & \text{psudo-real rep} \\ 0 & \text{complex.} \end{cases}$$

Extra-degeneracy of $\hat{H}$ due to time-reversal inv.

$$\hat{H} \psi_{nl} = E_n \psi_{nl} \quad l = 1 \cdots d$$

$\psi_{nl} \in \mathcal{H}^{(n)}$ unitary irrep. of symmetry group G

$l = 1 \cdots d$ $\quad \dim \mathcal{H}^{(n)} = d$

find degeneracy at E_n, deg. at E_n is either d or $2d$ (spin-zero)

if $\hat{H}$ is real and not explicitly dep on t

$$\hat{H} \psi_{nl} = E_n \psi_{nl}$$

$$\hat{H} \psi_{nl}^* = E_n \psi_{nl}^* \qquad \psi_{nl}^*(\vec{r}) = \hat{T}_0 \psi_{nl}(\vec{r})$$

- if $\forall \psi_{nl}^*$, $\psi_{nl}^* \in \mathcal{H}^{(n)}$ spanned by ψ_{nl}

then $\mathcal{H}^{(n)}$ is the final eigenspace i.e. $\mathcal{H}^{(n)} = \mathcal{H}^{(n)*}$

degeneracy at E_n is d

- otherwise, eigenspace $= \mathcal{H}^{(n)} \oplus \mathcal{H}^{(n)*}$

degeneracy at $E_n = 2d$ (extra degeneracy)

<u>spin-zero</u>

$$\hat{H} \psi_i = E \psi_i \qquad \psi_i \in \underline{\underline{\mathcal{H}}}^{(n)}$$

$$D(g) \psi_i = \sum_j \psi_j D_{ji}(g) \quad \forall g \in G$$

$\hat{H}$ is real

$$\hat{H} \psi_i^* = E \psi_i^* \qquad \psi_i^* \in \mathcal{H}^{(n)*}$$

$$D^*(g) \psi_i^* = \sum_j \psi_j^* D_{ji}(g)^* \quad \forall g \in G$$

$\mathcal{H}^{(n)}$ carries irrep D of G

$\mathcal{H}^{(n)*}$ carries complex conjugate irrep D^* of G

$\mathcal{H}^{(n)} \simeq \mathcal{H}^{(n)*}$ or not relates to $D \simeq D^*$ or not

firstly if $D \not\simeq D^*$ complex irrep, $\mathcal{H}^{(n)} \perp \mathcal{H}^{(n)*}$ by orthogonality theorem.

$\Rightarrow$ extra degeneracy $2d$

Let's look at case (1) real rep and (2) pseudo-real rep.

$\exists$ unitary Z, $D^*(g) = Z D(g) Z^{-1}$, $ZZ^* = \begin{cases} 1 & \text{real} \\ -1 & \text{pseudo-real} \end{cases}$

<u>Lemma</u> if $\mathcal{H}^{(n)} = \mathcal{H}^{(n)*}$, then D is of case (1) i.e. real rep.

pf. $\underline{\underline{\mathcal{H}^{(n)} = \mathcal{H}^{(n)*}}}$, $\hat{H}\psi_i = E\psi_i$

$\hat{H}\psi_i^* = E\psi_i^*$

both $\{\psi_i\}$, $\{\psi_i^*\}$ are both orthonormal basis

then $\exists$ unitary U s.t. $\psi_i = \sum_k \psi_k^* U_{ki}$

$\psi_i^* = \sum_k \psi_k U_{ki}^*$

$\Rightarrow \psi_i = \sum_{kl} \psi_l U_{lk}^* U_{ki}$ i.e. $UU^* = \mathbb{1}$

$D(g)\psi_i = \sum_j \psi_j D_{ji}(g)$

$\| \qquad\qquad \|$

$D(g) \sum_k \psi_k^* U_{ki} \qquad \sum_j \sum_k \psi_k^* U_{kj} D_{ji}(g)$

$\times U^{-1}_{il}, \sum_i$

$D(g)\psi_l^* = \sum_k \psi_k^* (U D(g) U^{-1})_{kl}$

$\uparrow$

same as $D^*(g)\psi_l^*$

Compare to $D^*(g)\psi_l^* = \sum_j \psi_j^* D_{jl}^*(g)$

$D^*(g) = U D(g) U^{-1}$ $\}$ $\Rightarrow$ D is real

$$U U^* = 1 \qquad \rfloor$$

$\square$

$$U = Z$$

<u>Lemma</u> if D is real, then $\mathcal{H}^{(n)} = \mathcal{H}^{(n)*}$

pf D is real, $\underline{Z D Z^{-1} = D^*}$ & $Z Z^* = 1$

$$D^*(g)\psi_i^* = \sum_j \psi_j D^*_{ji}(g)$$

$$= \sum_j \psi_j \sum_{k,l} Z_{jk} D_{kl}(g) Z^{-1}_{li}$$

$\times Z_{im}$ and $\sum_i$

$$\Rightarrow D^*(g)\Big(\underline{\underline{\sum_i \psi_i^* Z_{im}}}\Big) = \sum_k \Big(\underline{\underline{\sum_j \psi_j^* Z_{jk}}}\Big)\underline{\underline{D_{km}(g)}}$$

$$D(g)\psi_m = \sum_k \psi_k D_{km}(g)$$

$$\Rightarrow \sum_i \psi_i^* Z_{im} = \psi_m$$

$$\underbrace{\psi_i^* \in \mathcal{H}^{(n)*} \qquad \psi_i}_{} \in \mathcal{H}^{(n)}$$

they are linear dep. by $\sum_i \psi_i^* Z_{\underline{\underline{im}}} = \psi_m$

$$\Rightarrow \mathcal{H}^{(n)} = \mathcal{H}^{(n)*}$$

above 2 lemmas $\Rightarrow$ $\mathcal{H}^{(n)} = \mathcal{H}^{(n)*}$ iff D is real

(no extra degeneracy) d

then if D is psudo-real then $\mathcal{H}^{(n)} \neq \mathcal{H}^{(n)*}$

$\Rightarrow$ extra degeneracy $2d$

$$\text{if } D \text{ is } \begin{cases} \text{real} & \text{degeneracy} = d = \dim(D) \\ \text{psudo-real} & \text{degeneracy} = 2d \\ \text{complex} & \text{degeneracy} = 2d \end{cases}$$

<u>Examples</u> (1) 1d free particle $\hat{H} = \frac{1}{2m}\hat{p}^2$

$$\hat{p} = -i\hbar \frac{\partial}{\partial x}$$

symmetry : transl. inv. $Q(\lambda)\, x = x + \lambda \qquad \lambda \in \mathbb{R}$

$$D(\lambda) = e^{-\frac{i}{\hbar}\lambda \hat{p}}$$

$$D(\lambda)\psi(x) = \psi(x+\lambda)$$

$G = \mathbb{R} = \{\lambda\}$ group multiplication : $+$: $\lambda_1 + \lambda_2$

irrep of $\mathbb{R}$: $D^{(k)}(\lambda) = e^{ik\lambda}$

$$\mathcal{H}^{(k)} = \mathbb{C} \qquad \dim(D^{(k)}) = 1$$

all irrep of G are 1-dim

$$\underline{D^{(k)}(\lambda_1)\, D^{(k)}(\lambda_2) = e^{ik(\lambda_1+\lambda_2)} = D^{(k)}(\lambda_1+\lambda_2)}$$

Complex conjugate of $D^{(k)}$: $D^{(k)}(\lambda)^* = e^{-ik\lambda}$

$$\neq e^{ik\lambda}$$

$$D^{(k)} \neq D^{(k)*}$$

cannot transf between them by unitary on $\mathbb{C}$

Complex irrep.

energy level degeneracy $= 2 \dim(D^{(k)})$

$= 2$

eigenstates of H : $\hat{H}\psi_k = \frac{\hbar^2 k^2}{2m}\psi_k$

$$\underset{\substack{\cap \\ D^{(k)}}}{\psi_k = e^{ikx}}, \quad \underset{\substack{\cap \\ D^{(k)*}}}{\psi_{-k} = e^{-ikx}}$$

indeed degeneracy $= 2$

$$U e^{ik\lambda} U^{-1} = e^{-ikx}$$

$U \curvearrowright \mathbb{C}$

$$e^{ik\lambda} = e^{ik\lambda} \neq e^{-ik\lambda} \Rightarrow D^{(k)} \neq D^{(k)*}$$