Path integral of fermions

Dirac fermion in 4-dim : $\underbrace{\text{spinor field}}^{\text{4-component}}$ $\begin{pmatrix} \psi_1(x) \\ \psi_2(x) \\ \psi_3(x) \\ \psi_4(x) \end{pmatrix} \equiv \psi(x)$ ← complex

(electrons)

$$\bar\psi(x) = \psi(x)^\dagger \gamma^0$$

$\gamma^{\mu=0..3}$: $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}\mathbb{1}_{4\times4}$

Dirac eqn : $(i\gamma^\mu\partial_\mu - m)\psi = 0$

HW (1) show that S is Lorentz inv.

(2) derive EOM from S

Dirac action $S = \int d^4x\, \bar\psi(i\gamma^\mu\partial_\mu - m)\psi$

↑ Grassmann variable ↑

$xy = -y\cdot x$

spin-statistics thm + positivity of Hamiltonian $\Rightarrow \{\psi(t,\vec{x}), \bar\psi(t,\vec{x}')\} = i\hbar\, \delta^{(3)}(\vec{x}-\vec{x}')$

anti commutation rel. $\{A, B\} = AB + BA$

$\hbar\to 0$ $\{\psi, \bar\psi\} = 0 \to \psi\bar\psi + \bar\psi\psi = 0$

$$\int D\psi D\bar\psi\, e^{i\int d^4x\, \bar\psi(i\gamma^\mu\partial_\mu - m)\psi}$$

$$\langle 0|T\, \hat{O}_1(x_1)\cdots\hat{O}_n(x_n)|0\rangle = \frac{\int D\psi D\bar\psi\, O_1(x_1)\cdots O_n(x_n)\, e^{i\int d^4x\, \bar\psi(i\gamma^\mu\partial_\mu - m)\psi}}{\int D\psi D\bar\psi\, e^{i\int d^4x\, \bar\psi(i\gamma^\mu\partial_\mu - m)\psi}}$$

↑ different from bosonic fields

$O_i(x_i) = O_i[\psi(x), \bar\psi(x)]$

$dx_1\wedge dx_2 = -dx_2\wedge dx_1$

rules of Grassmann variables : x, y Grassmann numbers (fermionic) (diff. form, $\wedge$)

(1) $xy = -yx \rightarrow x^2 = 0$

(2) $ax = xa \quad \forall a \in \mathbb{C}$ bosonic

(3) $(x_1 x_2 \cdots x_m)^* = x_m^* x_{m-1}^* \cdots x_2^* x_1^*$

function of Grassmann numbers : $f(x) = f_0 + f_1 x + \cancel{f_2 x^2}$

$f(x)$ ↑ bosonic, f_0 ↑ bosonic, f_1 ↑ fermionic, x ↑ fermionic

all functions of $\underbrace{\text{Grassmann variable}}_{\text{single variable}}$ is linear

$$\rightarrow f(x_1 \cdots x_m) = f_0 + f_i x_i + f_{ij} \underset{\sim}{x_i x_j} + f_{ijk} x_i x_j x_k + \cdots + f_{1 \cdots m} x_1 \cdots x_m$$

all functions are polynomials

e.g. $f(x_1, x_2) = f_0 + f_1 x_1 + f_2 x_2 + f_{12} x_1 x_2$

$$\rightarrow f_{ij} x_i x_j = -f_{ij} x_j x_i = -f_{ji} x_i x_j \Rightarrow f_{ij} = -f_{ji}$$

similar $f_{ijk} = -f_{jik} = -f_{ikj} = -f_{kji}$

$f_{i_1 \cdots i_m} x_{i_1} \cdots x_{i_m}$ $\quad m=3 \quad f_{ijk} x_i x_j x_k$

$$= \underline{f_{123}} x_1 x_2 x_3 + \underline{f_{213}} x_2 x_1 x_3 + \cdots$$

$$= 6 f_{123} x_1 x_2 x_3$$

derivative : $\frac{d}{dx} f(x) = \frac{d}{dx}(f_0 + x f_1) := f_1$ $\quad : \boxed{\frac{d}{dx} x = 1}$ $\qquad \frac{d}{dx} yx = -\frac{d}{dx} xy = -y$

$x \frac{d}{dx} f(x) = x f_1$ $\qquad \frac{d}{dx} x f(x) = \frac{d}{dx} x (f_0 + x f_1) = \frac{d}{dx} x f_0 = f_0$

$$\left(x \frac{d}{dx} + \frac{d}{dx} x\right) f(x) = x f_1 + f_0 = f(x)$$

$$\Rightarrow \left\{x, \frac{d}{dx}\right\} = 1$$

$yxy = -y^2 x = 0$

2-variables . $f(x,y) = f_0 + x f_1 + y f_2 + xy f_{12}$

$\frac{\partial}{\partial x} f(x,y) = f_1 + y f_{12}$ $\qquad \frac{\partial}{\partial y} f(x,y) = f_2 - x f_{12}$

$$\left(y \frac{\partial}{\partial x} + \frac{\partial}{\partial x} y\right) f(x,y) = y f_1 + \frac{\partial}{\partial x}(y f_0 + y x f_1)$$

$$= y f_1 - y f_1 = 0$$

$$\left\{y, \frac{\partial}{\partial x}\right\} = 0$$

$$\left(\frac{\partial}{\partial x}\frac{\partial}{\partial y} + \frac{\partial}{\partial y}\frac{\partial}{\partial x}\right) f(x,y) = -f_{12} + f_{12} = 0$$

$$\left\{ \frac{\partial}{\partial y}, \frac{\partial}{\partial x} \right\} = 0$$

$x_{i=1 \cdots m}$ Grassmann

Summary : $\left\{ x_i, \frac{\partial}{\partial x_i} \right\} = \delta_{ij}$ $\quad \left\{ \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right\} = 0$

$$\{ x_i, x_j \} = 0$$

integral : $\int dx\, f(x) = f_0 \int dx\, 1 + \left(\int dx\, x \right) f_1$

$$f(x) = f_0 + x f_1$$

$$\Delta x \Delta y = -\Delta y \Delta x$$

$$\left(\int dx\, 1 \right)^2 = \int dx \int dy\, 1 = -\int dy \int dx\, 1 = -\left(\int dx\, 1 \right)^2$$

$$\Rightarrow \int dx\, 1 = 0$$

define $\boxed{\int dx\, x = 1}$ $\Rightarrow \int dx\, f(x) = f_1$ same as $\frac{d}{dx} f(x)$

Gaussian integral : $\int dx^* dx\, e^{-x^* x}$ treat x, x^* indep.

$$= \int dx^* dx\, (1 - x^* x) = -\int dx^* dx\, x^* x$$

$$= \int dx^* \underline{\int dx\, x}\, x^* = \int dx^* x^* = 1$$

$(x_1^* x_1)(x_2^* x_2) = (x_2^* x_2)(x_1^* x_1)$

$$\int \prod_{i=1}^{m} dx_i^* dx_i\; e^{-x^\dagger M x}$$

any matrix, bosonic

$x = \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix}$

$x^\dagger = (x_1^* \cdots x_m^*)$

$$y = Mx$$

<u>Lemma</u> $\prod_{i=1}^{m} dy_i = (\det M)^{-1} \prod_{i=1}^{m} dx_i$

1-dim : $y = \underset{\substack{\uparrow \\ \text{bosonic}}}{a} x$ $\quad \int dy\, y = \int dx\, x = 1$

$$\int dy\, y = \int dy\, a x$$

$\Rightarrow a\, dy = dx$

$dy = a^{-1} dx$

2-dim : $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = M \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$\int dy_1\, dy_2\, y_1 y_2 = \underbrace{\int dx_1\, dx_2\, x_1 x_2} = -1$$

$$= \int dy_1\, dy_2 (ax_1 + bx_2)(cx_1 + dx_2)$$

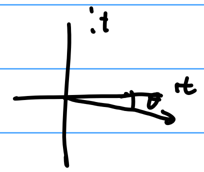
$$= \underbrace{\int dy_1\, dy_2 (ad - cb)\, x_1 x_2} \quad \Rightarrow \quad dy_1\, dy_2 = (\det M)^{-1} dx_1\, dx_2$$

$$-dx_1^* dx_2^* dx_1\, dx_2 = dx_1^* dx_1\, dx_2^* dx_2$$

$$\int \underbrace{\prod_i dx_i^* dx_i} e^{-x^\dagger M x} = \int \prod_i (dx_i^*\, dy_i)\, e^{-\sum_i x_i^* y_i} (\det M)$$

$$\underbrace{(-1)^{\#} \prod_i dx_i^* \underbrace{\prod_i dx_i}_{(\det M) \prod_i dy_i}}_{(\det M) \prod_i (dx_i^*\, dy_i)} \qquad = \prod_i \int dx_i^*\, dy_i\, e^{-x_i^* y_i} (\det M) = \det M$$

$$e^{-\sum_i x_i^* y_i} = \prod_i e^{-x_i^* y_i}$$



$t \to t e^{-i\theta}$

$$S = \int d^4x\, \bar{\psi}(x)(i\gamma^\mu \partial_\mu - m)\psi(x) \qquad \psi(x) = \int \psi(k)\, e^{-ik\cdot x} \frac{d^4k}{(2\pi)^4}$$

$$\bar{\psi}(x) = \int \bar{\psi}(k)\, e^{ik\cdot x} \frac{d^4k}{(2\pi)^4}$$

$$= \int \underline{dt}\, d^3x \int \frac{d^4k}{(2\pi)^4} \frac{d^4k'}{(2\pi)^4} \bar{\psi}(k')\, e^{-i\theta} (\gamma^0 k_0 e^{i\theta} + \gamma^i k_i - m)\, \psi(k)\, \underline{e^{i(k'-k)\cdot x}}$$

$$= \int \frac{d^4k}{(2\pi)^4} \bar{\psi}(k) \underbrace{e^{-i\theta}(\gamma^0 k_0 e^{i\theta} + \gamma^i k_i - m)}_{K} \psi(k)$$

Generating function : $Z[\bar{\eta}, \eta] = \int D\bar{\psi} D\psi\, e^{i\int \frac{d^4k}{(2\pi)^4} (\bar{\psi} K \psi + \bar{\eta}\psi + \bar{\psi}\eta)}$

$\psi, \bar{\psi}, \eta, \bar{\eta}$ Grassmann-valued field.

$$\psi = \psi' - K^{-1}\eta \qquad \bar{\psi} = \bar{\psi}' - \bar{\eta} K^{-1} \qquad D\psi = D\psi'$$
$$D\bar{\psi} = D\bar{\psi}'$$

$$\bar{\psi} K \psi + \bar{\eta}\psi + \bar{\psi}\eta = \bar{\psi}' K \psi' - \cancel{\bar{\eta} K^{-1} K \psi'} - \cancel{\bar{\psi}' K K^{-1} \eta} + \cancel{\bar{\eta} K^{-1} K K^{-1} \eta}$$
$$+ \cancel{\bar{\eta}\psi'} + \cancel{\bar{\psi}'\eta} - \cancel{\bar{\eta} K^{-1} \eta} - \bar{\eta} K^{-1} \eta$$

$$= \bar{\psi}' K \psi' - \bar{\eta} K^{-1} \eta$$

$$Z[\bar{\eta}, \eta] = \det(-iK)\, e^{-i \int d^4x\, \bar{\eta} K^{-1} \eta} \qquad Z[0] = \det(-iK)$$

$$\frac{\int D\bar{\psi} D\psi\, \psi_a(k) \bar{\psi}_b(k')\, e^{iS[\bar{\psi},\psi]}}{\int D\bar{\psi} D\psi\, e^{iS[\bar{\psi},\psi]}} = \frac{1}{Z[0]} \frac{\delta}{\delta(i\bar{\eta}_a(k))} \underline{\frac{\delta}{\delta(-i\eta_b(k'))}} Z[\bar{\eta},\eta] \Big|_{\bar{\eta}=\eta=0}$$

note: $$\frac{\delta}{\delta(-i\eta_a(k))} e^{i\int \frac{d^4k}{(2\pi)^4} \bar{\psi}\eta} = \frac{\delta}{\delta(-i\eta_a(k))} \Big(1 + \underbrace{i\int \bar{\psi}\eta}_{-i\int \eta^T \bar{\psi}^T} + \cdots \Big)$$

$$\bar{\psi}\eta = \bar{\psi}_a \eta_a = -\eta_a \bar{\psi}_a = -\eta^T \bar{\psi}^T$$

$$= \bar{\psi}_a(k) + \cdots$$

$$\frac{\delta}{\delta\bar{\eta}_a(k)} \frac{\delta}{\delta\eta_b(k')} \underbrace{e^{-i\int \frac{d^4k}{(2\pi)^4} \bar{\eta} K^{-1} \eta}}_{1 - i\int \bar{\eta}_a K^{-1}_{ab} \eta_b + \cdots} \Bigg|_{\bar{\eta}=\eta=0} \qquad \frac{\delta}{\delta\eta(k')} \eta(k) = \delta(k-k')$$

$$= i K^{-1}_{ab}(k)\, \delta^{(4)}(k-k') = \underbrace{\left[\frac{i}{e^{-i\theta}(\gamma^0 k_0 e^{i\theta} + \gamma^i k_i - m)} \right]_{ab}} \delta^{(4)}(k-k')$$