CHAPTER 9: NUMERICAL METHODS

9.1 LU-Decompositions

1. Step 1. Rewrite the system as $\underbrace{\begin{bmatrix} 3 & 0 \\ -2 & 1 \end{bmatrix}}_{L}\underbrace{\begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}}_{U}\underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\mathbf{x}} = \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{\mathbf{b}}$

Step 2. Define y_1 and y_2 by $\underbrace{\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}}_{\mathbf{y}} = \underbrace{\begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}}_{U}\underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\mathbf{x}}$ and solve $\underbrace{\begin{bmatrix} 3 & 0 \\ -2 & 1 \end{bmatrix}}_{L}\underbrace{\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}}_{\mathbf{y}} = \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{\mathbf{b}}$ by forward substitution to obtain $y_1 = 0$, $y_2 = 1$.

Step 3. Solve $\underbrace{\begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}}_{U}\underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\mathbf{x}} = \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{\mathbf{y}}$ by back substitution to find $x_1 = 2$, $x_2 = 1$.

2. Step 1. Rewrite the system as $\underbrace{\begin{bmatrix} 3 & 0 & 0 \\ 2 & 4 & 0 \\ -4 & -1 & 2 \end{bmatrix}}_{L}\underbrace{\begin{bmatrix} 1 & -2 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}}_{U}\underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\mathbf{x}} = \underbrace{\begin{bmatrix} -3 \\ -22 \\ 3 \end{bmatrix}}_{\mathbf{b}}$

Step 2. Define y_1, y_2, and y_3 by $\underbrace{\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}}_{\mathbf{y}} = \underbrace{\begin{bmatrix} 1 & -2 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}}_{U}\underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\mathbf{x}}$ and solve $\underbrace{\begin{bmatrix} 3 & 0 & 0 \\ 2 & 4 & 0 \\ -4 & -1 & 2 \end{bmatrix}}_{L}\underbrace{\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}}_{\mathbf{y}} = \underbrace{\begin{bmatrix} -3 \\ -22 \\ 3 \end{bmatrix}}_{\mathbf{b}}$ by forward substitution to obtain $y_1 = -1$, $y_2 = -5$, $y_3 = -3$.

Step 3. Solve $\underbrace{\begin{bmatrix} 1 & -2 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}}_{U}\underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\mathbf{x}} = \underbrace{\begin{bmatrix} -1 \\ -5 \\ -3 \end{bmatrix}}_{\mathbf{y}}$ by back substitution to find $x_1 = -2$, $x_2 = 1$, $x_3 = -3$.

3. $A = \begin{bmatrix} 2 & 8 \\ -1 & -1 \end{bmatrix}$ $\qquad \begin{bmatrix} \bullet & 0 \\ \bullet & \bullet \end{bmatrix}$ (we follow the procedure of Example 3)

$\begin{bmatrix} 1 & 4 \\ -1 & -1 \end{bmatrix} \leftarrow \text{multiplier} = \frac{1}{2}$ $\qquad \begin{bmatrix} 2 & 0 \\ \bullet & \bullet \end{bmatrix}$

$\begin{bmatrix} 1 & 4 \\ 0 & 3 \end{bmatrix} \leftarrow \text{multiplier} = 1$ $\qquad \begin{bmatrix} 2 & 0 \\ -1 & \bullet \end{bmatrix}$

$U = \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix} \leftarrow \text{multiplier} = \frac{1}{3}$ $\qquad L = \begin{bmatrix} 2 & 0 \\ -1 & 3 \end{bmatrix}$

Step 1. Rewrite the system as $\underbrace{\begin{bmatrix} 2 & 0 \\ -1 & 3 \end{bmatrix}}_{L}\underbrace{\begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}}_{U}\underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\mathbf{x}}=\underbrace{\begin{bmatrix} -2 \\ -2 \end{bmatrix}}_{\mathbf{b}}$

Step 2. Define y_1 and y_2 by $\underbrace{\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}}_{\mathbf{y}}=\underbrace{\begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}}_{U}\underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\mathbf{x}}$ and solve $\underbrace{\begin{bmatrix} 2 & 0 \\ -1 & 3 \end{bmatrix}}_{L}\underbrace{\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}}_{\mathbf{y}}=\underbrace{\begin{bmatrix} -2 \\ -2 \end{bmatrix}}_{\mathbf{b}}$ by forward substitution to obtain $y_1=-1$, $y_2=-1$.

Step 3. Solve $\underbrace{\begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}}_{U}\underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\mathbf{x}}=\underbrace{\begin{bmatrix} -1 \\ -1 \end{bmatrix}}_{\mathbf{y}}$ by back substitution to find $x_1=3$, $x_2=-1$.

4. $A=\begin{bmatrix} -5 & -10 \\ 6 & 5 \end{bmatrix}$ $\qquad\begin{bmatrix} \bullet & 0 \\ \bullet & \bullet \end{bmatrix}$ (we follow the procedure of Example 3)

$\begin{bmatrix} 1 & 2 \\ 6 & 5 \end{bmatrix} \leftarrow \text{multiplier} = -\tfrac{1}{5}$ $\qquad\begin{bmatrix} -5 & 0 \\ \bullet & \bullet \end{bmatrix}$

$\begin{bmatrix} 1 & 2 \\ 0 & -7 \end{bmatrix} \leftarrow \text{multiplier} = -6$ $\qquad\begin{bmatrix} -5 & 0 \\ 6 & \bullet \end{bmatrix}$

$U=\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \leftarrow \text{multiplier} = -\tfrac{1}{7}$ $\qquad L=\begin{bmatrix} -5 & 0 \\ 6 & -7 \end{bmatrix}$

Step 1. Rewrite the system as $\underbrace{\begin{bmatrix} -5 & 0 \\ 6 & -7 \end{bmatrix}}_{L}\underbrace{\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}}_{U}\underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\mathbf{x}}=\underbrace{\begin{bmatrix} -10 \\ 19 \end{bmatrix}}_{\mathbf{b}}$

Step 2. Define y_1 and y_2 by $\underbrace{\underbrace{\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}}_{\mathbf{y}}=\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}}_{U}\underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\mathbf{x}}$ and solve $\underbrace{\begin{bmatrix} -5 & 0 \\ 6 & -7 \end{bmatrix}}_{L}\underbrace{\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}}_{\mathbf{y}}=\underbrace{\begin{bmatrix} -10 \\ 19 \end{bmatrix}}_{\mathbf{b}}$ by forward substitution to obtain $y_1=2$, $y_2=-1$.

Step 3. Solve $\underbrace{\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}}_{U}\underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\mathbf{x}}=\underbrace{\begin{bmatrix} 2 \\ -1 \end{bmatrix}}_{\mathbf{y}}$ by back substitution to find $x_1=4$, $x_2=-1$.

5. $A=\begin{bmatrix} 2 & -2 & -2 \\ 0 & -2 & 2 \\ -1 & 5 & 2 \end{bmatrix}$ $\qquad\begin{bmatrix} \bullet & 0 & 0 \\ \bullet & \bullet & 0 \\ \bullet & \bullet & \bullet \end{bmatrix}$

$$\begin{bmatrix} 1 & -1 & -1 \\ 0 & -2 & 2 \\ -1 & 5 & 2 \end{bmatrix} \begin{matrix} \leftarrow \text{multiplier} = \frac{1}{2} \\ \\ \\ \end{matrix} \qquad \begin{bmatrix} 2 & 0 & 0 \\ \bullet & \bullet & 0 \\ \bullet & \bullet & \bullet \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & -1 \\ 0 & -2 & 2 \\ 0 & 4 & 1 \end{bmatrix} \begin{matrix} \\ \leftarrow \text{multiplier} = 0 \\ \leftarrow \text{multiplier} = 1 \end{matrix} \qquad \begin{bmatrix} 2 & 0 & 0 \\ 0 & \bullet & 0 \\ -1 & \bullet & \bullet \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 4 & 1 \end{bmatrix} \begin{matrix} \\ \leftarrow \text{multiplier} = -\frac{1}{2} \\ \\ \end{matrix} \qquad \begin{bmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ -1 & \bullet & \bullet \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 5 \end{bmatrix} \begin{matrix} \\ \\ \leftarrow \text{multiplier} = -4 \end{matrix} \qquad \begin{bmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ -1 & 4 & \bullet \end{bmatrix}$$

$$U = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{matrix} \\ \\ \leftarrow \text{multiplier} = \frac{1}{5} \end{matrix} \qquad L = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ -1 & 4 & 5 \end{bmatrix}$$

Step 1. Rewrite the system as $\underbrace{\begin{bmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ -1 & 4 & 5 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\mathbf{x}} = \underbrace{\begin{bmatrix} -4 \\ -2 \\ 6 \end{bmatrix}}_{\mathbf{b}}$

Step 2. Define y_1, y_2, and y_3 by $\underbrace{\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}}_{\mathbf{y}} = \underbrace{\begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\mathbf{x}}$ and solve $\underbrace{\begin{bmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ -1 & 4 & 5 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}}_{\mathbf{y}} = \underbrace{\begin{bmatrix} -4 \\ -2 \\ 6 \end{bmatrix}}_{\mathbf{b}}$ by forward substitution to obtain $y_1 = -2$, $y_2 = 1$, $y_3 = 0$.

Step 3. Solve $\underbrace{\begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\mathbf{x}} = \underbrace{\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}}_{\mathbf{y}}$ by back substitution to find $x_1 = -1$, $x_2 = 1$, $x_3 = 0$.

6. $$A = \begin{bmatrix} -3 & 12 & -6 \\ 1 & -2 & 2 \\ 0 & 1 & 1 \end{bmatrix} \qquad \begin{bmatrix} \bullet & 0 & 0 \\ \bullet & \bullet & 0 \\ \bullet & \bullet & \bullet \end{bmatrix}$$

$$\begin{bmatrix} 1 & -4 & 2 \\ 1 & -2 & 2 \\ 0 & 1 & 1 \end{bmatrix} \begin{matrix} \leftarrow \text{multiplier} = -\frac{1}{3} \\ \\ \\ \end{matrix} \qquad \begin{bmatrix} -3 & 0 & 0 \\ \bullet & \bullet & 0 \\ \bullet & \bullet & \bullet \end{bmatrix}$$

$$\begin{bmatrix} 1 & -4 & 2 \\ 0 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{matrix} \\ \leftarrow \text{multiplier} = -1 \\ \leftarrow \text{multiplier} = 0 \end{matrix} \qquad \begin{bmatrix} -3 & 0 & 0 \\ 1 & \bullet & 0 \\ 0 & \bullet & \bullet \end{bmatrix}$$

$$\begin{bmatrix} 1 & -4 & 2 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{matrix} \\ \leftarrow \text{multiplier} = \frac{1}{2} \\ \\ \end{matrix} \qquad \begin{bmatrix} -3 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & \bullet & \bullet \end{bmatrix}$$

$$\begin{bmatrix} 1 & -4 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{matrix} \\ \\ \leftarrow \text{multiplier} = -1 \end{matrix} \qquad \begin{bmatrix} -3 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & \bullet \end{bmatrix}$$

$$U = \begin{bmatrix} 1 & -4 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{matrix} \\ \\ \leftarrow \text{multiplier} = 1 \end{matrix} \qquad L = \begin{bmatrix} -3 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

Step 1. Rewrite the system as $\underbrace{\begin{bmatrix} -3 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 1 & -4 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\mathbf{x}} = \underbrace{\begin{bmatrix} -33 \\ 7 \\ -1 \end{bmatrix}}_{\mathbf{b}}$

Step 2. Define y_1, y_2, and y_3 by $\underbrace{\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}}_{\mathbf{y}} = \underbrace{\begin{bmatrix} 1 & -4 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\mathbf{x}}$ and solve $\underbrace{\begin{bmatrix} -3 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}}_{\mathbf{y}} = \underbrace{\begin{bmatrix} -33 \\ 7 \\ -1 \end{bmatrix}}_{\mathbf{b}}$ by forward substitution to obtain $y_1 = 11$, $y_2 = -2$, $y_3 = 1$.

Step 3. Solve $\underbrace{\begin{bmatrix} 1 & -4 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\mathbf{x}} = \underbrace{\begin{bmatrix} 11 \\ -2 \\ 1 \end{bmatrix}}_{\mathbf{y}}$ by back substitution to find $x_1 = 1$, $x_2 = -2$, $x_3 = 1$.

7. **(a)** $L^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$; $U^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{8} & -\frac{7}{48} \\ 0 & \frac{1}{4} & \frac{5}{24} \\ 0 & 0 & \frac{1}{6} \end{bmatrix}$ **(b)** $A^{-1} = U^{-1}L^{-1} = \begin{bmatrix} \frac{5}{48} & -\frac{1}{48} & -\frac{7}{48} \\ -\frac{7}{24} & \frac{11}{24} & \frac{5}{24} \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \end{bmatrix}$

8. **(a)** $L^{-1} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ \frac{3}{2} & 1 & 0 \\ \frac{5}{14} & \frac{3}{7} & \frac{1}{7} \end{bmatrix}$; $U^{-1} = \begin{bmatrix} 1 & -3 & 8 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix}$ **(b)** $A^{-1} = U^{-1}L^{-1} = \begin{bmatrix} -\frac{8}{7} & \frac{3}{7} & \frac{8}{7} \\ \frac{3}{7} & -\frac{2}{7} & -\frac{3}{7} \\ \frac{5}{14} & \frac{3}{7} & \frac{1}{7} \end{bmatrix}$

9. **(a)** Reduce A to upper triangular form.

$$\begin{bmatrix} 2 & 1 & -1 \\ -2 & -1 & 2 \\ 2 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & \frac{1}{2} & -\frac{1}{2} \\ -2 & -1 & 2 \\ 2 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} = U$$

The multipliers used were $\frac{1}{2}$, 2, and –2, which leads to $L = \begin{bmatrix} 2 & 0 & 0 \\ -2 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$ where the 1's on the diagonal reflect that no multiplication was required on the 2nd and 3rd diagonal entries.

(b) To change the 2 on the diagonal of *L* to a 1, the first column of *L* is divided by 2 and the diagonal matrix has a 2 as the 1, 1 entry.

$A = L_1 D U_1 = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ is the *LDU*-decomposition of *A*.

(c) Let $U_2 = DU$, and $L_2 = L_1$, then $A = L_2 U_2 = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$.

10. **(a)** Setting $\begin{bmatrix} a & 0 \\ b & c \end{bmatrix}\begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ yields $\begin{bmatrix} a & ad \\ b & bd+c \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Equating entries in the first row leads to a contradiction, since we cannot simultaneously have $a = 0$ and $ad = 1$. We conclude that the matrix has no LU-decomposition. (This matrix cannot be reduced to a row echelon form without interchanging rows.)

(b) By inspection, $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{P}\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{L}\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{U}$.

11. $P^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $P^{-1}\mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$, so the system $P^{-1}A\mathbf{x} = P^{-1}\mathbf{b}$ is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & -5 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & 4 \\ 0 & 0 & 17 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}.$$

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & -5 & 1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$ is

$$\begin{aligned} y_1 \phantom{{}-5y_2+y_3} &= 1 \\ y_2 \phantom{{}+y_3} &= 2 \\ 3y_1 - 5y_2 + y_3 &= 5 \end{aligned}$$

which has the solution $y_1 = 1,\ y_2 = 2,\ y_3 = 12.$

$$\begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & 4 \\ 0 & 0 & 17 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 12 \end{bmatrix} \text{ is}$$

$$\begin{aligned} x_1 + 2x_2 + 2x_3 &= 1 \\ x_2 + 4x_3 &= 2 \\ 17x_3 &= 12 \end{aligned}$$

which gives the solution of the original system: $x_1 = \frac{21}{17},\ x_2 = -\frac{14}{17},\ x_3 = \frac{12}{17}.$

12. The inverse of $P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ is $P^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ thus $P^{-1}\mathbf{b} = \begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix}$.

We rewrite $A\mathbf{x} = \mathbf{b}$ as $P^{-1}(PLU)\mathbf{x} = P^{-1}\mathbf{b}$, i.e., $\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}}_{L}\underbrace{\begin{bmatrix} 4 & 1 & 2 \\ 0 & -1 & 4 \\ 0 & 0 & 9 \end{bmatrix}}_{U}\underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\mathbf{x}} = \underbrace{\begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix}}_{P^{-1}\mathbf{b}}$.

Solving $\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}}_{L}\underbrace{\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}}_{\mathbf{y}} = \underbrace{\begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix}}_{P^{-1}\mathbf{b}}$ by forward substitution yields $y_1 = 3$, $y_2 = 0$, $y_3 = 0$.

Solving $\underbrace{\begin{bmatrix} 4 & 1 & 2 \\ 0 & -1 & 4 \\ 0 & 0 & 9 \end{bmatrix}}_{U}\underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\mathbf{x}} = \underbrace{\begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}}_{\mathbf{y}}$ by back substitution yields $x_1 = \frac{3}{4}$, $x_2 = 0$, $x_3 = 0$.

13. $A = \begin{bmatrix} 2 & 2 \\ 4 & 1 \end{bmatrix}$ $\qquad \begin{bmatrix} \bullet & 0 \\ \bullet & \bullet \end{bmatrix}$

$\begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \leftarrow \text{multiplier} = \frac{1}{2}$ $\qquad \begin{bmatrix} 2 & 0 \\ \bullet & \bullet \end{bmatrix}$

$\begin{bmatrix} 1 & 1 \\ 0 & -3 \end{bmatrix} \leftarrow \text{multiplier} = -4$ $\qquad \begin{bmatrix} 2 & 0 \\ 4 & \bullet \end{bmatrix}$

$U = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \leftarrow \text{multiplier} = -\frac{1}{3}$ $\qquad \begin{bmatrix} 2 & 0 \\ 4 & -3 \end{bmatrix}$

A general 2×2 lower triangular matrix with nonzero main diagonal entries can be factored as $\begin{bmatrix} a_{11} & 0 \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ a_{21}/a_{11} & 1 \end{bmatrix}\begin{bmatrix} a_{11} & 0 \\ 0 & a_{22} \end{bmatrix}$ therefore $\begin{bmatrix} 2 & 0 \\ 4 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}$. We conclude that an LDU -decomposition of A is $A = \begin{bmatrix} 2 & 2 \\ 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = LDU$.

14. Reduce A to upper triangular form:

$$\begin{bmatrix} 3 & -12 & 6 \\ 0 & 2 & 0 \\ 6 & -28 & 13 \end{bmatrix} \to \begin{bmatrix} 1 & -4 & 2 \\ 0 & 2 & 0 \\ 6 & -28 & 13 \end{bmatrix} \to \begin{bmatrix} 1 & -4 & 2 \\ 0 & 2 & 0 \\ 0 & -4 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & -4 & 2 \\ 0 & 1 & 0 \\ 0 & -4 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & -4 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = U$$

The multipliers used were $\frac{1}{3}$, -6, $\frac{1}{2}$, and 4, which leads to $L_1 = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 6 & -4 & 1 \end{bmatrix}$.

Since $L_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & -2 & 1 \end{bmatrix}\begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, we conclude that $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & -2 & 1 \end{bmatrix}\begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -4 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is the LDU-decomposition of A.

15. If rows 2 and 3 of A are interchanged, then the resulting matrix has an LU-decomposition.

For $P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, $PA = \begin{bmatrix} 3 & -1 & 0 \\ 0 & 2 & 1 \\ 3 & -1 & 1 \end{bmatrix}$. Reduce PA to upper triangular form:

$$\begin{bmatrix} 3 & -1 & 0 \\ 0 & 2 & 1 \\ 3 & -1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & -\frac{1}{3} & 0 \\ 0 & 2 & 1 \\ 3 & -1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & -\frac{1}{3} & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & -\frac{1}{3} & 0 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix} = U$$

The multipliers used were $\frac{1}{3}$, -3, and $\frac{1}{2}$, so $L = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 3 & 0 & 1 \end{bmatrix}$. Since $P = P^{-1}$, we have

$$A = PLU = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 3 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -\frac{1}{3} & 0 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix}.$$

Since $P\mathbf{b} = \begin{bmatrix} -2 \\ 4 \\ 1 \end{bmatrix}$, the system to solve is $\begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 3 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -\frac{1}{3} & 0 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \\ 1 \end{bmatrix}$.

$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 3 & 0 & 1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \\ 1 \end{bmatrix}$ is

$$\begin{aligned} 3y_1 \qquad\quad &= -2 \\ 2y_2 \quad &= 4 \\ 3y_1 \qquad + y_3 &= 1 \end{aligned}$$

which has the solution $y_1 = -\frac{2}{3},\ y_2 = 2,\ y_3 = 3.$

$\begin{bmatrix} 1 & -\frac{1}{3} & 0 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -\frac{2}{3} \\ 2 \\ 3 \end{bmatrix}$ is

$$\begin{aligned} x_1 - \frac{1}{3}x_2 &= -\frac{2}{3} \\ x_2 + \frac{1}{2}x_3 &= 2 \\ x_3 &= 3 \end{aligned}$$

which gives the solution to the original system: $x_1 = -\frac{1}{2},\ x_2 = \frac{1}{2},\ x_3 = 3.$

16. As discussed in the last subsection of Section 9.1, we introduce a permutation matrix $Q = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Multiplying QA results in interchanging the first two rows of A, so that an LU-decomposition can be found.

$$QA = \begin{bmatrix} 1 & 1 & 4 \\ 0 & 3 & -2 \\ 2 & 2 & 5 \end{bmatrix} \begin{matrix} \leftarrow \text{multiplier} = 1 \\ \\ \\ \end{matrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ \bullet & \bullet & 0 \\ \bullet & \bullet & \bullet \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 4 \\ 0 & 3 & -2 \\ 0 & 0 & -3 \end{bmatrix} \begin{matrix} \\ \leftarrow \text{multiplier} = 0 \\ \leftarrow \text{multiplier} = -2 \end{matrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & \bullet & 0 \\ 2 & \bullet & \bullet \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 4 \\ 0 & 1 & -\frac{2}{3} \\ 0 & 0 & -3 \end{bmatrix} \begin{matrix} \\ \leftarrow \text{multiplier} = \frac{1}{3} \\ \\ \end{matrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 2 & \bullet & \bullet \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 4 \\ 0 & 1 & -\frac{2}{3} \\ 0 & 0 & -3 \end{bmatrix} \begin{matrix} \\ \\ \leftarrow \text{multiplier} = 0 \end{matrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 2 & 0 & \bullet \end{bmatrix}$$

$$U = \begin{bmatrix} 1 & 1 & 4 \\ 0 & 1 & -\frac{2}{3} \\ 0 & 0 & 1 \end{bmatrix} \begin{matrix} \\ \\ \leftarrow \text{multiplier} = -\frac{1}{3} \end{matrix} \qquad L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 2 & 0 & -3 \end{bmatrix}$$

Since $P = Q^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, we obtain a PLU-decomposition of A:

$$A = \begin{bmatrix} 0 & 3 & -2 \\ 1 & 1 & 4 \\ 2 & 2 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 2 & 0 & -3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 4 \\ 0 & 1 & -\frac{2}{3} \\ 0 & 0 & 1 \end{bmatrix} = PLU.$$

Using $P^{-1}\mathbf{b} = \begin{bmatrix} 5 \\ 7 \\ -2 \end{bmatrix}$, we rewrite $A\mathbf{x} = \mathbf{b}$ as $P^{-1}A\mathbf{x} = P^{-1}\mathbf{b}$, i.e., $\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 2 & 0 & -3 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 1 & 1 & 4 \\ 0 & 1 & -\dfrac{2}{3} \\ 0 & 0 & 1 \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\mathbf{x}} = \underbrace{\begin{bmatrix} 5 \\ 7 \\ -2 \end{bmatrix}}_{P^{-1}\mathbf{b}}$.

Solving $\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 2 & 0 & -3 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}}_{\mathbf{y}} = \underbrace{\begin{bmatrix} 5 \\ 7 \\ -2 \end{bmatrix}}_{P^{-1}\mathbf{b}}$ by forward substitution yields $y_1 = 5$, $y_2 = \frac{7}{3}$, $y_3 = 4$.

Solving $\underbrace{\begin{bmatrix} 1 & 1 & 4 \\ 0 & 1 & -\frac{2}{3} \\ 0 & 0 & 1 \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\mathbf{x}} = \underbrace{\begin{bmatrix} 5 \\ \frac{7}{3} \\ 4 \end{bmatrix}}_{\mathbf{y}}$ by back substitution yields $x_1 = -16$, $x_2 = 5$, $x_3 = 4$.

17. Approximately $\frac{2}{3}n^3$ additions and multiplications are required – see Section 9.3.

18. **(a)** If A has such an LU-decomposition, it can be written as

$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ w & 1 \end{bmatrix} \begin{bmatrix} x & y \\ 0 & z \end{bmatrix} = \begin{bmatrix} x & y \\ wx & wy + z \end{bmatrix}$ which leads to the equations

$$\begin{aligned} x &= a \\ y &= b \\ wx &= c \\ wy + z &= d \end{aligned}$$

Since $a \neq 0$, the system has the unique solution $x = a$, $y = b$, $w = \frac{c}{a}$, and $z = d - \frac{bc}{a} = \frac{ad-bc}{a}$. Because the solution is unique, the LU-decomposition is also unique.

(b) From part (a) the LU-decomposition is $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{c}{a} & 1 \end{bmatrix} \begin{bmatrix} a & b \\ 0 & \frac{ad-bc}{a} \end{bmatrix}$.

True-False Exercises

(a) False. If the matrix cannot be reduced to row echelon form without interchanging rows, then it does not have an LU-decomposition.

(b) False. If the row equivalence of A and U requires interchanging rows of A, then A does not have an LU-decomposition.

(c) True. This follows from part (b) of Theorem 1.7.1.

(d) True. (Refer to the subsection "LDU-Decompositions" for the relevant result.)

(e) True. The procedure for obtaining a PLU-decomposition of a matrix A has been described in the subsection "PLU-Decompositions".

9.2 The Power Method

1. **(a)** $\lambda_3 = -8$ is the dominant eigenvalue since $|\lambda_3| = 8$ is greater than the absolute values of all remaining eigenvalues

(b) $|\lambda_1| = |\lambda_4| = 5$; no dominant eigenvalue

2. **(a)** $\lambda_3 = -3$ is the dominant eigenvalue since $|\lambda_3| = 3$ is greater than the absolute values of all remaining eigenvalues

(b) $|\lambda_1| = |\lambda_4| = 3$; no dominant eigenvalue

3. $A\mathbf{x}_0 = \begin{bmatrix} 5 & -1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \end{bmatrix}$ $\qquad$ $\mathbf{x}_1 = \frac{A\mathbf{x}_0}{\|A\mathbf{x}_0\|} = \frac{1}{\sqrt{26}}\begin{bmatrix} 5 \\ -1 \end{bmatrix} \approx \begin{bmatrix} 0.98058 \\ -0.19612 \end{bmatrix}$

$A\mathbf{x}_1 \approx \begin{bmatrix} 5 & -1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} 0.98058 \\ -0.19612 \end{bmatrix} \approx \begin{bmatrix} 5.09902 \\ -0.78446 \end{bmatrix}$ $\qquad$ $\mathbf{x}_2 = \frac{A\mathbf{x}_1}{\|A\mathbf{x}_1\|} \approx \begin{bmatrix} 0.98837 \\ -0.15206 \end{bmatrix}$

$A\mathbf{x}_2 \approx \begin{bmatrix} 5 & -1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} 0.98837 \\ -0.15206 \end{bmatrix} \approx \begin{bmatrix} 5.09391 \\ -0.83631 \end{bmatrix}$ $\qquad$ $\mathbf{x}_3 = \frac{A\mathbf{x}_2}{\|A\mathbf{x}_2\|} \approx \begin{bmatrix} 0.98679 \\ -0.16201 \end{bmatrix}$

$A\mathbf{x}_3 \approx \begin{bmatrix} 5 & -1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} 0.98679 \\ -0.16201 \end{bmatrix} \approx \begin{bmatrix} 5.09596 \\ -0.82478 \end{bmatrix}$ $\qquad$ $\mathbf{x}_4 = \frac{A\mathbf{x}_3}{\|A\mathbf{x}_3\|} \approx \begin{bmatrix} 0.98715 \\ -0.15977 \end{bmatrix}$

$$\lambda^{(1)} = A\mathbf{x}_1 \cdot \mathbf{x}_1 = \left(A\mathbf{x}_1\right)^T \mathbf{x}_1 \approx 5.15385$$

$$\lambda^{(2)} = A\mathbf{x}_2 \cdot \mathbf{x}_2 = \left(A\mathbf{x}_2\right)^T \mathbf{x}_2 \approx 5.16185$$

$$\lambda^{(3)} = A\mathbf{x}_3 \cdot \mathbf{x}_3 = \left(A\mathbf{x}_3\right)^T \mathbf{x}_3 \approx 5.16226$$

$$\lambda^{(4)} = A\mathbf{x}_4 \cdot \mathbf{x}_4 = \left(A\mathbf{x}_4\right)^T \mathbf{x}_4 \approx 5.16228$$

$\det(\lambda I - A) = \begin{vmatrix} \lambda-5 & 1 \\ 1 & \lambda+1 \end{vmatrix} = \lambda^2 - 4\lambda - 6 = \left(\lambda - 2 - \sqrt{10}\right)\left(\lambda - 2 + \sqrt{10}\right)$; the dominant eigenvalue is $2 + \sqrt{10} \approx 5.16228$.

The reduced row echelon form of $\left(2+\sqrt{10}\right)I-A$ is $\begin{bmatrix} 1 & 3+\sqrt{10} \\ 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda = 2+\sqrt{10}$ contains vectors (x_1, x_2) where $x_1 = -\left(3+\sqrt{10}\right)t$, $x_2 = t$,. A vector $\left(-3-\sqrt{10},1\right)$ forms a basis for this eigenspace. We see that $\mathbf{x}_4$ approximates a unit eigenvector $\frac{1}{\sqrt{20+6\sqrt{10}}}\left(3\sqrt{10},-1\right) \approx (0.98709,-0.16018)$ and $\lambda^{(4)}$ approximates the dominant eigenvalue $2+\sqrt{10} \approx 5.16228$.

4. $A\mathbf{x}_0 = \begin{bmatrix} 7 & -2 & 0 \\ -2 & 6 & -2 \\ 0 & -2 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 7 \\ -2 \\ 0 \end{bmatrix}$ $\qquad$ $\mathbf{x}_1 = \frac{A\mathbf{x}_0}{\|A\mathbf{x}_0\|} = \frac{1}{\sqrt{53}}\begin{bmatrix} 7 \\ -2 \\ 0 \end{bmatrix} \approx \begin{bmatrix} 0.96152 \\ -0.27472 \\ 0.00000 \end{bmatrix}$

$A\mathbf{x}_1 \approx \begin{bmatrix} 7 & -2 & 0 \\ -2 & 6 & -2 \\ 0 & -2 & 5 \end{bmatrix}\begin{bmatrix} 0.96152 \\ -0.27472 \\ 0.00000 \end{bmatrix} \approx \begin{bmatrix} 7.28011 \\ -3.57137 \\ 0.54944 \end{bmatrix}$ $\qquad$ $\mathbf{x}_2 = \frac{A\mathbf{x}_1}{\|A\mathbf{x}_1\|} \approx \frac{1}{8.12752}\begin{bmatrix} 7.28011 \\ -3.57137 \\ 0.54944 \end{bmatrix} \approx \begin{bmatrix} 0.89574 \\ -0.43942 \\ 0.06760 \end{bmatrix}$

$A\mathbf{x}_2 \approx \begin{bmatrix} 7 & -2 & 0 \\ -2 & 6 & -2 \\ 0 & -2 & 5 \end{bmatrix}\begin{bmatrix} 0.89574 \\ -0.43942 \\ 0.06760 \end{bmatrix} \approx \begin{bmatrix} 7.14898 \\ -4.56318 \\ 1.21685 \end{bmatrix}$ $\qquad$ $\mathbf{x}_3 = \frac{A\mathbf{x}_2}{\|A\mathbf{x}_2\|} \approx \frac{1}{8.56804}\begin{bmatrix} 7.14898 \\ -4.56318 \\ 1.21685 \end{bmatrix} \approx \begin{bmatrix} 0.83438 \\ -0.53258 \\ 0.14202 \end{bmatrix}$

$A\mathbf{x}_3 \approx \begin{bmatrix} 7 & -2 & 0 \\ -2 & 6 & -2 \\ 0 & -2 & 5 \end{bmatrix}\begin{bmatrix} 0.83438 \\ -0.53258 \\ 0.14202 \end{bmatrix} \approx \begin{bmatrix} 6.90581 \\ -5.14829 \\ 1.77527 \end{bmatrix}$ $\qquad$ $\mathbf{x}_4 = \frac{A\mathbf{x}_3}{\|A\mathbf{x}_3\|} \approx \frac{1}{8.7947}\begin{bmatrix} 6.90581 \\ -5.14829 \\ 1.77527 \end{bmatrix} \approx \begin{bmatrix} 0.78522 \\ -0.58539 \\ 0.20186 \end{bmatrix}$

$$\lambda^{(1)} = A\mathbf{x}_1 \cdot \mathbf{x}_1 = \left(A\mathbf{x}_1\right)^T \mathbf{x}_1 \approx \begin{bmatrix} 7.28011 & -3.57137 & 0.54944 \end{bmatrix}\begin{bmatrix} 0.96152 \\ -0.27472 \\ 0.00000 \end{bmatrix} \approx 7.98113$$

$$\lambda^{(2)} = A\mathbf{x}_2 \cdot \mathbf{x}_2 = \left(A\mathbf{x}_2\right)^T \mathbf{x}_2 \approx \begin{bmatrix} 7.14898 & -4.56318 & 1.21685 \end{bmatrix}\begin{bmatrix} 0.89574 \\ -0.43942 \\ 0.06760 \end{bmatrix} \approx 8.49100$$

$$\lambda^{(3)} = A\mathbf{x}_3 \cdot \mathbf{x}_3 = \left(A\mathbf{x}_3\right)^T \mathbf{x}_3 \approx \begin{bmatrix} 6.90581 & -5.14829 & 1.77527 \end{bmatrix}\begin{bmatrix} 0.83438 \\ -0.53258 \\ 0.14202 \end{bmatrix} \approx 8.75607$$

$$\lambda^{(4)} = A\mathbf{x}_4 \cdot \mathbf{x}_4 = \left(A\mathbf{x}_4\right)^T \mathbf{x}_4 \approx \begin{bmatrix} 6.66734 & -5.48648 & 2.18006 \end{bmatrix}\begin{bmatrix} 0.78522 \\ -0.58539 \\ 0.20186 \end{bmatrix} \approx 8.88712$$

$$\det(\lambda I - A) = \begin{vmatrix} \lambda-7 & 2 & 0 \\ 2 & \lambda-6 & 2 \\ 0 & 2 & \lambda-5 \end{vmatrix} = (\lambda-3)(\lambda-6)(\lambda-9)$$; the dominant eigenvalue is 9.

The reduced row echelon form of $9I - A$ is $\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda = 9$ contains vectors (x_1, x_2, x_3) where $x_1 = 2t$, $x_2 = -2t$, $x_3 = t$. A vector $(2,-2,1)$ forms a basis for this eigenspace. We see that $\mathbf{x}_4$ approximates the unit eigenvector $\left(\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}\right) \approx (0.66667, -0.66667, 0.33333)$ and $\lambda^{(4)}$ approximates the dominant eigenvalue 9.

5. $A\mathbf{x}_0 = \begin{bmatrix} 1 & -3 \\ -3 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$ $\qquad$ $\mathbf{x}_1 = \frac{A\mathbf{x}_0}{\max(A\mathbf{x}_0)} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$A\mathbf{x}_1 \approx \begin{bmatrix} 1 & -3 \\ -3 & 5 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -4 \\ 8 \end{bmatrix}$ $\qquad$ $\mathbf{x}_2 = \frac{A\mathbf{x}_1}{\max(A\mathbf{x}_1)} = \begin{bmatrix} -0.5 \\ 1 \end{bmatrix}$

$A\mathbf{x}_2 \approx \begin{bmatrix} 1 & -3 \\ -3 & 5 \end{bmatrix}\begin{bmatrix} -0.5 \\ 1 \end{bmatrix} \approx \begin{bmatrix} -3.5 \\ 6.5 \end{bmatrix}$ $\qquad$ $\mathbf{x}_3 = \frac{A\mathbf{x}_2}{\max(A\mathbf{x}_2)} \approx \begin{bmatrix} -0.53846 \\ 1 \end{bmatrix}$

$A\mathbf{x}_3 \approx \begin{bmatrix} 1 & -3 \\ -3 & 5 \end{bmatrix}\begin{bmatrix} -0.53846 \\ 1 \end{bmatrix} \approx \begin{bmatrix} -3.53846 \\ 6.61538 \end{bmatrix}$ $\qquad$ $\mathbf{x}_4 = \frac{A\mathbf{x}_3}{\max(A\mathbf{x}_3)} \approx \begin{bmatrix} -0.53488 \\ 1 \end{bmatrix}$

$$\lambda^{(1)} = \frac{A\mathbf{x}_1 \cdot \mathbf{x}_1}{\mathbf{x}_1 \cdot \mathbf{x}_1} = 6$$
$$\lambda^{(2)} = \frac{A\mathbf{x}_2 \cdot \mathbf{x}_2}{\mathbf{x}_2 \cdot \mathbf{x}_2} = 6.6$$
$$\lambda^{(3)} = \frac{A\mathbf{x}_3 \cdot \mathbf{x}_3}{\mathbf{x}_3 \cdot \mathbf{x}_3} \approx 6.60550$$
$$\lambda^{(4)} = \frac{A\mathbf{x}_4 \cdot \mathbf{x}_4}{\mathbf{x}_4 \cdot \mathbf{x}_4} \approx 6.60555$$

$\det(\lambda I - A) = \begin{vmatrix} \lambda-1 & 3 \\ 3 & \lambda-5 \end{vmatrix} = \lambda^2 - 6\lambda - 4$, so the eigenvalues of A are $\lambda = 3 \pm \sqrt{13}$. The dominant eigenvalue is $3 + \sqrt{13} \approx 6.60555$ with corresponding scaled eigenvector $\begin{bmatrix} \frac{2-\sqrt{13}}{3} \\ 1 \end{bmatrix} \approx \begin{bmatrix} -0.53518 \\ 1 \end{bmatrix}$.

6. $A\mathbf{x}_0 = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 2 & 0 \\ 2 & 0 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \\ 6 \end{bmatrix}$ $\qquad$ $\mathbf{x}_1 = \frac{A\mathbf{x}_0}{\max(A\mathbf{x}_0)} = \frac{1}{7}\begin{bmatrix} 7 \\ 4 \\ 6 \end{bmatrix} \approx \begin{bmatrix} 1.00000 \\ 0.57143 \\ 0.85714 \end{bmatrix}$

$$A\mathbf{x}_1 \approx \begin{bmatrix} 3 & 2 & 2 \\ 2 & 2 & 0 \\ 2 & 0 & 4 \end{bmatrix}\begin{bmatrix} 1.00000 \\ 0.57143 \\ 0.85714 \end{bmatrix} \approx \begin{bmatrix} 5.85714 \\ 3.14286 \\ 5.42857 \end{bmatrix} \qquad \mathbf{x}_2 = \frac{A\mathbf{x}_1}{\max(A\mathbf{x}_1)} \approx \frac{1}{5.85714}\begin{bmatrix} 5.85714 \\ 3.14286 \\ 5.42857 \end{bmatrix} \approx \begin{bmatrix} 1.00000 \\ 0.53659 \\ 0.92683 \end{bmatrix}$$

$$A\mathbf{x}_2 \approx \begin{bmatrix} 3 & 2 & 2 \\ 2 & 2 & 0 \\ 2 & 0 & 4 \end{bmatrix}\begin{bmatrix} 1.00000 \\ 0.53659 \\ 0.92683 \end{bmatrix} \approx \begin{bmatrix} 5.92683 \\ 3.07317 \\ 5.70732 \end{bmatrix} \qquad \mathbf{x}_3 = \frac{A\mathbf{x}_2}{\max(A\mathbf{x}_2)} \approx \frac{1}{5.92683}\begin{bmatrix} 5.92683 \\ 3.07317 \\ 5.70732 \end{bmatrix} \approx \begin{bmatrix} 1.00000 \\ 0.51852 \\ 0.96296 \end{bmatrix}$$

$$A\mathbf{x}_3 \approx \begin{bmatrix} 3 & 2 & 2 \\ 2 & 2 & 0 \\ 2 & 0 & 4 \end{bmatrix}\begin{bmatrix} 1.00000 \\ 0.51852 \\ 0.96296 \end{bmatrix} \approx \begin{bmatrix} 5.96296 \\ 3.03704 \\ 5.85185 \end{bmatrix} \qquad \mathbf{x}_4 = \frac{A\mathbf{x}_3}{\max(A\mathbf{x}_3)} \approx \frac{1}{5.96296}\begin{bmatrix} 5.96296 \\ 3.03704 \\ 5.85185 \end{bmatrix} \approx \begin{bmatrix} 1.00000 \\ 0.50932 \\ 0.98137 \end{bmatrix}$$

$$\lambda^{(1)} = \frac{A\mathbf{x}_1 \cdot \mathbf{x}_1}{\mathbf{x}_1 \cdot \mathbf{x}_1} = \frac{(A\mathbf{x}_1)^T \mathbf{x}_1}{\mathbf{x}_1^T \mathbf{x}_1} \approx \frac{12.30612}{2.06122} \approx 5.97030$$

$$\lambda^{(2)} = \frac{A\mathbf{x}_2 \cdot \mathbf{x}_2}{\mathbf{x}_2 \cdot \mathbf{x}_2} = \frac{(A\mathbf{x}_2)^T \mathbf{x}_2}{\mathbf{x}_2^T \mathbf{x}_2} \approx \frac{12.86556}{2.14694} \approx 5.99252$$

$$\lambda^{(3)} = \frac{A\mathbf{x}_3 \cdot \mathbf{x}_3}{\mathbf{x}_3 \cdot \mathbf{x}_3} = \frac{(A\mathbf{x}_3)^T \mathbf{x}_3}{\mathbf{x}_3^T \mathbf{x}_3} \approx \frac{13.17284}{2.19616} \approx 5.99813$$

$$\lambda^{(4)} = \frac{A\mathbf{x}_4 \cdot \mathbf{x}_4}{\mathbf{x}_4 \cdot \mathbf{x}_4} = \frac{(A\mathbf{x}_4)^T \mathbf{x}_4}{\mathbf{x}_4^T \mathbf{x}_4} \approx \frac{13.33386}{2.22248} \approx 5.99953$$

$\det(\lambda I - A) = \begin{vmatrix} \lambda-3 & -2 & -2 \\ -2 & \lambda-2 & 0 \\ -2 & 0 & \lambda-4 \end{vmatrix} = \lambda(\lambda-3)(\lambda-6)$; the dominant eigenvalue is 6.

The reduced row echelon form of $6I - A$ is $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda = 6$ contains vectors (x_1, x_2, x_3) where $x_1 = t$, $x_2 = \frac{1}{2}t$, $x_3 = t$. A vector $(1, \frac{1}{2}, 1)$ forms a basis for this eigenspace. We see that $\mathbf{x}_4$ approximates the eigenvector $(1, \frac{1}{2}, 1)$ and $\lambda^{(4)}$ approximates the dominant eigenvalue 6.

7. (a) $A\mathbf{x}_0 = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ $\qquad \mathbf{x}_1 = \frac{A\mathbf{x}_0}{\max(A\mathbf{x}_0)} = \begin{bmatrix} 1 \\ -0.5 \end{bmatrix}$

$A\mathbf{x}_1 = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ -0.5 \end{bmatrix} = \begin{bmatrix} 2.5 \\ 2 \end{bmatrix}$ $\qquad \mathbf{x}_2 = \frac{A\mathbf{x}_1}{\max(A\mathbf{x}_1)} = \begin{bmatrix} 1 \\ -0.8 \end{bmatrix}$

$$A\mathbf{x}_2 = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ -0.8 \end{bmatrix} = \begin{bmatrix} 2.8 \\ -2.6 \end{bmatrix} \qquad \mathbf{x}_3 = \frac{A\mathbf{x}_2}{\max(A\mathbf{x}_2)} \approx \begin{bmatrix} 1 \\ -0.929 \end{bmatrix}$$

(b) $\lambda^{(1)} = \frac{A\mathbf{x}_1 \cdot \mathbf{x}_1}{\mathbf{x}_1 \cdot \mathbf{x}_1} = 2.8$; $\lambda^{(2)} = \frac{A\mathbf{x}_2 \cdot \mathbf{x}_2}{\mathbf{x}_2 \cdot \mathbf{x}_2} \approx 2.976$; $\lambda^{(3)} = \frac{A\mathbf{x}_3 \cdot \mathbf{x}_3}{\mathbf{x}_3 \cdot \mathbf{x}_3} \approx 2.997$

(c) $\det(\lambda I - A) = \begin{vmatrix} \lambda-2 & 1 \\ 1 & \lambda-2 \end{vmatrix} = (\lambda-3)(\lambda-1)$; the dominant eigenvalue is 3.

The reduced row echelon form of $3I - A$ is $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda = 3$ contains vectors (x_1, x_2) where $x_1 = -t$, $x_2 = t$. A vector $(-1,1)$ forms a basis for this eigenspace. We see that $\mathbf{x}_3$ approximates the eigenvector $(1,-1)$ and $\lambda^{(3)}$ approximates the dominant eigenvalue 3.

(d) The percentage error is $\left|\frac{\lambda - \lambda^{(3)}}{\lambda}\right| \approx \left|\frac{3-2.997}{3}\right| = 0.001 = 0.1\%$.

8. **(a)**

$$A\mathbf{x}_0 = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 10 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 10 \end{bmatrix} \qquad \mathbf{x}_1 = \frac{A\mathbf{x}_0}{\max(A\mathbf{x}_0)} = \frac{1}{10}\begin{bmatrix} 3 \\ 3 \\ 10 \end{bmatrix} = \begin{bmatrix} 0.3 \\ 0.3 \\ 1.0 \end{bmatrix}$$

$$A\mathbf{x}_1 = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 10 \end{bmatrix}\begin{bmatrix} 0.3 \\ 0.3 \\ 1.0 \end{bmatrix} = \begin{bmatrix} 0.9 \\ 0.9 \\ 10.0 \end{bmatrix} \qquad \mathbf{x}_2 = \frac{A\mathbf{x}_1}{\max(A\mathbf{x}_1)} = \frac{1}{10}\begin{bmatrix} 0.9 \\ 0.9 \\ 10.0 \end{bmatrix} = \begin{bmatrix} 0.09 \\ 0.09 \\ 1.00 \end{bmatrix}$$

$$A\mathbf{x}_2 = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 10 \end{bmatrix}\begin{bmatrix} 0.09 \\ 0.09 \\ 1.00 \end{bmatrix} = \begin{bmatrix} 0.27 \\ 0.27 \\ 10.00 \end{bmatrix} \qquad \mathbf{x}_3 = \frac{A\mathbf{x}_2}{\max(A\mathbf{x}_2)} = \frac{1}{10}\begin{bmatrix} 0.27 \\ 0.27 \\ 10.00 \end{bmatrix} = \begin{bmatrix} 0.027 \\ 0.027 \\ 1.000 \end{bmatrix}$$

(b)

$$\lambda^{(1)} = \frac{A\mathbf{x}_1 \cdot \mathbf{x}_1}{\mathbf{x}_1 \cdot \mathbf{x}_1} = \frac{(A\mathbf{x}_1)^T \mathbf{x}_1}{\mathbf{x}_1^T \mathbf{x}_1} \approx \frac{10.54}{1.18} \approx 8.932$$

$$\lambda^{(2)} = \frac{A\mathbf{x}_2 \cdot \mathbf{x}_2}{\mathbf{x}_2 \cdot \mathbf{x}_2} = \frac{(A\mathbf{x}_2)^T \mathbf{x}_2}{\mathbf{x}_2^T \mathbf{x}_2} \approx \frac{10.049}{1.016} \approx 9.889$$

$$\lambda^{(3)} = \frac{A\mathbf{x}_3 \cdot \mathbf{x}_3}{\mathbf{x}_3 \cdot \mathbf{x}_3} = \frac{(A\mathbf{x}_3)^T \mathbf{x}_3}{\mathbf{x}_3^T \mathbf{x}_3} \approx \frac{10.004}{1.001} \approx 9.990$$

(c) $\det(\lambda I - A) = \begin{vmatrix} \lambda-2 & -1 & 0 \\ -1 & \lambda-2 & 0 \\ 0 & 0 & \lambda-10 \end{vmatrix} = (\lambda-1)(\lambda-3)(\lambda-10)$; the dominant eigenvalue is 10.

The reduced row echelon form of $10I - A$ is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda = 10$ contains vectors (x_1, x_2, x_3) where $x_1 = 0$, $x_2 = 0$, $x_3 = t$. A vector $(0,0,1)$ forms a basis for this eigenspace. We see that $\mathbf{x}_3$ approximates the eigenvector $(0,0,1)$ and $\lambda^{(3)}$ approximates the dominant eigenvalue 10.

(d) The percentage error in the approximation $\lambda^{(3)} \approx 9.99$ of the dominant eigenvalue $\lambda = 10$ is

$\left|\frac{\lambda-\lambda^{(3)}}{\lambda}\right| = \left|\frac{10-9.99}{10}\right| = 0.001 = 0.1\%$

9. By Formula (10), $\mathbf{x}_5 = \frac{A^5\mathbf{x}_0}{\max(A^5\mathbf{x}_0)} \approx \begin{bmatrix} 0.99180 \\ 1 \end{bmatrix}$. Thus $\lambda^{(5)} = \frac{A\mathbf{x}_5 \cdot \mathbf{x}_5}{\mathbf{x}_5 \cdot \mathbf{x}_5} \approx 2.99993.$

10. By Formula (10), $\mathbf{x}_5 = \frac{A^5\mathbf{x}_0}{\max(A^5\mathbf{x}_0)} \approx \begin{bmatrix} 1 \\ 0.99180 \end{bmatrix}$. Thus $\lambda^{(5)} = \frac{A\mathbf{x}_5 \cdot \mathbf{x}_5}{\mathbf{x}_5 \cdot \mathbf{x}_5} \approx 2.99993.$

11. By inspection, A is symmetric and has a dominant eigenvalue -1.
Assuming $a \neq 0$, the power sequence is

$A\mathbf{x}_0 = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -a \\ 0 \end{bmatrix}$ $\qquad$ $\mathbf{x}_1 = \frac{A\mathbf{x}_0}{\|A\mathbf{x}_0\|} = \frac{1}{|a|}\begin{bmatrix} -a \\ 0 \end{bmatrix} = \begin{bmatrix} -a/|a| \\ 0 \end{bmatrix}$

$A\mathbf{x}_1 \approx \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} -a/|a| \\ 0 \end{bmatrix} = \begin{bmatrix} a/|a| \\ 0 \end{bmatrix}$ $\qquad$ $\mathbf{x}_2 = \frac{A\mathbf{x}_1}{\|A\mathbf{x}_1\|} = \frac{1}{1}\begin{bmatrix} a/|a| \\ 0 \end{bmatrix} = \begin{bmatrix} a/|a| \\ 0 \end{bmatrix}$

$A\mathbf{x}_2 \approx \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} a/|a| \\ 0 \end{bmatrix} = \begin{bmatrix} -a/|a| \\ 0 \end{bmatrix}$ $\qquad$ $\mathbf{x}_3 = \frac{A\mathbf{x}_2}{\|A\mathbf{x}_2\|} = \frac{1}{1}\begin{bmatrix} -a/|a| \\ 0 \end{bmatrix} = \begin{bmatrix} -a/|a| \\ 0 \end{bmatrix}$

$A\mathbf{x}_3 \approx \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} -a/|a| \\ 0 \end{bmatrix} \approx \begin{bmatrix} a/|a| \\ 0 \end{bmatrix}$ $\qquad$ $\mathbf{x}_4 = \frac{A\mathbf{x}_3}{\|A\mathbf{x}_3\|} = \frac{1}{1}\begin{bmatrix} a/|a| \\ 0 \end{bmatrix} = \begin{bmatrix} a/|a| \\ 0 \end{bmatrix}$

$\vdots$ $\qquad$ $\vdots$

The quantity $a/|a|$ is equal to 1 if $a > 0$ and -1 if $a < 0$. Since the power sequence continues to oscillate between $\begin{bmatrix} -1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, it does not converge.

12. **(a)** E.g., choose $\mathbf{x}_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

$$\mathbf{x}_1 = \frac{A\mathbf{x}_0}{\|A\mathbf{x}_0\|} \approx \begin{bmatrix} 0.28604 \\ -0.09535 \\ 0.95346 \end{bmatrix} \qquad \lambda^{(1)} = A\mathbf{x}_1 \cdot \mathbf{x}_1 \approx 10.86364$$

$$\mathbf{x}_2 = \frac{A\mathbf{x}_1}{\|A\mathbf{x}_1\|} \approx \begin{bmatrix} 0.26286 \\ -0.04381 \\ 0.96384 \end{bmatrix} \qquad \lambda^{(2)} = A\mathbf{x}_2 \cdot \mathbf{x}_2 \approx 10.90211 \qquad \left|\frac{\lambda^{(2)} - \lambda^{(1)}}{\lambda^{(2)}}\right| \approx 0.00353 = 0.353\%$$

$$\mathbf{x}_3 = \frac{A\mathbf{x}_2}{\|A\mathbf{x}_2\|} \approx \begin{bmatrix} 0.27723 \\ -0.03214 \\ 0.96027 \end{bmatrix} \qquad \lambda^{(3)} = A\mathbf{x}_3 \cdot \mathbf{x}_3 \approx 10.90765 \qquad \left|\frac{\lambda^{(3)} - \lambda^{(2)}}{\lambda^{(3)}}\right| \approx 0.00051 = 0.051\%$$

(b) E.g., choose $\mathbf{x}_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$.

$$\mathbf{x}_1 = \frac{A\mathbf{x}_0}{\|A\mathbf{x}_0\|} \approx \begin{bmatrix} 0.12217 \\ 0.12217 \\ 0.12217 \\ 0.97736 \end{bmatrix} \qquad \lambda^{(1)} = A\mathbf{x}_1 \cdot \mathbf{x}_1 \approx 8.46269$$

$$\mathbf{x}_2 = \frac{A\mathbf{x}_1}{\|A\mathbf{x}_1\|} \approx \begin{bmatrix} 0.14413 \\ 0.12971 \\ 0.17295 \\ 0.96565 \end{bmatrix} \qquad \lambda^{(2)} = A\mathbf{x}_2 \cdot \mathbf{x}_2 \approx 8.50187 \qquad \left|\frac{\lambda^{(2)} - \lambda^{(1)}}{\lambda^{(2)}}\right| \approx 0.05467 = 5.467\%$$

$$\mathbf{x}_3 = \frac{A\mathbf{x}_2}{\|A\mathbf{x}_2\|} \approx \begin{bmatrix} 0.15083 \\ 0.12371 \\ 0.19658 \\ 0.96089 \end{bmatrix} \qquad \lambda^{(3)} = A\mathbf{x}_3 \cdot \mathbf{x}_3 \approx 8.51040 \qquad \left|\frac{\lambda^{(3)} - \lambda^{(2)}}{\lambda^{(3)}}\right| \approx 0.00461 = 0.461\%$$

$$\mathbf{x}_4 = \frac{A\mathbf{x}_3}{\|A\mathbf{x}_3\|} \approx \begin{bmatrix} 0.15372 \\ 0.11887 \\ 0.20847 \\ 0.95853 \end{bmatrix} \qquad \lambda^{(4)} = A\mathbf{x}_4 \cdot \mathbf{x}_4 \approx 8.51272 \qquad \left|\frac{\lambda^{(4)} - \lambda^{(3)}}{\lambda^{(4)}}\right| \approx 0.00027 = 0.027\%$$

13. **(a)** Starting with $\mathbf{x}_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, it takes 8 iterations.

$$\mathbf{x}_1 \approx \begin{bmatrix} 0.229 \\ 0.668 \\ 0.668 \end{bmatrix}, \quad \lambda^{(1)} \approx 7.632$$

$$\mathbf{x}_2 \approx \begin{bmatrix} 0.507 \\ 0.320 \\ 0.800 \end{bmatrix}, \quad \lambda^{(2)} \approx 9.968$$

$$\mathbf{x}_3 \approx \begin{bmatrix} 0.380 \\ 0.197 \\ 0.904 \end{bmatrix}, \quad \lambda^{(3)} \approx 10.622$$

$$\mathbf{x}_4 \approx \begin{bmatrix} 0.344 \\ 0.096 \\ 0.934 \end{bmatrix}, \quad \lambda^{(4)} \approx 10.827$$

$$\mathbf{x}_5 \approx \begin{bmatrix} 0.317 \\ 0.044 \\ 0.948 \end{bmatrix}, \quad \lambda^{(5)} \approx 10.886$$

$$\mathbf{x}_6 \approx \begin{bmatrix} 0.302 \\ 0.016 \\ 0.953 \end{bmatrix}, \quad \lambda^{(6)} \approx 10.903$$

$$\mathbf{x}_7 \approx \begin{bmatrix} 0.294 \\ 0.002 \\ 0.956 \end{bmatrix}, \quad \lambda^{(7)} \approx 10.908$$

$$\mathbf{x}_8 \approx \begin{bmatrix} 0.290 \\ -0.006 \\ 0.957 \end{bmatrix}, \quad \lambda^{(8)} \approx 10.909$$

(b) Starting with $\mathbf{x}_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, it takes 8 iterations.

$$\mathbf{x}_1 \approx \begin{bmatrix} 0.577 \\ 0 \\ 0.577 \\ 0.577 \end{bmatrix}, \quad \lambda^{(1)} \approx 6.333$$

$$\mathbf{x}_2 \approx \begin{bmatrix} 0.249 \\ 0 \\ 0.498 \\ 0.830 \end{bmatrix}, \quad \lambda^{(2)} \approx 8.062$$

$$\mathbf{x}_3 = \begin{bmatrix} 0.193 \\ 0.041 \\ 0.376 \\ 0.905 \end{bmatrix}, \quad \lambda^{(3)} \approx 8.382$$

$$\mathbf{x}_4 \approx \begin{bmatrix} 0.175 \\ 0.073 \\ 0.305 \\ 0.933 \end{bmatrix}, \quad \lambda^{(4)} \approx 8.476$$

$$\mathbf{x}_5 \approx \begin{bmatrix} 0.167 \\ 0.091 \\ 0.266 \\ 0.945 \end{bmatrix}, \quad \lambda^{(5)} \approx 8.503$$

$$\mathbf{x}_6 \approx \begin{bmatrix} 0.162 \\ 0.101 \\ 0.245 \\ 0.951 \end{bmatrix}, \quad \lambda^{(6)} \approx 8.511$$

$$\mathbf{x}_7 \approx \begin{bmatrix} 0.159 \\ 0.107 \\ 0.234 \\ 0.953 \end{bmatrix}, \quad \lambda^{(7)} \approx 8.513$$

$$\mathbf{x}_8 \approx \begin{bmatrix} 0.158 \\ 0.110 \\ 0.228 \\ 0.954 \end{bmatrix}, \quad \lambda^{(8)} \approx 8.513$$

14. (a) E.g., choose $\mathbf{x}_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

$$\mathbf{x}_1 = \frac{A\mathbf{x}_0}{\max(A\mathbf{x}_0)} \approx \begin{bmatrix} 0.3 \\ -0.1 \\ 1.0 \end{bmatrix} \qquad \lambda^{(1)} = \frac{A\mathbf{x}_1 \cdot \mathbf{x}_1}{\mathbf{x}_1 \cdot \mathbf{x}_1} \approx 10.86364$$

$$\mathbf{x}_2 = \frac{A\mathbf{x}_1}{\max(A\mathbf{x}_1)} \approx \begin{bmatrix} 0.27273 \\ -0.04545 \\ 1.00000 \end{bmatrix} \qquad \lambda^{(2)} = \frac{A\mathbf{x}_2 \cdot \mathbf{x}_2}{\mathbf{x}_2 \cdot \mathbf{x}_2} \approx 10.90211 \qquad \left|\frac{\lambda^{(2)}-\lambda^{(1)}}{\lambda^{(2)}}\right| \approx 0.00353 = 0.353\%$$

$$\mathbf{x}_3 = \frac{A\mathbf{x}_2}{\max(A\mathbf{x}_2)} \approx \begin{bmatrix} 0.28870 \\ -0.03347 \\ 1.00000 \end{bmatrix} \qquad \lambda^{(3)} = \frac{A\mathbf{x}_3 \cdot \mathbf{x}_3}{\mathbf{x}_3 \cdot \mathbf{x}_3} \approx 10.90765 \qquad \left|\frac{\lambda^{(3)}-\lambda^{(2)}}{\lambda^{(3)}}\right| \approx 0.00051 = 0.051\%$$

(b) E.g., choose $\mathbf{x}_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$.

$$\mathbf{x}_1 = \frac{A\mathbf{x}_0}{\max(A\mathbf{x}_0)} = \begin{bmatrix} 0.125 \\ 0.125 \\ 0.125 \\ 1.000 \end{bmatrix} \qquad \lambda^{(1)} = \frac{A\mathbf{x}_1 \cdot \mathbf{x}_1}{\mathbf{x}_1 \cdot \mathbf{x}_1} \approx 8.46269$$

$$\mathbf{x}_2 = \frac{A\mathbf{x}_1}{\max(A\mathbf{x}_1)} \approx \begin{bmatrix} 0.14925 \\ 0.13433 \\ 0.17910 \\ 1.00000 \end{bmatrix} \qquad \lambda^{(2)} = \frac{A\mathbf{x}_2 \cdot \mathbf{x}_2}{\mathbf{x}_2 \cdot \mathbf{x}_2} \approx 8.50187 \qquad \left|\frac{\lambda^{(2)}-\lambda^{(1)}}{\lambda^{(2)}}\right| \approx 0.05467 = 5.467\%$$

$$\mathbf{x}_3 = \frac{A\mathbf{x}_2}{\max(A\mathbf{x}_2)} \approx \begin{bmatrix} 0.15697 \\ 0.12875 \\ 0.20459 \\ 1.00000 \end{bmatrix} \qquad \lambda^{(3)} = \frac{A\mathbf{x}_3 \cdot \mathbf{x}_3}{\mathbf{x}_3 \cdot \mathbf{x}_3} \approx 8.51040 \qquad \left|\frac{\lambda^{(3)}-\lambda^{(2)}}{\lambda^{(3)}}\right| \approx 0.00461 = 0.461\%$$

$$\mathbf{x}_4 = \frac{A\mathbf{x}_3}{\max(A\mathbf{x}_3)} \approx \begin{bmatrix} 0.16037 \\ 0.12401 \\ 0.21749 \\ 1.00000 \end{bmatrix} \qquad \lambda^{(4)} = \frac{A\mathbf{x}_4 \cdot \mathbf{x}_4}{\mathbf{x}_4 \cdot \mathbf{x}_4} \approx 8.51272 \qquad \left|\frac{\lambda^{(4)}-\lambda^{(3)}}{\lambda^{(4)}}\right| \approx 0.00027 = 0.027\%$$

9.3 Comparison of Procedures for Solving Linear Systems

1. **(a)** For $n = 1000 = 10^3$, the flops for both phases is $\frac{2}{3}(10^3)^3 + \frac{3}{2}(10^3)^2 - \frac{7}{6}\left(10^3\right) = 668{,}165{,}500$, which is 0.6681655 gigaflops, so it will take $0.6681655 \times 10^{-1} \approx 0.067$ second.

(b) $n = 10{,}000 = 10^4$: $\frac{2}{3}(10^4)^3 + \frac{3}{2}(10^4)^2 - \frac{7}{6}\left(10^4\right) = 666{,}816{,}655{,}000$ flops or 666.816655 gigaflops. The time is about 66.68 seconds.

(c) $n = 100{,}000 = 10^5$; $\frac{2}{3}(10^5)^3 + \frac{3}{2}(10^5)^2 - \frac{7}{6}\left(10^5\right) \approx 666{,}682 \times 10^9$ flops or 666,682 gigaflops. The time is about 66,668 seconds which is about 18.5 hours.

2. **(a)** The number of gigaflops required is $\left(\frac{2}{3}n^3 + \frac{3}{2}n^2 - \frac{7}{6}n\right)10^{-9} \approx 666.817$. At 100 gigaflops per second, the time required to solve the system is approximately 6.66817 seconds.

(b) The number of gigaflops required is $\left(\frac{2}{3}n^3 + \frac{3}{2}n^2 - \frac{7}{6}n\right)10^{-9} \approx 666{,}681.667$. At 100 gigaflops per second, the time required to solve the system is approximately 6666.817 seconds (i.e., 1 hour, 51 minutes, and 6.817 seconds).

(c) The number of gigaflops required is $\left(\frac{2}{3}n^3 + \frac{3}{2}n^2 - \frac{7}{6}n\right)10^{-9} \approx 6.666682 \times 10^8$. At 100 gigaflops per second, the time required to solve the system is approximately $6{,}666{,}682$ seconds (i.e., 77 days, 3 hours, 51 minutes, and 22 seconds).

3. $n = 10{,}000 = 10^4$

(a) $\frac{2}{3}n^3 \approx \frac{2}{3}\left(10^{12}\right) \approx 666.67 \times 10^9$;
666.67 gigaflops are required, which will take $\frac{666.67}{70} \approx 9.52$ seconds.

(b) $n^2 \approx 10^8 = 0.1 \times 10^9$; 0.1 gigaflop is required, which will take about 0.0014 second.

(c) This is the same as part (a); about 9.52 seconds.

(d) $2n^3 \approx 2 \times 10^{12} = 2000 \times 10^9$;
2000 gigaflops are required, which will take about 28.57 seconds.

4. **(a)** The number of petaflops required is approximately $\frac{2}{3}n^3 10^{-15} \approx 0.66667$, therefore the time required for the forward phase of Gauss-Jordan elimination is approximately 0.041667 seconds.

(b) The number of petaflops required is approximately $n^2 10^{-15} \approx 0.00001$, therefore the time required for the backward phase of Gauss-Jordan elimination is approximately 0.000000625 seconds.

(c) The number of petaflops required is approximately $\frac{2}{3}n^3 10^{-15} \approx 0.66667$, therefore the time required for the LU-decomposition is approximately 0.041667 seconds.

(d) The number of petaflops required is approximately $2n^3 10^{-15} \approx 2$, therefore the time required for the computation of A^{-1} by reducing $[A\,|\,I]$ to $[I\,|\,A^{-1}]$ is approximately 0.125 seconds.

5. **(a)** $n = 100{,}000 = 10^5$; $\frac{2}{3}n^3 \approx \frac{2}{3} \times 10^{15} \approx 0.667 \times 10^{15} = 6.67 \times 10^5 \times 10^9$;

Thus, the forward phase would require about 6.67×10^5 seconds.
$n^2 = 10^{10} = 10 \times 10^9$; The backward phase would require about 10 seconds.

(b) $n = 10{,}000 = 10^4$; $\frac{2}{3}n^3 \approx \frac{2}{3} \times 10^{12} \approx 0.667 \times 10^{12} \approx 6.67 \times 10^2 \times 10^9$;
About 667 gigaflops are required, so the computer would have to execute 2(667) = 1334 gigaflops per second.

6. The number of teraflops required is approximately $2n^3 10^{-12} \approx 2000$. A computer must be able to execute more than 4000 teraflops per second to be able to find A^{-1} in less than 0.5 seconds.

7. Multiplying each of the n^2 entries of A by c requires n^2 flops.

8. n^2 flops are required to compute $A + B$.

9. Let $C = \left[c_{ij}\right] = AB$. Computing c_{ij} requires first multiplying each of the n entries a_{ik} by the corresponding entry b_{kj}, which requires n flops. Then the n terms $a_{ik}b_{kj}$ must be summed, which requires $n - 1$ flops. Thus, each of the n^2 entries in AB requires $2n - 1$ flops, for a total of $n^2(2n-1) = 2n^3 - n^2$ flops. Note that adding two numbers requires 1 flop, adding three numbers requires 2 flops, and in general, $n - 1$ flops are required to add n numbers.

10. Each diagonal entry can be obtained using $k-1$ multiplications, so the computation of A^k would involve $n(k-1)$ flops overall. (Note that the number of flops can be reduced to $n \log_2 k$.)

9.4 Singular Value Decomposition

1. The characteristic polynomial of $A^TA = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 4 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ is $\lambda^2(\lambda - 5)$; thus the eigenvalues of A^TA are $\lambda_1 = 5$ and $\lambda_2 = 0$, and $\sigma_1 = \sqrt{5}$ and $\sigma_2 = 0$ are singular values of A.

2. $A^TA = \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 9 & 0 \\ 0 & 16 \end{bmatrix}$; $\det(\lambda I - A^TA) = \begin{vmatrix} \lambda - 9 & 0 \\ 0 & \lambda - 16 \end{vmatrix} = (\lambda - 9)(\lambda - 16)$;
the eigenvalues of A^TA are $\lambda_1 = 16$ and $\lambda_2 = 9$ therefore the singular values of A are $\sigma_1 = \sqrt{\lambda_1} = 4$ and $\sigma_2 = \sqrt{\lambda_2} = 3$.

3. The eigenvalues of $A^TA = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$ are $\lambda_1 = 5$ and $\lambda_2 = 5$ (i.e., $\lambda = 5$ is an eigenvalue of multiplicity 2); thus the singular value of A is $\sigma_1 = \sqrt{5}$.

4. $A^TA = \begin{bmatrix} \sqrt{2} & 1 \\ 0 & \sqrt{2} \end{bmatrix}\begin{bmatrix} \sqrt{2} & 0 \\ 1 & \sqrt{2} \end{bmatrix} = \begin{bmatrix} 3 & \sqrt{2} \\ \sqrt{2} & 2 \end{bmatrix}$; $\det(\lambda I - A^TA) = \begin{vmatrix} \lambda - 3 & -\sqrt{2} \\ -\sqrt{2} & \lambda - 2 \end{vmatrix} = (\lambda - 1)(\lambda - 4)$;

the eigenvalues of A^TA are $\lambda_1 = 4$ and $\lambda_2 = 1$ therefore the singular values of A are $\sigma_1 = \sqrt{\lambda_1} = 2$ and $\sigma_2 = \sqrt{\lambda_2} = 1$.

5. The only eigenvalue of $A^TA = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ is $\lambda = 2$ (multiplicity 2), and the vectors $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ form an orthonormal basis for the eigenspace (which is all of R^2).

The singular values of A are $\sigma_1 = \sqrt{2}$ and $\sigma_2 = \sqrt{2}$. We have $\mathbf{u}_1 = \frac{1}{\sigma_1}A\mathbf{v}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$, and

$\mathbf{u}_2 = \frac{1}{\sigma_2}A\mathbf{v}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$.

This results in the following singular value decomposition of A:

$$A = U\Sigma V^T = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{2} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

6. $A^TA = \begin{bmatrix} -3 & 0 \\ 0 & -4 \end{bmatrix}\begin{bmatrix} -3 & 0 \\ 0 & -4 \end{bmatrix} = \begin{bmatrix} 9 & 0 \\ 0 & 16 \end{bmatrix}$; $\det(\lambda I - A^TA) = \begin{vmatrix} \lambda - 9 & 0 \\ 0 & \lambda - 16 \end{vmatrix} = (\lambda - 9)(\lambda - 16)$;

the eigenvalues of A^TA are $\lambda_1 = 16$ and $\lambda_2 = 9$ therefore the singular values of A are $\sigma_1 = \sqrt{\lambda_1} = 4$ and $\sigma_2 = \sqrt{\lambda_2} = 3$.

The reduced row echelon form of $16I - A^TA$ is $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda_1 = 16$ consists of vectors $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ where $x_1 = 0$, $x_2 = t$. A vector $\mathbf{v}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ forms an orthonormal basis for this eigenspace.

The reduced row echelon form of $9I - A^TA$ is $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda_2 = 9$ consists of vectors $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ where $x_1 = t$, $x_2 = 0$. A vector $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ forms an orthonormal basis for this eigenspace.

The matrix $V = \left[\mathbf{v}_1 \mid \mathbf{v}_2\right] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ orthogonally diagonalizes $A^T A$: $V^T\left(A^T A\right)V = \begin{bmatrix} 16 & 0 \\ 0 & 9 \end{bmatrix}$.

From part (d) of Theorem 9.4.4,

$\mathbf{u}_1 = \frac{1}{\sigma_1} A\mathbf{v}_1 = \frac{1}{4}\begin{bmatrix} -3 & 0 \\ 0 & -4 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$ and $\mathbf{u}_2 = \frac{1}{\sigma_2} A\mathbf{v}_2 = \frac{1}{3}\begin{bmatrix} -3 & 0 \\ 0 & -4 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$.

A singular value decomposition of A is

$$\underbrace{\begin{bmatrix} -3 & 0 \\ 0 & -4 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}}_{U}\underbrace{\begin{bmatrix} 4 & 0 \\ 0 & 3 \end{bmatrix}}_{\Sigma}\underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{V^T}$$

7. The eigenvalues of $A^T A = \begin{bmatrix} 4 & 0 \\ 6 & 4 \end{bmatrix}\begin{bmatrix} 4 & 6 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 16 & 24 \\ 24 & 52 \end{bmatrix}$ are $\lambda_1 = 64$ and $\lambda_2 = 4$, with corresponding unit eigenvectors $\mathbf{v}_1 = \begin{bmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} -\frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix}$ respectively. The singular values of A are $\sigma_1 = 8$ and $\sigma_2 = 2$. We have $\mathbf{u}_1 = \frac{1}{\sigma_1} A\mathbf{v}_1 = \frac{1}{8}\begin{bmatrix} 4 & 6 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{bmatrix} = \begin{bmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix}$, and $\mathbf{u}_2 = \frac{1}{\sigma_2} A\mathbf{v}_2 = \frac{1}{2}\begin{bmatrix} 4 & 6 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} -\frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix} = \begin{bmatrix} -\frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{bmatrix}$.

This results in the following singular value decomposition:

$$A = U\Sigma V^T = \begin{bmatrix} \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix}\begin{bmatrix} 8 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix}$$

8. $A^T A = \begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix}\begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix} = \begin{bmatrix} 18 & 18 \\ 18 & 18 \end{bmatrix}$; $\det\left(\lambda I - A^T A\right) = \begin{vmatrix} \lambda - 18 & -18 \\ -18 & \lambda - 18 \end{vmatrix} = (\lambda - 36)\lambda$;

the eigenvalues of $A^T A$ are $\lambda_1 = 36$ and $\lambda_2 = 0$ therefore the singular values of A are $\sigma_1 = \sqrt{\lambda_1} = 6$ and $\sigma_2 = \sqrt{\lambda_2} = 0$.

The reduced row echelon form of $36I - A^T A$ is $\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda_1 = 36$ consists of vectors $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ where $x_1 = t$, $x_2 = t$. A vector $\mathbf{v}_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$ forms an orthonormal basis for this eigenspace.

The reduced row echelon form of $0I - A^T A$ is $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda_2 = 0$ consists of vectors $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ where $x_1 = -t$, $x_2 = t$. A vector $\mathbf{v}_2 = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$ forms an orthonormal basis for this eigenspace.

The matrix $V = [\mathbf{v}_1 \mid \mathbf{v}_2] = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$ orthogonally diagonalizes A^TA: $V^T\left(A^TA\right)V = \begin{bmatrix} 36 & 0 \\ 0 & 0 \end{bmatrix}$.

From part (d) of Theorem 9.4.4,

$\mathbf{u}_1 = \frac{1}{\sigma_1} A\mathbf{v}_1 = \frac{1}{6}\begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$.

To obtain $\mathbf{u}_2$, we extend the set $\{\mathbf{u}_1\}$ to an orthonormal basis for R^2. To simplify the computations, we consider $\sqrt{2}\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. A vector $\mathbf{u}_2$ orthogonal to this vector must be a solution of $\begin{bmatrix} 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix}$.

An orthonormal basis for the solution space is formed by $\mathbf{u}_2 = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$.

A singular value decomposition of A is

$$\underbrace{\begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} 6 & 0 \\ 0 & 0 \end{bmatrix}}_{\Sigma} \underbrace{\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}}_{V^T}$$

9. The eigenvalues of $A^TA = \begin{bmatrix} -2 & -1 & 2 \\ 2 & 1 & -2 \end{bmatrix}\begin{bmatrix} -2 & 2 \\ -1 & 1 \\ 2 & -2 \end{bmatrix} = \begin{bmatrix} 9 & -9 \\ -9 & 9 \end{bmatrix}$ are $\lambda_1 = 18$ and $\lambda_2 = 0$, with corresponding unit eigenvectors $\mathbf{v}_1 = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$ respectively. The only nonzero singular value of A is $\sigma_1 = \sqrt{18} = 3\sqrt{2}$, and we have $\mathbf{u}_1 = \frac{1}{\sigma_1}A\mathbf{v}_1 = \frac{1}{3\sqrt{2}}\begin{bmatrix} -2 & 2 \\ -1 & 1 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \\ -\frac{2}{3} \end{bmatrix}$. We must choose the vectors $\mathbf{u}_2$ and $\mathbf{u}_3$ so that $\{\mathbf{u}_1,\ \mathbf{u}_2,\ \mathbf{u}_3\}$ is an orthonormal basis R^3.

A possible choice is $\mathbf{u}_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}$ and $\mathbf{u}_3 = \begin{bmatrix} \frac{\sqrt{2}}{6} \\ -\frac{2\sqrt{2}}{3} \\ -\frac{\sqrt{2}}{6} \end{bmatrix}$. This results in the following singular value decomposition: $A = U\Sigma V^T = \begin{bmatrix} \frac{2}{3} & \frac{1}{\sqrt{2}} & \frac{\sqrt{2}}{6} \\ \frac{1}{3} & 0 & -\frac{2\sqrt{2}}{3} \\ -\frac{2}{3} & \frac{1}{\sqrt{2}} & -\frac{\sqrt{2}}{6} \end{bmatrix}\begin{bmatrix} 3\sqrt{2} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$.

Note: The singular value decomposition is not unique. It depends on the choice of the (extended) orthonormal basis for R^3. This is just one possibility.

10. $A^T A = \begin{bmatrix} -2 & 2 \\ -1 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} -2 & -1 & 2 \\ 2 & 1 & -2 \end{bmatrix} = \begin{bmatrix} 8 & 4 & -8 \\ 4 & 2 & -4 \\ -8 & -4 & 8 \end{bmatrix}$;

$\det\left(\lambda I - A^T A\right) = \begin{vmatrix} \lambda - 8 & -4 & 8 \\ -4 & \lambda - 2 & 4 \\ 8 & 4 & \lambda - 8 \end{vmatrix} = (\lambda - 18)\lambda^2$; the eigenvalues of $A^T A$ are $\lambda_1 = 18$ and

$\lambda_2 = \lambda_3 = 0$ therefore the singular values of A are $\sigma_1 = \sqrt{\lambda_1} = 3\sqrt{2}$, $\sigma_2 = \sqrt{\lambda_2} = 0$, and $\sigma_3 = \sqrt{\lambda_3} = 0$.

The reduced row echelon form of $18I - A^T A$ is $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda = 18$

contains vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ where $x_1 = -t$, $x_2 = -\frac{1}{2}t$, $x_3 = t$. A vector $\mathbf{v}_1 = \begin{bmatrix} -\frac{2}{3} \\ -\frac{1}{3} \\ \frac{2}{3} \end{bmatrix}$ forms an orthonormal basis

for this eigenspace.

The reduced row echelon form of $0I - A^T A$ is $\begin{bmatrix} 1 & \frac{1}{2} & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda = 0$

contains vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ where $x_1 = -\frac{1}{2}s + t$, $x_2 = s$, $x_3 = t$. Vectors $\mathbf{p}_1 = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$ and $\mathbf{p}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ form a basis for

this eigenspace. We apply the Gram-Schmidt process to find an orthogonal basis for this eigenspace:

$\mathbf{q}_1 = \mathbf{p}_1 = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$ and $\mathbf{q}_2 = \mathbf{p}_2 - \frac{\langle \mathbf{p}_2, \mathbf{q}_1 \rangle}{\|\mathbf{q}_1\|^2}\mathbf{q}_1 = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} - \frac{-1}{5}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{4}{5} \\ \frac{2}{5} \\ 1 \end{bmatrix}$, then proceed to normalize the two vectors to

yield an orthonormal basis: $\mathbf{v}_2 = \frac{\mathbf{q}_1}{\|\mathbf{q}_1\|} = \begin{bmatrix} -\frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \\ 0 \end{bmatrix}$ and $\mathbf{v}_3 = \frac{\mathbf{q}_2}{\|\mathbf{q}_2\|} = \begin{bmatrix} \frac{4}{3\sqrt{5}} \\ \frac{2}{3\sqrt{5}} \\ \frac{5}{3\sqrt{5}} \end{bmatrix}$.

The matrix $V = \left[\mathbf{v}_1 \mid \mathbf{v}_2 \mid \mathbf{v}_3\right] = \begin{bmatrix} -\frac{2}{3} & -\frac{1}{\sqrt{5}} & \frac{4}{3\sqrt{5}} \\ -\frac{1}{3} & \frac{2}{\sqrt{5}} & \frac{2}{3\sqrt{5}} \\ \frac{2}{3} & 0 & \frac{5}{3\sqrt{5}} \end{bmatrix}$ orthogonally diagonalizes $A^T A$:

$V^T\left(A^T A\right)V = \begin{bmatrix} 18 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

From part (d) of Theorem 9.4.4,

$$\mathbf{u}_1 = \tfrac{1}{\sigma_1} A\mathbf{v}_1 = \tfrac{1}{3\sqrt{2}}\begin{bmatrix} -2 & -1 & 2 \\ 2 & 1 & -2 \end{bmatrix}\begin{bmatrix} -\frac{2}{3} \\ -\frac{1}{3} \\ \frac{2}{3} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$$

To obtain $\mathbf{u}_2$, we extend the set $\{\mathbf{u}_1\}$ to an orthonormal basis for R^2. To simplify the computations, we consider $\sqrt{2}\mathbf{u}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. A vector $\mathbf{u}_2$ orthogonal to this vector must be a solution of $\begin{bmatrix} 1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix}$.

An orthonormal basis for the solution space is formed by $\mathbf{u}_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$.

A singular value decomposition of A is

$$\underbrace{\begin{bmatrix} -2 & -1 & 2 \\ 2 & 1 & -2 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} 3\sqrt{2} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\Sigma} \underbrace{\begin{bmatrix} -\frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \\ -\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} & 0 \\ \frac{4}{3\sqrt{5}} & \frac{2}{3\sqrt{5}} & \frac{5}{3\sqrt{5}} \end{bmatrix}}_{V^T}$$

11. The eigenvalues of $A^T A = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$ are $\lambda_1 = 3$ and $\lambda_2 = 2,$ with corresponding unit eigenvectors $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ respectively. The singular values of A are $\sigma_1 = \sqrt{3}$ and $\sigma_2 = \sqrt{2}.$ We have $\mathbf{u}_1 = \tfrac{1}{\sigma_1} A\mathbf{v}_1 = \tfrac{1}{\sqrt{3}}\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \end{bmatrix}$ and $\mathbf{u}_2 = \tfrac{1}{\sqrt{2}}\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$. We choose $\mathbf{u}_3 = \begin{bmatrix} \frac{2}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{bmatrix}$ so that $\{\mathbf{u}_1,\ \mathbf{u}_2,\ \mathbf{u}_3\}$ is an orthonormal basis for R^3. This results in the following singular value decomposition: $A = U\Sigma V^T = \begin{bmatrix} \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{bmatrix}\begin{bmatrix} \sqrt{3} & 0 \\ 0 & \sqrt{2} \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

12. $A^TA=\begin{bmatrix}6&0&4\\4&0&0\end{bmatrix}\begin{bmatrix}6&4\\0&0\\4&0\end{bmatrix}=\begin{bmatrix}52&24\\24&16\end{bmatrix}$; $\det\left(\lambda I-A^TA\right)=\begin{vmatrix}\lambda-52&-24\\-24&\lambda-16\end{vmatrix}=(\lambda-64)(\lambda-4)$

the eigenvalues of A^TA are $\lambda_1=64$ and $\lambda_2=4$ therefore the singular values of A are $\sigma_1=\sqrt{\lambda_1}=8$ and $\sigma_2=\sqrt{\lambda_2}=2$.

The reduced row echelon form of $64I-A^TA$ is $\begin{bmatrix}1&-2\\0&0\end{bmatrix}$ so that the eigenspace corresponding to $\lambda_1=64$ consists of vectors $\begin{bmatrix}x_1\\x_2\end{bmatrix}$ where $x_1=2t$, $x_2=t$. A vector $\mathbf{v}_1=\begin{bmatrix}\frac{2}{\sqrt5}\\\frac{1}{\sqrt5}\end{bmatrix}$ forms an orthonormal basis for this eigenspace.

The reduced row echelon form of $4I-A^TA$ is $\begin{bmatrix}1&\frac12\\0&0\end{bmatrix}$ so that the eigenspace corresponding to $\lambda_2=4$ consists of vectors $\begin{bmatrix}x_1\\x_2\end{bmatrix}$ where $x_1=-\frac12t$, $x_2=t$. A vector $\mathbf{v}_2=\begin{bmatrix}-\frac{1}{\sqrt5}\\\frac{2}{\sqrt5}\end{bmatrix}$ forms an orthonormal basis for this eigenspace.

The matrix $V=[\mathbf{v}_1\mid\mathbf{v}_2]=\begin{bmatrix}\frac{2}{\sqrt5}&-\frac{1}{\sqrt5}\\\frac{1}{\sqrt5}&\frac{2}{\sqrt5}\end{bmatrix}$ orthogonally diagonalizes A^TA: $V^T\left(A^TA\right)V=\begin{bmatrix}64&0\\0&4\end{bmatrix}$.

From part (d) of Theorem 9.4.4,

$$\mathbf{u}_1=\tfrac{1}{\sigma_1}A\mathbf{v}_1=\tfrac18\begin{bmatrix}6&4\\0&0\\4&0\end{bmatrix}\begin{bmatrix}\frac{2}{\sqrt5}\\\frac{1}{\sqrt5}\end{bmatrix}=\begin{bmatrix}\frac{2}{\sqrt5}\\0\\\frac{1}{\sqrt5}\end{bmatrix}\text{ and }\mathbf{u}_2=\tfrac{1}{\sigma_2}A\mathbf{v}_2=\tfrac12\begin{bmatrix}6&4\\0&0\\4&0\end{bmatrix}\begin{bmatrix}-\frac{1}{\sqrt5}\\\frac{2}{\sqrt5}\end{bmatrix}=\begin{bmatrix}\frac{1}{\sqrt5}\\0\\-\frac{2}{\sqrt5}\end{bmatrix}.$$

To obtain $\mathbf{u}_3$, we extend the set $\{\mathbf{u}_1,\mathbf{u}_2\}$ to an orthonormal basis for R^2. To simplify the computations, we consider $\sqrt5\mathbf{u}_1=\begin{bmatrix}2\\0\\1\end{bmatrix}$ and $\sqrt5\mathbf{u}_2=\begin{bmatrix}1\\0\\-2\end{bmatrix}$. A vector $\mathbf{u}_3$ orthogonal to both of these vectors must be a solution of the homogeneous linear system $\begin{bmatrix}2&0&1\\1&0&-2\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$.

The augmented matrix of this system has the reduced row echelon form $\begin{bmatrix}1&0&0&0\\0&0&1&0\end{bmatrix}$ therefore an orthonormal basis for the solution space is formed by $\mathbf{u}_3=\begin{bmatrix}0\\1\\0\end{bmatrix}$.

A singular value decomposition of A is

$$\underbrace{\begin{bmatrix} 6 & 4 \\ 0 & 0 \\ 4 & 0 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} & 0 \\ 0 & 0 & 1 \\ \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} & 0 \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} 8 & 0 \\ 0 & 2 \\ 0 & 0 \end{bmatrix}}_{\Sigma} \underbrace{\begin{bmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ -\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix}}_{V^T}$$

19. (b) In the solution of Exercise 5, we obtained a singular value decomposition

$$A = U\Sigma V^T = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{2} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

A polar decomposition of A is

$$\begin{aligned} A &= \left(U\Sigma U^T\right)\left(UV^T\right) \\ &= \left(\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{2} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}\right)\left(\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) \\ &= \begin{bmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{2} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \end{aligned}$$

True-False Exercises

(a) False. If A is an $m \times n$ matrix, then A^T is an $n \times m$ matrix, and A^TA is an $n \times n$ matrix.

(b) True. $\left(A^TA\right)^T = A^T\left(A^T\right)^T = A^TA$.

(c) False. A^TA may have eigenvalues that are 0.

(d) False. A would have to be symmetric to be orthogonally diagonalizable.

(e) True. This follows since A^TA is a symmetric $n \times n$ matrix.

(f) False. The eigenvalues of A^TA are the squares of the singular values of A.

(g) True. This follows from Theorem 9.4.3.

9.5 Data Compression Using Singular Value Decomposition

1. From Exercise 9 in Section 9.4, A has the singular value decomposition

$$A=\begin{bmatrix} \frac{2}{3} & \frac{1}{\sqrt{2}} & \frac{\sqrt{2}}{6} \\ \frac{1}{3} & 0 & -\frac{2\sqrt{2}}{3} \\ -\frac{2}{3} & \frac{1}{\sqrt{2}} & -\frac{\sqrt{2}}{6} \end{bmatrix}\begin{bmatrix} 3\sqrt{2} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}.$$

Thus the reduced singular value decomposition of A is $A=U_1\Sigma_1V_1^T=\begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \\ -\frac{2}{3} \end{bmatrix}\begin{bmatrix} 3\sqrt{2} \end{bmatrix}\begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$.

2. $\begin{bmatrix} -2 & -1 & 2 \\ 2 & 1 & -2 \end{bmatrix}=\begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} 3\sqrt{2} \end{bmatrix}\begin{bmatrix} -\frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \end{bmatrix}$

3. From Exercise 11 in Section 9.4, A has the singular value decomposition

$$A=U\Sigma V^T=\begin{bmatrix} \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{bmatrix}\begin{bmatrix} \sqrt{3} & 0 \\ 0 & \sqrt{2} \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Thus the reduced singular value decomposition of A is $A=U_1\Sigma_1V_1^T=\begin{bmatrix} \frac{1}{\sqrt{3}} & 0 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} \sqrt{3} & 0 \\ 0 & \sqrt{2} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

4. $\begin{bmatrix} 6 & 4 \\ 0 & 0 \\ 4 & 0 \end{bmatrix}=\begin{bmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ 0 & 0 \\ \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \end{bmatrix}\begin{bmatrix} 8 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ -\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix}$

5. The reduced singular value expansion of A is $3\sqrt{2}\begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \\ -\frac{2}{3} \end{bmatrix}\begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$.

6. $\begin{bmatrix} -2 & -1 & 2 \\ 2 & 1 & -2 \end{bmatrix}=\sigma_1\mathbf{u}_1\mathbf{v}_1^T=3\sqrt{2}\begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} -\frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \end{bmatrix}$

7. The reduced singular value decomposition of A is $\sqrt{3}\begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \end{bmatrix}\begin{bmatrix} 1 & 0 \end{bmatrix}+\sqrt{2}\begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} 0 & 1 \end{bmatrix}$.

8. $\begin{bmatrix} 6 & 4 \\ 0 & 0 \\ 4 & 0 \end{bmatrix}=\sigma_1\mathbf{u}_1\mathbf{v}_1^T+\sigma_2\mathbf{u}_2\mathbf{v}_2^T=8\begin{bmatrix} \frac{2}{\sqrt{5}} \\ 0 \\ \frac{1}{\sqrt{5}} \end{bmatrix}\begin{bmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix}+2\begin{bmatrix} \frac{1}{\sqrt{5}} \\ 0 \\ -\frac{2}{\sqrt{5}} \end{bmatrix}\begin{bmatrix} -\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix}$

9. A rank 100 approximation of A requires storage space for 100(200 + 500 + 1) = 70,100 numbers, while A has 200(500) = 100,000 entries.

True-False Exercises

(a) True. This follows from the definition of a reduced singular value decomposition.

(b) True. This follows from the definition of a reduced singular value decomposition.

(c) False. V_1 has size $n \times k$ so that V_1^T has size $k \times n$.

Chapter 9 Supplementary Exercises

1. Reduce A to upper triangular form:

$$\begin{bmatrix} -6 & 2 \\ 6 & 0 \end{bmatrix} \to \begin{bmatrix} -3 & 1 \\ 6 & 0 \end{bmatrix} \to \begin{bmatrix} -3 & 1 \\ 0 & 2 \end{bmatrix} = U$$

The multipliers used were $\frac{1}{2}$ and 2, so $L = \begin{bmatrix} 2 & 0 \\ -2 & 1 \end{bmatrix}$ and $A = \begin{bmatrix} 2 & 0 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} -3 & 1 \\ 0 & 2 \end{bmatrix}$.

2.

$$A = \begin{bmatrix} -6 & 2 \\ 6 & 0 \end{bmatrix} \qquad \begin{bmatrix} \bullet & 0 \\ \bullet & \bullet \end{bmatrix}$$

$$\begin{bmatrix} 1 & -\frac{1}{3} \\ 6 & 0 \end{bmatrix} \leftarrow \text{multiplier} = -\tfrac{1}{6} \qquad \begin{bmatrix} -6 & 0 \\ \bullet & \bullet \end{bmatrix}$$

$$\begin{bmatrix} 1 & -\frac{1}{3} \\ 0 & 2 \end{bmatrix} \leftarrow \text{multiplier} = -6 \qquad \begin{bmatrix} -6 & 0 \\ 6 & \bullet \end{bmatrix}$$

$$U = \begin{bmatrix} 1 & -\frac{1}{3} \\ 0 & 1 \end{bmatrix} \leftarrow \text{multiplier} = \tfrac{1}{2} \qquad \begin{bmatrix} -6 & 0 \\ 6 & 2 \end{bmatrix}$$

A general 2×2 lower triangular matrix with nonzero main diagonal entries can be factored as $\begin{bmatrix} a_{11} & 0 \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ a_{21}/a_{11} & 1 \end{bmatrix}\begin{bmatrix} a_{11} & 0 \\ 0 & a_{22} \end{bmatrix}$ therefore $\begin{bmatrix} -6 & 0 \\ 6 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} -6 & 0 \\ 0 & 2 \end{bmatrix}$. We conclude that an LDU-decomposition of A is $A = \begin{bmatrix} -6 & 2 \\ 6 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} -6 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & -\frac{1}{3} \\ 0 & 1 \end{bmatrix} = LDU$.

3. Reduce A to upper triangular form.

$$\begin{bmatrix}2&4&6\\1&4&7\\1&3&7\end{bmatrix}\to\begin{bmatrix}1&2&3\\1&4&7\\1&3&7\end{bmatrix}\to\begin{bmatrix}1&2&3\\0&2&4\\1&3&7\end{bmatrix}\to\begin{bmatrix}1&2&3\\0&2&4\\0&1&4\end{bmatrix}$$

$$\to\begin{bmatrix}1&2&3\\0&1&2\\0&1&4\end{bmatrix}\to\begin{bmatrix}1&2&3\\0&1&2\\0&0&2\end{bmatrix}\to\begin{bmatrix}1&2&3\\0&1&2\\0&0&1\end{bmatrix}=U$$

The multipliers used were $\frac{1}{2}$, $-1, -1$, $\frac{1}{2}$, -1, and $\frac{1}{2}$ so $L=\begin{bmatrix}2&0&0\\1&2&0\\1&1&2\end{bmatrix}$ and $A=\begin{bmatrix}2&0&0\\1&2&0\\1&1&2\end{bmatrix}\begin{bmatrix}1&2&3\\0&1&2\\0&0&1\end{bmatrix}$.

4. It was shown in the solution of Exercise 3 that $A=\begin{bmatrix}2&0&0\\1&2&0\\1&1&2\end{bmatrix}\begin{bmatrix}1&2&3\\0&1&2\\0&0&1\end{bmatrix}$.

A general 3×3 lower triangular matrix with nonzero main diagonal entries can be factored as

$$\begin{bmatrix}a_{11}&0&0\\a_{21}&a_{22}&0\\a_{31}&a_{32}&a_{33}\end{bmatrix}=\begin{bmatrix}1&0&0\\a_{21}/a_{11}&1&0\\a_{31}/a_{11}&a_{32}/a_{22}&1\end{bmatrix}\begin{bmatrix}a_{11}&0&0\\0&a_{22}&0\\0&0&a_{33}\end{bmatrix}$$

therefore $\begin{bmatrix}2&0&0\\1&2&0\\1&1&2\end{bmatrix}=\begin{bmatrix}1&0&0\\\frac{1}{2}&1&0\\\frac{1}{2}&\frac{1}{2}&1\end{bmatrix}\begin{bmatrix}2&0&0\\0&2&0\\0&0&2\end{bmatrix}$. We conclude that an LDU-decomposition of A is

$$A=\begin{bmatrix}2&4&6\\1&4&7\\1&3&7\end{bmatrix}=\begin{bmatrix}1&0&0\\\frac{1}{2}&1&0\\\frac{1}{2}&\frac{1}{2}&1\end{bmatrix}\begin{bmatrix}2&0&0\\0&2&0\\0&0&2\end{bmatrix}\begin{bmatrix}1&2&3\\0&1&2\\0&0&1\end{bmatrix}=LDU.$$

5. **(a)** The characteristic equation of A is $\lambda^2-4\lambda+3=(\lambda-3)(\lambda-1)=0$ so the dominant eigenvalue of A is $\lambda_1=3$, with corresponding positive unit eigenvector

$$\mathbf{v}=\begin{bmatrix}\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\end{bmatrix}\approx\begin{bmatrix}0.7071\\0.7071\end{bmatrix}.$$

(b) $A\mathbf{x}_0=\begin{bmatrix}2&1\\1&2\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}2\\1\end{bmatrix}$

$$\mathbf{x}_1 = \frac{A\mathbf{x}_0}{\| A\mathbf{x}_0 \|} = \frac{1}{\sqrt{5}}\begin{bmatrix} 2 \\ 1 \end{bmatrix} \approx \begin{bmatrix} 0.8944 \\ 0.4472 \end{bmatrix}$$

$$\mathbf{x}_2 = \frac{A\mathrm{x}_1}{\| A\mathbf{x}_1 \|} \approx \begin{bmatrix} 0.7809 \\ 0.6247 \end{bmatrix}$$

$$\mathbf{x}_3 = \frac{A\mathbf{x}_2}{\| A\mathbf{x}_2 \|} \approx \begin{bmatrix} 0.7328 \\ 0.6805 \end{bmatrix}$$

$$\mathbf{x}_4 = \frac{A\mathbf{x}_3}{\| A\mathbf{x}_3 \|} \approx \begin{bmatrix} 0.7158 \\ 0.6983 \end{bmatrix}$$

$$\mathbf{x}_5 = \frac{A\mathbf{x}_4}{\| A\mathbf{x}_4 \|} \approx \begin{bmatrix} 0.7100 \\ 0.7042 \end{bmatrix}$$

$\mathbf{x}_5 \approx \begin{bmatrix} 0.7100 \\ 0.7042 \end{bmatrix}$ as compared to $\mathbf{v} \approx \begin{bmatrix} 0.7071 \\ 0.7071 \end{bmatrix}$.

(c) $A\mathbf{x}_0 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$$\mathbf{x}_1 = \frac{A\mathbf{x}_0}{\max(A\mathbf{x}_0)} = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix}$$

$$\mathbf{x}_2 = \frac{A\mathbf{x}_1}{\max(A\mathbf{x}_1)} = \begin{bmatrix} 1 \\ 0.8 \end{bmatrix}$$

$$\mathbf{x}_3 = \frac{A\mathbf{x}_2}{\max(A\mathbf{x}_2)} \approx \begin{bmatrix} 1 \\ 0.9286 \end{bmatrix}$$

$$\mathbf{x}_4 = \frac{A\mathrm{x}_3}{\max(A\mathbf{x}_3)} \approx \begin{bmatrix} 1 \\ 0.9756 \end{bmatrix}$$

$$\mathbf{x}_5 = \frac{A\mathbf{x}_4}{\max(A\mathbf{x}_4)} \approx \begin{bmatrix} 1 \\ 0.9918 \end{bmatrix}$$

$\mathbf{x}_5 \approx \begin{bmatrix} 1 \\ 0.9918 \end{bmatrix}$ as compared to the exact eigenvector $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

7. The Rayleigh quotients will converge to the dominant eigenvalue $\lambda_4 = -8.1$. However, since the ratio $\frac{|\lambda_4|}{|\lambda_1|} = \frac{8.1}{8} = 1.0125$ is very close to 1, the rate of convergence is likely to be quite slow.

8. $A^T A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$; $\det(\lambda I - A^T A) = \begin{vmatrix} \lambda - 2 & -2 \\ -2 & \lambda - 2 \end{vmatrix} = (\lambda - 4)\lambda$;

the eigenvalues of $A^T A$ are $\lambda_1 = 4$ and $\lambda_2 = 0$ therefore the singular values of A are $\sigma_1 = \sqrt{\lambda_1} = 2$ and $\sigma_2 = \sqrt{\lambda_2} = 0$.

The reduced row echelon form of $4I - A^T A$ is $\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda_1 = 4$

consists of vectors $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ where $x_1 = t$, $x_2 = t$. A vector $\mathbf{v}_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$ forms an orthonormal basis for this eigenspace.

The reduced row echelon form of $0I - A^T A$ is $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda_2 = 0$ consists of vectors $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ where $x_1 = -t$, $x_2 = t$. A vector $\mathbf{v}_2 = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$ forms an orthonormal basis for this eigenspace.

The matrix $V = [\mathbf{v}_1 \mid \mathbf{v}_2] = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$ orthogonally diagonalizes $A^T A$: $V^T \left(A^T A\right) V = \begin{bmatrix} 4 & 0 \\ 0 & 0 \end{bmatrix}$.

From part (d) of Theorem 9.4.4, $\mathbf{u}_1 = \frac{1}{\sigma_1} A\mathbf{v}_1 = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$.

To obtain $\mathbf{u}_2$, we extend the set $\{\mathbf{u}_1\}$ to an orthonormal basis for R^2. To simplify the computations, we consider $\sqrt{2}\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. A vector $\mathbf{u}_2$ orthogonal to this vector must be a solution of $\begin{bmatrix} 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix}$.

An orthonormal basis for the solution space is formed by $\mathbf{u}_2 = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$.

A singular value decomposition of A is

$$\underbrace{\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}}_{\Sigma} \underbrace{\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}}_{V^T}$$

9. The eigenvalues of $A^T A = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$ are $\lambda_1 = 4$ and $\lambda_2 = 0$ with corresponding unit eigenvectors

$\mathbf{v}_1 = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$, respectively. The only nonzero singular value A is $\sigma_1 = \sqrt{4} = 2$, and we have

$\mathbf{u}_1 = \frac{1}{\sigma_1} A\mathbf{v}_1 = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$.

We must choose the vectors $\mathbf{u}_2$ and $\mathbf{u}_3$ so that $\{\mathbf{u}_1,\ \mathbf{u}_2,\ \mathbf{u}_3\}$ is an orthonormal basis for R^3. A possible choice is $\mathbf{u}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ and $\mathbf{u}_3 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$. This results in the following singular value decomposition:

$$A = U\Sigma V^T = \begin{bmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}.$$

10. $A^T A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$; $\det\left(\lambda I - A^T A\right) = \begin{vmatrix} \lambda - 2 & -2 \\ -2 & \lambda - 2 \end{vmatrix} = (\lambda - 4)\lambda$;

the eigenvalues of $A^T A$ are $\lambda_1 = 4$ and $\lambda_2 = 0$ therefore the singular values of A are $\sigma_1 = \sqrt{\lambda_1} = 2$ and $\sigma_2 = \sqrt{\lambda_2} = 0$.

The reduced row echelon form of $4I - A^T A$ is $\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$ so that the eigenspace corresponding to $\lambda_1 = 4$ consists of vectors $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ where $x_1 = t$, $x_2 = t$. A vector $\mathbf{v}_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$ forms an orthonormal basis for this eigenspace.

From part (d) of Theorem 9.4.4, $\mathbf{u}_1 = \frac{1}{\sigma_1} A\mathbf{v}_1 = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}$.

A reduced singular value decomposition of A is $\begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix} [2] \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$.

A reduced singular value expansion is $\begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 1 & 1 \end{bmatrix} = \sigma_1 \mathbf{u}_1 \mathbf{v}_1^T = 2 \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$.

11. A has rank 2, thus $U_1 = [\mathbf{u}_1 \quad \mathbf{u}_2]$ and $V_1^T = \begin{bmatrix} \mathbf{v}_1^T \\ \mathbf{v}_2^T \end{bmatrix}$ and the reduced singular value decomposition of A is

$$A = U_1 \Sigma_1 V_1^T = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 24 & 0 \\ 0 & 12 \end{bmatrix} \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \end{bmatrix}.$$

14. Since A has rank 1 it can be written as $A = \mathbf{u}\mathbf{v}^T$. Thus, $A^2 = \left(\mathbf{u}\mathbf{v}^T\right)\left(\mathbf{u}\mathbf{v}^T\right) = \mathbf{u}\left(\mathbf{v}^T\mathbf{u}\right)\mathbf{v}^T$.
But $\mathbf{v}^T\mathbf{u}$ is a scalar, say k. Thus, $A^2 = \mathbf{u}\left(\mathbf{v}^T\mathbf{u}\right)\mathbf{v}^T = k\mathbf{u}\mathbf{v}^T = kA$.