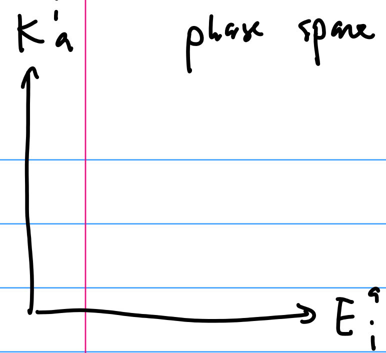
Extended ADM phase space



$$E^a_i = \sqrt{\det q}\, e^a_i$$

triad

$$\underline{\{ E^a_i(x),\ K^j_b(y) \} = \frac{\kappa}{2} \delta^j_i \delta^a_b \delta^{(3)}(x,y)}$$

$$\{E, E\} = \{K, K\} = 0$$

$$\boxed{\begin{aligned} E^a_i &\to M^i{}_j E^a_j \\ K^i_b &\to M^i{}_j K^j_b \end{aligned}}$$

$M \in SO(3)$

$$(E^a_i \ \ K^i_b) \longrightarrow (\underline{q_{ab} \ \ P^{ab}})$$

K_{kl}

Contraint (Gauss constraint), $\underline{G_j(x) = \varepsilon_{jkl} K^k_a E^a_l = \sqrt{\det q}\, \varepsilon_{jkl} K^k_a e^a_l = 0}$

Smeared Constraint $\underline{G(\Lambda)} = \int d^3x\, G_j(x) \Lambda^j(x)$

HW $\delta_\Lambda E^a_i(x) := \{ E^a_i(x), G(\Lambda) \}$ show δ_Λ is the infinitesimal gauge transf.

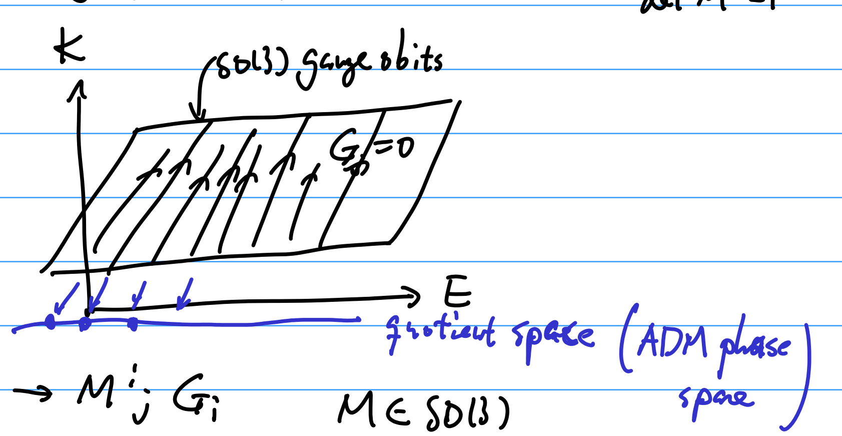
$\delta_\Lambda K^i_a(x) := \{ K^i_a(x), G(\Lambda) \}$

$$\begin{cases} E^a_i \to M^i{}_j E^a_j \\ K^i_a \to M^i{}_j K^j_a \end{cases} \quad M \in SO(3)$$

$$\begin{cases} \delta_{jk} M^j{}_i M^k{}_l = \delta_{il} \\ \det M = 1 \end{cases}$$

i.e. G_j generates $SO(3)$ gauge transf.

Relation to ADM phase space



SO(3) gauge transf. $G_j \to M^i{}_j G_i$ $\quad M \in SO(3)$

$G_j = 0 \to 0$ leave constraint surface inv.

Ashtekar New Variable :

coordinate transf. in phase space

Ashtekar – Barbero connection

(gauge potential with gauge group SU(2))

$$\left(E^a_i,\ \underline{\underline{K^i_a}}\right) \longrightarrow \left(\frac{E^a_i}{\beta},\ A^i_a\right)$$

It is a canonical transformation

$$A^i_a = \underline{\underline{\Gamma^i_a}} + \beta\, \underline{\underline{K^i_a}}$$

i.e. $\frac{E^a_i}{\beta}$ A^i_a are canonical conjugate variables

real Barbero – Immirzi parameter

Thm $\underline{\{E^a_i(x),\ A^j_b(y)\}} = \frac{\kappa\beta}{2}\delta^i_j\,\delta^a_b\,\delta^{(3)}(x,y)$

$\Gamma^i_a = \underline{\underline{\Gamma^i_a(E)}}$

$\{E, E\} = \{A, A\} = 0$

so, we can write poisson bracket equivallently as

$$\{f, f'\} = \frac{\kappa\beta}{2}\int\left(\frac{\delta f}{\delta E^a_i(x)}\frac{\delta f'}{\delta A^i_a(x)} - \frac{\delta f'}{\delta E^a_i(x)}\frac{\delta f}{\delta A^i_a(x)}\right) d^3x$$

what is Γ^i_a : spin connection 1-form

satisfying $D_a e^j_b \equiv \underline{\underline{\partial_a e^j_b - \Gamma^c{}_{ab}\, e^j_c} + \varepsilon_{jkl}\Gamma^k_a\, e^l_b} = 0$

covariant derivative w.r.t both indices a and j

$$\Rightarrow \underline{\varepsilon_{jkl}\Gamma^k_a = -e^b_l\left(\partial_a e^j_b - \Gamma^c{}_{ab}\, e^j_c\right)}$$

see Thiemann's book Eq (4.2.18) for $\Gamma^i_a(E)$

HW compute $\{E^a_i(x), A^j_b(y)\}$, $\{A^j_a(x)\ A^k_b(y)\}$ using poisson bracket in terms of E^a_i, K^j_b

Constraint in terms of new variables

Gauss constraint $0 \approx G_j = 0 + \beta\,\varepsilon_{jkl}K^k_a E^a_l$

$D_a E^a_j = 0$

$$= \underbrace{\partial_a E^a_j + \varepsilon_{jkl}\Gamma^k_a E^a_l}_{0} + \beta\,\varepsilon_{jkl} K^k_a E^a_l$$

HW check this

(careful: E^a_j is a density of weight 1

$$G_j = \underline{\partial_a E^a_j + \varepsilon_{jkl} A^k_a E^a_l}$$

$$\equiv \mathcal{D}_a E^a_j$$

analog of $\vec{\nabla}\cdot\vec{E} = 0$ in EM

Diffeo. constraint $\quad C_a = \frac{2}{\kappa\beta} F^i_{ab} E^b_j - \frac{2}{\kappa} K^i_a G_j \approx \frac{2}{\kappa\beta} F^i_{ab} E^b_j$

curvature 2-form

$$F^j_{ab} := \partial_a A^j_b - \partial_b A^j_a + \varepsilon_{jkl} A^k_a A^l_b$$

$$F_{ab} = \partial_a A_b - \partial_b A_a + \underline{\underline{[A_a, A_b]}}$$

(field strength, by viewing A^j_b is gauge potential in adj. rep. of $SU(2)$ or $SO(3)$)

Hamiltonian constraint:

"$-$" for Euclidean theory

$$C = \underbrace{\frac{1}{\kappa} F^j_{ab} \frac{\varepsilon_{jkl} E^a_k E^b_l}{\sqrt{\det q}}}_{\text{Euclidean term}} \underbrace{- \frac{1}{\kappa}(\beta^2+1)\,\varepsilon_{jmn} K^m_a K^n_b \times \frac{\varepsilon_{jkl} E^a_k E^b_l}{\sqrt{\det q}}}_{\text{Lorentz term}}$$

$+$ terms proportional to G_j

$$K^m_a = \frac{A^m_a - \Gamma^m_a}{\beta}$$

HW derive C_a, C in terms of A, E

Ref: Thiemann's book

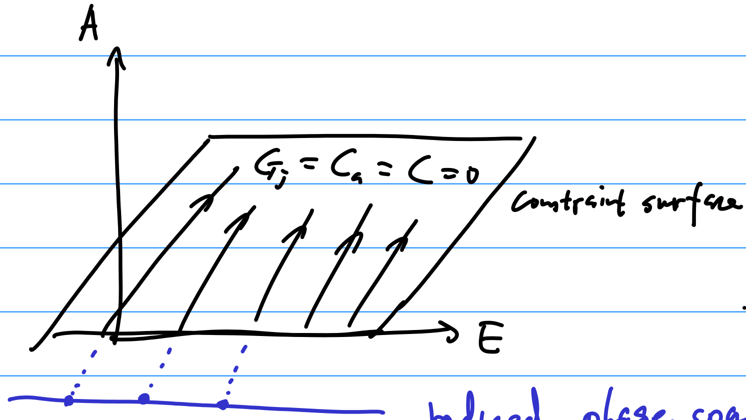
Ashtekar–Barbero Hamiltonian $H = \int d^3x \, \underline{[\Lambda^i G_i + N^a C_a + N C]}$ + boundary terms

EOM : $\dot{f}(A,E) = \{ f(A,E), H \}$

Holst action

$\downarrow$ 3+1 decomposition

Ashtekar-Barbero. H.



HW count. the dim of reduced phase space

Gauge transformation :

① Gauss constraint $\quad G(\Lambda) = \int d^3x \, \underline{\Lambda}^i G_i \qquad G_i = \mathcal{D}_a E^a_i$

$$\begin{cases} \delta_\Lambda A^i_a \equiv \{ A^i_a(x), G(\Lambda) \} = \kappa\beta \mathcal{D}_a \Lambda^i = \kappa\beta \left(\partial_a \Lambda^i + \varepsilon_{ijk} A^j_a \Lambda^k \right) \\ \delta_\Lambda E^a_i \equiv \{ E^a_i(x), G(\Lambda) \} = -\kappa\beta \, \underline{\varepsilon_{ijk} \Lambda^j E^a_k} \end{cases}$$

HW check them, and show $(\delta_\Lambda A, \delta_\Lambda E)$ is the infinitesimal transf.

of $\underline{(A_a, E^a) \to \left(g A_a g^{-1} + g \partial_a g^{-1}, \; g E^a g^{-1} \right)}$

$A_a = A^j_a \frac{\tau_j}{2}$, $E^a = E^a_j \frac{\tau_j}{2}$ $\qquad \frac{\tau_{j=1,2,3}}{2}$ su(2) Lie alg. generator.

$\tau_j = -i \sigma_j \qquad \sigma_{j=1,2,3}$ is Pauli matrix

$A_a = A_a(x)$, $g = g(x)$ SU(2) valued function

(local SU(2) gauge transf.)

$\vec{A} \to \vec{A} + \vec{\nabla} \chi(x)$

② Diff. constraint $\quad \delta_{\vec{N}} A^i_a(x) \equiv \{ A^i_a(x), C(\vec{N}) \} = \mathcal{L}_{\vec{N}} A^i_a$

$\delta_{\vec{N}} E^a_i(x) \equiv \{ E^a_i(x), C(\vec{N}) \} = \mathcal{L}_{\vec{N}} E^a_i$

use EOM

③ Ham. constraint $\quad \delta_N A^i_a(x) \equiv \{ A^i_a(x), C(N) \} = \mathcal{L}_{N n^a} A^i_a$

$\delta_N E^a_i(x) \equiv \{ E^a_i(x), C(N) \} = \mathcal{L}_{N n^a} E^a_i$

They are all 1st class

$$\{G(\Lambda), G(\Lambda')\} = G([\Lambda, \Lambda'])$$ $\Lambda^j \frac{\tau_j}{2}$

$$\{G(\Lambda), C(\vec{N})\} = G(\mathcal{L}_{\vec{N}} \Lambda)$$

$$\{G(\Lambda), C(N)\} = 0$$

$$\{C(\vec{N}), C(\vec{N}')\} = C(\mathcal{L}_{\vec{N}} \vec{N}')$$

$$\{C(\vec{N}), \underline{C(N')}\} = -C(\mathcal{L}_{\vec{N}} N')$$

$$\{C(N), C(N')\} = C\left((N\partial_a N' - N'\partial_b N)\, q^{ab}\right)$$

+ terms propotional to Gauss constraint

In terms of Ashtekar variable, GR is reformulated as a SU(2) <u>gauge theory with gauge potential A_a</u>

differences from usual nonabelian gauge theories

① additional constraints : Diffeo. & Hamil. constraints

② vanishing bulk Hamiltonian

③ background independent.

quantizing GR using method of nonabelian gauge theory

$\longrightarrow$ Loop Quantum Gravity (LQG)