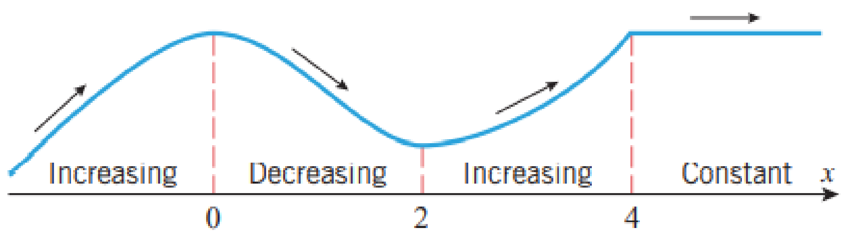
ANALYSIS OF FUNCTIONS-I

INCREASE, DECREASE, AND CONCAVITY

INCREASING AND DECREASING FUNCTIONS

The terms *increasing*, *decreasing*, and *constant* are used to describe the behavior of a function as we travel left to right along its graph. For example, the function graphed in Figure 4.1.1 can be described as increasing to the left of $x = 0$, decreasing from $x = 0$ to $x = 2$, increasing from $x = 2$ to $x = 4$, and constant to the right of $x = 4$.



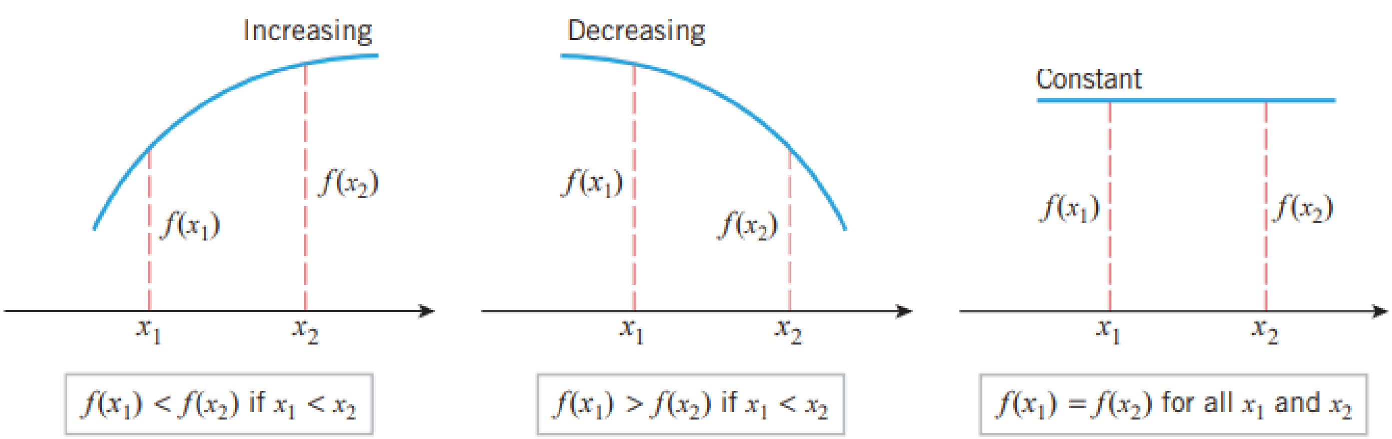
▶ **Figure 4.1.1**

4.1.1 DEFINITION Let f be defined on an interval, and let x_1 and x_2 denote points in that interval.

(a) f is ***increasing*** on the interval if $f(x_1) < f(x_2)$ whenever $x_1 < x_2$.

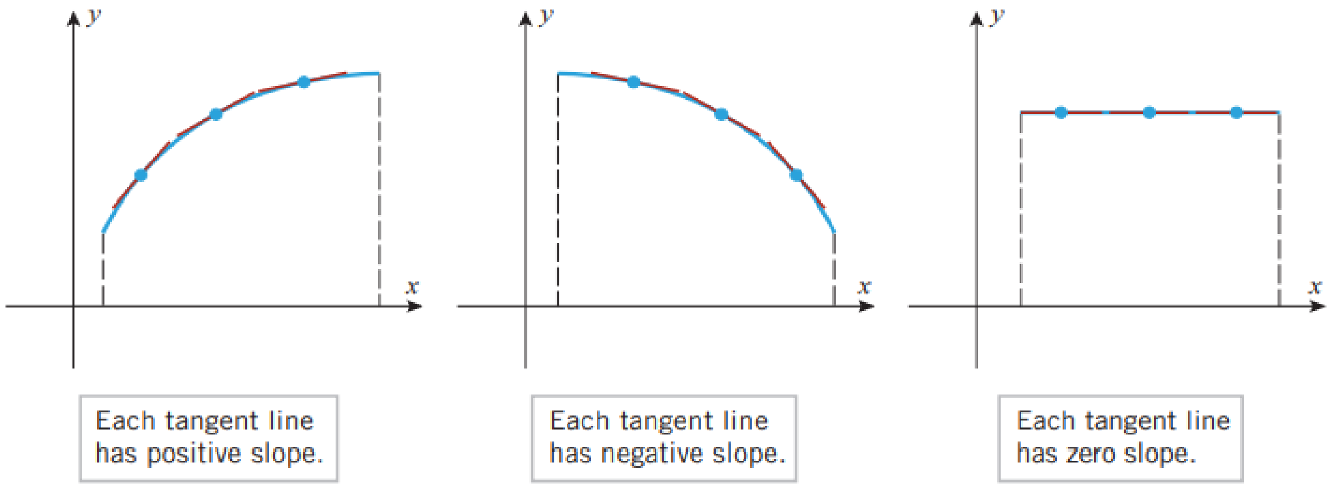
(b) f is ***decreasing*** on the interval if $f(x_1) > f(x_2)$ whenever $x_1 < x_2$.

(c) f is ***constant*** on the interval if $f(x_1) = f(x_2)$ for all points x_1 and x_2.



(*a*) (*b*) (*c*)

Figure 4.1.3 suggests that a differentiable function f is increasing on any interval where each tangent line to its graph has positive slope, is decreasing on any interval where each tangent line to its graph has negative slope, and is constant on any interval where each tangent line to its graph has zero slope. This intuitive observation suggests the following important theorem that will be proved in Section 4.8.

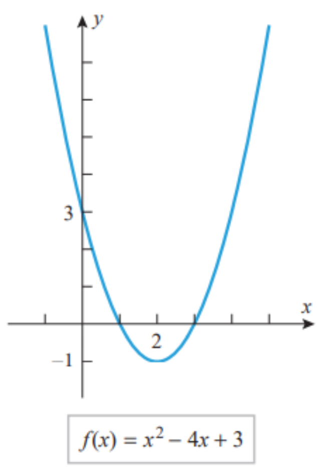


4.1.2 THEOREM *Let f be a function that is continuous on a closed interval $[a, b]$ and differentiable on the open interval (a, b).*

(a) If $f'(x) > 0$ for every value of x in (a, b), then f is increasing on $[a, b]$.

(b) If $f'(x) < 0$ for every value of x in (a, b), then f is decreasing on $[a, b]$.

(c) If $f'(x) = 0$ for every value of x in (a, b), then f is constant on $[a, b]$.



▸ **Figure 4.1.4**

▸ **Example 1** Find the intervals on which $f(x) = x^2 - 4x + 3$ is increasing and the intervals on which it is decreasing.

Solution. The graph of f in Figure 4.1.4 suggests that f is decreasing for $x \le 2$ and increasing for $x \ge 2$. To confirm this, we analyze the sign of f'. The derivative of f is

$$f'(x) = 2x - 4 = 2(x - 2)$$

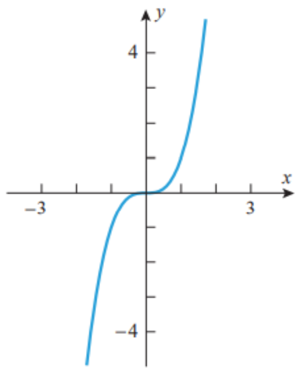
It follows that

$$f'(x) < 0 \quad \text{if} \quad x < 2$$
$$f'(x) > 0 \quad \text{if} \quad 2 < x$$

Since f is continuous everywhere, it follows from the comment after Theorem 4.1.2 that

$$f \text{ is decreasing on } (-\infty, 2]$$
$$f \text{ is increasing on } [2, +\infty)$$

These conclusions are consistent with the graph of f in Figure 4.1.4. ◂



$f(x) = x^3$

▶ **Figure 4.1.5**

▶ **Example 2** Find the intervals on which $f(x) = x^3$ is increasing and the intervals on which it is decreasing.

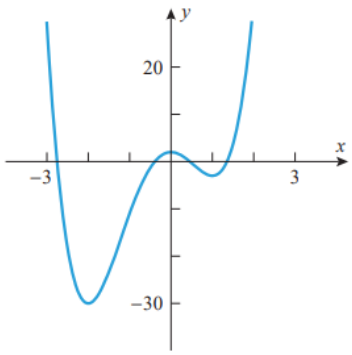
Solution. The graph of f in Figure 4.1.5 suggests that f is increasing over the entire x-axis. To confirm this, we differentiate f to obtain $f'(x) = 3x^2$. Thus,

$$f'(x) > 0 \quad \text{if} \quad x < 0$$
$$f'(x) > 0 \quad \text{if} \quad 0 < x$$

Since f is continuous everywhere,

$$f \text{ is increasing on } (-\infty, 0]$$
$$f \text{ is increasing on } [0, +\infty)$$

Since f is increasing on the adjacent intervals $(-\infty, 0]$ and $[0, +\infty)$, it follows that f is increasing on their union $(-\infty, +\infty)$ (see Exercise 59). ◀



$f(x) = 3x^4 + 4x^3 - 12x^2 + 2$

▶ **Figure 4.1.6**

▶ **Example 3**

(a) Use the graph of $f(x) = 3x^4 + 4x^3 - 12x^2 + 2$ in Figure 4.1.6 to make a conjecture about the intervals on which f is increasing or decreasing.

(b) Use Theorem 4.1.2 to determine whether your conjecture is correct.

Solution (a). The graph suggests that the function f is decreasing if $x \le -2$, increasing if $-2 \le x \le 0$, decreasing if $0 \le x \le 1$, and increasing if $x \ge 1$.

Solution (b). Differentiating f we obtain

$$f'(x) = 12x^3 + 12x^2 - 24x = 12x(x^2 + x - 2) = 12x(x+2)(x-1)$$

The sign analysis of f' in Table 4.1.1 can be obtained using the method of test points discussed in Web Appendix E. The conclusions in Table 4.1.1 confirm the conjecture in part (a). ◀

Table 4.1.1

INTERVAL	$(12x)(x+2)(x-1)$	$f'(x)$	CONCLUSION
$x < -2$	$(-)(-)(-)$	$-$	f is decreasing on $(-\infty, -2]$
$-2 < x < 0$	$(-)(+)(-)$	$+$	f is increasing on $[-2, 0]$
$0 < x < 1$	$(+)(+)(-)$	$-$	f is decreasing on $[0, 1]$
$1 < x$	$(+)(+)(+)$	$+$	f is increasing on $[1, +\infty)$

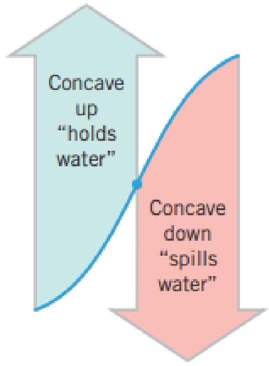
CONCAVITY

Although the sign of the derivative of f reveals where the graph of f is increasing or decreasing, it does not reveal the direction of *curvature*. For example, the graph is increasing on both sides of the point in Figure 4.1.7, but on the left side it has an upward curvature ("holds water") and on the right side it has a downward curvature ("spills water"). On intervals where the graph of f has upward curvature we say that f is *concave up*, and on intervals where the graph has downward curvature we say that f is *concave down*.

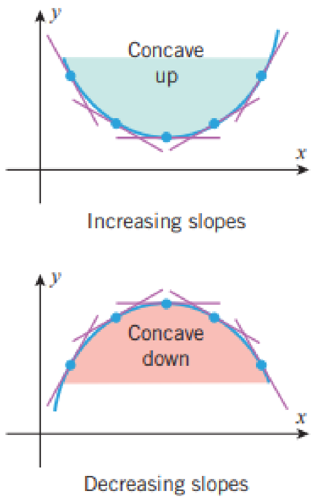
Figure 4.1.8 suggests two ways to characterize the concavity of a differentiable function f on an open interval:

- f is concave up on an open interval if its tangent lines have increasing slopes on that interval and is concave down if they have decreasing slopes.
- f is concave up on an open interval if its graph lies above its tangent lines on that interval and is concave down if it lies below its tangent lines.

Our formal definition for "concave up" and "concave down" corresponds to the first of these characterizations.



▶ **Figure 4.1.7**



▶ **Figure 4.1.8**

> **4.1.3 DEFINITION** If f is differentiable on an open interval, then f is said to be ***concave up*** on the open interval if f' is increasing on that interval, and f is said to be ***concave down*** on the open interval if f' is decreasing on that interval.

Since the slopes of the tangent lines to the graph of a differentiable function f are the values of its derivative f', it follows from Theorem 4.1.2 (applied to f' rather than f) that f' will be increasing on intervals where f'' is positive and that f' will be decreasing on intervals where f'' is negative. Thus, we have the following theorem.

> **4.1.4 THEOREM** *Let f be twice differentiable on an open interval.*
>
> (*a*) *If $f''(x) > 0$ for every value of x in the open interval, then f is concave up on that interval.*
>
> (*b*) *If $f''(x) < 0$ for every value of x in the open interval, then f is concave down on that interval.*

INFLECTION POINTS

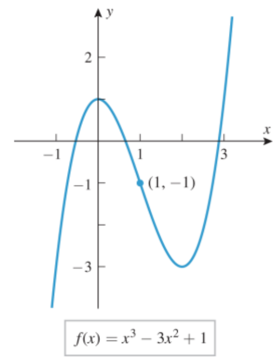
We see from Example 4 and Figure 4.1.5 that the graph of $f(x) = x^3$ changes from concave down to concave up at $x = 0$. Points where a curve changes from concave up to concave down or vice versa are of special interest, so there is some terminology associated with them.

> **4.1.5 DEFINITION** If f is continuous on an open interval containing a value x_0, and if f changes the direction of its concavity at the point $(x_0, f(x_0))$, then we say that f has an ***inflection point at*** $\boldsymbol{x_0}$, and we call the point $(x_0, f(x_0))$ on the graph of f an ***inflection point*** of f (Figure 4.1.9).

▶ **Example 5** Figure 3.1.10 shows the graph of the function $f(x) = x^3 - 3x^2 + 1$. Use the first and second derivatives of f to determine the intervals on which f is increasing, decreasing, concave up, and concave down. Locate all inflection points and confirm that your conclusions are consistent with the graph.

Solution. Calculating the first two derivatives of f we obtain

$$f'(x) = 3x^2 - 6x = 3x(x-2)$$
$$f''(x) = 6x - 6 = 6(x-1)$$



▲ Figure 3.1.10

The sign analysis of these derivatives is shown in the following tables:

INTERVAL	$(3x)(x-2)$	$f'(x)$	CONCLUSION
$x < 0$	$(-)(-)$	$+$	f is increasing on $(-\infty, 0]$
$0 < x < 2$	$(+)(-)$	$-$	f is decreasing on $[0, 2]$
$x > 2$	$(+)(+)$	$+$	f is increasing on $[2, +\infty)$

INTERVAL	$6(x-1)$	$f''(x)$	CONCLUSION
$x < 1$	$(-)$	$-$	f is concave down on $(-\infty, 1)$
$x > 1$	$(+)$	$+$	f is concave up on $(1, +\infty)$

The second table shows that there is an inflection point at $x = 1$, since f changes from concave down to concave up at that point. The inflection point is $(1, f(1)) = (1, -1)$. All of these conclusions are consistent with the graph of f. ◀

▶ **Example 6** Figure 3.1.11 shows the graph of the function $f(x) = x + 2\sin x$ over the interval $[0, 2\pi]$. Use the first and second derivatives of f to determine where f is increasing, decreasing, concave up, and concave down. Locate all inflection points and confirm that your conclusions are consistent with the graph.

Solution. Calculating the first two derivatives of f we obtain

$$f'(x) = 1 + 2\cos x$$
$$f''(x) = -2\sin x$$

Since f' is a continuous function, it changes sign on the interval $(0, 2\pi)$ only at points where $f'(x) = 0$ (why?). These values are solutions of the equation

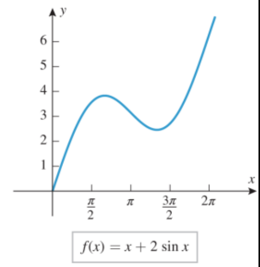
$$1 + 2\cos x = 0 \quad \text{or equivalently} \quad \cos x = -\tfrac{1}{2}$$

There are two solutions of this equation in the interval $(0, 2\pi)$, namely, $x = 2\pi/3$ and $x = 4\pi/3$ (verify). Similarly, f'' is a continuous function, so its sign changes in the interval $(0, 2\pi)$ will occur only at values of x for which $f''(x) = 0$. These values are solutions of the equation

$$-2\sin x = 0$$

There is one solution of this equation in the interval $(0, 2\pi)$, namely, $x = \pi$. With the help of these "sign transition points" we obtain the sign analysis shown in the following tables:

INTERVAL	$f'(x) = 1 + 2\cos x$	CONCLUSION
$0 < x < 2\pi/3$	$+$	f is increasing on $[0, 2\pi/3]$
$2\pi/3 < x < 4\pi/3$	$-$	f is decreasing on $[2\pi/3, 4\pi/3]$
$4\pi/3 < x < 2\pi$	$+$	f is increasing on $[4\pi/3, 2\pi]$



▲ Figure 3.1.11

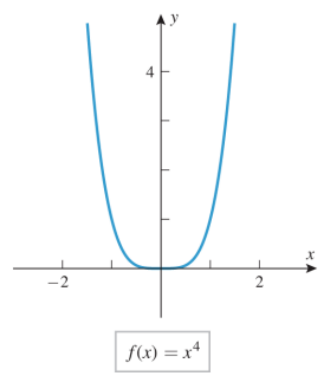
INTERVAL	$f''(x) = -2\sin x$	CONCLUSION
$0 < x < \pi$	$-$	f is concave down on $(0, \pi)$
$\pi < x < 2\pi$	$+$	f is concave up on $(\pi, 2\pi)$

The second table shows that there is an inflection point at $x = \pi$, since f changes from concave down to concave up at that point. All of these conclusions are consistent with the graph of f. ◀

▶ Example 7 Find the inflection points, if any, of $f(x) = x^4$.

Solution. Calculating the first two derivatives of f we obtain

$$f'(x) = 4x^3$$
$$f''(x) = 12x^2$$



Since $f''(x)$ is positive for $x < 0$ and for $x > 0$, the function f is concave up on the interval $(-\infty, 0)$ and on the interval $(0, +\infty)$. Thus, there is no change in concavity and hence no inflection point at $x = 0$, even though $f''(0) = 0$ (Figure 3.1.12). ◀

Question

15–32 Find: (a) the intervals on which f is increasing, (b) the intervals on which f is decreasing, (c) the open intervals on which f is concave up, (d) the open intervals on which f is concave down, and (e) the x-coordinates of all inflection points.

30. $f(x) = x^3 \ln x$

Solution

$f'(x) = x^2(1 + 3\ln x)$, $f''(x) = x(5 + 6\ln x)$.

(a) $[e^{-1/3}, +\infty)$ **(b)** $(0, e^{-1/3}]$ **(c)** $(e^{-5/6}, +\infty)$ **(d)** $(0, e^{-5/6})$ **(e)** $e^{-5/6}$

Question

15–32 Find: (a) the intervals on which f is increasing, (b) the intervals on which f is decreasing, (c) the open intervals on which f is concave up, (d) the open intervals on which f is concave down, and (e) the x-coordinates of all inflection points.

28. $f(x) = xe^{x^2}$

Solution

$f'(x) = (2x^2 + 1)e^{x^2}$, $f''(x) = 2x(2x^2 + 3)e^{x^2}$.

(a) $(-\infty, +\infty)$ **(b)** none **(c)** $(0, +\infty)$ **(d)** $(-\infty, 0)$ **(e)** 0

EXERCISE SET 3.1

15–26 Find: (a) the intervals on which f is increasing, (b) the intervals on which f is decreasing, (c) the open intervals on which f is concave up, (d) the open intervals on which f is concave down, and (e) the x-coordinates of all inflection points.

15. $f(x) = x^2 - 3x + 8$

16. $f(x) = 5 - 4x - x^2$

17. $f(x) = (2x + 1)^3$

18. $f(x) = 5 + 12x - x^3$

19. $f(x) = 3x^4 - 4x^3$

20. $f(x) = x^4 - 5x^3 + 9x^2$

21. $f(x) = \dfrac{x - 2}{(x^2 - x + 1)^2}$

22. $f(x) = \dfrac{x}{x^2 + 2}$

23. $f(x) = \sqrt[3]{x^2 + x + 1}$

24. $f(x) = x^{4/3} - x^{1/3}$

25. $f(x) = (x^{2/3} - 1)^2$

26. $f(x) = x^{2/3} - x$

SOLUTION SET:

15. $f'(x) = 2(x - 3/2)$, $f''(x) = 2$.

(a) $[3/2, +\infty)$ **(b)** $(-\infty, 3/2]$ **(c)** $(-\infty, +\infty)$ **(d)** nowhere **(e)** none

17. $f'(x) = 6(2x + 1)^2$, $f''(x) = 24(2x + 1)$.

(a) $(-\infty, +\infty)$ **(b)** nowhere **(c)** $(-1/2, +\infty)$ **(d)** $(-\infty, -1/2)$ **(e)** $-1/2$

19. $f'(x) = 12x^2(x - 1)$, $f''(x) = 36x(x - 2/3)$.

(a) $[1, +\infty)$ **(b)** $(-\infty, 1]$ **(c)** $(-\infty, 0)$, $(2/3, +\infty)$ **(d)** $(0, 2/3)$ **(e)** $0, 2/3$

21. $f'(x) = -\dfrac{3(x^2 - 3x + 1)}{(x^2 - x + 1)^3}$, $f''(x) = \dfrac{6x(2x^2 - 8x + 5)}{(x^2 - x + 1)^4}$.

(a) $\left[\dfrac{3 - \sqrt{5}}{2}, \dfrac{3 + \sqrt{5}}{2}\right]$ **(b)** $\left(-\infty, \dfrac{3 - \sqrt{5}}{2}\right], \left[\dfrac{3 + \sqrt{5}}{2}, +\infty\right)$ **(c)** $\left(0, 2 - \dfrac{\sqrt{6}}{2}\right), \left(2 + \dfrac{\sqrt{6}}{2}, +\infty\right)$

(d) $(-\infty, 0), \left(2 - \dfrac{\sqrt{6}}{2}, 2 + \dfrac{\sqrt{6}}{2}\right)$ **(e)** $0, 2 - \dfrac{\sqrt{6}}{2}, 2 + \dfrac{\sqrt{6}}{2}$

23. $f'(x) = \dfrac{2x + 1}{3(x^2 + x + 1)^{2/3}}$, $f''(x) = -\dfrac{2(x + 2)(x - 1)}{9(x^2 + x + 1)^{5/3}}$.

(a) $[-1/2, +\infty)$ **(b)** $(-\infty, -1/2]$ **(c)** $(-2, 1)$ **(d)** $(-\infty, -2), (1, +\infty)$ **(e)** $-2, 1$

25. $f'(x) = \dfrac{4(x^{2/3} - 1)}{3x^{1/3}}$, $f''(x) = \dfrac{4(x^{5/3} + x)}{9x^{7/3}}$.

(a) $[-1, 0], [1, +\infty)$ **(b)** $(-\infty, -1], [0, 1]$ **(c)** $(-\infty, 0)$, $(0, +\infty)$ **(d)** nowhere **(e)** none