PRINCIPLES OF INTEGRAL EVALUATION

Trigonometric Substitutions:

TRIGONOMETRIC SUBSTITUTIONS

EXPRESSION IN THE INTEGRAND	SUBSTITUTION	RESTRICTION ON θ	SIMPLIFICATION
$\sqrt{a^2-x^2}$	$x = a\sin\theta$	$-\pi/2 \le \theta \le \pi/2$	$a^2-x^2 = a^2 - a^2\sin^2\theta = a^2\cos^2\theta$
$\sqrt{a^2+x^2}$	$x = a\tan\theta$	$-\pi/2 < \theta < \pi/2$	$a^2+x^2 = a^2 + a^2\tan^2\theta = a^2\sec^2\theta$
$\sqrt{x^2-a^2}$	$x = a\sec\theta$	$\begin{cases} 0 \le \theta < \pi/2 & (\text{if } x \ge a) \\ \pi/2 < \theta \le \pi & (\text{if } x \le -a) \end{cases}$	$x^2-a^2 = a^2\sec^2\theta - a^2 = a^2\tan^2\theta$

Example 1 Evaluate $\displaystyle\int \frac{dx}{x^2\sqrt{4-x^2}}$.

Solution. To eliminate the radical we make the substitution

$$x = 2\sin\theta, \quad dx = 2\cos\theta\, d\theta$$

This yields

$$\int \frac{dx}{x^2\sqrt{4-x^2}} = \int \frac{2\cos\theta\, d\theta}{(2\sin\theta)^2\sqrt{4-4\sin^2\theta}}$$

$$= \int \frac{2\cos\theta\, d\theta}{(2\sin\theta)^2(2\cos\theta)} = \frac{1}{4}\int \frac{d\theta}{\sin^2\theta}$$

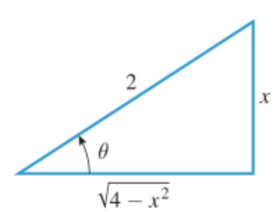
$$= \frac{1}{4}\int \csc^2\theta\, d\theta = -\frac{1}{4}\cot\theta + C \qquad (2)$$

At this point we have completed the integration; however, because the original integral was expressed in terms of x, it is desirable to express $\cot\theta$ in terms of x as well. This can be done using trigonometric identities, but the expression can also be obtained by writing the substitution $x = 2\sin\theta$ as $\sin\theta = x/2$ and representing it geometrically as in Figure 7.4.1. From that figure we obtain

$$\cot\theta = \frac{\sqrt{4-x^2}}{x}$$

Substituting this in (2) yields

$$\int \frac{dx}{x^2\sqrt{4-x^2}} = -\frac{1}{4}\frac{\sqrt{4-x^2}}{x} + C$$



$x = 2\sin\theta$

▶ Example 2 Evaluate $\displaystyle\int_1^{\sqrt{2}} \frac{dx}{x^2\sqrt{4-x^2}}$.

Solution. There are two possible approaches: we can make the substitution in the indefinite integral (as in Example 1) and then evaluate the definite integral using the x-limits of integration, or we can make the substitution in the definite integral and convert the x-limits to the corresponding θ-limits.

Method 1.

Using the result from Example 1 with the x-limits of integration yields

$$\int_1^{\sqrt{2}} \frac{dx}{x^2\sqrt{4-x^2}} = -\frac{1}{4}\left[\frac{\sqrt{4-x^2}}{x}\right]_1^{\sqrt{2}} = -\frac{1}{4}\left[1-\sqrt{3}\right] = \frac{\sqrt{3}-1}{4}$$

Method 2.

The substitution $x = 2\sin\theta$ can be expressed as $x/2 = \sin\theta$ or $\theta = \sin^{-1}(x/2)$, so the θ-limits that correspond to $x = 1$ and $x = \sqrt{2}$ are

$$\begin{aligned} x = 1&: \quad \theta = \sin^{-1}(1/2) = \pi/6 \\ x = \sqrt{2}&: \quad \theta = \sin^{-1}(\sqrt{2}/2) = \pi/4 \end{aligned}$$

Thus, from (2) in Example 1 we obtain

$$\int_1^{\sqrt{2}} \frac{dx}{x^2\sqrt{4-x^2}} = \frac{1}{4}\int_{\pi/6}^{\pi/4} \csc^2\theta\, d\theta \qquad \boxed{\text{Convert } x\text{-limits to } \theta\text{-limits.}}$$

$$= -\frac{1}{4}\left[\cot\theta\right]_{\pi/6}^{\pi/4} = -\frac{1}{4}\left[1-\sqrt{3}\right] = \frac{\sqrt{3}-1}{4} \quad ◀$$

INTEGRALS INVOLVING $ax^2 + bx + c$

Integrals that involve a quadratic expression $ax^2 + bx + c$, where $a \neq 0$ and $b \neq 0$, can often be evaluated by first completing the square, then making an appropriate substitution. The following example illustrates this idea.

Example 6 Evaluate $\displaystyle\int \frac{x}{x^2 - 4x + 8}\, dx$.

Solution. Completing the square yields

$$x^2 - 4x + 8 = (x^2 - 4x + 4) + 8 - 4 = (x - 2)^2 + 4$$

Thus, the substitution

$$u = x - 2, \quad du = dx$$

yields

$$\begin{aligned}
\int \frac{x}{x^2 - 4x + 8}\, dx = \int \frac{x}{(x-2)^2 + 4}\, dx &= \int \frac{u + 2}{u^2 + 4}\, du \\
&= \int \frac{u}{u^2 + 4}\, du + 2 \int \frac{du}{u^2 + 4} \\
&= \frac{1}{2} \int \frac{2u}{u^2 + 4}\, du + 2 \int \frac{du}{u^2 + 4} \\
&= \frac{1}{2} \ln(u^2 + 4) + 2\left(\frac{1}{2}\right) \tan^{-1} \frac{u}{2} + C \\
&= \frac{1}{2} \ln[(x-2)^2 + 4] + \tan^{-1}\left(\frac{x-2}{2}\right) + C
\end{aligned}$$

EXERCISE SET 7.4

1–26 Evaluate the integral.

1. $\int \sqrt{4-x^2}\,dx$

2. $\int \sqrt{1-4x^2}\,dx$

3. $\int \frac{x^2}{\sqrt{16-x^2}}\,dx$

4. $\int \frac{dx}{x^2\sqrt{9-x^2}}$

5. $\int \frac{dx}{(4+x^2)^2}$

6. $\int \frac{x^2}{\sqrt{5+x^2}}\,dx$

7. $\int \frac{\sqrt{x^2-9}}{x}\,dx$

8. $\int \frac{dx}{x^2\sqrt{x^2-16}}$

9. $\int \frac{3x^3}{\sqrt{1-x^2}}\,dx$

10. $\int x^3\sqrt{5-x^2}\,dx$

11. $\int \frac{dx}{x^2\sqrt{9x^2-4}}$

12. $\int \frac{\sqrt{1+t^2}}{t}\,dt$

13. $\int \frac{dx}{(1-x^2)^{3/2}}$

14. $\int \frac{dx}{x^2\sqrt{x^2+25}}$

15. $\int \frac{dx}{\sqrt{x^2-9}}$

16. $\int \frac{dx}{1+2x^2+x^4}$

17. $\int \frac{dx}{(4x^2-9)^{3/2}}$

18. $\int \frac{3x^3}{\sqrt{x^2-25}}\,dx$

19. $\int e^x\sqrt{1-e^{2x}}\,dx$

20. $\int \frac{\cos\theta}{\sqrt{2-\sin^2\theta}}\,d\theta$

21. $\int_0^1 5x^3\sqrt{1-x^2}\,dx$

22. $\int_0^{1/2} \frac{dx}{(1-x^2)^2}$

23. $\int_{\sqrt{2}}^2 \frac{dx}{x^2\sqrt{x^2-1}}$

24. $\int_{\sqrt{2}}^2 \frac{\sqrt{2x^2-4}}{x}\,dx$

25. $\int_1^3 \frac{dx}{x^4\sqrt{x^2+3}}$

37–48 Evaluate the integral.

37. $\displaystyle\int \frac{dx}{x^2 - 4x + 5}$

38. $\displaystyle\int \frac{dx}{\sqrt{2x - x^2}}$

39. $\displaystyle\int \frac{dx}{\sqrt{3 + 2x - x^2}}$

40. $\displaystyle\int \frac{dx}{16x^2 + 16x + 5}$

41. $\displaystyle\int \frac{dx}{\sqrt{x^2 - 6x + 10}}$

42. $\displaystyle\int \frac{x}{x^2 + 2x + 2}\,dx$

43. $\displaystyle\int \sqrt{3 - 2x - x^2}\,dx$

44. $\displaystyle\int \frac{e^x}{\sqrt{1 + e^x + e^{2x}}}\,dx$

45. $\displaystyle\int \frac{dx}{2x^2 + 4x + 7}$

46. $\displaystyle\int \frac{2x + 3}{4x^2 + 4x + 5}\,dx$

47. $\displaystyle\int_1^2 \frac{dx}{\sqrt{4x - x^2}}$

48. $\displaystyle\int_0^4 \sqrt{x(4 - x)}\,dx$

SOLUTION SET

1. $x = 2\sin\theta$, $dx = 2\cos\theta\, d\theta$, $4\int \cos^2\theta\, d\theta = 2\int (1+\cos 2\theta)d\theta = 2\theta + \sin 2\theta + C = 2\theta + 2\sin\theta\cos\theta + C = 2\sin^{-1}(x/2) + \frac{1}{2}x\sqrt{4-x^2} + C$.

3. $x = 4\sin\theta$, $dx = 4\cos\theta\, d\theta$, $16\int \sin^2\theta\, d\theta = 8\int (1-\cos 2\theta)d\theta = 8\theta - 4\sin 2\theta + C = 8\theta - 8\sin\theta\cos\theta + C = 8\sin^{-1}(x/4) - \frac{1}{2}x\sqrt{16-x^2} + C$.

5. $x = 2\tan\theta$, $dx = 2\sec^2\theta\, d\theta$, $\frac{1}{8}\int \frac{1}{\sec^2\theta}d\theta = \frac{1}{8}\int \cos^2\theta\, d\theta = \frac{1}{16}\int (1+\cos 2\theta)d\theta = \frac{1}{16}\theta + \frac{1}{32}\sin 2\theta + C = \frac{1}{16}\theta + \frac{1}{16}\sin\theta\cos\theta + C = \frac{1}{16}\tan^{-1}\frac{x}{2} + \frac{x}{8(4+x^2)} + C$.

7. $x = 3\sec\theta$, $dx = 3\sec\theta\tan\theta\, d\theta$, $3\int \tan^2\theta\, d\theta = 3\int (\sec^2\theta - 1)d\theta = 3\tan\theta - 3\theta + C = \sqrt{x^2-9} - 3\sec^{-1}\frac{x}{3} + C$.

9. $x = \sin\theta$, $dx = \cos\theta\, d\theta$, $3\int \sin^3\theta\, d\theta = 3\int \left[1-\cos^2\theta\right]\sin\theta\, d\theta = 3\left(-\cos\theta + \cos^3\theta\right) + C = -3\sqrt{1-x^2} + (1-x^2)^{3/2} + C$.

11. $x = \frac{2}{3}\sec\theta$, $dx = \frac{2}{3}\sec\theta\tan\theta\, d\theta$, $\frac{3}{4}\int \frac{1}{\sec\theta}d\theta = \frac{3}{4}\int \cos\theta\, d\theta = \frac{3}{4}\sin\theta + C = \frac{1}{4x}\sqrt{9x^2-4} + C$.

13. $x = \sin\theta$, $dx = \cos\theta\, d\theta$, $\int \frac{1}{\cos^2\theta}d\theta = \int \sec^2\theta\, d\theta = \tan\theta + C = x/\sqrt{1-x^2} + C$.

15. $x = 3\sec\theta$, $dx = 3\sec\theta\tan\theta\, d\theta$, $\int \sec\theta\, d\theta = \ln|\sec\theta + \tan\theta| + C = \ln\left|\frac{1}{3}x + \frac{1}{3}\sqrt{x^2-9}\right| + C$.

17. $x = \frac{3}{2}\sec\theta$, $dx = \frac{3}{2}\sec\theta\tan\theta\, d\theta$, $\frac{3}{2}\int \frac{\sec\theta\tan\theta\, d\theta}{27\tan^3\theta} = \frac{1}{18}\int \frac{\cos\theta}{\sin^2\theta}d\theta = -\frac{1}{18}\frac{1}{\sin\theta} + C = -\frac{1}{18}\csc\theta + C = -\frac{x}{9\sqrt{4x^2-9}} + C$.

19. $e^x = \sin\theta$, $e^x dx = \cos\theta\, d\theta$, $\int \cos^2\theta\, d\theta = \frac{1}{2}\int (1+\cos 2\theta)d\theta = \frac{1}{2}\theta + \frac{1}{4}\sin 2\theta + C = \frac{1}{2}\sin^{-1}(e^x) + \frac{1}{2}e^x\sqrt{1-e^{2x}} + C$.

21. $x = \sin\theta$, $dx = \cos\theta\, d\theta$, $5\int_0^1 \sin^3\theta\cos^2\theta\, d\theta = 5\left[-\frac{1}{3}\cos^3\theta + \frac{1}{5}\cos^5\theta\right]_0^{\pi/2} = 5(1/3 - 1/5) = 2/3$.

23. $x = \sec\theta$, $dx = \sec\theta\tan\theta\, d\theta$, $\int_{\pi/4}^{\pi/3} \frac{1}{\sec\theta}d\theta = \int_{\pi/4}^{\pi/3} \cos\theta\, d\theta = \sin\theta\Big]_{\pi/4}^{\pi/3} = (\sqrt{3} - \sqrt{2})/2$.

25. $x = \sqrt{3}\tan\theta$, $dx = \sqrt{3}\sec^2\theta\, d\theta$, $\frac{1}{9}\int_{\pi/6}^{\pi/3} \frac{\sec\theta}{\tan^4\theta}d\theta = \frac{1}{9}\int_{\pi/6}^{\pi/3} \frac{\cos^3\theta}{\sin^4\theta}d\theta = \frac{1}{9}\int_{\pi/6}^{\pi/3} \frac{1-\sin^2\theta}{\sin^4\theta}\cos\theta\, d\theta$

$$= \frac{1}{9}\int_{1/2}^{\sqrt{3}/2} \frac{1-u^2}{u^4}du \text{ (with } u = \sin\theta) = \frac{1}{9}\int_{1/2}^{\sqrt{3}/2} (u^{-4} - u^{-2})du = \frac{1}{9}\left[-\frac{1}{3u^3} + \frac{1}{u}\right]_{1/2}^{\sqrt{3}/2} = \frac{10\sqrt{3}+18}{243}.$$

37. $\displaystyle\int \frac{1}{(x-2)^2+1}dx = \tan^{-1}(x-2)+C.$

39. $\displaystyle\int \frac{1}{\sqrt{4-(x-1)^2}}dx = \sin^{-1}\left(\frac{x-1}{2}\right)+C$

41. $\displaystyle\int \frac{1}{\sqrt{(x-3)^2+1}}dx = \ln\left(x-3+\sqrt{(x-3)^2+1}\right)+C.$

43. $\displaystyle\int \sqrt{4-(x+1)^2}\,dx,$ let $x+1=2\sin\theta$, $\displaystyle\int 4\cos^2\theta\,d\theta = \int 2(1+\cos 2\theta)\,d\theta = 2\theta+\sin 2\theta+C = 2\sin^{-1}\left(\frac{x+1}{2}\right)+ \frac{1}{2}(x+1)\sqrt{3-2x-x^2}+C.$

45. $\displaystyle\int \frac{1}{2(x+1)^2+5}dx = \frac{1}{2}\int \frac{1}{(x+1)^2+5/2}dx = \frac{1}{\sqrt{10}}\tan^{-1}\sqrt{2/5}(x+1)+C.$

47. $\displaystyle\int_1^2 \frac{1}{\sqrt{4x-x^2}}\,dx = \int_1^2 \frac{1}{\sqrt{4-(x-2)^2}}\,dx = \sin^{-1}\frac{x-2}{2}\Big]_1^2 = \pi/6.$