$$V(\phi) = m^2\phi^2 + \lambda\phi^4 + \phi^6 + \phi^8 + \dots$$

Wilsonian RG   2 step   (1) integrate out high energy mode

(2) rescaling $k' = k/b$, $x' = x b$ $b = \frac{\mu}{\Lambda_0}$

UV $\mathcal{L}_{UV}$

$\downarrow$ μ $\mathcal{L}_\mu = \mathcal{L}_{eff}$

μ' $\mathcal{L}_{\mu'}$

$\mathcal{L}_{eff}$ dep k' $|k'| \le \Lambda_0$

$$S_\mu = \int d^dx\, \mathcal{L}_{eff} = \int d^dx'\, b^{-d}\Big[\frac{1}{2}(1+\Delta Z)\, b^2 (\partial'_\mu\phi)^2 + \frac{1}{2}(m_0^2+\Delta m^2)\phi^2 + \frac{1}{4!}(\lambda_0+\Delta\lambda)\phi^4 + \Delta C\, b^2 \underline{(\partial'_\mu\phi)^2\phi^2} + \Delta D\underline{\phi^6} + \dots\Big]$$

wave function renormalization $\phi' = \left[b^{2-d}(1+\Delta Z)\right]^{\frac{1}{2}}\phi$

renormalized coupling

$$m'^2 = (m_0^2+\Delta m^2)(1+\Delta Z)^{-1} b^{-2}$$

$$\lambda' = (\lambda_0+\Delta\lambda)(1+\Delta Z)^{-2} b^{d-4}$$

$$C' = (C_0+\Delta C)(1+\Delta Z)^{-2} b^{d-2}$$

$$D' = (D_0+\Delta D)(1+\Delta Z)^{-3} b^{+d-6}$$

$$\dots$$

$$S_\mu = \int d^dx' \Big[\frac{1}{2}(\partial'_\mu\phi')^2 + \frac{1}{2}m'^2\phi'^2 + \frac{1}{4}\lambda'\phi'^4 + C'(\partial'_\mu\phi')^2\phi'^2 + D'\phi'^6 + \dots\Big]$$

UV Λ

μ $\rightarrow$ $m' = m(b)$, $\lambda' = \lambda(b)$, $C' = C(b)$, $D' = D(b)$ ... $b = \frac{\mu}{\Lambda_0}$

$g(b) \nearrow$ as $b \downarrow$ $\longrightarrow$ g is relevant

$g(b) \downarrow$ as $b \downarrow$ $\longrightarrow$ g is irrelevant

$g(b)$ unchange as $b \downarrow$ $\longrightarrow$ g is marginal if RG leave S_M inv. (e.g all $g(b)$ are marginal) : RG fix pt.

UV

IR

$$\mathcal{L}_0 = \frac{1}{2}(\partial_\mu \phi)^2$$ is a fix point in RG (Gaussian fix pt)

semiclassical limit :

$$\left.\begin{aligned} m'^2 &= (m_0^2 + \Delta m^2)(1+\Delta Z)^{-1} b^{-2} \rightarrow m_0^2 b^{-2} \\ \lambda' &= (\lambda_0 + \Delta\lambda)(1+\Delta Z)^{-2} b^{d-4} \rightarrow \lambda_0 b^{d-4} \\ C' &= (C_0 + \Delta C)(1+\Delta Z)^{-2} b^{d-2} \rightarrow C_0 b^{d-2} \\ D' &= (D_0 + \Delta D)(1+\Delta Z)^{-3} b^{2d-6} \rightarrow D_0 b^{2d-6} \end{aligned}\right\} \text{semiclassical RG}$$

...

$\Delta m^2, \Delta\lambda, \Delta C, \Delta D \dots$ are all $O(\hbar)$ $\quad d=4$

in general $\quad C_{N,M} \int d^d x \, (\phi^N, \partial^M)$, $\quad C_{4,2} \int d^4 x \, (\partial_\mu \phi)^2 \phi^2$

$$\Rightarrow C'_{N,M} = b^{N(\frac{d}{2}-1)+M-d} C_{N,M} = b^{-[C_{N,M}]} C_{N,M}$$

<u>HW</u> show this

$[\phi] = \frac{d}{2} - 1$ $\qquad [C_{N,M}] = -\left(N(\frac{d}{2}-1) + M - d\right)$

$[x] = -1$

$[\partial] = 1 \qquad [S_M] = 0$

$\Rightarrow$ $[C_{N,M}] > 0$, $C_{N,M} \nearrow$ as $b \downarrow$, relevant coupling

$[C_{N,M}] < 0$, $C_{N,M} \downarrow$ as $b \downarrow$, irrelevant coupling

$[C_{N,M}] = 0$, semiclassically marginal

we need quantum correction to judge

$\rightarrow$ irrelevant coupling has negative mass dimension

quantum gravity : coupling const , $\int e^{\frac{i}{\hbar}\frac{1}{G}\int R\sqrt{g}\, d^4x} \, Dg_{\mu\nu}$ $\qquad \hbar G = l_p^2$

$\frac{1}{g^2} F_{\mu\nu} F^{\mu\nu}$

$[l_p^2] = -2 \rightarrow$ all gravity interaction are irrelevant

gravity is a free spin-2 theory at IR fix pt

$(\partial_\mu \phi)^2 + \lambda \phi^4$

weakly coupled theory at IR

but coupling grows as $b \nearrow$, strongly coupled at UV

HW show l_p^2 is a coupling const.

perturbative QFT fails.

$g_{\mu\nu} = \eta_{\mu\nu} + l_p h_{\mu\nu}$

$m^2\phi^2 + \phi^4 + \phi^6 + \phi^8 + \partial^2\phi^2 \cdot \phi^2 + \partial^4 \phi^4 - - \ldots$

$$S[\phi] = \int d^dx \left[(\partial_\mu \phi)^2 + V(\phi) \right]$$

QFTs are usually low energy effective theories.

QFT tells us :

symmetry principle $\rightarrow$ local Lagrangian $\rightarrow$ all local interaction terms respecting the symmetry

∞ number of terms

(classial) RG analysis $\longrightarrow$ what is relevant at IR

what is irrelevant at IR

$\longrightarrow$ you have less terms in Lagrangian.

$\uparrow$

low energy effective theory

for scalar theory : all higher order & higher derivative interactions are irrelevant.

$$\longrightarrow S = \int d^4x \left[\frac{1}{2} (\partial_\mu \phi)^2 + \underset{\substack{\uparrow \\ \text{relevant}}}{\frac{1}{2} m^2 \phi^2} + \underset{\substack{\uparrow \\ \text{classically marginal.}}}{\frac{\lambda}{4!} \phi^4} \right] \qquad d=4$$

β - function $\qquad \lambda' = (\lambda_0 + \Delta\lambda)(1 + \Delta Z)^{-2} b^{d-4} \qquad d \to 4$

$$\Delta\lambda = 3 = -\frac{3\lambda_0^2}{16\pi^2} \ln \frac{1}{b} \qquad \Delta Z \sim O(\lambda_0^2)$$

$\lambda' \left\{ \lambda_0 \downarrow \mu \right.$

$d \to 4 \qquad \lambda' = \left[\lambda_0 + \frac{3\lambda_0^2}{16\pi^2} \ln b \right] \left[1 + O(\lambda_0^2) \right] \qquad b = \frac{\mu}{\Lambda_0}$

$$\beta = \mu \frac{\partial}{\partial \mu} \lambda' = b \frac{\partial}{\partial b} \lambda' = \frac{\partial}{\partial \ln b} \lambda' = \frac{3\lambda_0^2}{16\pi^2} \simeq \frac{3\lambda'^2}{16\pi^2}$$

same β-function as in perturbative renormalization

$\beta > 0$, $b \downarrow \Rightarrow \lambda' \downarrow$ irrelevant coupling

(marginal irrelevant.)

interacting scalar theory

UV fix pt ? ← (UV completion)

$\mu \downarrow$

← strongly coupled, cannot be defined as perturbation theory when μ is large + you need to take into account higher-order & higher derivative interactions

← weakly coupled $\lambda\phi^4$ theory

IR fix pt. : free massless scalar

QG : UV fix pt ← UV completion of QG.

$\mu \downarrow$

← strongly coupled regime, perturbation theory fails, you need higher order & higher der. interactions. (nonlocal)

← graviton + weakly coupled interactions, perturbation theory from Einstein gravity

↑ low energy effective theory.

IR fix pt : free spin-2 theory (graviton)