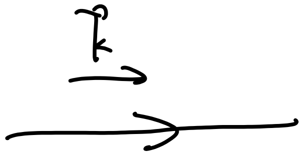
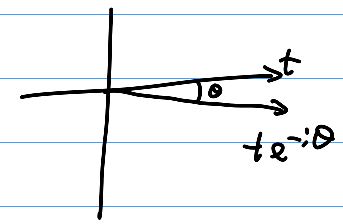


$$\int \phi(x)^4 \, d^4x$$

$$\int d^4x \, d^4y \; \phi(x) \, \underline{G(x-y)} \, \phi(y)$$



$$\frac{\int D\bar{\psi} D\psi \; \psi_a(k) \bar{\psi}_b(k') \, e^{iS}}{\int D\bar{\psi} D\psi \; e^{iS}} = i K^{-1}_{ab}(k) \, \delta^{(4)}(k-k')$$

$a, b = 1 \ldots 4$

$$iK^{-1} = \frac{i}{e^{-i\theta}(\gamma^0 k_0 e^{i\theta} + \gamma^i k_i - m)} = \left.\frac{i e^{i\theta}}{\gamma^\mu k_\mu - m}\right|_{k_0 \to k_0 e^{i\theta}}$$

$$\frac{1}{\gamma^\mu k_\mu - m} = (\gamma^\mu k_\mu - m)^{-1} = \underbrace{(\gamma^\mu k_\mu - m)^{-1} (\gamma^\mu k_\mu + m)^{-1}}_{[(\gamma^\mu k_\mu + m)(\gamma^\mu k_\mu - m)]^{-1} = \frac{1}{k^2 - m^2}} (\gamma^\mu k_\mu + m)$$

$$\gamma^\mu k_\mu \gamma^\nu k_\nu - \cancel{m \gamma^\mu k_\mu} + \cancel{m \gamma^\mu k_\mu} - m^2$$

$$= \frac{1}{2} \gamma^\mu k_\mu \gamma^\nu k_\nu + \frac{1}{2} \gamma^\nu k_\nu \gamma^\mu k_\mu - m^2$$

$$= \frac{1}{2} \{\gamma^\mu \gamma^\nu\} k_\mu k_\nu - m^2 = \eta^{\mu\nu} k_\mu k_\nu - m^2$$

$$= \frac{\gamma^\mu k_\mu + m}{k^2 - m^2} \equiv \frac{\not{k} + m}{k^2 - m^2}$$

$$iK^{-1} = \frac{i e^{i\theta} (\gamma^0 k_0 e^{i\theta} + \gamma^i k^i + m)}{\underbrace{(k^0)^2 e^{2i\theta}}_{(k^0)^2 + 2i\theta (k^0)^2 = (k^0)^2 + i\varepsilon} - (k^i)^2 - m^2} = \frac{i(\not{k} + m + O(\varepsilon))}{k^2 - m^2 + i\varepsilon} \simeq \frac{i(\not{k} + m)}{k^2 - m^2 + i\varepsilon}$$

$$= \frac{i}{\not{k} - m + i\varepsilon} \equiv S_F(k)$$

$$\frac{\int D\psi D\bar\psi\, \psi(k)\bar\psi(k')\, e^{iS}}{\int D\psi D\bar\psi\, e^{iS}} = S_F(k)\,\delta^{(4)}(k-k')$$

$$\frac{\int D\psi D\bar\psi\, \psi(x)\bar\psi(x')\, e^{iS}}{\int D\psi D\bar\psi\, e^{iS}} = \int \frac{d^4k}{(2\pi)^4}\frac{d^4k'}{(2\pi)^4}\, e^{-ik\cdot x}\, e^{ik'x'}\, S_F(k)\,\delta^{(4)}(k-k')$$

$$= \int \frac{d^4k}{(2\pi)^4}\, S_F(k)\, e^{-ik(x-x')}$$

$$T\,\psi(t,\vec{x})\bar\psi(t',\vec{x}') = \begin{cases} \psi(t,\vec{x})\bar\psi(t',\vec{x}') & t>t' \\ -\bar\psi(t',\vec{x}')\psi(t,\vec{x}) & t<t' \end{cases}$$

$$\psi(x) = e^{iH(t-t_{-\infty})}\psi(\vec{x})\underbrace{e^{-iH(t-t_{-\infty})}}_{U(t,-\infty)}$$

$t>t'$

$$\langle 0|T\,\psi(t,\vec{x})\bar\psi(t',\vec{x}')|0\rangle = \langle 0|\,\psi(x)\bar\psi(x')\,|0\rangle$$

$$= \frac{\langle 0|U(\infty,t)\,\psi(\vec{x})\,U(t,t')\,\bar\psi(\vec{x}')\,U(t',-\infty)|0\rangle}{\langle 0|U(\infty,-\infty)|0\rangle}$$

$$= \frac{\int D\bar\psi D\psi\, \psi(x)\bar\psi(x')\, e^{iS}}{\int D\bar\psi D\psi\, e^{iS}}$$

$t<t'$

$$\langle 0|T\,\psi(x)\bar\psi(x')|0\rangle = -\langle 0|\,\bar\psi(x')\psi(x)|0\rangle$$

$$= -\frac{\langle 0|U(\infty,t')\,\bar\psi(\vec{x}')\,U(t'-t)\,\psi(\vec{x})\,U(t,-\infty)|0\rangle}{\langle 0|U(\infty,-\infty)|0\rangle}$$

$$= -\frac{\int D\bar\psi D\psi\, \bar\psi(x')\psi(x)\, e^{iS}}{\int D\bar\psi D\psi\, e^{iS}} = \frac{\int D\bar\psi D\psi\, \psi(x)\bar\psi(x')\, e^{iS}}{\int D\bar\psi D\psi\, e^{iS}}$$

$$\langle 0 | T \psi(x) \bar{\psi}(x') | 0 \rangle = \frac{\int D\bar{\psi} D\psi \, \psi(x) \bar{\psi}(x') e^{iS}}{\int D\bar{\psi} D\psi \, e^{iS}}$$

$$= \int \frac{d^4k}{(2\pi)^4} \frac{i}{\not{k} - m + i\varepsilon} e^{-ik(x-x')}$$

Gauge theories

$$\mathcal{L}_D = \bar{\psi} (i\gamma^\mu \partial_\mu - m) \psi \qquad \psi \to e^{i\alpha} \psi$$
$$\bar{\psi} \to e^{-i\alpha} \bar{\psi} \qquad \alpha \text{ is const}$$

$\mathcal{L}_D$ is inv. $\longrightarrow$ this is a global symmetry

local transf. $\psi(x) \to e^{i\alpha(x)} \psi(x)$
$\bar{\psi}(x) \to e^{-i\alpha(x)} \bar{\psi}(x)$

$$i\gamma^\mu \partial_\mu \psi \to i\gamma^\mu \partial_\mu \left(e^{i\alpha(x)} \psi(x) \right) = e^{i\alpha(x)} i\gamma^\mu \partial_\mu \psi(x)$$
$$+ i\gamma^\mu \psi(x) \underline{\underline{e^{i\alpha(x)}}} \partial_\mu \alpha(x)$$

$$\mathcal{L}_D \to \mathcal{L}_D + \bar{\psi} i\gamma^\mu \psi \, \partial_\mu \alpha(x)$$

to make it inv. by gauge the global symm. :

$$\partial_\mu \longrightarrow D_\mu = \partial_\mu + ieA_\mu \qquad \text{"covariant derivative"}$$

gauge transf. $\psi(x) \to e^{i\alpha(x)} \psi(x)$
$\bar{\psi}(x) \to e^{-i\alpha(x)} \bar{\psi}(x)$
$A_\mu(x) \to A_\mu(x) - \frac{1}{e} \partial_\mu \alpha(x)$

$$D_\mu \psi = (\partial_\mu + ieA_\mu) \psi \to \left(\partial_\mu + ieA_\mu - i\partial_\mu \alpha \right) e^{i\alpha} \psi$$
$$= e^{i\alpha} \partial_\mu \psi + \cancel{ie^{i\alpha} \psi \partial_\mu \alpha} + ieA_\mu e^{i\alpha} \psi$$
$$\cancel{-i(\partial_\mu \alpha) e^{i\alpha} \psi}$$

$$= e^{i\alpha}(\partial_\mu + ieA_\mu)\psi$$
$$= e^{i\alpha(x)} D_\mu \psi$$

$D_\mu\psi$ transf. in the same way as ψ.

$$\mathcal{L}_D = \bar{\psi}(i\gamma^\mu D_\mu - m)\psi \rightarrow \mathcal{L}_D \text{ is gauge inv.}$$

this is a local symm. (gauge symm.)

A_μ is dynamical $\quad \mathcal{L}_A = -\frac{1}{4} F_{\mu\nu}F^{\mu\nu}$ $\quad F$: field strength.

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$F_{\mu\nu}$ is gauge inv.:

$$\begin{aligned}
&[D_\mu, D_\nu]\psi \rightarrow e^{i\alpha}[D_\mu D_\nu]\psi \\
&\quad \parallel \\
&\cancel{[\partial_\mu, \partial_\nu]}^{0}\psi + ie\left([\partial_\mu, A_\nu] - [\partial_\nu, A_\mu]\right)\psi - e^2\cancel{[A_\mu, A_\nu]}^{0}\psi \\
&= ie(\partial_\mu A_\nu - \partial_\nu A_\mu)\psi = ieF_{\mu\nu}\psi
\end{aligned}$$

$$[D_\mu, D_\nu] = ieF_{\mu\nu}$$

$$[D_\mu D_\nu]\psi = ieF_{\mu\nu}\psi \rightarrow e^{i\alpha}[D_\mu, D_\nu]\psi = ieF'_{\mu\nu}e^{i\alpha}\psi$$
$$\rightarrow [D_\mu D_\nu]\psi = ieF'_{\mu\nu}\psi$$
$$\rightarrow F'_{\mu\nu} = F_{\mu\nu} \quad \text{gauge inv.}$$

$\rightarrow \mathcal{L}_A$ is gauge inv.

$\cancel{(F_{\mu\nu}F^{\mu\nu})^2}$

$$\mathcal{L}_{QED} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(i\gamma^\mu D_\mu - m)\psi$$
$$\parallel$$
$$\bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi - \underline{e}\,\overbrace{\bar{\psi}\gamma^\mu\psi}^{J^\mu} A_\mu$$

$$\int d^4x \, \underline{\partial A} \, \underline{\partial A} \qquad [A] = 1 \qquad \left[\int d^4x \, \bar\psi \gamma^\mu \psi A_\mu \right] = 0$$

$$\int d^4x \, \bar\psi \partial \psi \qquad [\psi] = [\bar\psi] = \frac{3}{2}$$

$\mathcal{L}_{QED}$ is an abelian gauge theory

$$\psi \xrightarrow[\alpha]{} e^{i\alpha}\psi \xrightarrow[\beta]{} \underline{e^{i\alpha} e^{i\beta}\psi}$$

$$A_\mu \to A_\mu - \frac{1}{e}\partial_\mu \alpha \to A_\mu - \frac{1}{e}\partial_\mu \beta - \frac{1}{e}\partial_\mu \alpha$$

$$\psi \xrightarrow[\beta]{} e^{i\beta}\psi \xrightarrow[\alpha]{} e^{i\beta} e^{i\alpha}\psi$$

$$A_\mu \to A_\mu - \frac{1}{e}\partial_\mu \beta \to A_\mu - \frac{1}{e}\partial_\mu \alpha - \frac{1}{e}\partial_\mu \beta$$

$$e^{i\alpha} \in U(1) \qquad \left. \begin{array}{l} e^{i\alpha} \cdot e^{i\beta} = e^{i(\alpha+\beta)} \\ (e^{i\alpha})^{-1} = e^{-i\alpha} \\ e^{i0} = 1 \end{array} \right\} U(1)$$

$\mathcal{L}_{QED}$ is $U(1)$ gauge theory, $U(1)$ is the gauge group.

$SU(2)$ nonabelian group : $\begin{pmatrix} a, & -\bar b \\ b, & \bar a \end{pmatrix} \equiv u \in SU(2)$ special unitary group on $\mathbb{C}^2$

$$a, b \in \mathbb{C} \qquad \det u = a\bar a + b\bar b = 1$$

$$u^\dagger = u^{-1}$$

$$u_1 u_2 \neq u_2 u_1 \qquad \text{nonabelian}$$

$$\forall u \in SU(2), \qquad u = e^{i \sum_{i=1}^{3} \alpha_i \frac{\sigma^i}{2}} \quad \swarrow \text{generators}$$

$\sigma^{1,2,3}$ are Pauli matrices

a pair (doublet) of fermion $\psi^{a=1,2}(x)$ $\quad a$: flavor index

$$\psi^{a=1,2}_{\alpha=1\ldots4}(x) \qquad \psi \in \mathbb{C}^2 \otimes \mathbb{C}^4$$

$$\mathcal{L}_D = \bar\psi^a (i\gamma^\mu \partial_\mu - m)\psi^a \qquad \begin{array}{l} u \in SU(2) \\ \underline{\psi^a \to u^a{}_b \psi^b(x)} \\ \bar\psi^a \to \bar\psi^b (u^a{}_b)^* \end{array}$$

$$\to \bar{\psi}^b (u^{-1}{}_b{}^a)(i\gamma^\mu \partial_\mu - m)\, u^a{}_c \psi^c \qquad = \bar{\psi}^b (u^{-1}{}_b{}^a)$$

$$\to \bar{\psi}^a (i\gamma^\mu \partial_\mu - m) \psi^a = \mathcal{L}_D$$

$\mathcal{L}_D$ is globally $SU(2)$ inv.

" gauge this flavor symm " let $u = u(x) = e^{i\alpha^i(x) \frac{\sigma^i}{2}}$

$$\partial_\mu (u^a{}_b \psi^b) = u^a{}_b \partial_\mu \psi^b + \psi^b \partial_\mu u^a{}_b$$

Covariant derivative : $D_\mu = \partial_\mu - ig \sum_{i=1}^{3} A_\mu^i \frac{\sigma^i}{2}$

3 A_μ's $\longleftrightarrow$ 3 generators.

non abelian gauge transf. $\psi^a(x) \to u^a{}_b(x)\, \psi^b(x)$

$$\bar{\psi}^a(x) \to \bar{\psi}^b(x) (u^{-1}{}_b{}^a)(x)$$

$$A_\mu(x) \equiv A_\mu^i(x) \frac{\sigma^i}{2} \to u(x) A_\mu(x) u(x)^{-1} + \frac{i}{g} u(x) \partial_\mu u(x)^{-1}$$

$u \to e^{i\alpha}$

$$D_\mu \psi \to (\partial_\mu - ig\, u A_\mu u^{-1} + u \partial_\mu u^{-1})\, u \psi$$

$$= u \partial_\mu \psi + (\partial_\mu u)\psi - ig\, u A_\mu \psi + u(\partial_\mu u^{-1}) u \psi$$

$$\partial(u u^{-1}) = 0$$
$$\parallel$$
$$(\partial u) u^{-1} + u \partial u^{-1}$$

$$= u \partial_\mu \psi + \cancel{(\partial_\mu u)\psi} - ig\, u A_\mu \psi - \cancel{(\partial_\mu u) u^{-1}} u \psi$$

$$= u(\partial_\mu - ig A_\mu)\psi = u D_\mu \psi$$

$D_\mu \psi$ transf. in the same way as ψ

$$\mathcal{L}_D = \bar{\psi}^a \left[(i\gamma^\mu D_\mu - m)\psi\right]^a = \bar{\psi}^a (i\gamma^\mu D_\mu^{ab} - m)\psi^b$$

$$\equiv \bar{\Psi}\,(i\gamma^\mu D_\mu - m)\,\Psi \qquad \text{is gauge inv.}$$

make A_μ^i dynamical $\qquad \mathcal{L}_A = -\frac{1}{2} F_{\mu\nu}^i F^{\mu\nu i}$

$$F_{\mu\nu} = \sum_{i=1}^{3} F_{\mu\nu}^i \frac{\sigma^i}{2}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig\,[A_\mu A_\nu]$$

$$F_{\mu\nu}^k \frac{\sigma^k}{2} = (\partial_\mu A_\nu^i - \partial_\nu A_\mu^i)\frac{\sigma^i}{2} - ig\,A_\mu^i A_\nu^j \underbrace{\left[\frac{\sigma^i}{2}, \frac{\sigma^j}{2}\right]}_{i\varepsilon^{ijk}\frac{\sigma^k}{2}}$$

$$= \left(\partial_\mu A_\nu^k - \partial_\nu A_\mu^k + g\,\varepsilon^{ijk} A_\mu^i A_\nu^j\right)\frac{\sigma^k}{2}$$

$$F_{\mu\nu}^k = \partial_\mu A_\nu^k - \partial_\nu A_\mu^k + g\,\varepsilon^{ijk} A_\mu^i A_\nu^j$$