nonabelian gauge group : $SU(2)$ $\qquad D_\mu = \partial_\mu \psi - i g A_\mu \psi$

$$D_\mu = \partial_\mu - i \underset{\substack{\uparrow \\ \text{coupling const}}}{g} A_\mu^i \frac{\sigma_i}{2} \qquad i = 1,2,3$$

$$\psi^a \to u^a{}_b(x)\, \psi^b(x) \qquad a = 1,2$$

$$\bar\psi^a \to \bar\psi^b(x) (u^{-1}{}_b{}^a)(x) \qquad i = 1,2,3$$

$$\underline{A_\mu = A_\mu^i \frac{\sigma^i}{2}} \to u(x) A_\mu u(x)^{-1} + \frac{i}{g} u(x) \partial_\mu u(x)^{-1}$$

$$D_\mu \psi \to u D_\mu \psi$$

$$\mathcal{L}_D = \bar\psi (i \gamma^\mu D_\mu^{(A)} - m) \psi \quad \text{gauge inv.}$$

make A dynamical $\qquad \mathcal{L}_A = -\frac{1}{4} F_{\mu\nu}^i F^{\mu\nu i}$

$$F_{\mu\nu} = F_{\mu\nu}^i \frac{\sigma^i}{2} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig [A_\mu, A_\nu]$$

$$F_{\mu\nu}^i = \partial_\mu A_\nu^i - \partial_\nu A_\mu^i + \underline{g\, \varepsilon^{ijk} A_\mu^j A_\nu^k}$$

gauge transf. $\qquad [D_\mu, D_\nu]^{ab} \psi^b = -ig F_{\mu\nu}^{ab} \psi^b \qquad \psi \in \mathbb{C}^2 \otimes \mathbb{C}^4, \quad \psi = \psi^{a=1,2}_{\alpha=1..4}$

HW prove this

$$\text{left} \to [D_\mu, D_\nu] u \psi = u [D_\mu, D_\nu] \psi = -ig\, \underline{u F_{\mu\nu} \psi}$$

$$\text{right} \to -ig\, \underline{F'_{\mu\nu} u \psi}$$

$$\Rightarrow F'_{\mu\nu} = u F_{\mu\nu} u^{-1}$$

$$\mathcal{L}_A = -\frac{1}{2} \operatorname{tr}(F_{\mu\nu} F^{\mu\nu}) \to -\frac{1}{2} \operatorname{tr}(u F_{\mu\nu} u^{-1} u F^{\mu\nu} u^{-1})$$

$$= -\frac{1}{2} F_{\mu\nu}^i F^{\mu\nu j} \operatorname{tr}\left(\frac{\sigma^i}{2} \frac{\sigma^j}{2}\right) \qquad = -\frac{1}{2} \operatorname{tr}(F_{\mu\nu} F^{\mu\nu}) \quad \text{gauge inv.}$$

$$= -\frac{1}{4} F_{\mu\nu}^{i} F^{\mu\nu i}$$

$$\psi_\alpha^a$$

SU(2) gauge theory ; $\mathcal{L}_{SU(2)} = \underbrace{-\frac{1}{2} tr\left(F_{\mu\nu} F^{\mu\nu}\right)}_{\text{free gluon + self interaction of gluons "g"}} + \bar{\psi}(i\gamma^\mu D_\mu - m)\psi$

free quark + gluon-quark interaction ($ig\bar{\psi}A\psi$)

$m A_\mu A^\nu$

$A_\mu \to A_\mu + \partial_\mu \chi$

SU(N) gauge group. $u \in SU(N)$ $N \times N$ unitary matrix $\underline{u^\dagger = u^{-1}}$

$\det(u) = 1$

generators

$u = e^{i\alpha^i t^i}$ $i = 1 \ldots \dim(SU(N))$

$\alpha^i \in \mathbb{R}$ $\dim(SU(N)) = N^2 - 1$

$SU(2)$: $t^i = \frac{\sigma^i}{2}$ $SU(N)$ t^i : basis of trassless Hermitian $N \times N$ matrices.

$SU(3)$: t^i $\underbrace{\text{Gel-Man}}_{8}$ matrices

α^i are small $u = 1 + i\alpha^i t^i$ $u u^\dagger = 1 \to \underbrace{(1 + i\alpha^i t^i)(1 - i\alpha^i t^{i\dagger})}_{=1}$

$= 1 + i\alpha^i(t^i - t^{i\dagger})$

$\Rightarrow t^i = t^{i\dagger}$ Hermitian

$1 = \det(u) = 1 + i\alpha^i tr(t^i)$ $\Rightarrow tr(t^i) = 0$ traceless.

$\psi^{a=1\ldots N}$ N flavors of quark, $\psi^a \to u^a{}_b \psi^b$ global symm

of $\mathcal{L}_D = \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi$ if u indep. of x

upgrade $u \to u(x)$

$$\partial_\mu \to D_\mu = \partial_\mu - ig A_\mu$$

$N \times N$ matrix $\sum_{i=1}^{\dim(SU(N))} A_\mu^i t^i \equiv A_\mu$

$$(A_\mu)^a{}_b = A_\mu^i (t^i)^a{}_b$$

$$A_\mu \to u A_\mu u^{-1} + \frac{i}{g} u \partial_\mu u^{-1}$$

$$\psi \to u\psi \qquad \psi^a_\alpha(x) \to u^a{}_b(x)\, \psi^b_\alpha(x)$$

$$\bar\psi \to \bar\psi u^{-1} \qquad \bar\psi^a_\alpha(x) \to \bar\psi^b_\alpha(x) (u^{-1}{}_b{}^a)(x)$$

$$\mathcal{L}_{SU(N)} = -\frac{1}{2} \mathrm{tr}_N (F_{\mu\nu} F^{\mu\nu}) + \underline{\bar\psi (i\gamma^\mu D_\mu - m)\psi}$$

$$\underline{D_\mu \psi \to D_\mu (u\psi) = u D_\mu \psi}$$

<u>HW</u> check this

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig [A_\mu, A_\nu]$$

$$F^i_{\mu\nu} = \partial_\mu A^i_\nu - \partial_\nu A^i_\mu + g f^{ijk} A^j_\mu A^k_\nu \quad , \quad i,j,k = 1 \cdots \underset{N^2-1}{\dim(SU(N))}$$

structure const.

$$[t^i, t^j] = i f^{ijk} t^k \quad , \quad f^{ijk} \text{ is totally skew}$$

$$[D_\mu, D_\nu]\psi = -ig F_{\mu\nu}\psi \quad \to \quad F_{\mu\nu} \to u F_{\mu\nu} u^{-1}$$

<u>HW</u> check this

<u>HW</u> check this

QCD : $N = 3$

Path integral of gauge theory

gauge index : $i = 1 \cdots \dim(SU(N)) \longrightarrow a = 1 \cdots \dim(SU(N))$

rep. index : $a = 1 \cdots N \longrightarrow i = 1 \cdots N$

generator : $t^{i=1\cdots \dim(SU(N))} \longrightarrow T^{a=1\cdots\dim(SU(N))}$

gauge transf : $u(x)\in SU(N) \longrightarrow U(x)\in SU(N)$

$$S = \int d^dx \left[-\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu a} + \bar\psi^i (\gamma^\mu D^{ij}_\mu - m)\psi^j \right]$$

$$A^a_\mu T^a \equiv A_\mu(x) \to U(x) A_\mu(x) U(x)^{-1} + \frac{i}{g} U(x)\partial_\mu U(x)^{-1} \equiv A'$$

$$\psi^i(x) \to U^i{}_j(x)\psi^j(x) \equiv \psi'$$

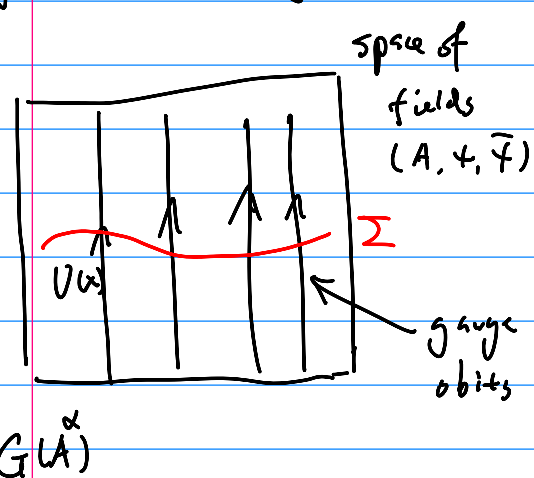
Naively $\underline{Z = \int DA^a_\mu D\psi^i_\alpha D\bar\psi^i_\alpha \; e^{iS}}$

assume $DA\, D\psi\, D\bar\psi$ gauge inv.

$\hookrightarrow DA' D\psi' D\bar\psi' = DA D\psi D\bar\psi \cdot J$

field indep

Why Z is wrong?



$$Z \sim \underset{\substack{\Sigma \\ \perp \text{ to gauge obits}}}{\int D\mu} \underset{\substack{\text{along gauge obits}}}{\int DU(x)} \; e^{iS}$$

$$= \underbrace{\mathrm{Vol}[\text{gauge obits}]}_{\infty} \underbrace{\int_\Sigma D\mu \; e^{iS}}$$

Z is divergent.

We want to define " $\dfrac{Z}{\mathrm{Vol}[\text{gauge obits}]} = \displaystyle\int_\Sigma D\mu\; e^{iS}$ "

secondly Z violate unitarity (P&S sec 16.1)

solution : Faddeev-Popov trick : a gauge fixed version of path integral

gauge fixing condition : $G(A) = 0$

e.g. $G(A) = \partial^\mu A_\mu^a(x) - \omega^a(x)$

any function $u = e^{i\alpha^a T^a}$

$\partial^\mu A_\mu^a = 0$ covariant gauge

finite gauge transf

gauge transf : $A_\mu \to \underline{u\left[A_\mu + \frac{i}{g}\partial_\mu\right]u^{-1}} = uA_\mu u^{-1} + \frac{i}{g}u\partial_\mu u^{-1}$

$\partial^\mu A_\mu \to \partial^\mu A_\mu + \cdots \partial_\mu u$ or $\partial_\mu u^{-1}$

infinitesimal gauge transf. when $\alpha^a(x)$ is small : $u(x) = 1 + i\alpha^a(x)T^a + O(\alpha^2)$

$$A_\mu^a \to (A^\alpha)_\mu^a = A_\mu^a + \underline{\frac{1}{g}\partial_\mu\alpha^a + f^{abc}A_\mu^b\alpha^c} + O(\alpha^2)$$

$$= A_\mu^a + \frac{1}{g}D_\mu^{ac}\alpha^c + O(\alpha^2)$$

HW check this

$D_\mu^{ac} = \partial_\mu\delta^{ac} + g f^{abc}A_\mu^b$

covariant derivative w.r.t. adj. rep.

$G = \partial^\mu A_\mu^a - \omega^a$

$$\partial^\mu A_\mu^a \to \partial^\mu(A^\alpha)_\mu^a = \partial^\mu A_\mu^a + \underline{\frac{1}{g}\partial^\mu D_\mu^{ac}\alpha^c}$$

FP trick : insert in path integral Z an identity

$$1 = \int DG\, \delta(G)$$

$$1 = \underbrace{\int \delta(G(A^\alpha))\, D\alpha}_{\text{along gauge orbit}} \det\left(\frac{\delta G(A^\alpha)}{\delta\alpha}\right)$$

✓ FP determinant Δ_{FP}

$$G^a(A^\alpha(y)) = \partial^\mu A^a_\mu(y) + \frac{1}{g}\partial^\mu D^{ac}_\mu \alpha^c(y) - \omega^a(y)$$

$$\frac{\delta G^a(A^\alpha(y))}{\delta \alpha^c(x)} = \frac{1}{g}\partial^\mu D^{ac}_\mu \delta^{(4)}(y-x)$$

$$Z = \int DAD\psi D\bar\psi\, e^{iS} = \int D\alpha \int \underline{DAD\psi D\bar\psi}\; \underline{e^{iS[A,\psi,\bar\psi]}}\; \delta(G(A^\alpha))\det\left(\frac{\delta G}{\delta\alpha}\right)(A^\alpha)$$

gauge inv. of S & $DAD\psi D\bar\psi$

$$= \int D\alpha \underbrace{\int DA^\alpha D\psi^\alpha D\bar\psi^\alpha\, e^{iS[A^\alpha,\psi^\alpha,\bar\psi^\alpha]}\,\delta(G(A^\alpha))\det\left(\frac{\delta G}{\delta\alpha}\right)(A^\alpha)}$$

change of variable $A^\alpha \to A$ or $A \to A^{\alpha^{-1}}$

$$= \underbrace{\int D\alpha}_{\text{vol [gauge obits]}} \underbrace{\int DAD\psi D\bar\psi\, e^{iS[A,\psi,\bar\psi]}\,\delta(G(A))\det\left(\frac{\delta G}{\delta\alpha}\right)(A)}_{\int_\Sigma D\mu\, e^{iS}}$$

$$\int f(x)\,dx = \int f(y)\,dy$$