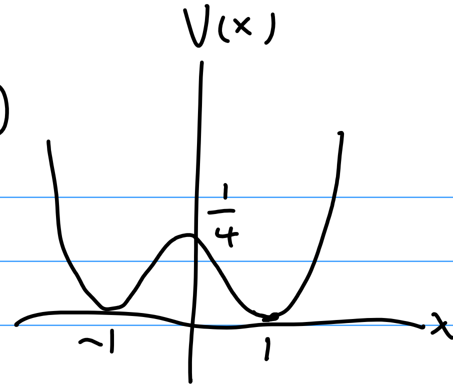
$$L(\dot{x}, x) = \frac{1}{2} m \left(\frac{dx}{d\tau}\right)^2 - V(x)$$

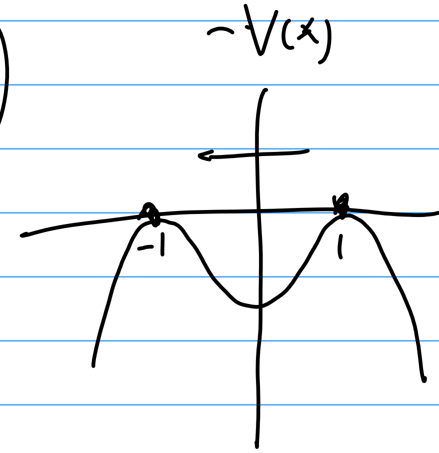


$$V(x) = \frac{1}{4}(x^2 - 1)^2$$

$$A(x_b, x_a, \tau) = \langle x_b | e^{-H\tau} | x_a \rangle$$

$$= \int D\underline{\underline{x(\tau)}}\, e^{-\frac{1}{\hbar} S_E[x(\tau)]}$$

$$S_E = \int_{-\infty}^{\infty} d\tau \left(\frac{1}{2} m \dot{x}^2 + V(x) \right)$$



$$\boxed{\begin{aligned} x_a &= x(\tau \to -\infty) = x_0 = 1 \\ x_b &= x(\tau \to \infty) = -x_0 = -1 \end{aligned}}$$

An <u>instanton</u> is a classical solution from Euclidean path integral, with a finite nonzero S_E

EOM :
$$m \frac{d^2x}{d\tau^2} = \frac{dV}{dx} = x(x^2 - 1)$$

$$x(\tau) = x_0 = 1 \qquad x(\tau) = -x_0 = -1$$

$$\Rightarrow S_E = 0$$

$$S_E = m \int d\tau\, \frac{1}{2} \left[\left(\frac{dx}{d\tau}\right)^2 + \left(\sqrt{\frac{2V}{m}}\right)^2 \right]$$

$$= m\int_{-\infty}^{\infty} d\tau \, \frac{1}{2}\underbrace{\left[\left(\frac{dx}{d\tau}\right) - \sqrt{\frac{2V}{m}}\,\right]^2}_{\to 0 \; : \; \text{minimum}} + \underbrace{m\int_{-\infty}^{\infty} d\tau \, \frac{dx}{d\tau}\sqrt{\frac{2V}{m}}}_{\sqrt{m}\int_{1}^{-1} dx \, \frac{1}{\sqrt{2}}(x^2-1)}$$

$$\geq \sqrt{m}\,\frac{2\sqrt{2}}{3} \qquad\qquad = \sqrt{m}\,\frac{2\sqrt{2}}{3}$$

Bogomol'nyi – Prasad – Sommerfield (BPS) bound

Saturation of the BPS bound : $\frac{dx}{d\tau} = \sqrt{\frac{2V}{m}} \leftarrow$

minimum of S_E : solution is "BPS state"

$$m\frac{d^2x}{d\tau} = m\frac{d}{d\tau}\left(\frac{dx}{d\tau}\right) = m\frac{d}{d\tau}\left(\sqrt{\frac{2V}{m}}\right) = m\sqrt{\frac{2}{m}}\,\frac{1}{2\sqrt{V}}\,\frac{dV}{dx}\,\frac{dx}{d\tau}$$

$$= \sqrt{\frac{m}{2V}}\sqrt{\frac{2V}{m}}\,\frac{dV}{dx} = \frac{dV}{dx}$$

BPS bound implies EOM.

$\therefore$

$$\frac{dx}{d\tau} = \sqrt{\frac{2}{m}}\,\frac{1}{2}(x^2-1)$$

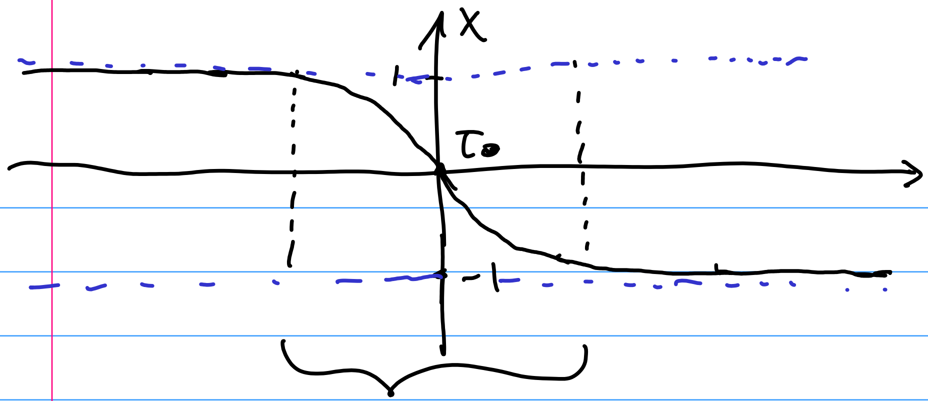
$$\Rightarrow x(\tau) = \tanh\left(-\frac{1}{\sqrt{2m}}(\tau-\tau_0)\right)$$

τ_0 is an integration const

HW

solve BPS eqn

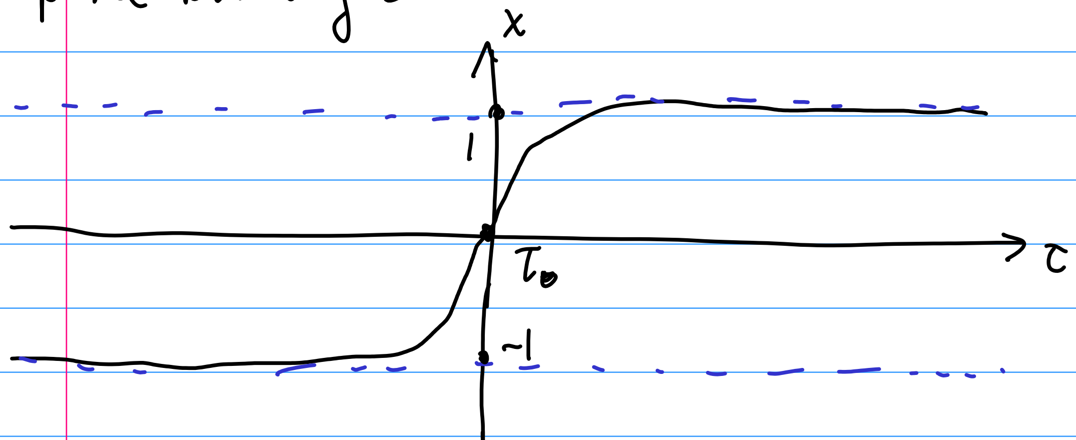
for kink & anti-kink



$\Delta\tau \sim \sqrt{2m}$ transition time / tunneling time

kink (instanton)

flip the boundary condition



antikink (anti-instanton)

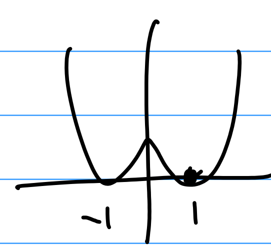
HW: show S_E of antikink $= \sqrt{m}\,\frac{2\sqrt{2}}{3} \equiv S_0$

$$\langle x_0|\, e^{-H(\tau_b-\tau_a)}\, |-x_0\rangle \qquad x_0 = 1$$

$$= \int Dx(\tau)\, e^{-\frac{1}{\hbar}S_E[x(\tau)]} = \sum\!\!\!\!\!\!\int_{x_c} \frac{e^{-\frac{1}{\hbar}S_E[x_c(\tau)]}}{\sqrt{\det(M)}}\,(1+O(\hbar))$$

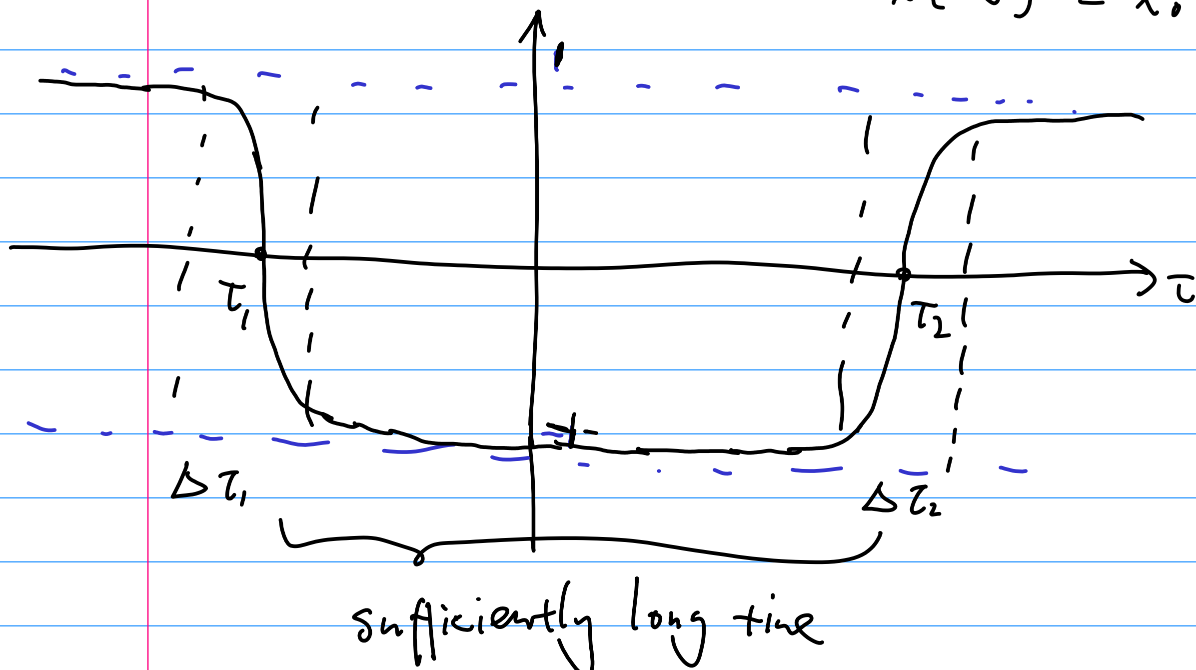
$$= \int_{\tau_a}^{\tau_b} d\tau_0\, \underline{\underline{K}}\, \underline{\underline{e^{-\frac{S_0}{\hbar}}}} \qquad K = \sqrt{\frac{1}{\det M}}$$

amplitude of quantum tunneling.

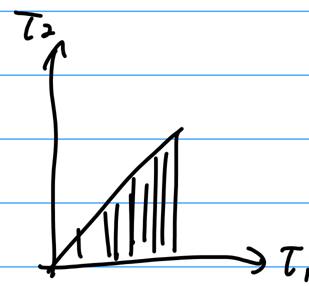


multi-instanton solution : b.c $x(-\infty) = x_0 = 1$

$x(\infty) = x_0 = 1$



2-kink solution



$$\text{amplitude} = \int_{-\infty}^{+\infty} d\tau_2 \int_{-\infty}^{\tau_2} d\tau_1 \underbrace{K e^{-S_0}}_{\text{at } \tau_2} \underbrace{K e^{-S_0}}_{\text{at } \tau_1}$$

"n-instanton gas"

$$\text{amplitude} = \int_{-\infty}^{\infty} d\tau_n \int_{-\infty}^{\tau_n} d\tau_{n-1} \cdots \int_{-\infty}^{\tau_2} d\tau_1 \underbrace{\left(K e^{-S_0}\right)}_{\text{const.}}{}^{n}$$

$$\langle x_0 | e^{-H(\tau_b - \tau_a)} | x_0 \rangle$$

$$= e^{-\frac{\omega_0}{2}(\tau_b - \tau_a)} \sum_{\substack{n \text{ even} \\ n=0}} \frac{(\tau_b - \tau_a)^n}{n!} \left(K e^{-S_0}\right)^n \qquad \hbar = 1$$

$E_n = (n + \frac{1}{2})\hbar\omega_0$

$$\langle x_0 | e^{-HT} | x_0 \rangle \simeq \sum_n e^{-E_n T} \langle x_0 | n \rangle \langle n | x_0 \rangle$$

like a simple oscillator $= e^{-E_0 T} \underline{\langle x_0|0\rangle\langle 0|x_0\rangle}$

$$E_0 = \frac{1}{2}\hbar\omega_0$$

$$= e^{-\frac{\omega_0}{2}(\tau_b-\tau_a)} \cosh\left[(\tau_b-\tau_a) K e^{-S_0}\right]$$

$$\langle x_0| e^{-H(\tau_b-\tau_a)} |-x_0\rangle$$

$$= e^{-\frac{\omega_0}{2}(\tau_b-\tau_a)} \sum_{n \text{ odd}} \frac{(\tau_b-\tau_a)^n}{n!} \left(K e^{-S_0}\right)^n$$

$$= e^{-\frac{\omega_0}{2}(\tau_b-\tau_a)} \sinh\left[(\tau_b-\tau_a) K e^{-S_0}\right]$$

$$|\psi_\pm\rangle = \frac{|x_0\rangle \pm |-x_0\rangle}{\sqrt{2}}$$

$$\langle\psi_\pm| e^{-TH} |\psi_\pm\rangle = e^{-T\left(\frac{\omega_0}{2} \mp K e^{-S_0}\right)}$$

$$\langle\psi_\mp| e^{-TH} |\psi_\pm\rangle = 0$$

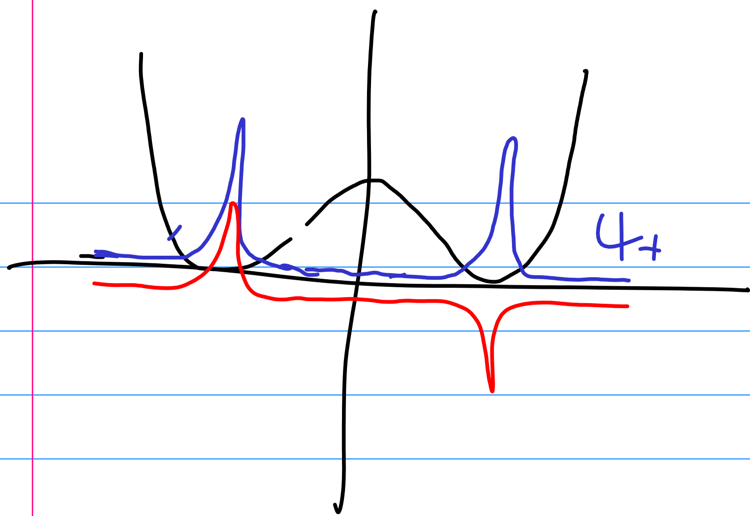
$$\rightarrow \quad H|\psi_\pm\rangle = \left(\frac{\omega_0}{2} \mp K e^{-S_0}\right)|\psi_\pm\rangle$$

ground state : $\psi_+ = \dfrac{|x_0\rangle + |-x_0\rangle}{\sqrt{2}}$

ground state energy : $\frac{\omega_0 \hbar}{2} - K e^{-S_0/\hbar}$

instanton effect

simple Harmonic oscillator



$\hat{x}(t) \quad |x(t)\rangle$

path integral of real scalar field

$\hat{\phi}(x,t) \, | \phi(x,t)\rangle$

$$\langle x_b, t_b | e^{-iHt} | x_a \, t_a\rangle = \int Dx(t) \, e^{\frac{i}{\hbar} S[x(t)]}$$

$$\langle \phi(x_b \, t_b) | e^{-iHt} | \phi(x_a, t_a)\rangle$$

$$= \int D\phi(\vec{x},t) \; e^{\frac{i}{\hbar} S[\phi(\vec{x},t)]}$$

$1 >> \theta > 0$

$$\langle \phi(x_b \, t_b) | e^{-iHte^{-i\theta}} | \phi(x_a, t_a)\rangle \qquad t \to \infty$$

$$= \langle 0 | e^{-iHte^{-i\theta}} | 0\rangle$$

$$= \int D\phi(\vec{x},t) \, e^{\frac{i}{\hbar} S_\theta[\phi(\vec{x},t)]} \qquad \hbar = 1$$

$$\langle 0 | T(\phi(x_1) \cdots \phi(x_n)) | 0\rangle = \frac{\int D\phi \, e^{iS_\theta[\phi(x)]} \, \phi(x_1) \cdots \phi(x_n)}{\int D\phi \, e^{iS_\theta[\phi(x)]}}$$

$x = (\vec{x}, t)$

$$Z[J] = \int D\phi \; e^{iS_\theta[\phi(x)] + i\int d^4x\, J(x)\phi(x)}$$

$$\langle 0| T(\phi(x_1)\cdots\phi(x_n))|0\rangle$$

$$= \frac{1}{Z[0]} \frac{\delta}{i\delta J(x_1)} \cdots \frac{\delta}{i\delta J(x_n)} Z[J] \Big|_{J\to 0}$$

free real scalar field $\quad S[\phi] = -\frac{1}{2}\int d^4x \left(\partial_\mu\phi\partial^\mu\phi - m^2\phi^2\right)$

$d=4$

$t \to t e^{-i\theta} \qquad \eta_{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$

$$S[\phi] = -\frac{1}{2}\int dt\, d^3x\; e^{-i\theta}\Big[\underbrace{(\partial_t\phi)^2}_{e^{2i\theta}} - \vec\nabla\phi\cdot\vec\nabla\phi - m^2\phi^2\Big]$$

$$= -\frac{1}{2}\int dt\, d^3x \left[(\partial_t\phi)^2 e^{i\theta} - e^{-i\theta}\vec\nabla\phi\cdot\vec\nabla\phi - m^2\phi^2 \times e^{-i\theta}\right]$$

$$S = \int \phi\hat{A}\phi$$

$$= \frac{1}{2}\int dt\, d^3x\left[\phi\,\partial_t^2\phi\, e^{i\theta} - e^{-i\theta}\phi\nabla^2\phi + e^{-i\theta}m^2\phi^2\right]$$

$$= \frac{-1}{2}\int dt\, d^3x\; \phi(\vec x,t)\;\hat{A}\;\phi(\vec x,t)$$

$$\hat{A} = \frac{-1}{2}\left(e^{i\theta}\partial_t^2 - e^{-i\theta}\nabla^2 + e^{-i\theta}m^2\right)$$

$$Z[J] = \int D\phi \; e^{-i\int d^4x \; \phi \hat{A} \phi \; + i \int J(x)\phi(x) d^4x}$$

$$= N \sqrt{\frac{1}{\mathrm{Det}(A)}} \; e^{-\frac{i}{4}\int d^4x_1 d^4x_2 \; J(x_1) A^{-1}(x_1, x_2) J(x_2)}$$

$$\langle 0 | T \phi(x_1) \phi(x_2) | 0 \rangle = \frac{i}{2} A^{-1}(x_1, x_2)$$

$$\hat{A} A^{-1}(x, x_2) = \delta^{(4)}(x, x_2)$$

$$-\frac{1}{2}\left(e^{i\theta}\partial_t^2 - e^{-i\theta}\nabla^2 + e^{-i\theta} m^2\right) A^{-1}(x_1, x_2) = \delta^{(4)}(x_1 - x_2)$$

$$A^{-1}(x_1 x_2) = \int \frac{d^4k}{(2\pi)^4} \; e^{-ik(x_1 - x_2)} A^{-1}(k)$$

$$k \cdot x = kx = k_\mu x^\mu = (Et - \vec{k}\cdot\vec{x})$$

$$\delta^{(4)}(x_1 - x_2) = \int \frac{d^4k}{(2\pi)^4} \; e^{-ik(x_1 - x_2)}$$

$$-\frac{1}{2}\left[e^{i\theta}(-iE)^2 - e^{-i\theta}(i\vec{k})^2 + m^2 e^{-i\theta}\right] A^{-1}(k) = 1$$

$$A^{-1}(k) = \frac{-2}{-E^2 \underset{\substack{\uparrow \\ 1+i\theta}}{e^{i\theta}} + \vec{k}^2 e^{-i\theta} + m^2 e^{-i\theta}}$$

$$= \frac{-2}{-E^2 - i\theta E^2 + |\vec{k}|^2 - i\theta |\vec{k}|^2 + m^2 - i\theta m^2}$$

$$k^\mu k_\mu = \frac{-2}{-k^2 + m^2 \underbrace{- i\theta\,(\underbrace{E^2 + |\vec{k}|^2 + m^2}_{>0})}_{\sim i\varepsilon}}$$

$$= E^2 - |\vec{k}|^2$$

$$0 < \varepsilon \ll 1$$

$$= \frac{2}{k^2 - m^2 + i\varepsilon}$$

$$\langle 0 | T(\phi(x_1)\,\phi(x_2)) | 0 \rangle$$

$$= \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2 + i\varepsilon} e^{-ik(x_1 - x_2)}$$

Feynman propagator of free scalar field.