Reduced phase space quantization

Ref: 0711.0119
0711.0115

Motivation:

issues of canonical LQG:

(1) $\hat{C}\Psi = 0$, $\hat{C}_j\Psi = 0$ hard to solve, have quantum anomaly

(2) $H = \int d^3x \, (NC + N^a C_a) = 0$ on the constraint surface

→ problem of time (time evolution is gauge transf.)

→ issue of unitarity in quantum theory

idea: quantize the reduced phase space

unconstrained phase space P

(A, E, matter DOF)

advantages:

- remove constraints & gauge classically
- introduce physical-time + physical Hamiltonian

functions on P_{red} are gauge inv observables

(Dirac observables)

deparametrized gravity model (gravity + matter)

1. gravity + Brown–Kuchař (BK) dust : $S = S_{GR} + S_{BK}$

$$S_{BK}[\rho, g_{\mu\nu}, T, S^j, W_j] = -\frac{1}{2}\int d^4x \sqrt{|\det g|}\, \rho \left[g^{\mu\nu} U_\mu U_\nu + 1 \right]$$

$j = 1,2,3$

ρ: Lagrangian multiplier (dust density)

T, S^j: scalars (clock fields)

W_j: Lagrangian multiplier

$$U_\mu = -\partial_\mu T + W_j \partial_\mu S^j$$

$$\delta S = 0 \Rightarrow G_{\mu\nu} = 8\pi G\, T_{\mu\nu}^{dust}$$

$$T_{\mu\nu}^{dust} = -\frac{2}{\sqrt{|\det g|}} \frac{\delta S_{BK}}{\delta g^{\mu\nu}} = \rho\, U_\mu U_\nu$$ perfect fluid with no pressure

perfect fluid $T_{\mu\nu} = \rho U_\mu U_\nu + p(g_{\mu\nu} + U_\mu U_\nu)$

we have used : $0 = \frac{\delta S}{\delta \rho} \Rightarrow \underline{g^{\mu\nu} U_\mu U_\nu = -1}$ $\quad U^\mu$ is time like

2. Gaussian dust :

$$S_{GD} = -\int d^4x \sqrt{|\det g|} \left[\frac{\rho}{2} (g^{\mu\nu} \partial_\mu T \partial_\nu T + 1) + g^{\mu\nu} \partial_\mu T (W_j \partial_\nu S^j) \right]$$

3. scalar fields

$$S_\phi = -\frac{1}{2} \int d^4x \sqrt{|\det g|}\, g^{\mu\nu} \partial_\mu \phi^I \partial_\nu \phi^I$$

BK dust : Hamiltonian analysis : $\{P(x)\ T(x')\} = \delta^{(3)}(x, x')$

$\{P^j(x)\ S_k(x')\} = \delta^{(3)}(x, x')$

$$H_{tot} = \int d^3x \left(N C^{tot} + N^a C_a^{tot} \right)$$

$$C^{tot} = C + P\sqrt{1 + q^{ab} C_a C_b / P^2} \approx 0$$

$$C_a^{tot} = C_a + P \partial_a T + P_j \partial_a S^j \approx 0$$

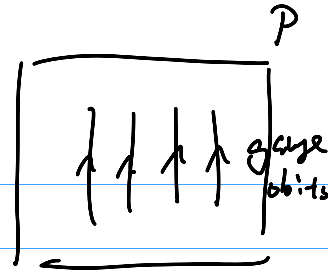
they can be solved by $P = h \equiv \sqrt{C^2 - q^{ab} C_a C_b}$

$P_j = -S_j^a (C_a - h \partial_a T)$ $\quad S_j^a$ is the inverse of $\partial_a S^j$

$$\tilde{C}_\mu(x) \begin{cases} \tilde{C} = P - h \approx 0 \\ \tilde{C}_j = P_j + S_j^a (C_a - h \partial_a T) \approx 0 \end{cases}$$ abelian constraints

i.e. $\{ \tilde{C}_\mu(x), \tilde{C}_\nu(x') \} = 0$

Dirac observables (functions on P_{red})



$\forall\ f$ function on P (unconstrained phase space)

finite gauge transf of f $\quad \alpha_{\vec{\beta}}(f) := \alpha_{\vec{\beta}}^{*} f$

$\uparrow$ diff. group on P generated by $\int d^3x \left(\beta^0_{(x)} \tilde{C}_{(x)} + \beta^j_{(x)} \tilde{C}_{j(x)} \right)$

$$f(\tau, \vec{\sigma}) := \left[\alpha_{\vec{\beta}}(f) \right]_{\alpha_{\vec{\beta}}(T) = \tau,\ \alpha_{\vec{\beta}}(S^j) = \sigma^j}$$

these determine $\beta^0(\vec{x}) = \tau - T(\vec{x})$

$\vec{\beta}(\vec{x}) = \vec{\sigma} - \vec{S}(\vec{x})$

- $f(\tau, \vec{\sigma})$ is a Dirac observable

$$\{ C^{tot}, f \} \approx 0$$

$$\{ C^{tot}_a, f \} \approx 0$$

f is gauge inv. on the constraint surface
$\rightarrow$ f is a function on P_{red}

- f is parametrized by values of dust fields (clock fields)

(T, S^j) dust reference frame

$T \to \tau$ physical time, $S^j \to \sigma^j$ physical space

Example : $g_{ab}(x^\mu)$

$$g_{ij}(\tau, \vec{\sigma}) = \left[\alpha_{\vec{\beta}}\left(g_{ab}(x^\mu) \right) \right]_{\beta^0 = \tau - T(x),\ \vec{\beta} = \vec{\sigma} - \vec{S}(x)}$$

$(\beta^0, \vec{\beta}) = 0$ if we set $\left. \begin{array}{l} T(x) = \tau \\ \vec{S}(x) = \vec{\sigma} \end{array} \right\}$

$$= \left[q_{ab}(x^\mu) \right]_{\substack{T(x)=\tau \\ \vec{S}(x)=\vec{\sigma}}}$$

evaluate gravity fields at point x where dust fields take values

$$T(x) = \tau, \quad \vec{S}(x) = \vec{\sigma}$$

- Dirac observables are constructed relationally, taking dust fields as references
- $$\{ P^{ij}(\tau, \vec{\sigma}), \quad q_{kl}(\tau, \vec{\sigma}') \} = \kappa\, \delta^{(i}_k \delta^{j)}_l \delta^{(3)}(\vec{\sigma}, \vec{\sigma}')$$

or $$\{ E^j_a(\tau, \vec{\sigma}) \quad A^b_k(\tau, \vec{\sigma}') \} = \frac{\kappa\beta}{2} \delta^j_k \delta^b_a \delta^{(3)}(\vec{\sigma}, \vec{\sigma}')$$

here, a, b are $SU(2)$ indices

Canonical conjugate pairs in $\mathcal{P}_{red}$.

$$\mathcal{P} : (A, E), (T, P), (S^j, P_j)$$

constraint surface : $(A, \hat{E}), \; T, \; S^j$

$$\mathcal{P}_{red} : (A(\tau, \sigma), \; E(\tau, \sigma))$$

- physical Hamiltonian

$$H = \int d^3\sigma \, h(\tau, \sigma), \quad h(\tau, \sigma) = \sqrt{C(\tau, \sigma)^2 - q^{ij}(\tau, \sigma)\, C_i(\sigma) C_j(\sigma)}$$

$$C(\tau, \vec{\sigma}) = -\frac{2}{\kappa} \mathrm{tr}\left(F_{ij}(\tau, \sigma) \frac{[E^i(\tau, \sigma), E^j(\tau, \sigma)]}{\sqrt{\det q(\tau, \sigma)}} \right) + \cdots$$

$$C_j(\tau, \vec{\sigma}) = \frac{2}{\beta\kappa} F^a_{jk} E^k_a(\tau, \sigma)$$

$$q^{ij} = e^i_a e^j_a$$

$$C_a = 2 e^i_a C_i$$

$$\frac{\partial}{\partial \tau} f(\tau,\sigma) = \{ f(\tau,\sigma), H \},$$ ↳ before we call it C_j

H generates physical time evolution

the dynamics is free of Hamiltonian & diffeo. constraints

- quantization . γ : cubic lattice in the space of $\vec{\sigma}$ (dust space S)

$$A(\tau,\vec{\sigma}) \rightsquigarrow h(e) \rightsquigarrow \hat{h}(e)$$

$$E(\tau,\vec{\sigma}) \rightsquigarrow p^a(e) \rightsquigarrow \hat{p}^a(e) = \frac{i \ell_p^2}{2} \hat{R}_e^a$$

} quantization of Dirac observables

Gauss constraint $\rightsquigarrow$ $\mathcal{H}_\gamma$: Hilbert space of gauge inv. functions of $h(e)$

$\partial_j E^j_a + \varepsilon^{abc} A^b_j E^j_c = 0$

basis: spin-networks

physical Hilbert space

physical Hamiltonian $H \rightsquigarrow \hat{H} \circlearrowright \mathcal{H}_\gamma$

$$\hat{H} = \sum_v \sqrt{\left| \hat{C}_v^2 - \frac{1}{4}\sum_{j=1}^{3} \hat{C}_{j,v}^2 \right|}$$

Hamiltonian & Diff constraint operators earlier

$$\sqrt{|\hat{O}|} = \left(\hat{O}^\dagger \hat{O}\right)^{\frac{1}{4}}$$

Reduced phase space quantization makes LQG similar to lattice gauge theory with a more complicated Hamiltonian

$\hat{H}$ is self-adj $\longrightarrow$ dynamics is manifestly unitary, problem of time is resolved there is no quantum constraint to be solved

we resolve the problem of constraints.

$\hat{H}$ is complicated, but the quantum dynamics can be studied.

see. e.g. 1910.03763

2005.00988

or my recent ILQGS talk