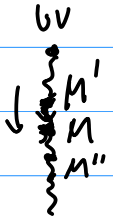
Wilsonian RG (1) integrating out high energy mode

$$Z = \int D\phi_{UV}\, e^{-S_{UV}[\phi_{UV}]} \longrightarrow Z = \int [D\phi]_{\Lambda}\, e^{-S_{\Lambda}[\phi]}$$

$$\phi_{UV} = \phi(x) + \hat\phi(x)$$

$$\phi(x) = \int_{|k|\le\Lambda} \phi(k)\, e^{ikx} \frac{d^dk}{(2\pi)^d}, \qquad \hat\phi(x) = \int_{\Lambda<|k|\le\Lambda_0} \phi(k)\, e^{ikx}$$




$$= \int D\phi\, e^{-S_0[\phi]} \underbrace{\int D\hat\phi\, e^{-\int \mathcal{L}[\phi,\hat\phi]}}$$

$$S_0[\phi] = \int d^dx \left[ \frac{1}{2}(\partial\phi)^2 + \frac{1}{2} m_0^2 \phi^2 + \frac{\lambda_0}{4!}\phi^4 \right]$$

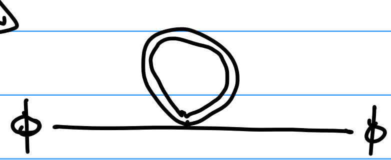
$$\int D\hat\phi\, e^{-\int\mathcal{L}[\phi,\hat\phi]} = \int D\hat\phi \left( 1 - \int d^dx \left[ \frac{1}{2} m_0^2 \hat\phi^2 + \lambda_0 \left( \cancel{\frac{1}{6}\phi^3\hat\phi} + \frac{1}{4}\phi^2\hat\phi^2 + \cancel{\frac{1}{6}\phi\hat\phi^3} + \frac{1}{4!}\hat\phi^4 \right) \right] + \cdots \right) e^{-\int \frac{1}{2}(\partial\hat\phi)^2}$$

const const

$O(\lambda_0^2, m_0^4, \lambda_0 m_0^2)$

$$\int \frac{1}{2}(\partial\hat\phi)^2 = \int \frac{d^dk}{(2\pi)^d} \hat\phi^*(k)\, k^2 \hat\phi(k)$$

$$\overbrace{\hat\phi(k)\hat\phi(p)} = \frac{\int D\hat\phi\, e^{-\int\frac{1}{2}(\partial\hat\phi)^2}\, \hat\phi(k)\hat\phi(p)}{\int D\hat\phi\, e^{-\int\frac{1}{2}(\partial\hat\phi)^2}} = \frac{1}{k^2}(2\pi)^d \delta^{(d)}(k+p)\, \Theta(k) \equiv$$

$1 +$  $+ \cdots = 1 + \int d^dx \left(-\frac{1}{2}\right)\mu\, \phi(x)^2 + \cdots$

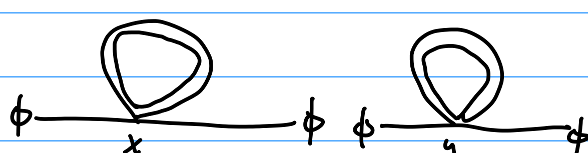
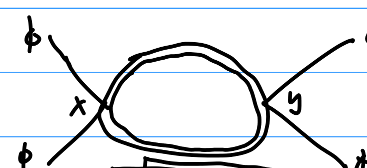

$O(\lambda_0)$

$$\mu = \frac{\lambda_0}{(4\pi)^{d/2}\Gamma(\frac{d}{2})} \frac{1-b^{d-2}}{d-2} \Lambda_0^{d-2}$$

$$\int e^{-x^2} xxxx\, dx$$

$$b = \frac{\Lambda}{\Lambda_0}$$

$$O(\lambda_0^2): \quad \int D\hat\phi\, \frac{1}{2} \int d^dx \frac{\lambda_0}{4}\phi^2\hat\phi\hat\phi \int d^dy \frac{\lambda_0}{4}\phi^2\hat\phi\hat\phi\, e^{-\int\frac{1}{2}(\partial\hat\phi)^2} \qquad (1)$$

$=$  $+$ 


$\int dx\, \hat\phi\hat\phi \int dy\, \hat\phi\hat\phi \qquad \int dx\, \hat\phi\hat\phi \int dy\, \hat\phi\hat\phi$

$$\frac{1}{2}\left[\int d^dx \left(-\frac{1}{2}\mu\right)\phi^2\right]^2$$

$$\uparrow$$

$$\int d^dx\, d^dy\ \phi(x)^2\phi(y)^2\, G(x-y)$$

nonlocal

$$1+\int -\frac{1}{2}\mu\phi^2+\frac{1}{2}\left(\int -\frac{1}{2}\mu\phi^2\right)^2+\dots$$

$$= e^{-\frac{1}{2}\mu\int\phi^2}$$

$$= \underline{\zeta\int d^dx\ \phi(x)^4}$$

$$+\left(-\frac{\eta}{4}\right)\int d^dx\ \phi^2(\partial_\mu\phi)^2$$

$$+\ O(\partial^4) \qquad (2)$$

$$\zeta = -\frac{3}{2}\lambda_0^2\int_{\mu<|k|\le\Lambda_0}\frac{d^dk}{(2\pi)^d}\left(\frac{1}{k^2}\right)^2 = \frac{-3\lambda_0^2}{(4\pi)^{d/2}\Gamma\left(\frac{d}{2}\right)}\ \frac{1-b^{d-4}}{d-4}\ \Lambda_0^{d-4} \qquad (3)$$

$$\xrightarrow{d\to4}\ -\frac{3\lambda_0^2}{16\pi^2}\ln\frac{1}{b} \qquad b=\frac{\mu}{\Lambda_0}$$

HW find the term (1) in the expansion
show (2) & (3)

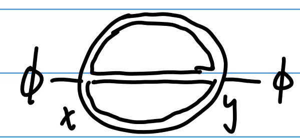
$$O(\lambda_0^2):\ \frac{1}{2}\int d^dx\ \frac{\lambda_0}{6}\phi^3\hat\phi\int d^dy\ \frac{\lambda_0}{6}\phi^3\hat\phi \sim \int d^dx\ \phi^6 + \text{derivatives}$$

you can keep going

e.g. $O(\lambda_0^3)$
$$\int d^dx\ \phi^2\hat\phi^2\int d^dy\ \phi^2\hat\phi^2\int d^dz\ \phi^2\hat\phi^2 \sim \int d^dx\,\phi^6 + \text{derivatives}$$

$$\int d^dx\ \phi^2\hat\phi^2\int d^dy\ \phi^3\hat\phi\int d^dz\ \phi^3\hat\phi \sim \int d^dx\,\phi^8 + \text{derivatives}$$

$$O(\lambda_0^2):\ \frac{1}{2}\int d^dx\ \frac{\lambda_0}{6}\phi\hat\phi^3\int d^dy\ \frac{\lambda_0}{6}\phi\hat\phi^3 \sim$$




$$\sim \alpha\int\phi^2+\beta\int(\partial_\mu\phi)^2+\dots$$

correct mass term in $S_0$

correct kinetic term in $S_0$

$$Z=\int[D\phi]_\mu e^{-S_0[\phi]}\ \underline{\int D\hat\phi\ e^{-\int\mathcal{L}[\hat\phi,\phi]}} = \int[D\phi]_\mu e^{-S_\mu[\phi]}$$

$$S_M[\phi] = \int d^dx \; \mathcal{L}_{eff}$$

$$= \int d^dx \Big[ \frac{1}{2}(1+\underbrace{\Delta Z}_{\beta+\cdots})(\partial_\mu\phi)^2 + \frac{1}{2}(m_0^2 + \underbrace{\Delta m^2}_{\mu+\cdots})\phi^2 + \frac{1}{4!}(\lambda_0 + \underbrace{\Delta\lambda}_{\zeta+\cdots})\phi^4$$

$$+ \underbrace{\Delta C}_{\eta+\cdots}\, \phi^2(\partial_\mu\phi)^2 + \Delta D\, \phi^6 + \cdots \Big]$$

higher order interaction $O(\phi^8)$

higher order derivatives $O(\partial^4)$

$\mathcal{L}_{eff}$ : (1) renormalizes $S_0$

(2) add $\infty$ many high order interaction to $S_0$

(3) add $\infty$ many higher derivative interactions to $S_0$ $\rightarrow$ $\mathcal{L}_{eff}$ is non-local.

$\partial_\mu \sim k_\mu$

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$$\int dx\, dy\, \phi(x)^2 \underline{\phi(y)^2}\, G(x-y) = \int dx\, \overset{d(x-y)}{dy}\, \phi(x)^2 \big(\phi(x) + \underline{\partial_\mu\phi(x)}\,\underline{(y^\mu - x^\mu)} + \cdots\big)^2 \underline{G(x-y)}$$

$$= \zeta \int dx\, \phi(x)^4 + \eta \int dx\, \phi^2 (\partial\phi)^2$$

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Step (2) in Wilsonian RG (rescaling $k$)

we want to compare $\mathcal{L}_{eff}$ with the original $\mathcal{L}_{UV}$

but they are not in the same regime

$\mathcal{L}_{eff}$: $|k| \leq M$ $\qquad$ $\mathcal{L}_{UV}$: $|k| \leq \Lambda_0$

$b = \frac{M}{\Lambda_0} < 1$

Now we rescale : $k' = k/b \qquad x' = xb$

changing the unit

$|k| \leq M$

$\rightarrow |k'| = \frac{|k|}{b} \leq \frac{M}{b} = \Lambda_0$

e.g. $\begin{cases} 1\,\text{mm} \;@\; \text{UV} \;\rightarrow x \\ 1\,\text{cm} \;@\; \text{IR} \;\rightarrow x' \end{cases}$

$x' < x$

$$S_M = \int d^dx\, \mathcal{L}_{eff} = \int d^dx'\, b^{-d}\Big[\frac{1}{2}(1+\Delta Z)\, b^2(\partial'_\mu\phi)^2 + \frac{1}{2}(m_0^2+\Delta m)\,\phi^2 + \frac{1}{4!}(\lambda_0+\Delta\lambda)\phi^4 + \Delta C\, b^2(\partial'_\mu\phi)^2\phi^2 + \Delta D\,\phi^6 + \cdots\Big]$$

wave function renormalization $\phi' = \left[b^{2-d}(1+\Delta Z)\right]^{\frac{1}{2}}\phi$

$$S_M = \int d^dx'\Big[\frac{1}{2}(\partial'_\mu\phi')^2 + \frac{1}{2}m'^2\phi'^2 + \frac{1}{4}\lambda'\phi'^4 + c'(\partial'_\mu\phi')^4 + D'\phi'^6 + \cdots\Big]$$

Renormalized couplings :

$$m'^2 = (m_0^2+\Delta m^2)(1+\Delta Z)^{-1}b^{-2}$$
$$\lambda' = (\lambda_0+\Delta\lambda)(1+\Delta Z)^{-2}b^{d-4}$$
$$c' = (C_0+\Delta C)(1+\Delta Z)^{-2}b^{d-2}$$
$$D' = (D_0+\Delta D)(1+\Delta Z)^{-3}b^{2d-6}$$
$$\vdots$$

$C_0 = D_0 = 0$ is our starting point, but we can start with $C_0 \neq 0$, $D_0 \neq 0$

UV

$M_1 \to m' = m(b_1)$, $\lambda' = \lambda(b_1)$, $c' = C(b_1)$, $D' = D(b_1)$ ... $\qquad b_1 = \frac{M_1}{\Lambda_0}$

$M_2 \to m(b_1)$, $\lambda(b_2)$, $C(b_2)$, $D(b_2)$ .... $\qquad b_2 = \frac{M_2}{\Lambda_0}$

IR

RG is a semi-group, only flows from UV to IR, no inverse.

Let $M$ change continuously, $b$ changes continuously

$\Rightarrow m(b)$, $\lambda(b)$, $C(b)$, $D(b)$ .... flow continuously

$\to$ RG flow.

i.e. $S_M$ renormalized action flows continuously

Definition : (1) Given a RG flow from UV to IR, consider a coupling $g(b)$

$b = \frac{M}{\Lambda_0}$

if $g(b)$ grows as $b$ becomes small,

$\longrightarrow$ $g(b)$ is relevant coupling

if $g(b)$ becomes small as $b$ becomes small

$\longrightarrow$ $g(b)$ is irrelevant coupling

if $g(b)$ unchange as $b$ becomes small

$\longrightarrow$ $g(b)$ is marginal

(2) if RG leave $S_M$ invariant, $S_M$ is a fix point of RG.

HW show that $\mathcal{L}_0 = \frac{1}{2}(\partial_\mu \phi)^2$ is a fix point (Gaussian fix-pt)

$S_M, S_{UV} = \mathcal{L}_0$ + power expansion in $\phi$

$\longrightarrow$ we are close to Gaussian fix-pt.