IMPROPER INTEGRAL

It is assumed in the definition of the definite integral

$$\int_a^b f(x)\,dx$$

that $[a, b]$ is a finite interval and that the limit that defines the integral exists; that is, the function f is integrable. We observed in Theorems 5.5.2 and 5.5.8 that continuous functions are integrable, as are bounded functions with finitely many points of discontinuity. We also observed in Theorem 5.5.8 that functions that are not bounded on the interval of integration are not integrable. Thus, for example, a function with a vertical asymptote within the interval of integration would not be integrable.

Our main objective in this section is to extend the concept of a definite integral to allow for infinite intervals of integration and integrands with vertical asymptotes within the interval of integration. We will call the vertical asymptotes ***infinite discontinuities***, and we will call integrals with infinite intervals of integration or infinite discontinuities within the interval of integration ***improper integrals***. Here are some examples:

- Improper integrals with infinite intervals of integration:

$$\int_1^{+\infty} \frac{dx}{x^2}, \quad \int_{-\infty}^{0} e^x\,dx, \quad \int_{-\infty}^{+\infty} \frac{dx}{1+x^2}$$

- Improper integrals with infinite discontinuities in the interval of integration:

$$\int_{-3}^{3} \frac{dx}{x^2}, \quad \int_1^2 \frac{dx}{x-1}, \quad \int_0^{\pi} \tan x\,dx$$

- Improper integrals with infinite discontinuities and infinite intervals of integration:

$$\int_0^{+\infty} \frac{dx}{\sqrt{x}}, \quad \int_{-\infty}^{+\infty} \frac{dx}{x^2-9}, \quad \int_1^{+\infty} \sec x\,dx$$

The improper integral of f over the interval $[a, +\infty)$

7.8.1 DEFINITION The ***improper integral of f over the interval*** $[a, +\infty)$ is defined to be

$$\int_a^{+\infty} f(x)\,dx = \lim_{b\to+\infty} \int_a^b f(x)\,dx$$

In the case where the limit exists, the improper integral is said to ***converge***, and the limit is defined to be the value of the integral. In the case where the limit does not exist, the improper integral is said to ***diverge***, and it is not assigned a value.

▶ Example 1 Evaluate

$$\text{(a)} \int_1^{+\infty} \frac{dx}{x^3} \qquad \text{(b)} \int_1^{+\infty} \frac{dx}{x}$$

Solution (a). Following the definition, we replace the infinite upper limit by a finite upper limit b, and then take the limit of the resulting integral. This yields

$$\int_1^{+\infty} \frac{dx}{x^3} = \lim_{b\to+\infty} \int_1^b \frac{dx}{x^3} = \lim_{b\to+\infty} \left[-\frac{1}{2x^2}\right]_1^b = \lim_{b\to+\infty} \left(\frac{1}{2} - \frac{1}{2b^2}\right) = \frac{1}{2}$$

Since the limit is finite, the integral converges and its value is $1/2$.

Solution (b).

$$\int_1^{+\infty} \frac{dx}{x} = \lim_{b\to+\infty} \int_1^b \frac{dx}{x} = \lim_{b\to+\infty} \left[\ln x\right]_1^b = \lim_{b\to+\infty} \ln b = +\infty$$

In this case the integral diverges and hence has no value. ◀

The improper integral of f over the interval $(-\infty, b]$

7.8.3 DEFINITION The ***improper integral of f over the interval*** $(-\infty, b]$ is defined to be

$$\int_{-\infty}^{b} f(x)\,dx = \lim_{a\to-\infty}\int_{a}^{b} f(x)\,dx \qquad (2)$$

The integral is said to ***converge*** if the limit exists and ***diverge*** if it does not.

The ***improper integral of f over the interval*** $(-\infty, +\infty)$ is defined as

$$\int_{-\infty}^{+\infty} f(x)\,dx = \int_{-\infty}^{c} f(x)\,dx + \int_{c}^{+\infty} f(x)\,dx \qquad (3)$$

where c is any real number. The improper integral is said to ***converge*** if *both* terms converge and ***diverge*** if *either* term diverges.

▶ **Example 4** Evaluate $\displaystyle\int_{-\infty}^{+\infty} \frac{dx}{1+x^2}$.

Solution. We will evaluate the integral by choosing $c = 0$ in (3). With this value for c we obtain

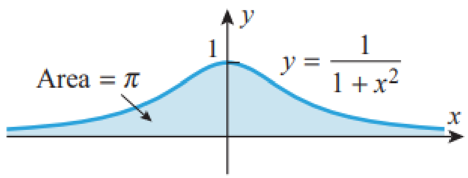
$$\int_{0}^{+\infty} \frac{dx}{1+x^2} = \lim_{b\to+\infty}\int_{0}^{b}\frac{dx}{1+x^2} = \lim_{b\to+\infty}\Big[\tan^{-1}x\Big]_0^b = \lim_{b\to+\infty}(\tan^{-1}b) = \frac{\pi}{2}$$

$$\int_{-\infty}^{0} \frac{dx}{1+x^2} = \lim_{a\to-\infty}\int_{a}^{0}\frac{dx}{1+x^2} = \lim_{a\to-\infty}\Big[\tan^{-1}x\Big]_a^0 = \lim_{a\to-\infty}(-\tan^{-1}a) = \frac{\pi}{2}$$

Thus, the integral converges and its value is

$$\int_{-\infty}^{+\infty} \frac{dx}{1+x^2} = \int_{-\infty}^{0}\frac{dx}{1+x^2} + \int_{0}^{+\infty}\frac{dx}{1+x^2} = \frac{\pi}{2}+\frac{\pi}{2} = \pi$$

Since the integrand is nonnegative on the interval $(-\infty, +\infty)$, the integral represents the area of the region shown in Figure 7.8.6. ◀



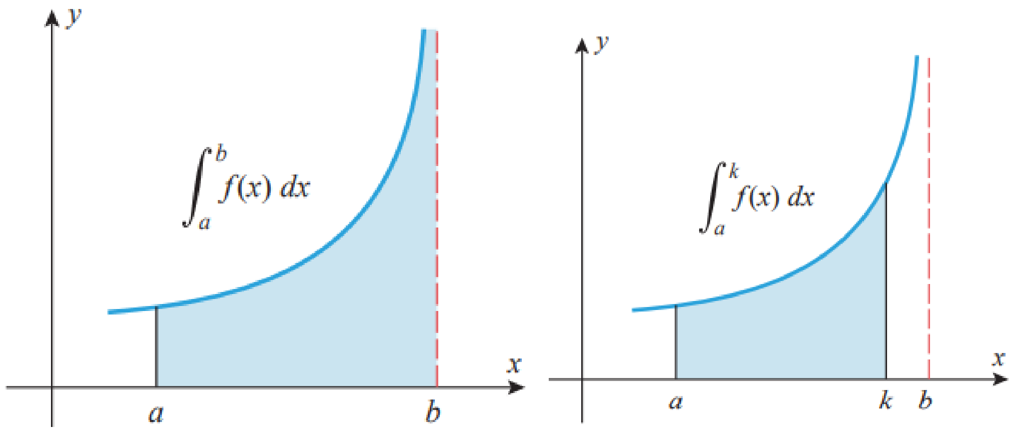
▲ **Figure 7.8.6**

The improper integral of f over the interval $[a, b]$

7.8.4 DEFINITION If f is continuous on the interval $[a, b]$, except for an infinite discontinuity at b, then the ***improper integral of f over the interval $[a, b]$*** is defined as

$$\int_a^b f(x)\,dx = \lim_{k \to b^-} \int_a^k f(x)\,dx \tag{4}$$

In the case where the indicated limit exists, the improper integral is said to ***converge***, and the limit is defined to be the value of the integral. In the case where the limit does not exist, the improper integral is said to ***diverge***, and it is not assigned a value.



Example 5 Evaluate $\displaystyle\int_0^1 \frac{dx}{\sqrt{1-x}}$.

Solution. The integral is improper because the integrand approaches $+\infty$ as x approaches the upper limit 1 from the left (Figure 7.8.8). From (4),

$$\int_0^1 \frac{dx}{\sqrt{1-x}} = \lim_{k \to 1^-} \int_0^k \frac{dx}{\sqrt{1-x}} = \lim_{k \to 1^-} \Big[-2\sqrt{1-x}\Big]_0^k$$

$$= \lim_{k \to 1^-} \Big[-2\sqrt{1-k} + 2\Big] = 2$$

The improper integral of f over the interval $[a, b]$

7.8.5 DEFINITION If f is continuous on the interval $[a, b]$, except for an infinite discontinuity at a, then the ***improper integral of f over the interval* [a, b]** is defined as

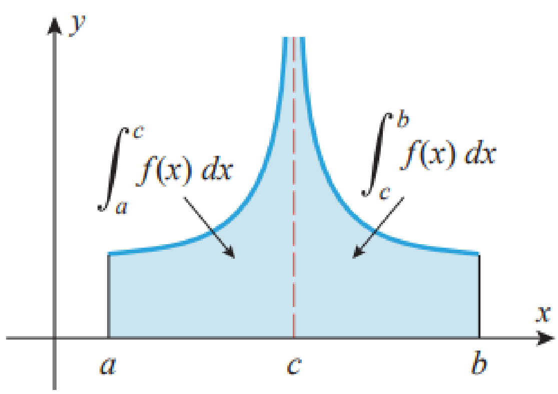
$$\int_a^b f(x)\,dx = \lim_{k \to a^+} \int_k^b f(x)\,dx \qquad (5)$$

The integral is said to ***converge*** if the indicated limit exists and ***diverge*** if it does not.

If f is continuous on the interval $[a, b]$, except for an infinite discontinuity at a point c in (a, b), then the ***improper integral of f over the interval* [a, b]** is defined as

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx \qquad (6)$$

where the two integrals on the right side are themselves improper. The improper integral on the left side is said to ***converge*** if *both* terms on the right side converge and ***diverge*** if *either* term on the right side diverges (Figure 7.8.9).



$\int_a^b f(x)\,dx$ is improper.

Example 6 Evaluate

$$\text{(a)} \int_1^2 \frac{dx}{1-x} \qquad \text{(b)} \int_1^4 \frac{dx}{(x-2)^{2/3}}$$

Solution (a). The integral is improper because the integrand approaches $-\infty$ as x approaches the lower limit 1 from the right (Figure 7.8.10). From Definition 7.8.5 we obtain

$$\int_1^2 \frac{dx}{1-x} = \lim_{k\to 1^+} \int_k^2 \frac{dx}{1-x} = \lim_{k\to 1^+} \Big[-\ln|1-x|\Big]_k^2$$

$$= \lim_{k\to 1^+} \Big[-\ln|-1| + \ln|1-k|\Big] = \lim_{k\to 1^+} \ln|1-k| = -\infty$$

so the integral diverges.

Solution (b). The integral is improper because the integrand approaches $+\infty$ at $x = 2$, which is inside the interval of integration. From Definition 7.8.5 we obtain

$$\int_1^4 \frac{dx}{(x-2)^{2/3}} = \int_1^2 \frac{dx}{(x-2)^{2/3}} + \int_2^4 \frac{dx}{(x-2)^{2/3}} \tag{7}$$

and we must investigate the convergence of both improper integrals on the right. Since

$$\int_1^2 \frac{dx}{(x-2)^{2/3}} = \lim_{k\to 2^-} \int_1^k \frac{dx}{(x-2)^{2/3}} = \lim_{k\to 2^-} \Big[3(k-2)^{1/3} - 3(1-2)^{1/3}\Big] = 3$$

$$\int_2^4 \frac{dx}{(x-2)^{2/3}} = \lim_{k\to 2^+} \int_k^4 \frac{dx}{(x-2)^{2/3}} = \lim_{k\to 2^+} \Big[3(4-2)^{1/3} - 3(k-2)^{1/3}\Big] = 3\sqrt[3]{2}$$

we have from (7) that

$$\int_1^4 \frac{dx}{(x-2)^{2/3}} = 3 + 3\sqrt[3]{2}$$

The improper integral of f over the interval $(-\infty, +\infty)$

The ***improper integral of f over the interval*** $(-\infty, +\infty)$ is defined as

$$\int_{-\infty}^{+\infty} f(x)\,dx = \int_{-\infty}^{c} f(x)\,dx + \int_{c}^{+\infty} f(x)\,dx \tag{3}$$

where c is any real number. The improper integral is said to ***converge*** if *both* terms converge and ***diverge*** if *either* term diverges.

▶ Example 4 Evaluate $\displaystyle\int_{-\infty}^{+\infty} \frac{dx}{1+x^2}$.

Solution. We will evaluate the integral by choosing $c = 0$ in (3). With this value for c we obtain

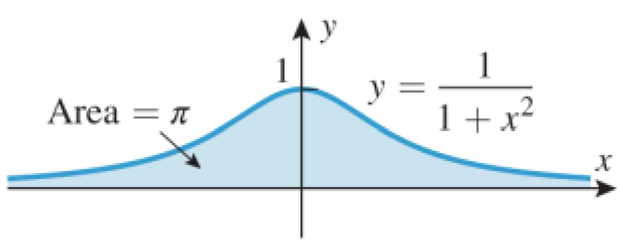
$$\int_{0}^{+\infty} \frac{dx}{1+x^2} = \lim_{b\to+\infty} \int_{0}^{b} \frac{dx}{1+x^2} = \lim_{b\to+\infty} \Big[\tan^{-1} x\Big]_0^b = \lim_{b\to+\infty} (\tan^{-1} b) = \frac{\pi}{2}$$

$$\int_{-\infty}^{0} \frac{dx}{1+x^2} = \lim_{a\to-\infty} \int_{a}^{0} \frac{dx}{1+x^2} = \lim_{a\to-\infty} \Big[\tan^{-1} x\Big]_a^0 = \lim_{a\to-\infty} (-\tan^{-1} a) = \frac{\pi}{2}$$

Thus, the integral converges and its value is

$$\int_{-\infty}^{+\infty} \frac{dx}{1+x^2} = \int_{-\infty}^{0} \frac{dx}{1+x^2} + \int_{0}^{+\infty} \frac{dx}{1+x^2} = \frac{\pi}{2} + \frac{\pi}{2} = \pi$$

Since the integrand is nonnegative on the interval $(-\infty, +\infty)$, the integral represents the area of the region shown in Figure 7.8.6. ◀



▲ Figure 7.8.6

EXERCISE SET 7.8

3–32 Evaluate the integrals that converge.

3. $\displaystyle\int_0^{+\infty} e^{-2x}\,dx$

4. $\displaystyle\int_{-1}^{+\infty} \frac{x}{1+x^2}\,dx$

5. $\displaystyle\int_3^{+\infty} \frac{2}{x^2-1}\,dx$

6. $\displaystyle\int_0^{+\infty} xe^{-x^2}\,dx$

7. $\displaystyle\int_e^{+\infty} \frac{1}{x\ln^3 x}\,dx$

8. $\displaystyle\int_2^{+\infty} \frac{1}{x\sqrt{\ln x}}\,dx$

9. $\displaystyle\int_{-\infty}^{0} \frac{dx}{(2x-1)^3}$

10. $\displaystyle\int_{-\infty}^{3} \frac{dx}{x^2+9}$

11. $\displaystyle\int_{-\infty}^{0} e^{3x}\,dx$

12. $\displaystyle\int_{-\infty}^{0} \frac{e^x\,dx}{3-2e^x}$

13. $\displaystyle\int_{-\infty}^{+\infty} x\,dx$

14. $\displaystyle\int_{-\infty}^{+\infty} \frac{x}{\sqrt{x^2+2}}\,dx$

15. $\displaystyle\int_{-\infty}^{+\infty} \frac{x}{(x^2+3)^2}\,dx$

16. $\displaystyle\int_{-\infty}^{+\infty} \frac{e^{-t}}{1+e^{-2t}}\,dt$

17. $\displaystyle\int_0^{4} \frac{dx}{(x-4)^2}$

18. $\displaystyle\int_0^{8} \frac{dx}{\sqrt[3]{x}}$

19. $\displaystyle\int_0^{\pi/2} \tan x\,dx$

20. $\displaystyle\int_0^{4} \frac{dx}{\sqrt{4-x}}$

21. $\displaystyle\int_0^{1} \frac{dx}{\sqrt{1-x^2}}$

22. $\displaystyle\int_{-3}^{1} \frac{x\,dx}{\sqrt{9-x^2}}$

23. $\displaystyle\int_{\pi/3}^{\pi/2} \frac{\sin x}{\sqrt{1 - 2\cos x}}\, dx$

24. $\displaystyle\int_{0}^{\pi/4} \frac{\sec^2 x}{1 - \tan x}\, dx$

25. $\displaystyle\int_{0}^{3} \frac{dx}{x - 2}$

26. $\displaystyle\int_{-2}^{2} \frac{dx}{x^2}$

27. $\displaystyle\int_{-1}^{8} x^{-1/3}\, dx$

28. $\displaystyle\int_{0}^{1} \frac{dx}{(x - 1)^{2/3}}$

29. $\displaystyle\int_{0}^{+\infty} \frac{1}{x^2}\, dx$

30. $\displaystyle\int_{1}^{+\infty} \frac{dx}{x\sqrt{x^2 - 1}}$

31. $\displaystyle\int_{0}^{1} \frac{dx}{\sqrt{x}(x + 1)}$

32. $\displaystyle\int_{0}^{+\infty} \frac{dx}{\sqrt{x}(x + 1)}$

SOLUTION SET

3. $\lim_{\ell\to+\infty}\left(-\frac{1}{2}e^{-2x}\right)\Big]_0^\ell = \frac{1}{2}\lim_{\ell\to+\infty}(-e^{-2\ell}+1) = \frac{1}{2}.$

5. $\lim_{\ell\to+\infty} -2\coth^{-1}x\Big]_3^\ell = \lim_{\ell\to+\infty}\left(2\coth^{-1}3 - 2\coth^{-1}\ell\right) = 2\coth^{-1}3.$

7. $\lim_{\ell\to+\infty} -\frac{1}{2\ln^2 x}\Big]_e^\ell = \lim_{\ell\to+\infty}\left[-\frac{1}{2\ln^2\ell}+\frac{1}{2}\right] = \frac{1}{2}.$

9. $\lim_{\ell\to-\infty} -\frac{1}{4(2x-1)^2}\Big]_\ell^0 = \lim_{\ell\to-\infty}\frac{1}{4}[-1+1/(2\ell-1)^2] = -1/4.$

11. $\lim_{\ell\to-\infty}\frac{1}{3}e^{3x}\Big]_\ell^0 = \lim_{\ell\to-\infty}\left[\frac{1}{3}-\frac{1}{3}e^{3\ell}\right] = \frac{1}{3}.$

13. $\int_{-\infty}^{+\infty} x\,dx$ converges if $\int_{-\infty}^{0} x\,dx$ and $\int_{0}^{+\infty} x\,dx$ both converge; it diverges if either (or both) diverges. $\int_0^{+\infty} x\,dx = \lim_{\ell\to+\infty}\frac{1}{2}x^2\Big]_0^\ell = \lim_{\ell\to+\infty}\frac{1}{2}\ell^2 = +\infty$, so $\int_{-\infty}^{+\infty} x\,dx$ is divergent.

15. $\int_0^{+\infty}\frac{x}{(x^2+3)^2}dx = \lim_{\ell\to+\infty} -\frac{1}{2(x^2+3)}\Big]_0^\ell = \lim_{\ell\to+\infty}\frac{1}{2}[-1/(\ell^2+3)+1/3] = \frac{1}{6}$, similarly $\int_{-\infty}^{0}\frac{x}{(x^2+3)^2}dx = -1/6$, so $\int_{-\infty}^{\infty}\frac{x}{(x^2+3)^2}dx = 1/6+(-1/6) = 0.$

17. $\lim_{\ell\to4^-} -\frac{1}{x-4}\Big]_0^\ell = \lim_{\ell\to4^-}\left[-\frac{1}{\ell-4}-\frac{1}{4}\right] = +\infty$, divergent.

19. $\lim_{\ell\to\pi/2^-} -\ln(\cos x)\Big]_0^\ell = \lim_{\ell\to\pi/2^-} -\ln(\cos\ell) = +\infty$, divergent.

21. $\lim_{\ell\to1^-}\sin^{-1}x\Big]_0^\ell = \lim_{\ell\to1^-}\sin^{-1}\ell = \pi/2.$

23. $\lim_{\ell\to\pi/3^+}\sqrt{1-2\cos x}\Big]_\ell^{\pi/2} = \lim_{\ell\to\pi/3^+}(1-\sqrt{1-2\cos\ell}) = 1.$

25. $\int_0^2\frac{dx}{x-2} = \lim_{\ell\to2^-}\ln|x-2|\Big]_0^\ell = \lim_{\ell\to2^-}(\ln|\ell-2|-\ln 2) = -\infty$, so $\int_0^3\frac{dx}{x-2}$ is divergent.

27. $\int_0^8 x^{-1/3}dx = \lim_{\ell\to0^+}\frac{3}{2}x^{2/3}\Big]_\ell^8 = \lim_{\ell\to0^+}\frac{3}{2}(4-\ell^{2/3}) = 6$, $\int_{-1}^0 x^{-1/3}dx = \lim_{\ell\to0^-}\frac{3}{2}x^{2/3}\Big]_{-1}^\ell = \lim_{\ell\to0^-}\frac{3}{2}(\ell^{2/3}-1) = -3/2$, so $\int_{-1}^8 x^{-1/3}dx = 6+(-3/2) = 9/2.$

29. $\int_0^{+\infty} \frac{1}{x^2}dx = \int_0^a \frac{1}{x^2}dx + \int_a^{+\infty} \frac{1}{x^2}dx$ where $a > 0$; take $a = 1$ for convenience, $\int_0^1 \frac{1}{x^2}dx = \lim_{\ell\to 0+} (-1/x)\Big]_\ell^1 = \lim_{\ell\to 0+} (1/\ell - 1) = +\infty$ so $\int_0^{+\infty} \frac{1}{x^2}dx$ is divergent.

31. Let $u = \sqrt{x}, x = u^2, dx = 2u\,du$. Then $\int \frac{dx}{\sqrt{x}(x+1)} = \int 2\frac{du}{u^2+1} = 2\tan^{-1} u + C = 2\tan^{-1}\sqrt{x} + C$ and $\int_0^1 \frac{dx}{\sqrt{x}(x+1)} = 2\lim_{\epsilon\to 0^+} \tan^{-1}\sqrt{x}\Big]_\epsilon^1 = 2\lim_{\epsilon\to 0^+} (\pi/4 - \tan^{-1}\sqrt{\epsilon}) = \pi/2.$