CHAPTER WEB PROJECTS: Expanding the Calculus Horizon (online only)

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THE ROOTS OF CALCULUS

Today's exciting applications of calculus have roots that can be traced to the work of the Greek mathematician Archimedes, but the actual discovery of the fundamental principles of calculus was made independently by Isaac Newton (English) and Gottfried Leibniz (German) in the late seventeenth century. The work of Newton and Leibniz was motivated by four major classes of scientific and mathematical problems of the time:

- Find the tangent line to a general curve at a given point.
- Find the area of a general region, the length of a general curve, and the volume of a general solid.
- Find the maximum or minimum value of a quantity—for example, the maximum and minimum distances of a planet from the Sun, or the maximum range attainable for a projectile by varying its angle of fire.
- Given a formula for the distance traveled by a body in any specified amount of time, find the velocity and acceleration of the body at any instant. Conversely, given a formula that specifies the acceleration of velocity at any instant, find the distance traveled by the body in a specified period of time.

Newton and Leibniz found a fundamental relationship between the problem of finding a tangent line to a curve and the problem of determining the area of a region. Their realization of this connection is considered to be the "discovery of calculus." Though Newton saw how these two problems are related ten years before Leibniz did, Leibniz published his work twenty years before Newton. This situation led to a stormy debate over who was the rightful discoverer of calculus. The debate engulfed Europe for half a century, with the scientists of the European continent supporting Leibniz and those from England supporting Newton. The conflict was extremely unfortunate because Newton's inferior notation badly hampered scientific development in England, and the Continent in turn lost the benefit of Newton's discoveries in astronomy and physics for nearly fifty years. In spite of it all, Newton and Leibniz were sincere admirers of each other's work.

[*Image: Public domain image from http://commons.wikimedia.org/wiki/File:Hw-newton.jpg. Image provided courtesy of the University of Texas Libraries, The University of Texas at Austin.*]

ISAAC NEWTON (1642–1727)

Newton was born in the village of Woolsthorpe, England. His father died before he was born and his mother raised him on the family farm. As a youth he showed little evidence of his later brilliance, except for an unusual talent with mechanical devices—he apparently built a working water clock and a toy flour mill powered by a mouse. In 1661 he entered Trinity College in Cambridge with a deficiency in geometry. Fortunately, Newton caught the eye of Isaac Barrow, a gifted mathematician and teacher. Under Barrow's guidance Newton immersed himself in mathematics and science, but he graduated without any special distinction. Because the bubonic plague was spreading rapidly through London, Newton returned to his home in Woolsthorpe and stayed there during the years of 1665 and 1666. In those two momentous years the entire framework of modern science was miraculously created in Newton's mind. He discovered calculus, recognized the underlying principles of planetary motion and gravity, and determined that "white" sunlight was composed of all colors, red to violet. For whatever reasons he kept his discoveries to himself. In 1667 he returned to Cambridge to obtain his Master's degree and upon graduation became a teacher at Trinity. Then in 1669 Newton succeeded his teacher, Isaac Barrow, to the Lucasian chair of mathematics at Trinity, one of the most honored chairs of mathematics in the world.

Thereafter, brilliant discoveries flowed from Newton steadily. He formulated the law of gravitation and used it to explain the motion of the moon, the planets, and the tides; he formulated basic theories of light, thermodynamics, and hydrodynamics; and he devised and constructed the first modern reflecting telescope. Throughout his life Newton was hesitant to publish his major discoveries, revealing them only to a select circle of friends, perhaps because of a fear of criticism or controversy. In 1687, only after intense coaxing by the astronomer, Edmond Halley (discoverer of Halley's comet), did Newton publish his masterpiece, *Philosophiae Naturalis Principia Mathematica* (The Mathematical Principles of Natural Philosophy). This work is generally considered to be the most important and influential scientific book ever written. In it Newton explained the workings of the solar system and formulated the basic laws of motion, which to this day are fundamental in engineering and physics. However, not even the pleas of his friends could convince Newton to publish his discovery of calculus. Only after Leibniz published his results did Newton relent and publish his own work on calculus.

After twenty-five years as a professor, Newton suffered depression and a nervous breakdown. He gave up research in 1695 to accept a position as warden and later master of the London mint. During the twenty-five years that he worked at the mint, he did virtually no scientific or mathematical work. He was knighted in 1705 and on his death was buried in Westminster Abbey with all the honors his country could bestow. It is interesting to note that Newton was a learned theologian who viewed the primary value of his work to be its support of the existence of God. Throughout his life he worked passionately to date biblical events by relating them to astronomical phenomena. He was so consumed with this passion that he spent years searching the Book of Daniel for clues to the end of the world and the geography of hell.

Newton described his brilliant accomplishments as follows: "I seem to have been only like a boy playing on the seashore and diverting myself in now and then finding a smoother pebble or prettier shell than ordinary, whilst the great ocean of truth lay all undiscovered before me."

[*Image: Public domain image from http://commons.wikimedia.org/wiki/File:Gottfried_Wilhelm_von_ Leibniz.jpg*]

GOTTFRIED WILHELM LEIBNIZ (1646–1716)

This gifted genius was one of the last people to have mastered most major fields of knowledge—an impossible accomplishment in our own era of specialization. He was an expert in law, religion, philosophy, literature, politics, geology, metaphysics, alchemy, history, and mathematics.

Leibniz was born in Leipzig, Germany. His father, a professor of moral philosophy at the University of Leipzig, died when Leibniz was six years old. The precocious boy then gained access to his father's library and began reading voraciously on a wide range of subjects, a habit that he maintained throughout his life. At age fifteen he entered the University of Leipzig as a law student and by the age of twenty received a doctorate from the University of Altdorf. Subsequently, Leibniz followed a career in law and international politics, serving as counsel to kings and princes. During his numerous foreign missions, Leibniz came in contact with outstanding mathematicians and scientists who stimulated his interest in mathematics—most notably, the physicist Christian Huygens. In mathematics Leibniz was self-taught, learning the subject by reading papers and journals. As a result of this fragmented mathematical education, Leibniz often rediscovered the results of others, and this helped to fuel the debate over the discovery of calculus.

Leibniz never married. He was moderate in his habits, quick-tempered but easily appeased, and charitable in his judgment of other people's work.

In spite of his great achievements, Leibniz never received the honors showered on Newton, and he spent his final years as a lonely embittered man. At his funeral there was one mourner, his secretary. An eyewitness stated, "He was buried more like a robber than what he really was—an ornament of his country."

Kevin Morroun and William O'Nieal

Air resistance prevents the velocity of a skydiver from increasing indefinitely. The velocity approaches a limit, called the "terminal velocity."

1

LIMITS AND CONTINUITY

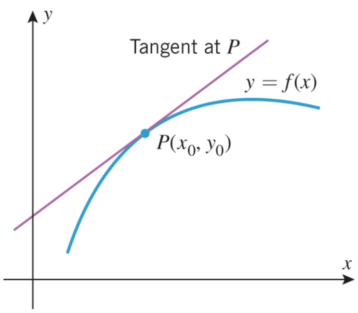
The development of calculus in the seventeenth century by Newton and Leibniz provided scientists with their first real understanding of what is meant by an "instantaneous rate of change" such as velocity and acceleration. Once the idea was understood conceptually, efficient computational methods followed, and science took a quantum leap forward. The fundamental building block on which rates of change rest is the concept of a "limit," an idea that is so important that all other calculus concepts are now based on it.

In this chapter we will develop the concept of a limit in stages, proceeding from an informal, intuitive notion to a precise mathematical definition. We will also develop theorems and procedures for calculating limits, and use limits to study "continuous" curves. We will conclude by studying continuity and other properties of trigonometric, inverse trigonometric, logarithmic, and exponential functions.

1.1 LIMITS (AN INTUITIVE APPROACH)

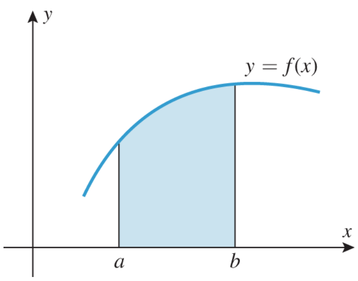
The concept of a "limit" is the fundamental building block on which all calculus concepts are based. In this section we will study limits informally, with the goal of developing an intuitive feel for the basic ideas. In the next three sections we will focus on computational methods and precise definitions.

Many of the ideas of calculus originated with the following two geometric problems:



▲ **Figure 1.1.1**

THE TANGENT LINE PROBLEM Given a function f and a point $P(x_0, y_0)$ on its graph, find an equation of the line that is tangent to the graph at P (Figure 1.1.1).



▲ **Figure 1.1.2**

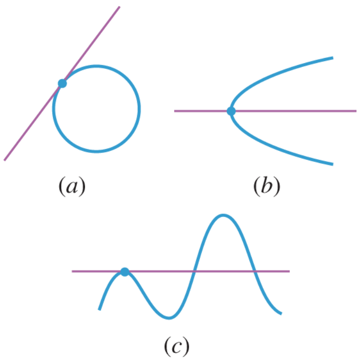
THE AREA PROBLEM Given a function f, find the area between the graph of f and an interval $[a, b]$ on the x-axis (Figure 1.1.2).

Traditionally, that portion of calculus arising from the tangent line problem is called ***differential calculus*** and that arising from the area problem is called ***integral calculus***. However, we will see later that the tangent line and area problems are so closely related that the distinction between differential and integral calculus is somewhat artificial.

TANGENT LINES AND LIMITS

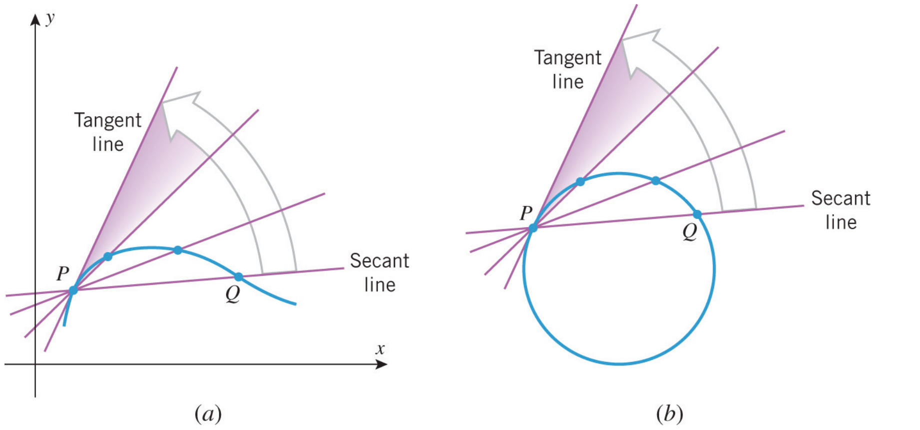
In plane geometry, a line is called *tangent* to a circle if it meets the circle at precisely one point (Figure 1.1.3*a*). Although this definition is adequate for circles, it is not appropriate for more general curves. For example, in Figure 1.1.3*b*, the line meets the curve exactly once but is obviously not what we would regard to be a tangent line; and in Figure 1.1.3*c*, the line appears to be tangent to the curve, yet it intersects the curve more than once.

To obtain a definition of a tangent line that applies to curves other than circles, we must view tangent lines another way. For this purpose, suppose that we are interested in the tangent line at a point P on a curve in the xy-plane and that Q is any point that lies on



▲ **Figure 1.1.3**

the curve and is different from P. The line through P and Q is called a ***secant line*** for the curve at P. Intuition suggests that if we move the point Q along the curve toward P, then the secant line will rotate toward a *limiting position*. The line in this limiting position is what we will consider to be the ***tangent line*** at P (Figure 1.1.4*a*). As suggested by Figure 1.1.4*b*, this new concept of a tangent line coincides with the traditional concept when applied to circles.



▶ **Figure 1.1.4**

▶ Example 1 Find an equation for the tangent line to the parabola $y = x^2$ at the point $P(1, 1)$.

Solution. If we can find the slope $m_{\tan}$ of the tangent line at P, then we can use the point P and the point-slope formula for a line (Web Appendix H) to write the equation of the tangent line as

$$y - 1 = m_{\tan}(x - 1) \tag{1}$$

To find the slope $m_{\tan}$, consider the secant line through P and a point $Q(x, x^2)$ on the parabola that is distinct from P. The slope $m_{\sec}$ of this secant line is

Why are we requiring that P and Q be distinct?

$$m_{\sec} = \frac{x^2 - 1}{x - 1} \tag{2}$$

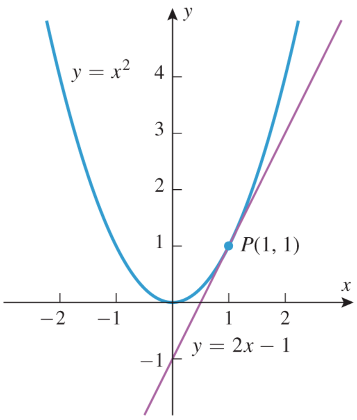
Figure 1.1.4*a* suggests that if we now let Q move along the parabola, getting closer and closer to P, then the limiting position of the secant line through P and Q will coincide with that of the tangent line at P. This in turn suggests that the value of $m_{\sec}$ will get closer and closer to the value of $m_{\tan}$ as P moves toward Q along the curve. However, to say that $Q(x, x^2)$ gets closer and closer to $P(1, 1)$ is algebraically equivalent to saying that x gets closer and closer to 1. Thus, the problem of finding $m_{\tan}$ reduces to finding the "limiting value" of $m_{\sec}$ in Formula (2) as x gets closer and closer to 1 (but with $x \neq 1$ to ensure that P and Q remain distinct).

We can rewrite (2) as

$$m_{\sec} = \frac{x^2 - 1}{x - 1} = \frac{(x - 1)(x + 1)}{(x - 1)} = x + 1$$

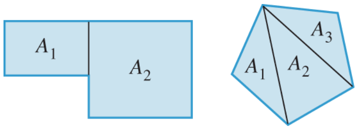
where the cancellation of the factor $(x - 1)$ is allowed because $x \neq 1$. It is now evident that $m_{\sec}$ gets closer and closer to 2 as x gets closer and closer to 1. Thus, $m_{\tan} = 2$ and (1) implies that the equation of the tangent line is

$$y - 1 = 2(x - 1) \quad \text{or equivalently} \quad y = 2x - 1$$



▲ **Figure 1.1.5**

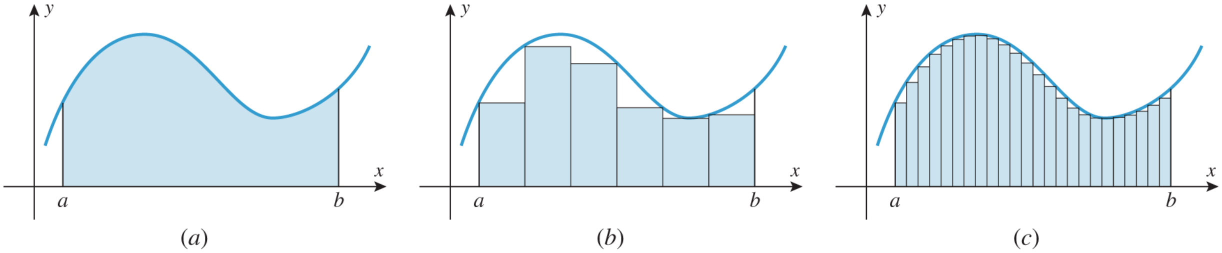
Figure 1.1.5 shows the graph of $y = x^2$ and this tangent line. ◀



▲ **Figure 1.1.6**

AREAS AND LIMITS

Just as the general notion of a tangent line leads to the concept of *limit*, so does the general notion of area. For plane regions with straight-line boundaries, areas can often be calculated by subdividing the region into rectangles or triangles and adding the areas of the constituent parts (Figure 1.1.6). However, for regions with curved boundaries, such as that in Figure 1.1.7*a*, a more general approach is needed. One such approach is to begin by approximating the area of the region by inscribing a number of rectangles of equal width under the curve and adding the areas of these rectangles (Figure 1.1.7*b*). Intuition suggests that if we repeat that approximation process using more and more rectangles, then the rectangles will tend to fill in the gaps under the curve, and the approximations will get closer and closer to the exact area under the curve (Figure 1.1.7*c*). This suggests that we can define the area under the curve to be the limiting value of these approximations. This idea will be considered in detail later, but the point to note here is that once again the concept of a limit comes into play.



▲ **Figure 1.1.7**

DECIMALS AND LIMITS

Limits also arise in the familiar context of decimals. For example, the decimal expansion of the fraction $\frac{1}{3}$ is

$$\frac{1}{3} = 0.33333\ldots \tag{3}$$

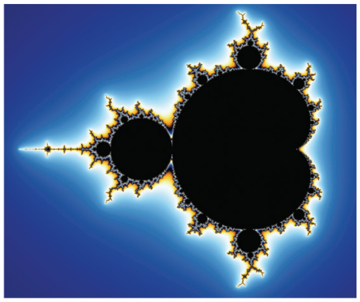
in which the dots indicate that the digit 3 repeats indefinitely. Although you may not have thought about decimals in this way, we can write (3) as

$$\frac{1}{3} = 0.33333\ldots = 0.3 + 0.03 + 0.003 + 0.0003 + 0.00003 + \cdots \tag{4}$$

which is a sum with "infinitely many" terms. As we will discuss in more detail later, we interpret (4) to mean that the succession of finite sums

$$0.3,\quad 0.3 + 0.03,\quad 0.3 + 0.03 + 0.003,\quad 0.3 + 0.03 + 0.003 + 0.0003,\ \ldots$$

gets closer and closer to a limiting value of $\frac{1}{3}$ as more and more terms are included. Thus, limits even occur in the familiar context of decimal representations of real numbers.



Andreas Nilsson/Shutterstock

This figure shows a region called the ***Mandelbrot Set****. It illustrates how complicated a region in the plane can be and why the notion of area requires careful definition.*

LIMITS

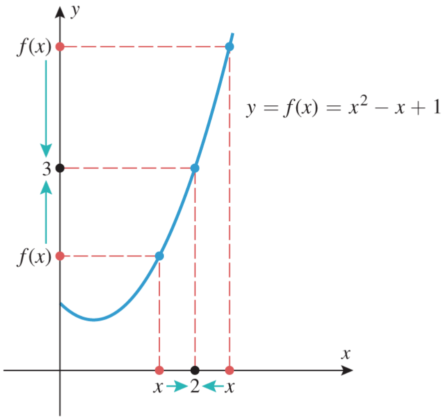
Now that we have seen how limits arise in various ways, let us focus on the limit concept itself.

The most basic use of limits is to describe how a function behaves as the independent variable approaches a given value. For example, let us examine the behavior of the function

$$f(x) = x^2 - x + 1$$

for x-values closer and closer to 2. It is evident from the graph and table in Figure 1.1.8 that the values of $f(x)$ get closer and closer to 3 as values of x are selected closer and closer to 2 on either the left or the right side of 2. We describe this by saying that the "limit of $x^2 - x + 1$ is 3 as x approaches 2 from either side," and we write

$$\lim_{x \to 2} (x^2 - x + 1) = 3 \tag{5}$$



x	1.0	1.5	1.9	1.95	1.99	1.995	1.999	2	2.001	2.005	2.01	2.05	2.1	2.5	3.0
$f(x)$	1.000000	1.750000	2.710000	2.852500	2.970100	2.985025	2.997001		3.003001	3.015025	3.030100	3.152500	3.310000	4.750000	7.000000

Left side → ← Right side

▲ **Figure 1.1.8**

This leads us to the following general idea.

> Since x is required to be different from a in (6), the value of f at a, or even whether f is defined at a, has no bearing on the limit L. The limit describes the behavior of f *close to* a but not *at* a.

1.1.1 LIMITS (AN INFORMAL VIEW) If the values of $f(x)$ can be made as close as we like to L by taking values of x sufficiently close to a (but not equal to a), then we write

$$\lim_{x\to a} f(x) = L \tag{6}$$

which is read "the limit of $f(x)$ as x approaches a is L" or "$f(x)$ approaches L as x approaches a." The expression in (6) can also be written as

$$f(x)\to L \quad \text{as} \quad x\to a \tag{7}$$

▶ **Example 2** Use numerical evidence to make a conjecture about the value of

$$\lim_{x\to 1} \frac{x-1}{\sqrt{x}-1} \tag{8}$$

Solution. Although the function

$$f(x) = \frac{x-1}{\sqrt{x}-1} \tag{9}$$

is undefined at $x = 1$, this has no bearing on the limit. Table 1.1.1 shows sample x-values approaching 1 from the left side and from the right side. In both cases the corresponding values of $f(x)$, calculated to six decimal places, appear to get closer and closer to 2, and hence we conjecture that

$$\lim_{x\to 1} \frac{x-1}{\sqrt{x}-1} = 2$$

This is consistent with the graph of f shown in Figure 1.1.9. In the next section we will show how to obtain this result algebraically. ◀

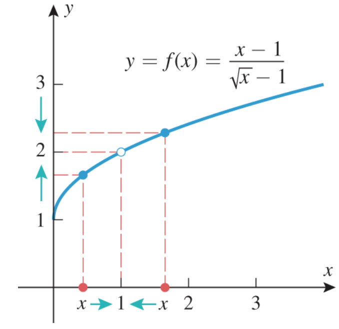
TECHNOLOGY MASTERY

> Use a graphing utility to generate the graph of the equation $y = f(x)$ for the function in (9). Find a window containing $x = 1$ in which all values of $f(x)$ are within 0.5 of $y = 2$ and one in which all values of $f(x)$ are within 0.1 of $y = 2$.

Table 1.1.1

x	0.99	0.999	0.9999	0.99999		1.00001	1.0001	1.001	1.01
$f(x)$	1.994987	1.999500	1.999950	1.999995		2.000005	2.000050	2.000500	2.004988

Left side → ← Right side



▲ **Figure 1.1.9**

Use numerical evidence to determine whether the limit in (11) changes if x is measured in degrees.

Table 1.1.2

x (RADIANS)	$y = \dfrac{\sin x}{x}$
± 1.0	0.84147
± 0.9	0.87036
± 0.8	0.89670
± 0.7	0.92031
± 0.6	0.94107
± 0.5	0.95885
± 0.4	0.97355
± 0.3	0.98507
± 0.2	0.99335
± 0.1	0.99833
± 0.01	0.99998

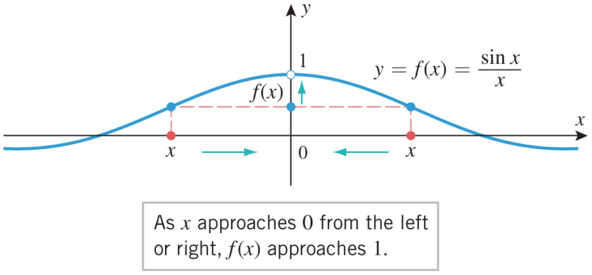
▶ **Example 3** Use numerical evidence to make a conjecture about the value of

$$\lim_{x\to 0} \frac{\sin x}{x} \tag{10}$$

Solution. With the help of a calculating utility set in radian mode, we obtain Table 1.1.2. The data in the table suggest that

$$\lim_{x\to 0} \frac{\sin x}{x} = 1 \tag{11}$$

The result is consistent with the graph of $f(x) = (\sin x)/x$ shown in Figure 1.1.10. Later in this chapter we will give a geometric argument to prove that our conjecture is correct. ◀



▶ **Figure 1.1.10**

SAMPLING PITFALLS

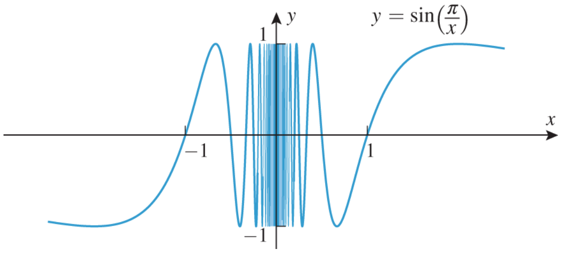
Numerical evidence can sometimes lead to incorrect conclusions about limits because of roundoff error or because the sample values chosen do not reveal the true limiting behavior. For example, one might *incorrectly* conclude from Table 1.1.3 that

$$\lim_{x\to 0} \sin\left(\frac{\pi}{x}\right) = 0$$

The fact that this is not correct is evidenced by the graph of f in Figure 1.1.11. The graph reveals that the values of f oscillate between -1 and 1 with increasing rapidity as $x \to 0$ and hence do not approach a limit. The data in the table deceived us because the x-values selected all happened to be x-intercepts for $f(x)$. This points out the need for having alternative methods for corroborating limits conjectured from numerical evidence.

Table 1.1.3

x	$\dfrac{\pi}{x}$	$f(x) = \sin\left(\dfrac{\pi}{x}\right)$
$x = \pm 1$	$\pm\pi$	$\sin(\pm\pi) = 0$
$x = \pm 0.1$	$\pm 10\pi$	$\sin(\pm 10\pi) = 0$
$x = \pm 0.01$	$\pm 100\pi$	$\sin(\pm 100\pi) = 0$
$x = \pm 0.001$	$\pm 1000\pi$	$\sin(\pm 1000\pi) = 0$
$x = \pm 0.0001$	$\pm 10{,}000\pi$	$\sin(\pm 10{,}000\pi) = 0$
$\vdots$	$\vdots$	$\vdots$



▲ **Figure 1.1.11**

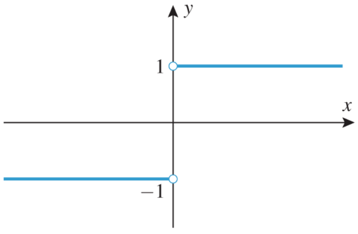
ONE-SIDED LIMITS

The limit in (6) is called a ***two-sided limit*** because it requires the values of $f(x)$ to get closer and closer to L as values of x are taken from *either* side of $x = a$. However, some functions exhibit different behaviors on the two sides of an x-value a, in which case it is necessary to distinguish whether values of x near a are on the left side or on the right side of a for purposes of investigating limiting behavior. For example, consider the function

$$f(x) = \frac{|x|}{x} = \begin{cases} 1, & x > 0 \\ -1, & x < 0 \end{cases} \tag{12}$$

which is graphed in Figure 1.1.12. As x approaches 0 from the *right*, the values of $f(x)$ approach a limit of 1 [in fact, the values of $f(x)$ are exactly 1 for all such x], and similarly, as x approaches 0 from the *left*, the values of $f(x)$ approach a limit of -1. We denote these limits by writing

$$\lim_{x\to 0^+} \frac{|x|}{x} = 1 \quad \text{and} \quad \lim_{x\to 0^-} \frac{|x|}{x} = -1 \tag{13}$$



$y = \dfrac{|x|}{x}$

▲ Figure 1.1.12

With this notation, the superscript "+" indicates a limit from the right and the superscript "−" indicates a limit from the left.

This leads to the general idea of a ***one-sided limit***.

> **1.1.2 ONE-SIDED LIMITS (AN INFORMAL VIEW)** If the values of $f(x)$ can be made as close as we like to L by taking values of x sufficiently close to a (but greater than a), then we write
>
> $$\lim_{x\to a^+} f(x) = L \tag{14}$$
>
> and if the values of $f(x)$ can be made as close as we like to L by taking values of x sufficiently close to a (but less than a), then we write
>
> $$\lim_{x\to a^-} f(x) = L \tag{15}$$
>
> Expression (14) is read "the limit of $f(x)$ as x approaches a from the right is L" or "$f(x)$ approaches L as x approaches a from the right." Similarly, expression (15) is read "the limit of $f(x)$ as x approaches a from the left is L" or "$f(x)$ approaches L as x approaches a from the left."

As with two-sided limits, the one-sided limits in (14) and (15) can also be written as

$$f(x) \to L \quad \text{as} \quad x \to a^+$$

and

$$f(x) \to L \quad \text{as} \quad x \to a^-$$

respectively.

THE RELATIONSHIP BETWEEN ONE-SIDED LIMITS AND TWO-SIDED LIMITS

In general, there is no guarantee that a function f will have a two-sided limit at a given point a; that is, the values of $f(x)$ may not get closer and closer to any *single* real number L as $x \to a$. In this case we say that

$$\lim_{x\to a} f(x) \quad \textit{does not exist}$$

Similarly, the values of $f(x)$ may not get closer and closer to a single real number L as $x \to a^+$ or as $x \to a^-$. In these cases we say that

$$\lim_{x\to a^+} f(x) \quad \textit{does not exist}$$

or that

$$\lim_{x\to a^-} f(x) \quad \textit{does not exist}$$

In order for the two-sided limit of a function $f(x)$ to exist at a point a, the values of $f(x)$ must approach some real number L as x approaches a, and this number must be the same regardless of whether x approaches a from the left or the right. This suggests the following result, which we state without formal proof.

> **1.1.3 THE RELATIONSHIP BETWEEN ONE-SIDED AND TWO-SIDED LIMITS** The two-sided limit of a function $f(x)$ exists at a if and only if both of the one-sided limits exist at a and have the same value; that is,
>
> $$\lim_{x\to a} f(x) = L \quad \text{if and only if} \quad \lim_{x\to a^-} f(x) = L = \lim_{x\to a^+} f(x)$$

▶ Example 4 Explain why

$$\lim_{x\to 0} \frac{|x|}{x}$$

does not exist.

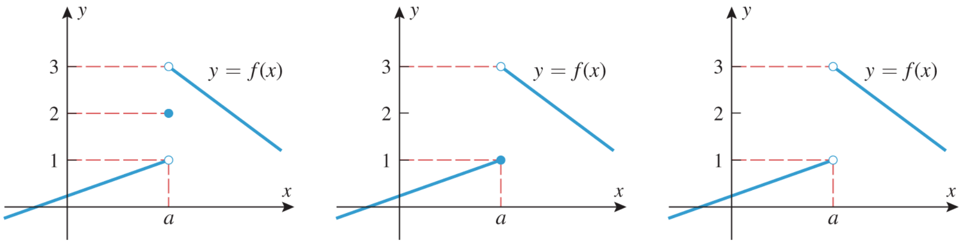
Solution. As x approaches 0, the values of $f(x) = |x|/x$ approach -1 from the left and approach 1 from the right [see (13)]. Thus, the one-sided limits at 0 are not the same. ◀

▶ Example 5 For the functions in Figure 1.1.13, find the one-sided and two-sided limits at $x = a$ if they exist.

Solution. The functions in all three figures have the same one-sided limits as $x \to a$, since the functions are identical, except at $x = a$. These limits are

$$\lim_{x \to a^+} f(x) = 3 \quad \text{and} \quad \lim_{x \to a^-} f(x) = 1$$

In all three cases the two-sided limit does not exist as $x \to a$ because the one-sided limits are not equal. ◀



▶ **Figure 1.1.13**

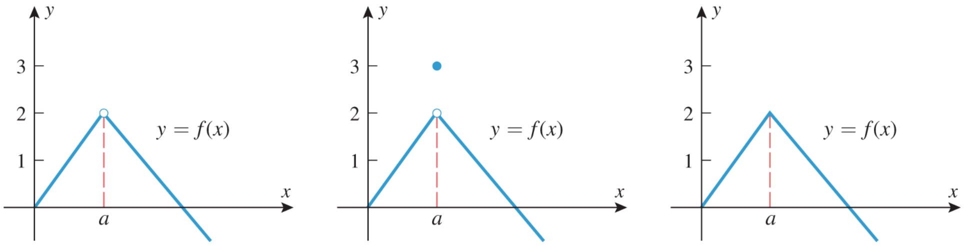
▶ Example 6 For the functions in Figure 1.1.14, find the one-sided and two-sided limits at $x = a$ if they exist.

Solution. As in the preceding example, the value of f at $x = a$ has no bearing on the limits as $x \to a$, so in all three cases we have

$$\lim_{x \to a^+} f(x) = 2 \quad \text{and} \quad \lim_{x \to a^-} f(x) = 2$$

Since the one-sided limits are equal, the two-sided limit exists and

$$\lim_{x \to a} f(x) = 2 \quad ◀$$



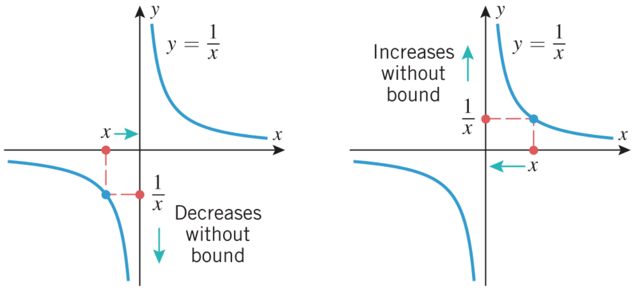
▲ **Figure 1.1.14**

INFINITE LIMITS

Sometimes one-sided or two-sided limits fail to exist because the values of the function increase or decrease without bound. For example, consider the behavior of $f(x) = 1/x$ for values of x near 0. It is evident from the table and graph in Figure 1.1.15 that as x-values are taken closer and closer to 0 from the right, the values of $f(x) = 1/x$ are positive and increase without bound; and as x-values are taken closer and closer to 0 from the left, the values of $f(x) = 1/x$ are negative and decrease without bound. We describe these limiting behaviors by writing

$$\lim_{x \to 0^+} \frac{1}{x} = +\infty \quad \text{and} \quad \lim_{x \to 0^-} \frac{1}{x} = -\infty$$

The symbols $+\infty$ and $-\infty$ here are *not* real numbers; they simply describe particular ways in which the limits fail to exist. Do not make the mistake of manipulating these symbols using rules of algebra. For example, it is *incorrect* to write $(+\infty) - (+\infty) = 0$.



x	−1	−0.1	−0.01	−0.001	−0.0001	0	0.0001	0.001	0.01	0.1	1
$\frac{1}{x}$	−1	−10	−100	−1000	−10,000		10,000	1000	100	10	1

Left side → ← Right side

▲ **Figure 1.1.15**

1.1.4 INFINITE LIMITS (AN INFORMAL VIEW) The expressions

$$\lim_{x \to a^-} f(x) = +\infty \quad \text{and} \quad \lim_{x \to a^+} f(x) = +\infty$$

denote that $f(x)$ increases without bound as x approaches a from the left and from the right, respectively. If both are true, then we write

$$\lim_{x \to a} f(x) = +\infty$$

Similarly, the expressions

$$\lim_{x \to a^-} f(x) = -\infty \quad \text{and} \quad \lim_{x \to a^+} f(x) = -\infty$$

denote that $f(x)$ decreases without bound as x approaches a from the left and from the right, respectively. If both are true, then we write

$$\lim_{x \to a} f(x) = -\infty$$

▶ **Example 7** For the functions in Figure 1.1.16, describe the limits at $x = a$ in appropriate limit notation.

Solution (a). In Figure 1.1.16*a*, the function increases without bound as x approaches a from the right and decreases without bound as x approaches a from the left. Thus,

$$\lim_{x \to a^+} \frac{1}{x-a} = +\infty \quad \text{and} \quad \lim_{x \to a^-} \frac{1}{x-a} = -\infty$$

Solution (b). In Figure 1.1.16*b*, the function increases without bound as x approaches a from both the left and right. Thus,

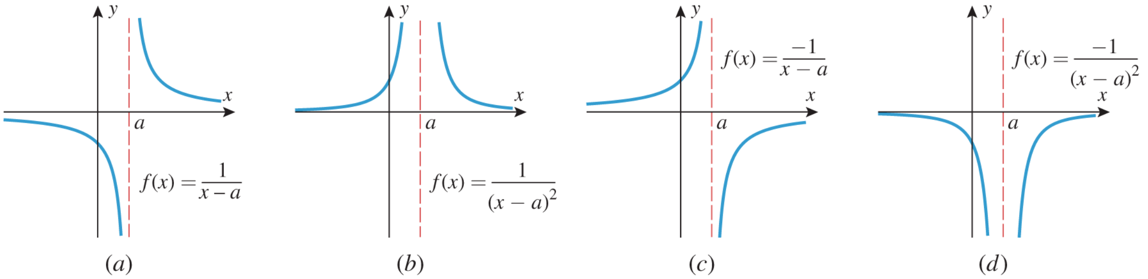
$$\lim_{x \to a} \frac{1}{(x-a)^2} = \lim_{x \to a^+} \frac{1}{(x-a)^2} = \lim_{x \to a^-} \frac{1}{(x-a)^2} = +\infty$$

Solution (c). In Figure 1.1.16*c*, the function decreases without bound as x approaches a from the right and increases without bound as x approaches a from the left. Thus,

$$\lim_{x \to a^+} \frac{-1}{x-a} = -\infty \quad \text{and} \quad \lim_{x \to a^-} \frac{-1}{x-a} = +\infty$$

Solution (d). In Figure 1.1.16*d*, the function decreases without bound as x approaches a from both the left and right. Thus,

$$\lim_{x\to a} \frac{-1}{(x-a)^2} = \lim_{x\to a^+} \frac{-1}{(x-a)^2} = \lim_{x\to a^-} \frac{-1}{(x-a)^2} = -\infty \blacktriangleleft$$



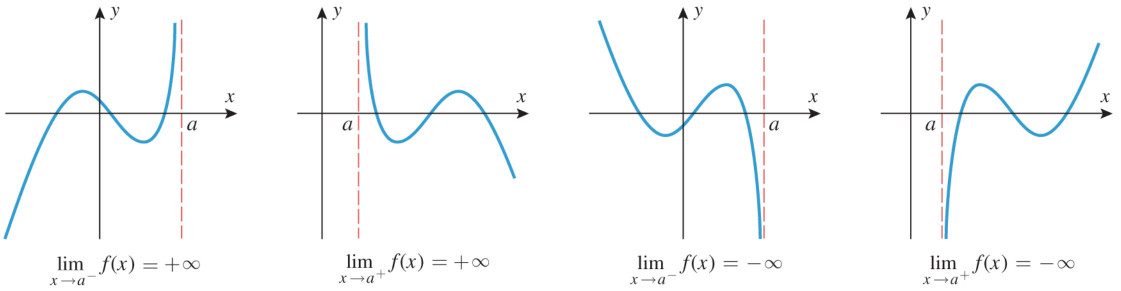
▲ **Figure 1.1.16**

VERTICAL ASYMPTOTES

Figure 1.1.17 illustrates geometrically what happens when any of the following situations occur:

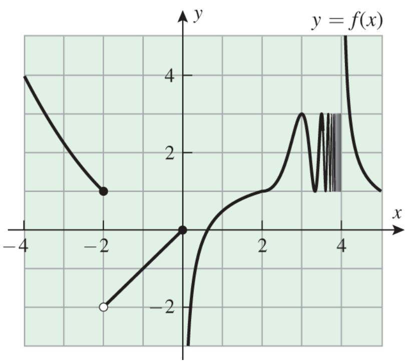
$$\lim_{x\to a^-} f(x) = +\infty, \quad \lim_{x\to a^+} f(x) = +\infty, \quad \lim_{x\to a^-} f(x) = -\infty, \quad \lim_{x\to a^+} f(x) = -\infty$$

In each case the graph of $y = f(x)$ either rises or falls without bound, squeezing closer and closer to the vertical line $x = a$ as x approaches a from the side indicated in the limit. The line $x = a$ is called a ***vertical asymptote*** of the curve $y = f(x)$ (from the Greek word *asymptotos*, meaning "nonintersecting").



▲ **Figure 1.1.17**

In general, the graph of a single function can display a wide variety of limits.



▲ **Figure 1.1.18**

▶ **Example 8** For the function f graphed in Figure 1.1.18, find

(a) $\lim_{x\to -2^-} f(x)$ (b) $\lim_{x\to -2^+} f(x)$ (c) $\lim_{x\to 0^-} f(x)$ (d) $\lim_{x\to 0^+} f(x)$
(e) $\lim_{x\to 4^-} f(x)$ (f) $\lim_{x\to 4^+} f(x)$ (g) the vertical asymptotes of the graph of f.

Solution (a) and (b).

$$\lim_{x\to -2^-} f(x) = 1 = f(-2) \quad \text{and} \quad \lim_{x\to -2^+} f(x) = -2$$

Solution (c) and (d).

$$\lim_{x\to 0^-} f(x) = 0 = f(0) \quad \text{and} \quad \lim_{x\to 0^+} f(x) = -\infty$$

Solution (e) and (f).

$$\lim_{x\to 4^-} f(x) \text{ does not exist due to oscillation} \quad \text{and} \quad \lim_{x\to 4^+} f(x) = +\infty$$

Solution (g). The y-axis and the line $x = 4$ are vertical asymptotes for the graph of f. ◀

✓ QUICK CHECK EXERCISES 1.1

(See page 13 for answers.)

1. We write $\lim_{x\to a} f(x) = L$ provided the values of ________ can be made as close to ________ as desired, by taking values of ________ sufficiently close to ________ but not ________.

2. We write $\lim_{x\to a^-} f(x) = +\infty$ provided ________ increases without bound, as ________ approaches ________ from the left.

3. State what must be true about
$$\lim_{x\to a^-} f(x) \quad \text{and} \quad \lim_{x\to a^+} f(x)$$
in order for it to be the case that
$$\lim_{x\to a} f(x) = L$$

4. Use the accompanying graph of $y = f(x)$ $(-\infty < x < 3)$ to determine the limits.
   (a) $\lim_{x\to 0} f(x) =$ ________
   (b) $\lim_{x\to 2^-} f(x) =$ ________
   (c) $\lim_{x\to 2^+} f(x) =$ ________
   (d) $\lim_{x\to 3^-} f(x) =$ ________

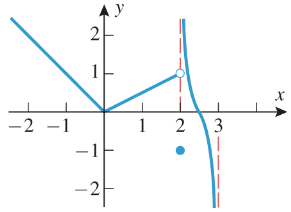


Figure Ex-4

5. The slope of the secant line through $P(2, 4)$ and $Q(x, x^2)$ on the parabola $y = x^2$ is $m_{\text{sec}} = x + 2$. It follows that the slope of the tangent line to this parabola at the point P is ________.

EXERCISE SET 1.1 Graphing Utility C CAS

1–10 In these exercises, make reasonable assumptions about the graph of the indicated function outside of the region depicted.

1. For the function g graphed in the accompanying figure, find
   (a) $\lim_{x\to 0^-} g(x)$ (b) $\lim_{x\to 0^+} g(x)$
   (c) $\lim_{x\to 0} g(x)$ (d) $g(0)$.

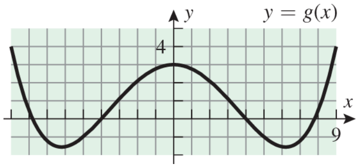


Figure Ex-1

2. For the function G graphed in the accompanying figure, find
   (a) $\lim_{x\to 0^-} G(x)$ (b) $\lim_{x\to 0^+} G(x)$
   (c) $\lim_{x\to 0} G(x)$ (d) $G(0)$.

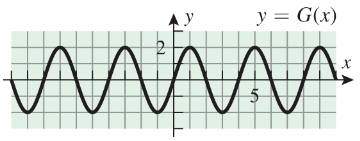


Figure Ex-2

3. For the function f graphed in the accompanying figure, find
   (a) $\lim_{x\to 3^-} f(x)$ (b) $\lim_{x\to 3^+} f(x)$
   (c) $\lim_{x\to 3} f(x)$ (d) $f(3)$.

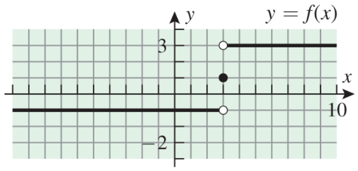


Figure Ex-3

4. For the function f graphed in the accompanying figure, find
   (a) $\lim_{x\to 2^-} f(x)$ (b) $\lim_{x\to 2^+} f(x)$
   (c) $\lim_{x\to 2} f(x)$ (d) $f(2)$.

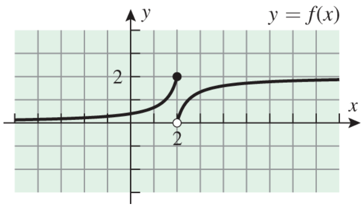


Figure Ex-4

5. For the function F graphed in the accompanying figure, find
   (a) $\lim_{x\to -2^-} F(x)$ (b) $\lim_{x\to -2^+} F(x)$
   (c) $\lim_{x\to -2} F(x)$ (d) $F(-2)$.

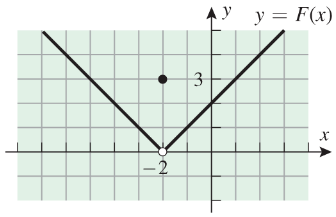
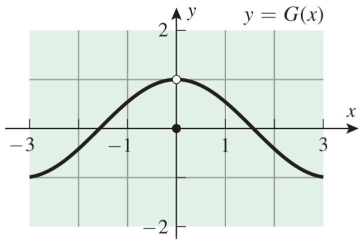


Figure Ex-5

6. For the function G graphed in the accompanying figure, find

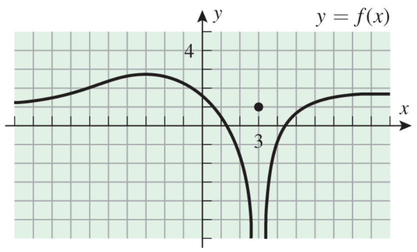
(a) $\lim_{x \to 0^-} G(x)$ (b) $\lim_{x \to 0^+} G(x)$
(c) $\lim_{x \to 0} G(x)$ (d) $G(0)$.



◀ **Figure Ex-6**

7. For the function f graphed in the accompanying figure, find

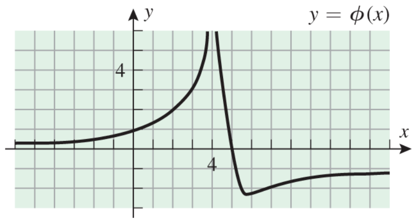
(a) $\lim_{x \to 3^-} f(x)$ (b) $\lim_{x \to 3^+} f(x)$
(c) $\lim_{x \to 3} f(x)$ (d) $f(3)$.



◀ **Figure Ex-7**

8. For the function ϕ graphed in the accompanying figure, find

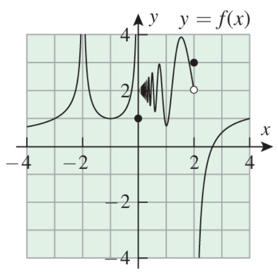
(a) $\lim_{x \to 4^-} \phi(x)$ (b) $\lim_{x \to 4^+} \phi(x)$
(c) $\lim_{x \to 4} \phi(x)$ (d) $\phi(4)$.



◀ **Figure Ex-8**

9. For the function f graphed in the accompanying figure, find

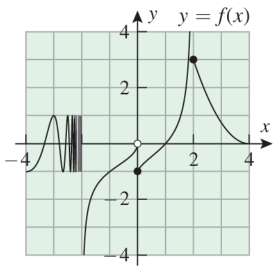
(a) $\lim_{x \to -2} f(x)$ (b) $\lim_{x \to 0^-} f(x)$
(c) $\lim_{x \to 0^+} f(x)$ (d) $\lim_{x \to 2^-} f(x)$
(e) $\lim_{x \to 2^+} f(x)$
(f) the vertical asymptotes of the graph of f.



◀ **Figure Ex-9**

10. For the function f graphed in the accompanying figure, find

(a) $\lim_{x \to -2^-} f(x)$ (b) $\lim_{x \to -2^+} f(x)$ (c) $\lim_{x \to 0^-} f(x)$
(d) $\lim_{x \to 0^+} f(x)$ (e) $\lim_{x \to 2^-} f(x)$ (f) $\lim_{x \to 2^+} f(x)$
(g) the vertical asymptotes of the graph of f.



◀ **Figure Ex-10**

11–12 (i) Complete the table and make a guess about the limit indicated. (ii) Confirm your conclusions about the limit by graphing a function over an appropriate interval. [*Note:* For the trigonometric functions, be sure to put your calculating and graphing utilities in radian mode.]

11. $f(x) = \dfrac{\sin 2x}{x}$; $\lim_{x \to 0} f(x)$

x	-0.1	-0.01	-0.001	0.001	0.01	0.1
$f(x)$						

▲ **Table Ex-11**

12. $f(x) = \dfrac{\cos x - 1}{x^2}$; $\lim_{x \to 0} f(x)$

x	-0.5	-0.05	-0.005	0.005	0.05	0.5
$f(x)$						

▲ **Table Ex-12**

13–16 (i) Make a guess at the limit (if it exists) by evaluating the function at the specified x-values. (ii) Confirm your conclusions about the limit by graphing the function over an appropriate interval. (iii) If you have a CAS, then use it to find the limit. [*Note:* For the trigonometric functions, be sure to put your calculating and graphing utilities in radian mode.]

13. (a) $\lim_{x \to 1} \dfrac{x - 1}{x^3 - 1}$; $x = 2, 1.5, 1.1, 1.01, 1.001, 0, 0.5, 0.9, 0.99, 0.999$

(b) $\lim_{x \to 1^+} \dfrac{x + 1}{x^3 - 1}$; $x = 2, 1.5, 1.1, 1.01, 1.001, 1.0001$

(c) $\lim_{x \to 1^-} \dfrac{x + 1}{x^3 - 1}$; $x = 0, 0.5, 0.9, 0.99, 0.999, 0.9999$

14. (a) $\lim_{x \to 0} \dfrac{\sqrt{x+1} - 1}{x}$; $x = \pm 0.25, \pm 0.1, \pm 0.001, \pm 0.0001$

(b) $\lim_{x \to 0^+} \dfrac{\sqrt{x+1} + 1}{x}$; $x = 0.25, 0.1, 0.001, 0.0001$

(cont.)

(c) $\lim_{x\to 0^-} \dfrac{\sqrt{x+1}+1}{x}$; $x = -0.25, -0.1, -0.001, -0.0001$

15. (a) $\lim_{x\to 0} \dfrac{\sin 3x}{x}$; $x = \pm 0.25, \pm 0.1, \pm 0.001, \pm 0.0001$

(b) $\lim_{x\to -1} \dfrac{\cos x}{x+1}$; $x = 0, -0.5, -0.9, -0.99, -0.999, -1.5, -1.1, -1.01, -1.001$

16. (a) $\lim_{x\to -1} \dfrac{\tan(x+1)}{x+1}$; $x = 0, -0.5, -0.9, -0.99, -0.999, -1.5, -1.1, -1.01, -1.001$

(b) $\lim_{x\to 0} \dfrac{\sin(5x)}{\sin(2x)}$; $x = \pm 0.25, \pm 0.1, \pm 0.001, \pm 0.0001$

17–20 True–False Determine whether the statement is true or false. Explain your answer.

17. If $f(a) = L$, then $\lim_{x\to a} f(x) = L$.

18. If $\lim_{x\to a} f(x)$ exists, then so do $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^+} f(x)$.

19. If $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^+} f(x)$ exist, then so does $\lim_{x\to a} f(x)$.

20. If $\lim_{x\to a^+} f(x) = +\infty$, then $f(a)$ is undefined.

21–26 Sketch a possible graph for a function f with the specified properties. (Many different solutions are possible.)

21. (i) the domain of f is $[-1, 1]$
(ii) $f(-1) = f(0) = f(1) = 0$
(iii) $\lim_{x\to -1^+} f(x) = \lim_{x\to 0} f(x) = \lim_{x\to 1^-} f(x) = 1$

22. (i) the domain of f is $[-2, 1]$
(ii) $f(-2) = f(0) = f(1) = 0$
(iii) $\lim_{x\to -2^+} f(x) = 2$, $\lim_{x\to 0} f(x) = 0$, and $\lim_{x\to 1^-} f(x) = 1$

23. (i) the domain of f is $(-\infty, 0]$
(ii) $f(-2) = f(0) = 1$
(iii) $\lim_{x\to -2} f(x) = +\infty$

24. (i) the domain of f is $(0, +\infty)$
(ii) $f(1) = 0$
(iii) the y-axis is a vertical asymptote for the graph of f
(iv) $f(x) < 0$ if $0 < x < 1$

25. (i) $f(-3) = f(0) = f(2) = 0$
(ii) $\lim_{x\to -2^-} f(x) = +\infty$ and $\lim_{x\to -2^+} f(x) = -\infty$
(iii) $\lim_{x\to 1} f(x) = +\infty$

26. (i) $f(-1) = 0$, $f(0) = 1$, $f(1) = 0$
(ii) $\lim_{x\to -1^-} f(x) = 0$ and $\lim_{x\to -1^+} f(x) = +\infty$
(iii) $\lim_{x\to 1^-} f(x) = 1$ and $\lim_{x\to 1^+} f(x) = +\infty$

27–30 Modify the argument of Example 1 to find the equation of the tangent line to the specified graph at the point given.

27. the graph of $y = x^2$ at $(-1, 1)$

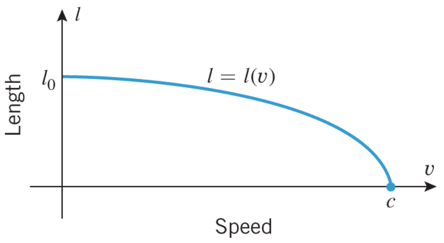
28. the graph of $y = x^2$ at $(0, 0)$

29. the graph of $y = x^4$ at $(1, 1)$

30. the graph of $y = x^4$ at $(-1, 1)$

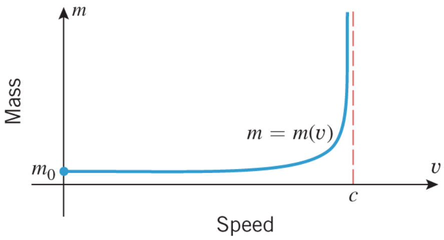
FOCUS ON CONCEPTS

31. In the special theory of relativity the length l of a narrow rod moving longitudinally is a function $l = l(v)$ of the rod's speed v. The accompanying figure, in which c denotes the speed of light, displays some of the qualitative features of this function.
(a) What is the physical interpretation of l_0?
(b) What is $\lim_{v\to c^-} l(v)$? What is the physical significance of this limit?



▲ **Figure Ex-31**

32. In the special theory of relativity the mass m of a moving object is a function $m = m(v)$ of the object's speed v. The accompanying figure, in which c denotes the speed of light, displays some of the qualitative features of this function.
(a) What is the physical interpretation of m_0?
(b) What is $\lim_{v\to c^-} m(v)$? What is the physical significance of this limit?



▲ **Figure Ex-32**

33. Let

$$f(x) = \left(1 + x^2\right)^{1.1/x^2}$$

(a) Graph f in the window

$$[-1, 1] \times [2.5, 3.5]$$

and use the calculator's trace feature to make a conjecture about the limit of $f(x)$ as $x \to 0$.

(b) Graph f in the window

$$[-0.001, 0.001] \times [2.5, 3.5]$$

and use the calculator's trace feature to make a conjecture about the limit of $f(x)$ as $x \to 0$.

(c) Graph f in the window

$$[-0.000001, 0.000001] \times [2.5, 3.5]$$

and use the calculator's trace feature to make a conjecture about the limit of $f(x)$ as $x \to 0$.

(cont.)

(d) Later we will be able to show that

$$\lim_{x \to 0} \left(1 + x^2\right)^{1.1/x^2} \approx 3.00416602$$

What flaw do your graphs reveal about using numerical evidence (as revealed by the graphs you obtained) to make conjectures about limits?

34. **Writing** Two students are discussing the limit of $\sqrt{x}$ as x approaches 0. One student maintains that the limit is 0, while the other claims that the limit does not exist. Write a short paragraph that discusses the pros and cons of each student's position.

35. **Writing** Given a function f and a real number a, explain informally why

$$\lim_{x \to 0} f(x + a) = \lim_{x \to a} f(x)$$

(Here "equality" means that either both limits exist and are equal or that both limits fail to exist.)

✓**QUICK CHECK ANSWERS 1.1** **1.** $f(x)$; L; x; a; a **2.** $f(x)$; x; a **3.** Both one-sided limits must exist and equal L. **4.** (a) 0 (b) 1 (c) $+\infty$ (d) $-\infty$ **5.** 4

1.2 COMPUTING LIMITS

In this section we will discuss techniques for computing limits of many functions. We base these results on the informal development of the limit concept discussed in the preceding section. A more formal derivation of these results is possible after Section 1.4.

SOME BASIC LIMITS

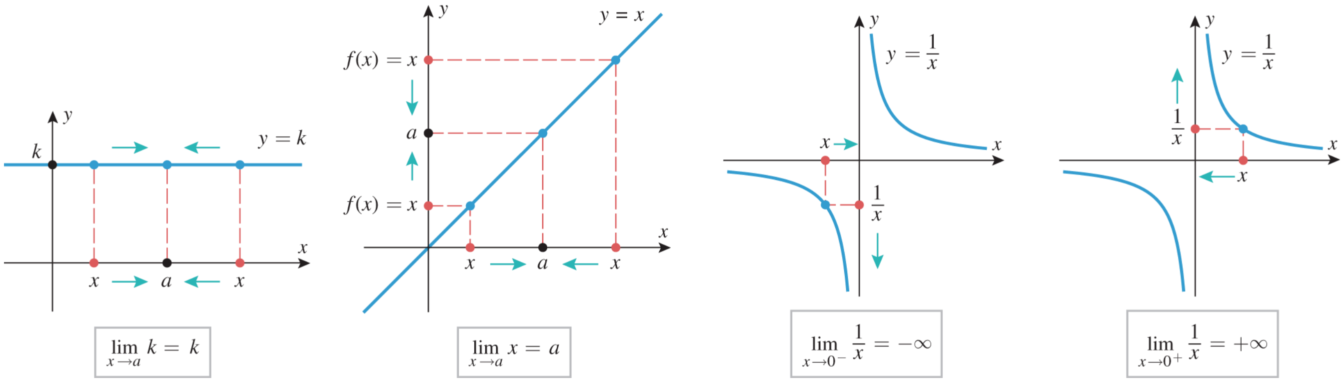
Our strategy for finding limits algebraically has two parts:

- First we will obtain the limits of some simple functions.
- Then we will develop a repertoire of theorems that will enable us to use the limits of those simple functions as building blocks for finding limits of more complicated functions.

We start with the following basic results, which are illustrated in Figure 1.2.1.

1.2.1 THEOREM *Let a and k be real numbers.*

(a) $\displaystyle\lim_{x \to a} k = k$ *(b)* $\displaystyle\lim_{x \to a} x = a$ *(c)* $\displaystyle\lim_{x \to 0^-} \frac{1}{x} = -\infty$ *(d)* $\displaystyle\lim_{x \to 0^+} \frac{1}{x} = +\infty$



▲ **Figure 1.2.1**

The following examples explain these results further.

> Do not confuse the algebraic size of a number with its closeness to zero. For positive numbers, the smaller the number the closer it is to zero, but for negative numbers, the larger the number the closer it is to zero. For example, -2 is larger than -4, but it is closer to zero.

▶ Example 1 If $f(x) = k$ is a constant function, then the values of $f(x)$ remain fixed at k as x varies, which explains why $f(x) \to k$ as $x \to a$ for all values of a. For example,

$$\lim_{x\to -25} 3 = 3, \qquad \lim_{x\to 0} 3 = 3, \qquad \lim_{x\to \pi} 3 = 3 \quad ◀$$

▶ Example 2 If $f(x) = x$, then as $x \to a$ it must also be true that $f(x) \to a$. For example,

$$\lim_{x\to 0} x = 0, \qquad \lim_{x\to -2} x = -2, \qquad \lim_{x\to \pi} x = \pi \quad ◀$$

▶ Example 3 You should know from your experience with fractions that for a fixed nonzero numerator, the closer the denominator is to zero, the larger the absolute value of the fraction. This fact and the data in Table 1.2.1 suggest why $1/x \to +\infty$ as $x \to 0^+$ and why $1/x \to -\infty$ as $x \to 0^-$. ◀

Table 1.2.1

	VALUES						CONCLUSION
x $1/x$	-1 -1	-0.1 -10	-0.01 -100	-0.001 -1000	-0.0001 $-10{,}000$	$\cdots$ $\cdots$	As $x \to 0^-$ the value of $1/x$ decreases without bound.
x $1/x$	1 1	0.1 10	0.01 100	0.001 1000	0.0001 $10{,}000$	$\cdots$ $\cdots$	As $x \to 0^+$ the value of $1/x$ increases without bound.

The following theorem, parts of which are proved in Web Appendix L, will be our basic tool for finding limits algebraically.

1.2.2 THEOREM *Let a be a real number, and suppose that*

$$\lim_{x\to a} f(x) = L_1 \quad \textit{and} \quad \lim_{x\to a} g(x) = L_2$$

That is, the limits exist and have values L_1 and L_2, respectively. Then:

(*a*) $\displaystyle \lim_{x\to a} [f(x) + g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x) = L_1 + L_2$

(*b*) $\displaystyle \lim_{x\to a} [f(x) - g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x) = L_1 - L_2$

(*c*) $\displaystyle \lim_{x\to a} [f(x)g(x)] = \left(\lim_{x\to a} f(x)\right)\left(\lim_{x\to a} g(x)\right) = L_1L_2$

(*d*) $\displaystyle \lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \frac{L_1}{L_2}$, *provided $L_2 \neq 0$*

(*e*) $\displaystyle \lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)} = \sqrt[n]{L_1}$, *provided $L_1 > 0$ if n is even.*

Moreover, these statements are also true for the one-sided limits as $x \to a^-$ or as $x \to a^+$.

> Theorem 1.2.2(e) remains valid for n even and $L_1 = 0$, provided $f(x)$ is nonnegative for x near a with $x \neq a$.

This theorem can be stated informally as follows:

(*a*) *The limit of a sum is the sum of the limits.*

(*b*) *The limit of a difference is the difference of the limits.*

(*c*) *The limit of a product is the product of the limits.*

(*d*) *The limit of a quotient is the quotient of the limits, provided the limit of the denominator is not zero.*

(*e*) *The limit of an nth root is the nth root of the limit.*

For the special case of part (*c*) in which $f(x) = k$ is a constant function, we have

$$\lim_{x\to a} (kg(x)) = \lim_{x\to a} k \cdot \lim_{x\to a} g(x) = k \lim_{x\to a} g(x) \tag{1}$$

and similarly for one-sided limits. This result can be rephrased as follows:

A constant factor can be moved through a limit symbol.

Although parts (*a*) and (*c*) of Theorem 1.2.2 are stated for two functions, the results hold for any finite number of functions. Moreover, the various parts of the theorem can be used in combination to reformulate expressions involving limits.

▶ Example 4

$$\lim_{x\to a} [f(x) - g(x) + 2h(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x) + 2 \lim_{x\to a} h(x)$$

$$\lim_{x\to a} [f(x)g(x)h(x)] = \left(\lim_{x\to a} f(x)\right)\left(\lim_{x\to a} g(x)\right)\left(\lim_{x\to a} h(x)\right)$$

$$\lim_{x\to a} [f(x)]^3 = \left(\lim_{x\to a} f(x)\right)^3$$

Take $g(x) = h(x) = f(x)$ in the last equation.

$$\lim_{x\to a} [f(x)]^n = \left(\lim_{x\to a} f(x)\right)^n$$

The extension of Theorem 1.2.2(*c*) in which there are n factors, each of which is $f(x)$

$$\lim_{x\to a} x^n = \left(\lim_{x\to a} x\right)^n = a^n$$

Apply the previous result with $f(x) = x$. ◀

LIMITS OF POLYNOMIALS AND RATIONAL FUNCTIONS AS $x \to a$

▶ Example 5 Find $\lim_{x\to 5} (x^2 - 4x + 3)$.

Solution.

$$\begin{aligned} \lim_{x\to 5} (x^2 - 4x + 3) &= \lim_{x\to 5} x^2 - \lim_{x\to 5} 4x + \lim_{x\to 5} 3 && \text{Theorem 1.2.2}(a), (b) \\ &= \lim_{x\to 5} x^2 - 4 \lim_{x\to 5} x + \lim_{x\to 5} 3 && \text{A constant can be moved through a limit symbol.} \\ &= 5^2 - 4(5) + 3 && \text{The last part of Example 4} \\ &= 8 \end{aligned}$$

◀

Observe that in Example 5 the limit of the polynomial $p(x) = x^2 - 4x + 3$ as $x \to 5$ turned out to be the same as $p(5)$. This is not an accident. The next result shows that, in general, the limit of a polynomial $p(x)$ as $x \to a$ is the same as the value of the polynomial at a. Knowing this fact allows us to reduce the computation of limits of polynomials to simply evaluating the polynomial at the appropriate point.

1.2.3 THEOREM *For any polynomial*

$$p(x) = c_0 + c_1 x + \cdots + c_n x^n$$

and any real number a,

$$\lim_{x \to a} p(x) = c_0 + c_1 a + \cdots + c_n a^n = p(a)$$

PROOF

$$\begin{aligned}
\lim_{x \to a} p(x) &= \lim_{x \to a} (c_0 + c_1 x + \cdots + c_n x^n) \\
&= \lim_{x \to a} c_0 + \lim_{x \to a} c_1 x + \cdots + \lim_{x \to a} c_n x^n \\
&= \lim_{x \to a} c_0 + c_1 \lim_{x \to a} x + \cdots + c_n \lim_{x \to a} x^n \\
&= c_0 + c_1 a + \cdots + c_n a^n = p(a) \quad \blacksquare
\end{aligned}$$

► Example 6 Find $\lim_{x \to 1} (x^7 - 2x^5 + 1)^{35}$.

Solution. The function involved is a polynomial (why?), so the limit can be obtained by evaluating this polynomial at $x = 1$. This yields

$$\lim_{x \to 1} (x^7 - 2x^5 + 1)^{35} = 0 \quad ◄$$

Recall that a rational function is a ratio of two polynomials. The following example illustrates how Theorems 1.2.2(d) and 1.2.3 can sometimes be used in combination to compute limits of rational functions.

► Example 7 Find $\lim_{x \to 2} \dfrac{5x^3 + 4}{x - 3}$.

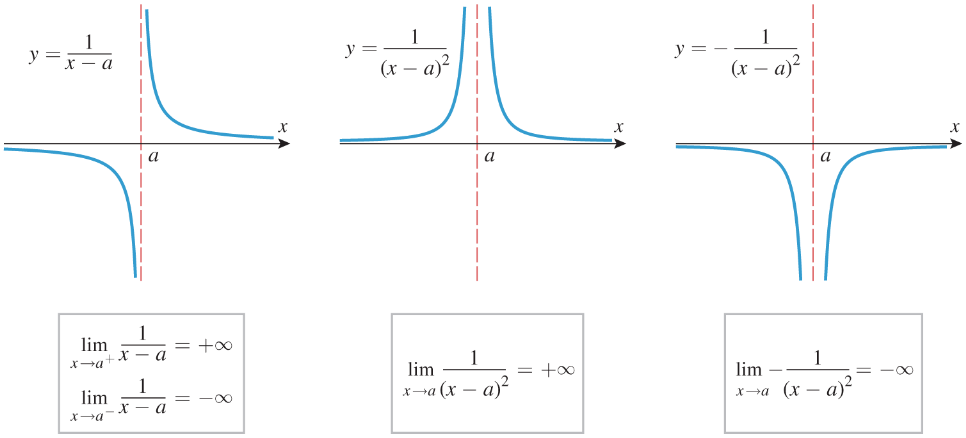
Solution.

$$\begin{aligned}
\lim_{x \to 2} \frac{5x^3 + 4}{x - 3} &= \frac{\lim_{x \to 2} (5x^3 + 4)}{\lim_{x \to 2} (x - 3)} && \boxed{\text{Theorem 1.2.2}(d)} \\
&= \frac{5 \cdot 2^3 + 4}{2 - 3} = -44 && \boxed{\text{Theorem 1.2.3}} \quad ◄
\end{aligned}$$

The method used in the last example will not work for rational functions in which the limit of the denominator is zero because Theorem 1.2.2(d) is not applicable. There are two cases of this type to be considered—the case where the limit of the denominator is zero and the limit of the numerator is not, and the case where the limits of the numerator and denominator are both zero. If the limit of the denominator is zero but the limit of the numerator is not, then one can prove that the limit of the rational function does not exist and that one of the following situations occurs:

- The limit may be $-\infty$ from one side and $+\infty$ from the other.
- The limit may be $+\infty$.
- The limit may be $-\infty$.

Figure 1.2.2 illustrates these three possibilities graphically for rational functions of the form $1/(x - a)$, $1/(x - a)^2$, and $-1/(x - a)^2$.



▲ **Figure 1.2.2**

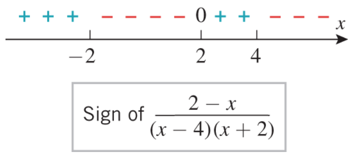
▶ **Example 8** Find

$$\text{(a)} \lim_{x\to 4^+} \frac{2-x}{(x-4)(x+2)} \qquad \text{(b)} \lim_{x\to 4^-} \frac{2-x}{(x-4)(x+2)} \qquad \text{(c)} \lim_{x\to 4} \frac{2-x}{(x-4)(x+2)}$$

Solution. In all three parts the limit of the numerator is -2, and the limit of the denominator is 0, so the limit of the ratio does not exist. To be more specific than this, we need to analyze the sign of the ratio. The sign of the ratio, which is given in Figure 1.2.3, is determined by the signs of $2-x$, $x-4$, and $x+2$. (The method of test points, discussed in Web Appendix F, provides a way of finding the sign of the ratio here.) It follows from this figure that as x approaches 4 from the right, the ratio is always negative; and as x approaches 4 from the left, the ratio is eventually positive. Thus,

$$\lim_{x\to 4^+} \frac{2-x}{(x-4)(x+2)} = -\infty \quad \text{and} \quad \lim_{x\to 4^-} \frac{2-x}{(x-4)(x+2)} = +\infty$$

Because the one-sided limits have opposite signs, all we can say about the two-sided limit is that it does not exist. ◀



▲ **Figure 1.2.3**

In the case where $p(x)/q(x)$ is a rational function for which $p(a) = 0$ and $q(a) = 0$, the numerator and denominator must have one or more common factors of $x - a$. In this case the limit of $p(x)/q(x)$ as $x \to a$ can be found by canceling all common factors of $x - a$ and using one of the methods already considered to find the limit of the simplified function. Here is an example.

In Example 9(a), the simplified function $x - 3$ is defined at $x = 3$, but the original function is not. However, this has no effect on the limit as x *approaches* 3 since the two functions are identical if $x \neq 3$ (Exercise 50).

▶ **Example 9** Find

$$\text{(a)} \lim_{x\to 3} \frac{x^2-6x+9}{x-3} \qquad \text{(b)} \lim_{x\to -4} \frac{2x+8}{x^2+x-12} \qquad \text{(c)} \lim_{x\to 5} \frac{x^2-3x-10}{x^2-10x+25}$$

Solution (a). The numerator and the denominator both have a zero at $x = 3$, so there is a common factor of $x - 3$. Then

$$\lim_{x\to 3} \frac{x^2-6x+9}{x-3} = \lim_{x\to 3} \frac{(x-3)^2}{x-3} = \lim_{x\to 3} (x-3) = 0$$

Solution (b). The numerator and the denominator both have a zero at $x = -4$, so there is a common factor of $x - (-4) = x + 4$. Then

$$\lim_{x\to -4} \frac{2x+8}{x^2+x-12} = \lim_{x\to -4} \frac{2(x+4)}{(x+4)(x-3)} = \lim_{x\to -4} \frac{2}{x-3} = -\frac{2}{7}$$

Solution (c). The numerator and the denominator both have a zero at $x = 5$, so there is a common factor of $x - 5$. Then

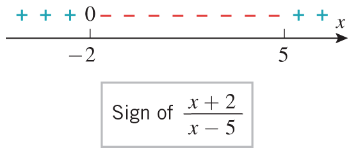
$$\lim_{x\to 5} \frac{x^2 - 3x - 10}{x^2 - 10x + 25} = \lim_{x\to 5} \frac{(x-5)(x+2)}{(x-5)(x-5)} = \lim_{x\to 5} \frac{x+2}{x-5}$$

However,

$$\lim_{x\to 5}(x+2) = 7 \neq 0 \quad \text{and} \quad \lim_{x\to 5}(x-5) = 0$$

so

$$\lim_{x\to 5} \frac{x^2 - 3x - 10}{x^2 - 10x + 25} = \lim_{x\to 5} \frac{x+2}{x-5}$$



▲ **Figure 1.2.4**

does not exist. More precisely, the sign analysis in Figure 1.2.4 implies that

$$\lim_{x\to 5^+} \frac{x^2 - 3x - 10}{x^2 - 10x + 25} = \lim_{x\to 5^+} \frac{x+2}{x-5} = +\infty$$

and

$$\lim_{x\to 5^-} \frac{x^2 - 3x - 10}{x^2 - 10x + 25} = \lim_{x\to 5^-} \frac{x+2}{x-5} = -\infty \quad ◀$$

Discuss the logical errors in the following statement: An indeterminate form of type $0/0$ must have a limit of zero because zero divided by anything is zero.

A quotient $f(x)/g(x)$ in which the numerator and denominator both have a limit of zero as $x \to a$ is called an ***indeterminate form of type* 0/0**. The problem with such limits is that it is difficult to tell by inspection whether the limit exists, and, if so, its value. Informally stated, this is because there are two conflicting influences at work. The value of $f(x)/g(x)$ would tend to zero as $f(x)$ approached zero if $g(x)$ were to remain at some fixed nonzero value, whereas the value of this ratio would tend to increase or decrease without bound as $g(x)$ approached zero if $f(x)$ were to remain at some fixed nonzero value. But with both $f(x)$ and $g(x)$ approaching zero, the behavior of the ratio depends on precisely how these conflicting tendencies offset one another for the particular f and g.

Sometimes, limits of indeterminate forms of type $0/0$ can be found by algebraic simplification, as in the last example, but frequently this will not work and other methods must be used. We will study such methods in later sections.

The following theorem summarizes our observations about limits of rational functions.

1.2.4 THEOREM *Let*

$$f(x) = \frac{p(x)}{q(x)}$$

be a rational function, and let a be any real number.

(*a*) *If* $q(a) \neq 0$, *then* $\lim_{x\to a} f(x) = f(a)$.

(*b*) *If* $q(a) = 0$ *but* $p(a) \neq 0$, *then* $\lim_{x\to a} f(x)$ *does not exist.*

LIMITS INVOLVING RADICALS

▶ **Example 10** Find $\lim_{x\to 1} \dfrac{x-1}{\sqrt{x}-1}$.

Solution. In Example 2 of Section 1.1 we used numerical evidence to conjecture that this limit is 2. Here we will confirm this algebraically. Since this limit is an indeterminate form of type $0/0$, we will need to devise some strategy for making the limit (if it exists) evident. One such strategy is to rationalize the denominator of the function. This yields

$$\frac{x-1}{\sqrt{x}-1} = \frac{(x-1)(\sqrt{x}+1)}{(\sqrt{x}-1)(\sqrt{x}+1)} = \frac{(x-1)(\sqrt{x}+1)}{x-1} = \sqrt{x}+1 \quad (x \neq 1)$$

Confirm the limit in Example 10 by factoring the numerator.

Therefore,

$$\lim_{x\to 1} \frac{x-1}{\sqrt{x}-1} = \lim_{x\to 1} (\sqrt{x}+1) = 2 \blacktriangleleft$$

LIMITS OF PIECEWISE-DEFINED FUNCTIONS

For functions that are defined piecewise, a two-sided limit at a point where the formula changes is best obtained by first finding the one-sided limits at that point.

▶ Example 11 Let

$$f(x) = \begin{cases} 1/(x+2), & x < -2 \\ x^2 - 5, & -2 < x \leq 3 \\ \sqrt{x+13}, & x > 3 \end{cases}$$

Find

$$\text{(a)} \lim_{x\to -2} f(x) \qquad \text{(b)} \lim_{x\to 0} f(x) \qquad \text{(c)} \lim_{x\to 3} f(x)$$

Solution (a). We will determine the stated two-sided limit by first considering the corresponding one-sided limits. For each one-sided limit, we must use that part of the formula that is applicable on the interval over which x varies. For example, as x approaches -2 from the left, the applicable part of the formula is

$$f(x) = \frac{1}{x+2}$$

and as x approaches -2 from the right, the applicable part of the formula near -2 is

$$f(x) = x^2 - 5$$

Thus,

$$\lim_{x\to -2^-} f(x) = \lim_{x\to -2^-} \frac{1}{x+2} = -\infty$$

$$\lim_{x\to -2^+} f(x) = \lim_{x\to -2^+} (x^2 - 5) = (-2)^2 - 5 = -1$$

from which it follows that $\lim_{x\to -2} f(x)$ does not exist.

Solution (b). The applicable part of the formula is $f(x) = x^2 - 5$ on both sides of 0, so there is no need to consider one-sided limits here. We see directly that

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} (x^2 - 5) = 0^2 - 5 = -5$$

Solution (c). Using the applicable parts of the formula for $f(x)$, we obtain

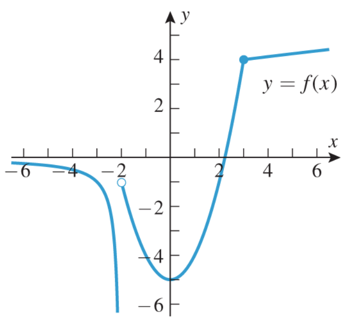
$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} (x^2 - 5) = 3^2 - 5 = 4$$

$$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} \sqrt{x+13} = \sqrt{\lim_{x\to 3^+} (x+13)} = \sqrt{3+13} = 4$$

Since the one-sided limits are equal, we have

$$\lim_{x\to 3} f(x) = 4$$

We note that the limit calculations in parts (a), (b), and (c) are consistent with the graph of f shown in Figure 1.2.5. ◀



▲ Figure 1.2.5

✓ QUICK CHECK EXERCISES 1.2 (See page 21 for answers.)

1. In each part, find the limit by inspection.
(a) $\lim_{x \to 8} 7 =$ ______ (b) $\lim_{y \to 3^+} 12y =$ ______
(c) $\lim_{x \to 0^-} \dfrac{x}{|x|} =$ ______ (d) $\lim_{w \to 5} \dfrac{w}{|w|} =$ ______
(e) $\lim_{z \to 1^-} \dfrac{1}{1-z} =$ ______

2. Given that $\lim_{x \to a} f(x) = 1$ and $\lim_{x \to a} g(x) = 2$, find the limits.
(a) $\lim_{x \to a} [3f(x) + 2g(x)] =$ ______
(b) $\lim_{x \to a} \dfrac{2f(x)+1}{1 - f(x)g(x)} =$ ______
(c) $\lim_{x \to a} \dfrac{\sqrt{f(x)+3}}{g(x)} =$ ______

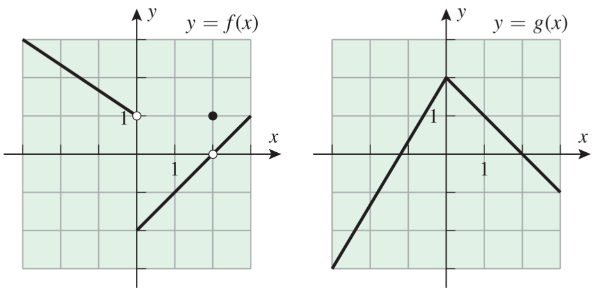
3. Find the limits.
(a) $\lim_{x \to -1} (x^3 + x^2 + x)^{101} =$ ______
(b) $\lim_{x \to 2^-} \dfrac{(x-1)(x-2)}{x+1} =$ ______
(c) $\lim_{x \to -1^+} \dfrac{(x-1)(x-2)}{x+1} =$ ______
(d) $\lim_{x \to 4} \dfrac{x^2-16}{x-4} =$ ______

4. Let
$$f(x) = \begin{cases} x+1, & x \le 1 \\ x-1, & x > 1 \end{cases}$$
Find the limits that exist.
(a) $\lim_{x \to 1^-} f(x) =$ ______
(b) $\lim_{x \to 1^+} f(x) =$ ______
(c) $\lim_{x \to 1} f(x) =$ ______

EXERCISE SET 1.2

1. Given that
$$\lim_{x \to a} f(x) = 2, \quad \lim_{x \to a} g(x) = -4, \quad \lim_{x \to a} h(x) = 0$$
find the limits.
(a) $\lim_{x \to a} [f(x) + 2g(x)]$
(b) $\lim_{x \to a} [h(x) - 3g(x) + 1]$
(c) $\lim_{x \to a} [f(x)g(x)]$ (d) $\lim_{x \to a} [g(x)]^2$
(e) $\lim_{x \to a} \sqrt[3]{6 + f(x)}$ (f) $\lim_{x \to a} \dfrac{2}{g(x)}$

2. Use the graphs of f and g in the accompanying figure to find the limits that exist. If the limit does not exist, explain why.
(a) $\lim_{x \to 2} [f(x) + g(x)]$ (b) $\lim_{x \to 0} [f(x) + g(x)]$
(c) $\lim_{x \to 0^+} [f(x) + g(x)]$ (d) $\lim_{x \to 0^-} [f(x) + g(x)]$
(e) $\lim_{x \to 2} \dfrac{f(x)}{1 + g(x)}$ (f) $\lim_{x \to 2} \dfrac{1 + g(x)}{f(x)}$
(g) $\lim_{x \to 0^+} \sqrt{f(x)}$ (h) $\lim_{x \to 0^-} \sqrt{f(x)}$



▲ **Figure Ex-2**

3–30 Find the limits.

3. $\lim_{x \to 2} x(x-1)(x+1)$

4. $\lim_{x \to 3} x^3 - 3x^2 + 9x$

5. $\lim_{x \to 3} \dfrac{x^2 - 2x}{x+1}$

6. $\lim_{x \to 0} \dfrac{6x - 9}{x^3 - 12x + 3}$

7. $\lim_{x \to 1^+} \dfrac{x^4 - 1}{x - 1}$

8. $\lim_{t \to -2} \dfrac{t^3 + 8}{t + 2}$

9. $\lim_{x \to -1} \dfrac{x^2 + 6x + 5}{x^2 - 3x - 4}$

10. $\lim_{x \to 2} \dfrac{x^2 - 4x + 4}{x^2 + x - 6}$

11. $\lim_{x \to -1} \dfrac{2x^2 + x - 1}{x + 1}$

12. $\lim_{x \to 1} \dfrac{3x^2 - x - 2}{2x^2 + x - 3}$

13. $\lim_{t \to 2} \dfrac{t^3 + 3t^2 - 12t + 4}{t^3 - 4t}$

14. $\lim_{t \to 1} \dfrac{t^3 + t^2 - 5t + 3}{t^3 - 3t + 2}$

15. $\lim_{x \to 3^+} \dfrac{x}{x - 3}$

16. $\lim_{x \to 3^-} \dfrac{x}{x - 3}$

17. $\lim_{x \to 3} \dfrac{x}{x - 3}$

18. $\lim_{x \to 2^+} \dfrac{x}{x^2 - 4}$

19. $\lim_{x \to 2^-} \dfrac{x}{x^2 - 4}$

20. $\lim_{x \to 2} \dfrac{x}{x^2 - 4}$

21. $\lim_{y \to 6^+} \dfrac{y + 6}{y^2 - 36}$

22. $\lim_{y \to 6^-} \dfrac{y + 6}{y^2 - 36}$

23. $\lim_{y \to 6} \dfrac{y + 6}{y^2 - 36}$

24. $\lim_{x \to 4^+} \dfrac{3 - x}{x^2 - 2x - 8}$

25. $\lim_{x \to 4^-} \dfrac{3 - x}{x^2 - 2x - 8}$

26. $\lim_{x \to 4} \dfrac{3 - x}{x^2 - 2x - 8}$

27. $\lim_{x \to 2^+} \dfrac{1}{|2 - x|}$

28. $\lim_{x \to 3^-} \dfrac{1}{|x - 3|}$

29. $\lim_{x \to 9} \dfrac{x - 9}{\sqrt{x} - 3}$

30. $\lim_{y \to 4} \dfrac{4 - y}{2 - \sqrt{y}}$

31. Let
$$f(x) = \begin{cases} x - 1, & x \le 3 \\ 3x - 7, & x > 3 \end{cases}$$
Find
(a) $\lim_{x \to 3^-} f(x)$ (b) $\lim_{x \to 3^+} f(x)$ (c) $\lim_{x \to 3} f(x)$.

32. Let

$$g(t) = \begin{cases} t-2, & t<0 \\ t^2, & 0 \le t \le 2 \\ 2t, & t>2 \end{cases}$$

Find
(a) $\lim_{t\to 0} g(t)$ (b) $\lim_{t\to 1} g(t)$ (c) $\lim_{t\to 2} g(t)$.

33–36 True–False Determine whether the statement is true or false. Explain your answer. ■

33. If $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist, then so does $\lim_{x\to a}[f(x)+g(x)]$.

34. If $\lim_{x\to a} g(x) = 0$ and $\lim_{x\to a} f(x)$ exists, then $\lim_{x\to a}[f(x)/g(x)]$ does not exist.

35. If $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ both exist and are equal, then $\lim_{x\to a}[f(x)/g(x)] = 1$.

36. If $f(x)$ is a rational function and $x=a$ is in the domain of f, then $\lim_{x\to a} f(x) = f(a)$.

37–38 First rationalize the numerator and then find the limit. ■

37. $\lim_{x\to 0} \dfrac{\sqrt{x+4}-2}{x}$ **38.** $\lim_{x\to 0} \dfrac{\sqrt{x^2+4}-2}{x}$

39. Let

$$f(x) = \frac{x^3-1}{x-1}$$

(a) Find $\lim_{x\to 1} f(x)$.
(b) Sketch the graph of $y=f(x)$.

40. Let

$$f(x) = \begin{cases} \dfrac{x^2-9}{x+3}, & x \neq -3 \\ k, & x=-3 \end{cases}$$

(a) Find k so that $f(-3) = \lim_{x\to -3} f(x)$.
(b) With k assigned the value $\lim_{x\to -3} f(x)$, show that $f(x)$ can be expressed as a polynomial.

FOCUS ON CONCEPTS

41. (a) Explain why the following calculation is incorrect.

$$\lim_{x\to 0^+}\left(\frac{1}{x}-\frac{1}{x^2}\right) = \lim_{x\to 0^+}\frac{1}{x} - \lim_{x\to 0^+}\frac{1}{x^2}$$
$$= +\infty - (+\infty) = 0$$

(b) Show that $\lim_{x\to 0^+}\left(\dfrac{1}{x}-\dfrac{1}{x^2}\right) = -\infty$.

42. (a) Explain why the following argument is incorrect.

$$\lim_{x\to 0}\left(\frac{1}{x}-\frac{2}{x^2+2x}\right) = \lim_{x\to 0}\frac{1}{x}\left(1-\frac{2}{x+2}\right)$$
$$= \infty \cdot 0 = 0$$

(b) Show that $\lim_{x\to 0}\left(\dfrac{1}{x}-\dfrac{2}{x^2+2x}\right) = \dfrac{1}{2}$.

43. Find all values of a such that

$$\lim_{x\to 1}\left(\frac{1}{x-1}-\frac{a}{x^2-1}\right)$$

exists and is finite.

44. (a) Explain informally why

$$\lim_{x\to 0^-}\left(\frac{1}{x}+\frac{1}{x^2}\right) = +\infty$$

(b) Verify the limit in part (a) algebraically.

45. Let $p(x)$ and $q(x)$ be polynomials, with $q(x_0)=0$. Discuss the behavior of the graph of $y=p(x)/q(x)$ in the vicinity of $x=x_0$. Give examples to support your conclusions.

46. Suppose that f and g are two functions such that $\lim_{x\to a} f(x)$ exists but $\lim_{x\to a}[f(x)+g(x)]$ does not exist. Use Theorem 1.2.2 to prove that $\lim_{x\to a} g(x)$ does not exist.

47. Suppose that f and g are two functions such that both $\lim_{x\to a} f(x)$ and $\lim_{x\to a}[f(x)+g(x)]$ exist. Use Theorem 1.2.2 to prove that $\lim_{x\to a} g(x)$ exists.

48. Suppose that f and g are two functions such that

$$\lim_{x\to a} g(x) = 0 \quad \text{and} \quad \lim_{x\to a}\frac{f(x)}{g(x)}$$

exists. Use Theorem 1.2.2 to prove that $\lim_{x\to a} f(x) = 0$.

49. Writing According to Newton's Law of Universal Gravitation, the gravitational force of attraction between two masses is inversely proportional to the square of the distance between them. What results of this section are useful in describing the gravitational force of attraction between the masses as they get closer and closer together?

50. Writing Suppose that f and g are two functions that are equal except at a finite number of points and that a denotes a real number. Explain informally why both

$$\lim_{x\to a} f(x) \quad \text{and} \quad \lim_{x\to a} g(x)$$

exist and are equal, or why both limits fail to exist. Write a short paragraph that explains the relationship of this result to the use of "algebraic simplification" in the evaluation of a limit.

✓ **QUICK CHECK ANSWERS 1.2** **1.** (a) 7 (b) 36 (c) −1 (d) 1 (e) $+\infty$ **2.** (a) 7 (b) −3 (c) 1
3. (a) −1 (b) 0 (c) $+\infty$ (d) 8 **4.** (a) 2 (b) 0 (c) does not exist

1.3 LIMITS AT INFINITY; END BEHAVIOR OF A FUNCTION

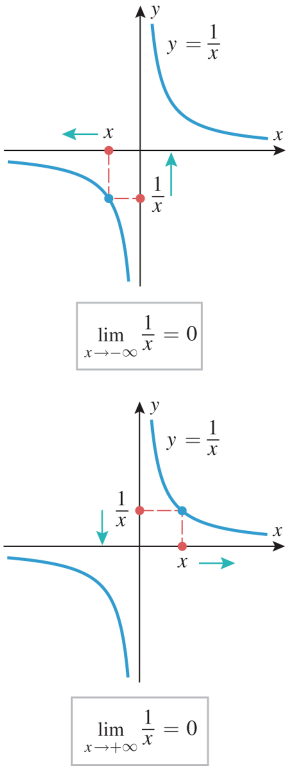
Up to now we have been concerned with limits that describe the behavior of a function $f(x)$ as x approaches some real number a. In this section we will be concerned with the behavior of $f(x)$ as x increases or decreases without bound.

LIMITS AT INFINITY AND HORIZONTAL ASYMPTOTES

If the values of a variable x increase without bound, then we write $x \to +\infty$, and if the values of x decrease without bound, then we write $x \to -\infty$. The behavior of a function $f(x)$ as x increases without bound or decreases without bound is sometimes called the ***end behavior*** of the function. For example,

$$\lim_{x \to -\infty} \frac{1}{x} = 0 \quad \text{and} \quad \lim_{x \to +\infty} \frac{1}{x} = 0 \tag{1–2}$$

are illustrated numerically in Table 1.3.1 and geometrically in Figure 1.3.1.



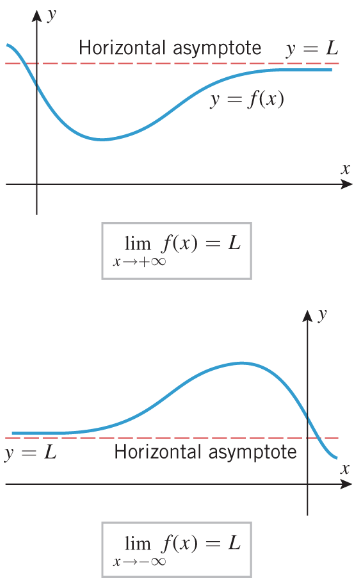
▲ **Figure 1.3.1**

Table 1.3.1

	VALUES						CONCLUSION
x $1/x$	-1 -1	-10 -0.1	-100 -0.01	-1000 -0.001	$-10{,}000$ -0.0001	$\cdots$ $\cdots$	As $x \to -\infty$ the value of $1/x$ increases toward zero.
x $1/x$	1 1	10 0.1	100 0.01	1000 0.001	$10{,}000$ 0.0001	$\cdots$ $\cdots$	As $x \to +\infty$ the value of $1/x$ decreases toward zero.

In general, we will use the following notation.

> **1.3.1 LIMITS AT INFINITY (AN INFORMAL VIEW)** If the values of $f(x)$ eventually get as close as we like to a number L as x increases without bound, then we write
>
> $$\lim_{x \to +\infty} f(x) = L \quad \text{or} \quad f(x) \to L \text{ as } x \to +\infty \tag{3}$$
>
> Similarly, if the values of $f(x)$ eventually get as close as we like to a number L as x decreases without bound, then we write
>
> $$\lim_{x \to -\infty} f(x) = L \quad \text{or} \quad f(x) \to L \text{ as } x \to -\infty \tag{4}$$



▲ **Figure 1.3.2**

Figure 1.3.2 illustrates the end behavior of a function f when

$$\lim_{x \to +\infty} f(x) = L \quad \text{or} \quad \lim_{x \to -\infty} f(x) = L$$

In the first case the graph of f eventually comes as close as we like to the line $y = L$ as x increases without bound, and in the second case it eventually comes as close as we like to the line $y = L$ as x decreases without bound. If either limit holds, we call the line $y = L$ a ***horizontal asymptote*** for the graph of f.

▶ **Example 1** It follows from (1) and (2) that $y = 0$ is a horizontal asymptote for the graph of $f(x) = 1/x$ in both the positive and negative directions. This is consistent with the graph of $y = 1/x$ shown in Figure 1.3.1. ◀

remains constant, what happens to the current in a superconductor as $R \to 0^+$?

76. Writing Compare informal Definition 1.1.1 with Definition 1.4.1.
(a) What portions of Definition 1.4.1 correspond to the expression "values of $f(x)$ can be made as close as we like to L" in Definition 1.1.1? Explain.
(b) What portions of Definition 1.4.1 correspond to the expression "taking values of x sufficiently close to a (but not equal to a)" in Definition 1.1.1? Explain.

77. Writing Compare informal Definition 1.3.1 with Definition 1.4.2.
(a) What portions of Definition 1.4.2 correspond to the expression "values of $f(x)$ eventually get as close as we like to a number L" in Definition 1.3.1? Explain.
(b) What portions of Definition 1.4.2 correspond to the expression "as x increases without bound" in Definition 1.3.1? Explain.

✓QUICK CHECK ANSWERS 1.4 **1.** $\epsilon > 0$; $\delta > 0$; $0 < |x - a| < \delta$ **2.** $\lim_{x \to 1} f(x) = 5$ **3.** $\delta = \epsilon/5$ **4.** $\epsilon > 0$; N; $x > N$ **5.** $N = 10{,}000$

1.5 CONTINUITY

A thrown baseball cannot vanish at some point and reappear someplace else to continue its motion. Thus, we perceive the path of the ball as an unbroken curve. In this section, we translate "unbroken curve" into a precise mathematical formulation called continuity, and develop some fundamental properties of continuous curves.

Joseph Helfenberger/iStockphoto

A baseball moves along a "continuous" trajectory after leaving the pitcher's hand.

DEFINITION OF CONTINUITY

Intuitively, the graph of a function can be described as a "continuous curve" if it has no breaks or holes. To make this idea more precise we need to understand what properties of a function can cause breaks or holes. Referring to Figure 1.5.1, we see that the graph of a function has a break or hole if any of the following conditions occur:

- The function f is undefined at c (Figure 1.5.1*a*).
- The limit of $f(x)$ does not exist as x approaches c (Figures 1.5.1*b*, 1.5.1*c*).
- The value of the function and the value of the limit at c are different (Figure 1.5.1*d*).

This suggests the following definition.

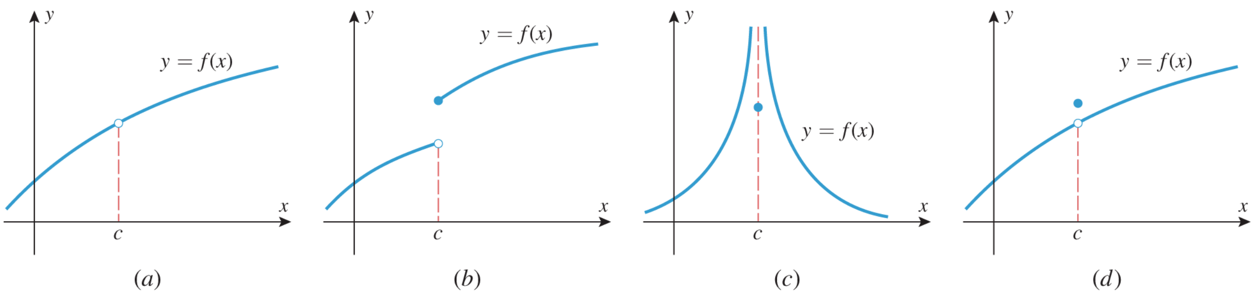
> The third condition in Definition 1.5.1 actually implies the first two, since it is tacitly understood in the statement
> $$\lim_{x \to c} f(x) = f(c)$$
> that the limit exists and the function is defined at c. Thus, when we want to establish continuity at c our usual procedure will be to verify the third condition only.

1.5.1 DEFINITION A function f is said to be ***continuous at $x = c$*** provided the following conditions are satisfied:

1. $f(c)$ is defined.

2. $\lim_{x \to c} f(x)$ exists.

3. $\lim_{x \to c} f(x) = f(c)$.



▲ **Figure 1.5.1**

If one or more of the conditions of this definition fails to hold, then we will say that f has a ***discontinuity at*** $\boldsymbol{x = c}$. Each function drawn in Figure 1.5.1 illustrates a discontinuity at $x = c$. In Figure 1.5.1a, the function is not defined at c, violating the first condition of Definition 1.5.1. In Figure 1.5.1b, the one-sided limits of $f(x)$ as x approaches c both exist but are not equal. Thus, $\lim_{x\to c} f(x)$ does not exist, and this violates the second condition of Definition 1.5.1. We will say that a function like that in Figure 1.5.1b has a ***jump discontinuity*** at c. In Figure 1.5.1c, the one-sided limits of $f(x)$ as x approaches c are infinite. Thus, $\lim_{x\to c} f(x)$ does not exist, and this violates the second condition of Definition 1.5.1. We will say that a function like that in Figure 1.5.1c has an ***infinite discontinuity*** at c. In Figure 1.5.1d, the function is defined at c and $\lim_{x\to c} f(x)$ exists, but these two values are not equal, violating the third condition of Definition 1.5.1. We will say that a function like that in Figure 1.5.1d has a ***removable discontinuity*** at c. Exercises 33 and 34 help to explain why discontinuities of this type are given this name.

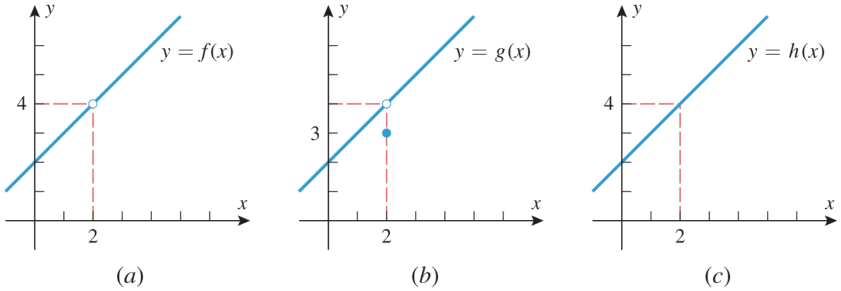
▶ Example 1 Determine whether the following functions are continuous at $x = 2$.

$$f(x) = \frac{x^2-4}{x-2}, \qquad g(x) = \begin{cases} \dfrac{x^2-4}{x-2}, & x \neq 2 \\ 3, & x = 2, \end{cases} \qquad h(x) = \begin{cases} \dfrac{x^2-4}{x-2}, & x \neq 2 \\ 4, & x = 2 \end{cases}$$

Solution. In each case we must determine whether the limit of the function as $x \to 2$ is the same as the value of the function at $x = 2$. In all three cases the functions are identical, except at $x = 2$, and hence all three have the same limit at $x = 2$, namely,

$$\lim_{x\to 2} f(x) = \lim_{x\to 2} g(x) = \lim_{x\to 2} h(x) = \lim_{x\to 2} \frac{x^2-4}{x-2} = \lim_{x\to 2} (x+2) = 4$$

The function f is undefined at $x = 2$, and hence is not continuous at $x = 2$ (Figure 1.5.2a). The function g is defined at $x = 2$, but its value there is $g(2) = 3$, which is not the same as the limit as x approaches 2; hence, g is also not continuous at $x = 2$ (Figure 1.5.2b). The value of the function h at $x = 2$ is $h(2) = 4$, which is the same as the limit as x approaches 2; hence, h is continuous at $x = 2$ (Figure 1.5.2c). (Note that the function h could have been written more simply as $h(x) = x + 2$, but we wrote it in piecewise form to emphasize its relationship to f and g.) ◀



▲ Figure 1.5.2

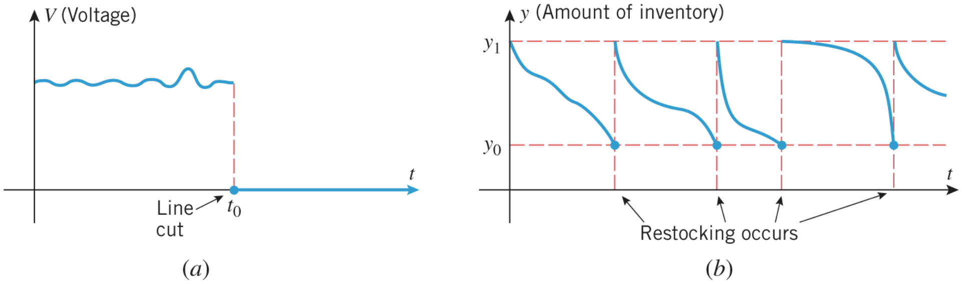
CONTINUITY IN APPLICATIONS

In applications, discontinuities often signal the occurrence of important physical events. For example, Figure 1.5.3a is a graph of voltage versus time for an underground cable that is accidentally cut by a work crew at time $t = t_0$ (the voltage drops to zero when the line is cut). Figure 1.5.3b shows the graph of inventory versus time for a company that restocks its warehouse to y_1 units when the inventory falls to y_0 units. The discontinuities occur at those times when restocking occurs.

CONTINUITY ON AN INTERVAL

If a function f is continuous at each number in an open interval (a, b), then we say that f is ***continuous on*** $\boldsymbol{(a, b)}$. This definition applies to infinite open intervals of the form $(a, +\infty)$, $(-\infty, b)$, and $(-\infty, +\infty)$. In the case where f is continuous on $(-\infty, +\infty)$, we will say that f is ***continuous everywhere***.

A poor connection in a transmission cable can cause a discontinuity in the electrical signal it carries.



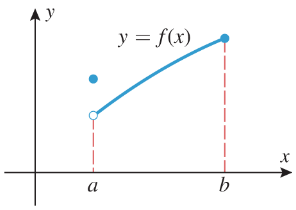
▲ **Figure 1.5.3**

Because Definition 1.5.1 involves a two-sided limit, that definition does not generally apply at the endpoints of a closed interval $[a, b]$ or at the endpoint of an interval of the form $[a, b)$, $(a, b]$, $(-\infty, b]$, or $[a, +\infty)$. To remedy this problem, we will agree that a function is continuous at an endpoint of an interval if its value at the endpoint is equal to the appropriate one-sided limit at that endpoint. For example, the function graphed in Figure 1.5.4 is continuous at the right endpoint of the interval $[a, b]$ because

$$\lim_{x \to b^-} f(x) = f(b)$$

but it is not continuous at the left endpoint because

$$\lim_{x \to a^+} f(x) \neq f(a)$$



▲ **Figure 1.5.4**

In general, we will say a function f is ***continuous from the left*** at c if

$$\lim_{x \to c^-} f(x) = f(c)$$

and is ***continuous from the right*** at c if

$$\lim_{x \to c^+} f(x) = f(c)$$

Using this terminology we define continuity on a closed interval as follows.

1.5.2 DEFINITION A function f is said to be ***continuous on a closed interval* [*a, b*]** if the following conditions are satisfied:

1. f is continuous on (a, b).
2. f is continuous from the right at a.
3. f is continuous from the left at b.

Modify Definition 1.5.2 appropriately so that it applies to intervals of the form $[a, +\infty)$, $(-\infty, b]$, $(a, b]$, and $[a, b)$.

▶ **Example 2** What can you say about the continuity of the function $f(x) = \sqrt{9 - x^2}$?

Solution. Because the natural domain of this function is the closed interval $[-3, 3]$, we will need to investigate the continuity of f on the open interval $(-3, 3)$ and at the two endpoints. If c is any point in the interval $(-3, 3)$, then it follows from Theorem 1.2.2(*e*) that

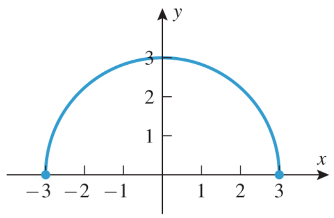
$$\lim_{x \to c} f(x) = \lim_{x \to c} \sqrt{9 - x^2} = \sqrt{\lim_{x \to c} (9 - x^2)} = \sqrt{9 - c^2} = f(c)$$

which proves f is continuous at each point in the interval $(-3, 3)$. The function f is also continuous at the endpoints since

$$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} \sqrt{9 - x^2} = \sqrt{\lim_{x \to 3^-} (9 - x^2)} = 0 = f(3)$$

$$\lim_{x \to -3^+} f(x) = \lim_{x \to -3^+} \sqrt{9 - x^2} = \sqrt{\lim_{x \to -3^+} (9 - x^2)} = 0 = f(-3)$$

Thus, f is continuous on the closed interval $[-3, 3]$ (Figure 1.5.5). ◀



$f(x) = \sqrt{9 - x^2}$

▲ **Figure 1.5.5**

SOME PROPERTIES OF CONTINUOUS FUNCTIONS

The following theorem, which is a consequence of Theorem 1.2.2, will enable us to reach conclusions about the continuity of functions that are obtained by adding, subtracting, multiplying, and dividing continuous functions.

1.5.3 THEOREM *If the functions f and g are continuous at c, then*

(*a*) *$f+g$ is continuous at c.*

(*b*) *$f-g$ is continuous at c.*

(*c*) *fg is continuous at c.*

(*d*) *f/g is continuous at c if $g(c) \neq 0$ and has a discontinuity at c if $g(c)=0$.*

We will prove part (*d*). The remaining proofs are similar and will be left to the exercises.

PROOF First, consider the case where $g(c)=0$. In this case $f(c)/g(c)$ is undefined, so the function f/g has a discontinuity at c.

Next, consider the case where $g(c) \neq 0$. To prove that f/g is continuous at c, we must show that

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{f(c)}{g(c)} \tag{1}$$

Since f and g are continuous at c,

$$\lim_{x \to c} f(x) = f(c) \quad \text{and} \quad \lim_{x \to c} g(x) = g(c)$$

Thus, by Theorem 1.2.2(*d*)

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{\lim_{x \to c} f(x)}{\lim_{x \to c} g(x)} = \frac{f(c)}{g(c)}$$

which proves (1). ■

CONTINUITY OF POLYNOMIALS AND RATIONAL FUNCTIONS

The general procedure for showing that a function is continuous everywhere is to show that it is continuous at an *arbitrary* point. For example, we know from Theorem 1.2.3 that if $p(x)$ is a polynomial and a is *any* real number, then

$$\lim_{x \to a} p(x) = p(a)$$

This shows that polynomials are continuous everywhere. Moreover, since rational functions are ratios of polynomials, it follows from part (*d*) of Theorem 1.5.3 that rational functions are continuous at points other than the zeros of the denominator, and at these zeros they have discontinuities. Thus, we have the following result.

1.5.4 THEOREM

(*a*) *A polynomial is continuous everywhere.*

(*b*) *A rational function is continuous at every point where the denominator is nonzero, and has discontinuities at the points where the denominator is zero.*

TECHNOLOGY MASTERY

If you use a graphing utility to generate the graph of the equation in Example 3, there is a good chance you will see the discontinuity at $x = 2$ but not at $x = 3$. Try it, and explain what you think is happening.

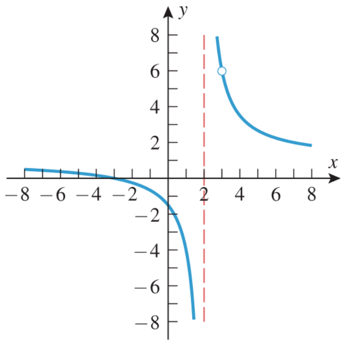
▶ Example 3 For what values of x is there a discontinuity in the graph of

$$y = \frac{x^2 - 9}{x^2 - 5x + 6}?$$

Solution. The function being graphed is a rational function, and hence is continuous at every number where the denominator is nonzero. Solving the equation

$$x^2 - 5x + 6 = 0$$

yields discontinuities at $x = 2$ and at $x = 3$ (Figure 1.5.6). ◀



$$y = \frac{x^2 - 9}{x^2 - 5x + 6}$$

▲ **Figure 1.5.6**

▶ Example 4 Show that $|x|$ is continuous everywhere.

Solution. We can write $|x|$ as

$$|x| = \begin{cases} x & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -x & \text{if } x < 0 \end{cases}$$

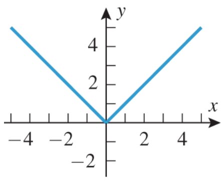
so $|x|$ is the same as the polynomial x on the interval $(0, +\infty)$ and is the same as the polynomial $-x$ on the interval $(-\infty, 0)$. But polynomials are continuous everywhere, so $x = 0$ is the only possible discontinuity for $|x|$. Since $|0| = 0$, to prove the continuity at $x = 0$ we must show that

$$\lim_{x \to 0} |x| = 0 \tag{2}$$

Because the piecewise formula for $|x|$ changes at 0, it will be helpful to consider the one-sided limits at 0 rather than the two-sided limit. We obtain

$$\lim_{x \to 0^+} |x| = \lim_{x \to 0^+} x = 0 \quad \text{and} \quad \lim_{x \to 0^-} |x| = \lim_{x \to 0^-} (-x) = 0$$

Thus, (2) holds and $|x|$ is continuous at $x = 0$ (Figure 1.5.7). ◀



$y = |x|$

▲ **Figure 1.5.7**

CONTINUITY OF COMPOSITIONS

The following theorem, whose proof is given in Web Appendix L, will be useful for calculating limits of compositions of functions.

In words, Theorem 1.5.5 states that a limit symbol can be moved through a function sign provided the limit of the expression inside the function sign exists and the function is continuous at this limit.

1.5.5 THEOREM *If $\lim_{x \to c} g(x) = L$ and if the function f is continuous at L, then $\lim_{x \to c} f(g(x)) = f(L)$. That is,*

$$\lim_{x \to c} f(g(x)) = f\left(\lim_{x \to c} g(x)\right)$$

This equality remains valid if $\lim_{x \to c}$ is replaced everywhere by one of $\lim_{x \to c^+}$, $\lim_{x \to c^-}$, $\lim_{x \to +\infty}$, or $\lim_{x \to -\infty}$.

In the special case of this theorem where $f(x) = |x|$, the fact that $|x|$ is continuous everywhere allows us to write

$$\lim_{x \to c} |g(x)| = \left|\lim_{x \to c} g(x)\right| \tag{3}$$

provided $\lim_{x \to c} g(x)$ exists. Thus, for example,

$$\lim_{x \to 3} |5 - x^2| = \left|\lim_{x \to 3} (5 - x^2)\right| = |-4| = 4$$

The following theorem is concerned with the continuity of compositions of functions; the first part deals with continuity at a specific number and the second with continuity everywhere.

> **1.5.6 THEOREM**
>
> (*a*) *If the function g is continuous at c, and the function f is continuous at $g(c)$, then the composition $f \circ g$ is continuous at c.*
>
> (*b*) *If the function g is continuous everywhere and the function f is continuous everywhere, then the composition $f \circ g$ is continuous everywhere.*

PROOF We will prove part (*a*) only; the proof of part (*b*) can be obtained by applying part (*a*) at an arbitrary number c. To prove that $f \circ g$ is continuous at c, we must show that the value of $f \circ g$ and the value of its limit are the same at $x = c$. But this is so, since we can write

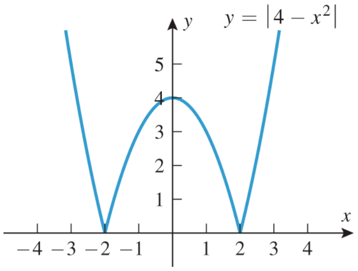
$$\lim_{x \to c} (f \circ g)(x) = \lim_{x \to c} f(g(x)) \underset{\text{Theorem 1.5.5}}{=} f\left(\lim_{x \to c} g(x)\right) \underset{g \text{ is continuous at } c.}{=} f(g(c)) = (f \circ g)(c) \quad \blacksquare$$

Can the absolute value of a function that is not continuous everywhere be continuous everywhere? Justify your answer.

We know from Example 4 that the function $|x|$ is continuous everywhere. Thus, if $g(x)$ is continuous at c, then by part (*a*) of Theorem 1.5.6, the function $|g(x)|$ must also be continuous at c; and, more generally, if $g(x)$ is continuous everywhere, then so is $|g(x)|$. Stated informally:

> *The absolute value of a continuous function is continuous.*

For example, the polynomial $g(x) = 4 - x^2$ is continuous everywhere, so we can conclude that the function $|4 - x^2|$ is also continuous everywhere (Figure 1.5.8).



▲ **Figure 1.5.8**

CONTINUITY OF INVERSE FUNCTIONS

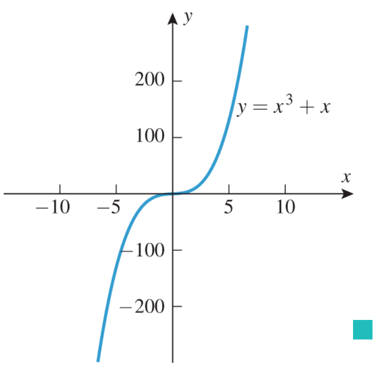
Since the graphs of a one-to-one function f and its inverse f^{-1} are reflections of one another about the line $y = x$, it is clear geometrically that if the graph of f has no breaks or holes in it, then neither does the graph of f^{-1}. This, and the fact that the range of f is the domain of f^{-1}, suggests the following result, which we state without formal proof.

To paraphrase Theorem 1.5.7, *the inverse of a continuous function is continuous.*

> **1.5.7 THEOREM** *If f is a one-to-one function that is continuous at each point of its domain, then f^{-1} is continuous at each point of its domain; that is, f^{-1} is continuous at each point in the range of f.*

▶ **Example 5** Show that the function $f(x) = x^3 + x$ has an inverse. Is f^{-1} continuous at each point in $(-\infty, +\infty)$?

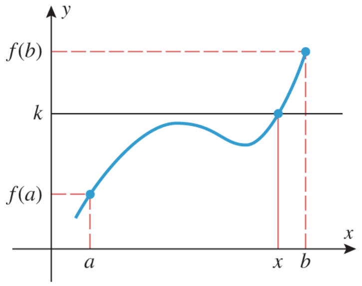
Solution. Note that f is increasing for all x (why?), so f is one-to-one and has an inverse. Also notice that f is continuous everywhere since it is a polynomial. From Figure 1.5.9 we infer that the range of f is $(-\infty, +\infty)$, so the domain of f^{-1} is also $(-\infty, +\infty)$. Although a formula for $f^{-1}(x)$ cannot be found easily, we can use Theorem 1.5.7 to conclude that f^{-1} is continuous on $(-\infty, +\infty)$. ◀



▲ **Figure 1.5.9**

THE INTERMEDIATE-VALUE THEOREM

Figure 1.5.10 shows the graph of a function that is continuous on the closed interval $[a, b]$. The figure suggests that if we draw any horizontal line $y = k$, where k is between $f(a)$ and $f(b)$, then that line will cross the curve $y = f(x)$ at least once over the interval $[a, b]$. Stated in numerical terms, if f is continuous on $[a, b]$, then the function f must take on



▲ Figure 1.5.10

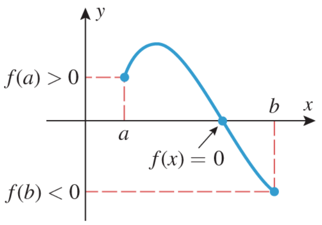
every value k between $f(a)$ and $f(b)$ at least once as x varies from a to b. For example, the polynomial $p(x) = x^5 - x + 3$ has a value of 3 at $x = 1$ and a value of 33 at $x = 2$. Thus, it follows from the continuity of p that the equation $x^5 - x + 3 = k$ has at least one solution in the interval $[1, 2]$ for every value of k between 3 and 33. This idea is stated more precisely in the following theorem.

> **1.5.8 THEOREM (*Intermediate-Value Theorem*)** *If f is continuous on a closed interval $[a, b]$ and k is any number between $f(a)$ and $f(b)$, inclusive, then there is at least one number x in the interval $[a, b]$ such that $f(x) = k$.*

Although this theorem is intuitively obvious, its proof depends on a mathematically precise development of the real number system, which is beyond the scope of this text.

APPROXIMATING ROOTS USING THE INTERMEDIATE-VALUE THEOREM

A variety of problems can be reduced to solving an equation $f(x) = 0$ for its roots. Sometimes it is possible to solve for the roots exactly using algebra, but often this is not possible and one must settle for decimal approximations of the roots. One procedure for approximating roots is based on the following consequence of the Intermediate-Value Theorem.



▲ Figure 1.5.11

> **1.5.9 THEOREM** *If f is continuous on $[a, b]$, and if $f(a)$ and $f(b)$ are nonzero and have opposite signs, then there is at least one solution of the equation $f(x) = 0$ in the interval (a, b).*

This result, which is illustrated in Figure 1.5.11, can be proved as follows.

PROOF Since $f(a)$ and $f(b)$ have opposite signs, 0 is between $f(a)$ and $f(b)$. Thus, by the Intermediate-Value Theorem there is at least one number x in the interval $[a, b]$ such that $f(x) = 0$. However, $f(a)$ and $f(b)$ are nonzero, so x must lie in the interval (a, b), which completes the proof. ■

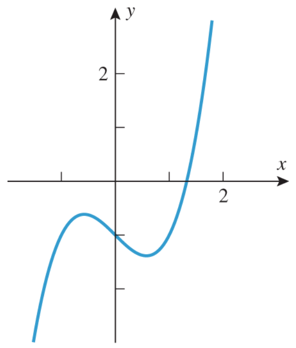
Before we illustrate how this theorem can be used to approximate roots, it will be helpful to discuss some standard terminology for describing errors in approximations. If x is an approximation to a quantity x_0, then we call

$$\epsilon = |x - x_0|$$

the ***absolute error*** or (less precisely) the ***error*** in the approximation. The terminology in Table 1.5.1 is used to describe the size of such errors.

Table 1.5.1

ERROR	DESCRIPTION
$\|x - x_0\| \le 0.1$	x approximates x_0 with an error of at most 0.1.
$\|x - x_0\| \le 0.01$	x approximates x_0 with an error of at most 0.01.
$\|x - x_0\| \le 0.001$	x approximates x_0 with an error of at most 0.001.
$\|x - x_0\| \le 0.0001$	x approximates x_0 with an error of at most 0.0001.
$\|x - x_0\| \le 0.5$	x approximates x_0 to the nearest integer.
$\|x - x_0\| \le 0.05$	x approximates x_0 to 1 decimal place (i.e., to the nearest tenth).
$\|x - x_0\| \le 0.005$	x approximates x_0 to 2 decimal places (i.e., to the nearest hundredth).
$\|x - x_0\| \le 0.0005$	x approximates x_0 to 3 decimal places (i.e., to the nearest thousandth).



$y = x^3 - x - 1$

▲ Figure 1.5.12

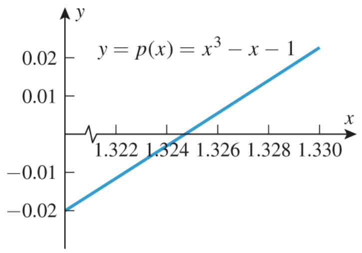
▶ **Example 6** The equation
$$x^3 - x - 1 = 0$$
cannot be solved algebraically very easily because the left side has no simple factors. However, if we graph $p(x) = x^3 - x - 1$ with a graphing utility (Figure 1.5.12), then we are led to conjecture that there is one real root and that this root lies inside the interval $[1, 2]$. The existence of a root in this interval is also confirmed by Theorem 1.5.9, since $p(1) = -1$ and $p(2) = 5$ have opposite signs. Approximate this root to two decimal-place accuracy.

Solution. Our objective is to approximate the unknown root x_0 with an error of at most 0.005. It follows that if we can find an interval of length 0.01 that contains the root, then the midpoint of that interval will approximate the root with an error of at most $\frac{1}{2}(0.01) = 0.005$, which will achieve the desired accuracy.

We know that the root x_0 lies in the interval $[1, 2]$. However, this interval has length 1, which is too large. We can pinpoint the location of the root more precisely by dividing the interval $[1, 2]$ into 10 equal parts and evaluating p at the points of subdivision using a calculating utility (Table 1.5.2). In this table $p(1.3)$ and $p(1.4)$ have opposite signs, so we know that the root lies in the interval $[1.3, 1.4]$. This interval has length 0.1, which is still too large, so we repeat the process by dividing the interval $[1.3, 1.4]$ into 10 parts and evaluating p at the points of subdivision; this yields Table 1.5.3, which tells us that the root is inside the interval $[1.32, 1.33]$ (Figure 1.5.13). Since this interval has length 0.01, its midpoint 1.325 will approximate the root with an error of at most 0.005. Thus, $x_0 \approx 1.325$ to two decimal-place accuracy. ◀

Table 1.5.2

x	1	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2
$p(x)$	−1	−0.77	−0.47	−0.10	0.34	0.88	1.50	2.21	3.03	3.96	5



▲ Figure 1.5.13

Table 1.5.3

x	1.3	1.31	1.32	1.33	1.34	1.35	1.36	1.37	1.38	1.39	1.4
$p(x)$	−0.103	−0.062	−0.020	0.023	0.066	0.110	0.155	0.201	0.248	0.296	0.344

REMARK

To say that x approximates x_0 to n decimal places does *not* mean that the first n decimal places of x and x_0 will be the same when the numbers are rounded to n decimal places. For example, $x = 1.084$ approximates $x_0 = 1.087$ to two decimal places because $|x - x_0| = 0.003\ (< 0.005)$. However, if we round these values to two decimal places, then we obtain $x \approx 1.08$ and $x_0 \approx 1.09$. Thus, if you approximate a number to n decimal places, then you should display that approximation to at least $n + 1$ decimal places to preserve the accuracy.

TECHNOLOGY MASTERY

Use a graphing or calculating utility to show that the root x_0 in Example 6 can be approximated as $x_0 \approx 1.3245$ to three decimal-place accuracy.

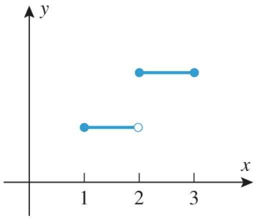
✓ QUICK CHECK EXERCISES 1.5

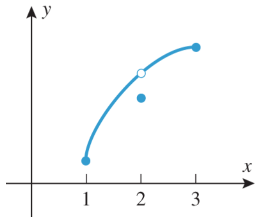
(See page 50 for answers.)

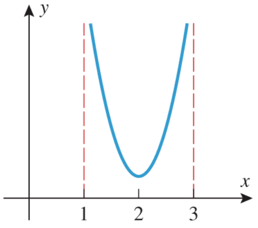
1. What three conditions are satisfied if f is continuous at $x = c$?
2. Suppose that f and g are continuous functions such that $f(2) = 1$ and $\lim_{x \to 2} [f(x) + 4g(x)] = 13$. Find
   (a) $g(2)$
   (b) $\lim_{x \to 2} g(x)$.
3. Suppose that f and g are continuous functions such that $\lim_{x \to 3} g(x) = 5$ and $f(3) = -2$. Find $\lim_{x \to 3} [f(x)/g(x)]$.
4. For what values of x, if any, is the function
$$f(x) = \frac{x^2 - 16}{x^2 - 5x + 4}$$
discontinuous?
5. Suppose that a function f is continuous everywhere and that $f(-2) = 3$, $f(-1) = -1$, $f(0) = -4$, $f(1) = 1$, and $f(2) = 5$. Does the Intermediate-Value Theorem guarantee that f has a root on the following intervals?
   (a) $[-2, -1]$ (b) $[-1, 0]$ (c) $[-1, 1]$ (d) $[0, 2]$

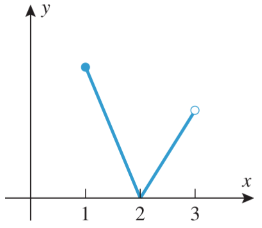
EXERCISE SET 1.5 Graphing Utility

1–4 Let f be the function whose graph is shown. On which of the following intervals, if any, is f continuous?
(a) $[1, 3]$ (b) $(1, 3)$ (c) $[1, 2]$
(d) $(1, 2)$ (e) $[2, 3]$ (f) $(2, 3)$
For each interval on which f is not continuous, indicate which conditions for the continuity of f do not hold. ■

1.


2.


3.


4.


5. Consider the functions

$$f(x) = \begin{cases} 1, & x \neq 4 \\ -1, & x = 4 \end{cases} \quad \text{and} \quad g(x) = \begin{cases} 4x - 10, & x \neq 4 \\ -6, & x = 4 \end{cases}$$

In each part, is the given function continuous at $x = 4$?
(a) $f(x)$ (b) $g(x)$ (c) $-g(x)$ (d) $|f(x)|$
(e) $f(x)g(x)$ (f) $g(f(x))$ (g) $g(x) - 6f(x)$

6. Consider the functions

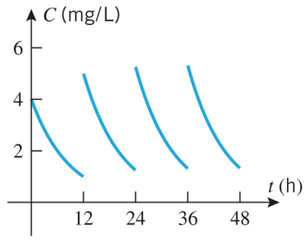
$$f(x) = \begin{cases} 1, & 0 \leq x \\ 0, & x < 0 \end{cases} \quad \text{and} \quad g(x) = \begin{cases} 0, & 0 \leq x \\ 1, & x < 0 \end{cases}$$

In each part, is the given function continuous at $x = 0$?
(a) $f(x)$ (b) $g(x)$ (c) $f(-x)$ (d) $|g(x)|$
(e) $f(x)g(x)$ (f) $g(f(x))$ (g) $f(x) + g(x)$

FOCUS ON CONCEPTS

7. In each part sketch the graph of a function f that satisfies the stated conditions.
(a) f is continuous everywhere except at $x = 3$, at which point it is continuous from the right.
(b) f has a two-sided limit at $x = 3$, but it is not continuous at $x = 3$.
(c) f is not continuous at $x = 3$, but if its value at $x = 3$ is changed from $f(3) = 1$ to $f(3) = 0$, it becomes continuous at $x = 3$.
(d) f is continuous on the interval $[0, 3)$ and is defined on the closed interval $[0, 3]$; but f is not continuous on the interval $[0, 3]$.

8. The accompanying figure models the concentration C of medication in the bloodstream of a patient over a 48-hour period of time. Discuss the significance of the discontinuities in the graph.



◀ **Figure Ex-8**

9. A student parking lot at a university charges \$2.00 for the first half hour (or any part) and \$1.00 for each subsequent half hour (or any part) up to a daily maximum of \$10.00.
(a) Sketch a graph of cost as a function of the time parked.
(b) Discuss the significance of the discontinuities in the graph to a student who parks there.

10. In each part determine whether the function is continuous or not, and explain your reasoning.
(a) The Earth's population as a function of time.
(b) Your exact height as a function of time.
(c) The cost of a taxi ride in your city as a function of the distance traveled.
(d) The volume of a melting ice cube as a function of time.

11–22 Find values of x, if any, at which f is not continuous. ■

11. $f(x) = 5x^4 - 3x + 7$

12. $f(x) = \sqrt[3]{x - 8}$

13. $f(x) = \dfrac{x + 2}{x^2 + 4}$

14. $f(x) = \dfrac{x + 2}{x^2 - 4}$

15. $f(x) = \dfrac{x}{2x^2 + x}$

16. $f(x) = \dfrac{2x + 1}{4x^2 + 4x + 5}$

17. $f(x) = \dfrac{3}{x} + \dfrac{x - 1}{x^2 - 1}$

18. $f(x) = \dfrac{5}{x} + \dfrac{2x}{x + 4}$

19. $f(x) = \dfrac{x^2 + 6x + 9}{|x| + 3}$

20. $f(x) = \left|4 - \dfrac{8}{x^4 + x}\right|$

21. $f(x) = \begin{cases} 2x + 3, & x \leq 4 \\ 7 + \dfrac{16}{x}, & x > 4 \end{cases}$

22. $f(x) = \begin{cases} \dfrac{3}{x - 1}, & x \neq 1 \\ 3, & x = 1 \end{cases}$

23–28 True–False Determine whether the statement is true or false. Explain your answer. ■

23. If $f(x)$ is continuous at $x = c$, then so is $|f(x)|$.

24. If $|f(x)|$ is continuous at $x = c$, then so is $f(x)$.

25. If f and g are discontinuous at $x = c$, then so is $f + g$.

26. If f and g are discontinuous at $x = c$, then so is fg.

27. If $\sqrt{f(x)}$ is continuous at $x = c$, then so is $f(x)$.

28. If $f(x)$ is continuous at $x = c$, then so is $\sqrt{f(x)}$.

29–30 Find a value of the constant k, if possible, that will make the function continuous everywhere.

29. (a) $f(x) = \begin{cases} 7x - 2, & x \le 1 \\ kx^2, & x > 1 \end{cases}$

(b) $f(x) = \begin{cases} kx^2, & x \le 2 \\ 2x + k, & x > 2 \end{cases}$

30. (a) $f(x) = \begin{cases} 9 - x^2, & x \ge -3 \\ k/x^2, & x < -3 \end{cases}$

(b) $f(x) = \begin{cases} 9 - x^2, & x \ge 0 \\ k/x^2, & x < 0 \end{cases}$

31. Find values of the constants k and m, if possible, that will make the function f continuous everywhere.

$$f(x) = \begin{cases} x^2 + 5, & x > 2 \\ m(x+1) + k, & -1 < x \le 2 \\ 2x^3 + x + 7, & x \le -1 \end{cases}$$

32. On which of the following intervals is

$$f(x) = \frac{1}{\sqrt{x - 2}}$$

continuous?

(a) $[2, +\infty)$ (b) $(-\infty, +\infty)$ (c) $(2, +\infty)$ (d) $[1, 2)$

33–36 A function f is said to have a ***removable discontinuity*** at $x = c$ if $\lim_{x \to c} f(x)$ exists but f is not continuous at $x = c$, either because f is not defined at c or because the definition for $f(c)$ differs from the value of the limit. This terminology will be needed in these exercises.

33. (a) Sketch the graph of a function with a removable discontinuity at $x = c$ for which $f(c)$ is undefined.
(b) Sketch the graph of a function with a removable discontinuity at $x = c$ for which $f(c)$ is defined.

34. (a) The terminology *removable discontinuity* is appropriate because a removable discontinuity of a function f at $x = c$ can be "removed" by redefining the value of f appropriately at $x = c$. What value for $f(c)$ removes the discontinuity?
(b) Show that the following functions have removable discontinuities at $x = 1$, and sketch their graphs.

$$f(x) = \frac{x^2 - 1}{x - 1} \quad \text{and} \quad g(x) = \begin{cases} 1, & x > 1 \\ 0, & x = 1 \\ 1, & x < 1 \end{cases}$$

(c) What values should be assigned to $f(1)$ and $g(1)$ to remove the discontinuities?

35–36 Find the values of x (if any) at which f is not continuous, and determine whether each such value is a removable discontinuity.

35. (a) $f(x) = \dfrac{|x|}{x}$ (b) $f(x) = \dfrac{x^2 + 3x}{x + 3}$

(c) $f(x) = \dfrac{x - 2}{|x| - 2}$

36. (a) $f(x) = \dfrac{x^2 - 4}{x^3 - 8}$ (b) $f(x) = \begin{cases} 2x - 3, & x \le 2 \\ x^2, & x > 2 \end{cases}$

(c) $f(x) = \begin{cases} 3x^2 + 5, & x \ne 1 \\ 6, & x = 1 \end{cases}$

37. (a) Use a graphing utility to generate the graph of the function $f(x) = (x + 3)/(2x^2 + 5x - 3)$, and then use the graph to make a conjecture about the number and locations of all discontinuities.
(b) Check your conjecture by factoring the denominator.

38. (a) Use a graphing utility to generate the graph of the function $f(x) = x/(x^3 - x + 2)$, and then use the graph to make a conjecture about the number and locations of all discontinuities.
(b) Use the Intermediate-Value Theorem to approximate the locations of all discontinuities to two decimal places.

39. Prove that $f(x) = x^{3/5}$ is continuous everywhere, carefully justifying each step.

40. Prove that $f(x) = 1/\sqrt{x^4 + 7x^2 + 1}$ is continuous everywhere, carefully justifying each step.

41. Prove:
(a) part (*a*) of Theorem 1.5.3
(b) part (*b*) of Theorem 1.5.3
(c) part (*c*) of Theorem 1.5.3.

42. Prove part (*b*) of Theorem 1.5.4.

43. (a) Use Theorem 1.5.5 to prove that if f is continuous at $x = c$, then $\lim_{h \to 0} f(c + h) = f(c)$.
(b) Prove that if $\lim_{h \to 0} f(c + h) = f(c)$, then f is continuous at $x = c$. [*Hint:* What does this limit tell you about the continuity of $g(h) = f(c + h)$?]
(c) Conclude from parts (a) and (b) that f is continuous at $x = c$ if and only if $\lim_{h \to 0} f(c + h) = f(c)$.

44. Prove: If f and g are continuous on $[a, b]$, and $f(a) > g(a)$, $f(b) < g(b)$, then there is at least one solution of the equation $f(x) = g(x)$ in (a, b). [*Hint:* Consider $f(x) - g(x)$.]

FOCUS ON CONCEPTS

45. Give an example of a function f that is defined on a closed interval, and whose values at the endpoints have opposite signs, but for which the equation $f(x) = 0$ has no solution in the interval.

46. Let f be the function whose graph is shown in Exercise 2. For each interval, determine (i) whether the hypothesis of the Intermediate-Value Theorem is satisfied, and (ii) whether the conclusion of the Intermediate-Value Theorem is satisfied.
(a) $[1, 2]$ (b) $[2, 3]$ (c) $[1, 3]$

47. Show that the equation $x^3 + x^2 - 2x = 1$ has at least one solution in the interval $[-1, 1]$.

48. Prove: If $p(x)$ is a polynomial of odd degree, then the equation $p(x) = 0$ has at least one real solution.

49. The accompanying figure on the next page shows the graph of the equation $y = x^4 + x - 1$. Use the method of