time reversal

$$i\hbar \frac{\partial}{\partial t}\psi(\vec{r},t) = \hat{H}\psi(\vec{r},t)$$

1) $\hat{H}$ is real, e.g. $\hat{H} = \frac{\hat{p}^2}{2m} + V(\vec{r})$

no explicit i

2) doesn't explicitly dep. on t

time reversal $t \to -t$

$$i\hbar \frac{\partial}{\partial(-t)}\psi(\vec{r},-t) \doteq \hat{H}\psi(\vec{r},-t)$$

complex conjugate

$$i\hbar \frac{\partial}{\partial t}\psi(\vec{r},-t)^* = \hat{H}\psi(\vec{r},-t)^*$$

Schrödinger eqn. is time reversal inv.

up to transformation of wave function $\psi(\vec{r},t) \to \psi(\vec{r},-t)^*$

time reversal operator, $\hat{T}_0 : \psi(\vec{r},t) \to \hat{T}_0\psi(\vec{r},t)$

$$= \psi(\vec{r},-t)^*$$

$\hat{T}_0^2 = 1$ i.e. $\hat{T}_0^{-1} = \hat{T}_0$

time reversal of operators

$$\hat{\vec{P}}' = \hat{T}_0 \hat{\vec{P}} \hat{T}_0 = -\hat{\vec{P}}$$

since $\hat{T}_0 \hat{\vec{P}} \hat{T}_0 \psi(\vec{r},t) = \hat{T}_0 \left(-i\hbar\vec{\nabla}\right) \psi^*(\vec{r},-t)$

$$= i\hbar\vec{\nabla}\psi(\vec{r},t)$$

$$= -\hat{\vec{P}}\psi(\vec{r},t)$$

$$\hat{\vec{R}}' = \hat{T}_0 \hat{\vec{R}} \hat{T}_0 = \hat{\vec{R}}$$

$$\hat{\vec{L}}' = \hat{T}_0 \hat{\vec{L}} \hat{T}_0 = -\hat{\vec{L}}$$

$$TL = -LT$$

$\psi(\vec{r},t)$ satisfies Schrödinger eqn.

$\Rightarrow \psi'(\vec{r},t) = \hat{T}_0 \psi(\vec{r},t)$ satisfies the same Schrödinger eqn.

$$\hat{H}|\psi_{ni}\rangle = E_n|\psi_{ni}\rangle \quad i = 1 \cdots d(n)$$

general sol. of Schrödinger eqn.

$$\psi(\vec{r},t) = \sum_{n,i} a_{ni}\psi_{ni}(\vec{r})\, e^{-\frac{i}{\hbar}E_n t}$$

$$\hat{T}_0 \Psi(\vec{r},t) = \sum_{n,i} a_{ni}^* \Psi_{ni}(\vec{r})^* e^{-\frac{i}{\hbar}E_n t}$$

$$\hat{T}_0 : \Psi_{ni}(\vec{r}) \mapsto \Psi_{ni}(\vec{r})^*$$

$$\mathcal{H}^{(n)} \mapsto \mathcal{H}^{(n)*}$$

<u>properties of $\hat{T}_0$</u>

$$\hat{T}_0 = \hat{K}\hat{T}_1$$

$$\hat{T}_1 \Psi(\vec{r},t) = \Psi(\vec{r},-t)$$

$$\hat{K}\Psi(\vec{r},t) = \Psi^*(\vec{r},t)$$

1) antilinear :

$$\hat{O}(a\Psi_1 + b\Psi_2) = a^*\underline{\hat{O}\Psi_1} + b^*\underline{\hat{O}\Psi_2}$$

$$\hat{T}_0(\alpha\Psi_1 + \beta\Psi_2) = \hat{T}_0(\alpha\Psi_1) + \hat{T}_0(\beta\Psi_2)$$

$$= \alpha^*\Psi_1(\vec{r},-t)^* + \beta^*\Psi_2(\vec{r},-t)$$

$$= \alpha^*\hat{T}_0\Psi_1 + \beta^*\hat{T}_0\Psi_2$$

2) anti-unitarity $\langle \hat{T}_0 \psi_1 | \hat{T}_0 \psi_2 \rangle$

$$= \int d^3r \left(\psi_1^*(\vec{r}, -t) \right)^* \psi_2(\vec{r}, -t)^*$$

$$= \int d^3r \; \psi_1(\vec{r}, -t)\, \psi_2(\vec{r}, -t)^*$$

$$= \langle \psi_2 | \psi_1 \rangle = \langle \psi_1 | \psi_2 \rangle^*$$

<u>time reversal of spin-$\frac{1}{2}$</u>

$$\mathcal{H} = \mathcal{H}_0 \otimes \mathbb{C}^2 \qquad \psi = \begin{pmatrix} \psi_1(\vec{r}, t) \\ \psi_2(\vec{r}, t) \end{pmatrix} \in \mathcal{H}$$

$\mathcal{H}_0$: $L^2(\mathbb{R}^3)$

$\hat{T}$ time reversal s.t. $\underline{\underline{\hat{T} \hat{\vec{S}} \hat{T}^{-1} = -\hat{\vec{S}}}}$

$\hat{\vec{S}}$: spin operator

$$\hat{\vec{S}} = \frac{\hbar}{2} \vec{\sigma}$$

$$\hat{T} = U \hat{T}_0 \qquad \hat{T}_0 = \hat{K} \hat{T}_1$$

U: 2x2 unitary matrix

$$\hat{T}\begin{pmatrix}\psi_1(\vec{r},t)\\ \psi_2(\vec{r},t)\end{pmatrix} = U\underline{\begin{pmatrix}\psi_1(\vec{r},-t)^*\\ \psi_2(\vec{r},-t)^*\end{pmatrix}}$$

$$\hat{T}\hat{\vec{S}}\hat{T}^{-1} = U T_0 \vec{S} T_0^{-1} U^{-1} = U \vec{S}^* U^{-1}$$

S_x, S_z real, S_y imaginary

$$U S_x U^{-1} = -S_x$$

$$\underline{\sigma_i \sigma_j + \sigma_j \sigma_i = 2\,\mathbb{1}_{2\times 2}\delta_{ij}}$$

$$U S_y U^{-1} = S_y$$

$$\sigma_y^2 = 1$$

$$U S_z U^{-1} = -S_z$$

$$U = e^{i\varphi}\sigma_y$$

φ is free parameter

$$U^{-1} = e^{-i\varphi}\sigma_y$$

$$T^2 = U T_0 U T_0 \begin{pmatrix}\psi_1(\vec{r}\ t)\\ \psi_2(\vec{r},t)\end{pmatrix}$$

$$= U T_0 e^{i\varphi}\sigma_y\begin{pmatrix}\psi_1^*(\vec{r},-t)\\ \psi_2^*(\vec{r},-t)\end{pmatrix}$$

$$= U e^{-i\varphi} (-\sigma_y) \begin{pmatrix} \psi_1(\vec{r}, t) \\ \psi_2(\vec{r}, t) \end{pmatrix}$$

$\uparrow$

$e^{i\varphi\sigma_y}$

$$= (-1) \begin{pmatrix} \psi_1(\vec{r}, t) \\ \psi_2(\vec{r}, t) \end{pmatrix}$$

$T^2 = -1$ for spin-$\frac{1}{2}$ fermion

$\hat{T}$ is an antiunitary Z_2 symmetry generator for fermion. (for boson $T^2 = 1$)

$$Z_2 = \{1, -1\}$$

if $[H, T] = 0 \Rightarrow H|\psi\rangle = E|\psi\rangle$

$$HT|\psi\rangle = ET|\psi\rangle$$

if $T^2 = -1$, $\langle\psi|T\psi\rangle = 0$

"Kramers Thm"

(due to T reverse the angular momentum)

every energy level is at least doubly degenerate.