Renormalization group (RG)

$m, \lambda, \delta_\lambda, \delta_m, \delta_Z$ depend on M , $\mathcal{L}_{Ren} = \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 + \frac{\lambda\mu^{4-d}}{4!}\phi^4$

$$\beta(M) = M\frac{\partial}{\partial M}\lambda \qquad \lambda = \lambda_0 Z^2 - \delta_\lambda$$

$$= -M\frac{\partial}{\partial M}\delta_\lambda \qquad \text{1-loop: } Z = 1+\delta_Z = 1$$

massless $m^2 = 0$

$$\delta_\lambda = \frac{3\lambda^2}{16\pi^2\varepsilon} - \frac{\lambda^2}{32\pi^2}\left[3\gamma + 3F(-M^2, 0, \mu)\right]$$

note: $-\frac{\partial\lambda}{\partial M} = \frac{\partial}{\partial M}\delta_\lambda$

$$F = \int_0^1 dz \ln\left[\frac{-M^2 z(1-z)}{4\pi\mu^2}\right]$$

$= \left[\frac{3\lambda}{8\pi^2\varepsilon} - \frac{3\lambda}{16\pi^2}(\gamma+F)\right]\frac{\partial\lambda}{\partial M} - \frac{3\lambda^2}{16\pi^2 M}$

$$\frac{\partial}{\partial M}\delta_\lambda = -\frac{3\lambda^2}{32\pi^2}\int_0^1 dz \frac{4\pi\mu^2}{-M^2 z(1-z)}\cdot\frac{(-2M)z(1-z)}{4\pi\mu^2}$$

$$= -\frac{3\lambda^2}{32\pi^2}\cdot\frac{2}{M} = -\frac{3\lambda^2}{16\pi^2 M}$$

$\Rightarrow -\left[1 + \frac{3\lambda}{8\pi^2\varepsilon} - \frac{3\lambda}{16\pi^2}(\gamma+F)\right]\frac{\partial\lambda}{\partial M} = -\frac{3\lambda^2}{16\pi^2 M}$

$\Rightarrow M\frac{\partial\lambda}{\partial M} = \frac{3\lambda^2}{16\pi^2}(1+O(\lambda))$

$$0 < \beta(M) = \frac{3\lambda^2}{16\pi^2} = M\frac{\partial}{\partial M}\lambda \rightarrow M\nearrow \ \lambda\nearrow$$

$\uparrow$

1-loop β-function of $\lambda\phi^4$ theory

$\lambda\phi^4_{UV} \leftarrow$ large λ

$\lambda\phi^4_{IR} \leftarrow$ small λ

perturbative expansion valid at IR

Wilsonian RG (k space)

Ref. Peskin & Schroeder QFT

UV $\leftarrow S_{UV}$

$M\downarrow$ flow of QFT Lagrangian $\mathcal{L}_M$

IR

2 step procedure

step (1): integrate out high energy modes

$$Z = \int \underbrace{D\phi_{UV}} \, e^{-S_{UV}[\phi_{UV}]}$$ ← Eucl. QFT

$\phi_{UV} = \phi_0$ $\quad S_{UV}[\phi_{UV}] = S[\phi_0]$

bare quantities

$$= \int [D\phi]_M \, e^{-S_M[\phi]}$$

renormalized action

by integrating out high energy modes

ϕ : relate to renormalized field (but not yet)

$$\underline{\underline{\phi_0(x)}} = \int_{|k| \le \Lambda_0} \frac{d^dk}{(2\pi)^d} \, \underline{\underline{\phi_0(k)}} \, e^{ik\cdot x}$$

Λ_0 : fundamental cut-off (UV cut-off)

regularize the integral

$\Lambda_0 \to \infty$

$$D\phi_{UV} = D\phi_0(k) = \prod_{|k| \le \Lambda_0} d\phi_0(k) = \overbrace{\prod_{|k| \le M} d\phi_0(k)}^{\text{low energy}} \; \overbrace{\prod_{M < |k| \le \Lambda_0} d\phi_0(k)}^{\text{high energy modes}}$$

we introduce a scale M

$$\phi(k) := \begin{cases} \phi_0(k) , & |k| \le M \\ 0 , & M < |k| \le \Lambda_0 \end{cases}$$

$$\hat{\phi}(k) = \begin{cases} 0 & |k| \le M \\ \phi_0(k) & M < |k| \le \Lambda_0 \end{cases}$$

$$D\phi_{UV} = \prod_{|k| \le M} d\phi(k) \prod_{M < |k| < \Lambda_0} d\hat{\phi}(k)$$

$$= [D\phi]_M \, D\hat{\phi}$$

$$\phi_0(x) = \underbrace{\int_{|k| \le M} \frac{d^dk}{(2\pi)^d} \, \phi(k) \, e^{ik\cdot x}}_{\phi(x)} + \underbrace{\int_{M < |k| \le \Lambda_0} \frac{d^dk}{(2\pi)^d} \, \hat{\phi}(k) \, e^{ikx}}_{\hat{\phi}(x)}$$

low energy field $\quad$ high energy field.

$$S_{UV}[\phi_0] = \int d^dx \left[\frac{1}{2} (\partial_\mu \phi_0)^2 + \frac{1}{2} m_0^2 \phi_0^2 + \frac{\lambda_0}{4!} \phi_0^4 \right]$$

(we remove μ^{4-d} ; λ_0 is dimensionful)

$$\phi_0 = \phi + \hat{\phi}$$

$$Z = \int [D\phi]_M \int D\hat{\phi}\; e^{-\int d^dx \left[\frac{1}{2}(\partial_\mu \phi + \partial_\mu \hat{\phi})^2 + \frac{1}{2} m_0^2 (\phi + \hat{\phi})^2 + \frac{\lambda_0}{4!}(\phi+\hat{\phi})^4 \right]}$$

$$= \underline{\int [D\phi]_M\, e^{-S_0[\phi]}} \underset{\uparrow}{\int D\hat{\phi}}\; e^{-\int d^dx \left[(\partial\hat{\phi})^2 \frac{1}{2} + \frac{1}{2} m_0^2 \hat{\phi}^2 + \lambda_0 \left(\frac{1}{6}\phi^3\hat{\phi} + \frac{1}{4}\phi^2\hat{\phi}^2 + \frac{1}{6}\phi\hat{\phi}^3 + \frac{1}{4!}\hat{\phi}^4 \right) \right]}$$

$$S_0[\phi] = \int d^dx \left[\frac{1}{2}(\partial\phi)^2 + \frac{1}{2} m_0^2 \phi^2 + \frac{\lambda_0}{4!}\phi^4 \right]$$

Note : $\int d^dx\, \phi\hat{\phi} = \int \frac{d^dk}{(2\pi)^d} \phi(k)^* \hat{\phi}(k) = 0$

$\int d^dx\, \partial\phi\, \partial\hat{\phi} = \int \frac{d^dk}{(2\pi)^d} k^2 \phi(k)^* \hat{\phi}(k) = 0$

$$= \int [D\phi]_M\, e^{-S_0[\phi]} \underbrace{\int D\hat{\phi}\; e^{-\int d^dx\, f[\phi,\hat{\phi}]}}_{e^{-W[\phi]}}$$

$\sim W[\phi] \sim$

$\sim \mu \int \phi^2$

$$= \int [D\phi]_M\, e^{-S_0[\phi] - \underline{\underline{W[\phi]}}} \qquad -W[\phi] = \ln \int d\hat{\phi}\, e^{-\int d^dx f[\phi,\hat{\phi}]}$$

$$S_0[\phi] + W[\phi] \equiv S_M[\phi] \equiv \int d^dx\, \mathcal{L}_{eff}[\phi]$$

low energy effective Lagrangian

$$= \int [D\phi]_M\, e^{-S_M[\phi]}$$

renormalized action

$$\int D\hat{\phi}\, e^{-\int d^dx\, f[\phi,\hat{\phi}]} = \int D\hat{\phi} \left\{ 1 - \int d^dx \left[\frac{1}{2}\underline{m_0^2 \hat{\phi}^2} + \lambda_0 \left(\frac{1}{6}\phi^3\hat{\phi} + \frac{1}{4}\phi^2_{(x)}\hat{\phi}^2_{(x)} + \frac{1}{6}\phi\hat{\phi}^3 + \frac{1}{4!}\hat{\phi}^4 \right) \right] + \cdots \right\} \times e^{-\int d^dx \frac{1}{2}(\partial_\mu\hat{\phi})^2}$$

we make perturbative expansion w.r.t. m_0, λ_0

$$\phi^3 \int_{-\infty}^{\infty} D\hat{\phi}\; \hat{\phi}\; e^{-\int (\partial\hat{\phi})^2/2} = 0 \qquad \phi \int D\hat{\phi}\; \hat{\phi}^3\, e^{-\int (\partial\hat{\phi})^2/2} = 0$$

$$\frac{\int D\hat{\phi}\, e^{-\int d^dx \frac{1}{2}(\partial_\mu \hat{\phi})^2}\, \hat{\phi}(k)\hat{\phi}(p)}{\int D\hat{\phi}\, e^{-\int d^dx \frac{1}{2}(\partial_\mu \hat{\phi})^2}} \equiv \hat{\phi}(k)\hat{\phi}(p)$$

const.

$$= \frac{1}{k^2}(2\pi)^d \delta^{(d)}(k+p)\, \Theta(k)$$

HW derive this

$$\Theta(k) = \begin{cases} 1 & \mu < |k| \le \Lambda_0 \\ 0 & \text{otherwise} \end{cases}$$

$$-\int d^dx\, \frac{\lambda_0}{4}\phi^2 \hat{\phi}\hat{\phi} = -\frac{\lambda_0}{4}\int d^dx \underbrace{\int \frac{d^dk_1}{(2\pi)^d}\int\frac{d^dk_2}{(2\pi)^d}}_{|k_{1,2}|\le \mu} \underbrace{\int\frac{d^dp_1}{(2\pi)^d}\int\frac{d^dp_2}{(2\pi)^d}}_{\mu<|p_{1,2}|\le\Lambda_0}$$

$$\cdot\, \phi(k_1)\phi(k_2)\hat{\phi}(p_1)\hat{\phi}(p_2)$$

$$\cdot\, e^{i(k_1+k_2+p_1+p_2)\cdot x}$$

$$= -\frac{\lambda_0}{4}\int \frac{d^dk_1}{(2\pi)^d}\cdots\frac{d^dp_2}{(2\pi)^d}\,(2\pi)^d\delta^{(d)}(k_1+k_2+p_1+p_2)\,\frac{(2\pi)^d}{p_1^2}\delta^{(d)}(p_1+p_2)$$

$$\phi(k_1)\phi(k_2)\,\Theta(p_1)$$

$$p_1+p_2=0 \qquad k_1+k_2=0$$

$$= -\frac{\lambda_0}{4}\int_{|k_1|\le\mu}\frac{d^dk_1}{(2\pi)^d}\int_{\mu<|p_1|\le\Lambda_0}\frac{d^dp_1}{(2\pi)^d}\,\frac{1}{p_1^2}\,\phi(k_1)\phi(-k_1)$$

$$= -\frac{1}{2}\mu\int\frac{d^dk}{(2\pi)^d}\,\phi(k)\phi(-k)$$

$$\mu = \frac{\lambda_0}{2}\int_{\mu<|p|\le\Lambda_0}\frac{d^dp}{(2\pi)^d}\,\frac{1}{p^2} = \frac{\lambda_0}{(4\pi)^{d/2}\Gamma(\frac{d}{2})}\,\frac{1-\left(\frac{\mu}{\Lambda_0}\right)^{d-2}}{d-2}\,\Lambda_0^{d-2}$$

quadratic div. $\Lambda_0 \to \infty$

$$\int D\hat{\phi} \left(1 - \frac{\lambda_0}{4}\int d^dx\, \phi^2 \hat{\phi}\hat{\phi}\right) e^{-\int d^dx\, \frac{1}{2}(\partial_\mu \hat{\phi})^2}$$

$$= \underbrace{\int D\hat{\phi}\, e^{-\int d^dx\, \frac{1}{2}(\partial_\mu \hat{\phi})^2}} \left(1 - \frac{\int D\hat{\phi} \left(\frac{\lambda_0}{4}\int d^dx\, \phi^2 \hat{\phi}^2\right) e^{-\int \frac{1}{2}(\partial \hat{\phi})^2}}{\int D\hat{\phi}\, e^{-\int d^dx\, \frac{1}{2}(\partial_\mu \hat{\phi})^2}}\right)$$

$$= \mathcal{N}\left(1 - \int d^dx\, \frac{\lambda_0}{4}\phi^2 \hat{\phi}\hat{\phi}\right)$$

$$= \mathcal{N}\left(1 - \frac{1}{2}\mu \int \frac{d^dk}{(2\pi)^d}\, \phi(k)\phi(-k)\right)$$

$$= \mathcal{N}\left(1 - \frac{1}{2}\mu \int d^dx\, \phi^2\right) \simeq \mathcal{N}\, e^{\overbrace{-\frac{1}{2}\mu \int d^dx\, \phi^2}^{\text{one term in } W[\phi]}}$$

$$\mathcal{L}_{\text{eff}} = \frac{1}{2}(\partial\phi)^2 + \frac{1}{2}m_0^2\phi^2 + \frac{\lambda_0}{4!}\phi^4 + \frac{1}{2}\mu\phi^2$$

$$= \frac{1}{2}(\partial\phi)^2 + \frac{1}{2}\underbrace{(m_0^2 + \mu)}_{m'^2}\phi^2 + \frac{\lambda_0}{4!}\phi^4$$