boson no spin $\quad T_0^2 = 1 \qquad T_0 \psi(t, \vec{r}) = \psi^*(-t, \vec{r})$

fermion spin $\frac{1}{2}$ $\quad T^2 = -1 \qquad T \begin{pmatrix} \psi_1(t, \vec{r}) \\ \psi_2(t, \vec{r}) \end{pmatrix} = e^{i\varphi} \sigma_y \begin{pmatrix} \psi_1^*(-t) \\ \psi_2^*(-t) \end{pmatrix}$

zero spin $\quad \hat{H} \psi_{nl} = E_n \psi_{nl} \qquad l = 1 .. d(n)$

general sol of Schrödingen eqn $\quad \psi(\vec{r}, t) = \sum_n a_n \psi_{nl}(\vec{r}) e^{-\frac{i}{\hbar} E_n t}$

($\hat{H}$ is real, not explicitly dep. on t) $\quad \hat{T}_0 \psi(\vec{r}, t) = \sum_n a_n^* \psi_{nl}^*(\vec{r}) e^{-\frac{i}{\hbar} E_n t}$

effectively $\quad \hat{T}_0 : \psi_{nl}(\vec{r}) \to \psi_{nl}^*(\vec{r})$

if $\hat{H}$ is real, $\quad \hat{H} \psi_{nl} = E_n \psi_{nl} \qquad \psi_{nl}$ is eigenstate

$\Rightarrow \hat{H} \psi_{nl}^* = E_n \psi_{nl}^* \qquad \psi_{nl}^*$ is eigenstate

ψ_{nl} basis in $\mathcal{H}^{(n)}$, ψ_{nl}^* is basis in $\mathcal{H}^{(n)*}$

whether $\mathcal{H}^{(n)}$ and $\mathcal{H}^{(n)*}$ <u>the same</u> ?

(1) $\quad \mathcal{H}^{(n)} \simeq \mathcal{H}^{(n)*} \qquad$ equivalent rep of symm. group

(2) $\quad \mathcal{H}^{(n)} \not\simeq \mathcal{H}^{(n)*} \qquad$ not equivalent ....

$$\text{eigenspace} = \mathcal{H}^{(n)} \oplus \mathcal{H}^{(n)*}$$

<u>real rep, pseudo real rep, and complex rep</u>

Given a symm. group G, unitary irrep $D: g \mapsto D(g) \circlearrowright \mathcal{H}^{(n)}$

Def $D(g)$ is unitary on $\mathcal{H}^{(n)}$, ψ_{ni} orthonormal basis in $\mathcal{H}^{(n)}$

(complex conjugate irrep)

$$D(g)\,\psi_{ni}(\vec{r}) = \sum_j \psi_{nj}(\vec{r})\, D_{ji}(g)$$

$D_{ji}(g)$ unitary matrix

complex conjugate :
$$D(g)^*\,\psi^*_{ni}(\vec{r}) = \sum_j \psi_{nj}(\vec{r})^*\, D_{ji}(g)^*$$

$$\psi^*_{ni}(\vec{r}) = T_0\,\psi_{ni}(\vec{r})$$

$D_{ij}(g)$ unitary irrep matrix $\xrightarrow{T_0}$ $D^*_{ij}(g)$ unitary irrep matrix

(irrep D) irrep D^* : complex conjugate irrep

relation between D and D^*

(1) if $\exists$ unitary transf. $U : \mathcal{H}^{(n)} \to \mathcal{H}^{(n)}$ s.t

$$U\,D(g)\,U^{-1} = \underset{\text{real matrix}}{D_0(g)} \qquad \forall g \in G$$

$\Rightarrow$ D is equivalent to D^*, since $U^* D(g)^* U^{*-1} = D_0(g)$

$\Rightarrow$ $D^*(g) = U^{*-1} D_0(g)\, U^* = U^{*-1} U\, D(g)\, U^{-1} U^*$

$U^{*-1}U$ is unitary

we say $D \simeq D^*$ is a real rep.

(2) D is $\underbrace{\text{equivalent}}_{\text{unitarily}}$ to D^*, but they are not equivalent to real rep.

$$\nexists\, U \text{ s.t. } D(g) = U \underset{\substack{\uparrow \\ \text{real matrix}}}{D_0(g)} U^{-1} \quad \forall g \in G$$

We say, $D \simeq D^*$ is pseudo-real

(3) D is inequivalent to D^*.

We say, D and D^* are complex reps

distinguish (3) from (1) and (2), check character

if $D \simeq D^*$, then $\chi(g) = \chi^*(g) \quad \forall g \in G$

if $\chi(g) \neq \chi^*(g)$, then D & D^* are complex reps.

distinguish between (1) & (2)

$$D^* \simeq D \rightarrow D^*(g) = \underset{\substack{\uparrow \\ \text{unitary matrix}}}{Z} D(g) Z^{-1} \quad \forall g \in G$$

Complex conjugate $D(g) = Z^* D^*(g) Z^{*-1}$

$$\Rightarrow D^*(g) = Z Z^* D^*(g) (Z Z^*)^{-1}$$

i.e. $[Z Z^*, D^*(g)] = 0 \quad \forall g \in G$

by Schur's lemma, when D^* is irrep, $ZZ^* = c\mathbb{1}$
$c \in \mathbb{C}$

Z is unitary, $Z^* = (Z^T)^{-1}$

$$\Rightarrow \quad Z(Z^T)^{-1} = c\mathbb{1} \Rightarrow \underline{Z = cZ^T}$$

$$\xrightarrow[\text{transpose}]{} Z^T = cZ$$

$$\Rightarrow \quad Z = c^2 Z \quad \Rightarrow \quad c^2 = 1, \quad c = \pm 1$$

$$D^* \simeq D \quad \Rightarrow \quad ZZ^* = \pm\mathbb{1}$$

$\underline{\text{Thm}}$: $ZZ^* = \mathbb{1}$ iff D is real

$ZZ^* = -\mathbb{1}$ iff D is pseudo-real

$\underline{\text{Pf}}$ if D is real $\longrightarrow$ $D^*(g) = (U^*)^{-1} D_0(g) U^*$

$$= (U^*)^{-1} U D(g) U^{-1} U^*$$

on the other hand $D^*(g) = Z D(g) Z^{-1}$

$$\Rightarrow \quad (U^*)^{-1} U = bZ \qquad b \in \mathbb{C}$$

Complex conjugate $U^{-1}U^* = b^* Z^*$

$$\underbrace{U^{-1}U^*(U^*)^{-1}U}_{= 1} = b^* b Z^* Z = |b|^2 Z^* Z$$

$$\underline{Z^* Z = |b|^{-2} \mathbb{1}}$$

$\underline{ZZ^* = c\mathbb{1}} \Rightarrow |b|^2 c = 1$

$c = 1$ or -1 $\quad c = 1 \quad |b|^2 > 0$

conversely if $c = 1$ $\quad (ZZ^* = \mathbb{1})$

Z unitary, $Z = e^{iA}$ $\quad A$ hermitian

$Z^* = Z^{-1} \Leftrightarrow Z^T = Z$, $\quad Z$ symmetric

$Z^* = (Z^T)^{-1}$

def. $U = Z^{\frac{1}{2}} = e^{iA/2}$ unitary

$U^2 = Z$

$$\underline{U^* Z = e^{-iA/2} e^{iA} = e^{\frac{1}{2}iA} = U}$$

$\forall g \in G$, $[U D(g) U^{-1}]^* = U^* D(g)^* U^{*-1}$

$= U^* Z D(g) Z^{-1} U^{*-1}$

$= U D(g) U^{-1} = D_0(g)$ real

$\Rightarrow D$ is real rep.

$ZZ^* = \mathbb{1} \Leftrightarrow D$ is real rep

$\Rightarrow ZZ^* = -\mathbb{1} \Leftrightarrow D$ is psudo-real rep. $\quad \square$

Distinguish $\underbrace{\text{real \& pseudo-real}}$ reps by characters

$$\underline{D^*(g) = Z D(g) Z^{-1}}$$

orthogonality $\displaystyle\sum_g D^*_{\alpha\delta}(g)\, D_{\beta\gamma}(g) = \delta_{\alpha\beta}\,\delta_{\gamma\delta}\,\frac{h}{\dim(D)=d}$

$$\parallel$$

$$\sum_{\alpha,\delta} \underline{\underline{Z^{-1}_{\tau\alpha}}} \times \sum_g \sum_{\sigma,\rho} \underline{\underline{Z_{\alpha\sigma}}}\, D_{\sigma\rho}(g)\, Z^{-1}_{\rho\delta}\, D_{\beta\gamma}(g) \times Z_{\delta\chi}$$

$$\Rightarrow \sum_g D_{\tau\chi}(g)\, D_{\beta\gamma}(g) = Z_{\gamma\chi}\, Z^{-1}_{\tau\beta}\,\frac{h}{d}$$

let $\chi = \beta$. $\sum_\beta$, $\displaystyle\sum_g \sum_\beta D_{\tau\beta}(g)\, D_{\beta\gamma}(g)$

$$\parallel$$

$$\sum_g D_{\tau\gamma}(g^2) = \left(Z Z^*\right)_{\tau\gamma}\frac{h}{d}$$

$$= \pm\, \delta_{\tau\gamma}\,\frac{h}{d}$$

let $\tau = \gamma$, $\sum_\tau$,

$$\boxed{\sum_g \chi(g^2) = \pm\, h}$$

for complex rep. $D \neq D^*$ $\Rightarrow$ $\displaystyle\sum_g \left[D^*_{\alpha\beta}(g)\right]^* D_{\gamma\delta}(g) = 0$

$$\parallel$$

$$\sum_g D_{\alpha\beta}(g) D_{\gamma\delta}(g)$$

$$\Rightarrow \sum_g \chi(g^2) = 0$$

summary : $$\frac{1}{h}\sum_{g\in G}\chi(g^2) = \begin{cases} 1 & \text{real rep} \\ -1 & \text{psudo-real rep} \\ 0 & \text{complex.} \end{cases}$$

Extra-degeneracy of $\hat{H}$ due to time-reversal inv.

$$\hat{H}\psi_{nl} = E_n \psi_{nl} \qquad l = 1 \cdots d$$

$\psi_{nl} \in \mathcal{H}^{(n)}$ unitary irrep. of symmetry group G

$l = 1 \cdots d \qquad \dim \mathcal{H}^{(n)} = d$

find degeneracy at E_n, deg. at E_n is either d or $2d$ (spin-zero)

if $\hat{H}$ is real and not explicitly dep on t

$$\hat{H}\psi_{nl} = E_n \psi_{nl}$$

$$\hat{H}\psi^*_{nl} = E_n \psi^*_{nl} \qquad \psi^*_{nl}(\vec{r}) = \hat{T}_0 \psi_{nl}(\vec{r})$$

- if $\forall\, \psi^*_{nl}$, $\psi^*_{nl} \in \mathcal{H}^{(n)}$ spanned by ψ_{nl}

then $\mathcal{H}^{(n)}$ is the final eigenspace i.e. $\mathcal{H}^{(n)} = \mathcal{H}^{(n)*}$

degeneracy at E_n is d

- otherwise, eigenspace $= \mathcal{H}^{(n)} \oplus \mathcal{H}^{(n)*}$

  degeneracy at E_n $= 2d$ (extra degeneracy)

<u>spin-zero</u>

$$\hat{H}\psi_i = E\psi_i \qquad \psi_i \in \underline{\mathcal{H}}^{(n)}$$

$$D(g)\psi_i = \sum_j \psi_j D_{ji}(g) \quad \forall g \in G$$

$\hat{H}$ is real $\quad \hat{H}\psi_i^* = E\psi_i^* \qquad \psi_i^* \in \mathcal{H}^{(n)*}$

$$D^*(g)\psi_i^* = \sum_j \psi_j^* D_{ji}(g)^* \quad \forall g \in G$$

$\mathcal{H}^{(n)}$ carries irrep D of G

$\mathcal{H}^{(n)*}$ carries complex conjugate irrep D^* of G

$\mathcal{H}^{(n)} \simeq \mathcal{H}^{(n)*}$ or not relates to $D \simeq D^*$ or not

firstly if $D \not\simeq D^*$ complex irrep, $\mathcal{H}^{(n)} \perp \mathcal{H}^{(n)*}$ by orthogonality theorem.

$\Rightarrow$ extra degeneracy $2d$

Let's look at case (1) real rep and (2) pseudo-real rep.

$\exists$ unitary Z, $D^*(g) = Z D(g) Z^{-1}$, $ZZ^* = \begin{cases} 1 & \text{real} \\ -1 & \text{pseudo-real} \end{cases}$

<u>Lemma</u> if $\mathcal{H}^{(n)} = \mathcal{H}^{(n)*}$, then D is of case (1) i.e. real rep.

pf. $\underline{\underline{\mathcal{H}^{(n)} = \mathcal{H}^{(n)*}}}$, $\hat{H}\psi_i = E\psi_i$

$\hat{H}\psi_i^* = E\psi_i^*$

both $\{\psi_i\}$, $\{\psi_i^*\}$ are both orthonormal basis

then $\exists$ unitary U s.t. $\psi_i = \sum_k \psi_k^* U_{ki}$

$\psi_i^* = \sum_k \psi_k U_{ki}^*$

$\Rightarrow \psi_i = \sum_{kl} \psi_l U_{lk}^* U_{ki}$ i.e. $UU^* = \mathbb{1}$

$D(g)\psi_i = \sum_j \psi_j D_{ji}(g)$

$\| \qquad\qquad \|$

$D(g)\sum_k \psi_k^* U_{ki} \qquad \sum_j \sum_k \psi_k^* U_{kj} D_{ji}(g)$

$\times U^{-1}_{il}, \sum_i$

$D(g)\psi_l^* = \sum_k \psi_k^* \left(U D(g) U^{-1}\right)_{kl}$

$\uparrow$

same as $D^*(g)\psi_l^*$

Compare to $D^*(g)\psi_l^* = \sum_j \psi_j^* D_{jl}^*(g)$

$\left. D^*(g) = U D(g) U^{-1} \right\} \Rightarrow D$ is real

$$U U^* = 1 \qquad \rfloor$$

$\square$

$$U = Z$$

Lemma if D is real, then $\mathcal{H}^{(n)} = \mathcal{H}^{(n)*}$

pf D is real, $\underline{Z D Z^{-1} = D^*}$ & $Z Z^* = 1$

$$D^*(g) \psi_i^* = \sum_j \psi_j D^*_{ji}(g)$$

$$= \sum_j \psi_j \sum_{k,l} Z_{jk} D_{kl}(g) Z^{-1}_{li}$$

$\times Z_{im}$ and $\sum_i$

$$\Rightarrow D^*(g) \left(\underline{\underline{\sum_i \psi_i^* Z_{im}}} \right) = \sum_k \left(\underline{\underline{\sum_j \psi_j^* Z_{jk}}} \right) \underline{\underline{D_{km}(g)}}$$

$$D(g) \psi_m = \sum_k \psi_k D_{km}(g)$$

$$\Rightarrow \sum_i \psi_i^* Z_{im} = \psi_m$$

$$\underbrace{\psi_i^* \in \mathcal{H}^{(n)*} \qquad \psi_i}_{\text{they are linear dep. by } \sum_i \psi_i^* Z_{\underline{\underline{im}}} = \psi_m} \in \mathcal{H}^{(n)}$$

$$\Rightarrow \mathcal{H}^{(n)} = \mathcal{H}^{(n)*}$$

above 2 lemmas $\Rightarrow$ $\mathcal{H}^{(n)} = \mathcal{H}^{(n)*}$ iff D is real

(no extra degeneracy) d

then if D is psudo-real then $\mathcal{H}^{(n)} \neq \mathcal{H}^{(n)*}$

$\Rightarrow$ extra degeneracy $2d$

if D is $\begin{cases} \text{real} & \text{degeneracy} = d = \dim(D) \\ \text{psudo-real} & \text{degeneracy} = 2d \\ \text{complex} & \text{degeneracy} = 2d \end{cases}$

Examples (1) 1d free particle $\hat{H} = \frac{1}{2m}\hat{p}^2$

$$\hat{p} = -i\hbar\frac{\partial}{\partial x}$$

symmetry : transl. inv. $Q(\lambda)\,x = x + \lambda \quad \lambda \in \mathbb{R}$

$$D(\lambda) = e^{-\frac{i}{\hbar}\lambda\hat{p}}$$

$$D(\lambda)\,\psi(x) = \psi(x+\lambda)$$

$G = \mathbb{R} = \{\lambda\}$ group multiplication : $+$: $\lambda_1 + \lambda_2$

irrep of $\mathbb{R}$: $D^{(k)}(\lambda) = e^{ik\lambda}$

$$\mathcal{H}^{(k)} = \mathbb{C} \qquad \dim(D^{(k)}) = 1$$

all irrep of G are 1-dim

$$D^{(k)}(\lambda_1)\, D^{(k)}(\lambda_2) = e^{ik(\lambda_1+\lambda_2)} = \underline{D^{(k)}(\lambda_1+\lambda_2)}$$

Complex conjugate of $D^{(k)}$: $D^{(k)}(\lambda)^* = e^{-ik\lambda}$

$\neq e^{ik\lambda}$

$$D^{(k)} \not\simeq D^{(k)*}$$

cannot transf between them by unitary on $\mathbb{C}$

Complex irrep.

energy level degeneracy $= 2 \dim(D^{(k)})$

$= 2$

eigenstates of H : $\hat{H}\psi_k = \frac{\hbar^2 k^2}{2m}\psi_k$

$$\psi_k = \underset{\substack{\cap \\ D^{(k)}}}{e^{ikx}}, \quad \psi_{-k} = \underset{\substack{\cap \\ D^{(k)*}}}{e^{-ikx}}$$

indeed degeneracy $= 2$

$$U e^{ik\lambda} U^{-1} = e^{-ikx}$$

$U \in \mathbb{C}$, $U e^{ik\lambda} U^{-1} = e^{ik\lambda}$

$$e^{ik\lambda} \neq e^{-ik\lambda} \Rightarrow D^{(k)} \not\simeq D^{(k)*}$$

(2) central potential : $\hat{H}\psi_{nlm}(\vec{r}) = E_{nl}\psi_{nlm}$

$l = 0, 1, 2 \cdots$

$m = -l, -l+1, \dots, l$

symmetry group $G = SO(3)$

eigenspace $\mathcal{H}^{(n,l)}$ relates to irrep of $SO(3)$

$\forall$ Rotation $Q(\alpha, \beta, \gamma) \in SO(3)$

↖ Euler angles

irrep of $SO(3)$ $\overset{D^{(l)}}{,}$ $\mathcal{H}^{(l)}$ is spanned by $Y_{lm}(\theta, \varphi)$
is labelled by l,

$m = -l, \dots l$

$(l = 0, 1, 2, 3 \dots)$ $\dim(\mathcal{H}^{(l)}) = 2l+1$

$$\langle l m | D^{(l)}(\alpha, \beta, \gamma) | l m' \rangle = \int_{S^2} d\theta d\varphi \sin\theta \, Y_{lm}^{*} \hat{D}^{l} Y_{lm'}$$

$$= D^{(l)}_{mm'}(\alpha, \beta, \gamma) = e^{-im'\alpha} d^{l}_{mm'}(\beta) e^{-im\gamma}$$

↑ Wigner D-function ↑ Wigner d-func

$d(0) = 1$

$$\chi^{l}(\alpha) = \operatorname{tr} D^{(l)}(\alpha, \beta, \gamma) = \operatorname{tr} D^{(l)}(\alpha, 0, 0)$$

$$= \sum_{m=-l}^{l} e^{-im\alpha} = \frac{\sin\left((l+\frac{1}{2})\alpha\right)}{\sin \alpha/2} \quad \text{real}$$

$\chi^{l*} = \chi^{l}$. D^{l} is not complex basis in $\mathcal{H}^{(l)}$: $Y_{lm}(\theta,\varphi)$, $Y^*_{lm}(\theta,\varphi) = Y_{l,-m}(\theta,\varphi)$

$$Y^*_{lm} \in \mathcal{H}^{(l)}$$

$$\Rightarrow \mathcal{H}^{(l)*} = \mathcal{H}^{(l)}$$

$\Rightarrow D^{(l)}$. $l=0,1\ldots$ are all real rep. , no extra degeneracy

$\mathcal{H}^{(n,l)}$ is the eigenspace , deg. $= 2l+1$

single spin-$\frac{1}{2}$ particle : $T^2 = -1$

<u>lemma</u> $\langle \psi | T\psi \rangle = 0$

pf. Let $\varphi = T\psi$ $\forall \psi \in \mathcal{H}^{(0)} \otimes \mathbb{C}^2$

$$\langle \psi | \varphi \rangle = \langle \psi | T\psi \rangle \underset{\substack{\uparrow \\ T \text{ is antiunitary}}}{=} \langle T\psi | T^2\psi \rangle^*$$

$$= -\langle T\psi | \psi \rangle^* = -\langle \psi | \varphi \rangle$$

$$\Rightarrow \langle \psi | \varphi \rangle = 0$$

ψ , $T\psi$ are orthogonal. $\square$

$\hat{H}\psi_i = E\psi_i$ $i = 1..d$ $\psi_i \in \mathcal{H}^{(d)}$ carry irrep $D^{(d)}$ of G

$\hat{H}(T\psi_i) = E(T\psi_i)$ $\varphi_i \equiv T\psi_i$

(1) $\langle \psi_i | \varphi_i \rangle = 0$

(2) $\langle \psi_i | \psi_j \rangle = \delta_{ij}$

(3) $\langle \varphi_i | \varphi_j \rangle = \delta_{ij}$

<u>Thm</u> (Kramer's Thm) single spin-$\frac{1}{2}$ particle, $\forall$ energy level degeneracy d' is always even and

$$d \leq d' \leq 2d$$

pf we have (1), (2), & (3) whether $\{\psi_i, \varphi_i\}$ form a complete basis

but it's possible $\langle \psi_i | \varphi_j \rangle \neq 0 \quad i \neq j$

if $|\varphi_k\rangle = \sum_l |\psi_l\rangle c_l$ the φ_k should be removed from the set of basis

but the $|\psi_k\rangle$ should be removed as well

since $T^2 = -1$

$$T|\varphi_k\rangle = \sum_l T(|\psi_l\rangle c_l)$$

$$T|\varphi_k\rangle = T^2|\psi_k\rangle = -|\psi_k\rangle$$

$$\sum_l T(|\psi_l\rangle c_l) = \sum_l c_l^* |\varphi_l\rangle$$

$$\Rightarrow |\psi_k\rangle = -\sum_l c_l^* |\varphi_l\rangle$$

$$\psi_1 \cdots \widehat{\psi_k} \cdots \psi_d \quad \varphi_1 \cdots \widehat{\varphi_k} \cdots \varphi_d$$

removed in pair.

find degeneracy $\quad d' = 2d - 2n$ (# of removed pairs) $\quad d'$ is even

$$d \leq d' \leq 2d$$

$\square$

Theory of angular momentum

spatial rotation & SO(3) group

finite rotation : $\quad Q$ 3x3 matrix s.t. $Q\vec{r}_1 \cdot Q\vec{r}_2 = \vec{r}_1 \cdot \vec{r}_2$

$$\Rightarrow Q^T Q = 1 \qquad Q \in O(3)$$

$$\det(Q^T Q) = 1$$

$$\det(Q)^2 = 1$$

$$\det Q = \pm 1$$

$SO(3)$: $Q^T Q = 1$ & $\det Q = 1$ $\qquad P = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

HW tutorial

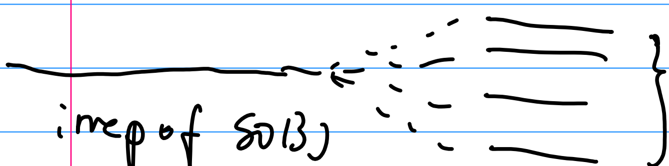
problem 1.
$$\hat{H} = \hat{H}_0 + V_{lattice}(\vec{r})$$

$\hat{H}_0$: central force, SO(3) symmetry

eigen space of H_0 : $\mathcal{H}_{l=0,1,\dots}$

carries irrep of SO(3)

$V_{lattice}$ breaks SO(3) to O (cubic lattice group)



irrep of SO(3)

$l = 0, 1 \dots$

irreps of O

$T_{i=1\dots5}$

$$\underset{\text{irrep of } SO(3)}{D^l} = \bigoplus_T \underset{\text{irreps of } O}{D_T}$$

$$\forall g \in O, \quad \chi_l(g) = \sum_T \chi_T(g)$$

character of O

$\chi_T(g)$	E	$8C_3$	$6C_2$	$2C_2$	$6C_4$
T_1	1	1	1	1	1
T_2	1	1	-1	1	-1
T_3	2	-1	0	2	0
T_4	3	0	-1	-1	1
T_5	3	0	1	-1	-1

character of SO(3) evaluated at elements in O

$\chi_l(g)$	E	$8C_3$	$6C_2$	$2C_2$	$6C_4$
$l=0$	1	1	1	1	1
$l=1$	3	0	-1	-1	1
$l=2$	5	-1	1	1	-1
$l=3$	7	1	-1	-1	-1
$l=4$	9	0	1	1	1

- $l=0$: $D^{l=0} = T_1$
- $l=1$ $D^{l=1} = T_4$
- $l=2$: $D^{l=2} = T_3 \oplus T_5$ $\qquad \chi_{l=2}(g) = \chi_{T_3}(g) + \chi_{T_5}(g)$ $\quad \forall g \in O$

check : E : $\chi_{l=2}(E) = 5$

$$\chi_{T_3}(E) + \chi_{T_5}(E) = 2+3 = 5$$

$8C_3$: $\chi_2(8C_3) = -1$

$$\chi_{T_3}(8C_3) + \chi_{T_5}(8C_3) = -1 + 0 = -1$$

$\underline{D_l = \bigoplus_T D_T}$ $\qquad \forall g \in O$, $D_l(g) = \bigoplus_T D_T(g)$

$$\begin{pmatrix} \boxed{D_{T_1}(g)} & & 0 \\ & \boxed{D_{T_2}(g)} & \\ 0 & & \boxed{D_{T_3}(g)} \end{pmatrix} \begin{vmatrix} - \\ - \\ - \end{vmatrix}$$

$$\chi_l(g) := \mathrm{tr}\, D_l(g) = \sum_T \mathrm{tr}\, D_T(g) = \sum_T \chi_T(g)$$

$6C_2$ $\chi_{l=2}(6C_2) = 1$

$\chi_{T_3}(6C_2) + \chi_{T_5}(6C_2) = 0 + 1 = 1$

$2C_2$: $\chi_{l=2}(2C_2) = 1$

$\chi_{T_3}(2C_2) + \chi_{T_5}(2C_2) = 2 + (-1) = 1$

$6C_4$: $\chi_{l=2}(6C_4) = -1$

$\chi_{T_3}(6C_4) + \chi_{T_5}(6C_4) = 0 + (-1) = -1$

- $l=3$ $D_3 = T_2 \oplus T_4 \oplus T_5$

E: $\chi_3 = 7$

$\chi_{T_2} + \chi_{T_4} + \chi_{T_5} = 1 + 3 + 3 = 7$

$8C_3$: $\chi_3 = 1$

$\chi_{T_2} + \chi_{T_4} + \chi_{T_5} = 1 + 0 + 0 = 1$

$6C_2$: $\chi_3 = -1$

$\chi_{T_2} + \chi_{T_4} + \chi_{T_5} = -1 + (-1) + 1 = -1$

$3C_2$: $\chi_3 = -1$

$\chi_{T_2} + \chi_{T_4} + \chi_{T_5} = 1 + (-1) + (-1) = -1$

$6C_4$: $\chi_3 = -1$

$\chi_{T_2} + \chi_{T_4} + \chi_{T_5} = -1 + 1 + (-1) = -1$

• $l=4$ $\quad D_4 = T_1 \oplus T_3 \oplus T_4 \oplus T_5$

$E:$ $\quad \chi_4 = 9$

$$\chi_{T_1} + \chi_{T_3} + \chi_{T_4} + \chi_{T_5} = 1+2+3+3 = 9$$

$8C_3:$ $\quad \chi_4 = 0$

$$\chi_{T_1} + \chi_{T_3} + \chi_{T_4} + \chi_{T_5} = 1+(-1)+0+0 = 0$$

$6C_2:$ $\quad \chi_4 = 1$

$$\chi_{T_1} + \chi_{T_3} + \chi_{T_4} + \chi_{T_5} = 1+0+(-1)+1 = 1$$

$3C_2:$ $\quad \chi_4 = 1$

$$\chi_{T_1} + \chi_{T_3} + \chi_{T_4} + \chi_{T_5} = 1+2+(-1)+(-1) = 1$$

$6C_4:$ $\quad \chi_4 = 1$

$$\chi_{T_1} + \chi_{T_3} + \chi_{T_4} + \chi_{T_5} = 1+0+1+(-1) = 1$$

$$\vec{r} = \sum_{i=1}^{3} r_i \vec{e}_i \qquad \text{choose basis } \vec{e}_{i=1,2,3}$$

$$\underline{\vec{r}\,' = Q\vec{r} \equiv \sum_{i=1}^{3} r_i' \vec{e}_i}$$

Rotation of basis $\quad \underline{\vec{r}\,' = Q\vec{r}} = \sum_{i=1}^{3} r_i Q \vec{e}_i = \underline{\sum_{i=1}^{3} r_i \vec{e}_i\,'}$

$$\vec{e}_i{}' = Q\vec{e}_i = \sum_j \vec{e}_j \, (\vec{e}_j \cdot \underbrace{Q\vec{e}_i}_{\vec{e}_i{}'}) = \sum_{j=1}^{3} \vec{e}_j \, Q_{ji}$$

$$\vec{r}\,' = \sum_{i=1}^{3} r_i{}' \, \underline{\underline{\vec{e}_i}}$$

$$\shortparallel$$

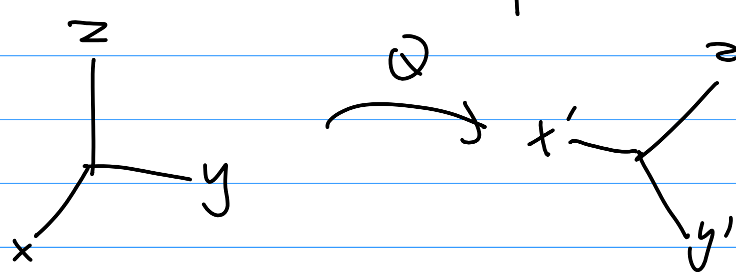
$$\sum_{i=1}^{3} r_i \sum_{j=1}^{3} \vec{e}_j \, Q_{ji} = \sum_{j=1}^{3} \left(\sum_{i=1}^{3} Q_{ji} r_i \right) \vec{e}_j$$

$$r_j{}' = \sum_{i=1}^{3} Q_{ji} \, r_i$$

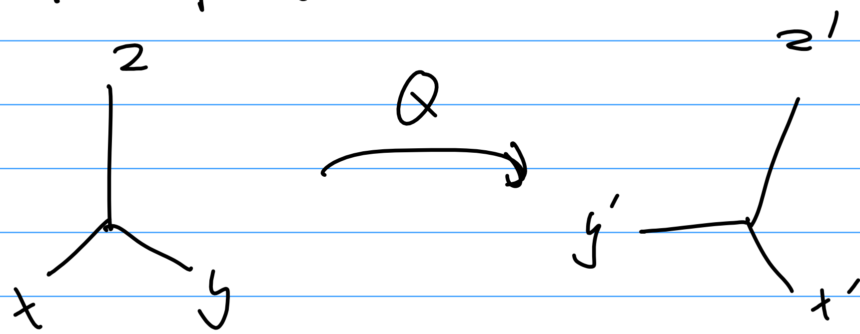
Parity $P : \vec{r} \to -\vec{r}$ $\quad P_{ij} = -\delta_{ij}$ $\quad \det P = -1$

$P \notin SO(3)$ $\quad P \in O(3)$

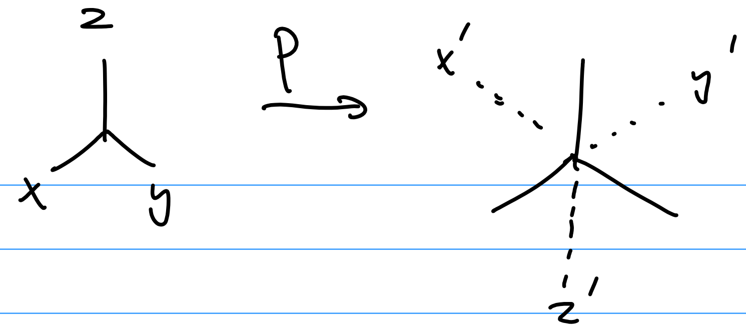
$Q \in SO(3)$ transf. right-hand frame to right-hand frame



$Q \notin SO(3)$, $Q \in O(3)$ transf. right-hand frame to left-hand frame



P:



Proof: Let $\vec{z} = \vec{x} \times \vec{y}$ we shall show that

$$Q\vec{x} \times Q\vec{y} = Q\vec{z} \quad \forall\, Q \in SO(3)$$

$$Q\vec{x} \times Q\vec{y} = -Q\vec{z} \quad \forall\, Q \notin SO(3),\ Q \in O(3)$$

$\forall$ vector $\vec{u} \in \mathbb{R}^3$

$$Q\vec{u} \cdot (Q\vec{x} \times Q\vec{y}) = \sum_{ijk} \varepsilon_{ijk} (Q\vec{u})_i (Q\vec{x})_j (Q\vec{y})_k$$

$$= \sum_{\substack{ijk \\ \alpha\beta\gamma}} \varepsilon_{ijk} Q_{i\alpha} u_\alpha Q_{j\beta} x_\beta Q_{k\gamma} y_\gamma$$

$$= \sum_{\alpha\beta\gamma} \underbrace{\left(\sum_{ijk} \varepsilon_{ijk} Q_{i\alpha} Q_{j\beta} Q_{k\gamma} \right)}_{\varepsilon_{\alpha\beta\gamma} \det Q} u_\alpha x_\beta y_\gamma$$

$$= \det Q \sum_{\alpha\beta\gamma} \varepsilon_{\alpha\beta\gamma} u_\alpha x_\beta y_\gamma$$

$$= \det Q \, (\vec{u} \cdot (\vec{x} \times \vec{y}))$$

$$= \det Q \, (\vec{u} \cdot \vec{z}) = \det Q \, (Q\vec{u} \cdot Q\vec{z})$$

$$Q\vec{x} \times Q\vec{y} = \underbrace{\det Q}_{\substack{\shortparallel \\ \pm 1}} \left(Q \vec{z} \right)$$

all rotations of rigid body are SO(3)

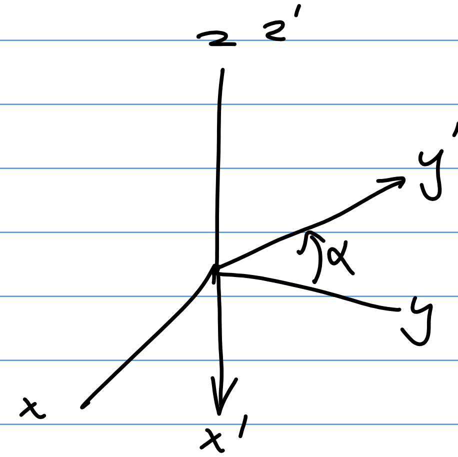
$\forall\, Q \in SO(3)$ can be composed by 2 types of simple rotations

$Q(\hat{k}, \alpha)$, $Q(\hat{j}, \beta)$ $\quad$ $\hat{i}, \hat{j}, \hat{k}$ basis of x, y, z axis

rotation around z-axis $\quad$ rotation around y-axis

$$Q(\hat{k}, \alpha) = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \vec{e}_i' = \sum_j \vec{e}_j Q_{ji}$$

$$\hat{i} \to \hat{i}' = \hat{i}\cos\alpha + \hat{j}\sin\alpha$$
$$\hat{j} \to \hat{j}' = \hat{i}(-\sin\alpha) + \hat{j}\cos\alpha$$
$$\hat{k} \to \hat{k}' = \hat{k}$$



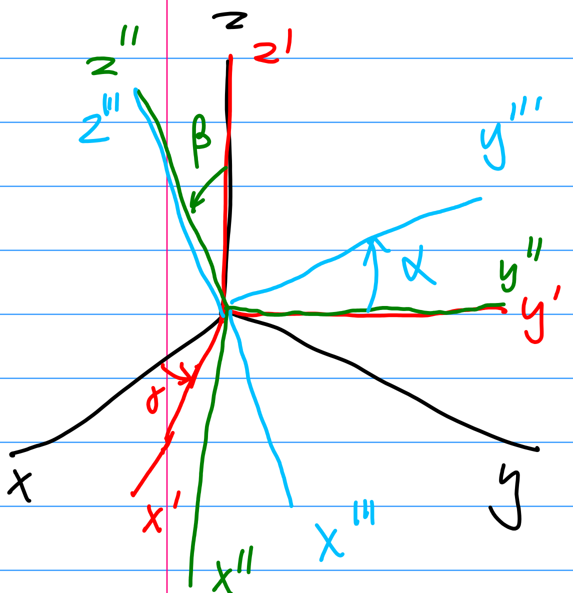
$$Q(\hat{j}, \beta) = \begin{pmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{pmatrix}$$

Euler angles : (1) rotation around z : γ } compose any rotations

give a basis (2) rotation around y : β

$\hat{i}, \hat{j}, \hat{k}$: (3) rotation around z : α

$$\alpha \in [0, 2\pi), \quad \beta \in [0, \pi], \quad \gamma \in [0, 2\pi)$$



$$Q(\alpha, \beta, \gamma)$$

$$= Q(\hat{k}, \alpha)\, Q(\hat{j}, \beta)\, Q(\hat{k}, \gamma)$$

$$\vec{e}_i' = \sum_j \vec{e}_j\, Q_{ji}(\alpha\,\beta\,\gamma)$$

$$Q(\alpha, \beta\, \gamma) = \begin{pmatrix} \cos\alpha\cos\beta\cos\gamma - \sin\alpha\sin\gamma & -\cos\alpha\cos\beta\sin\gamma - \sin\alpha\cos\gamma & - \\ \cdot & \cdot\cdot & \cdot \\ \cdot & \cdot\,\cdot & \cdot\cdot \end{pmatrix}$$

$\forall\, Q(\alpha, \beta, \gamma) \in SO(3)$

all group elements in SO(3) can be parametrized by 3 Euler angles

SO(3) is a 3-dim Lie group

SO(3) and SU(2)

in QM, it's better to work with SU(2), reps of SU(2)

since 1) {irreps of SU(2)} ⊃ {irreps of SO(3)}

2) we have spin.

SU(2) group: special unitary transf. on $\mathbb{C}^2$

$$SU(2) \ni u = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix}, \quad \det(u) = aa^* + bb^* = 1$$

$u^\dagger = u^{-1}$ $\quad a, b \in \mathbb{C}$ $\quad$ 2 complex = 4 reals

1 constraint.

3 real parameters

$\Rightarrow$ SU(2) 3 dim Lie group

$aa^* + bb^* = 1$ we write $a = \cos\eta \, e^{i\xi}$

$b = -\sin\eta \, e^{-i\zeta}$

$$u = \begin{pmatrix} \cos\eta \, e^{-i\xi} & -\sin\eta \, e^{-i\zeta} \\ \sin\eta \, e^{i\zeta} & \cos\eta \, e^{i\xi} \end{pmatrix} = u(\underbrace{\eta, \xi, \zeta}_{\text{3 real parameters}})$$

$\xi, \zeta \in [0, 2\pi)$ $\quad \eta \in [0, \frac{\pi}{2}]$

homomorphism between SU(2) and SO(3)

$$\vec{r} \in \mathbb{R}^3 \quad \rightarrow \quad \vec{r}\cdot\vec{\sigma} = \begin{pmatrix} z & x-iy \\ x+iy & -z \end{pmatrix}$$

$\vec{r} = (x, y, z)$, $\quad \vec{r}\cdot\vec{\sigma} \equiv h$

$$\det(h) = x^2+y^2+z^2 = |\vec{r}|^2$$

$u \in SU(2)$, view h an operator on $\mathbb{C}^2$

$$h \rightarrow h' = \underline{u h u^{-1} \equiv \vec{r}'\cdot\vec{\sigma}} = \begin{pmatrix} z' & x'-iy' \\ x'+iy' & -z' \end{pmatrix}$$

$$\det(h') = \det(u h u^{-1}) = \det(h)$$

$\det(h') = |\vec{r}'|^2$, $\quad \det(h) = |\vec{r}|^2$

This action by u from $\vec{r}$ to $\vec{r}'$ corresponds to a rotation $Q(u) \in SO(3)$, s.t. $Q(u)\vec{r} = \vec{r}'$

$$u(\vec{\sigma}\cdot\vec{r})u^{-1} = \vec{\sigma}\cdot(Q(u)\vec{r})$$

we find a map from $SU(2)$ to $SO(3)$

homomorphism : $\forall u_1, u_2 \in SU(2)$

$$\begin{aligned}(u_1u_2)(\vec{\sigma}\cdot\vec{r})(u_1u_2)^{-1} &= u_1u_2(\vec{\sigma}\cdot\vec{r})u_2^{-1}u_1^{-1} \\ &= u_1(\vec{\sigma}\cdot Q(u_2)\vec{r})u_1^{-1} \\ &= \vec{\sigma}\cdot(Q(u_1)Q(u_2)\vec{r})\end{aligned}$$

$\forall$ rep D of $SO(3)$

$$SU(2) \xrightarrow[\text{hom}]{i} SO(3) \xrightarrow[\text{hom}]{D} L(\mathcal{H})$$

$D \circ i$ is a rep of $SU(2)$

$$u(\vec{\sigma}\cdot\vec{r})u^{-1} = \vec{\sigma}\cdot(Q(u)\vec{r})$$

$$u = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix}$$

$$\underline{aa^* + bb^* = 1}$$

$\downarrow$ $u \to -u \in SU(2)$

$$(-u)(\vec{\sigma}\cdot\vec{r})(-u)^{-1}$$

$$-u = \begin{pmatrix} -a & -b \\ b^* & -a^* \end{pmatrix}$$

$$\underline{aa^* + bb^* = 1}$$

$$\begin{array}{ccc} SU(2) & & SO(3) \\ u & \mapsto & \\ & & Q(u) \\ -u & \mapsto & \end{array}$$

2-to-1 homomorphism

"$SU(2)$ is double covering group of $SO(3)$"

Example (1) $u_1(\alpha) = \begin{pmatrix} e^{-i\alpha/2} & 0 \\ 0 & e^{i\alpha/2} \end{pmatrix}$

$$h = \vec{r}\cdot\vec{\sigma} = \begin{pmatrix} z & x-iy \\ x+iy & -z \end{pmatrix}$$

$$h' = u_1(\alpha)\, h\, u_1(\alpha)^{-1} = \begin{pmatrix} z & (x-iy)e^{-i\alpha} \\ (x+iy)e^{i\alpha} & -z \end{pmatrix}$$

$\parallel$

$$\vec{\sigma}\cdot\vec{r}' = \begin{pmatrix} z' & x'-iy' \\ x'+iy' & -z' \end{pmatrix} \Rightarrow \begin{cases} x' = x\cos\alpha - y\sin\alpha \\ y' = x\sin\alpha + y\cos\alpha \\ z' = z \end{cases}$$

$$\vec{r}' = \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \underbrace{\begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{Q(u_1(\alpha)) \;=\; Q(\hat{k},\alpha)} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$\pm u_1(\alpha) \longmapsto Q(\hat{k}, \alpha)$ rotation around z-axis

(2) $u_2(\beta) = \begin{pmatrix} \cos\beta/2 & -\sin\beta/2 \\ \sin\beta/2 & \cos\beta/2 \end{pmatrix} \in SU(2)$

<u>HW</u> show $\pm u_2(\beta) \longmapsto Q(\hat{j}, \beta)$ rotation around y-axis

$$u(\alpha, \beta, \gamma) = u_1(\alpha)\, u_2(\beta)\, u_1(\gamma)$$

$$\downarrow$$

$$Q(\alpha, \beta, \gamma) = Q(\hat{k}, \alpha)\, Q(\hat{j}, \beta)\, Q(\hat{k}, \gamma)$$

α, β, γ are Euler angles

$$u(\alpha, \beta, \gamma) = \begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}} \cos\frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}} \sin\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}} \sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}} \cos\frac{\beta}{2} \end{pmatrix}$$

compare with 3.4.3

$$u = \begin{pmatrix} \cos\eta \, e^{-i\xi} & -\sin\eta \, e^{-i\zeta} \\ \sin\eta \, e^{i\zeta} & \cos\eta \, e^{i\xi} \end{pmatrix}$$

$$\alpha = \xi + \zeta \, , \quad \beta = 2\eta \, , \quad \gamma = \xi - \zeta$$

$$\xi, \zeta \in [0, 2\pi) \, , \quad \eta \in [0, \tfrac{\pi}{2}]$$

$$\Rightarrow \alpha \in [0, 4\pi) \, , \quad \beta \in [0, \pi] \, , \quad \gamma \in [0, 2\pi)$$

$\uparrow$

relate to 2-to-1 homomorphism $SU(2) \to SO(3)$

$$\Big(\mathrm{Vol}(SU(2)) = 2 \, \mathrm{Vol}(SO(3)) \Big)$$

<u>Reps of SU(2)</u>

fundamental rep of $SU(2)$ (defining rep) on $\mathbb{C}^2$

(spin Hilbert space)

$$v = \begin{pmatrix} \xi \\ \eta \end{pmatrix} \in \mathbb{C}^2$$

$$v' = \begin{pmatrix} \xi' \\ \eta' \end{pmatrix} = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} a\xi + b\eta \\ -b^*\xi + a\eta \end{pmatrix}$$

$$\xi \to \xi' = a\xi + b\eta$$

$$\eta \to \eta' = -b^*\xi + a^*\eta$$

homogeneous polynomials of ξ, η

basis

degree-0 $\quad 1 \iff \xi^0 \eta^0$

degree-1 $\quad \xi, \eta \iff \xi^{\frac{1}{2}+\frac{1}{2}} \eta^{\frac{1}{2}-\frac{1}{2}}, \ \xi^{\frac{1}{2}-\frac{1}{2}} \eta^{\frac{1}{2}+\frac{1}{2}}$

degree-2 $\quad \xi^2, \xi\eta, \eta^2 \iff \underline{\xi^{1+1}\eta^{1-1}, \ \xi^{1+0}\eta^{1+0}, \ \xi^{1-1}\eta^{1+1}}$

$\vdots$

degree-2j $\quad \xi^{j+j}\eta^{j-j}, \ \xi^{j+(j-1)}\eta^{j-(j-1)} \cdots$

$\xi^{j+m}\eta^{j-m} \ \cdots \ \xi^{j-j}\eta^{j+j}$

$$m = j, j-1, j-2, \ldots -j$$

$$\underline{\xi^{j+j}\eta^{j-j}} \sim \underline{|j, j\rangle}$$

$$\vdots \qquad\qquad \vdots$$

$$\xi^{j+m}\eta^{j-m} \sim |j, m\rangle$$

$$\vdots \qquad\qquad \vdots$$

$$\xi^{j-j}\eta^{j+j} \sim |j, -j\rangle$$

any degree-2j homogeneous polynomial $\displaystyle\sum_{m=-j}^{j} a_m \underline{\underline{\xi^{j+m}\eta^{j-m}}}$

$$\sim \sum_{m=-j}^{j} a_m |j, m\rangle$$

span $2j$-dim vector space $\mathcal{H}_j$. rep of $SU(2)$

$SU(2)$ action : $\displaystyle\sum_{m=-j}^{j} a_m \xi^{j+m} \eta^{j-m}$

$$\rightarrow \sum_{m=-j}^{j} a_m \underline{(a\xi + b\eta)}^{j+m} \underline{(-b^*\xi + a^*\eta)}^{j-m}$$

still homogeneous polynormial of $2j$

basis vector $\qquad f_m^j(\xi,\eta) = \dfrac{\xi^{j-m}\eta^{j+m}}{\sqrt{(j-m)!(j+m)!}} \quad \leftarrow$

$$m = -j, -j+1 \ldots j$$

$\{f_m^j\}_{m=-j}^{j}$ span $\mathcal{H}_j$ irrep. of $SU(2)$

$\forall u \in SU(2) \qquad D_j(u) f_m^j(\xi,\eta) := f_m^j\left(u^{-1}\begin{pmatrix}\xi\\ \eta\end{pmatrix}\right)$

$$u^{-1}\begin{pmatrix}\xi\\ \eta\end{pmatrix} = u^{+}\begin{pmatrix}\xi\\ \eta\end{pmatrix} = \begin{pmatrix} a^* & -b \\ b^* & a\end{pmatrix}\begin{pmatrix}\xi\\ \eta\end{pmatrix} = \begin{pmatrix} a^*\xi - b\eta \\ b^*\xi + a\eta\end{pmatrix}$$

$$D_j(u) f_m^j = \frac{(a^*\xi - b\eta)^{j-m}(b^*\xi + a\eta)^{j+m}}{\sqrt{(j-m)!(j+m)!}}$$

$$= \sum_{m'=-j}^{j} f_{m'}^j D_{m'm}^j(u)$$

rep. matrix $\qquad (2j+1)\times(2j+1)$ Wigner D-matrix derive $D^j_{mm'}(u)$

$$(x+y)^n = \sum_{r=0}^{n} \frac{n!}{r!\,(n-r)!} x^r y^{n-r}$$

$$D_j(u) f^j_m = \sum_{n=0}^{j-m} \sum_{m'=n-j}^{n+m} (-1)^n \frac{\sqrt{(j-m)!\,(j+m)!\,(j-m')!\,(j+m')!}}{n!\,(j-m-n)!\,(n+m-m')!\,(j+m'-m)!}$$

$$(a^*)^{j-m-n} b^n (b^*)^{n+m-m'} a^{j+m'-m} f^j_{m'}$$

$$\sum_{n=0}^{j-m} \sum_{m'=n-j}^{n+m} (\cdots) = \sum_{m'=-j}^{j} \sum_{n} (\cdots)$$

over all integers s.t. for all $(\cdots)!$

non negative.

$$D^j_{m'm}(a,b) = \sum_n (-1)^n \frac{\sqrt{(j-m)!\,(j+m)!\,(j-m')!\,(j+m')!}}{(j+m'-n)!\,(j-m-n)!\,n!\,(n+m-m')!}$$

$$(a^*)^{j-m-n} a^{j+m'-n} b^n (b^*)^{n+m-m'}$$

$$j = 0, \tfrac{1}{2}, 1 \ldots$$

$$m, m' = -j \ldots j$$

<u>properties of D-matrix</u>

1) Unitary : <u>Lemma</u> : $D(u) D(u)^{\dagger} = 1 \quad \forall u \in SU(2)$

i.e. $\underline{\sum_m D^j_{m_1 m}(u) D^j_{m_2 m}(u)^* = \delta_{m_1 m_2}}$

Pf: let's look at $\sum_m f^j_m(\xi,\eta)^* f^j_m(\xi,\eta)$

$$= \sum_{m=-j}^{j} \frac{(\xi^*\xi)^{j-m}(\eta^*\eta)^{j+m}}{(j-m)!\,(j+m)!}$$

$$= \frac{(\xi^*\xi+\eta^*\eta)^{2j}}{(2j)!} \leftarrow SU(2) \text{ inv.}$$

$\forall\, u \in SU(2)$

$$u\begin{pmatrix}\xi\\ \eta\end{pmatrix} = \begin{pmatrix}\xi'\\ \eta'\end{pmatrix} \qquad (\xi',\eta')^*\begin{pmatrix}\xi'\\ \eta'\end{pmatrix} = \xi'^*\xi' + \eta'^*\eta'$$

$$= (\xi.\eta)\,u^\dagger u\begin{pmatrix}\xi\\ \eta\end{pmatrix}$$

$$= (\xi\ \eta)\begin{pmatrix}\xi\\ \eta\end{pmatrix} = \xi^*\xi+\eta^*\eta$$

$$\sum_m \left(D_j(u) f^j_m\right)^*\left(D_j(u) f^j_m\right) = \underline{\underline{\sum_m f^{j*}_m f^j_m}}$$

$$\parallel$$

$$\underline{\sum_{m'm''} \underline{\underline{f^{j*}_{m''} f^j_{m'}}} \left(\sum_m D^j_{m''m}(u)^* D^j_{m'm}(u)\right)}$$

if $f^{j*}_{m''} f^j_{m'} \equiv b^j_{m''m'}$ are linear indep

then $\underline{\sum_m D^j_{m'm}(u)\, D^j_{m''m}(u)^* = \delta_{m''m'}}$

$$DD^\dagger = 1$$

To prove they are linearly indep.

solve eqn $\sum_{m'',m'} C_{m''m'} f^{j*}_{m''} f^{j}_{m'} = 0$

show $C_{m''m'} = 0 \quad \forall\, m', m'' = -j \cdots j$

HW

$\square$

Lemma D_j is irreducible.

Pf. Recall Schur's lemma; There is no matrix commuting with all $D_j(u)$ except $\lambda \mathbb{1}$, then D_j is irrep

assume $\exists\, M$ s.t. $M D^j(a,b) = D^j(a,b) M \quad \forall\, u = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix}$

choose $a = e^{i\alpha/2}$, $b = 0$, $\alpha \in [0, 4\pi]$, $D^j_{m'm}(e^{i\alpha/2}, 0)$

$= \underset{\text{const.}}{C}\ \underline{\delta_{m'm} e^{im\alpha}}$

$$\sum_{m'} M_{mm'} D^j_{m'm''}(e^{i\alpha/2}, 0)$$

$$= \sum_{m'} D^j_{mm'}(e^{i\alpha/2}, 0) M_{m'm''}$$

$\Leftrightarrow M_{mm''} e^{im''\alpha} = M_{mm''} e^{im\alpha} \Leftrightarrow M_{mm''}(e^{im''\alpha} - e^{im\alpha}) = 0$

$\Rightarrow M_{mm''} = 0$ if $m \neq m''$ i.e. $M_{mm''} = M_m \delta_{mm''}$

for other u s.t. D^j has nonzero off-diagonals

$MD^j = D^j M \Leftrightarrow M_m D_{mm''} = D_{mm''} M_{m''}$

$\Rightarrow M_m = M_{m''} \Rightarrow M \propto \mathbb{1}$

$\Rightarrow D^j$ is irreducible $\square$

We have classified all irreps of $SU(2)$

D-matrix in terms of Euler angles

$$u(\alpha,\beta,\gamma) = \begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \end{pmatrix}$$

$$a = e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \qquad b = -e^{-i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2}$$

$$D^j_{mm'}(\alpha,\beta,\gamma) = D^j_{mm'}\left(a = e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2},\ b = -e^{-i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2}\right)$$

$$\alpha \in [0,4\pi),\quad \beta \in [0,\pi],\quad \gamma \in [0,2\pi)$$

$$\alpha \to \alpha + 2\pi \quad \Rightarrow \quad a \to -a \quad b \to -b$$

$$u = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix} \longrightarrow u(-a,-b) = -u(a,b)$$

$$u(\vec{r}\cdot\vec{\sigma})u^{-1} = \vec{\sigma}\cdot Q(u)\vec{r} \qquad \begin{matrix} u \in SU(2) \\ Q(u) \in SO(3) \end{matrix}$$

$$\underbrace{u(a,b) \qquad u(-a,-b)}_{\text{the same element in } SO(3)}$$

$$D^j_{mm'}(-a,-b) = (-1)^{2j} D^j_{mm'}(a,b)$$

if j is integer

$$\begin{tikzcd} & u(\alpha,\beta,\gamma) \arrow[dr] & \\ \begin{matrix} SO(3) \text{ rotation} \\ Q(\alpha,\beta,\gamma) \\ \alpha \in [0,2\pi) \end{matrix} \arrow[ur] \arrow[dr] & & D^j_{mm'}(\alpha,\beta,\gamma) \\ & \begin{matrix} -u(\alpha,\beta,\gamma) \\ \shortparallel \\ u(\alpha+2\pi,\beta,\gamma) \end{matrix} \arrow[ur] & \end{tikzcd}$$

1-to-1 map between $Q(\alpha,\beta,\gamma)$ and $D^j(\alpha,\beta,\gamma)$

if j is half-integer

$$\text{SO(3) rotation } Q(\alpha,\beta,\gamma),\quad \alpha\in[0,2\pi)$$

$$u(\alpha,\beta,\gamma) \longrightarrow D^j_{mm'}(\alpha,\beta,\gamma)$$

$$u(\alpha+4\pi,\beta,\gamma) \longrightarrow D^j_{mm'}(\alpha+4\pi,\beta,\gamma)$$

not the same

1-to-2 map between $Q(\alpha,\beta,\gamma)$ and $D^j(\alpha,\beta,\gamma)$

double valued rep of SO(3)

HW compute $D^{j=1/2}(\alpha,\beta,\gamma)$ and $D^{j=1}(\alpha,\beta,\gamma)$

$\forall\, u_{2\times 2}\in SU(2)$

$$u = u_0\mathbb{1} + \sum_{i=1}^{3} u_i\sigma_i = \begin{pmatrix} u_0+u_3 & u_1-iu_2 \\ u_1+iu_2 & u_0-u_3\end{pmatrix}$$

$$\det(u) = u_0^2 - \sum_{i=1}^{3} u_i^2 = 1 \qquad u^+ = u^{-1}$$

$$\Rightarrow u_0\in\mathbb{R},\quad u_{i=1,2,3}\in i\mathbb{R}$$

$$= u_0^2 + \sum_{i=1}^{3} v_i^2 = 1 \qquad u_{i=1,2,3} = i\,v_{i=1,2,3}$$

$$\begin{pmatrix} u_0\\ v_1\\ v_2\\ v_3\end{pmatrix} \qquad u_0 = \cos\theta/2$$

$$\vec{u} = (u_1\,u_2\,u_3) = i\,\vec{n}\sin\theta/2 \qquad \vec{n}\cdot\vec{n}=1$$

$$u = \cos\frac{\theta}{2}\mathbb{1} + i\vec{n}\cdot\vec{\sigma}\sin\frac{\theta}{2} = \underline{\underline{e^{i\theta\vec{n}\cdot\vec{\sigma}/2}}}$$

$\Rightarrow \forall\, u\in SU(2)$ $\exists$ an angle $\theta\in[0,4\pi)$, and unit vector $\vec{n}$ (axis)

s.t. $\underline{u = e^{i\theta\vec{n}\cdot\vec{\sigma}/2}}$

conjugacy classes of SU(2):

$\forall\, u_1\in SU(2)$

$$u\to u_1 u u_1^{-1} = u_1 e^{i\theta\vec{n}\cdot\vec{\sigma}/2} u_1^{-1}$$

$$= e^{i\theta u_1(\vec{n}\cdot\vec{\sigma})u_1^{-1}/2}$$

recall $u_1(\vec{n}\cdot\vec{\sigma})u_1^{-1} = \vec{\sigma}\cdot(Q(u_1)\vec{n})$

$$u_1 u u_1^{-1} = e^{i\theta\vec{\sigma}\cdot(Q(u_1)\vec{n})/2}$$

rotation around new axis $Q(u_1)\vec{n}$, w/ same angle

$Q(u_1)\vec{n}$ $\forall u_1$ cover all directions in $\mathbb{R}^3$

A conjugacy class $\longleftrightarrow$ an angle θ

$\uparrow$

contain all axises

character of $SU(2)$: $\chi_j(u) \leftarrow$ class function , $\chi_j(u) = \chi_j(\theta)$

to compute $\chi_j(\theta)$: choose axis to be $\hat{k}$ (z-direction) : $Q(\hat{k}, \theta)$

$$D^j_{mm'}(\theta, 0, 0) = e^{-im'\theta}\delta_{m'm}$$

$$\chi_j(\theta) = \mathrm{tr}\, D^j(\theta,0,0) = \sum_{m=-j}^{j} e^{-im\theta} = \frac{\sin[(j+\frac{1}{2})\theta]}{\sin\theta/2}$$

Thm Orthogonality of D and χ

integral on $SU(2)$

$$\frac{1}{16\pi^2}\int_0^{4\pi} d\alpha \int_0^{\pi} d\beta \sin\beta \int_0^{2\pi} d\gamma\, D^j_{mn}(\alpha,\beta,\gamma)^* D^{j'}_{m'n'}(\alpha,\beta,\gamma)$$

$$= \delta_{jj'}\delta_{mm'}\delta_{nn'}\frac{1}{2j+1}$$

$$d\mu_H = \frac{1}{16\pi^2} d\alpha\, d\gamma\, d\beta \sin\beta$$ Haar measure of $SU(2)$

$$\int_0^{4\pi} d\theta \cdot (1-\cos\theta)\, \chi^j(\theta)\,\chi^{j'}(\theta) = \pi\,\delta_{jj'}$$

$$D^j(\alpha,\beta,\gamma) = D^j\big(u_1(\alpha)\, u_2(\beta)\, u_1(\gamma)\big)$$

$$= D^j(u_1(\alpha))\, D^j(u_2(\beta))\, D^j(u_1(\gamma))$$

$$D^j_{m'm}(\alpha,\beta,\gamma) = \sum_{n,n'} D^j_{m'n'}(u_1(\alpha))\, D^j_{n'n}(u_2(\beta))\, D^j_{nm}(u_1(\gamma))$$

$$u_1(\alpha) = \begin{pmatrix} e^{-i\alpha/2} & 0 \\ 0 & e^{i\alpha/2}\end{pmatrix} = e^{-i\frac{\alpha}{2}\sigma_3}$$

$$D^j_{mn}(u_1(\alpha)) = e^{-im\alpha}\delta_{mn}$$

$$D^j_{m'm}(\alpha,\beta,\gamma) = e^{-im'\alpha}\, d^j_{m'm}(\beta)\, e^{-im\gamma}$$

$$d^j_{m'm}(\beta) = D^j_{m'm}(u_2(\beta)) \quad \text{Wigner } d\text{-matrix}$$

SU(2) irreps and angular momentum

Recall group of unitary operators $D(\vec{n}.\varphi) = e^{-\frac{i}{\hbar}\varphi\vec{n}.\vec{J}}$

$\hat{\vec{J}} = \hat{\vec{L}} + \hat{\vec{S}}$ total angular momentum

$D(\vec{n},\varphi)$ is unitary rep of SU(2)

$$u(\vec{n}.\varphi) = e^{-i\varphi\vec{n}.\vec{\sigma}/2} \xrightarrow[\text{homomorphism}]{} D(\vec{n},\varphi) = e^{-\frac{i}{\hbar}\varphi\vec{n}.\vec{J}}$$

$\mathbb{C}^2$ $\qquad\qquad\qquad\qquad$ $\mathcal{H}$

$$\hat{\vec{S}} = \frac{\hbar}{2}\vec{\sigma}$$

$$u(\vec{n}.\varphi) = e^{-\frac{i}{\hbar}\varphi\vec{n}.\hat{\vec{S}}}$$

$$\underline{D(\vec{n},\varphi)}\, f^j_m = \sum_{m'} f^j_{m'} D^j_{m'm}(\vec{n}.\varphi) \qquad f^j_m = \frac{\xi^{j-m}\eta^{j+m}}{(j-m)!\,(j+m)!}$$

$m = -j, -j+1 \dots j$

(1) take $\vec{n} = \hat{k}$ (z-axis) $\quad D^j_{m'm}(\hat{k},\varphi) = \delta_{m'm} e^{-im\varphi}$ ← check by yourselves

$$D(\hat{k}.\varphi) = e^{-\frac{i}{\hbar}\varphi\hat{J}_z}$$

$$e^{-\frac{i}{\hbar}\varphi\hat{J}_z} f^j_m = e^{-im\varphi} f^j_m$$

$$\xrightarrow{\left.\frac{d}{d\varphi}\right|_{\varphi=0}} \quad -\frac{i}{\hbar}\hat{J}_z f^j_m = -im\, f^j_m \longrightarrow \hat{J}_z f^j_m = \hbar m\, f^j_m$$

(2) take $\vec{n} = \hat{j}$ (y-axis) $\quad D^j_{m'm}(0,\varphi,0)$

↖ in terms of Euler angles

check by yourselves

$$\frac{d}{d\varphi} D^j_{m'm}(0,\varphi,0)\Big|_{\varphi=0}$$

$$= -\frac{1}{2}\sqrt{(j-m)(j+m+1)}\;\delta_{m',m+1} + \frac{1}{2}\sqrt{(j+m)(j-m+1)}\;\delta_{m',m-1}$$

$$D(\hat{j},\varphi) = e^{-\frac{i}{\hbar}\varphi\hat{J}_y}$$

$$\Rightarrow \hat{J}_y f^j_m = -\frac{i\hbar}{2}\sqrt{(j-m)(j+m+1)}\, f^j_{m+1} + \frac{i\hbar}{2}\sqrt{(j+m)(j-m+1)}\, f^j_{m-1}$$