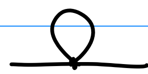
Ryder's QFT. chapter 9.

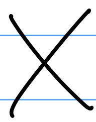
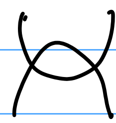
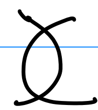
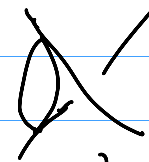
Peskin & schröder chapter 10 p325 – chapter 12

2pt func :  + 

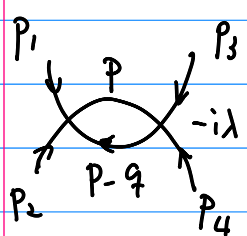
$O(\lambda^0)$ $O(\lambda)$

$$= \frac{i\lambda m^2}{16\pi\varepsilon} + \text{finite}.$$

$\varepsilon = 4-d$

4pt func. =  +  +  + 

$O(\lambda)$ $O(\lambda^2)$ $O(\lambda^2)$ $O(\lambda^2)$



$$= \frac{1}{2}\lambda^2(\mu^2)^{4-d}\int\frac{d^dp}{(2\pi)^d}\frac{1}{p^2-m^2}\frac{1}{(p-q)^2-m^2}$$

$q = p_1+p_2$

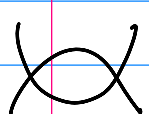
$$\text{integrand} = \frac{1}{a}\frac{1}{b} = \frac{1}{b-a}\left(\frac{1}{a}-\frac{1}{b}\right) = \frac{1}{b-a}\int_a^b\frac{dx}{x^2}$$

$$x = az+b(1-z) \qquad = \int_0^1\frac{dz}{[az+b(1-z)]^2}$$

$$a = p^2-m^2 \qquad b = (p-q)^2-m^2$$

$$\frac{1}{p^2-m^2}\frac{1}{(p-q)^2-m^2} = \int_0^1\frac{dz}{[p^2-m^2-2pq(1-z)+q^2(1-z)]^2}$$

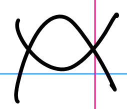
$$p = p' + q(1-z)$$

 $= \frac{1}{2}\lambda^2(\mu^2)^{4-d}\int\frac{d^dp'}{(2\pi)^d}\int_0^1 dz\,\frac{1}{[p'^2-m^2+q^2(1-z)z]^2}$

$$p'^0 \to ip'^0_E \quad \Rightarrow \quad p'^2 \to -p'^2_E = |p'_E|^2$$

$$d^dp \sim d|p|\,d\theta\cdots \quad (\ldots)$$

HW compute this integral

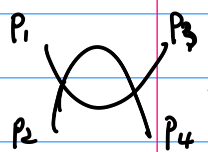


$$= \frac{i\lambda^2}{32\pi^2}(\mu^2)^{2-\frac{d}{2}}\,\underline{\Gamma\left(2-\frac{d}{2}\right)}\int_0^1 dz\left[\frac{q^2 z(1-z)-m^2}{4\pi\mu^2}\right]^{\frac{d}{2}-2}$$

$d=4-\varepsilon$ $\qquad \Gamma\left(2-\frac{d}{2}\right)=\Gamma\left(\frac{\varepsilon}{2}\right)=\frac{2}{\varepsilon}-\gamma+O(\varepsilon)$

pole at 0 $\Longleftrightarrow$ Logarithmic div.

$$= \frac{i\lambda^2\mu^\varepsilon}{32\pi^2}\left(\frac{2}{\varepsilon}-\gamma+O(\varepsilon)\right)\left(1-\frac{\varepsilon}{2}\int_0^1 dz\ln\left[\frac{q^2z(1-z)-m^2}{4\pi\mu^2}\right]+O(\varepsilon^2)\right)$$



$$= \underbrace{\frac{i\lambda^2\mu^\varepsilon}{16\pi^2\varepsilon}}_{\text{div.}} \underbrace{- \frac{i\lambda\mu^\varepsilon}{32\pi^2}\left(\gamma+\int_0^1 dz\ln\left[\frac{q^2z(1-z)-m^2}{4\pi\mu^2}\right]\right)}_{\text{finite}} + O(\varepsilon)$$

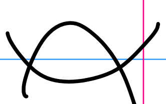
s-chanel

$$F(s,m,\mu) = \int_0^1 dz\ln\left[\frac{sz(1-z)-m^2}{4\pi\mu^2}\right]$$

$q = p_1+p_2$
$\quad = p_3+p_4$

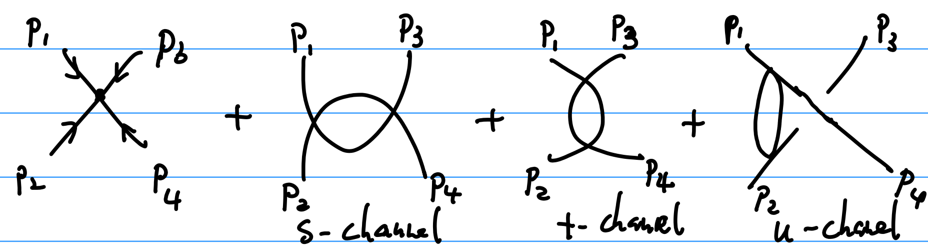
$q^2 = (p_1+p_2)^2 = s$ $\qquad$ Lorentz inv.

Mandelstam variables: $\quad s=(p_1+p_2)^2, \quad t=(p_1+p_3)^2, \quad u=(p_1+p_4)^2$

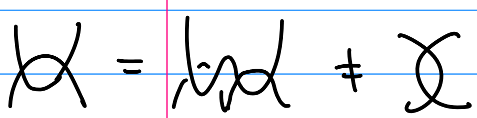


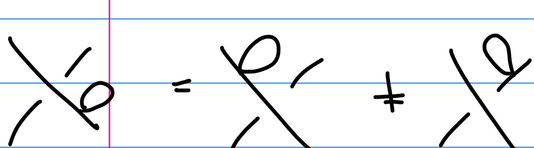
$$= \frac{i\lambda^2\mu^\varepsilon}{16\pi^2\varepsilon} - \frac{i\lambda\mu^\varepsilon}{32\pi^2}\left[\gamma+F(s,m,\mu)\right]$$

4-pt amplitude: $\Gamma^{(4)}(p_1\cdots p_4) =$



$$= -i\lambda\mu^\varepsilon + \frac{3i\lambda^2\mu^\varepsilon}{16\pi^2\varepsilon} - \frac{i\lambda\mu^\varepsilon}{32\pi^3}\Big[3\gamma + F(s,m,\mu) + F(t,m,\mu) + F(u,m,\mu)\Big] + O(\varepsilon)$$





$$\Gamma^{(4)} = -i\lambda\mu^\varepsilon\left(1-\frac{3\lambda}{16\pi^2\varepsilon}\right) + \text{finite}$$

"pre-baby" renormalization : $\lambda = \lambda_0$ bare coupling (div.)

$$\lambda_0 \mu^{\varepsilon}\left(1 - \frac{3\lambda_0}{16\pi^2 \varepsilon}\right) \equiv \lambda_{phys} \text{ (finite)}$$ renormalized coupling

$$\Gamma^{(4)} = -i \underline{\underline{\lambda_{phys}}} + \text{finite}$$

Perturbative renormalization of $\lambda\phi^4$ $(d = 4-\varepsilon)$ (renormalization for babies)

the interacting Lagrangians dep. on bare quantities, instead of physical quantities

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi_0 \partial^\mu \phi_0 - \frac{1}{2} m_0^2 \phi_0^2 - \frac{\lambda_0 \mu^{4-d}}{4!}\phi_0^4$$

ϕ_0 : bare field (wave function)

m_0 : bare mass

λ_0 : bare coupling

Renormalized field $\phi_0(x) = Z^{\frac{1}{2}} \phi(x)$

$$\mathcal{L} = \frac{1}{2} Z (\partial_\mu \phi)^2 - \frac{1}{2} Z m_0^2 \phi^2 - \frac{\lambda_0 \mu^{4-d}}{4!}\phi^4 Z^2$$

Define renormalized mass & coupling

$$\delta_m = m_0^2 Z - m^2 , \quad \delta_\lambda = \lambda_0 Z^2 - \lambda$$

$$\delta_Z = Z - 1$$

renormalized mass (m^2) renormalized coupling (λ)

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2} m^2 \phi^2 - \frac{\lambda \mu^{4-d}}{4!}\phi^4 \leftarrow \mathcal{L}_r \text{ renormalized Lagrangian}$$

$$+ \frac{1}{2}\delta_Z (\partial_\mu \phi)^2 - \frac{1}{2}\delta_m \phi^2 - \underbrace{\frac{\delta_\lambda \mu^{4-d}}{4!}}\phi^4(x) \leftarrow \underline{\text{local}} \text{ counter terms } \delta\mathcal{L}$$

$$= \mathcal{L}_r + \delta\mathcal{L}$$

$$Z[J] = e^{\lambda \int \left(\frac{\delta}{\delta J}\right)^4} Z_0[J]$$

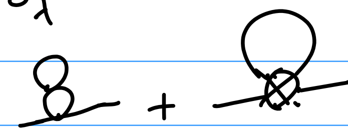
nonlocal term : $\int d^4x\, d^4y\ \phi(x)\, G(x-y)\, \phi(y)$

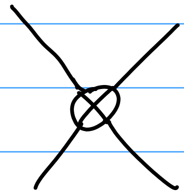
$\phi_0 \to \phi$ wavefunction renormalization : δ_Z

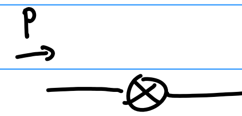
$m_0 \to m$ mass renormalization : δ_m

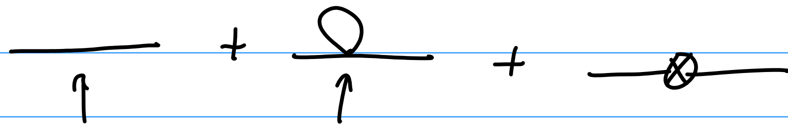
$\lambda_0 \to \lambda$ coupling const. renormalization : δ_λ

$\Bigg\}$ renormalization const.

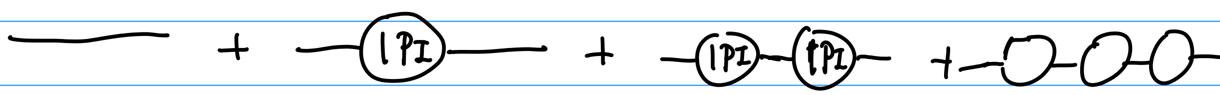
$\delta\mathcal{L}$: new Feynman rules (in momentum space) 

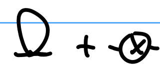
 $= -i\delta_\lambda \mu^{4-d}$

 $= i(p^2\delta_Z - \delta_m)$

2pt function = 

renormalized quantities

= 

 $\equiv -i\Sigma^2(p^2)$ + ...

$$= \frac{i}{p^2-m^2+i\varepsilon} + \frac{i}{p^2-m^2+i\varepsilon}\left(-i\Sigma^2(p^2)\right)\frac{i}{p^2-m^2+i\varepsilon}$$

$$+ \frac{i}{p^2-m^2+i\varepsilon}\left(-i\Sigma^2(p^2)\right)\frac{i}{p^2-m^2+i\varepsilon}\left(-i\Sigma^2(p^2)\right)\frac{i}{p^2-m^2+i\varepsilon}$$

$$+ \cdots$$

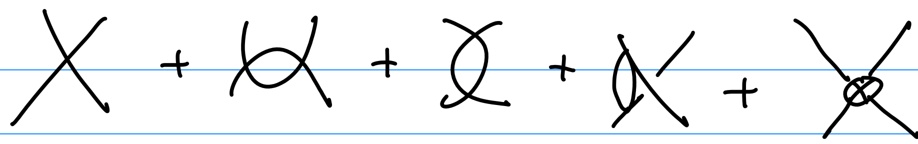
$$= \frac{i}{p^2-m^2-\Sigma^2(p^2)+i\varepsilon}$$

$$\frac{i}{p^2-m^2-\Sigma^2} = \frac{i}{(p^2-m^2)\left(1-\frac{\Sigma^2}{p^2-m^2}\right)} = \frac{i}{p^2-m^2}\left(1+\frac{\Sigma^2}{p^2-m^2}\right)$$

$$= \frac{i}{p^2-m^2} + \frac{i}{p^2-m^2}\left(-i\Sigma^2\right)\frac{i}{p^2-m^2}$$

$$-i\Sigma^2(p^2) = \frac{i\lambda m^2}{16\pi^2\varepsilon} + \frac{i\lambda m^2}{32\pi^2}\left[1-\gamma+\ln\left(\frac{4\pi\mu^2}{-m^2}\right)\right] + O(\varepsilon) + ip^2\delta_Z - \delta_m$$

idea : we can set δ_Z, δ_m to cancel $\frac{1}{\varepsilon}$ - pole $\rightarrow$ everything is finite

4pt func =  (we only consider 1-loop renormalization)

$$= -i\lambda\mu^{\varepsilon} + \frac{3i\lambda^2\mu^{\varepsilon}}{16\pi^2\varepsilon} - \frac{i\lambda^2\mu^{\varepsilon}}{32\pi^2}\left[3\gamma + F(s,m,\mu) + F(t,m,\mu) + F(u,m,\mu)\right] + O(\varepsilon)$$

$\varepsilon = 4-d$

$$-i\delta_\lambda\mu^{\varepsilon}$$

How to determine renormalization const. $\delta_m, \delta_Z, \delta_\lambda$?

Renormalization conditions: we set an energy scale M (the energy scale of scattering experiments)

(1) 2-pt func $= \dfrac{i}{p^2 - m^2 + i\varepsilon}$ physical propagator

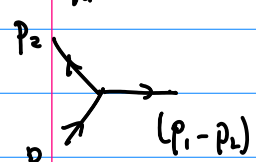
when massive $(m^2 > 0)$: $p^2 \rightarrow m^2$ (propagator has the pole at physical mass)

$\uparrow$ mass is a prefered energy scale

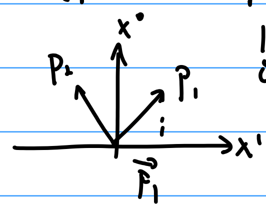
massless $(m^2 = 0)$: $p^2 \rightarrow -M^2$

$\uparrow$ arbitrary scale

$m^2 = 0$



$$(p_1 - p_2)^2 = \underset{0}{\underset{\parallel}{p_1^2}} + \underset{0}{\underset{\parallel}{p_2^2}} - 2p_1 \cdot p_2 < 0$$



$p_1 = |\vec{p}_1|(1,1,0,0)$

$p_2 = |\vec{p}_2|(1,-1,0,0)$

$$\eta = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

$p_1 \cdot p_2 \propto 1 + 1 = 2 > 0$

(2) 4-pt function $= \underline{\underline{-i\lambda\mu^{\varepsilon}}}$

$p_1^2 = p_2^2 = m^2$

when massive $(m^2 > 0)$: $s = 4m^2$, $t = u = 0$

$$s = (p_1 + p_2)^2 = p_1^2 + p_2^2 + 2p_1 \cdot p_2 = 4m^2 \quad \text{rest particle}$$

massless ($m^2=0$) : $\underset{\underset{(p_1+p_2)^2}{\shortparallel}}{s} = \underset{\underset{(p_1+p_3)^2}{\shortparallel}}{t} = \underset{\underset{(p_1+p_4)^2}{\shortparallel}}{u} = -\mu^2$

(λ is the physical coupling at the energy scale μ)

! the perturbative QFT is well-defined only when specifying an energy scale !

i.e. perturbative QFTs depend on a choice of energy scale.

2pt-function =

HW : derive 1-loop renormalization for QED

derive β-function

Due before 9/30