Topics in Differentiation

Exercise Set 3.1

1. (a) $1 + y + x\dfrac{dy}{dx} - 6x^2 = 0$, $\dfrac{dy}{dx} = \dfrac{6x^2 - y - 1}{x}$.

(b) $y = \dfrac{2 + 2x^3 - x}{x} = \dfrac{2}{x} + 2x^2 - 1$, $\dfrac{dy}{dx} = -\dfrac{2}{x^2} + 4x$.

(c) From part (a), $\dfrac{dy}{dx} = 6x - \dfrac{1}{x} - \dfrac{1}{x}y = 6x - \dfrac{1}{x} - \dfrac{1}{x}\left(\dfrac{2}{x} + 2x^2 - 1\right) = 4x - \dfrac{2}{x^2}$.

3. $2x + 2y\dfrac{dy}{dx} = 0$ so $\dfrac{dy}{dx} = -\dfrac{x}{y}$.

5. $x^2\dfrac{dy}{dx} + 2xy + 3x(3y^2)\dfrac{dy}{dx} + 3y^3 - 1 = 0$, $(x^2 + 9xy^2)\dfrac{dy}{dx} = 1 - 2xy - 3y^3$, so $\dfrac{dy}{dx} = \dfrac{1 - 2xy - 3y^3}{x^2 + 9xy^2}$.

7. $-\dfrac{1}{2x^{3/2}} - \dfrac{\frac{dy}{dx}}{2y^{3/2}} = 0$, so $\dfrac{dy}{dx} = -\dfrac{y^{3/2}}{x^{3/2}}$.

9. $\cos(x^2y^2)\left[x^2(2y)\dfrac{dy}{dx} + 2xy^2\right] = 1$, so $\dfrac{dy}{dx} = \dfrac{1 - 2xy^2\cos(x^2y^2)}{2x^2y\cos(x^2y^2)}$.

11. $3\tan^2(xy^2 + y)\sec^2(xy^2 + y)\left(2xy\dfrac{dy}{dx} + y^2 + \dfrac{dy}{dx}\right) = 1$, so $\dfrac{dy}{dx} = \dfrac{1 - 3y^2\tan^2(xy^2 + y)\sec^2(xy^2 + y)}{3(2xy + 1)\tan^2(xy^2 + y)\sec^2(xy^2 + y)}$.

13. $4x - 6y\dfrac{dy}{dx} = 0$, $\dfrac{dy}{dx} = \dfrac{2x}{3y}$, $4 - 6\left(\dfrac{dy}{dx}\right)^2 - 6y\dfrac{d^2y}{dx^2} = 0$, so $\dfrac{d^2y}{dx^2} = -\dfrac{3\left(\frac{dy}{dx}\right)^2 - 2}{3y} = \dfrac{2(3y^2 - 2x^2)}{9y^3} = -\dfrac{8}{9y^3}$.

15. $\dfrac{dy}{dx} = -\dfrac{y}{x}$, $\dfrac{d^2y}{dx^2} = -\dfrac{x(dy/dx) - y(1)}{x^2} = -\dfrac{x(-y/x) - y}{x^2} = \dfrac{2y}{x^2}$.

17. $\dfrac{dy}{dx} = (1 + \cos y)^{-1}$, $\dfrac{d^2y}{dx^2} = -(1 + \cos y)^{-2}(-\sin y)\dfrac{dy}{dx} = \dfrac{\sin y}{(1 + \cos y)^3}$.

19. By implicit differentiation, $2x + 2y(dy/dx) = 0$, $\dfrac{dy}{dx} = -\dfrac{x}{y}$; at $(1/2, \sqrt{3}/2)$, $\dfrac{dy}{dx} = -\sqrt{3}/3$; at $(1/2, -\sqrt{3}/2)$, $\dfrac{dy}{dx} = +\sqrt{3}/3$. Directly, at the upper point $y = \sqrt{1 - x^2}$, $\dfrac{dy}{dx} = \dfrac{-x}{\sqrt{1 - x^2}} = -\dfrac{1/2}{\sqrt{3/4}} = -1/\sqrt{3}$ and at the lower point $y = -\sqrt{1 - x^2}$, $\dfrac{dy}{dx} = \dfrac{x}{\sqrt{1 - x^2}} = +1/\sqrt{3}$.

21. False; $x = y^2$ defines two functions $y = \pm\sqrt{x}$. See Definition 3.1.1.

23. False; the equation is equivalent to $x^2 = y^2$ which is satisfied by $y = |x|$.

25. $4x^3 + 4y^3\dfrac{dy}{dx} = 0$, so $\dfrac{dy}{dx} = -\dfrac{x^3}{y^3} = -\dfrac{1}{15^{3/4}} \approx -0.1312$.

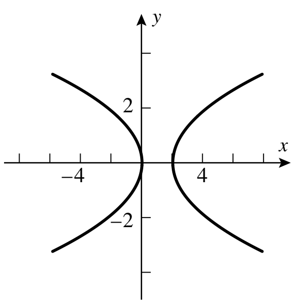
27. $4(x^2+y^2)\left(2x + 2y\dfrac{dy}{dx}\right) = 25\left(2x - 2y\dfrac{dy}{dx}\right)$, $\dfrac{dy}{dx} = \dfrac{x[25 - 4(x^2+y^2)]}{y[25 + 4(x^2+y^2)]}$; at $(3,1)$ $\dfrac{dy}{dx} = -9/13$.

29. $4a^3\dfrac{da}{dt} - 4t^3 = 6\left(a^2 + 2at\dfrac{da}{dt}\right)$, solve for $\dfrac{da}{dt}$ to get $\dfrac{da}{dt} = \dfrac{2t^3 + 3a^2}{2a^3 - 6at}$.

31. $2a^2\omega\dfrac{d\omega}{d\lambda} + 2b^2\lambda = 0$, so $\dfrac{d\omega}{d\lambda} = -\dfrac{b^2\lambda}{a^2\omega}$.

33. $2x + x\dfrac{dy}{dx} + y + 2y\dfrac{dy}{dx} = 0$. Substitute $y = -2x$ to obtain $-3x\dfrac{dy}{dx} = 0$. Since $x = \pm 1$ at the indicated points, $\dfrac{dy}{dx} = 0$ there.

35. **(a)**



(b) Implicit differentiation of the curve yields $(4y^3 + 2y)\dfrac{dy}{dx} = 2x - 1$, so $\dfrac{dy}{dx} = 0$ only if $x = 1/2$ but $y^4 + y^2 \geq 0$ so $x = 1/2$ is impossible.

(c) $x^2 - x - (y^4 + y^2) = 0$, so by the Quadratic Formula, $x = \dfrac{-1 \pm \sqrt{(2y^2+1)^2}}{2} = 1 + y^2$ or $-y^2$, and we have the two parabolas $x = -y^2, x = 1 + y^2$.

37. The point (1,1) is on the graph, so $1 + a = b$. The slope of the tangent line at (1,1) is $-4/3$; use implicit differentiation to get $\dfrac{dy}{dx} = -\dfrac{2xy}{x^2 + 2ay}$ so at (1,1), $-\dfrac{2}{1+2a} = -\dfrac{4}{3}$, $1 + 2a = 3/2$, $a = 1/4$ and hence $b = 1 + 1/4 = 5/4$.

39. We shall find when the curves intersect and check that the slopes are negative reciprocals. For the intersection solve the simultaneous equations $x^2 + (y-c)^2 = c^2$ and $(x-k)^2 + y^2 = k^2$ to obtain $cy = kx = \dfrac{1}{2}(x^2+y^2)$. Thus $x^2 + y^2 = cy + kx$, or $y^2 - cy = -x^2 + kx$, and $\dfrac{y-c}{x} = -\dfrac{x-k}{y}$. Differentiating the two families yields (black) $\dfrac{dy}{dx} = -\dfrac{x}{y-c}$, and (gray) $\dfrac{dy}{dx} = -\dfrac{x-k}{y}$. But it was proven that these quantities are negative reciprocals of each other.