position variables : $q_{ij}(x,t)$, N_i , N

momentum variables : $\tilde{p}^{ij}$ $\quad p^i_{\vec{N}}$ $\quad p_N$

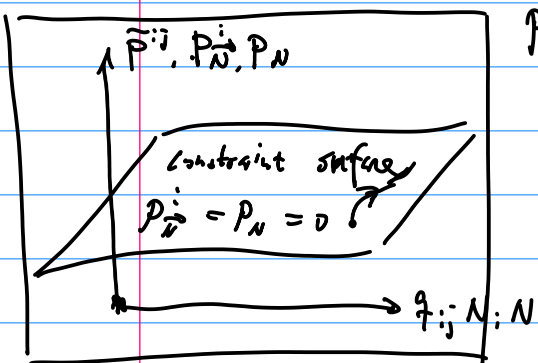
$\tilde{p}' = \frac{-1}{\kappa}\tilde{p}$

$\{q, \tilde{p}\} = \kappa\delta$

$$(\kappa=1) \longrightarrow \quad \tilde{p}^{ij} = \frac{\delta S}{\delta \dot{q}} \qquad p^i_{\vec{N}} = \frac{\delta S}{\delta \dot{N}_i} \qquad p_N = \frac{\delta S}{\delta \dot{N}}$$

$$\boxed{\begin{array}{ll} \text{Primary} & p^i_{\vec{N}} = 0 \\ \text{Constraint} & p_N = 0 \end{array}}$$

$$S = -\frac{1}{\kappa}\int \sqrt{-g}\, R\, d^3x$$



phase space

$(\mathcal{P}, \{,\})$

$\uparrow$

poisson bracket

$q_{ij}(\vec{x},t)$

$q_{11}(0,0) \in \mathbb{R}$

a function of phase space

$\{\,,\,\}$: bilinear map of two functions f, f' (phase space) to $\{f, f'\}$ a function.

$$f = f[q_{ij}, \tilde{p}_{ij} \cdots]$$

$$f' = f'[q_{ij}, \tilde{p}_{ij} \cdots]$$

equal-time poisson bracket

$$\{f, f'\} := \kappa \underbrace{\int}_{\Sigma} d^3x \left[\left(\frac{\delta f}{\delta q_{ij}(x)} \frac{\delta f'}{\delta \tilde{p}^{ij}(x)} - \frac{\delta f}{\delta \tilde{p}^{ij}(x)} \frac{\delta f'}{\delta q_{ij}(x)} \right) + \left(\frac{\delta f}{\delta N_i(x)} \frac{\delta f'}{\delta \tilde{p}^i_N(x)} - \frac{\delta f}{\delta \tilde{p}^i_N(x)} \frac{\delta f'}{\delta N_i(x)} \right) + \left(\frac{\delta f}{\delta N(x)} \frac{\delta f'}{\delta \tilde{p}_N(x)} - \frac{\delta f}{\delta \hat{p}_N(x)} \frac{\delta f'}{\delta N(x)} \right) \right]$$

$$\{f, f'\} = \sum_i \left(\frac{\partial f}{\partial q^i} \frac{\partial f'}{\partial p_i(t)} - \frac{\partial f}{\partial p_i} \frac{\partial f'}{\partial q^i} \right)$$

$$\{ f[p_i(t), q^i(t)] , f'[p_i(t), q^i(t)] \} =$$

$$\{q^i(t), p_j(t)\} = \delta^i_j$$

$$\underline{\{ q_{ij}(\vec{x},t), p^{kl}(\vec{x}',t) \} = \kappa\, \delta^k_{(i}\, \delta^l_{j)}\, \delta^{(3)}(x,y)} \longleftarrow \text{HW check this}$$

Hamilton's eqn. : for any phase space function $f[q_{ij}(\vec{x},t), \tilde{p}^{ij}(\vec{x},t) \cdots]$

$$\underline{\dot{f} = \{f, H\}}$$

Hamiltonian $H = \int_\Sigma d^3x \left[NC + N_i C^i \right]$ + boundary terms

$$C = \frac{1}{k}\left[\frac{1}{\sqrt{\det q}} \left(\tilde{p}^{ij} \tilde{p}_{ij} - \frac{1}{2} p^2 \right) - \sqrt{\det q}\, {}^3R \right]$$

$$C^i = -\frac{2}{k} D_i \tilde{p}^{ij}$$

$$\dot{q}_{ij}(\vec{x},t) = \frac{\delta H}{\delta \tilde{p}^{ij}(\vec{x},t)} \qquad \dot{\tilde{p}}^{ij} = -\frac{\delta H}{\delta q_{ij}}$$

$$\boxed{\dot{q} = \frac{\partial H}{\partial p} \qquad \dot{p} = -\frac{\partial H}{\partial q}}$$

$$\dot{N} = .. \qquad \dot{P}_N = ..$$

$$\dot{N}_i = .. \qquad \dot{P}_{\vec{N}} = --$$

$$S = \frac{1}{k}\int dt\, d^3x\, \tilde{p}^{ij} \dot{q}_{ij} - \int dt\, H \qquad \delta S = 0$$

primary constraints. $P_N \approx 0$, $P^i_{\vec{N}} \approx 0$

$$\left.\begin{aligned} \dot{P}_N &= \{P_N\ H\} = -\frac{\delta H}{\delta N} \approx 0 \\ \dot{P}^i_{\vec{N}} &= \{P^i_{\vec{N}}\ H\} = -\frac{\delta H}{\delta N_i} \approx 0 \end{aligned}\right\} \text{secondary constraints}$$

$C \approx 0$ Hamiltonian constraint (scalar constraint)

$C_i \approx 0$ Diffeomorphism constraint (vector constraint)

$$\left.\begin{aligned} \dot{C} &= \{C, H\} \approx 0 \\ \dot{C}^i &= \{C^i, H\} \approx 0 \end{aligned}\right\} \text{tertiary constraints}$$

but they are implied by $C = C^i = 0$

no new constraint.

Hamiltonian of GR : $H = \int_\Sigma d^3x \, \left(\underline{N C + N_i C^i} \right) + \underline{\text{boundary terms}}$

constraints : $C \approx 0 \, , \; C^i \approx 0 .$

up to boundary terms, H is a linear combination of constraints.

$\rightarrow$ $H \approx 0 + \text{boundary terms}$ on the constraint surface , N , N_i Lagrangian multipliers not dynamical

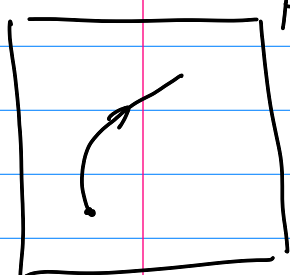
gravitional energy is nonlocal.

(because time evolution is just gauge transformation in GR)

EOM of ADM formalism : $\underline{\dot{q}_{ij} = \frac{\delta H}{\delta \tilde{p}^{ij}} \qquad \dot{p}^{ij} = - \frac{\delta H}{\delta q_{ij}}}$ ①

$C \approx 0 \; , \; C_i = q_{ij} C^j \approx 0$ ②

① and ② determine dynamics of gravity $\Longleftrightarrow$ Vaccum Einstein Eqn $G_{ab} = 0 .$



P

$$\begin{cases} C = 0 \iff G_{tt} = 0 \\ C^i = 0 \iff G_{ti} = 0 \end{cases}$$

① $\longleftrightarrow$ $G_{ij} = 0$

reduce 1st order PDE to 2nd order PDE

ADM is an initial value formulation of GR. : Given 3+1 decomposition of $M = R \times \Sigma$

initial value on $\Sigma_{t=0}$ $\left(q_{ij} , \tilde{p}^{ij} \right)_{t=0}$ satisfying $C = C_i = 0 .$

$\Rightarrow$ a solution $\underbrace{q_{ij} (t, \vec{x}) , \tilde{p}^{ij} (t, \vec{x})}_{\text{equivalent to } g_{\mu\nu}}$ is uniquely fixed by ①.

$C_i = q_{ij} C^j$

recover gauge transformation. , smeared constraints $C(N) := \int_\Sigma d^3x \, N C$

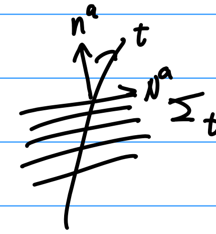
$$C(\vec{N}) := \int_\Sigma d^3x \, N^i C_i$$

$$C(\vec{x}) \longrightarrow C(N) \qquad \sum_A N_A C_A$$

HW check them

Hint: $\dot{q}_{ij} = \left(\mathcal{L}_{t^a} q\right)_{ij}$

$$\begin{cases} \delta_{\vec{N}} q_{ij}(x) \equiv \{ q_{ij}(x), \underline{C(\vec{N})} \} = \left(\mathcal{L}_{\vec{N}} q\right)_{ij}(x) \\ \delta_{\vec{N}} \tilde{P}^{ij}(x) \equiv \{ \tilde{P}_{ij}(x), \underline{C(\vec{N})} \} = \left(\mathcal{L}_{\vec{N}} \tilde{P}\right)^{ij}(x) \\ \qquad \text{infinitesimal spatial diffeomorphisms on } \Sigma \\ \delta_N q_{ij}(x) \equiv \{ q_{ij}(x), C(N) \} = \left(\mathcal{L}_{N n^a} q\right)_{ij}(x) \\ \delta_N \tilde{P}^{ij}(x) \equiv \{ \tilde{P}^{ij}(x), C(N) \} = \left(\mathcal{L}_{N n^a} \tilde{P}\right)^{ij}(x) \end{cases}$$



use Hamiltonian EOMs

infinitesimal diffeo. perpendicular to Σ.

Wald: "GR" Appendix