CONVERGENCE TESTS

The Divergence Test

9.4.1 THEOREM (*The Divergence Test*)

(*a*) *If* $\lim_{k\to+\infty} u_k \neq 0$, *then the series* $\sum u_k$ *diverges.*

(*b*) *If* $\lim_{k\to+\infty} u_k = 0$, *then the series* $\sum u_k$ *may either converge or diverge.*

9.4.2 THEOREM *If the series* $\sum u_k$ *converges, then* $\lim_{k\to+\infty} u_k = 0$.

▶ Example 1 The series

$$\sum_{k=1}^{\infty} \frac{k}{k+1} = \frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \cdots + \frac{k}{k+1} + \cdots$$

diverges since

$$\lim_{k\to+\infty} \frac{k}{k+1} = \lim_{k\to+\infty} \frac{1}{1+1/k} = 1 \neq 0 \quad ◀$$

▶ Example 2 Find the sum of the series

$$\sum_{k=1}^{\infty} \left(\frac{3}{4^k} - \frac{2}{5^{k-1}}\right)$$

Solution. The series

$$\sum_{k=1}^{\infty} \frac{3}{4^k} = \frac{3}{4} + \frac{3}{4^2} + \frac{3}{4^3} + \cdots$$

is a convergent geometric series $\left(a = \frac{3}{4}, r = \frac{1}{4}\right)$, and the series

$$\sum_{k=1}^{\infty} \frac{2}{5^{k-1}} = 2 + \frac{2}{5} + \frac{2}{5^2} + \frac{2}{5^3} + \cdots$$

is also a convergent geometric series $\left(a = 2, r = \frac{1}{5}\right)$. Thus, from Theorems 9.4.3(a) and 9.3.3 the given series converges and

$$\sum_{k=1}^{\infty}\left(\frac{3}{4^k} - \frac{2}{5^{k-1}}\right) = \sum_{k=1}^{\infty}\frac{3}{4^k} - \sum_{k=1}^{\infty}\frac{2}{5^{k-1}}$$

$$= \frac{\frac{3}{4}}{1-\frac{1}{4}} - \frac{2}{1-\frac{1}{5}} = -\frac{3}{2} \blacktriangleleft$$

9.4.3 THEOREM

(a) *If $\sum u_k$ and $\sum v_k$ are convergent series, then $\sum(u_k + v_k)$ and $\sum(u_k - v_k)$ are convergent series and the sums of these series are related by*

$$\sum_{k=1}^{\infty}(u_k + v_k) = \sum_{k=1}^{\infty}u_k + \sum_{k=1}^{\infty}v_k$$

$$\sum_{k=1}^{\infty}(u_k - v_k) = \sum_{k=1}^{\infty}u_k - \sum_{k=1}^{\infty}v_k$$

(b) *If c is a nonzero constant, then the series $\sum u_k$ and $\sum cu_k$ both converge or both diverge. In the case of convergence, the sums are related by*

$$\sum_{k=1}^{\infty}cu_k = c\sum_{k=1}^{\infty}u_k$$

(c) *Convergence or divergence is unaffected by deleting a finite number of terms from a series; in particular, for any positive integer K, the series*

$$\sum_{k=1}^{\infty}u_k = u_1 + u_2 + u_3 + \cdots$$

$$\sum_{k=K}^{\infty}u_k = u_K + u_{K+1} + u_{K+2} + \cdots$$

both converge or both diverge.

▸ Example 3 Determine whether the following series converge or diverge.

$$\text{(a)} \sum_{k=1}^{\infty} \frac{5}{k} = 5 + \frac{5}{2} + \frac{5}{3} + \cdots + \frac{5}{k} + \cdots \qquad \text{(b)} \sum_{k=10}^{\infty} \frac{1}{k} = \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \cdots$$

Solution. The first series is a constant times the divergent harmonic series, and hence diverges by part (*b*) of Theorem 9.4.3. The second series results by deleting the first nine terms from the divergent harmonic series, and hence diverges by part (*c*) of Theorem 9.4.3.

THE INTEGRAL TEST

9.4.4 THEOREM (*The Integral Test*) *Let $\sum u_k$ be a series with positive terms. If f is a function that is decreasing and continuous on an interval $[a, +\infty)$ and such that $u_k = f(k)$ for all $k \geq a$, then*

$$\sum_{k=1}^{\infty} u_k \quad \textit{and} \quad \int_a^{+\infty} f(x)\,dx$$

both converge or both diverge.

▸ Example 4 Show that the integral test applies, and use the integral test to determine whether the following series converge or diverge.

$$\text{(a)} \sum_{k=1}^{\infty} \frac{1}{k} \qquad \text{(b)} \sum_{k=1}^{\infty} \frac{1}{k^2}$$

Solution (a). We already know that this is the divergent harmonic series, so the integral test will simply illustrate another way of establishing the divergence.

Note first that the series has positive terms, so the integral test is applicable. If we replace k by x in the general term $1/k$, we obtain the function $f(x) = 1/x$, which is decreasing and continuous for $x \geq 1$ (as required to apply the integral test with $a = 1$). Since

$$\int_1^{+\infty} \frac{1}{x}\,dx = \lim_{b \to +\infty} \int_1^b \frac{1}{x}\,dx = \lim_{b \to +\infty} [\ln b - \ln 1] = +\infty$$

the integral diverges and consequently so does the series.

Solution (b). Note first that the series has positive terms, so the integral test is applicable. If we replace k by x in the general term $1/k^2$, we obtain the function $f(x) = 1/x^2$, which is decreasing and continuous for $x \geq 1$. Since

$$\int_1^{+\infty} \frac{1}{x^2}\,dx = \lim_{b\to+\infty} \int_1^b \frac{dx}{x^2} = \lim_{b\to+\infty} \left[-\frac{1}{x}\right]_1^b = \lim_{b\to+\infty} \left[1 - \frac{1}{b}\right] = 1$$

the integral converges and consequently the series converges by the integral test with $a = 1$.

P-Series

p-SERIES

The series in Example 4 are special cases of a class of series called ***p-series*** or ***hyperharmonic series***. A p-series is an infinite series of the form

$$\sum_{k=1}^{\infty} \frac{1}{k^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{k^p} + \cdots$$

where $p > 0$. Examples of p-series are

$$\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k} + \cdots \qquad \boxed{p = 1}$$

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{k^2} + \cdots \qquad \boxed{p = 2}$$

$$\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}} = 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{k}} + \cdots \qquad \boxed{p = \tfrac{1}{2}}$$

The following theorem tells when a p-series converges.

9.4.5 THEOREM (*Convergence of p-Series*)

$$\sum_{k=1}^{\infty} \frac{1}{k^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{k^p} + \cdots$$

converges if $p > 1$ *and diverges if* $0 < p \leq 1$.

▶ Example 5

$$1 + \frac{1}{\sqrt[3]{2}} + \frac{1}{\sqrt[3]{3}} + \cdots + \frac{1}{\sqrt[3]{k}} + \cdots$$

diverges since it is a p-series with $p = \frac{1}{3} < 1$. ◀

EXERCISE SET 9.4

3–4 For each given p-series, identify p and determine whether the series converges. ■

3. (a) $\sum_{k=1}^{\infty} \frac{1}{k^3}$ (b) $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$ (c) $\sum_{k=1}^{\infty} k^{-1}$ (d) $\sum_{k=1}^{\infty} k^{-2/3}$

4. (a) $\sum_{k=1}^{\infty} k^{-4/3}$ (b) $\sum_{k=1}^{\infty} \frac{1}{\sqrt[4]{k}}$ (c) $\sum_{k=1}^{\infty} \frac{1}{\sqrt[3]{k^5}}$ (d) $\sum_{k=1}^{\infty} \frac{1}{k^{\pi}}$

5–6 Apply the divergence test and state what it tells you about the series. ■

5. (a) $\sum_{k=1}^{\infty} \frac{k^2+k+3}{2k^2+1}$ (b) $\sum_{k=1}^{\infty} \left(1+\frac{1}{k}\right)^k$

(c) $\sum_{k=1}^{\infty} \cos k\pi$ (d) $\sum_{k=1}^{\infty} \frac{1}{k!}$

6. (a) $\sum_{k=1}^{\infty} \frac{k}{e^k}$ (b) $\sum_{k=1}^{\infty} \ln k$

(c) $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$ (d) $\sum_{k=1}^{\infty} \frac{\sqrt{k}}{\sqrt{k}+3}$

7–8 Confirm that the integral test is applicable and use it to determine whether the series converges. ■

7. (a) $\sum_{k=1}^{\infty} \frac{1}{5k+2}$ (b) $\sum_{k=1}^{\infty} \frac{1}{1+9k^2}$

8. (a) $\sum_{k=1}^{\infty} \frac{k}{1+k^2}$ (b) $\sum_{k=1}^{\infty} \frac{1}{(4+2k)^{3/2}}$

9–24 Determine whether the series converges.

9. $\sum_{k=1}^{\infty} \frac{1}{k+6}$

10. $\sum_{k=1}^{\infty} \frac{3}{5k}$

11. $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k+5}}$

12. $\sum_{k=1}^{\infty} \frac{1}{\sqrt[k]{e}}$

13. $\sum_{k=1}^{\infty} \frac{1}{\sqrt[3]{2k-1}}$

14. $\sum_{k=3}^{\infty} \frac{\ln k}{k}$

15. $\sum_{k=1}^{\infty} \frac{k}{\ln(k+1)}$

16. $\sum_{k=1}^{\infty} ke^{-k^2}$

17. $\sum_{k=1}^{\infty} \left(1+\frac{1}{k}\right)^{-k}$

18. $\sum_{k=1}^{\infty} \frac{k^2+1}{k^2+3}$

19. $\sum_{k=1}^{\infty} \frac{\tan^{-1} k}{1+k^2}$

20. $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k^2+1}}$

21. $\sum_{k=1}^{\infty} k^2 \sin^2\left(\frac{1}{k}\right)$

22. $\sum_{k=1}^{\infty} k^2 e^{-k^3}$

23. $\sum_{k=5}^{\infty} 7k^{-1.01}$

24. $\sum_{k=1}^{\infty} \operatorname{sech}^2 k$

SOLUTION SET

3. **(a)** $p=3>1$, converges. **(b)** $p=1/2\le 1$, diverges. **(c)** $p=1\le 1$, diverges. **(d)** $p=2/3\le 1$, diverges.

5. **(a)** $\lim\limits_{k\to+\infty}\dfrac{k^2+k+3}{2k^2+1}=\dfrac{1}{2}\ne 0$; the series diverges. **(b)** $\lim\limits_{k\to+\infty}\left(1+\dfrac{1}{k}\right)^k=e\ne 0$; the series diverges.

(c) $\lim\limits_{k\to+\infty}\cos k\pi$ does not exist; the series diverges. **(d)** $\lim\limits_{k\to+\infty}\dfrac{1}{k!}=0$; no information.

7. **(a)** $\displaystyle\int_1^{+\infty}\frac{1}{5x+2}=\lim_{\ell\to+\infty}\frac{1}{5}\ln(5x+2)\Big]_1^{\ell}=+\infty$, the series diverges by the Integral Test (which can be applied, because the series has positive terms, and f is decreasing and continuous).

(b) $\displaystyle\int_1^{+\infty}\frac{1}{1+9x^2}dx=\lim_{\ell\to+\infty}\frac{1}{3}\tan^{-1}3x\Big]_1^{\ell}=\frac{1}{3}\left(\pi/2-\tan^{-1}3\right)$, the series converges by the Integral Test (which can be applied, because the series has positive terms, and f is decreasing and continuous).

9. $\displaystyle\sum_{k=1}^{\infty}\frac{1}{k+6}=\sum_{k=7}^{\infty}\frac{1}{k}$, diverges because the harmonic series diverges.

11. $\displaystyle\sum_{k=1}^{\infty}\frac{1}{\sqrt{k+5}}=\sum_{k=6}^{\infty}\frac{1}{\sqrt{k}}$, diverges because the p-series with $p=1/2\le 1$ diverges.

13. $\displaystyle\int_1^{+\infty}(2x-1)^{-1/3}dx=\lim_{\ell\to+\infty}\frac{3}{4}(2x-1)^{2/3}\Big]_1^{\ell}=+\infty$, the series diverges by the Integral Test (which can be applied, because the series has positive terms, and f is decreasing and continuous).

15. $\displaystyle\lim_{k\to+\infty}\frac{k}{\ln(k+1)}=\lim_{k\to+\infty}\frac{1}{1/(k+1)}=+\infty$, the series diverges by the Divergence Test, because $\lim\limits_{k\to+\infty}u_k\ne 0$.

17. $\lim\limits_{k\to+\infty}(1+1/k)^{-k}=1/e\ne 0$, the series diverges by the Divergence Test.

19. $\displaystyle\int_1^{+\infty}\frac{\tan^{-1}x}{1+x^2}dx=\lim_{\ell\to+\infty}\frac{1}{2}\left(\tan^{-1}x\right)^2\Big]_1^{\ell}=3\pi^2/32$, the series converges by the Integral Test (which can be applied, because the series has positive terms, and f is decreasing and continuous), since $\dfrac{d}{dx}\dfrac{\tan^{-1}x}{1+x^2}=\dfrac{1-2x\tan^{-1}x}{(1+x^2)^2}<0$ for $x\ge 1$.

21. $\lim\limits_{k\to+\infty}k^2\sin^2(1/k)=1\ne 0$, the series diverges by the Divergence Test.

23. $7\displaystyle\sum_{k=5}^{\infty}k^{-1.01}$, p-series with $p=1.01>1$, converges.