ADM formalism of GR : Hamiltonian formulation of gravity

(Arnowitt - Deser - Misner)

$$S_{EH} = \frac{1}{K}\int_M (R-2\Lambda)\sqrt{-g}\, d^4x + \underset{\substack{\uparrow \\ \text{boundary term}}}{S_{GH}} \,, \qquad \delta S_{EH} = 0 \to \text{Einstein eqn}$$

Now we want to have $S_{EH} = $ " $\int p\dot{q} - H$ " so that Einstein eqn can be cast into Hamilton's Eqn

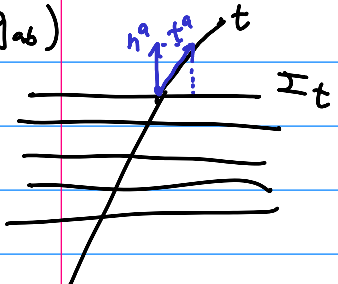
$$\dot{q} = \frac{\partial H}{\partial p} \qquad \dot{p} = -\frac{\partial H}{\partial q}$$

motivation :
- we need an initial value formulation of GR
- needed for quantization

one observation : we need to specify a time t (foliation) $\to$ 3+1 decomposition of spacetime

assume $M = \mathbb{R}\times\Sigma$, $\dim(\Sigma) = 3$

(M, g_{ab})



$$t^a = \left(\frac{\partial}{\partial t}\right)^a$$

$$t^a = N n^a + N^a \qquad N^a \text{ is tangent to } \Sigma$$

N: lapse function > 0 $\qquad n^a n_a = -1$ normalized

N^a: shift vector

break diff. invariance of GR but will be recovered.

coordinate system $(t, \underbrace{x^1, x^2, x^3}_{\text{coordinate on } \Sigma})$ $\qquad N^a = N^i\left(\frac{\partial}{\partial x^i}\right)^a$, $i = 1,2,3$

$$g_{tt} = g_{ab}\left(\frac{\partial}{\partial t}\right)^a\left(\frac{\partial}{\partial t}\right)^a$$

$$= g_{ab}(Nn^a + N^a)(Nn^b + N^b) = -N^2 + N^a N_a = -N^2 + N^i N_i$$

$$g_{ti} = g_{ab}\left(\frac{\partial}{\partial t}\right)^a\left(\frac{\partial}{\partial x^i}\right)^b = g_{ab}(Nn^a + N^a)\left(\frac{\partial}{\partial x^i}\right)^b$$

$$= N_b\left(\frac{\partial}{\partial x^i}\right)^b = N_i$$

$$g_{ij} = g_{ab}\left(\frac{\partial}{\partial x^i}\right)^a\left(\frac{\partial}{\partial x^j}\right)^b = q_{ij} \quad \text{induce metric on } \Sigma$$

$$q_{ab} = g_{ab} + n_a n_b$$

$$g_{\mu\nu} = (N, N_i, q_{ij}) \,, \quad 3+1 \text{ data of 4-metric.}$$

$$ds^2 = -N^2 dt^2 + q_{ij}(N^i dt + dx^i)(N^j dt + dx^j)$$

$$S_{EH} = \frac{1}{\kappa}\int_M R\sqrt{-g}\, d^4x$$ ignore Λ and boundary term

$$\sqrt{-g} = N\sqrt{\det q}$$

$$R\sqrt{-g} = N\sqrt{\det q}\left({}^3R + K_{ij}K^{ij} - K^2\right) + \text{boundary terms}$$

${}^3R$: 3d Ricci scalar of q_{ab} on Σ

extrisic curvature $\quad K_{ij} = \frac{1}{2}N^{-1}\left(\dot{q}_{ij} - 2D_{(i}N_{j)}\right) \qquad \dot{q}_{ij} = \frac{\partial}{\partial t}q_{ij}$

$$K = q^{ij}K_{ij}$$

D_a covariant derivation on Σ

$(\kappa \equiv 1)$ $\qquad D_a q_{bc} = 0$

$$S_{EH} = \int dt \int d^3x\; N\sqrt{\det q}\left({}^3R + K_{ij}K^{ij} - K^2\right)$$

<u>HW</u> derive this

position variables: $\quad q_{ij},\ N,\ N_i$

momentum variables: $\quad \left(P \equiv \frac{\delta S}{\delta \dot{q}}\right)$

$$\tilde{P}^{ij}(t,\vec{x}) \equiv \frac{\delta S_{EH}}{\delta \dot{q}_{ij}(t,\vec{x})} \qquad \tilde{P}_N(t,\vec{x}) = \frac{\delta S_{EH}}{\delta \dot{N}(t,\vec{x})}$$

$$\tilde{P}^i_{\vec{N}}(t,\vec{x}) = \frac{\delta S_{EH}}{\delta \dot{N}_i(t,\vec{x})}$$

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functional derivative: $\quad S = S[\phi_A] \qquad \phi_A = \phi_A(x^\mu)$

$$\delta S = \int d^Dx\; G^A(x)\,\delta\phi_A(x) \qquad\Big|\qquad dS = \sum_i G^A(i)\, d\phi_A(i)$$

$$\frac{\delta S}{\delta\phi_A(x)} = G^A(x) \qquad\Big|\qquad \frac{\partial S}{\partial\phi_A(i)} = G^A(i)$$

$$\phi^A(x) = \int d^Dx'\; \delta^D(x-x')\,\delta^A_B\,\phi^B(x')$$

$$\delta\phi^A(x) = \int d^Dx'\; \delta^D(x-x')\,\delta^A_B\,\delta\phi^B(x')$$

$$\frac{\delta\phi^A(x)}{\delta\phi^B(x')} = \delta^D(x-x')\,\delta^A_B \qquad\Big|\qquad \frac{\partial\phi^A(i)}{\partial\phi^B(j)} = \delta^A_B\,\delta^i_j$$

$$\Rightarrow \quad \tilde{p}^{ij} = \sqrt{\det q}\,\left(K^{ij} - K q^{ij}\right)$$

contain $\dot{q}_{ij}, N, N_i, q_{ij}$

- symmetric in $i \leftrightarrow j$
- not a spatial tensor, because of $\sqrt{\det q}$

$$\det q \longrightarrow \det q' \left(\underline{\det \frac{\partial x}{\partial x'}}\right)^2$$

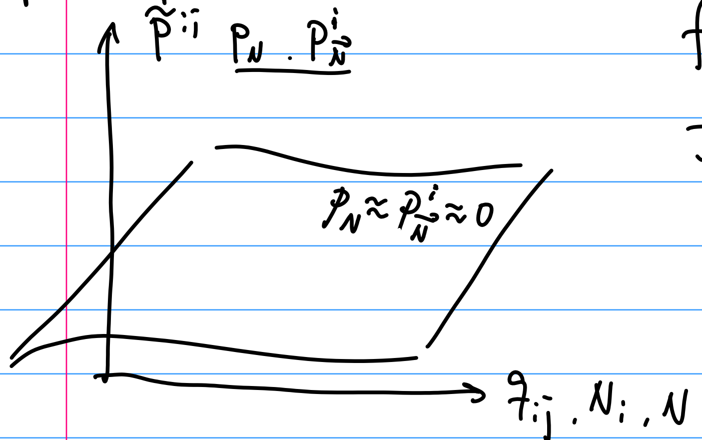
" tensor density of weight 1 "

$$\left\{ \begin{array}{l} \tilde{p}_N = 0 \\ \tilde{p}^i_{\vec{N}} = 0 \end{array} \right.$$

__HW__ derive them.

$\rightarrow$ primary constraints

phase space



$f = 0$ equal to zero on entire phase space.

$f \approx 0$ equal to zero on constraint surface

solve $\dot{q}_{ij}$ from $\tilde{p}^{ij}$:
$$\dot{q}_{ij} = \frac{2N}{\sqrt{\det q}}\left(\tilde{p}_{ij} - \frac{1}{2}\tilde{p}\, q_{ij}\right) + 2 D_{(i} N_{j)} \qquad (*)$$

$$\tilde{p} = \tilde{p}_{ij}\, q^{ij}$$

__HW__ derive this.

Legendre transf.

$$S_{EH} = \frac{1}{K}\int dt\, d^3x\, \underbrace{\mathcal{L}_{EH}}_{\text{Lagrangian density}} \overset{R\sqrt{-g}}{\equiv} \frac{1}{K}\int dt\, d^3x \left(\tilde{p}^{ij}\dot{q}_{ij} - \underbrace{\mathcal{H}}_{\text{Hamiltonian densitity}}(\tilde{p}^{ij}, q_{ij}, N, N_i)\right)$$

$$\mathcal{H} = \left(\tilde{p}^{ij}\dot{q}_{ij} - \mathcal{L}_{EH}\right)\Big|_{(*)}$$

__HW__ derive this

$$= \underline{N\left[-\sqrt{\det q}\;{}^3R + \frac{1}{\sqrt{\det q}}\left(\tilde{p}^{ij}\tilde{p}_{ij} - \frac{1}{2}\tilde{p}^2\right)\right]} + \underline{2\tilde{p}^{ij} D_i N_j} \quad ⓐ$$

Hamiltonian $H = \frac{1}{\kappa}\int_\Sigma d^3x\, \mathcal{H} = \int_\Sigma d^3x \left[N C + N_i C^i \right]$ + boundary terms.

$$C = \frac{1}{\kappa}\left[\frac{1}{\sqrt{\det q}}\left(\tilde{p}^{ij}\tilde{p}_{ij} - \frac{1}{2}p^2 \right) - \sqrt{\det q}\, {}^3R \right] \quad (b)$$

$$C^i = -\frac{2}{\kappa} D_j \tilde{p}^{ij}$$

covariant derivation of tensor density

HW derive (a) and (b)

(n,l) tensor density of weight m

$$\overset{(m)}{T}{}^{a_1\cdots a_n}{}_{b_1\cdots b_l} = \left(\sqrt{|\det g|}\right)^m T^{a_1\cdots a_n}{}_{b_1\cdots b_l}$$

T is a $(n;l)$ tensor, $m \in \mathbb{Z}$

Covariant derivative of densities

$$\nabla_\mu \overset{(m)}{T}{}^{\mu_1\cdots\mu_n}{}_{\nu_1\cdots\nu_l} = \partial_\mu \overset{(m)}{T}{}^{\mu_1\cdots\mu_n}{}_{\nu_1\cdots\nu_l} + \sum_j \Gamma^{\mu_j}{}_{\mu\alpha} \overset{(m)}{T}{}^{\mu_1\cdots \overset{j\text{th}}{\alpha} \cdots\mu_n}{}_{\nu_1\cdots\nu_l} - \sum_j \Gamma^{\alpha}{}_{\mu\nu_j} \overset{(m)}{T}{}^{\mu_1\cdots\mu_n}{}_{\nu_1\cdots \underset{j\text{th}}{\alpha}\cdots\nu_l} - m\,\Gamma^{\alpha}{}_{\alpha\mu} \overset{(m)}{T}{}^{\mu_1\cdots\mu_n}{}_{\nu_1\cdots\nu_l}$$

$$\mathcal{L}_v \overset{(m)}{T}{}^{a_1\cdots a_n}{}_{b_1\cdots b_l} = \left(\text{ordinary Lie derivative as if } \overset{(m)}{T} \text{ was a tensor}\right) + m\, \overset{(m)}{T}{}^{a_1\cdots a_n}{}_{b_1\cdots b_l} \nabla_a v^a \qquad \text{for all } \nabla_a$$

guarantee ∇_μ (tensor density) = (tensity density)

$\mathcal{L}_v$ (tensor density) = (tensity density)

$$\overset{(m)}{T}{}'^{\mu}{}_{\nu} = \underline{\left| \det\left(\frac{\partial x'}{\partial x}\right) \right|^m} \, \underline{\frac{\partial x'^\mu}{\partial x^\sigma}\frac{\partial x^\rho}{\partial x'^\nu}} \overset{(m)}{T}{}^{\sigma}{}_{\rho}$$