Solution of Exercise 6.5

7 $\lim\limits_{x\to 0} \dfrac{e^x}{\cos x} = 1.$

9 $\lim\limits_{\theta\to 0} \dfrac{\sec^2\theta}{1} = 1.$

11 $\lim\limits_{x\to \pi^+} \dfrac{\cos x}{1} = -1.$

13 $\lim\limits_{x\to +\infty} \dfrac{1/x}{1} = 0.$

15 $\lim\limits_{x\to 0^+} \dfrac{-\csc^2 x}{1/x} = \lim\limits_{x\to 0^+} \dfrac{-x}{\sin^2 x} = \lim\limits_{x\to 0^+} \dfrac{-1}{2\sin x\cos x} = -\infty.$

17 $\lim\limits_{x\to +\infty} \dfrac{100x^{99}}{e^x} = \lim\limits_{x\to +\infty} \dfrac{(100)(99)x^{98}}{e^x} = \cdots = \lim\limits_{x\to +\infty} \dfrac{(100)(99)(98)\cdots(1)}{e^x} = 0.$

19 $\lim\limits_{x\to +\infty} xe^{-x} = \lim\limits_{x\to +\infty} \dfrac{x}{e^x} = \lim\limits_{x\to +\infty} \dfrac{1}{e^x} = 0.$

21 $\lim\limits_{x\to +\infty} x\sin(\pi/x) = \lim\limits_{x\to +\infty} \dfrac{\sin(\pi/x)}{1/x} = \lim\limits_{x\to +\infty} \dfrac{(-\pi/x^2)\cos(\pi/x)}{-1/x^2} = \lim\limits_{x\to +\infty} \pi\cos(\pi/x) = \pi.$

23 $\lim\limits_{x\to (\pi/2)^-} \sec 3x\cos 5x = \lim\limits_{x\to (\pi/2)^-} \dfrac{\cos 5x}{\cos 3x} = \lim\limits_{x\to (\pi/2)^-} \dfrac{-5\sin 5x}{-3\sin 3x} = \dfrac{-5(+1)}{(-3)(-1)} = -\dfrac{5}{3}.$

25 $y = (1-3/x)^x,\ \lim\limits_{x\to +\infty} \ln y = \lim\limits_{x\to +\infty} \dfrac{\ln(1-3/x)}{1/x} = \lim\limits_{x\to +\infty} \dfrac{-3}{1-3/x} = -3,\ \lim\limits_{x\to +\infty} y = e^{-3}.$

27 $y = (e^x+x)^{1/x},\ \lim\limits_{x\to 0} \ln y = \lim\limits_{x\to 0} \dfrac{\ln(e^x+x)}{x} = \lim\limits_{x\to 0} \dfrac{e^x+1}{e^x+x} = 2,\ \lim\limits_{x\to 0} y = e^2.$

29 $y = (2-x)^{\tan(\pi x/2)},\ \lim\limits_{x\to 1} \ln y = \lim\limits_{x\to 1} \dfrac{\ln(2-x)}{\cot(\pi x/2)} = \lim\limits_{x\to 1} \dfrac{2\sin^2(\pi x/2)}{\pi(2-x)} = 2/\pi,\ \lim\limits_{x\to 1} y = e^{2/\pi}.$

31 $\lim_{x\to 0}\left(\frac{1}{\sin x}-\frac{1}{x}\right)=\lim_{x\to 0}\frac{x-\sin x}{x\sin x}=\lim_{x\to 0}\frac{1-\cos x}{x\cos x+\sin x}=\lim_{x\to 0}\frac{\sin x}{2\cos x-x\sin x}=0.$

33 $\lim_{x\to+\infty}\frac{(x^2+x)-x^2}{\sqrt{x^2+x}+x}=\lim_{x\to+\infty}\frac{x}{\sqrt{x^2+x}+x}=\lim_{x\to+\infty}\frac{1}{\sqrt{1+1/x}+1}=1/2.$

35 $\lim_{x\to+\infty}[x-\ln(x^2+1)]=\lim_{x\to+\infty}[\ln e^x-\ln(x^2+1)]=\lim_{x\to+\infty}\ln\frac{e^x}{x^2+1}, \lim_{x\to+\infty}\frac{e^x}{x^2+1}=\lim_{x\to+\infty}\frac{e^x}{2x}=\lim_{x\to+\infty}\frac{e^x}{2}=+\infty,$ so $\lim_{x\to+\infty}[x-\ln(x^2+1)]=+\infty$

37 $y=x^{\sin x}, \ln y=\sin x\ln x, \lim_{x\to 0^+}\ln y=\lim_{x\to 0^+}\frac{\ln x}{\csc x}=\lim_{x\to 0^+}\frac{1/x}{-\csc x\cot x}=\lim_{x\to 0^+}\left(\frac{\sin x}{x}\right)(-\tan x)=1(-0)=0,$ so $\lim_{x\to 0^+}x^{\sin x}=\lim_{x\to 0^+}y=e^0=1.$

39 $y=\left[-\frac{1}{\ln x}\right]^x, \ln y=x\ln\left[-\frac{1}{\ln x}\right], \lim_{x\to 0^+}\ln y=\lim_{x\to 0^+}\frac{\ln\left[-\frac{1}{\ln x}\right]}{1/x}=\lim_{x\to 0^+}\left(-\frac{1}{x\ln x}\right)(-x^2)=-\lim_{x\to 0^+}\frac{x}{\ln x}=0,$ so $\lim_{x\to 0^+}y=e^0=1.$

41 $y=(\ln x)^{1/x}, \ln y=(1/x)\ln\ln x, \lim_{x\to+\infty}\ln y=\lim_{x\to+\infty}\frac{\ln\ln x}{x}=\lim_{x\to+\infty}\frac{1/(x\ln x)}{1}=0,$ so $\lim_{x\to+\infty}y=1.$

43 $y=(\tan x)^{\pi/2-x}, \ln y=(\pi/2-x)\ln\tan x, \lim_{x\to(\pi/2)^-}\ln y=\lim_{x\to(\pi/2)^-}\frac{\ln\tan x}{1/(\pi/2-x)}=\lim_{x\to(\pi/2)^-}\frac{(\sec^2 x/\tan x)}{1/(\pi/2-x)^2}=\lim_{x\to(\pi/2)^-}\frac{(\pi/2-x)}{\cos x}\frac{(\pi/2-x)}{\sin x}=\lim_{x\to(\pi/2)^-}\frac{(\pi/2-x)}{\cos x}\lim_{x\to(\pi/2)^-}\frac{(\pi/2-x)}{\sin x}=1\cdot 0=0,$ so $\lim_{x\to(\pi/2)^-}y=1.$