boson no spin $\quad T_0^2 = 1 \qquad T_0 \psi(t,\vec{r}) = \psi^*(-t,\vec{r})$

fermion spin $\frac{1}{2}$ $\quad T^2 = -1 \qquad T\begin{pmatrix}\psi_1(t,\vec{r})\\ \psi_2(t,\vec{r})\end{pmatrix} = e^{i\varphi}\sigma_y \begin{pmatrix}\psi_1^*(-t)\\ \psi_2^*(-t)\end{pmatrix}$

zero spin $\quad \hat{H}\psi_{nl} = E_n \psi_{nl} \qquad l = 1..d(n)$

general sol of Schrödinger eqn $\quad \psi(\vec{r},t) = \sum_n a_n \psi_{nl}(\vec{r}) e^{-\frac{i}{\hbar}E_n t}$

( $\hat{H}$ is real, not explicitly dep. on $t$ ) $\quad \hat{T}_0 \psi(\vec{r},t) = \sum_n a_n^* \psi_{nl}^*(\vec{r}) e^{-\frac{i}{\hbar}E_n t}$

effectively $\quad \hat{T}_0 : \psi_{nl}(\vec{r}) \to \psi_{nl}^*(\vec{r})$

if $\hat{H}$ is real, $\quad \hat{H}\psi_{nl} = E_n\psi_{nl} \qquad \psi_{nl}$ is eigenstate

$\Rightarrow \hat{H}\psi_{nl}^* = E_n \psi_{nl}^* \qquad \psi_{nl}^*$ is eigenstate

$\psi_{nl}$ basis in $\mathcal{H}^{(n)}$, $\quad \psi_{nl}^*$ is basis in $\mathcal{H}^{(n)*}$

whether $\mathcal{H}^{(n)}$ and $\mathcal{H}^{(n)*}$ the same ?

(1) $\quad \mathcal{H}^{(n)} \simeq \mathcal{H}^{(n)*} \qquad$ equivalent rep of symm. group

(2) $\quad \mathcal{H}^{(n)} \not\simeq \mathcal{H}^{(n)*} \qquad$ not equivalent ....

$$\text{eigenspace} = \mathcal{H}^{(n)} \oplus \mathcal{H}^{(n)*}$$

<u>real rep, pseudo real rep, and complex rep</u>

Given a symm. group $G$, unitary irrep $D: g \mapsto D(g) \circlearrowright \mathcal{H}^{(n)}$

**Def** (complex conjugate irrep) $D(g)$ is unitary on $\mathcal{H}^{(n)}$, $\psi_{ni}$ orthonormal basis in $\mathcal{H}^{(n)}$

$$D(g)\,\psi_{ni}(\vec{r}) = \sum_j \psi_{nj}(\vec{r})\, D_{ji}(g)$$

$D_{ji}(g)$ unitary matrix

complex conjugate : $D(g)^* \psi^*_{ni}(\vec{r}) = \sum_j \psi_{nj}(\vec{r})^* D_{ji}(g)^*$

$$\psi^*_{ni}(\vec{r}) = T_0\, \psi_{ni}(\vec{r})$$

$D_{ij}(g)$ unitary irrep matrix $\xrightarrow{T_0}$ $D^*_{ij}(g)$ unitary irrep matrix

( irrep $D$) irrep $D^*$ : complex conjugate irrep

relation between $D$ and $D^*$

(1) if $\exists$ unitary transf. $U : \mathcal{H}^{(n)} \to \mathcal{H}^{(n)}$ s.t

$$U D(g) U^{-1} = D_0(g) \quad \forall g \in G$$

real matrix

$\Rightarrow$ $D$ is equivalent to $D^*$, since $U^* D(g)^* U^{*-1} = D_0(g)$

$\Rightarrow$ $D^*(g) = U^{*-1} D_0(g) U^* = U^{*-1} U D(g) U^{-1} U^*$

$U^{*-1}U$ is unitary

we say $D \simeq D^*$ is a real rep.

(2) $D$ is equivalent to $D^*$, but they are not equivalent to real rep.
unitarily

$$\nexists\, U \text{ s.t. } D(g) = U D_0(g) U^{-1} \quad \forall g \in G$$
real matrix

We say, $D \simeq D^*$ is pseudo-real

(3) $D$ is inequivalent to $D^*$.

We say, $D$ and $D^*$ are complex reps

distinguish (3) from (1) and (2), check character

if $D \simeq D^*$, then $\chi(g) = \chi^*(g) \quad \forall g \in G$

if $\chi(g) \neq \chi^*(g)$, then $D$ & $D^*$ are complex reps.

distinguish between (1) & (2)

$$D^* \simeq D \rightarrow D^*(g) = Z D(g) Z^{-1} \quad \forall g \in G$$
unitary matrix

Complex conjugate $D(g) = Z^* D^*(g) Z^{*-1}$

$$\Rightarrow D^*(g) = Z Z^* D^*(g) (Z Z^*)^{-1}$$

i.e. $[Z Z^*, D^*(g)] = 0 \quad \forall g \in G$

by Schur's lemma, when $D^*$ is irrep, $ZZ^* = c\,\mathbb{1}$
$c \in \mathbb{C}$

$Z$ is unitary, $Z^* = (Z^T)^{-1}$

$\Rightarrow \quad Z(Z^T)^{-1} = c\,\mathbb{1} \Rightarrow \underline{Z = cZ^T}$

$\xrightarrow[\text{transpose}]{} Z^T = cZ$

$\Rightarrow \quad Z = c^2 Z \quad \Rightarrow \quad c^2 = 1, \quad c = \pm 1$

$D^* \simeq D \Rightarrow ZZ^* = \pm\mathbb{1}$

<u>Thm</u>: $ZZ^* = \mathbb{1}$ iff $D$ is real

$ZZ^* = -\mathbb{1}$ iff $D$ is pseudo-real

Pf if $D$ is real $\rightarrow$ $D^*(g) = (U^*)^{-1} D_0(g)\, U^*$

$= (U^*)^{-1} U D(g) U^{-1} U^*$

on the other hand $D^*(g) = Z D(g) Z^{-1}$

$\Rightarrow \quad (U^*)^{-1} U = b\,Z \qquad b \in \mathbb{C}$

Complex conjugate $U^{-1}U^* = b^* Z^*$

$U^{-1}U^*(U^*)^{-1}U = b^* b\, Z^* Z = |b|^2 Z^* Z$

$\|$

$1$ $\qquad \underline{Z^* Z = |b|^{-2}\,\mathbb{1}}$

$\underline{ZZ^* = c\mathbb{1}}$ $\Rightarrow$ $|b|^2 c = 1$

$c = 1$ or $-1$ $\qquad$ $c = 1$ $\quad |b|^2 > 0$

conversely if $c = 1$ $\quad (ZZ^* = \mathbb{1})$

$Z$ unitary, $Z = e^{iA}$ $\quad$ $A$ hermitian

$Z^* = Z^{-1} \Leftrightarrow Z^T = Z$, $\quad$ $Z$ symmetric

$Z^* = (Z^T)^{-1}$

def. $U = Z^{\frac{1}{2}} = e^{iA/2}$ $\quad$ unitary

$U^2 = Z$

$$\underline{U^* Z = e^{-iA/2} e^{iA} = e^{\frac{1}{2}iA} = U}$$

$\forall g \in G$, $[U D(g) U^{-1}]^* = U^* D(g)^* U^{*-1}$

$= U^* Z D(g) Z^{-1} U^{*-1}$

$= U D(g) U^{-1} = D_0(g)$ real

$\Rightarrow D$ is real rep.

$ZZ^* = \mathbb{1} \Leftrightarrow D$ is real rep

$\Rightarrow ZZ^* = -\mathbb{1} \Leftrightarrow D$ is psudo-real rep. $\quad \square$

Distinguish $\underbrace{\text{real \& pseudo-real}}$ reps by characters

$$\underline{D^*(g) = Z\, D(g)\, Z^{-1}}$$

orthogonality $\displaystyle\sum_g D^*_{\alpha\delta}(g)\, D_{\beta\gamma}(g) = \delta_{\alpha\beta}\,\delta_{\gamma\delta}\, \frac{h}{\dim(D) = d}$

$$\parallel$$

$$\sum_{\alpha,\delta} \underline{Z_{\tau\alpha}^{-1}} \times \sum_g \sum_{\sigma,\rho} \underline{Z_{\alpha\sigma}}\, D_{\sigma\rho}(g)\, Z^{-1}_{\rho\delta}\, D_{\beta\gamma}(g) \times Z_{\delta\chi}$$

$$\Rightarrow \sum_g D_{\tau\chi}(g)\, D_{\beta\gamma}(g) = Z_{\gamma\chi}\, Z^{-1}_{\tau\beta}\, \frac{h}{d}$$

let $\chi = \beta$. $\sum_\beta$, $\displaystyle\sum_g \sum_\beta D_{\tau\beta}(g)\, D_{\beta\gamma}(g)$

$$\parallel$$

$$\sum_g D_{\tau\gamma}(g^2) = (Z\, Z^*)_{\tau\gamma}\, \frac{h}{d}$$

$$= \pm\, \delta_{\tau\gamma}\, \frac{h}{d}$$

let $\tau = \gamma$, $\sum_\tau$,

$$\boxed{\sum_g \chi(g^2) = \pm h}$$

for complex rep. $D \neq D^*$ $\Rightarrow \displaystyle\sum_g \left[D^*_{\alpha\beta}(g)\right]^* D_{\gamma\delta}(g) = 0$

$$\parallel$$

$$\sum_g D_{\alpha\beta}(g)\, D_{\gamma\delta}(g)$$

$$\Rightarrow \sum_g \chi(g^2) = 0$$

summary :
$$\frac{1}{h}\sum_{g\in G}\chi(g^2) = \begin{cases} 1 & \text{real rep} \\ -1 & \text{psudo-real rep} \\ 0 & \text{complex.} \end{cases}$$

## Extra-degeneracy of $\hat{H}$ due to time-reversal inv.

$\hat{H}\psi_{nl} = E_n \psi_{nl}$ $\quad l = 1 \dots d$

$\psi_{nl} \in \mathcal{H}^{(n)}$ unitary irrep. of symmetry group $G$

$l = 1 \cdots d$ $\quad \dim \mathcal{H}^{(n)} = d$

find degeneracy at $E_n$, deg. at $E_n$ is either $d$ or $2d$

(spin-zero)

if $\hat{H}$ is real and not explicitly dep on $t$

$\hat{H}\psi_{nl} = E_n \psi_{nl}$

$\hat{H}\psi_{nl}^* = E_n \psi_{nl}^*$ $\quad \psi^*_{nl}(\vec{r}) = \hat{T}_0\, \psi_{nl}(\vec{r})$

- if $\forall \psi_{nl}^*$, $\psi^*_{nl} \in \mathcal{H}^{(n)}$ spanned by $\psi_{nl}$

then $\mathcal{H}^{(n)}$ is the final eigenspace i.e. $\mathcal{H}^{(n)} = \mathcal{H}^{(n)*}$

degeneracy at $E_n$ is $d$

- otherwise, eigenspace $= \mathcal{H}^{(n)} \oplus \mathcal{H}^{(n)*}$

  degeneracy at $E_n = 2d$ (extra degeneracy)

<u>spin-zero</u>

$$\hat{H}\psi_i = E\psi_i \qquad \psi_i \in \underline{\underline{\mathcal{H}^{(n)}}}$$

$$D(g)\psi_i = \sum_j \psi_j D_{ji}(g) \quad \forall g \in G$$

$\hat{H}$ is real $\quad \hat{H}\psi_i^* = E\psi_i^* \qquad \psi_i^* \in \mathcal{H}^{(n)*}$

$$D^*(g)\psi_i^* = \sum_j \psi_j^* D_{ji}(g)^* \quad \forall g \in G$$

$\mathcal{H}^{(n)}$ carries irrep $D$ of $G$

$\mathcal{H}^{(n)*}$ carries complex conjugate irrep $D^*$ of $G$

$\mathcal{H}^{(n)} \simeq \mathcal{H}^{(n)*}$ or not relates to $D \simeq D^*$ or not

firstly if $D \not\simeq D^*$ complex irrep, $\mathcal{H}^{(n)} \perp \mathcal{H}^{(n)*}$ by orthogonality theorem.

$\Rightarrow$ extra degeneracy $2d$

let's look at case (1) real rep and (2) pseudo-real rep.

$\exists$ unitary $Z$, $D^*(g) = Z D(g) Z^{-1}$, $ZZ^* = \begin{cases} \mathbb{1} & \text{real} \\ -\mathbb{1} & \text{pseudo-real} \end{cases}$

Lemma if $\mathcal{H}^{(n)} = \mathcal{H}^{(n)*}$, then $D$ is of case (1) i.e. real rep.

pf. $\underline{\underline{\mathcal{H}^{(n)} = \mathcal{H}^{(n)*}}}$, $\hat{H}\psi_i = E\psi_i$

$\hat{H}\psi_i^* = E\psi_i^*$

both $\{\psi_i\}$, $\{\psi_i^*\}$ are both orthonormal basis

then $\exists$ unitary $U$ s.t. $\psi_i = \sum_k \psi_k^* U_{ki}$

$\psi_i^* = \sum_k \psi_k U_{ki}^*$

$\Rightarrow \psi_i = \sum_{kl} \psi_l U_{lk}^* U_{ki}$ i.e. $UU^* = \mathbb{1}$

$D(g)\psi_i = \sum_j \psi_j D_{ji}(g)$

$\| \qquad\qquad\qquad \|$

$D(g)\sum_k \psi_k^* U_{ki} \qquad \sum_j \sum_k \psi_k^* U_{kj} D_{ji}(g)$

$\times U^{-1}_{il}, \sum_i$

$D(g)\psi_l^* = \sum_k \psi_k^* \left(U D(g) U^{-1}\right)_{kl}$

$\uparrow$

same as $D^*(g)\psi_l^*$

Compare to $D^*(g)\psi_l^* = \sum_j \psi_j^* D_{jl}^*(g)$

$D^*(g) = U D(g) U^{-1} \Big\} \Rightarrow D$ is real

$$U U^* = 1 \qquad J$$

$\square$

$$U = Z$$

Lemma if $D$ is real, then $\mathcal{H}^{(n)} = \mathcal{H}^{(n)*}$

pf $D$ is real, $\underline{Z D Z^{-1} = D^*}$ & $Z Z^* = 1$

$$D^*(g)\psi_i^* = \sum_j \psi_j D_{ji}^*(g)$$

$$= \sum_j \psi_j \sum_{k,l} Z_{jk} D_{kl}(g) Z^{-1}_{li}$$

$\times Z_{im}$

and $\sum_i$

$$\Rightarrow D^*(g)\left(\sum_i \psi_i^* Z_{im}\right) = \sum_k \left(\sum_j \psi_j^* Z_{jk}\right) D_{km}(g)$$

$$D(g)\psi_m = \sum_k \psi_k D_{km}(g)$$

$$\Rightarrow \sum_i \psi_i^* Z_{im} = \psi_m$$

$$\underbrace{\psi_i^* \in \mathcal{H}^{(n)*} \qquad \psi_i \in \mathcal{H}^{(n)}}_{}$$

they are linear dep. by $\sum_i \psi_i^* Z_{im} = \psi_m$

$$\Rightarrow \mathcal{H}^{(n)} = \mathcal{H}^{(n)*}$$

above 2 lemmas $\Rightarrow$ $\mathcal{H}^{(n)} = \mathcal{H}^{(n)*}$ iff $D$ is real

(no extra degeneracy) $d$

then if $D$ is psudo-real then $\mathcal{H}^{(n)} \neq \mathcal{H}^{(n)*}$

$\Rightarrow$ extra degeneracy $2d$

$$\text{if } D \text{ is } \begin{cases} \text{real} & \text{degeneracy} = d = \dim(D) \\ \text{psudo-real} & \text{degeneracy} = 2d \\ \text{complex} & \text{degeneracy} = 2d \end{cases}$$

<u>Examples</u> (1) 1d free particle $\hat{H} = \frac{1}{2m}\hat{p}^2$

$$\hat{p} = -i\hbar \frac{\partial}{\partial x}$$

symmetry : transl. inv. $Q(\lambda)\, x = x + \lambda$ $\lambda \in \mathbb{R}$

$$D(\lambda) = e^{-\frac{i}{\hbar}\lambda \hat{p}}$$

$$D(\lambda)\,\psi(x) = \psi(x+\lambda)$$

$G = \mathbb{R} = \{\lambda\}$ group multiplication : $+$ : $\lambda_1 + \lambda_2$

irrep of $\mathbb{R}$ : $D^{(k)}(\lambda) = e^{ik\lambda}$

$\mathcal{H}^{(k)} = \mathbb{C}$ $\dim(D^{(k)}) = 1$

all irrep of $G$ are 1-dim

$$\underline{D^{(k)}(\lambda_1)\, D^{(k)}(\lambda_2) = e^{ik(\lambda_1+\lambda_2)} = D^{(k)}(\lambda_1+\lambda_2)}$$

Complex conjugate of $D^{(k)}$ : $D^{(k)}(\lambda)^* = e^{-ik\lambda}$

$$\neq e^{ik\lambda}$$

$$D^{(k)} \not\simeq D^{(k)*}$$

Cannot transf between them by unitary on $\mathbb{C}$

Complex irrep.

energy level degeneracy $= 2 \dim(D^{(k)})$

$= 2$

eigenstates of $H$ : $\hat{H}\psi_k = \frac{\hbar^2 k^2}{2m}\psi_k$

$$\psi_k = \underset{\substack{\cap \\ D^{(k)}}}{e^{ikx}}, \quad \psi_{-k} = \underset{\substack{\cap \\ D^{(k)*}}}{e^{-ikx}}$$

indeed degeneracy $= 2$

$$\underset{\substack{= \\ \parallel}}{U}\, e^{ik\lambda}\, \underset{\substack{\circlearrowleft \\ \mathbb{C}}}{U^{-1}} = e^{-ikx}$$

$$e^{ik\lambda} \neq e^{-ik\lambda} \Rightarrow D^{(k)} \not\simeq D^{(k)*}$$

(2) central potential : $\hat{H}\psi_{nlm}(\vec{r}) = E_{nl}\psi_{nlm}$

$l = 0, 1, 2 \cdots$

$m = -l, -l+1, \dots, l$

symmetry group $G = SO(3)$

eigenspace $\mathcal{H}^{(n,l)}$ relates to irrep of $SO(3)$

$\forall$ Rotation $Q(\alpha, \beta, \gamma) \in SO(3)$

↑ Euler angles

irrep of $SO(3)$ $\underbrace{,}_{D^{(l)}}$ $\mathcal{H}^{(l)}$ is spanned by $Y_{lm}(\theta, \varphi)$

is labelled by $l$, $m = -l, \dots, l$

$(l = 0, 1, 2, 3 \dots)$ $\dim(\mathcal{H}^{(l)}) = 2l+1$

$$\langle l\, m \,|\, D^{(l)}(\alpha, \beta, \gamma) \,|\, l\, m' \rangle = \int_{S^2} d\theta\, d\varphi \sin\theta\, Y^*_{lm} \hat{D}^l Y_{lm'}$$

$$= D^{(l)}_{mm'}(\alpha, \beta, \gamma) = e^{-im'\alpha} d^l_{mm'}(\beta) e^{-im\gamma}$$

↑ Wigner D-function

↑ Wigner d-func

$d(0) = 1$

$$\chi^l(\alpha) = \mathrm{tr}\, D^{(l)}(\alpha, \beta, \gamma) = \mathrm{tr}\, D^{(l)}(\alpha, 0, 0)$$

$$= \sum_{m=-l}^{l} e^{-im\alpha} = \frac{\sin\left((l+\frac{1}{2})\alpha\right)}{\sin \alpha/2} \quad \text{real}$$

$\chi^{l*} = \chi^l$. $D^l$ is not complex basis in $\mathcal{H}^{(l)}$ : $Y_{lm}(\theta,\varphi)$ , $Y^*_{lm}(\theta,\varphi) = Y_{l,-m}(\theta,\varphi)$

$$Y^*_{lm} \in \mathcal{H}^{(l)}$$

$$\Rightarrow \mathcal{H}^{(l)*} = \mathcal{H}^{(l)}$$

$\Rightarrow D^{(l)}$ $\cdot l = 0, 1 \cdots$ are all real rep. , no extra degeneracy

$\mathcal{H}^{(n,l)}$ is the eigenspace , deg. $= 2l+1$

single spin-$\frac{1}{2}$ particle : $T^2 = -1$

<u>lemma</u> $\langle \psi | T\psi \rangle = 0$

pf. Let $\varphi = T\psi$ $\forall \psi \in \mathcal{H}^{(0)} \otimes \mathbb{C}^2$

$$\langle \psi | \varphi \rangle = \langle \psi | T\psi \rangle \underset{\substack{\uparrow \\ T \text{ is antiunitary}}}{=} \langle T\psi | T^2 \psi \rangle^*$$

$$= -\langle T\psi | \psi \rangle^* = -\langle \psi | \varphi \rangle$$

$$\Rightarrow \langle \psi | \varphi \rangle = 0$$

$\psi$ , $T\psi$ are orthogonal. $\square$

$\hat{H} \psi_i = E \psi_i$ $i = 1 \cdots d$ $\psi_i \in \mathcal{H}^{(d)}$ carry irrep $D^{(d)}$ of $G$

$\hat{H}(T\psi_i) = E(T\psi_i)$ $\varphi_i \equiv T\psi_i$

(1) $\langle \psi_i | \varphi_i \rangle = 0$

(2) $\langle \psi_i | \psi_j \rangle = \delta_{ij}$

(3) $\langle \varphi_i | \varphi_j \rangle = \delta_{ij}$

<u>Thm</u> (Kramer's Thm) single spin-$\frac{1}{2}$ particle, $\forall$ energy level degeneracy $d'$ is always even and

$$d \leq d' \leq 2d$$

pf we have (1), (2), & (3) whether $\{\psi_i, \varphi_i\}$ form a complete basis

but it's possible $\langle \psi_i | \varphi_j \rangle \neq 0$ $i \neq j$

if $|\varphi_k\rangle = \sum_l |\psi_l\rangle c_l$ the $\varphi_k$ should be removed from the set of basis

but the $|\psi_k\rangle$ should be removed as well

since $T^2 = -1$

$$\begin{array}{ccc} T|\varphi_k\rangle & = & \sum_l T(|\psi_l\rangle c_l) \\ \| & & \| \\ T^2|\psi_k\rangle & & \sum_l c_l^* |\varphi_l\rangle \\ \| & & \\ -|\psi_k\rangle & & \end{array}$$

$$\Rightarrow |\psi_k\rangle = -\sum_l c_l^* |\varphi_l\rangle$$

$$\psi_1 \cdots \widehat{\psi_k} \cdots \psi_d \ \varphi_1 \cdots \widehat{\varphi_k} \cdots \varphi_d$$

removed in pair.

\# of removed pairs

find degeneracy $\quad d' = 2d - 2n \quad$ $d'$ is even

$$d \le d' \le 2d$$

$\square$

## Theory of angular momentum

### spatial rotation & SO(3) group

finite rotation : $Q$ 3x3 matrix s.t. $Q\vec{r}_1 \cdot Q\vec{r}_2 = \vec{r}_1 \cdot \vec{r}_2$

$$\Rightarrow Q^T Q = 1 \qquad Q \in O(3)$$

$$\det(Q^T Q) = 1$$

$$\det(Q)^2 = 1$$

$$\det Q = \pm 1$$

SO(3) : $Q^T Q = 1$ & $\det Q = 1$ $\qquad P = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

HW tutorial

problem 1.

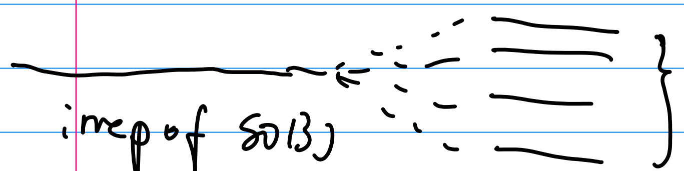
$$\hat{H} = \hat{H}_0 + V_{lattice}(\vec{r})$$

central force, SO(3) symmetry

eigenspace of $H_0$ : $\mathcal{H}_{l=0,1,\dots}$

carries irrep of SO(3)

$V_{lattice}$ breaks SO(3) to $O$ (cubic lattice group)




$$D^l = \bigoplus_T D_T$$

irrep of SO(3)

irreps of $O$

$$\forall g \in O, \quad \chi_l(g) = \sum_T \chi_T(g)$$

character of $O$

| $\chi_T(g)$ | $E$ | $8C_3$ | $6C_2$ | $2C_2$ | $6C_4$ |
|---|---|---|---|---|---|
| $T_1$ | 1 | 1 | 1 | 1 | 1 |
| $T_2$ | 1 | 1 | -1 | 1 | -1 |
| $T_3$ | 2 | -1 | 0 | 2 | 0 |
| $T_4$ | 3 | 0 | -1 | -1 | 1 |
| $T_5$ | 3 | 0 | 1 | -1 | -1 |

character of SO(3) evaluated at elements in $O$

| $\chi_l(g)$ | $E$ | $8C_3$ | $6C_2$ | $2C_2$ | $6C_4$ |
|---|---|---|---|---|---|
| $l=0$ | 1 | 1 | 1 | 1 | 1 |
| $l=1$ | 3 | 0 | $-1$ | $-1$ | 1 |
| $l=2$ | 5 | $-1$ | 1 | 1 | $-1$ |
| $l=3$ | 7 | 1 | $-1$ | $-1$ | $-1$ |
| $l=4$ | 9 | 0 | 1 | 1 | 1 |

- $l=0$ : $D^{l=0} = T_1$
- $l=1$ $D^{l=1} = T_4$
- $l=2$ : $D^{l=2} = T_3 \oplus T_5$ $\quad \chi_{l=2}(g) = \chi_{T_3}(g) + \chi_{T_5}(g)$ $\quad \forall g \in O$

check : $E$ : $\chi_{l=2}(E) = 5$

$$\chi_{T_3}(E) + \chi_{T_5}(E) = 2+3 = 5$$

$8C_3$ : $\chi_2(8C_3) = -1$

$$\chi_{T_3}(8C_3) + \chi_{T_5}(8C_3) = -1 + 0 = -1$$

$\underline{D_l = \bigoplus_T D_T}$ $\quad \forall g \in O$ , $D_l(g) = \bigoplus_T D_T(g)$

$$\begin{pmatrix} D_{T_1}(g) & & 0 \\ & D_{T_2}(g) & \\ 0 & & D_{T_3}(g) \\ & & & \ddots \end{pmatrix}$$

$$\chi_l(g) = \mathrm{tr}\, D_l(g) = \sum_T \mathrm{tr}\, D_T(g) = \sum_T \chi_T(g)$$

$6C_2$ $\chi_{l=2}(6C_2) = 1$

$\chi_{T_3}(6C_2) + \chi_{T_5}(6C_2) = 0 + 1 = 1$

$2C_2$ : $\chi_{l=2}(2C_2) = 1$

$\chi_{T_3}(2C_2) + \chi_{T_5}(2C_2) = 2 + (-1) = 1$

$6C_4$ : $\chi_{l=2}(6C_4) = -1$

$\chi_{T_3}(6C_4) + \chi_{T_5}(6C_4) = 0 + (-1) = -1$

- $l=3$ $D_3 = T_2 \oplus T_4 \oplus T_5$

$E$: $\chi_3 = 7$

$\chi_{T_2} + \chi_{T_4} + \chi_{T_5} = 1 + 3 + 3 = 7$

$8C_3$: $\chi_3 = 1$

$\chi_{T_2} + \chi_{T_4} + \chi_{T_5} = 1 + 0 + 0 = 1$

$6C_2$: $\chi_3 = -1$

$\chi_{T_2} + \chi_{T_4} + \chi_{T_5} = -1 + (-1) + 1 = -1$

$3C_2$: $\chi_3 = -1$

$\chi_{T_2} + \chi_{T_4} + \chi_{T_5} = 1 + (-1) + (-1) = -1$

$6C_4$: $\chi_3 = -1$

$\chi_{T_2} + \chi_{T_4} + \chi_{T_5} = -1 + 1 + (-1) = -1$

- $l=4$ $\quad D_4 = T_1 \oplus T_3 \oplus T_4 \oplus T_5$

$E$ : $\quad \chi_4 = 9$

$$\chi_{T_1} + \chi_{T_3} + \chi_{T_4} + \chi_{T_5} = 1 + 2 + 3 + 3 = 9$$

$8C_3$ : $\quad \chi_4 = 0$

$$\chi_{T_1} + \chi_{T_3} + \chi_{T_4} + \chi_{T_5} = 1 + (-1) + 0 + 0 = 0$$

$6C_2$ : $\quad \chi_4 = 1$

$$\chi_{T_1} + \chi_{T_3} + \chi_{T_4} + \chi_{T_5} = 1 + 0 + (-1) + 1 = 1$$

$3C_2$ : $\quad \chi_4 = 1$

$$\chi_{T_1} + \chi_{T_3} + \chi_{T_4} + \chi_{T_5} = 1 + 2 + (-1) + (-1)$$
$$= 1$$

$6C_4$ : $\quad \chi_4 = 1$

$$\chi_{T_1} + \chi_{T_3} + \chi_{T_4} + \chi_{T_5} = 1 + 0 + 1 + (-1) = 1$$

---

$$\vec{r} = \sum_{i=1}^{3} r_i \vec{e}_i$$ choose basis $\vec{e}_{i=1,2,3}$

$$\underline{\vec{r}' = Q\vec{r} \equiv \sum_{i=1}^{3} r_i' \vec{e}_i}$$

Rotation of basis $\quad \underline{\vec{r}' = Q\vec{r}} = \sum_{i=1}^{3} r_i Q \vec{e}_i = \underline{\sum_{i=1}^{3} r_i \vec{e}_i'}$

$$\vec{e}_i{}' = Q\vec{e}_i = \sum_j \vec{e}_j \left( \vec{e}_j \cdot \underbrace{Q\vec{e}_i}_{\vec{e}_i{}'} \right) = \sum_{j=1}^{3} \vec{e}_j \, Q_{ji}$$

$$\vec{r}\,' = \sum_{i=1}^{3} r_i{}' \, \underline{\underline{\vec{e}_i}}$$

$$\shortparallel$$

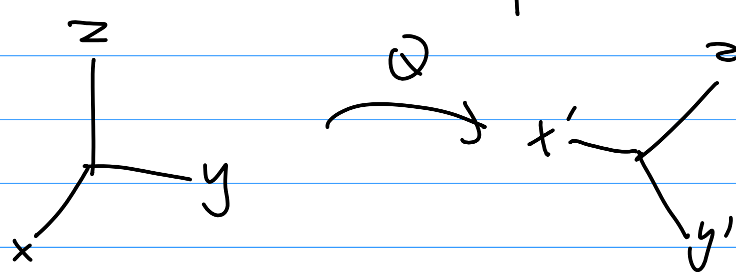
$$\sum_{i=1}^{3} r_i \sum_{j=1}^{3} \vec{e}_j \, Q_{ji} = \sum_{j=1}^{3} \left( \sum_{i=1}^{3} Q_{ji} \, r_i \right) \vec{e}_j$$

$$r_j{}' = \sum_{i=1}^{3} Q_{ji} \, r_i$$

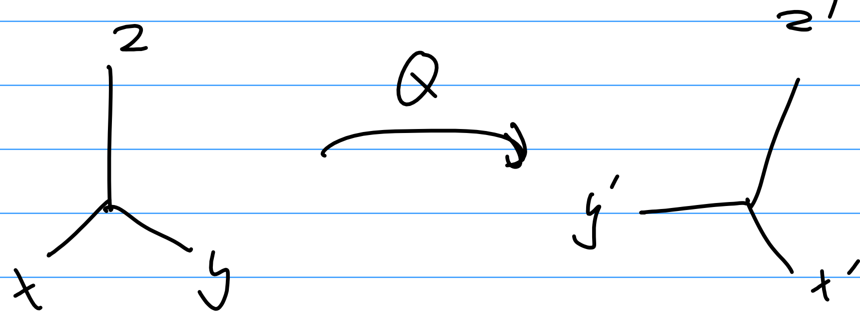
Parity $P : \vec{r} \to -\vec{r}$ $\quad P_{ij} = -\delta_{ij}$ $\quad \det P = -1$

$P \notin SO(3)$ $\quad P \in O(3)$

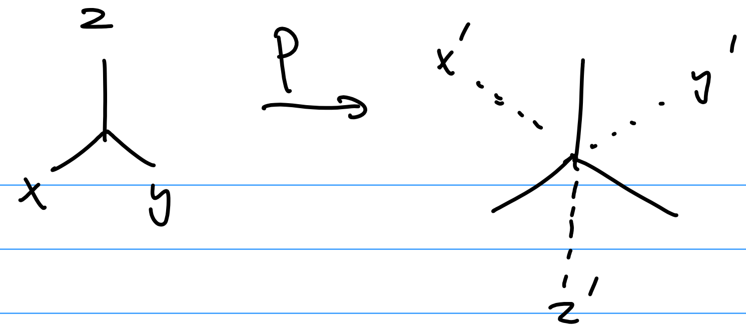
$Q \in SO(3)$ transf. right-hand frame to right-hand frame




$Q \notin SO(3)$, $Q \in O(3)$ transf. right-hand frame to left-hand frame



P:




Proof:

Let $\vec{z} = \vec{x} \times \vec{y}$ we shall show that

$$Q\vec{x} \times Q\vec{y} = Q\vec{z} \quad \forall\, Q \in SO(3)$$

$$Q\vec{x} \times Q\vec{y} = -Q\vec{z} \quad \forall\, Q \notin SO(3), \; Q \in O(3)$$

$\forall$ vector $\vec{u} \in \mathbb{R}^3$

$$Q\vec{u} \cdot \left(Q\vec{x} \times Q\vec{y}\right) = \sum_{ijk} \varepsilon_{ijk} (Q\vec{u})_i (Q\vec{x})_j (Q\vec{y})_k$$

$$= \sum_{\substack{ijk \\ \alpha\beta\gamma}} \varepsilon_{ijk}\, Q_{i\alpha} u_\alpha\, Q_{j\beta} x_\beta\, Q_{k\gamma} y_\gamma$$

$$= \sum_{\alpha\beta\gamma} \underbrace{\left( \sum_{ijk} \varepsilon_{ijk}\, Q_{i\alpha} Q_{j\beta} Q_{k\gamma} \right)}_{\varepsilon_{\alpha\beta\gamma} \det Q} u_\alpha x_\beta y_\gamma$$

$$= \det Q \sum_{\alpha\beta\gamma} \varepsilon_{\alpha\beta\gamma}\, u_\alpha x_\beta y_\gamma$$

$$= \det Q \left( \vec{u} \cdot (\vec{x} \times \vec{y}) \right)$$

$$= \det Q \left( \vec{u} \cdot \vec{z} \right) = \det Q \left( Q\vec{u} \cdot Q\vec{z} \right)$$

$$Q\vec{x} \times Q\vec{y} = \underbrace{\det Q}_{= \pm 1} \left( Q \vec{z} \right)$$

all rotations of rigid body are SO(3)

$\forall\, Q \in SO(3)$ can be composed by 2 types of simple rotations

$Q(\hat{k}, \alpha)$ , $Q(\hat{j}, \beta)$ $\quad$ $\hat{i}, \hat{j}, \hat{k}$ basis of x, y, z axis

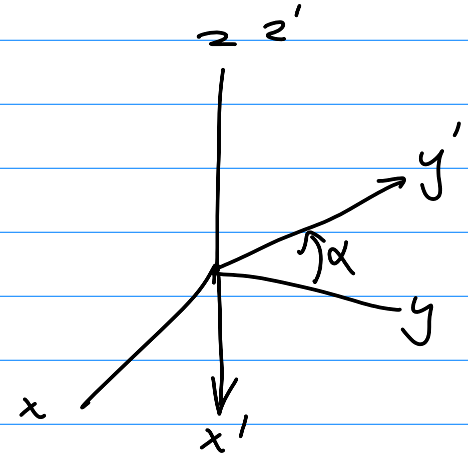
rotation around z-axis $\quad$ rotation around y-axis

$$Q(\hat{k}, \alpha) = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \vec{e}_i' = \sum_j \vec{e}_j Q_{ji}$$

$$\hat{i} \to \hat{i}' = \hat{i}\cos\alpha + \hat{j}\sin\alpha$$

$$\hat{j} \to \hat{j}' = \hat{i}(-\sin\alpha) + \hat{j}\cos\alpha$$

$$\hat{k} \to \hat{k}' = \hat{k}$$

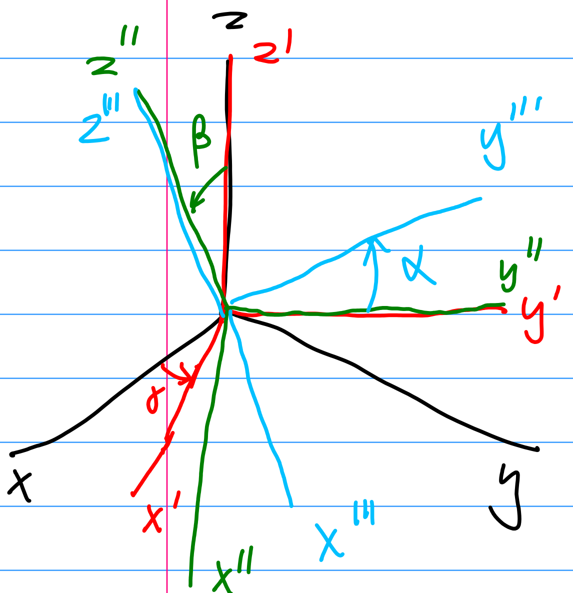



$$Q(\hat{j}, \beta) = \begin{pmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{pmatrix}$$

Euler angles: (1) rotation around z : $\gamma$
give a basis (2) rotation around y : $\beta$
$\hat{i}, \hat{j}, \hat{k}$ : (3) rotation around z : $\alpha$

compose any rotations

$$\alpha \in [0, 2\pi), \quad \beta \in [0, \pi], \quad \gamma \in [0, 2\pi)$$




$$Q(\alpha, \beta, \gamma)$$

$$= Q(\hat{k}, \alpha)\, Q(\hat{j}, \beta)\, Q(\hat{k}, \gamma)$$

$$\vec{e}_i' = \sum_j \vec{e}_j\, Q_{ji}(\alpha \beta \gamma)$$

$$Q(\alpha, \beta\, \gamma) = \begin{pmatrix} \cos\alpha\cos\beta\cos\gamma - \sin\alpha\sin\gamma & -\cos\alpha\cos\beta\sin\gamma - \sin\alpha\cos\gamma & - \\ \cdot & \cdot\cdot & \cdot \\ \cdot & \cdot\cdot & \cdot\cdot \end{pmatrix}$$

$\forall\, Q(\alpha, \beta, \gamma) \in SO(3)$

all group elements in SO(3) can be parametrized by 3 Euler angles

SO(3) is a 3-dim Lie group

# SO(3) and SU(2)

in QM, it's better to work with SU(2), reps of SU(2)

since 1) {irreps of SU(2)} ⊃ {irreps of SO(3)}

2) we have spin.

SU(2) group: special unitary transf. on $\mathbb{C}^2$

$$SU(2) \ni u = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix}, \quad \det(u) = \underline{aa^* + bb^*} = 1$$

$u^\dagger = u^{-1}$

$a, b \in \mathbb{C}$ 2 complex = 4 reals

1 constraint.

3 real parameters

$\Rightarrow$ SU(2) 3 dim Lie group

$aa^* + bb^* = 1$ we write $a = \cos\eta \, e^{i\xi}$

$b = -\sin\eta \, e^{-i\zeta}$

$$u = \begin{pmatrix} \cos\eta \, e^{-i\xi} & -\sin\eta \, e^{-i\zeta} \\ \sin\eta \, e^{i\zeta} & \cos\eta \, e^{i\xi} \end{pmatrix} = u(\underbrace{\eta, \xi, \zeta}_{\text{3 real parameters}})$$

$\xi, \zeta \in [0, 2\pi)$ $\quad \eta \in [0, \frac{\pi}{2}]$

homomorphism between SU(2) and SO(3)

$$\vec{r} \in \mathbb{R}^3 \quad \rightarrow \quad \vec{r}\cdot\vec{\sigma} = \begin{pmatrix} z & x-iy \\ x+iy & -z \end{pmatrix}$$

$$\vec{r} = (x, y, z), \qquad \vec{r}\cdot\vec{\sigma} \equiv h$$

$$\det(h) = x^2+y^2+z^2 = |\vec{r}|^2$$

$u \in SU(2)$, view $h$ an operator on $\mathbb{C}^2$

$$h \rightarrow \underline{h' = u h u^{-1} \equiv \vec{r}'\cdot\vec{\sigma}} = \begin{pmatrix} z' & x'-iy' \\ x'+iy' & -z' \end{pmatrix}$$

$$\underbrace{\det(h')}_{= |\vec{r}'|^2} = \det(u h u^{-1}) = \underbrace{\det(h)}_{= |\vec{r}|^2}$$

This action by $u$ from $\vec{r}$ to $\vec{r}'$ corresponds to a rotation $Q(u) \in SO(3)$, s.t. $Q(u)\vec{r} = \vec{r}'$

$$u(\vec{\sigma}\cdot\vec{r})u^{-1} = \vec{\sigma}\cdot(Q(u)\vec{r})$$

we find a map from $SU(2)$ to $SO(3)$

homomorphism: $\forall u_1, u_2 \in SU(2)$

$$(u_1u_2)(\vec{\sigma}\cdot\vec{r})(u_1u_2)^{-1} = u_1u_2(\vec{\sigma}\cdot\vec{r})u_2^{-1}u_1^{-1}$$

$$= u_1(\vec{\sigma}\cdot Q(u_2)\vec{r})u_1^{-1}$$

$$= \vec{\sigma}\cdot(Q(u_1)Q(u_2)\vec{r})$$

$\forall$ rep $D$ of $SO(3)$

$$SU(2) \xrightarrow[\text{hom}]{i} SO(3) \xrightarrow[\text{hom}]{D} L(\mathcal{H})$$

$D \circ i$ is a rep of $SU(2)$

$$u(\vec{\sigma}\cdot\vec{r})\,u^{-1} = \vec{\sigma}\cdot(Q(u)\,\vec{r})$$

$$u = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix}, \quad \underline{aa^* + bb^* = 1}$$

$\downarrow$ $u \to -u \in SU(2)$

$$(-u)(\vec{\sigma}\cdot\vec{r})(-u)^{-1}$$

$$-u = \begin{pmatrix} -a & -b \\ b^* & -a^* \end{pmatrix}, \quad \underline{aa^* + bb^* = 1}$$

$$\begin{array}{ccc} SU(2) & & SO(3) \\ u & \longmapsto & Q(u) \\ -u & \longmapsto & Q(u) \end{array}$$

2-to-1 homomorphism

"$SU(2)$ is double covering group of $SO(3)$"

Example (1) $u_1(\alpha) = \begin{pmatrix} e^{-i\alpha/2} & 0 \\ 0 & e^{i\alpha/2} \end{pmatrix}$

$$h = \vec{r}\cdot\vec{\sigma} = \begin{pmatrix} z & x-iy \\ x+iy & -z \end{pmatrix}$$

$$h' = u_1(\alpha)\, h\, u_1(\alpha)^{-1} = \begin{pmatrix} z & (x-iy)e^{-i\alpha} \\ (x+iy)e^{i\alpha} & -z \end{pmatrix}$$

$$\parallel$$

$$\vec{\sigma}\cdot\vec{r}\,' = \begin{pmatrix} z' & x'-iy' \\ x'+iy' & -z' \end{pmatrix} \Rightarrow \begin{cases} x' = x\cos\alpha - y\sin\alpha \\ y' = x\sin\alpha + y\cos\alpha \\ z' = z \end{cases}$$

$$\vec{r}\,' = \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \underbrace{\begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{Q(u_1(\alpha)) \;=\; Q(\hat{k},\alpha)} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$\pm u_1(\alpha) \longmapsto Q(\hat{k}, \alpha)$ rotation around $z$-axis

(2) $u_2(\beta) = \begin{pmatrix} \cos \beta/2 & -\sin\beta/2 \\ \sin\beta/2 & \cos\beta/2 \end{pmatrix} \in SU(2)$

HW show $\pm u_2(\beta) \longmapsto Q(\hat{j}, \beta)$ rotation around $y$-axis

$$u(\alpha, \beta, \gamma) = u_1(\alpha)\, u_2(\beta)\, u_1(\gamma)$$

$$\downarrow$$

$$Q(\alpha, \beta, \gamma) = Q(\hat{k}, \alpha)\, Q(\hat{j}, \beta)\, Q(\hat{k}, \gamma)$$

$\alpha, \beta, \gamma$ are Euler angles

$$u(\alpha,\beta,\gamma) = \begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}} \cos\frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}} \sin\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}} \sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}} \cos\frac{\beta}{2} \end{pmatrix}$$

compare with 3.4.3

$$u = \begin{pmatrix} \cos\eta \, e^{-i\xi} & -\sin\eta \, e^{-i\zeta} \\ \sin\eta \, e^{i\zeta} & \cos\eta \, e^{i\xi} \end{pmatrix}$$

$\alpha = \xi + \zeta$ , $\beta = 2\eta$ , $\gamma = \xi - \zeta$

$\xi, \zeta \in [0, 2\pi)$ , $\eta \in [0, \frac{\pi}{2}]$

$\Rightarrow$ $\alpha \in [0, 4\pi)$, $\beta \in [0, \pi]$ , $\gamma \in [0, 2\pi)$

relate to 2-to-1 homomorphism $SU(2) \to SO(3)$

$$\big( Vol(SU(2)) = 2 \, Vol(SO(3)) \big)$$

<u>Reps of SU(2)</u>

fundamental rep of $SU(2)$ (defining rep) on $\mathbb{C}^2$

(spin Hilbert space)

$$v = \begin{pmatrix} \xi \\ \eta \end{pmatrix} \in \mathbb{C}^2$$

$$v' = \begin{pmatrix} \xi' \\ \eta' \end{pmatrix} = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} a\xi + b\eta \\ -b^*\xi + a\eta \end{pmatrix}$$

$$\xi \to \xi' = a\xi + b\eta$$

$$\eta \to \eta' = -b^*\xi + a^*\eta$$

homogeneous polynomials of $\xi, \eta$

basis

degree - 0 $\quad 1 \iff \xi^0 \eta^0$

degree - 1 $\quad \xi, \eta \iff \xi^{\frac{1}{2}+\frac{1}{2}} \eta^{\frac{1}{2}-\frac{1}{2}}, \ \xi^{\frac{1}{2}-\frac{1}{2}} \eta^{\frac{1}{2}+\frac{1}{2}}$

degree - 2 $\quad \xi^2, \xi\eta, \eta^2 \iff \underline{\xi^{1+1}\eta^{1-1}, \ \xi^{1+0}\eta^{1+0}, \ \xi^{1-1}\eta^{1+1}}$

$\vdots$

degree - $2j$ $\quad \xi^{j+j}\eta^{j-j}, \ \xi^{j+(j-1)}\eta^{j-(j-1)} \cdots$

$\xi^{j+m}\eta^{j-m} \cdots \xi^{j-j}\eta^{j+j}$

$$m = j, j-1, j-2, \ldots -j$$

$$\underline{\xi^{j+j}\eta^{j-j}} \sim \underline{|j, j\rangle}$$

$$\vdots \qquad \vdots$$

$$\xi^{j+m}\eta^{j-m} \sim |j, m\rangle$$

$$\vdots \qquad \vdots$$

$$\xi^{j-j}\eta^{j+j} \sim |j, -j\rangle$$

any degree - $2j$ homogeneous polynomial $\displaystyle\sum_{m=-j}^{j} a_m \underline{\underline{\xi^{j+m}\eta^{j-m}}}$

$$\sim \sum_{m=-j}^{j} a_m |j, m\rangle$$

span 2j-dim vector space $\mathcal{H}_j$, rep of SU(2)

SU(2) action: $\sum_{m=-j}^{j} a_m \xi^{j+m} \eta^{j-m}$

$$\to \sum_{m=-j}^{j} a_m \underline{(a\xi + b\eta)}^{j+m} \underline{(-b^*\xi + a^*\eta)}^{j-m}$$

still homogeneous polynomial of 2j

basis vector $f^j_m(\xi,\eta) = \dfrac{\xi^{j-m}\eta^{j+m}}{\sqrt{(j-m)!\,(j+m)!}}$ ←

$$m = -j, -j+1 \ldots j$$

$\{f^j_m\}_{m=-j}^{j}$ span $\mathcal{H}_j$ irrep. of SU(2)

$\forall u \in SU(2)$ $\quad D_j(u)\, f^j_m(\xi,\eta) := f^j_m\left(u^{-1}\begin{pmatrix}\xi\\ \eta\end{pmatrix}\right)$

$$u^{-1}\begin{pmatrix}\xi\\ \eta\end{pmatrix} = u^{+}\begin{pmatrix}\xi\\ \eta\end{pmatrix} = \begin{pmatrix}a^* & -b\\ b^* & a\end{pmatrix}\begin{pmatrix}\xi\\ \eta\end{pmatrix} = \begin{pmatrix}a^*\xi - b\eta\\ b^*\xi + a\eta\end{pmatrix}$$

$$D_j(u)\, f^j_m = \frac{(a^*\xi - b\eta)^{j-m}(b^*\xi + a\eta)^{j+m}}{\sqrt{(j-m)!\,(j+m)!}}$$

$$= \sum_{m'=-j}^{j} f^j_{m'}\, \underset{\substack{\uparrow\\ \text{rep. matrix}}}{D^j_{m'm}(u)}$$

$(2j+1)\times(2j+1)$

rep. matrix Wigner D-matrix

derive $D^j_{mm'}(u)$

$$(x+y)^n = \sum_{r=0}^{n} \frac{n!}{r!\,(n-r)!} x^r y^{n-r}$$

$$D_j(u) f^j_m = \sum_{n=0}^{j-m} \sum_{m'=n-j}^{n+m} (-1)^n \frac{\sqrt{(j-m)!\,(j+m)!\,(j-m')!\,(j+m')!}}{n!\,(j-m-n)!\,(n+m-m')!\,(j+m'-m)!}$$

$$(a^*)^{j-m-n} b^n (b^*)^{n+m-m'} a^{j+m'-m} f^j_{m'}$$

$$\sum_{n=0}^{j-m} \sum_{m'=n-j}^{n+m} (\ldots) = \sum_{m'=-j}^{j} \sum_{n} (\ldots)$$

over all integers s.t. for all $(\ldots)!$

non negative.

$$D^j_{m'm}(a,b) = \sum_n (-1)^n \frac{\sqrt{(j-m)!\,(j+m)!\,(j-m')!\,(j+m')!}}{(j+m'-n)!\,(j-m-n)!\,n!\,(n+m-m')!}$$

$$(a^*)^{j-m-n} a^{j+m'-n} b^n (b^*)^{n+m-m'}$$

$$j = 0, \tfrac{1}{2}, 1 \ldots$$

$$m, m' = -j \ldots j$$

<u>properties of D-matrix</u>

1) Unitary : <u>Lemma</u> : $D(u) D(u)^\dagger = 1 \quad \forall u \in SU(2)$

i.e. $\underline{\sum_m D^j_{m_1 m}(u) D^j_{m_2 m}(u)^* = \delta_{m_1 m_2}}$

Pf: let's look at $\sum_m f^j_m(\xi,\eta)^* f^j_m(\xi,\eta)$

$$= \sum_{m=-j}^{j} \frac{(\xi^*\xi)^{j-m} (\eta^*\eta)^{j+m}}{(j-m)!\,(j+m)!}$$

$$= \frac{(\xi^*\xi + \eta^*\eta)^{2j}}{(2j)!} \leftarrow SU(2) \text{ inv.}$$

$\forall\, u \in SU(2)$

$$u\begin{pmatrix}\xi\\ \eta\end{pmatrix} = \begin{pmatrix}\xi'\\ \eta'\end{pmatrix} \qquad (\xi',\eta')^*\begin{pmatrix}\xi'\\ \eta'\end{pmatrix} = \xi'^*\xi' + \eta'^*\eta'$$

$$= (\xi,\eta)\, u^\dagger u \begin{pmatrix}\xi\\ \eta\end{pmatrix}$$

$$= (\xi\ \eta)\begin{pmatrix}\xi\\ \eta\end{pmatrix} = \xi^*\xi + \eta^*\eta$$

$$\sum_m \left(D_j(u) f^j_m\right)^* \left(D_j(u) f^j_m\right) = \underline{\underline{\sum_m f^{j*}_m f^j_m}}$$

$$\parallel$$

$$\sum_{m'm''} \underline{\underline{f^{j*}_{m''} f^j_{m'}}} \left(\sum_m D^j_{m''m}(u)^* D^j_{m'm}(u)\right)$$

---

if $f^{j*}_{m''} f^j_{m'} \equiv b^j_{m''m'}$ are linear indep

then $\underbrace{\sum_m D^j_{m'm}(u)\, D^j_{m''m}(u)^* = \delta_{m''m'}}_{DD^\dagger = 1}$

To prove they are linearly indep.

solve eqn $\sum_{m'',m'} C_{m''m'} f^{j*}_{m''} f^{j}_{m'} = 0$

show $C_{m''m'} = 0 \quad \forall m', m'' = -j \cdots j$

HW $\longrightarrow$

$\square$

Lemma $D_j$ is irreducible.

Pf. Recall Schur's lemma; There is no matrix commuting with all $D_j(u)$ except $\lambda \mathbb{1}$, then $D_j$ is irrep

assume $\exists M$ s.t. $M D^j(a,b) = D^j(a,b) M \quad \forall u = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix}$

choose $a = e^{i\alpha/2}$, $b = 0$, $\alpha \in [0, 4\pi]$, $D^j_{m'm}(e^{i\alpha/2}, 0)$

$= C \, \delta_{m'm} e^{im\alpha}$

$\uparrow$ const.

$$\sum_{m'} M_{mm'} D^j_{m'm''}(e^{i\alpha/2}, 0)$$

$$= \sum_{m'} D^j_{mm'}(e^{i\alpha/2}, 0) M_{m'm''}$$

$\Leftrightarrow M_{mm''} e^{im''\alpha} = M_{mm''} e^{im\alpha} \Leftrightarrow M_{mm''}(e^{im''\alpha} - e^{im\alpha}) = 0$

$\Rightarrow M_{mm''} = 0$ if $m \neq m''$ i.e. $M_{mm''} = M_m \delta_{mm''}$

for other $u$ s.t. $D^j$ has nonzero off-diagonals

$MD^j = D^j M \Leftrightarrow M_m D_{mm''} = D_{mm''} M_{m''}$

$\Rightarrow M_m = M_{m''} \Rightarrow M \propto \mathbb{1}$

$\Rightarrow D^j$ is irreducible $\square$

we have classified all irreps of $SU(2)$

## D-matrix in terms of Euler angles

$$u(\alpha,\beta,\gamma) = \begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \end{pmatrix}$$

$$a = e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \qquad b = -e^{-i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2}$$

$$D^j_{mm'}(\alpha,\beta,\gamma) = D^j_{mm'}\left(a = e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2},\ b = -e^{-i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2}\right)$$

$$\alpha\in[0,4\pi),\quad \beta\in[0,\pi],\quad \gamma\in[0,2\pi)$$

$$\alpha\to\alpha+2\pi \quad\Rightarrow\quad a\to -a \qquad b\to -b$$

$$u = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix} \longrightarrow u(-a,-b) = -u(a,b)$$

$$u(\vec{\sigma}\cdot\vec{r})\,u^{-1} = \vec{\sigma}\cdot Q(u)\,\vec{r} \qquad \begin{matrix} u\in SU(2) \\ Q(u)\in SO(3) \end{matrix}$$

$u(a,b)$ $\quad$ $u(-a,-b)$

the same element in $SO(3)$

$$D^j_{mm'}(-a,-b) = (-1)^{2j} D^j_{mm'}(a,b)$$

if $j$ is integer

$$\begin{tikzcd}
 & u(\alpha,\beta,\gamma) \arrow[dr] & \\
\begin{matrix} SO(3)\ \text{rotation} \\ Q(\alpha,\beta,\gamma) \\ \alpha\in[0,2\pi) \end{matrix} \arrow[ur] \arrow[dr] & & D^j_{mm'}(\alpha,\beta,\gamma) \\
 & \begin{matrix} -u(\alpha,\beta,\gamma) \\ \parallel \\ u(\alpha+2\pi,\beta,\gamma) \end{matrix} \arrow[ur] &
\end{tikzcd}$$

1-to-1 map between $Q(\alpha,\beta,\gamma)$ and $D^j(\alpha,\beta,\gamma)$

if $j$ is half-integer

SO(3) rotation $Q(\alpha,\beta,\gamma)$, $\alpha \in [0, 2\pi)$

$$\begin{cases} u(\alpha,\beta,\gamma) \longrightarrow D^{j}_{mm'}(\alpha,\beta,\gamma) \\ u(\alpha+4\pi,\beta,\gamma) \longrightarrow D^{j}_{mm'}(\alpha+4\pi,\beta,\gamma) \end{cases}$$

not the same

1-to-2 map between $Q(\alpha,\beta,\gamma)$ and $D^{j}(\alpha,\beta,\gamma)$

double valued rep of SO(3)

HW compute $D^{j=1/2}(\alpha,\beta,\gamma)$ and $D^{j=1}(\alpha,\beta,\gamma)$

$\forall\, u_{2\times 2} \in SU(2)$

$$u = u_0 \mathbb{1} + \sum_{i=1}^{3} u_i \sigma_i = \begin{pmatrix} u_0 + u_3 & u_1 - iu_2 \\ u_1 + iu_2 & u_0 - u_3 \end{pmatrix}$$

$$\det(u) = u_0^2 - \sum_{i=1}^{3} u_i^2 = 1 \qquad u^{\dagger} = u^{-1}$$

$\Rightarrow u_0 \in \mathbb{R}$, $u_{i=1,2,3} \in i\mathbb{R}$

$$= u_0^2 + \sum_{i=1}^{3} v_i^2 = 1 \qquad u_{i=1,2,3} = i\, v_{i=1,2,3}$$

$$\begin{pmatrix} u_0 \\ v_1 \\ v_2 \\ v_3 \end{pmatrix} \qquad u_0 = \cos\theta/2$$

$$\vec{u} = (u_1\ u_2\ u_3) = i\,\vec{n}\sin\theta/2 \qquad \vec{n}\cdot\vec{n} = 1$$

$$u = \cos\frac{\theta}{2}\,\mathbb{1} + i\,\vec{n}\cdot\vec{\sigma}\,\sin\frac{\theta}{2} = \underline{\underline{e^{i\theta\vec{n}\cdot\vec{\sigma}/2}}}$$

$\Rightarrow \forall\, u \in SU(2)$ $\exists$ an angle $\theta \in [0, 4\pi)$, and unit vector $\vec{n}$ (axis)

s.t. $u = e^{i\theta\vec{n}\cdot\vec{\sigma}/2}$