$\underline{(h(e),\ p^j(e))_{e\in E(\gamma)}}$ Lattice field $\in \big(\underbrace{SU(2)\times\mathbb{R}^3}_{T^*SU(2)}\big)^{|E(\gamma)|}$

$(g(v))_{v\in V(\gamma)}$ lattice gauge transf. $\in (SU(2))^{|V(\gamma)|}$

<u>Thm</u> holonomy-flux algebra:

$$\{h(e),\, h(e')\} = 0 \qquad (1)$$

$$\{p^j(e),\, h(e')\} = \kappa\,\delta_{ee'}\,\frac{\tau^j}{2}\,h(e') \qquad (2)$$

$$\{p^j(e)\ p^k(e')\} = -\kappa\,\delta_{ee'}\,\varepsilon^{jkl}\,p^l(e) \qquad (3)$$

proof of (2)

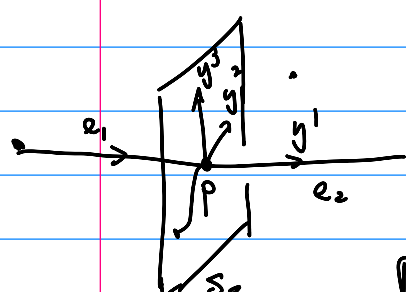
$$\{p^j(e),\ h(e')\} = \frac{\kappa\beta}{2}\int_\Sigma d^3x\left[\underbrace{\frac{\delta p^j(e)}{\delta E^a_i(x)}\frac{\delta h(e')}{\delta A^i_a(x)}} - \underbrace{\cancel{\frac{\delta p^j(e)}{\delta A^i_a(x)}\frac{\delta h(e')}{\delta E^a_i(x)}}}_{0}\right]$$

$$\frac{\delta p^j(e)}{\delta E^a_i(x)} = \frac{\delta}{\delta E^a_i(x)}\left[-\frac{1}{2\beta}\,\mathrm{tr}\left(\tau^j\int_{S_e} dy^b\wedge dy^c\,\varepsilon_{bcd}\,h(\rho_e(y))\,E^d_k(y)\,\tau^k\,h(\rho_e(y))^{-1}\right)\right]$$

$$= -\frac{1}{2\beta}\,\mathrm{tr}\left(\tau^j\int_{S_e} \underline{dy^b\wedge dy^c\,\varepsilon_{bcd}}\,h(\rho_e(y))\,\underbrace{\frac{\delta E^d_k(y)}{\delta E^a_i(x)}}_{\delta^d_a\delta^i_k\delta^{(3)}(y,x)}\,\tau^k\,h(\rho_e(y))^{-1}\right)$$

both $h(e)$, $p^j(e)$ coordinate indep.

we choose a coordinate



$p = (0,0,0)$

$e = e_1 \circ e_2$

$$\frac{\delta p^j(e)}{\delta E^a_i(x)} = -\frac{1}{2\beta}\,\mathrm{tr}\Big[\tau^j\int_{y^1=0}\Big(\overbrace{dy^2\wedge dy^3\,\varepsilon_{23d} + dy^3\wedge dy^2\,\varepsilon_{32d}}^{2\,dy^2\wedge dy^3}\Big)$$

$$h(\rho_e(y))\,\delta^d_a\,\delta^i_k\,\delta^{(3)}(\vec{y},\vec{x})\,\tau^k\,h(\rho_e(y))^{-1}\Big]$$

$$= -\frac{1}{\beta}\,\mathrm{tr}\Big[\tau^i\int_{y^1=0} dy^2dy^3\,\underbrace{\delta^{(3)}(\vec{y},\vec{x})}\,\delta^1_a$$

$$h(\rho_e(y))\,\tau^i\,h(\rho_e(y))^{-1}\Big]$$

$$\delta(y^1-x^1)\,\delta(y^2-x^2)\,\delta(y^3-x^3)$$

$$\overset{\shortparallel}{0}$$

$$= -\frac{1}{\beta}\,\delta^1_a\,\underline{\delta(x^1)}\,\mathrm{tr}\big[\tau^j\,h(\rho_e(x))\,\tau^i\,h(\rho_e(x))^{-1}\big]$$

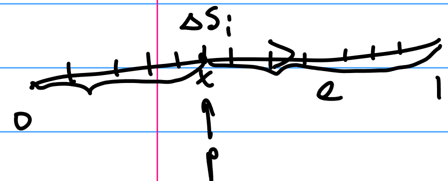
$$\frac{\delta h(e)}{\delta A_a^i(x)} = \frac{\delta}{\delta A_a^i(x)} \left[1+\int_{\Delta s_1} ds\, A(s)\right] \cdots \left[1+\int_{\Delta s_i} ds\, A(s)\right] \cdots \left[1+\int_{\Delta s_n} ds\, A(s)\right] \qquad n \to \infty$$

$$A(s) = A_b^j(y(s))\, \dot{e}^b \frac{\tau_i}{2}$$

if $x \notin e$ $\quad = 0$

if $x \in e$

$$= \underbrace{\left[1+\int_{\Delta s_1} ds\, A(s)\right] \cdots}_{h(0,x)} \int_{\Delta s_i} ds\, \delta^j_i \delta^a_b\, \dot{e}^b \frac{\tau_i}{2} \delta^{(3)}(\vec{x}, \vec{y}(s)) \cdots \underbrace{\left[1+\int_{\Delta s_n} A(s)\right]}_{h(x,1)}$$



$$= \int_0^1 ds\, \dot{e}^a \delta^{(3)}(\vec{x}, \vec{y}(s))\, h(0,x) \frac{\tau_i}{2} h(x,1)$$

$s = y^1$

$$= \int_0^1 dy^1 \left(\frac{\partial}{\partial y^1}\right)^a \delta(x^2 - \underset{0}{y^2})\, \underline{\underline{\delta(x^1-y^1)}}\, \delta(x^3 - \underset{0}{y^3})\, h(0,x)\frac{\tau_i}{2} h(x,1)$$

$$= \underbrace{\left(\frac{\partial}{\partial y^1}\right)^a}_{= \delta^a_1} \underline{\delta(x^2)\delta(x^3)}\; h(0,x) \frac{\tau_i}{2} h(x,1)$$

$$\{P^j(e), h(e)\} = \frac{k\beta}{2} \int d^3x \left(-\frac{1}{\beta}\right) \underbrace{\delta^1_a \delta^a_1}_{1} \overbrace{\delta(x^1)\delta(x^2)\delta(x^3)}^{\vec{x} = p}\, \mathrm{tr}\left[\tau^j \underset{h(e_1)}{h(\rho_e(x))}\, \tau^i \underset{h(e_1)^{-1}}{h(\rho_e(x))^{-1}}\right] h(0,x) \frac{\tau_i}{2} h(x,1)$$

$h(0,p) = h(e_1) \qquad h(p,1) = h(e_2)$

$$= -\frac{k}{2} \underbrace{\mathrm{tr}\left[\tau^j h(e_1) \tau^i h(e_1)^{-1}\right]}\, h(e_1) \frac{\tau_i}{2} h(e_2)$$

$h(e_1)^{-1} h(e_1)$

$h(e) \in SU(2) \qquad h(e_1)\, \tau^i\, h(e_1)^{-1} = \Lambda^i{}_k \tau^k \qquad \vec{\tau} = -i\vec{\sigma}$

$\tau^i \in su(2)$

3x3 matrix $\Lambda \in SO(3) \qquad \mathrm{tr}(\sigma^i \sigma^j) = 2\delta^{ij}$

$$= -\frac{k}{4} \Lambda^i{}_k \underbrace{\mathrm{tr}\left[\tau^j \tau^k\right]}_{-2\delta^{jk}} \Lambda^i{}_l \tau^l \underbrace{h(e_1) h(e_2)}_{h(e)}$$

$$\boxed{\delta_{ik} \Lambda^i{}_j \Lambda^k{}_l = \delta_{jl}}$$

$$= \frac{k}{2} \underbrace{\Lambda^i{}_j \Lambda^i{}_l}_{\delta_{jl}} \tau^l h(e)$$

$$= k \frac{\tau_j}{2} h(e) \qquad \blacksquare$$

HW prove ③ i.e. $\{P^i(e)\, P^k(e')\}$

Ref. hep-th/0005232

phase space : $P_\gamma = T^* SU(2)^{|E(\gamma)|} / SU(2)^{|V(\gamma)|}$

Poisson bracket on P_γ : holonomy-flux algebra.

Quantization : QM, phase space $T^*\mathbb{R} \simeq \mathbb{R}^2$ $(x, p) \rightsquigarrow (\hat{x}, \hat{p})$

$\mathcal{H} \simeq L^2(\mathbb{R}) \ni \psi(x)$ wave function

$\hat{x}\psi(x) = x\,\psi(x)$

$\hat{p}\,\psi(x) = -i\hbar \frac{\partial}{\partial x}\psi(x)$ $\quad [\hat{x}, \hat{p}] = i\hbar\{x, p\} = i\hbar$

Wave functions: functions of all holonomies on γ

$f(h(e_1) \cdots h(e_{|E(\gamma)|})) \in \mathcal{H}^0$

Hilbert space $\mathcal{H}^0_\gamma = \underbrace{L^2(SU(2), d\mu_H)^{|E(\gamma)|}}_{} \simeq L^2(SU(2)^{|E(\gamma)|}, d\mu_H^{|E(\gamma)|})$

Haar measure

$L^2(SU(2), d\mu_H)$: Hilbert space of square-integrable complex-valued function on $SU(2) \simeq S^3$

$\forall\, f_1\, f_2 \in \underline{L^2(SU(2), d\mu_H)}$

$\langle f_1, f_2 \rangle \equiv \int_{SU(2)} \underline{d\mu_H(h)}\ \overline{f_1(h)}\, f_2(h)$ is finite

functions on $SU(2)$

if we parametrize $\underbrace{h = e^{\alpha\tau_3/2}\, e^{\beta\tau_2/2}\, e^{\gamma\tau_3/2}}_{}$

α, β, γ Euler angles

$\forall f(h) \equiv f(\alpha, \beta, \gamma)$

$$\int d\mu_H(h)\, f(h) \equiv \frac{1}{8\pi^3}\int_0^{2\pi} d\alpha \int_0^{\pi} \sin\beta\, d\beta \int_0^{2\pi} d\gamma\, f(\alpha, \beta, \gamma)$$

Thm Haar measure is the unique measure on G such that

Give a compact Lie group G

- $\int_G d\mu_H = 1$

- $\int_G d\mu_H(h)\, f(h) = \int_G d\mu_H(h)\, f(hg) = \int_G d\mu_H(h)\, f(gh)$

$= \int_G d\mu_H(h)\, f(h^{-1})$

$\forall g \in G$

$\forall f \in C(G)$

Orthogonal basis in $L^2(SU(2), d\mu_H)$: Wigner D-functions

(matrix elements of Wigner D-matrix)

Ref. wikipedia "Wigner D-matrix"

$$\pi^j_{mn}(h) \equiv D^j_{mn}(\alpha, \beta, \gamma)$$

$$\simeq \langle j\, m |\, \underline{e^{-i\alpha \hat{J}_3} e^{-i\beta \hat{J}_2} e^{-i\gamma \hat{J}_3}}\, | j\, n \rangle$$

in QM

spin $j = 0, \frac{1}{2}, 1, \frac{3}{2} \dots$

magnetic quantum numbers

$m, n = -j, -j+1, \dots, j-1, j$

$|j, m\rangle \in V_j$

SU(2) spin j irrep.

$$\underline{\int d\mu_H(h)\, \overline{\pi^j_{mn}(h)}\, \pi^{j'}_{m'n'}(h) = \frac{1}{2j+1}\delta_{jj'}\delta_{mm'}\delta_{nn'}}$$

$\{\pi^j_{mn}\}^{j \in \frac{\mathbb{Z}_+}{2} \cup \{0\}}_{m,n = -j \dots j}$ is a othogonal basis in $L^2(SU(2), d\mu_H)$

$\forall f \in L^2(SU(2), d\mu_H)$, $\quad f(h) = \sum_{j,m,n} f_{jmn}\, \pi^j_{mn}(h)$

$f_{j,m,n} \in \mathbb{C}$

Orthogonal basis in $\mathcal{H}^0$: $\prod_{e\in E(\gamma)} \pi^{j_e}_{m_e n_e}(h_e) \equiv T_{\vec{j},\vec{m},\vec{n}}(\vec{h})$

inner product

$\vec{j} = \{j_e\}_{e\in E(\gamma)}$ $\vec{h} = \{h_e\}_{e\in E(\gamma)}$

$$\int \prod_e d\mu_H(h_e)\ \overline{T_{\vec{j},\vec{m},\vec{n}}(\vec{h})}\ T_{\vec{j}',\vec{m}',\vec{n}'}(\vec{h})$$

$$= \prod_{e\in E(\gamma)} \left(\frac{1}{2j_e+1}\, \delta_{j_e j'_e}\, \delta_{m_e m'_e}\, \delta_{n_e n'_e} \right)$$

$\forall f \in \mathcal{H}^0$, $f(\vec{h}) = \sum_{\vec{j},\vec{m},\vec{n}} f_{\vec{j}\vec{m}\vec{n}}\, T_{\vec{j}\vec{m}\vec{n}}(\vec{h})$