Boundary term in Hamiltonian (Wald "GR")

$$H = \int_\Sigma d^3x \,(N C + N^i C_i) + \int_{\partial\Sigma} B$$

2-form : all boundary terms
- from Gibbons-Hawking
- boundary terms from integration by parts

Consider $\dot{q}_{ij} = \frac{\delta H}{\delta p^{ij}}$ $\quad \dot{p}^{ij} = -\frac{\delta H}{\delta q_{ij}}$

these has to be well-defined

$$\delta H = \int_\Sigma d^3x \left[T^{ij}\delta q_{ij} + S_{ij}\,\delta p^{ij} \right] + \int_{\partial\Sigma} (\delta B + \delta B')$$

boundary terms from $\delta \int_\Sigma d^3x \,(N C + N^i C_i)$

in order that $\frac{\delta H}{\delta q_{ij}}, \frac{\delta H}{\delta p^{ij}}$ are well-defined, you must have $\delta B + \delta B' = 0$

from this you obtain δB and B

total energy of everything inside spacetime (gravity + matter)

$$H = \int_\Sigma d^3x \,(N C + N^i C_i) + \int_{\partial\Sigma} B$$

$$C = C_{grav.} + C_{matter}$$

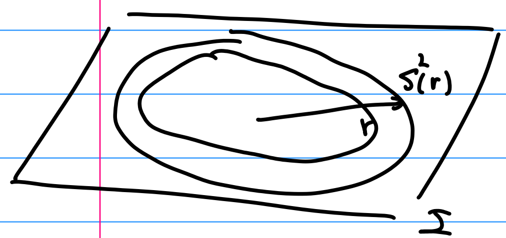
$$C_i = C_{i\,grav.} + C_{i\,matter}$$

$$H = \int_{\partial\Sigma} B \qquad \text{gravity energy is nonlocal}$$

Example : Asymptotically flat spacetime, (t, x^1, x^2, x^3) is asymptotically flat coordinate. i.e $g_{\mu\nu} \to \eta_{\mu\nu}$ as $r \to \infty$

ADM energy: $H = \lim_{r\to\infty} \int_{S^2(r)} \sum_{i,j=1}^{3} \left(\frac{\partial q_{ij}}{\partial x^j} - \frac{\partial q_{jj}}{\partial x^i} \right) \sqrt{\det h}\, d^2\sigma$

h is induced metric on S^2



Schwarzschild spacetime

$H = m$ mass of black hole

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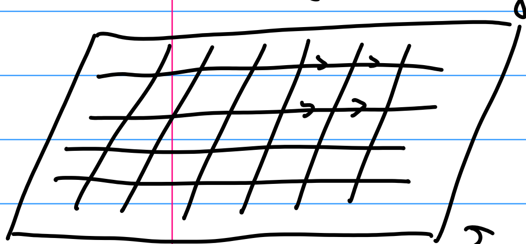
preparation of LQG quantization: Lattice discretization of A_a^i and E_j^a

(analog of Lattice gauge theory)

Loop Quantum Gravity: An attempt of nonperturbative, background indep. formulation of quantum gravity in 4 and higher dimensions

(a graph)

cubic lattice γ



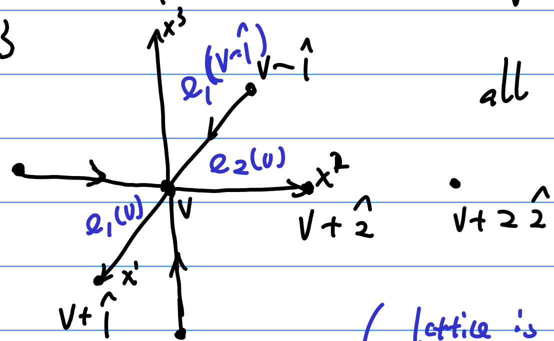
$\dim \Sigma = 3$

background = spacetime = g_{ab}

vertices of $\gamma \simeq \mathbb{Z}^3$

set of all oriented edges : $E(\gamma)$

set of all vertices : $V(\gamma)$



all vertices are 6-valent

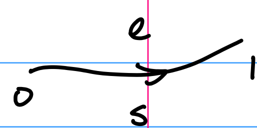
(lattice is adapted to ~~coordinate~~ arbitrary coordinate system)

smooth fields		Lattice variables
$A_a^i(x)$	$\longrightarrow$	$h(e) \in SU(2)$ holonomy (wilson line)
$E_i^a(x)$	$\longrightarrow$	$p^j(e) \in \mathbb{R}^3$ gauge covariant flux

$\forall e \in E(\gamma)$

Lattice discretization approximates a field theory by a system with finite # of DOFs, recover the field theory by Lattice refinement (continuum limit)

Def (holonomy) Give $A_a^i(x)$ an $SU(2)$ gauge potential (1-form), and an oriented edge $e : [0,1] \to \Sigma$ analytic curve,
$s \mapsto e(s) \in \Sigma$



Lie group; tangent vector of e is denoted by $\left(\frac{d}{ds}\right)^a \equiv \dot{e}^a = \sum_{\mu=1}^{3} \frac{dx^\mu}{ds}\left(\frac{\partial}{\partial x^\mu}\right)^a$

$$SU(2) \ni h(e) := \mathcal{P}\exp\left[\int_0^1 ds\, A_a^i\, \dot{e}^a \frac{\tau_i}{2}\right] \qquad \tau_i = -i\sigma_i \quad i=1,2,3$$

$A_a \in su(2)$ Lie alg.

$$= \mathcal{P}\exp\left[\int_0^1 ds\, A_\mu^i \frac{dx^\mu}{ds}\frac{\tau_i}{2}\right]$$

path ordered exponential

recall $\underline{S} = T\exp\left[-i\int_0^t dt' H_I(t')\right]$

matrix $A_a = A_a^i \frac{\tau_i}{2} \in su(2)$ Lie algebra

$$h(e) = \mathcal{P}e^{\int_e A}$$

$$h(e) = \mathcal{P}\left(\sum_{n=0}^{\infty}\frac{1}{n!}\left(\int_0^1 ds\, \underbrace{A_a\dot{e}^a}_{A(s)}\right)^n\right) = \sum_{n=0}^{\infty}\frac{1}{n!}\mathcal{P}\left(\int_0^1 ds\, A(s)\right)^n$$

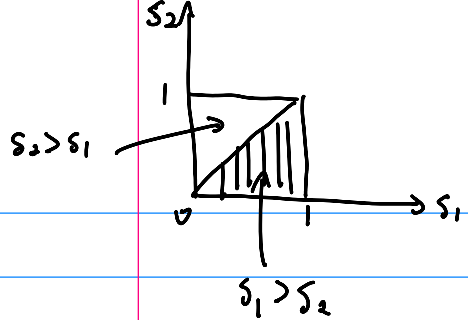
$$= 1_{2\times 2} + \int_0^1 ds\, A(s) + \frac{1}{2}\int_0^1 ds_1\int_0^1 ds_2\, \underline{\mathcal{P}\left[A(s_1)A(s_2)\right]} + \cdots$$

$$\mathcal{P}\left[A(s_1)A(s_2)\right] = \begin{cases} A(s_1)A(s_2) & s_1 < s_2 \\ A(s_2)A(s_1) & s_2 < s_1 \end{cases}$$

2 non commutative matrices

same convention as Thiemann's book.

$$\frac{1}{2}\int_0^1\int_0^1 ds_1 ds_2\, \mathcal{P}\left[A(s_1)A(s_2)\right] = \frac{1}{2}\iint_{s_1<s_2} ds_1 ds_2\, A(s_1)A(s_2) + \frac{1}{2}\iint_{s_2<s_1} ds_1 ds_2\, A(s_2)A(s_1)$$



$s_2 \leftrightarrow s_1$
change of variable

$$= \iint_{s_1 < s_2} ds_1 ds_2 \, A(s_1) A(s_2) = \underline{\int_0^1 ds_1 \int_{s_1}^1 ds_2 \, A(s_1) A(s_2)}$$

$$h(e) = \mathbb{1}_{2\times 2} + \sum_{n=1}^{\infty} \int_0^1 ds_1 \int_{s_1}^1 ds_2 \cdots \int_{s_{n-1}}^1 ds_n \, A(s_1) A(s_2) \cdots A(s_n) \qquad ①$$

$$A(s) \in su(2) \qquad h(e) \in SU(2)$$

Another way to compute $h(e)$

$$\underset{0 \;\; \Delta s_1 \; \Delta s_2 \;\cdots\; \Delta s_i \;\; \Delta s_n \;\; 1}{\overset{e}{\longrightarrow}}$$

$$h(e) = \lim_{\substack{n\to\infty \\ (\Delta s_i \to 0)}} \underline{\left[1 + \int_{\Delta s_1} ds\, A(s)\right]} \, \underline{\left[1 + \int_{\Delta s_2} ds\, A(s)\right]} \cdots \left[1 + \int_{\Delta s_n} ds\, A(s)\right] \qquad ②$$

<u>HW</u> check ① = ②

analog $\quad e^x = \lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n$