The Related rates

DIFFERENTIATING EQUATIONS TO RELATE RATES

Figure 2.8.1 shows a liquid draining through a conical filter. As the liquid drains, its volume V, height h, and radius r are functions of the elapsed time t, and at each instant these variables are related by the equation

$$V = \frac{\pi}{3} r^2 h$$

To find the rate of change of the volume V with respect to the time t, we can differentiate both sides of this equation with respect to t to obtain

$$\frac{dV}{dt} = \frac{\pi}{3}\left[r^2 \frac{dh}{dt} + h\left(2r\frac{dr}{dt}\right)\right] = \frac{\pi}{3}\left(r^2\frac{dh}{dt} + 2rh\frac{dr}{dt}\right)$$

Thus, to find dV/dt at a specific time t we need to have values for r, h, dh/dt, and dr/dt at that time. This is called a ***related rates problem*** because an unknown rate of change is related to other variables whose values and whose rates of change are known or can be found in some way. Let us begin with a simple example.

▶ Example 1 Suppose that x and y are differentiable functions of t and are related by the equation $y = x^3$. Find dy/dt at time $t = 1$ if $x = 2$ and $dx/dt = 4$ at time $t = 1$.

Solution. Differentiating both sides of the equation $y = x^3$ with respect to t yields

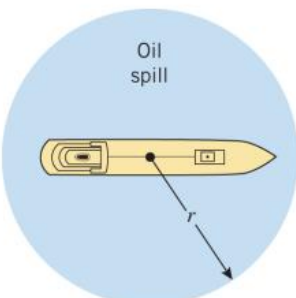
$$\frac{dy}{dt} = \frac{d}{dt}[x^3] = 3x^2\frac{dx}{dt}$$

Thus, the value of dy/dt at time $t = 1$ is

$$\left.\frac{dy}{dt}\right|_{t=1} = 3(2)^2 \left.\frac{dx}{dt}\right|_{t=1} = 12 \cdot 4 = 48 \blacktriangleleft$$

dehooks/Depositphotos

Oil spill from a ruptured tanker.



▲ **Figure 2.8.2**

▶ **Example 2** Assume that oil spilled from a ruptured tanker spreads in a circular pattern whose radius increases at a constant rate of 2 ft/s. How fast is the area of the spill increasing when the radius of the spill is 60 ft?

Solution. Let

t = number of seconds elapsed from the time of the spill
r = radius of the spill in feet after t seconds
A = area of the spill in square feet after t seconds

(Figure 2.8.2). We know the rate at which the radius is increasing, and we want to find the rate at which the area is increasing at the instant when $r = 60$; that is, we want to find

$$\left.\frac{dA}{dt}\right|_{r=60} \quad \text{given that} \quad \frac{dr}{dt} = 2 \text{ ft/s}$$

Since A is the area of a circle of radius r,

$$A = \pi r^2 \tag{1}$$

Differentiating both sides of (1) with respect to t yields

$$\frac{dA}{dt} = 2\pi r \frac{dr}{dt} \tag{2}$$

Thus, when $r = 60$ the area of the spill is increasing at the rate of

$$\left.\frac{dA}{dt}\right|_{r=60} = 2\pi(60)(2) = 240\pi \text{ ft}^2/\text{s} \approx 754 \text{ ft}^2/\text{s}$$ ◀

With some minor variations, the method used in Example 2 can be used to solve a variety of related rates problems. We can break the method down into five steps.

A Strategy for Solving Related Rates Problems

Step 1. Assign letters to all quantities that vary with time and any others that seem relevant to the problem. Give a definition for each letter.

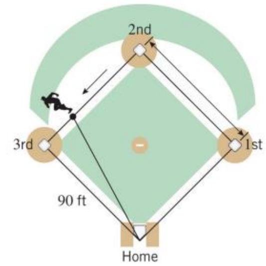
Step 2. Identify the rates of change that are known and the rate of change that is to be found. Interpret each rate as a derivative.

Step 3. Find an equation that relates the variables whose rates of change were identified in Step 2. To do this, it will often be helpful to draw an appropriately labeled figure that illustrates the relationship.

Step 4. Differentiate both sides of the equation obtained in Step 3 with respect to time to produce a relationship between the known rates of change and the unknown rate of change.

Step 5. *After* completing Step 4, substitute all known values for the rates of change and the variables, and then solve for the unknown rate of change.

► Example 3 A baseball diamond is a square whose sides are 90 ft long (Figure 2.8.3). Suppose that a player running from second base to third base has a speed of 30 ft/s at the instant when he is 20 ft from third base. At what rate is the player's distance from home plate changing at that instant?



▲ Figure 2.8.3

Solution. We are given a constant speed with which the player is approaching third base, and we want to find the rate of change of the distance between the player and home plate at a particular instant. Thus, let

t = number of seconds since the player left second base
x = distance in feet from the player to third base
y = distance in feet from the player to home plate

(Figure 2.8.4). Thus, we want to find

$$\left.\frac{dy}{dt}\right|_{x=20} \quad \text{given that} \quad \left.\frac{dx}{dt}\right|_{x=20} = -30 \text{ ft/s}$$

As suggested by Figure 2.8.4, the variables x and y are related by the Theorem of Pythagoras:

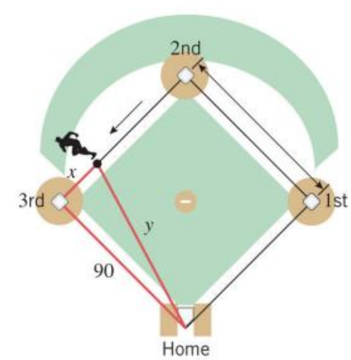
$$x^2 + 90^2 = y^2 \tag{3}$$

Differentiating both sides of this equation with respect to t yields

$$2x\frac{dx}{dt} = 2y\frac{dy}{dt}$$

from which we obtain

$$\frac{dy}{dt} = \frac{x}{y}\frac{dx}{dt}$$



▲ Figure 2.8.4

When $x = 20$, it follows from (3) that

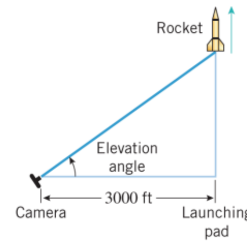
$$y = \sqrt{20^2 + 90^2} = \sqrt{8500} = 10\sqrt{85}$$

so that (4) yields

$$\left.\frac{dy}{dt}\right|_{x=20} = \frac{20}{10\sqrt{85}}(-30) = -\frac{60}{\sqrt{85}} \approx -6.51 \text{ ft/s}$$

The negative sign in the answer tells us that y is decreasing, which makes sense physically from Figure 2.8.4. ◄

▶ Example 4 Figure 2.8.5 shows a camera mounted at a point 3000 ft from the base of a rocket launching pad. If the rocket is rising vertically at 880 ft/s when it is 4000 ft above the launching pad, how fast must the camera elevation angle change at that instant to keep the camera aimed at the rocket?



▲ Figure 2.8.5

Solution. Let

t = number of seconds elapsed from the time of launch
ϕ = camera elevation angle in radians after t seconds
h = height of the rocket in feet after t seconds

(Figure 2.8.6). At each instant the rate at which the camera elevation angle must change is $d\phi/dt$, and the rate at which the rocket is rising is dh/dt. We want to find

$$\left.\frac{d\phi}{dt}\right|_{h=4000} \quad \text{given that} \quad \left.\frac{dh}{dt}\right|_{h=4000} = 880 \text{ ft/s}$$

From Figure 2.8.6 we see that

$$\tan\phi = \frac{h}{3000} \tag{5}$$

Differentiating both sides of (5) with respect to t yields

$$(\sec^2\phi)\frac{d\phi}{dt} = \frac{1}{3000}\frac{dh}{dt} \tag{6}$$

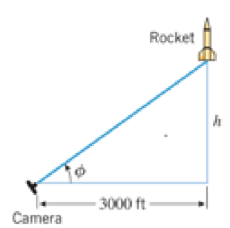
When $h = 4000$, it follows that

$$(\sec\phi)\Big|_{h=4000} = \frac{5000}{3000} = \frac{5}{3}$$

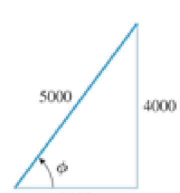
(see Figure 2.8.7), so that from (6)

$$\left(\frac{5}{3}\right)^2 \left.\frac{d\phi}{dt}\right|_{h=4000} = \frac{1}{3000}\cdot 880 = \frac{22}{75}$$

$$\left.\frac{d\phi}{dt}\right|_{h=4000} = \frac{22}{75}\cdot\frac{9}{25} = \frac{66}{625} \approx 0.11 \text{ rad/s} \approx 6.05 \text{ deg/s} \blacktriangleleft$$



▲ Figure 2.8.6



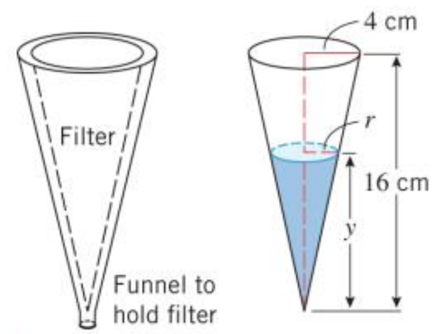
▲ Figure 2.8.7

► Example 5 Suppose that liquid is to be cleared of sediment by allowing it to drain through a conical filter that is 16 cm high and has a radius of 4 cm at the top (Figure 2.8.8). Suppose also that the liquid is forced out of the cone at a constant rate of 2 cm^3/min.

(a) Do you think that the depth of the liquid will decrease at a constant rate? Give a verbal argument that justifies your conclusion.

(b) Find a formula that expresses the rate at which the depth of the liquid is changing in terms of the depth, and use that formula to determine whether your conclusion in part (a) is correct.

(c) At what rate is the depth of the liquid changing at the instant when the liquid in the cone is 8 cm deep?



▲ Figure 2.8.8

Solution (a). For the volume of liquid to decrease by a *fixed amount*, it requires a greater decrease in depth when the cone is close to empty than when it is almost full (Figure 2.8.9). This suggests that for the volume to decrease at a constant rate, the depth must decrease at an increasing rate.

Solution (b). Let

t = time elapsed from the initial observation (min)
V = volume of liquid in the cone at time t (cm^3)
y = depth of the liquid in the cone at time t (cm)
r = radius of the liquid surface at time t (cm)

(Figure 2.8.8). At each instant the rate at which the volume of liquid is changing is dV/dt, and the rate at which the depth is changing is dy/dt. We want to express dy/dt in terms of y given that dV/dt has a constant value of $dV/dt = -2$. (We must use a minus sign here because V *decreases* as t increases.)

From the formula for the volume of a cone, the volume V, the radius r, and the depth y are related by

$$V = \tfrac{1}{3}\pi r^2 y \qquad (7)$$

If we differentiate both sides of (7) with respect to t, the right side will involve the quantity dr/dt. Since we have no direct information about dr/dt, it is desirable to eliminate r from (7) before differentiating. This can be done using similar triangles. From Figure 2.8.8 we see that

$$\frac{r}{y} = \frac{4}{16} \quad \text{or} \quad r = \frac{1}{4}y$$

Substituting this expression in (7) gives

$$V = \frac{\pi}{48}y^3 \tag{8}$$

Differentiating both sides of (8) with respect to t we obtain

$$\frac{dV}{dt} = \frac{\pi}{48}\left(3y^2\frac{dy}{dt}\right)$$

or

$$\frac{dy}{dt} = \frac{16}{\pi y^2}\frac{dV}{dt} = \frac{16}{\pi y^2}(-2) = -\frac{32}{\pi y^2} \tag{9}$$

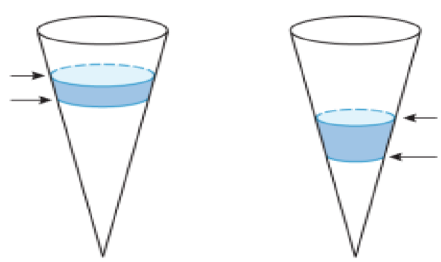
which expresses dy/dt in terms of y. The minus sign tells us that y is decreasing with time, and

$$\left|\frac{dy}{dt}\right| = \frac{32}{\pi y^2}$$

tells us how fast y is decreasing. From this formula we see that $|dy/dt|$ increases as y decreases, which confirms our conjecture in part (a) that the depth of the liquid decreases more quickly as the liquid drains through the filter.

Solution (c). The rate at which the depth is changing when the depth is 8 cm can be obtained from (9) with $y = 8$:

$$\left.\frac{dy}{dt}\right|_{y=8} = -\frac{32}{\pi(8^2)} = -\frac{1}{2\pi} \approx -0.16 \text{ cm/min} \blacktriangleleft$$



The same volume has drained, but the change in height is greater near the bottom than near the top.

▲ **Figure 2.8.9**

EXERCISE SET 2.8

10. Suppose that $z = x^3y^2$, where both x and y are changing with time. At a certain instant when $x = 1$ and $y = 2$, x is decreasing at the rate of 2 units/s, and y is increasing at the rate of 3 units/s. How fast is z changing at this instant? Is z increasing or decreasing?

11. The minute hand of a certain clock is 4 in long. Starting from the moment when the hand is pointing straight up, how fast is the area of the sector that is swept out by the hand increasing at any instant during the next revolution of the hand?

12. A stone dropped into a still pond sends out a circular ripple whose radius increases at a constant rate of 3 ft/s. How rapidly is the area enclosed by the ripple increasing at the end of 10 s?

13. Oil spilled from a ruptured tanker spreads in a circle whose area increases at a constant rate of 6 mi^2/h. How fast is the radius of the spill increasing when the area is 9 mi^2?

14. A spherical balloon is inflated so that its volume is increasing at the rate of 3 ft^3/min. How fast is the diameter of the balloon increasing when the radius is 1 ft?

15. A spherical balloon is to be deflated so that its radius decreases at a constant rate of 15 cm/min. At what rate must air be removed when the radius is 9 cm?

16. A 17 ft ladder is leaning against a wall. If the bottom of the ladder is pulled along the ground away from the wall at a constant rate of 5 ft/s, how fast will the top of the ladder be moving down the wall when it is 8 ft above the ground?

17. A 13 ft ladder is leaning against a wall. If the top of the ladder slips down the wall at a rate of 2 ft/s, how fast will the foot be moving away from the wall when the top is 5 ft above the ground?

18. A 10 ft plank is leaning against a wall. If at a certain instant the bottom of the plank is 2 ft from the wall and is being pushed toward the wall at the rate of 6 in/s, how fast is the acute angle that the plank makes with the ground increasing?

19. A softball diamond is a square whose sides are 60 ft long. Suppose that a player running from first to second base has a speed of 25 ft/s at the instant when she is 10 ft from second base. At what rate is the player's distance from home plate changing at that instant?

20. A rocket, rising vertically, is tracked by a radar station that is on the ground 5 mi from the launchpad. How fast is the rocket rising when it is 4 mi high and its distance from the radar station is increasing at a rate of 2000 mi/h?

Solution Set:

10. Find $\left.\frac{dz}{dt}\right|_{\substack{x=1,\\y=2}}$ given that $\left.\frac{dx}{dt}\right|_{\substack{x=1,\\y=2}} = -2$ and $\left.\frac{dy}{dt}\right|_{\substack{x=1,\\y=2}} = 3$. $\frac{dz}{dt} = 2x^3y\frac{dy}{dt} + 3x^2y^2\frac{dx}{dt}$, $\left.\frac{dz}{dt}\right|_{\substack{x=1,\\y=2}} = (4)(3) + (12)(-2) = -12$ units/s; z is decreasing.

11. Let A be the area swept out, and θ the angle through which the minute hand has rotated. Find $\frac{dA}{dt}$ given that $\frac{d\theta}{dt} = \frac{\pi}{30}$ rad/min; $A = \frac{1}{2}r^2\theta = 8\theta$, so $\frac{dA}{dt} = 8\frac{d\theta}{dt} = \frac{4\pi}{15}$ in^2/min.

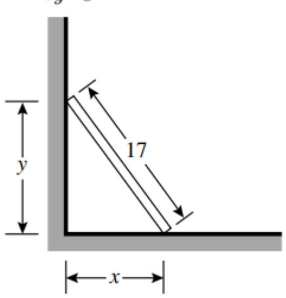
12. Let r be the radius and A the area enclosed by the ripple. We want $\left.\frac{dA}{dt}\right|_{t=10}$ given that $\frac{dr}{dt} = 3$. We know that $A = \pi r^2$, so $\frac{dA}{dt} = 2\pi r\frac{dr}{dt}$. Because r is increasing at the constant rate of 3 ft/s, it follows that $r = 30$ ft after 10 seconds so $\left.\frac{dA}{dt}\right|_{t=10} = 2\pi(30)(3) = 180\pi$ ft^2/s.

13. Find $\left.\frac{dr}{dt}\right|_{A=9}$ given that $\frac{dA}{dt} = 6$. From $A = \pi r^2$ we get $\frac{dA}{dt} = 2\pi r\frac{dr}{dt}$ so $\frac{dr}{dt} = \frac{1}{2\pi r}\frac{dA}{dt}$. If $A = 9$ then $\pi r^2 = 9$, $r = 3/\sqrt{\pi}$ so $\left.\frac{dr}{dt}\right|_{A=9} = \frac{1}{2\pi(3/\sqrt{\pi})}(6) = 1/\sqrt{\pi}$ mi/h.

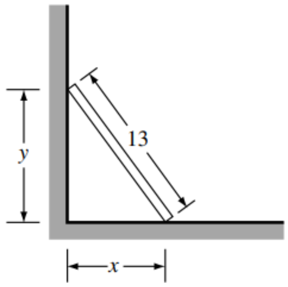
14. The volume V of a sphere of radius r is given by $V = \frac{4}{3}\pi r^3$ or, because $r = \frac{D}{2}$ where D is the diameter, $V = \frac{4}{3}\pi\left(\frac{D}{2}\right)^3 = \frac{1}{6}\pi D^3$. We want $\left.\frac{dD}{dt}\right|_{r=1}$ given that $\frac{dV}{dt} = 3$. From $V = \frac{1}{6}\pi D^3$ we get $\frac{dV}{dt} = \frac{1}{2}\pi D^2\frac{dD}{dt}$, $\frac{dD}{dt} = \frac{2}{\pi D^2}\frac{dV}{dt}$, so $\left.\frac{dD}{dt}\right|_{r=1} = \frac{2}{\pi(2)^2}(3) = \frac{3}{2\pi}$ ft/min.

15. Find $\left.\frac{dV}{dt}\right|_{r=9}$ given that $\frac{dr}{dt} = -15$. From $V = \frac{4}{3}\pi r^3$ we get $\frac{dV}{dt} = 4\pi r^2\frac{dr}{dt}$ so $\left.\frac{dV}{dt}\right|_{r=9} = 4\pi(9)^2(-15) = -4860\pi$. Air must be removed at the rate of 4860π cm^3/min.

16. Let x and y be the distances shown in the diagram. We want to find $\left.\frac{dy}{dt}\right|_{y=8}$ given that $\frac{dx}{dt} = 5$. From $x^2+y^2 = 17^2$ we get $2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$, so $\frac{dy}{dt} = -\frac{x}{y}\frac{dx}{dt}$. When $y = 8$, $x^2 + 8^2 = 17^2$, $x^2 = 289 - 64 = 225$, $x = 15$ so $\left.\frac{dy}{dt}\right|_{y=8} = -\frac{15}{8}(5) = -\frac{75}{8}$ ft/s; the top of the ladder is moving down the wall at a rate of 75/8 ft/s.

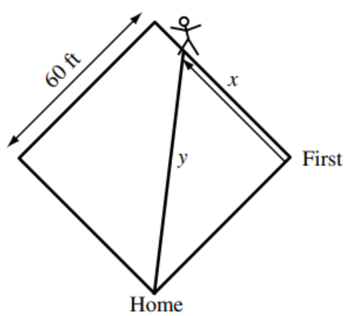


17. Find $\left.\frac{dx}{dt}\right|_{y=5}$ given that $\frac{dy}{dt} = -2$. From $x^2 + y^2 = 13^2$ we get $2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$ so $\frac{dx}{dt} = -\frac{y}{x}\frac{dy}{dt}$. Use $x^2 + y^2 = 169$ to find that $x = 12$ when $y = 5$ so $\left.\frac{dx}{dt}\right|_{y=5} = -\frac{5}{12}(-2) = \frac{5}{6}$ ft/s.



18. Let θ be the acute angle, and x the distance of the bottom of the plank from the wall. Find $\left.\frac{d\theta}{dt}\right|_{x=2}$ given that $\left.\frac{dx}{dt}\right|_{x=2} = -\frac{1}{2}$ ft/s. The variables θ and x are related by the equation $\cos\theta = \frac{x}{10}$ so $-\sin\theta\frac{d\theta}{dt} = \frac{1}{10}\frac{dx}{dt}$, $\frac{d\theta}{dt} = -\frac{1}{10\sin\theta}\frac{dx}{dt}$. When $x = 2$, the top of the plank is $\sqrt{10^2 - 2^2} = \sqrt{96}$ ft above the ground so $\sin\theta = \sqrt{96}/10$ and $\left.\frac{d\theta}{dt}\right|_{x=2} = -\frac{1}{\sqrt{96}}\left(-\frac{1}{2}\right) = \frac{1}{2\sqrt{96}} \approx 0.051$ rad/s.

19. Let x denote the distance from first base and y the distance from home plate. Then $x^2 + 60^2 = y^2$ and $2x\frac{dx}{dt} = 2y\frac{dy}{dt}$. When $x = 50$ then $y = 10\sqrt{61}$ so $\frac{dy}{dt} = \frac{x}{y}\frac{dx}{dt} = \frac{50}{10\sqrt{61}}(25) = \frac{125}{\sqrt{61}}$ ft/s.



20. Find $\left.\frac{dx}{dt}\right|_{x=4}$ given that $\left.\frac{dy}{dt}\right|_{x=4} = 2000$. From $x^2 + 5^2 = y^2$ we get $2x\frac{dx}{dt} = 2y\frac{dy}{dt}$ so $\frac{dx}{dt} = \frac{y}{x}\frac{dy}{dt}$. Use $x^2 + 25 = y^2$

to find that $y = \sqrt{41}$ when $x = 4$ so $\left.\frac{dx}{dt}\right|_{x=4} = \frac{\sqrt{41}}{4}(2000) = 500\sqrt{41}$ mi/h.