boson no spin $\quad T_0^2 = 1 \quad\quad T_0 \psi(t,\vec{r}) = \psi^*(-t,\vec{r})$

fermion spin $\frac{1}{2}$ $\quad T^2 = -1 \quad\quad T\begin{pmatrix}\psi_1(t,\vec{r})\\ \psi_2(t,\vec{r})\end{pmatrix} = e^{i\varphi}\sigma_y\begin{pmatrix}\psi_1^*(-t)\\ \psi_2^*(-t)\end{pmatrix}$

zero spin $\quad \hat{H}\psi_{nl} = E_n \psi_{nl} \quad l = 1..d(n)$

general sol of Schrödingen eqn $\quad \psi(\vec{r},t) = \sum_n a_n \psi_{nl}(\vec{r}) e^{-\frac{i}{\hbar}E_n t}$

( $\hat{H}$ is real, not explicitly dep. on t ) $\quad \hat{T}_0 \psi(\vec{r},t) = \sum_n a_n^* \psi_{nl}^*(\vec{r}) e^{-\frac{i}{\hbar}E_n t}$

effectively $\quad \hat{T}_0 : \psi_{nl}(\vec{r}) \to \psi_{nl}^*(\vec{r})$

if $\hat{H}$ is real, $\quad \hat{H}\psi_{nl} = E_n\psi_{nl} \quad \psi_{nl}$ is eigenstate

$\Rightarrow \hat{H}\psi_{nl}^* = E_n\psi_{nl}^* \quad \psi_{nl}^*$ is eigenstate

$\psi_{nl}$ basis in $\mathcal{H}^{(n)}$, $\psi_{nl}^*$ is basis in $\mathcal{H}^{(n)*}$

whether $\mathcal{H}^{(n)}$ and $\mathcal{H}^{(n)*}$ the same ?

(1) $\quad \mathcal{H}^{(n)} \simeq \mathcal{H}^{(n)*} \quad$ equivalent rep of symm. group

(2) $\quad \mathcal{H}^{(n)} \not\simeq \mathcal{H}^{(n)*} \quad$ not equivalent ....

$$\text{eigenspace} = \mathcal{H}^{(n)} \oplus \mathcal{H}^{(n)*}$$

<u>real rep, pseudo real rep, and complex rep</u>

Given a symm. group $G$, unitary irrep $D: g \mapsto D(g) \circlearrowright \mathcal{H}^{(n)}$

**Def** (complex conjugate irrep) $D(g)$ is unitary on $\mathcal{H}^{(n)}$, $\psi_{ni}$ orthonormal basis in $\mathcal{H}^{(n)}$

$$D(g)\,\psi_{ni}(\vec{r}) = \sum_j \psi_{nj}(\vec{r})\, D_{ji}(g)$$

$D_{ji}(g)$ unitary matrix

complex conjugate : $D(g)^* \psi_{ni}^*(\vec{r}) = \sum_j \psi_{nj}(\vec{r})^* D_{ji}(g)^*$

$$\psi_{ni}^*(\vec{r}) = T_0\, \psi_{ni}(\vec{r})$$

$D_{ij}(g)$ unitary irrep matrix $\xrightarrow{T_0}$ $D_{ij}^*(g)$ unitary irrep matrix

( irrep $D$) irrep $D^*$ : Complex conjugate irrep

relation between $D$ and $D^*$

(1) if $\exists$ unitary transf. $U : \mathcal{H}^{(n)} \to \mathcal{H}^{(n)}$ s.t

$$U\, D(g)\, U^{-1} = D_0(g) \qquad \forall g \in G$$

$\uparrow$ real matrix

$\Rightarrow$ $D$ is equivalent to $D^*$, since $U^* D(g)^* U^{*-1} = D_0(g)$

$\Rightarrow$ $D^*(g) = U^{*-1} D_0(g)\, U^* = U^{*-1} U\, D(g)\, U^{-1} U^*$

$U^{*-1} U$ is unitary

we say $D \simeq D^*$ is a real rep.

(2) $D$ is unitarily equivalent to $D^*$, but they are not equivalent to real rep.

$$\nexists\, U \text{ s.t. } D(g) = U\, \underset{\substack{\uparrow \\ \text{real matrix}}}{D_0(g)}\, U^{-1} \quad \forall g \in G$$

We say, $D \simeq D^*$ is pseudo-real

(3) $D$ is inequivalent to $D^*$.

We say, $D$ and $D^*$ are complex reps

distinguish (3) from (1) and (2), check character

if $D \simeq D^*$, then $\chi(g) = \chi^*(g) \quad \forall g \in G$

if $\chi(g) \neq \chi^*(g)$, then $D$ & $D^*$ are complex reps.

distinguish between (1) & (2)

$$D^* \simeq D \rightarrow D^*(g) = \underset{\substack{\uparrow \\ \text{unitary matrix}}}{Z}\, D(g) Z^{-1} \quad \forall g \in G$$

Complex conjugate $D(g) = Z^* D^*(g) Z^{*-1}$

$$\Rightarrow D^*(g) = Z Z^* D^*(g) (Z Z^*)^{-1}$$

i.e. $[Z Z^*, D^*(g)] = 0 \quad \forall g \in G$

by Schur's lemma, when $D^*$ is irrep, $ZZ^* = c\,\mathbb{1}$
$c \in \mathbb{C}$

$Z$ is unitary, $Z^* = (Z^T)^{-1}$

$$\Rightarrow \quad Z(Z^T)^{-1} = c\,\mathbb{1} \Rightarrow \underline{Z = c Z^T}$$

$$\xrightarrow[\text{transpose}]{} Z^T = cZ$$

$$\Rightarrow \quad Z = c^2 Z \Rightarrow c^2 = 1, \quad c = \pm 1$$

$$D^* \simeq D \Rightarrow ZZ^* = \pm\mathbb{1}$$

<u>Thm</u>: $ZZ^* = \mathbb{1}$ iff $D$ is real

$ZZ^* = -\mathbb{1}$ iff $D$ is pseudo-real

<u>Pf</u>