∞ - dim path integral

$$\int Dx(t)\, e^{\frac{i}{\hbar}S[x(t)]} = N \sum_{x_c} \sqrt{\frac{1}{\mathrm{Det}(-iM(x_c))}}\; e^{\frac{i}{\hbar}S[x_c]} \left(1 + O(\hbar)\right)$$

x_c : $\delta S[x_c] = 0$ classical trajectory

$M(x_c) := \delta^2 S[x_c]$ ∞ - dim matrix

$\mathrm{Det}(-iM(x_c))$ functional determinat

$\vec{x} = (x_1 \cdots x_n)$

$x(t) = x_c(t) + \delta x(t)$ $\quad x(t) \sim x_i \quad \sum_i \frac{\partial S}{\partial x^i}(x_c)\,\delta x^i$

classical trajectory $\underline{\delta S[x_c] = 0}$

$$\to \int D(\underline{x_c(t) + \delta x(t)})\; e^{\frac{i}{\hbar}S[x_c(t)+\delta x(t)]}$$

$$= \int D\,\underline{\underline{\delta x(t)}}\; e^{\frac{i}{\hbar}\left(S[x_c(t)] + \cancelto{0}{\int \frac{\delta S}{\delta x(t)}[x_c]\,\delta x(t)\,dt} + \frac{1}{2}\int dt_1 dt_2 \frac{\delta^2 S}{\delta x(t_1)\delta x(t_2)}[x_c]\, \underline{\underline{\delta x(t_1)\,\delta x(t_2)}} + \cdots\right)}$$

$$= e^{\frac{i}{\hbar}S[x_c(t)]} \underbrace{\int D\delta x(t)\; e^{\frac{i}{2\hbar}\int dt_1 dt_2\, M(t_1,t_2)\,\delta x(t_1)\,\delta x(t_2)}} + \underline{\underline{O(\delta x^3)}}$$

$$M(t_1, t_2) = \frac{\delta^2 S}{\delta x(t_1)\,\delta x(t_2)}[x_c] \quad \infty\text{-dim Hessian matrix}$$

$$\int d^n x\; e^{\frac{i}{2\hbar}\sum_{ij} M_{ij} x^i x^j}$$

$$= \underbrace{N \sqrt{\frac{1}{\mathrm{Det}(-iM)}}\; e^{\frac{i}{\hbar}S[\underline{\underline{x_c(t)}}]}} \left(1 + \underline{\underline{O(\hbar)}}\right)$$

$x(t) \qquad i,j \rightsquigarrow t \qquad$ QM

$\phi(x,t) \qquad i,j \rightsquigarrow x,t \qquad$ QFT

classical

$$= N \sum_{x_c} e^{\overbrace{\frac{i}{\hbar}S[x_c(t)]}^{\text{classical}} - \frac{1}{2}\ln(\mathrm{Det}(-iM)) + O(\hbar)} \qquad e^{O(\hbar)} = 1 + O(\hbar)$$

quantum effective action

NLO "1-loop determinant"

Thermal partition function & path integral

classical system with temprature T, $\beta = \frac{1}{T}$

partition function $\quad Z(\beta) = \sum_E e^{-\beta E} \quad$ (canonical ensemble)

quantum system ;

$$\underbrace{Z(\beta)} = \sum_E e^{-\beta E} = \sum_E \langle E | e^{-\beta \hat{H}} | E \rangle = \underbrace{Tr(e^{-\beta \hat{H}})}$$

we can use $\hat{x}$ - represent (1 dim QM)

$$Z(\beta) = \int dx \underbrace{\langle x | e^{-\beta \hat{H}} | x \rangle}_{\mathcal{A}(x, x, \beta)}$$

$$\left(\begin{aligned} &\text{transition amplitude ; } iA(x_b t_b, x_a t_a) \\ &= \langle x_b | \underline{e^{-i\hat{H}(t_b - t_a)}} | x_a \rangle \end{aligned} \right)$$

relation between $\mathcal{A}$ and iA

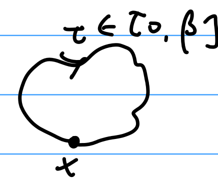
$$x = x_b = x_a \qquad \beta = i(t_b - t_a)$$

in $\mathcal{A}$: time is imaginary $\quad t = -i\tau \qquad \tau_a = 0, \ \tau_b = \beta$

$$\mathcal{A}(x, x, \beta) = \int \mathcal{D}x(t)\, e^{i\int_{-i0}^{-i\beta} \underbrace{\left(\frac{m}{2}\left(\frac{dx}{dt}\right)^2 - V(x)\right)}_{-\frac{m}{2}\left(\frac{dx}{d\tau}\right)^2 - V(x)} \underbrace{dt}_{-i d\tau}}$$

$$= \int_{\text{closed paths at } x} Dx(\tau)\, e^{-\int_0^\beta \left[\frac{m}{2}\left(\frac{dx}{d\tau}\right)^2 + V(x)\right] d\tau}$$

- close path ; $\quad x(\tau = 0) = x_a = x$
  $\quad x(\tau = \beta) = x_b = x$



- Euclidean action $\quad S_E := \oint_0^\beta d\tau \left(\frac{m}{2}\left(\frac{dx}{d\tau}\right)^2 + V(x) \right)$

$$ds^2 = -dt^2 + d\vec{x}^2 \qquad t = -i\tau$$
$$= d\tau^2 + d\vec{x}^2$$

$$Z(\beta) = \int dx\, A(x, x, \beta)$$

$$= \int dx \int_{\substack{\text{closed} \\ \text{paths at } x}} DX(\tau)\, e^{-S_E[x(\tau)]}$$

$$= \int_{\text{closed paths}} DX(\tau)\, e^{-S_E[x(\tau)]}$$ "Euclidean path integral"

sum over all close paths

$$X(\tau=0) = X(\tau=\beta)$$

for all quantum system with finite tempreture β

$Z(\beta)$ = Euclidean path integral on closed paths with period β.

$t_{-\infty} \equiv -T,\ T \to \infty$

Correlation function:

$$\to \mathcal{O}(t) = e^{iH(t-t_{-\infty})}\, \mathcal{O}\, e^{-iH(t-t_{-\infty})}$$

$$= U(t,-T)^{-1}\, \mathcal{O}\, U(t,-T)$$

$\langle 0 | \mathcal{O}_1(t_1)\, \mathcal{O}_2(t_2) | 0 \rangle$ 2 pt-func.

ground state

$$\langle 0 | T\big(\mathcal{O}_1(t_1)\, \mathcal{O}_2(t_2)\big) | 0 \rangle = \begin{cases} \langle 0 | \mathcal{O}_1(t_1)\, \mathcal{O}_2(t_2) | 0 \rangle & t_1 > t_2 \\ \langle 0 | \mathcal{O}_2(t_2)\, \mathcal{O}_1(t_1) | 0 \rangle & t_2 > t_1 \end{cases}$$

time ordered correlation function.

$t_1 > t_2$ $\langle 0 | \mathcal{O}_1(t_1)\, \mathcal{O}_2(t_2) | 0 \rangle = iG(t_1, t_2)$

$$= \langle 0 | e^{iH(t_1 - t_{-\infty})}\, \mathcal{O}_1\, \underline{e^{-iH(t_1 - t_{-\infty})}\, e^{iH(t_2 - t_{-\infty})}}\, \mathcal{O}_2\, e^{-iH(t_2 - t_{-\infty})} | 0 \rangle$$

$$= \langle 0 | e^{iH(t_1 - t_{-\infty})}\, \mathcal{O}_1\, e^{-iH(t_1 - t_2)}\, \mathcal{O}_2\, e^{-iH(t_2 - t_{-\infty})} | 0 \rangle$$

$$\langle 0 | \mathcal{O}_1(x_1 t_1)\, \mathcal{O}_2(x_2 t_2) | 0 \rangle = iG(x_1 t_1, x_2 t_2)$$

$$H|0\rangle = E_0 |0\rangle$$

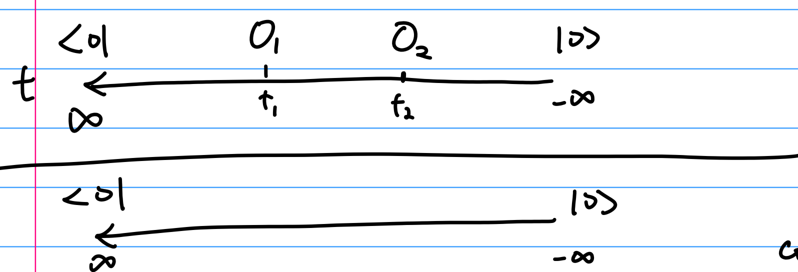
$$iG(t_1,t_2) = e^{iE_0(t_1-t_{-\infty})}\langle 0| O_1 e^{-iH(t_1-t_2)} O_2 |0\rangle e^{-iE_0(t_2-t_{-\infty})}$$

$$= \frac{e^{-iE_0(t_\infty - t_1)}\langle 0|O_1 e^{-iH(t_1-t_2)} O_2|0\rangle e^{-iE_0(t_2-t_{-\infty})}}{e^{-iE_0(t_\infty - t_{-\infty})}}$$

$t_\infty = T \to \infty$

$t_{-\infty} = -T \to -\infty$

$$= \frac{\langle 0| e^{-iH(t_\infty - t_1)} O_1 e^{-iH(t_1-t_2)} O_2 e^{-iH(t_2-t_{-\infty})} |0\rangle}{\langle 0| e^{-iH(t_\infty - t_{-\infty})} |0\rangle}$$



$t_\infty \to \infty$

$t_{-\infty} \to -\infty$

$iG(t_1,t_2)$ only dep. on t_1-t_2

Corelation function is time translation inv.