- Path integral in QM and QFT
- perturbation theory in QFT
- Renormalization
- Gauge field theory (QCD)
- Lattice QCD

Peskin & Schroeder "QFT"
Ryder "QFT"
Wen, X.G. "Many body Physics ..."
d' Fresesco ... "Conformal Field theory"

Path integral in QM

$$\hat{H} = \frac{\hat{p}^2}{2m} + V(x) \longrightarrow \hat{U}(t_b, t_a) = e^{-\frac{i}{\hbar}\hat{H}(t_b - t_a)} \qquad (\hbar = 1)$$

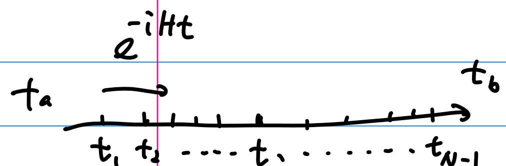
time evolution operator.

transition amplitude : $iA = \langle \psi_f | e^{-iH(t_b - t_a)} | \psi_i \rangle$

final state   initial state.

$|\psi_i\rangle = |x_a\rangle$ eigenstates of $\hat{x}$.
$|\psi_f\rangle = |x_b\rangle$

$$iA(x_b, t_b, x_a, t_a) = \langle x_b | e^{-iH(t_b - t_a)} | x_a \rangle$$

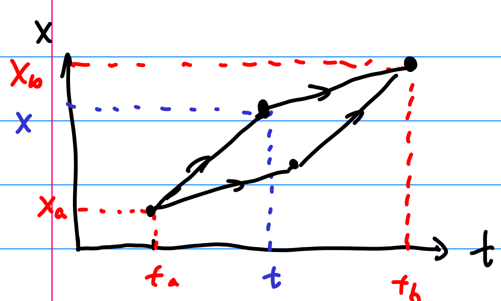


$$e^{-iH(t_b - t_a)} = e^{-iH(t_b - t)} e^{-iH(t - t_a)}$$

for arbitrary t.

$$iA(x_b, t_b, x_a, t_a) = \langle x_b | e^{-iH(t_b - t)} e^{-iH(t - t_a)} | x_a \rangle$$

$$\int dx\, |x\rangle\langle x| = 1 \qquad = \int dx\, \langle x_b | e^{-iH(t_b - t)} |x\rangle\langle x| e^{-iH(t - t_a)} | x_a \rangle$$



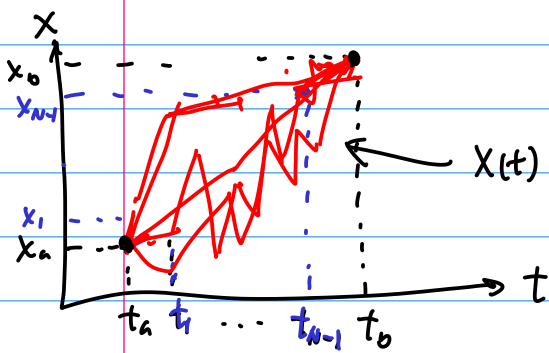
$$U(t_b, t_a) = U(t_b, t_{N-1})\, U(t_{N-1}, t_{N-2}) \cdots U(t_2\, t_1)\, U(t_1, t_a)$$

$$iA(x_b\, t_a, x_a\, t_a)$$

$$= \underbrace{\int dx_1 \cdots dx_{N-1}}_{\text{summing all paths}} \langle x_b | U(t_b\, t_{N-1}) | x_{N-1} \rangle \langle x_{N-1} | U(t_{N-1}\, t_{N-2}) | x_{N-2} \rangle \cdots$$

$$t_{i+1}-t_i = \Delta t = \frac{t_b - t_a}{N} \qquad \cdots \langle x_1 | U(t_1, t_a) | x_a \rangle$$

$N \to \infty \qquad \Delta t \to 0$



all paths have to satisfy

$$\left.\begin{array}{l} x_0 = x_a \\ x_N = x_b \end{array}\right\} \to \left\{\begin{array}{l} x(t=t_a) = x_a \\ x(t=t_b) = x_b \end{array}\right.$$

$$\langle x_i | \underset{\substack{\uparrow \\ \int dp |p\rangle\langle p|}}{e^{-iH\Delta t}} | x_{i-1} \rangle = \int dp_i \langle x_i | e^{-iH\Delta t} | p_i \rangle \underbrace{\langle p_i | x_{i-1} \rangle}_{\text{planewave}}$$

$$= \int dp_i \underbrace{\langle x_i | 1 - i\hat{H}\Delta t | p_i \rangle}\frac{1}{\sqrt{2\pi}} e^{-ip_i x_{i-1}}$$

$$\hat{H} = \frac{\hat{p}^2}{2m} + V(x)$$

$$\langle x_i | \hat{H} | p_i \rangle = \left(\frac{p_i^2}{2m} + V(x_i)\right) \langle x_i | p_i \rangle$$

$$= H(p_i, x_i) \langle x_i | p_i \rangle$$

$$= \int dp_i \left[1 - iH(p_i, x_i)\Delta t\right] \frac{1}{\sqrt{2\pi}} e^{ip_i x_i} \frac{1}{\sqrt{2\pi}} e^{-ip_i x_{i-1}}$$

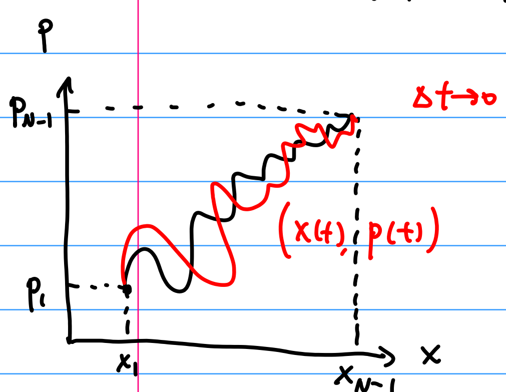
$$\langle x_i | e^{-iH\Delta t} | x_{i-1} \rangle = \int dp_i \, e^{-iH(p_i, x_i)\Delta t} \frac{1}{2\pi} e^{ip_i(x_i - x_{i-1})}$$

$$iA(x_b t_b\, x_a t_a) = \int \prod_{i=1}^{N-1} dx_i \prod_{i=1}^{N} \frac{dp_i}{2\pi} \, e^{\sum_{i=1}^{N} i\left[p_i \frac{x_i - x_{i-1}}{\Delta t} - H(p_i, x_i)\right]\Delta t}$$

$$N \to \infty \qquad = \int \underbrace{Dx(t)\,Dp(t)}_{\text{continuum limit of } \prod_{i=1}^{N-1} dx_i \prod_{i=1}^{N} \frac{dp_i}{2\pi}} \underbrace{e^{i\int_{t_a}^{t_b} dt \left[p(t)\dot{x}(t) - H(p(t), x(t))\right]}}$$

Feynman's path integral of QM

phase space



$Dx(t)\,Dp(t)$ integration measure of space of paths.

↑ infinite-dimensional

$$Dx(t) = \prod_{i=1}^{N-1} dx_i \qquad (N \to \infty)$$

$$Dp(t) = \prod_{i=1}^{N} dp_i / 2\pi \qquad (N \to \infty)$$

$$H(p,x) = \frac{p^2}{2m} + V(x)$$

$$iA(x_b, t_b, x_a, t_a) = \int Dx(t)\, Dp(t)\, e^{i\int_{t_b}^{t_a} dt \left[p\dot{x} - \frac{p^2}{2m} - V(x) \right]}$$

$$\int \prod_{i=1}^{N} \frac{dp_i}{2\pi}\, e^{i\sum_{i=1}^{N}\left(p_i \frac{x_i - x_{i-1}}{\Delta t} - \frac{p_i^2}{2m}\right)\Delta t}$$

Gaussian integral $\int_{-\infty}^{\infty} e^{-ax^2+bx+c}\, dx$ "exact localization"
"exact on-shell"

$$\downarrow$$

$$\int_{-\infty}^{\infty} e^{S(x)}\, dx \qquad \delta S(x) = S'(x) = -2ax + b = 0$$

$$x = x_0 = \frac{b}{2a}$$

$$S(x_0) = -a\left(\frac{b}{2a}\right)^2 + \frac{b^2}{2a} + c = -\frac{b^2}{4a} + \frac{b^2}{2a} + c = \frac{b^2}{4a} + c$$

$$e^{S(x_0)} = e^{\frac{b^2}{4a} + c}$$

$$\int_{-\infty}^{\infty} e^{S(x)}\, dx = \underline{\underline{\sqrt{\frac{\pi}{a}}}}\, \underline{\underline{e^{\frac{b^2}{4a}}}} + c$$ exact result for Gaussian integral

$$a = \frac{i}{2m}\Delta t \qquad b = i\dot{x}\Delta t \qquad c = -iV\Delta t$$

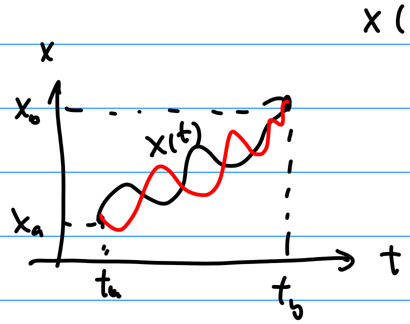
$$iA(x_a t_a x_b t_b) = N \int Dx(t)\, \underbrace{e^{i\int_{t_a}^{t_b} dt\left[\frac{m}{2}\dot{x}^2 - V(x)\right]}}_{e^{i\int_{t_a}^{t_b} dt\, L(x,\dot{x})} = e^{iS[x(t)]}}$$

"Path integral formula in configuration space"

$$N = \prod_{i=1}^{N} \sqrt{\frac{2\pi m}{i\Delta t}} \qquad (N \to \infty)$$

- the space of $x(t)$ contains all paths, satisfying boundary condition

$$x(t_a) = x_a, \quad x(t_b) = x_b$$



- $\int \mathcal{D}x(t)$ sum over all paths (infinite dim integral)
- the weight of each path is $e^{iS[x(t)]}$ (classical quantity)
- $iA(x_b, t_b, x_a, t_a)$ is an infinite dim integral over space of paths connecting x_a, x_b, integrand is classical quantity

both trajectory $x(t)$ and weight $e^{iS[x(t)]}$ are classical, but sum over all paths gives quantum quantities.

advantage of path integral: quantum derived from classical

path integral quantization: $S[q^{(i)}(t), \dot{q}^{(i)}(t)]$ $i = 1 \cdots M$

$$iA(q_a^{(i)} t_a, q_b^{(i)} t_b) := \int \mathcal{D} q^{(i)}(t)\, e^{iS[q^{(i)}(t), \dot{q}^{(i)}(t)]}$$

QFT

$$iA(\phi_f t_f, \phi_i t_i) := \int \mathcal{D}\phi(x,t)\, e^{iS[\phi(x,t), \partial_\mu \phi(x,t)]}$$

$\downarrow$

field theory $S[\phi(x,t), \partial_\mu \phi(x,t)]$

QM = 0+1 dim QFT

finite dim integral = 0+0 dim QFT

question: which path contributes more than others

answer: classical path (path satisfying classical EOM) and small fluctuations contribute dominantly in the path integral.

if recover $\hbar$: $\int \mathcal{D}x(t)\, e^{\frac{i}{\hbar} S[x(t)]}$

<u>Thm</u> (stationary phase approximation)

$$f(\hbar) = \int d^N x \; e^{\frac{i}{\hbar} S[x_1 \cdots x_N]} = \sum_{\vec{x}_c} \underbrace{\sqrt{\frac{(2\pi)^N \hbar^N}{\det(-i M(\vec{x}_c))}} \; e^{\frac{i}{\hbar} S[\vec{x}_c]} \left(1 + O(\hbar)\right)}$$

$$\cdots (f_0'' + f_1 \hbar \cdots)$$

$S \in \mathbb{R}$.

$\vec{x}_c$ saddle point of S.

- $\vec{x}_c$ is a saddle point, satisfying $\delta S(\vec{x}_c) = 0 \Leftrightarrow \partial_{x_i} S[\vec{x}_c] = 0 \quad \forall\, i = 1 \cdots N$
- $M(\vec{x})$ $N \times N$ Hessian matrix $\quad M_{ij}(\vec{x}) = \dfrac{\partial^2 S}{\partial x_i \partial x_j}(\vec{x})$