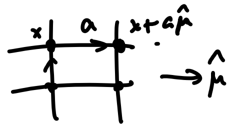
Lattice fermion



$\psi = \begin{pmatrix} \psi_1 \\ \vdots \\ \psi_4 \end{pmatrix}(x) \qquad \psi = \psi^i_\alpha$

$$S_F = a^4 \sum_x \Bigg\{ \sum_{\mu=0}^{3} \left(\frac{i}{2a}\right)\Big[\bar{\psi}(x)\gamma^\mu U_\mu(x)\,\psi(x+a\hat{\mu}) - \bar{\psi}(x+a\hat{\mu})\gamma^\mu U_\mu(x)^\dagger \psi(x) \Big] - m_0 \bar{\psi}(x)\psi(x)$$

$$+ \sum_{\mu=0}^{3} \left(\frac{r}{2a}\right)\Big[\bar{\psi}(x) U_\mu(x)\psi(x+a\hat{\mu}) + \bar{\psi}(x+a\hat{\mu}) U_\mu(x)^\dagger \psi(x) - 2\bar{\psi}(x)\psi(x) \Big] \Bigg\}$$

Wilson fermion

$$\int DU\, D\psi\, D\bar{\psi}\; e^{S - S_F}$$

$r > 0$ free parameter

coupling of Wilson term

$$\psi(x) \to g(x)\psi(x)$$
$$\bar{\psi}(x) \to \bar{\psi}(x) g(x)^\dagger \qquad U_\mu(x) \to g(x) U_\mu(x) g(x+a\hat{\mu})^\dagger$$

$$U_\mu(x) = 1 + iagA_\mu(x) + O(a^2)$$

$$\psi(x+a\hat{\mu}) = \psi(x) + a\partial_\mu \psi(x) + O(a^2)$$

$$\bar{\psi}(x)\gamma^\mu U_\mu(x)\psi(x+a\hat{\mu})$$
$$\simeq \bar{\psi}(x)\gamma^\mu (1 + iagA_\mu(x))(\psi(x) + a\partial_\mu\psi(x))$$
$$= \bar{\psi}(x)\gamma^\mu\psi(x) + a\bar{\psi}(x)\gamma^\mu\partial_\mu\psi(x) + iag\bar{\psi}(x)\gamma^\mu A_\mu(x)\psi(x) + O(a^2)$$
$$= \bar{\psi}(x)\gamma^\mu\psi(x) + a\bar{\psi}(x)\gamma^\mu D_\mu\psi(x) + O(a^2)$$

$$\bar{\psi}(x) U_\mu(x)\psi(x+a\hat{\mu})$$

$a\bar{\psi}(x)\partial_\mu\psi(x) + ia^2 g\bar{\psi}(x)A_\mu\psi(x)$

$$\simeq \bar{\psi}(x)\psi(x) + a\underline{\bar{\psi}(x) D_\mu\psi(x)} + O(a^2)$$

$$\bar{\psi}(x+a\hat{\mu}) U_\mu(x)^\dagger \psi(x)$$
$$\simeq \bar{\psi}(x)\psi(x) + \underbrace{a\partial_\mu\bar{\psi}(x)\psi(x) - iag\bar{\psi}(x)A_\mu\psi(x)}_{} + \underline{\underline{O(a^2)}}$$

$$\left(\frac{r}{2a}\right)\left[\cdots\right] \sim O(a) \longrightarrow 0 \quad \text{as } a \to 0$$

$$S_F = \int d^4x \left\{ \frac{i}{2}\left[\bar\psi(x)\gamma^\mu D_\mu \psi(x) - \overline{D_\mu \psi(x)}\,\gamma^\mu \psi(x)\right] - m_0\bar\psi\psi - \frac{r}{4}\cdot O(a) \right\}$$

$$\sum_x e^{-i(k-k')x} = \delta(k-k')$$

$$S_F^{(0)} = a^4 \sum_x \left\{ \sum_\mu \left(\frac{i}{2a}\right)\left[\bar\psi(x)\gamma^\mu\psi(x+a\hat\mu) - \bar\psi(x+a\hat\mu)\gamma^\mu\psi(x)\right] - m_0\bar\psi\psi\right]$$

Fourier transf. on the lattice

$$\uparrow$$

$$(a\mathbb{Z})^4 \subset \mathbb{R}^4$$

$$\psi(x) = \int_{-\frac{\pi}{2a}}^{\frac{3\pi}{2a}} \frac{d^4k}{(2\pi)^4} e^{-ik_\mu x^\mu}\psi(k)$$

$$x^\mu \in a\mathbb{Z} \qquad e^{-ik_\mu x^\mu} = e^{-ik_\mu a n^\mu}$$

$$x^\mu = a n^\mu$$

single period $\quad k_\mu \in \left[-\frac{\pi}{2a}, \frac{3\pi}{2a}\right]$

the 1st Brillouin zone

$$S_F = \int \frac{d^4k}{(2\pi)^4} \sum_\mu \bar\psi(k)\left[\left(\frac{i}{2a}\right)\left(\gamma^\mu e^{-iak_\mu} - \gamma^\mu e^{iak_\mu}\right) - m_0\right]\psi(x) + \cdots$$

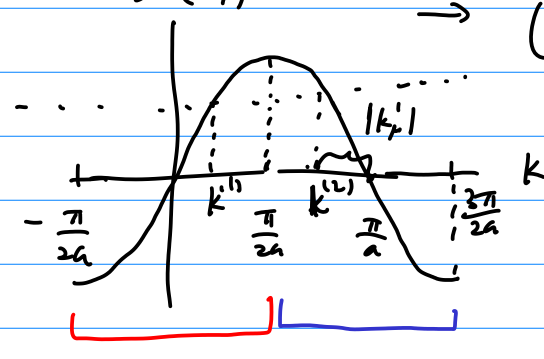
$$\longrightarrow \bar\psi\gamma^\mu k_\mu\psi$$

Feynman propagator

$$\langle 0|\psi(k)\bar\psi(k)|0\rangle = \left(\sum_\mu \frac{1}{a}\gamma^\mu \sin ak_\mu - m_0\right)^{-1}$$
$$\equiv G(k)$$

$$\longrightarrow (\gamma^\mu k_\mu - m_0)^{-1}$$

$$-\frac{\pi}{2a} \le k_\mu \le \frac{3\pi}{2a}$$



certain value of G

(1) $\quad -\frac{\pi}{2a} \le k_\mu^{(1)} \le \frac{\pi}{2a}$ $\qquad$ continuum limit $\quad G(k^{(1)}) \to (\gamma^\mu k_\mu^{(1)} - m_0)^{-1}$

(2) $\frac{\pi}{2a} \leq k_\mu^{(2)} \leq \frac{3\pi}{2a}$ $\quad k_\mu^{(2)} = \frac{\pi}{a} + k_\mu' \quad k_\mu' \in [-\frac{\pi}{2a}, \frac{\pi}{2a}]$

$$\sin(ak_\mu) = \sin(\pi + ak_\mu') = -\sin(ak_\mu')$$

continuum limit $G(k) \to (-\gamma^\mu k_\mu' - m_0)^{-1}$

$\Rightarrow$ this is called fermion doubling

$$a^4 \sum_x \sum_{\mu=0}^{3} \left(\frac{r}{2a}\right) \Big[\bar\psi(x) \psi(x+a\hat\mu) + \bar\psi(x+a\hat\mu) \psi(x) - 2\bar\psi(x)\psi(x) \Big]$$

$$\sum_x a^4 \int \frac{d^4k'}{(2\pi)^4} \int \frac{d^4k}{(2\pi)^4} \bar\psi(k) e^{ik\cdot x} \psi(k') e^{-ik'\cdot(x+a\hat\mu)} \qquad \sum_x e^{ik\cdot x} = (2\pi)^4 \delta(k)$$

$$= a^4 \int \frac{d^4k'}{(2\pi)^4} \int \frac{d^4k}{(2\pi)^4} \bar\psi(k)\psi(k') (2\pi)^4 \delta^{(4)}(k-k') e^{-ik' a\hat\mu}$$

$$= a^4 \int \frac{d^4k}{(2\pi)^4} \bar\psi(k)\psi(k) e^{-ika\hat\mu}$$

$$\Bigg(\underbrace{\sum_\mu \frac{1}{a}\gamma^\mu \sin(ak_\mu)}_{k+k^3} - m_0 - \underbrace{\sum_\mu \frac{r}{a}[1-\cos(ak_\mu)]}_{1+k^2} \Bigg)^{-1}$$

= new propagator.

- $k_\mu^{(1)} \in [-\frac{\pi}{2a}, \frac{\pi}{2a}]$ $\quad a \to 0 \quad (\gamma^\mu k_\mu^{(1)} - m_0)^{-1}$
- $k_\mu^{(2)} \in [\frac{\pi}{2a}, \frac{3\pi}{2a}]$ $\quad k_\mu^{(2)} = \frac{\pi}{a} + k' \quad \sin(ak_\mu^{(2)}) = \sin(\pi + k'a) = -\sin(k'a)$

$$\cos(ak_\mu^{(2)}) = \cos(\pi + k'a) = -\cos(k'a)$$

$a \to 0$ $\quad \Big(\sum_\mu -\gamma^\mu k'_\mu - m_0 \underbrace{- \sum_\mu \frac{2\gamma}{a}}_{m(a) \to \infty} \Big)^{-1}$

the spurious fermion mode become super massive