1. Symmetry of QM
2. Theory of Angular momentum.
3. Scattering theory.

Symmetry of transformation of space $\mathbb{R}^3$

wave function $\psi(\vec{r})$ $\quad \vec{r} \in \mathbb{R}^3$.

consider linear transf. $Q: \mathbb{R}^3 \to \mathbb{R}^3$

$$Q\vec{r} = \vec{r}' \qquad \vec{r}: \text{3d column vector}$$

$$Q: 3\times 3 \text{ matrix}$$

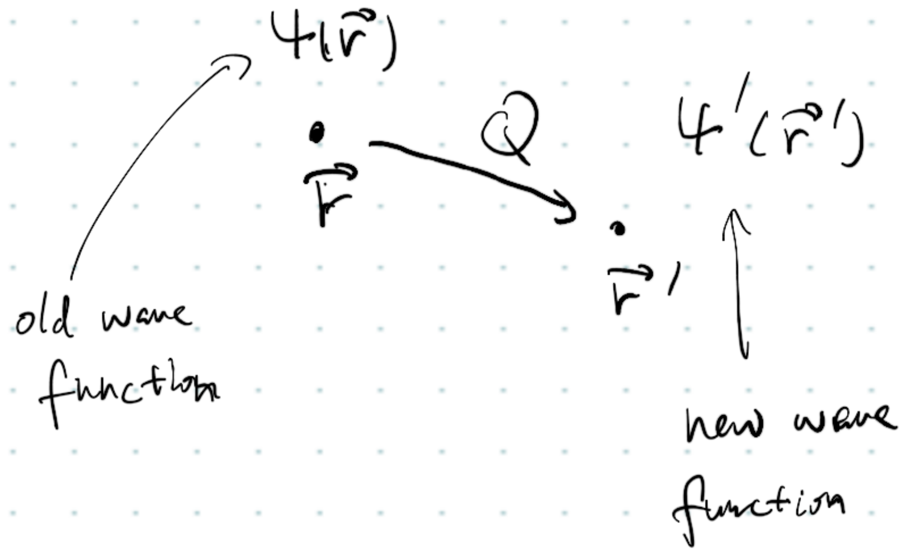
Q is a symmetry of the space, if it preserves the inner product

$$\vec{r}_1 \cdot \vec{r}_2 = (Q\vec{r}_1)\cdot(Q\vec{r}_2)$$

in particular, Q preserves the distance in $\mathbb{R}^3$ ✓

$$\vec{r}\cdot\vec{r} = (Q\vec{r})\cdot(Q\vec{r})$$

Q induces transformation of $\psi(\vec{r})$



s.t. $\psi'(\vec{r}\,') = \psi(\vec{r})$

$$\underbrace{\psi'(Q\vec{r})}_{\vec{r}\,' = Q\vec{r}} = \psi(\vec{r}), \qquad \vec{r} = Q^{-1}\vec{r}\,'$$

$$\boxed{\begin{aligned}\psi'(\vec{r}) &= \psi(Q^{-1}\vec{r}) \\ &\equiv \hat{D}(Q)\psi(\vec{r})\end{aligned}}$$

$Q : \mathbb{R}^3 \to \mathbb{R}^3$

$\hat{D}(Q) : \mathcal{H} \to \mathcal{H} \qquad \mathcal{H} = L^2(\mathbb{R}^3, d^3x)$

$|\psi\rangle \to |\psi'\rangle \qquad \psi'(\vec{r}) = D(Q)\psi(\vec{r})$

$$= \psi(Q^{-1}\vec{r})$$

properties of $D(Q)$: linear operator on $\mathcal{H}$.

- unitarity : $\langle \psi_1 | \psi_2 \rangle = \int d^3\vec{r}\, \overline{\psi_1(\vec{r})}\, \psi_2(\vec{r})$

change of variable $\vec{r} \to Q^{-1}\vec{r}$

$$= \int d^3(Q^{-1}\vec{r})\, \overline{\psi_1(Q^{-1}\vec{r})}\, \psi_2(Q^{-1}\vec{r})$$

Lemma $\det Q = \pm 1$

Pf: $\vec{r}\cdot\vec{r}' = \delta_{ij} r^i r'^j$

$\quad\;\; \|$

$Q\vec{r}\cdot Q\vec{r}' = \delta_{ij} Q^i{}_k r^k Q^j{}_l r'^l$

$\Rightarrow \delta_{ij} Q^i{}_k Q^j{}_l = \delta_{kl}$

$\Updownarrow$

$Q^T Q = 1_{3\times 3}$

$$\Rightarrow Q^T = Q^{-1}, \quad (\det Q)^2 = 1$$

$$\det Q = \pm 1$$

$$d^3(Q^{-1}\vec{r}) = d^3\vec{r}\ |\det Q^{-1}| = d^3\vec{r}$$

$$\langle \psi_1 | \psi_2 \rangle = \int d^3\vec{r}\ \overline{\psi_1(Q^{-1}\vec{r})}\ \psi_2(Q^{-1}\vec{r})$$

$$= \langle D(Q)\psi_1 | D(Q)\psi_2 \rangle$$

$\Rightarrow$ $D(Q)$ is unitary operator.

- Consider 2 transformations $Q_1, Q_2 : \mathbb{R}^3 \to \mathbb{R}^3$

$$Q = Q_1 \circ Q_2 \qquad Q\vec{r} = Q_1 \circ Q_2 \vec{r}$$

$$D(Q_1)\, D(Q_2)\, \psi(\vec{r}) = D(Q_1)\, \psi(Q_2^{-1}\vec{r})$$

$$= \psi\left(Q_2^{-1}(Q_1^{-1}\vec{r})\right)$$

$$= \psi(Q_2^{-1} Q_1^{-1}\vec{r}) = \psi\left((Q_1 Q_2)^{-1}\vec{r}\right)$$

$$= D(Q_1 Q_2)\, \psi(\vec{r})$$

$$\boxed{D(Q_1)\, D(Q_2) = D(Q_1 Q_2)}$$

if we view D to be a map from symmetry transf. on $\mathbb{R}^3$ to linear operators on $\mathcal{H}$.

D respects the product of transf.

- $D(Q)\, D(Q^{-1}) = D(Q Q^{-1}) = D(1_{3\times 3})$

$$D(1)\, \psi(\vec{r}) = \psi(1\vec{r}) = \psi(\vec{r})$$

$$D(1) = 1_{\mathcal{H}}$$

$$D(Q)D(Q^{-1}) = \mathbb{1}_{\mathcal{H}}$$

$$\boxed{D(Q)^{-1} = D(Q^{-1})}$$ D respects the inverse.

Def A group is a set G together with a binary operation

$a \cdot b \in G \quad \forall a, b \in G$

called group multiplication

s.t. (1) $(a \cdot b) \cdot c = a \cdot (b \cdot c) \quad \forall a, b, c \in G$

(associativity)

(2) $\exists!\, e \in G$, s.t. $e \cdot a = a \quad \forall a \in G$

(identity)

(3) $\forall a \in G$, $\exists\, a^{-1} \in G$ s.t. $a \cdot a^{-1} = a^{-1} \cdot a = e$

(inverse)

Example (1) $\mathbb{R}^1 \setminus \{0\}$ with multiplication

(2) set of symmetry transf. Q's on $\mathbb{R}^3$

multiplication ; composition of Q_1, Q_2

- Q ; 3×3 matrix $\rightarrow$ associativity
- identity $\mathbb{1}_{3\times 3}$
- inverse $Q^T Q = \mathbb{1}$ $\quad Q^{-1} = Q^T$

the set of all 3×3 matrices with $Q^{-1} = Q^T$ is called $O(3)$ group.

(orthogonal group on $\mathbb{R}^3$)

(3) We have found a set of $D(Q)$ unitary operators on $\mathcal{H}$

$$\{ D(Q) \mid Q \in O(3) \} \equiv D(O(3))$$

multiplication, composition of operators (product) $\underline{D(Q_1) D(Q_2)}$

- $D(Q)$ are operators $\rightarrow$ associativity
- identity operators $D(\mathbb{1}_{3\times 3}) = \mathbb{1}_{\mathcal{H}}$

- inverse $D(Q^{-1}) = D(Q)^{-1}$

$$\underbrace{D(Q_1)\,D(Q_2)}_{\substack{\text{multiplication} \\ \text{on } D(O(3))}} = D(\underbrace{Q_1 Q_2}_{\substack{\text{multiplication} \\ \text{on } O(3)}})$$

<u>Def</u>, given 2 groups G, H, a group homomorphism is a map $h: G \to H$ s.t.

$$\underbrace{h(g_1)\,h(g_2)}_{\substack{\text{multiplitcation} \\ \text{in } H}} = h(\underbrace{g_1 g_2}_{\substack{\text{multiplication} \\ \text{in } G}}) \quad \forall g_1, g_2 \in G$$

$D: O(3) \to D(O(3))$ is a homomorphism.

in terms of Dirac bra-ket, $|\psi\rangle \in \mathcal{H}$

$$\psi(\vec{r}) := \langle \vec{r} | \psi \rangle \qquad \hat{\vec{R}} |\vec{r}\rangle = \vec{r} |\vec{r}\rangle$$

$$\psi'(\vec{r}) := \langle \vec{r} | \psi' \rangle$$

$$|\psi'\rangle = D(Q) |\psi\rangle$$

$$\psi'(\vec{r}) = \langle \vec{r} | D(Q) | \psi \rangle = \langle D(Q)^{\dagger} \vec{r} | \psi \rangle$$

$$\shortparallel \qquad\qquad\qquad\qquad \shortparallel$$

$$\psi(Q^{-1}\vec{r}) = \langle Q^{-1} \vec{r} | \psi \rangle \qquad \langle D(Q^{-1}) \vec{r} | \psi \rangle$$

$$\langle \psi | \hat{O} | \phi \rangle = \langle \hat{O}^{\dagger} \psi | \phi \rangle$$

$$D(Q)^{\dagger} = D(Q)^{-1} = D(Q^{-1})$$

$$D(Q^{-1}) |\vec{r}\rangle = |Q^{-1} \vec{r}\rangle$$

$$\boxed{D(Q) |\vec{r}\rangle = |Q\vec{r}\rangle}$$

$$\underline{D(Q)\,\psi(\vec r) = \psi(Q^{-1}\vec r)}$$

$$Q \longrightarrow D(Q) \longrightarrow \text{transformations}$$

transf on $\mathbb{R}^3$ — transf on $\mathcal{H}$ — of operators on $\mathcal{H}$

$\hat A\, D(Q)^{-1}|\psi\rangle$ $\qquad$ $|\psi\rangle$

$$D(Q):\ \begin{tikzcd} \mathcal{H}_{(1)} \arrow[r] \arrow[loop below, "\hat A"] & \mathcal{H}_{(2)} \arrow[loop below, "\hat A'"] \end{tikzcd}$$

old $\qquad$ new

$\forall\, |\psi\rangle \in \mathcal{H}_{(2)}$, $\quad D(Q)^{-1}|\psi\rangle \in \mathcal{H}_{(1)}$

$$\hat A\, D(Q)^{-1}|\psi\rangle \in \mathcal{H}_{(1)}$$

$$\left(D(Q)\,\hat A\, D(Q)^{-1}\right)|\psi\rangle \in \mathcal{H}_{(2)}$$

$$\hat{A}' := D(Q)\,\hat{A}\,D(Q)^{-1}$$

$\hat{A} \mapsto \hat{A}'$ transf. of operators induced by Q.

$Q \longrightarrow D(Q) \longrightarrow$ transf. of operators

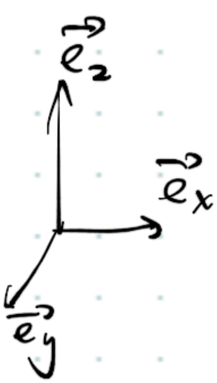
$Q\vec{r} = \vec{r}'$ $\quad D(Q)|\psi\rangle = |\psi'\rangle$ $\quad D(Q)\,\hat{A}\,D(Q)^{-1} = \hat{A}'$

conjugate.

position operator: $\hat{\vec{R}}$ $\quad \hat{\vec{R}}\,|\vec{r}\rangle = \vec{r}\,|\vec{r}\rangle$

$$\hat{\vec{R}} = \hat{R}_1 \vec{e}_x + \hat{R}_2 \vec{e}_y + \hat{R}_3 \vec{e}_z$$

$$= \sum_{i=1}^{3} \hat{R}_i\,\vec{e}_i$$



$$\hat{R}_i\,|\vec{r}\rangle = r_i\,|\vec{r}\rangle$$

$i = 1, 2, 3$

$$\hat{\vec{R}} \to D(Q)\,\hat{\vec{R}}\,D(Q)^{-1} \equiv \hat{\vec{R}}'$$

$$\hat{\vec{R}}\,|\vec{r}\rangle = \vec{r}\,|\vec{r}\rangle$$

$D(Q)$ $\quad D(Q)^{-1}D(Q)$

$$\Rightarrow \underbrace{D(Q)\,\hat{\vec{R}}\,D(Q)^{-1}}\,D(Q)\,|\vec{r}\rangle = D(Q)\,\vec{r}\,|\vec{r}\rangle$$

$$\hat{\vec{R}}'\,\underbrace{D(Q)|\vec{r}\rangle} = \vec{r}\,\underbrace{D(Q)|\vec{r}\rangle}$$

new eigenstate

$$\underline{\hat{\vec{R}}'\,|Q\vec{r}\rangle = \vec{r}\,|Q\vec{r}\rangle}$$

On the other hand, $\quad \hat{\vec{R}}\,|Q\vec{r}\rangle = Q\vec{r}\,|Q\vec{r}\rangle$

$$\Rightarrow \underline{\underbrace{Q^{-1}\,\hat{\vec{R}}}\,|Q\vec{r}\rangle = \vec{r}\,|Q\vec{r}\rangle}$$

$$Q^{-1}\hat{\vec{R}} = \sum_{i=1}^{3} \hat{R}_i\left(\underline{Q^{-1}\vec{e}_i}\right)$$

$$\hat{\vec{R}}' = Q^{-1} \hat{\vec{R}}$$

$$\boxed{D(Q)\, \hat{\vec{R}}\, D(Q)^{-1} = Q^{-1} \hat{\vec{R}}}$$