boson no spin $T_0^2 = 1$ $T_0 \psi(t,\vec{r}) = \psi^*(-t,\vec{r})$

fermion spin $\frac{1}{2}$ $T^2 = -1$ $T \begin{pmatrix} \psi_1(t,\vec{r}) \\ \psi_2(t,\vec{r}) \end{pmatrix} = e^{i\varphi} \sigma_y \begin{pmatrix} \psi_1^*(-t) \\ \psi_2^*(-t) \end{pmatrix}$

zero spin $\hat{H} \psi_{nl} = E_n \psi_{nl}$ $l = 1 .. d(n)$

general sol of Schrödingen eqn $\psi(\vec{r},t) = \sum_n a_n \psi_{nl}(\vec{r}) e^{-\frac{i}{\hbar} E_n t}$

($\hat{H}$ is real, not explicitly dep. on t) $\hat{T}_0 \psi(\vec{r},t) = \sum_n a_n^* \psi_{nl}^*(\vec{r}) e^{-\frac{i}{\hbar} E_n t}$

effectively $\hat{T}_0 : \psi_{nl}(\vec{r}) \to \psi_{nl}^*(\vec{r})$

if $\hat{H}$ is real, $\hat{H} \psi_{nl} = E_n \psi_{nl}$ ψ_{nl} is eigenstate

$\Rightarrow \hat{H} \psi_{nl}^* = E_n \psi_{nl}^*$ ψ_{nl}^* is eigenstate

ψ_{nl} basis in $\mathcal{H}^{(n)}$, ψ_{nl}^* is basis in $\mathcal{H}^{(n)*}$

whether $\mathcal{H}^{(n)}$ and $\mathcal{H}^{(n)*}$ the same ?

(1) $\mathcal{H}^{(n)} \simeq \mathcal{H}^{(n)*}$ equivalent rep of symm. group

(2) $\mathcal{H}^{(n)} \not\simeq \mathcal{H}^{(n)*}$ not equivalent ....

eigenspace $= \mathcal{H}^{(n)} \oplus \mathcal{H}^{(n)*}$

real rep, psudo real rep, and complex rep

Given a symm. group G, unitary irrep $D : g \mapsto D(g) \circlearrowright \mathcal{H}^{(n)}$

Def $D(g)$ is unitary on $\mathcal{H}^{(n)}$, ψ_{nl} orthonormal basis in $\mathcal{H}^{(n)}$

(complex conjugate irrep)

$$D(g)\psi_{nl}(\vec{r}) = \sum_j \psi_{nj}(\vec{r}) D_{ji}(g)$$

$D_{ji}(g)$ unitary matrix

complex conjugate : $D(g)^* \psi^*_{ni}(\vec{r}) = \sum_j \psi_{nj}(\vec{r})^* D_{ji}(g)^*$

$$\psi^*_{ni}(\vec{r}) = T_0 \psi_{ni}(\vec{r})$$

$D_{ij}(g)$ unitary irrep matrix $\xrightarrow{T_0}$ $D^*_{ij}(g)$ unitary irrep matrix

(irrep D) irrep D^* : complex conjugate irrep

relation between D and D^*

(1) if $\exists$ unitary transf. $U: \mathcal{H}^{(n)} \to \mathcal{H}^{(n)}$ s.t

$$U D(g) U^{-1} = \underset{\uparrow \atop \text{real matrix}}{D_0(g)} \qquad \forall g \in G$$

$\Rightarrow$ $\underline{D \text{ is equivalent to } D^*}$, since $U^* D(g)^* U^{*-1} = D_0(g)$

$\Rightarrow$ $\underline{D^*(g)} = U^{*-1} D_0(g) U^* = \underline{U^{*-1} U D(g) U^{-1} U^*}$

$U^{*-1}U$ is unitary

we say $D \simeq D^*$ is a real rep.

(2) D is $\underbrace{\text{equivalent}}_{\text{unitarily}}$ to D^*, but they are not equivalent to real rep.

$$\nexists\, U \text{ s.t. } D(g) = U \underset{\substack{\uparrow \\ \text{real matrix}}}{D_0(g)} U^{-1} \quad \forall g \in G$$

We say, $D \simeq D^*$ is pseudo-real

(3) D is inequivalent to D^*.

We say, D and D^* are complex reps

distinguish (3) from (1) and (2), check character

if $D \simeq D^*$, then $\chi(g) = \chi^*(g) \quad \forall g \in G$

if $\chi(g) \neq \chi^*(g)$, then D & D^* are complex reps.

distinguish between (1) & (2)

$$D^* \simeq D \rightarrow D^*(g) = \underset{\substack{\uparrow \\ \text{unitary matrix}}}{Z} D(g) Z^{-1} \quad \forall g \in G$$

Complex conjugate $D(g) = Z^* D^*(g) Z^{*-1}$

$$\Rightarrow D^*(g) = Z Z^* D^*(g) (Z Z^*)^{-1}$$

i.e. $[Z Z^*, D^*(g)] = 0 \quad \forall g \in G$

by Schur's lemma, when D^* is irrep, $ZZ^* = c\mathbb{1}$
$c \in \mathbb{C}$

Z is unitary, $Z^* = (Z^T)^{-1}$

$$\Rightarrow \quad Z(Z^T)^{-1} = c\mathbb{1} \Rightarrow \underline{Z = cZ^T}$$

$$\xrightarrow[\text{transpose}]{} Z^T = cZ$$

$$\Rightarrow \quad Z = c^2 Z \quad \Rightarrow \quad c^2 = 1, \quad c = \pm 1$$

$$D^* \simeq D \quad \Rightarrow \quad ZZ^* = \pm\mathbb{1}$$

<u>Thm</u>: $ZZ^* = \mathbb{1}$ iff D is real

$ZZ^* = -\mathbb{1}$ iff D is pseudo-real

Pf if D is real $\rightarrow$ $D^*(g) = (U^*)^{-1} D_0(g)\, U^*$

$$= (U^*)^{-1} U D(g)\, U^{-1} U^*$$

on the other hand $D^*(g) = Z D(g) Z^{-1}$

$$\Rightarrow \quad (U^*)^{-1} U = bZ \qquad b \in \mathbb{C}$$

Complex conjugate $U^{-1}U^* = b^* Z^*$

$$\underset{\substack{\shortparallel \\ 1}}{U^{-1}U^*(U^*)^{-1}U} = b^*b\, Z^* Z = |b|^2 Z^* Z$$

$$\underline{Z^* Z = |b|^{-2}\mathbb{1}}$$

$\underline{Z Z^* = c \mathbb{1}}$ $\Rightarrow$ $|b|^2 c = 1$

$c = 1$ or -1 $\quad$ $c = 1$ $\quad$ $|b|^2 > 0$

conversely if $c = 1$ $\quad (Z Z^* = \mathbb{1})$

Z unitary, $Z = e^{iA}$ A hermitian

$Z^* = Z^{-1} \Leftrightarrow Z^T = Z$, Z symmetric

$Z^* = (Z^T)^{-1}$

def. $U = Z^{\frac{1}{2}} = e^{iA/2}$ unitary

$U^2 = Z$

$$\underline{U^* Z = e^{-iA/2} e^{iA} = e^{\frac{1}{2} iA} = U}$$

$\forall g \in G$, $[U D(g) U^{-1}]^* = U^* D(g)^* U^{*-1}$

$= U^* Z D(g) Z^{-1} U^{*-1}$

$= U D(g) U^{-1} = D_0(g)$ real

$\Rightarrow D$ is real rep.

$Z Z^* = \mathbb{1} \Leftrightarrow D$ is real rep

$\Rightarrow Z Z^* = -\mathbb{1} \Leftrightarrow D$ is psudo-real rep. $\square$

$$\text{Distinguish } \underbrace{\text{real \& pseudo-real}}_{\underline{D^*(g) = Z D(g) Z^{-1}}} \text{ reps by characters}$$

orthogonality $\sum_g D^*_{\alpha\delta}(g)\, D_{\beta\gamma}(g) = \delta_{\alpha\beta}\, \delta_{\gamma\delta} \dfrac{h}{\dim(D) = d}$

$$\parallel$$

$$\sum_{\alpha,\delta} \underline{\underline{Z^{-1}_{\tau\alpha}}} \times \sum_g \sum_{\sigma,\rho} \underline{\underline{Z_{\alpha\sigma}}}\, D_{\sigma\rho}(g)\, \underline{Z^{-1}_{\rho\delta}}\, D_{\beta\gamma}(g) \times \underline{Z_{\delta\chi}}$$

$$\Rightarrow \sum_g D_{\tau\chi}(g)\, D_{\beta\gamma}(g) = Z_{\gamma\chi}\, Z^{-1}_{\tau\beta} \frac{h}{d}$$

let $\chi = \beta$. $\sum_\beta$, $\sum_g \sum_\beta D_{\tau\beta}(g)\, D_{\beta\gamma}(g)$

$$\parallel$$

$$\sum_g D_{\tau\gamma}(g^2) = (Z Z^*)_{\tau\gamma} \frac{h}{d}$$

$$= \pm\, \delta_{\tau\gamma} \frac{h}{d}$$

let $\tau = \gamma$, $\sum_\tau$,

$$\boxed{\sum_g \chi(g^2) = \pm h}$$

for complex rep. $D \neq D^*$ $\Rightarrow \sum_g \left[D^*_{\alpha\beta}(g)\right]^* D_{\gamma\delta}(g) = 0$

$$\parallel$$

$$\sum_g D_{\alpha\beta}(g) D_{\gamma\delta}(g)$$

$$\Rightarrow \sum_g \chi(g^2) = 0$$

summary : $\frac{1}{h}\sum_{g\in G} \chi(g^2) = \begin{cases} 1 & \text{real rep} \\ -1 & \text{psudo-real rep} \\ 0 & \text{complex.} \end{cases}$

Extra-degeneracy of $\hat{H}$ due to time-reversal inv.

$$\hat{H}\psi_{nl} = E_n \psi_{nl} \quad l = 1 \cdots d$$

$\psi_{nl} \in \mathcal{H}^{(n)}$ unitary irrep. of symmetry group G

$l = 1 \cdots d$ $\quad \dim \mathcal{H}^{(n)} = d$

find degeneracy at E_n , deg. at E_n is either d or $2d$ (spin-zero)

if $\hat{H}$ is real and not explicitly dep on t

$$\hat{H}\psi_{nl} = E_n \psi_{nl}$$

$$\hat{H}\psi^*_{nl} = E_n \psi^*_{nl} \qquad \psi^*_{nl}(\vec{r}) = \hat{T}_0 \psi_{nl}(\vec{r})$$

- if $\forall \psi^*_{nl}$, $\psi^*_{nl} \in \mathcal{H}^{(n)}$ spanned by ψ_{nl}

then $\mathcal{H}^{(n)}$ is the final eigenspace i.e. $\mathcal{H}^{(n)} = \mathcal{H}^{(n)*}$

degeneracy at E_n is d

- otherwise, eigenspace $= \mathcal{H}^{(n)} \oplus \mathcal{H}^{(n)*}$

  degeneracy at $E_n = 2d$ (extra degeneracy)

<u>spin-zero</u> $\hat{H}\psi_i = E\psi_i$ $\quad \psi_i \in \underline{\underline{\mathcal{H}^{(n)}}}$

$$D(g)\psi_i = \sum_j \psi_j D_{ji}(g) \quad \forall g \in G$$

$\hat{H}$ is real $\quad \hat{H}\psi_i^* = E\psi_i^* \quad \psi_i^* \in \mathcal{H}^{(n)*}$

$$D^*(g)\psi_i^* = \sum_j \psi_j^* D_{ji}(g)^* \quad \forall g \in G$$

$\mathcal{H}^{(n)}$ carries irrep D of G

$\mathcal{H}^{(n)*}$ carries complex conjugate irrep D^* of G

$\mathcal{H}^{(n)} \simeq \mathcal{H}^{(n)*}$ or not relates to $D \simeq D^*$ or not

firstly if $D \not\simeq D^*$ complex irrep, $\mathcal{H}^{(n)} \perp \mathcal{H}^{(n)*}$ by orthogonality theorem.

$\Rightarrow$ extra degeneracy $2d$

let's look at case (1) real rep and (2) psudo-real rep.

$\exists$ unitary Z, $D^*(g) = Z D(g) Z^{-1}$, $ZZ^* = \begin{cases} 1 & \text{real} \\ -1 & \text{psudo-real} \end{cases}$

$\underline{\text{Lemma}}$ if $\mathcal{H}^{(n)} = \mathcal{H}^{(n)*}$, then D is of case (1) i.e. real rep.

pf. $\underline{\underline{\mathcal{H}^{(n)} = \mathcal{H}^{(n)*}}}$, $\hat{H}\psi_i = E\psi_i$

$$\hat{H}\psi_i^* = E\psi_i^*$$

both $\{\psi_i\}$, $\{\psi_i^*\}$ are both orthonormal basis

then $\exists$ unitary U s.t. $\psi_i = \sum_k \psi_k^* U_{ki}$

$$\psi_i^* = \sum_k \psi_k U_{ki}^*$$

$$\Rightarrow \psi_i = \sum_{kl} \psi_l U_{lk}^* U_{ki} \quad \text{i.e. } UU^* = \mathbb{1}$$

$$D(g)\psi_i = \sum_j \psi_j D_{ji}(g)$$

$$\| \qquad\qquad \|$$

$$D(g)\sum_k \psi_k^* U_{ki} \qquad \sum_j \sum_k \psi_k^* U_{kj} D_{ji}(g)$$

$$\times U^{-1}_{il}, \sum_i$$

$$D(g)\psi_l^* = \sum_k \psi_k^* \left(U D(g) U^{-1}\right)_{kl}$$

$\uparrow$

same as $D^*(g)\psi_l^*$

Compare to $D^*(g)\psi_l^* = \sum_j \psi_j^* D_{ji}^*(g)$

$$D^*(g) = U D(g) U^{-1} \Big\} = D \text{ is real}$$

$U U^* = 1$ $\qquad$ J

□

$U = Z$

Lemma if D is real, then $\mathcal{H}^{(n)} = \mathcal{H}^{(n)*}$

pf D is real, $\underline{Z D Z^{-1} = D^*}$ & $Z Z^* = 1$

$$D^*(g)\psi_i^* = \sum_j \psi_j D^*_{ji}(g)$$

$$= \sum_j \psi_j \sum_{k,l} Z_{jk} D_{kl}(g) Z^{-1}_{li}$$

$\times Z_{im}$ and $\sum_i$

$$\Rightarrow D^*(g)\underline{\underline{\left(\sum_i \psi_i^* Z_{im}\right)}} = \sum_k \underline{\underline{\left(\sum_j \psi_j^* Z_{jk}\right)}} \underline{\underline{D_{km}(g)}}$$

$$D(g)\psi_m = \sum_k \psi_k D_{km}(g)$$

$$\Rightarrow \sum_i \psi_i^* Z_{im} = \psi_m$$

$$\underbrace{\psi_i^* \in \mathcal{H}^{(n)*} \qquad \psi_i \in \mathcal{H}^{(n)}}_{\text{they are linear dep. by } \sum_i \psi_i^* Z_{i\underline{\underline{m}}} = \psi_m}$$

$$\Rightarrow \mathcal{H}^{(n)} = \mathcal{H}^{(n)*}$$

above 2 lemmas $\Rightarrow$ $\mathcal{H}^{(n)} = \mathcal{H}^{(n)*}$ iff D is real

(no extra degeneracy) d

then if D is psudo-real then $\mathcal{H}^{(n)} \neq \mathcal{H}^{(n)*}$

$\Rightarrow$ extra degeneracy $2d$

if D is

- real: degeneracy $= d = \dim(D)$
- psudo-real: degeneracy $= 2d$
- complex: degeneracy $= 2d$

Examples (1) 1d free particle $\hat{H} = \frac{1}{2m}\hat{p}^2$

$$\hat{p} = -i\hbar\frac{\partial}{\partial x}$$

symmetry: transl. inv. $Q(\lambda)x = x + \lambda$ $\lambda \in \mathbb{R}$

$$D(\lambda) = e^{-\frac{i}{\hbar}\lambda\hat{p}}$$

$$D(\lambda)\psi(x) = \psi(x+\lambda)$$

$G = \mathbb{R} = \{\lambda\}$ group multiplication: $+$: $\lambda_1 + \lambda_2$

irrep of $\mathbb{R}$: $D^{(k)}(\lambda) = e^{ik\lambda}$

$\mathcal{H}^{(k)} = \mathbb{C}$ $\dim(D^{(k)}) = 1$

all irrep of G are 1-dim

$$\underline{D^{(k)}(\lambda_1)\, D^{(k)}(\lambda_2) = e^{ik(\lambda_1+\lambda_2)} = D^{(k)}(\lambda_1+\lambda_2)}$$

Complex conjugate of $D^{(k)}$: $D^{(k)}(\lambda)^* = e^{-ik\lambda}$

$$\neq e^{ik\lambda} \updownarrow$$

$$D^{(k)} \neq D^{(k)*}$$

cannot transf between them by unitary on $\mathbb{C}$

Complex irrep.

energy level degeneracy $= 2 \dim(D^{(k)})$

$= 2$

eigenstates of H : $\hat{H}\psi_k = \frac{\hbar^2 k^2}{2m}\psi_k$

$$\psi_k = \underset{\overset{\cap}{D^{(k)}}}{e^{ikx}}, \quad \psi_{-k} = \underset{\overset{\cap}{D^{(k)*}}}{e^{-ikx}}$$

indeed degeneracy $= 2$

$$U \underset{\|}{\underset{=}{e^{ik\lambda}}} \underset{\overset{\cap}{\mathbb{C}}}{U^{-1}} = e^{-ikx}$$

$$e^{ik\lambda} \neq e^{-ik\lambda} \Rightarrow D^{(k)} \neq D^{(k)*}$$

(2) central potential : $\hat{H}\psi_{nlm}(\vec{r}) = E_{nl}\psi_{nlm}$

$l = 0, 1, 2 \ldots$

$m = -l, -l+1, \dots, l$

symmetry group $G = SO(3)$

eigenspace $\mathcal{H}^{(n,l)}$ relates to irrep of $SO(3)$

$\forall$ Rotation $Q(\alpha, \beta, \gamma) \in SO(3)$

$\uparrow$ Euler angles

irrep of $SO(3)$ $D^{(l)}$, $\mathcal{H}^{(l)}$ is spanned by $Y_{lm}(\theta, \varphi)$
is labelled by l, $m = -l, \dots l$

$(l = 0, 1, 2, 3 \dots)$ $\dim(\mathcal{H}^{(l)}) = 2l+1$

$$\langle l m | D^{(l)}(\alpha, \beta, \gamma) | l m' \rangle = \int_{S^2} d\theta d\varphi \sin\theta \, Y^*_{lm} \hat{D}^l Y_{lm'}$$

$$= D^{(l)}_{mm'}(\alpha, \beta, \gamma) = e^{-im'\alpha} d^l_{mm'}(\beta) \, e^{-im\gamma}$$

$\uparrow$ Wigner D-function

$\uparrow$ Wigner d-func

$d(0) = 1$

$$\chi^l(\alpha) = \operatorname{tr} D^{(l)}(\alpha, \beta, \gamma) = \operatorname{tr} D^{(l)}(\alpha, 0, 0)$$

$$= \sum_{m=-l}^{l} e^{-im\alpha} = \frac{\sin\left((l+\frac{1}{2})\alpha\right)}{\sin \alpha/2}$$ real

$\chi^{l*} = \chi^l$. D^l is not complex basis in $\mathcal{H}^{(l)}$: $Y_{lm}(\theta,\varphi)$, $Y^{*}_{lm}(\theta,\varphi) = Y_{l,-m}(\theta,\varphi)$

$$Y^{*}_{lm} \in \mathcal{H}^{(l)}$$

$$\Rightarrow \mathcal{H}^{(l)*} = \mathcal{H}^{(l)}$$

$\Rightarrow$ $D^{(l)}$, $l = 0, 1 \ldots$ are all real rep. , no extra degeneracy

$\mathcal{H}^{(n,l)}$ is the eigenspace , deg. $= 2l+1$

single spin-$\frac{1}{2}$ particle : $T^2 = -1$

<u>Lemma</u> $\langle \psi | T\psi \rangle = 0$

pf. Let $\varphi = T\psi$ $\quad \forall \psi \in \mathcal{H}^{(0)} \otimes \mathbb{C}^2$

$$\langle \psi | \varphi \rangle = \langle \psi | T\psi \rangle \underset{\substack{\uparrow \\ T \text{ is antiunitary}}}{=} \langle T\psi | T^2 \psi \rangle^{*}$$

$$= -\langle T\psi | \psi \rangle^{*} = -\langle \psi | \varphi \rangle$$

$$\Rightarrow \langle \psi | \varphi \rangle = 0$$

ψ , $T\psi$ are orthogonal. $\square$

$\hat{H} \psi_i = E \psi_i$ $\quad i = 1 .. d$ $\quad \psi_i \in \mathcal{H}^{(d)}$ carry irrep $D^{(d)}$ of G

$\hat{H} (T\psi_i) = E (T\psi_i)$ $\quad \varphi_i \equiv T\psi_i$

(1) $\langle \psi_i | \varphi_i \rangle = 0$

(2) $\langle \psi_i | \psi_j \rangle = \delta_{ij}$

(3) $\langle \varphi_i | \varphi_j \rangle = \delta_{ij}$

Thm (Kramer's Thm) single spin-$\frac{1}{2}$ particle, $\forall$ energy level degeneracy d' is always even and

$$d \leq d' \leq 2d$$

Pf we have (1), (2), & (3) whether $\{\psi_i, \varphi_i\}$ form a complete basis

but it's possible $\langle \psi_i | \varphi_j \rangle \neq 0 \quad i \neq j$

if $|\varphi_k\rangle = \sum_l |\psi_l\rangle c_l$ the φ_k should be removed from the set of basis

but the $|\psi_k\rangle$ should be removed as well

since $T^2 = -1$

$$T|\varphi_k\rangle = \sum_l T\left(|\psi_l\rangle c_l\right)$$

$$T|\varphi_k\rangle = T^2|\psi_k\rangle = -|\psi_k\rangle$$

$$\sum_l T\left(|\psi_l\rangle c_l\right) = \sum_l c_l^* |\varphi_l\rangle$$

$$\Rightarrow \quad |\psi_k\rangle = -\sum_l c_l^* |\varphi_l\rangle$$

$$\psi_1 \cdots \underset{\uparrow}{\widehat{\psi_k}} \cdots \psi_d \;\; \varphi_1 \cdots \underset{\uparrow}{\widehat{\varphi_k}} \cdots \varphi_d$$

removed in pair.

find degeneracy $d' = 2d - 2\overset{\text{\# of removed pairs}}{\overset{\downarrow}{n}}$ $\quad$ d' is even

$$d \le d' \le 2d$$

$\square$

Theory of angular momentum

spatial rotation & SO(3) group

finite rotation : Q 3x3 matrix s.t. $Q\vec{r}_1 \cdot Q\vec{r}_2 = \vec{r}_1 \cdot \vec{r}_2$

$$\Rightarrow Q^T Q = 1 \qquad Q \in O(3)$$

$$\det(Q^T Q) = 1$$

$$\det(Q)^2 = 1$$

$$\det Q = \pm 1$$

$SO(3)$: $Q^T Q = 1$ & $\det Q = 1$ $\qquad P = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

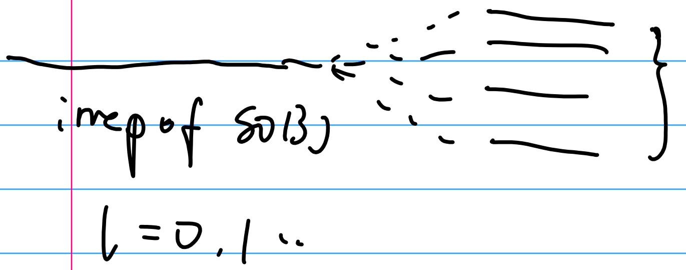
HW tutorial

problem 1. $\hat{H} = \hat{H}_0 + V_{lattice}(\vec{r})$

$$\underset{\substack{\uparrow \\ \text{central force, } SO(3) \text{ symmetry}}}{\hat{H}_0}$$

eigenspace of H_0 : $\mathcal{H}_{l=0,1,\ldots}$
carries irrep of $SO(3)$

$V_{lattice}$ breaks $SO(3)$ to O (cubic lattice group)



$$D^l = \bigoplus_T D_T$$

D^l ← irrep of SO(3)

D_T ← irreps of O

$$\forall g \in O, \quad \chi_l(g) = \sum_T \chi_T(g)$$

character of O

$\chi_T(g)$	E	$8C_3$	$6C_2$	$2C_2$	$6C_4$
T_1	1	1	1	1	1
T_2	1	1	-1	1	-1
T_3	2	-1	0	2	0
T_4	3	0	-1	-1	1
T_5	3	0	1	-1	-1

character of SO(3) evaluated at elements in O

$\chi_l(g)$	E	$8C_3$	$6C_2$	$2C_2$	$6C_4$
$l=0$	1	1	1	1	1
$l=1$	3	0	-1	-1	1
$l=2$	5	-1	1	1	-1
$l=3$	7	1	-1	-1	-1
$l=4$	9	0	1	1	1

- $l=0$: $D^{l=0} = T_1$ • $l=1$ $D^{l=1} = T_4$
- $l=2$: $D^{l=2} = T_3 \oplus T_5$ $\chi_{l=2}(g) = \chi_{T_3}(g) + \chi_{T_5}(g)$

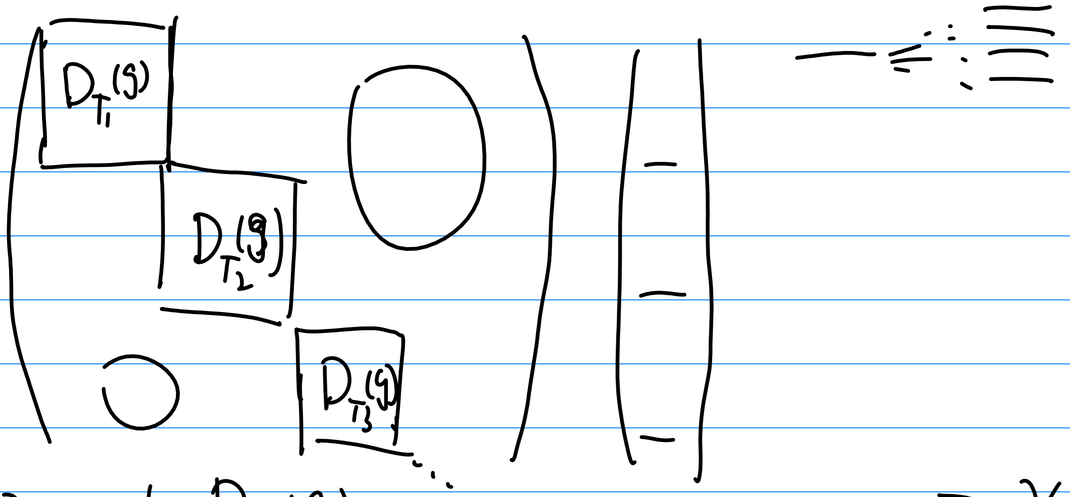
check : E : $\chi_{l=2}(E) = 5$ $\forall g \in O$

$$\chi_{T_3}(E) + \chi_{T_5}(E) = 2+3 = 5$$

$8C_3$: $\chi_2(8C_3) = -1$

$$\chi_{T_3}(8C_3) + \chi_{T_5}(8C_3) = -1 + 0 = -1$$

$\underline{D_l = \bigoplus_T D_T}$ $\forall g \in O$, $D_l(g) = \bigoplus_T D_T(g)$



$$\chi_l(g) = \mathrm{tr}\, D_l(g) = \sum_T \mathrm{tr}\, D_T(g) = \sum_T \chi_T(g)$$

$6C_2 \quad \chi_{l=2}(6C_2) = 1$

$\chi_{T_3}(6C_2) + \chi_{T_5}(6C_2) = 0 + 1 = 1$

$2C_2 : \quad \chi_{l=2}(2C_2) = 1$

$\chi_{T_3}(2C_2) + \chi_{T_5}(2C_2) = 2 + (-1) = 1$

$6C_4 : \quad \chi_{l=2}(6C_4) = -1$

$\chi_{T_3}(6C_4) + \chi_{T_5}(6C_4) = 0 + (-1) = -1$

- $l=3 \quad D_3 = T_2 \oplus T_4 \oplus T_5$

$E : \quad \chi_3 = 7$

$\chi_{T_2} + \chi_{T_4} + \chi_{T_5} = 1 + 3 + 3 = 7$

$8C_3 : \quad \chi_3 = 1$

$\chi_{T_2} + \chi_{T_4} + \chi_{T_5} = 1 + 0 + 0 = 1$

$6C_2 : \quad \chi_3 = -1$

$\chi_{T_2} + \chi_{T_4} + \chi_{T_5} = -1 + (-1) + 1 = -1$

$3C_2 : \quad \chi_3 = -1$

$\chi_{T_2} + \chi_{T_4} + \chi_{T_5} = 1 + (-1) + (-1) = -1$

$6C_4 : \quad \chi_3 = -1$

$\chi_{T_2} + \chi_{T_4} + \chi_{T_5} = -1 + 1 + (-1) = -1$

- $l = 4$ $\qquad D_4 = T_1 \oplus T_3 \oplus T_4 \oplus T_5$

E : $\quad \chi_4 = 9$

$$\chi_{T_1} + \chi_{T_3} + \chi_{T_4} + \chi_{T_5} = 1 + 2 + 3 + 3 = 9$$

$8C_3$: $\quad \chi_4 = 0$

$$\chi_{T_1} + \chi_{T_3} + \chi_{T_4} + \chi_{T_5} = 1 + (-1) + 0 + 0 = 0$$

$6C_2$: $\quad \chi_4 = 1$

$$\chi_{T_1} + \chi_{T_3} + \chi_{T_4} + \chi_{T_5} = 1 + 0 + (-1) + 1 = 1$$

$3C_2$: $\quad \chi_4 = 1$

$$\chi_{T_1} + \chi_{T_3} + \chi_{T_4} + \chi_{T_5} = 1 + 2 + (-1) + (-1)$$
$$= 1$$

$6C_4$: $\quad \chi_4 = 1$

$$\chi_{T_1} + \chi_{T_3} + \chi_{T_4} + \chi_{T_5} = 1 + 0 + 1 + (-1) = 1$$

$$\vec{r} = \sum_{i=1}^{3} r_i \vec{e}_i \qquad \text{choose basis } \vec{e}_{i=1,2,3}$$

$$\underline{\vec{r}\,' = Q\vec{r} \equiv \sum_{i=1}^{3}} r'_i \vec{e}_i$$

Rotation of basis $\qquad \underline{\vec{r}\,' = Q\vec{r}} = \sum_{i=1}^{3} r_i Q \vec{e}_i = \underline{\sum_{i=1}^{3} r_i \vec{e}_i\,'}$

$$\vec{e}_i' = Q\vec{e}_i = \sum_j \vec{e}_j \left(\vec{e}_j \cdot \underbrace{Q\vec{e}_i}_{\vec{e}_i'} \right) = \sum_{j=1}^{3} \vec{e}_j Q_{ji}$$

$$\vec{r}\,' = \sum_{i=1}^{3} r_i' \underline{\underline{\vec{e}_i}}$$

$$\shortparallel$$

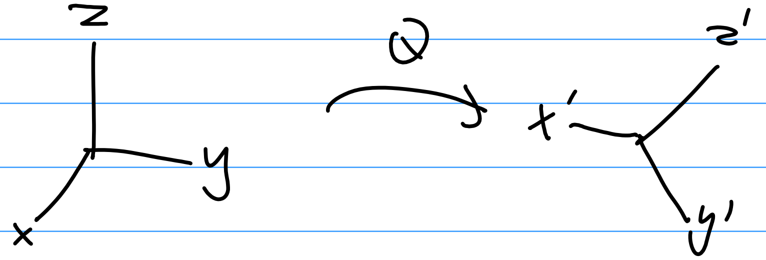
$$\sum_{i=1}^{3} r_i \sum_{j=1}^{3} \vec{e}_j Q_{ji} = \sum_{j=1}^{3} \left(\sum_{i=1}^{3} Q_{ji} r_i \right) \vec{e}_j$$

$$r_j' = \sum_{i=1}^{3} Q_{ji} r_i$$

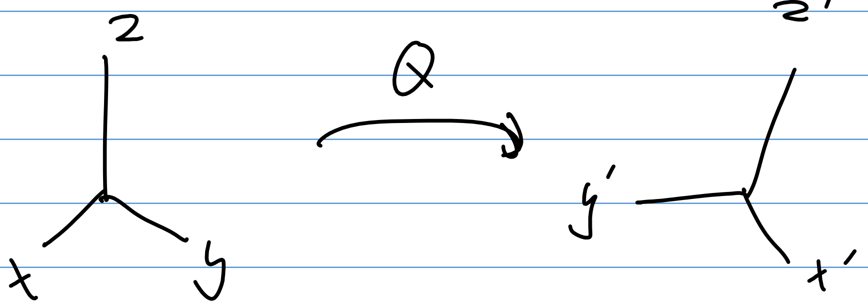
Parity $P : \vec{r} \to -\vec{r}$ $\quad P_{ij} = -\delta_{ij}$ $\quad \det P = -1$

$P \notin SO(3)$ $\quad P \in O(3)$

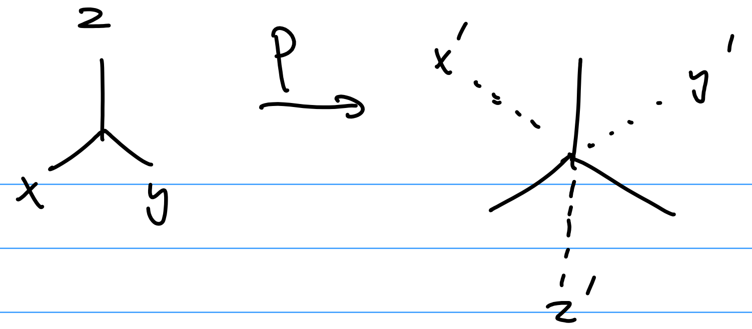
$Q \in SO(3)$ transf. right-hand frame to right-hand fram



$Q \notin SO(3)$, $Q \in O(3)$ transf. right-hand frame to left-hand frame



P:



Proof:

Let $\vec z = \vec x \times \vec y$ we should show that

$$Q\vec x \times Q\vec y = Q\vec z \quad \forall\, Q \in SO(3)$$

$$Q\vec x \times Q\vec y = -Q\vec z \quad \forall\, Q \notin SO(3)$$
$$Q \in O(3)$$

∀ vector $\vec u \in \mathbb{R}^3$

$$\underline{\underline{Q\vec u}} \cdot \left(Q\vec x \times Q\vec y\right) = \sum_{ijk} \varepsilon_{ijk} (\underline{Q\vec u})_i (Q\vec x)_j (Q\vec y)_k$$

$$= \sum_{\substack{ijk\\ \alpha\beta\gamma}} \varepsilon_{ijk}\, Q_{i\alpha} u_\alpha\, Q_{j\beta} x_\beta\, Q_{k\gamma} y_\gamma$$

$$= \sum_{\alpha\beta\gamma} \left(\underbrace{\sum_{ijk} \varepsilon_{ijk}\, Q_{i\alpha} Q_{j\beta} Q_{k\gamma}}_{\varepsilon_{\alpha\beta\gamma} \det Q} \right) u_\alpha x_\beta y_\gamma$$

$$= \det Q \sum_{\alpha\beta\gamma} \varepsilon_{\alpha\beta\gamma}\, u_\alpha x_\beta y_\gamma$$

$$= \det Q \left(\vec u \cdot (\vec x \times \vec y) \right)$$

$$= \det Q \left(\vec u \cdot \vec z \right) = \det Q \left(\underline{Q\vec u} \cdot Q\vec z \right)$$

$$Q\vec{x} \times Q\vec{y} = \underbrace{\det Q}_{\substack{\shortparallel \\ \pm 1}} \left(Q\vec{z} \right)$$

all rotations of rigid body are SO(3)

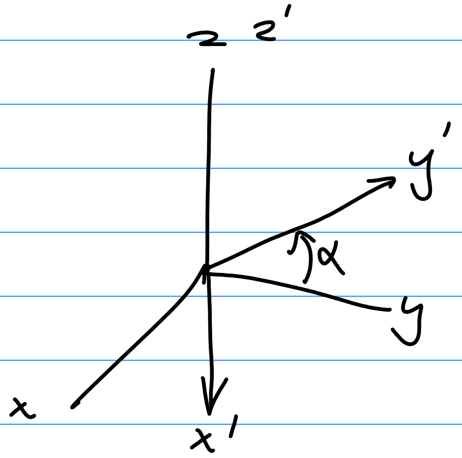
$\forall\, Q \in SO(3)$ can be composed by 2 types of simple rotations

$Q(\hat{k}, \alpha)$, $Q(\hat{j}, \beta)$ $\quad$ $\hat{i}, \hat{j}, \hat{k}$ basis of x, y, z axis

rotation around z-axis $\quad$ rotation around y-axis

$$Q(\hat{k}, \alpha) = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \vec{e}_i' = \sum_j \vec{e}_j Q_{ji}$$

$$\hat{i} \rightarrow \hat{i}' = \hat{i} \cos\alpha + \hat{j} \sin\alpha$$
$$\hat{j} \rightarrow \hat{j}' = \hat{i}(-\sin\alpha) + \hat{j} \cos\alpha$$
$$\hat{k} \rightarrow \hat{k}' = \hat{k}$$



$$Q(\hat{j}, \beta) = \begin{pmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{pmatrix}$$

Euler angles :

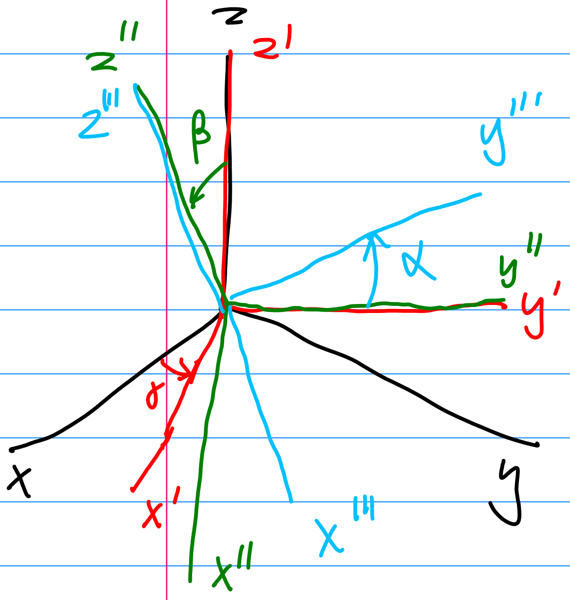
give a basis

$\hat{i}\ \hat{j}, \hat{k}$:

(1) rotation around z : γ
(2) rotation around y : β
(3) rotation around z : α

} compose any rotations

$$\alpha \in [0, 2\pi), \quad \beta \in [0, \pi], \quad \gamma \in [0, 2\pi)$$



$$Q(\alpha, \beta, \gamma)$$

$$= \underline{Q(\hat{k}, \alpha)\, Q(\hat{j}, \beta)\, Q(\hat{k}, \gamma)}$$

$$\vec{e}_i' = \sum_j \vec{e}_j\, Q_{ji}(\alpha \beta \gamma)$$

$$Q(\alpha, \beta\ \gamma) = \left(\begin{matrix} \cos\alpha \cos\beta \cos\gamma - \sin\alpha \sin\gamma & -\cos\alpha\cos\beta \sin\gamma - \sin\alpha \cos\gamma & - \\ \cdot & \cdot\cdot & \cdot \\ \cdot & -\cdot & \cdot\cdot \end{matrix} \right.$$

$\forall\ Q(\alpha, \beta, \gamma) \in SO(3)$

all group elements in SO(3) can be parametrized by 3 Euler angles

SO(3) is a 3-dim Lie group

SO(3) and SU(2)

in QM, it's better to work with SU(2), reps of SU(2)

since 1) { irreps of SU(2) } $\supset$ { irreps of SO(3) }

2) we have spin.

SU(2) group: special unitary transf. on $\mathbb{C}^2$

$$SU(2) \ni u = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix}, \quad \det(u) = \underline{aa^* + bb^*} = 1$$

$u^\dagger = u^{-1}$

$a, b \in \mathbb{C}$ 2 complex = 4 reals

1 constraint.

3 real parameters

$\Rightarrow$ SU(2) 3 dim Lie group

$aa^* + bb^* = 1$ we write $a = \cos\eta \, e^{i\xi}$

$b = -\sin\eta \, e^{-i\zeta}$

$$u = \begin{pmatrix} \cos\eta \, e^{-i\xi} & -\sin\eta \, e^{-i\zeta} \\ \sin\eta \, e^{i\zeta} & \cos\eta \, e^{i\xi} \end{pmatrix} = u(\underbrace{\eta, \xi, \zeta}_{\text{3 real parameters}})$$

$\xi, \zeta \in [0, 2\pi)$ $\quad \eta \in [0, \frac{\pi}{2}]$

homomorphism between SU(2) and SO(3)

$$\vec{r} \in \mathbb{R}^3 \quad \rightarrow \quad \vec{r}\cdot\vec{\sigma} = \begin{pmatrix} z & x-iy \\ x+iy & -z \end{pmatrix}$$

$$\vec{r} = (x, y, z), \qquad \vec{r}\cdot\vec{\sigma} \equiv h$$

$$\det(h) = x^2+y^2+z^2 = |\vec{r}|^2$$

$u \in SU(2)$, view h an operator on $\mathbb{C}^2$

$$h \rightarrow \underline{h' = u h u^{-1} \equiv \vec{r}'\cdot\vec{\sigma}} = \begin{pmatrix} z' & x'-iy' \\ x'+iy' & -z' \end{pmatrix}$$

$$\underset{|\vec{r}'|^2}{\underset{\|}{\det(h')}} = \det(u h u^{-1}) = \underset{|\vec{r}|^2}{\underset{\|}{\det(h)}}$$

This action by u from $\vec{r}$ to $\vec{r}'$ corresponds to a rotation $Q(u) \in SO(3)$, s.t. $Q(u)\vec{r} = \vec{r}'$

$$u(\vec{\sigma}\cdot\vec{r})u^{-1} = \vec{\sigma}\cdot(Q(u)\vec{r})$$

we find a map from $SU(2)$ to $SO(3)$

homomorphism : $\forall u_1, u_2 \in SU(2)$

$$(u_1u_2)(\vec{\sigma}\cdot\vec{r})(u_1u_2)^{-1} = u_1u_2(\vec{\sigma}\cdot\vec{r})u_2^{-1}u_1^{-1}$$

$$= u_1(\vec{\sigma}\cdot Q(u_2)\vec{r})u_1^{-1}$$

$$= \vec{\sigma}\cdot(Q(u_1)Q(u_2)\vec{r})$$

$\forall$ rep D of $SO(3)$

$$SU(2) \xrightarrow[\text{hom}]{i} SO(3) \xrightarrow[\text{hom}]{D} L(\mathcal{H})$$

$D \circ i$ is a rep of $SU(2)$

$$u(\vec{\sigma}\cdot\vec{r})u^{-1} = \vec{\sigma}\cdot(Q(u)\vec{r}) \qquad u = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix}$$

$$\underline{aa^* + bb^* = 1}$$

$\downarrow$ $u \to -u \in SU(2)$

$$(-u)(\vec{\sigma}\cdot\vec{r})(-u)^{-1} \qquad -u = \begin{pmatrix} -a & -b \\ b^* & -a^* \end{pmatrix}$$

$$\underline{aa^* + bb^* = 1}$$

$$\begin{tikzcd}
SU(2) & SO(3) \\
u \arrow[dr, mapsto] & \\
& Q(u) \\
-u \arrow[ur, mapsto] &
\end{tikzcd}$$

2-to-1 homomorphism

"SU(2) is double covering group of SO(3)"

Example (1) $u_1(\alpha) = \begin{pmatrix} e^{-i\alpha/2} & 0 \\ 0 & e^{i\alpha/2} \end{pmatrix}$

$$h = \vec{r}\cdot\vec{\sigma} = \begin{pmatrix} z & x-iy \\ x+iy & -z \end{pmatrix}$$

$$h' = u_1(\alpha)\, h\, u_1(\alpha)^{-1} = \begin{pmatrix} z & (x-iy)e^{-i\alpha} \\ (x+iy)e^{i\alpha} & -z \end{pmatrix}$$

$\|$

$$\vec{\sigma}\cdot\vec{r}' = \begin{pmatrix} z' & x'-iy' \\ x'+iy' & -z' \end{pmatrix} \Rightarrow \begin{cases} x' = x\cos\alpha - y\sin\alpha \\ y' = x\sin\alpha + y\cos\alpha \\ z' = z \end{cases}$$

$$\vec{r}\,' = \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \underbrace{\begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{Q(u_1(\alpha)) \;=\; Q(\hat{k},\alpha)} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$\pm u_1(\alpha) \longmapsto Q(\hat{k},\alpha)$ rotation around z-axis

(2) $u_2(\beta) = \begin{pmatrix} \cos\beta/2 & -\sin\beta/2 \\ \sin\beta/2 & \cos\beta/2 \end{pmatrix} \in SU(2)$

<u>HW</u> show $\pm u_2(\beta) \longmapsto Q(\hat{j},\beta)$ rotation around y-axis

$$u(\alpha,\beta,\gamma) = u_1(\alpha)\,u_2(\beta)\,u_1(\gamma)$$

$$\downarrow$$

$$Q(\alpha,\beta,\gamma) = Q(\hat{k},\alpha)\,Q(\hat{j},\beta)\,Q(\hat{k},\gamma)$$

α, β, γ are Euler angles

$$u(\alpha,\beta,\gamma) = \begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \end{pmatrix}$$

compare with 3.4.3

$$u = \begin{pmatrix} \cos\eta \, e^{-i\xi} & -\sin\eta \, e^{-i\zeta} \\ \sin\eta \, e^{i\zeta} & \cos\eta \, e^{i\xi} \end{pmatrix}$$

$$\alpha = \xi + \zeta \, , \quad \beta = 2\eta \, , \quad \gamma = \xi - \zeta$$

$$\xi, \zeta \in [0, 2\pi) \, , \quad \eta \in [0, \tfrac{\pi}{2}]$$

$$\Rightarrow \alpha \in [0, 4\pi) \, , \quad \beta \in [0, \pi] \, , \quad \gamma \in [0, 2\pi)$$

$\uparrow$

relate to 2-to-1 homomorphism $SU(2) \to SO(3)$

$$\left(\mathrm{Vol}(SU(2)) = 2 \, \mathrm{Vol}(SO(3)) \right)$$

Reps of SU(2)

fundamental rep of $SU(2)$ (defining rep) on $\mathbb{C}^2$ (spin Hilbert space)

$$v = \begin{pmatrix} \xi \\ \eta \end{pmatrix} \in \mathbb{C}^2$$

$$v' = \begin{pmatrix} \xi' \\ \eta' \end{pmatrix} = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} a\xi + b\eta \\ -b^*\xi + a\eta \end{pmatrix}$$

$$\xi \to \xi' = a\xi + b\eta$$

$$\eta \to \eta' = -b^*\xi + a^*\eta$$

homogeneous polynomials of ξ, η

basis

degree-0 $\quad 1 \iff \xi^0 \eta^0$

degree-1 $\quad \xi, \eta \iff \xi^{\frac{1}{2}+\frac{1}{2}} \eta^{\frac{1}{2}-\frac{1}{2}}, \ \xi^{\frac{1}{2}-\frac{1}{2}} \eta^{\frac{1}{2}+\frac{1}{2}}$

degree-2 $\quad \xi^2, \xi\eta, \eta^2 \iff \underline{\xi^{1+1}\eta^{1-1}, \ \xi^{1+0}\eta^{1+0}, \ \xi^{1-1}\eta^{1+1}}$

$\vdots$

degree-2j $\quad \xi^{j+j}\eta^{j-j}, \ \xi^{j+(j-1)}\eta^{j-(j-1)} \cdots$

$\xi^{j+m}\eta^{j-m} \cdots\cdots \xi^{j-j}\eta^{j+j}$

$m = j, j-1, j-2, \ldots -j$

$$\underline{\xi^{j+j}\eta^{j-j}} \sim \underline{|j,j\rangle}$$

$$\vdots \qquad \vdots$$

$$\xi^{j+m}\eta^{j-m} \sim |j,m\rangle$$

$$\vdots \qquad \vdots$$

$$\xi^{j-j}\eta^{j+j} \sim |j,-j\rangle$$

any degree-$2j$ homogeneous polynomial $\displaystyle\sum_{m=-j}^{j} a_m \underline{\underline{\xi^{j+m}\eta^{j-m}}}$

$$\sim \sum_{m=-j}^{j} a_m |j,m\rangle$$

span 2j-dim vector space $\mathcal{H}_j$. rep of SU(2)

SU(2) action : $\sum_{m=-j}^{j} a_m \xi^{j+m} \eta^{j-m}$

$$\to \sum_{m=-j}^{j} a_m \underline{(a\xi + b\eta)}^{j+m} \underline{(-b^*\xi + a^*\eta)}^{j-m}$$

still homogeneous polynomial of 2j

basis vector $f^j_m(\xi, \eta) = \frac{\xi^{j-m}\eta^{j+m}}{\sqrt{(j-m)!(j+m)!}}$ ←

$$m = -j, -j+1 \dots j$$

$\{f^j_m\}_{m=-j}^{j}$ span $\mathcal{H}_j$ irrep. of SU(2)

$\forall u \in SU(2)$ $D_j(u) f^j_m(\xi, \eta) := f^j_m\left(u^{-1}\begin{pmatrix}\xi \\ \eta\end{pmatrix}\right)$

$$u^{-1}\begin{pmatrix}\xi \\ \eta\end{pmatrix} = u^+\begin{pmatrix}\xi \\ \eta\end{pmatrix} = \begin{pmatrix}a^* & -b \\ b^* & a\end{pmatrix}\begin{pmatrix}\xi \\ \eta\end{pmatrix} = \begin{pmatrix}a^*\xi - b\eta \\ b^*\xi + a\eta\end{pmatrix}$$

$$D_j(u) f^j_m = \frac{(a^*\xi - b\eta)^{j-m}(b^*\xi + a\eta)^{j+m}}{\sqrt{(j-m)!(j+m)!}}$$

$$= \sum_{m'=-j}^{j} f^j_{m'} \underset{\uparrow}{D^j_{m'm}}(u)$$

rep. matrix (2j+1) × (2j+1) Wigner D-matrix derive $D^j_{mm'}(u)$

$$(x+y)^n = \sum_{r=0}^{n} \frac{n!}{r!\,(n-r)!} x^r y^{n-r}$$

$$D_j(u)\, f^j_m = \sum_{n=0}^{j-m} \sum_{m'=n-j}^{n+m} (-1)^n \frac{\sqrt{(j-m)!\,(j+m)!\,(j-m')!\,(j+m')!}}{n!\,(j-m-n)!\,(n+m-m')!\,(j+m'-m)!}$$

$$(a^*)^{j-m-n}\, b^n\, (b^*)^{n+m-m'}\, a^{j+m'-m}\, f^j_{m'}$$

$$\sum_{n=0}^{j-m} \sum_{m'=n-j}^{n+m} (\ldots) = \sum_{m'=-j}^{j} \sum_{n} (\ldots)$$

over all integers s.t. for all $(\ldots)!$ nonnegative.

$$D^j_{m'm}(a,b) = \sum_n (-1)^n \frac{\sqrt{(j-m)!\,(j+m)!\,(j-m')!\,(j+m')!}}{(j+m'-n)!\,(j-m-n)!\,n!\,(n+m-m')!}$$

$$(a^*)^{j-m-n}\, a^{j+m'-n}\, b^n\, (b^*)^{n+m-m'}$$

$$j = 0,\ \frac{1}{2},\ 1 \ldots$$

$$m, m' = -j \ldots j$$

properties of D-matrix

1) Unitary: **Lemma**: $D(u)\, D(u)^\dagger = 1 \quad \forall u \in SU(2)$

i.e. $\sum_m D^j_{m_1 m}(u)\, D^j_{m_2 m}(u)^* = \delta_{m_1 m_2}$

Pf: let's look at $\sum_m f^j_m(\xi,\eta)^* f^j_m(\xi,\eta)$

$$= \sum_{m=-j}^{j} \frac{(\xi^*\xi)^{j-m}(\eta^*\eta)^{j+m}}{(j-m)!\,(j+m)!}$$

$$= \frac{(\xi^*\xi+\eta^*\eta)^{2j}}{(2j)!} \leftarrow \text{SU(2) inv.}$$

$\forall\, u \in SU(2)$

$$u\begin{pmatrix}\xi\\ \eta\end{pmatrix} = \begin{pmatrix}\xi'\\ \eta'\end{pmatrix} \qquad (\xi',\eta')^*\begin{pmatrix}\xi'\\ \eta'\end{pmatrix} = \xi'^*\xi' + \eta'^*\eta'$$

$$= (\xi.\eta)\, u^\dagger u \begin{pmatrix}\xi\\ \eta\end{pmatrix}$$

$$= (\xi\ \ \eta)\begin{pmatrix}\xi\\ \eta\end{pmatrix} = \xi^*\xi+\eta^*\eta$$

$$\sum_m \left(D_j(u)\, f^j_m\right)^* \left(D_j(u)\, f^j_m\right) = \underline{\underline{\sum_m f^{j*}_m f^j_m}}$$

$$\|$$

$$\sum_{m'm''} \underline{\underline{f^{j*}_{m''} f^j_{m'}}} \left(\sum_m D^j_{m''m}(u)^* D^j_{m'm}(u)\right)$$

---

if $f^{j*}_{m''} f^j_{m'} \equiv b^j_{m''m'}$ are linear indep

then $\underline{\sum_m D^j_{m'm}(u)\, D^j_{m''m}(u)^* = \delta_{m''m'}}$

$$DD^\dagger = 1$$

To prove they are linearly indep.

solve eqn $\sum_{m'', m'} C_{m'' m'} f^{j\,*}_{m''} f^{j}_{m'} = 0$

show $C_{m'' m'} = 0 \quad \forall\, m', m'' = -j \cdots j$

HW →

$\square$