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Convex Optimization

Q1) $f(x,y,z) = x^4+y^4+z^4+x^2+y^2+z^2$

$f_x = 4x^3+2x \;,\; f_y = 4y^3+2y \;,\; f_z = 4z^3+2z$

$4x^3+2x=0$

$2x(2x^2+1)=0$

$\boxed{x=0} \quad 2x^2 \neq 1 \quad \boxed{y=0} \qquad \boxed{z=0}$

$f_{xx} = 12x^2+2 \;,\; f_{yy} = 12y^2+2 \;,\; f_{zz} = 12z^2+2$

$f_{xy} = f_{xz} = f_{yz} = 0$

$$H = \begin{bmatrix} 12x^2+2 & 0 & 0 \\ 0 & 12y^2+2 & 0 \\ 0 & 0 & 12z^2+2 \end{bmatrix}$$

CP $(0,0,0)$

$$H \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \quad D_1=2 \;,\; D_2=4 \;,\; D_3=8$$

$D_1, D_2, D_3 > 0$

so,

H is +ve definite & $(0,0,0)$ is global minima

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Q3) $f(x,y) = x^3 + y^3 - 3x - 2y$

$f_x = 3x^2 - 3$, $f_y = 3y^2 - 2$

$f_{xx} = 6x$, $f_{yy} = 6y$

$f_{xy} = f_{yx} = 0$

$H = \begin{bmatrix} 6x & 0 \\ 0 & 6y \end{bmatrix}$

if $(x,y) > 0$, then H is positive definite; ~~and as such~~

$3x^2 - 3 = 0$, $3y^2 - 2 = 0$

$x^2 - 1 = 0$, $y^2 = \frac{2}{3}$

$x = \pm 1$, $y = \pm\sqrt{\frac{2}{3}}$

ignore $x = -1$, $y = -\sqrt{2/3}$ global minimum $= \left(1, \sqrt{2/3}\right)$

Q3) $C(x,y) = 0.04x^2 - 0.01xy + 0.01y^2 + 4x + 2y + 500$

$\pi(x,y) = 13x + 8y - C(x,y)$

$\pi(x,y) = 9x + 6y - 0.04x^2 + 0.01xy - 0.01y^2 - 500$

$\pi_x = 9 - 0.08x + 0.01y \ , \ \pi_y = 6 + 0.01x - 0.02y$

$9 - 0.08x + 0.01y = 0 \ , \ 0.08x - 0.01y = 9$

$6 + 0.01x - 0.02y = 0 \ , \ 0.01x - 0.02y = -6$

$0.16x - 0.02y = 18 \quad — (1)$

$0.01x - 0.02y = -6 \quad — (2)$

$(1) - (2)$

$0.15x = 24$

$x = 160, \ y = 380$

$\pi_{xx} = -0.08, \ \pi_{yy} = -0.02$

$\pi_{xy} = \pi_{yx} = 0.01$

$$H = \begin{bmatrix} -0.08 & 0.01 \\ 0.01 & -0.02 \end{bmatrix} \qquad D_1 = -0.08 < 0, \quad D_2 = 1.5 \times 10^{-3} > 0$$

H is −ve definite, so CP (160, 380) is global maxima.

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Q4) $f(x,y,z) = x^2 + 2xy + y^2 + z^3$ $\quad S = \{(x,y,z) : z > 0\}$

$f_x = 2x + 2y$, $f_y = \cancel{2y} + 2x + 2y$, $f_z = 3z^2$

$f_{xx} = \cancel{2}2$, $f_{yy} = 2$, $f_{zz} = 6z$

$f_{xy} = 2$, $f_{xz} = 0$, $f_{yz} = 0$, $\cancel{f}$

$$H = \begin{bmatrix} 2 & \cancel{0}2 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 6z \end{bmatrix} \quad \begin{matrix} D_1 = 2 \\ D_2 = 0 \\ D_3 = 0 \end{matrix}$$

H is ~~+~~ +ve semi definite and as such convex but not strictly convex.

$f_x = 0$ $\quad \cancel{2x + 2y = 0}$

$f_y = 0$ $\quad \cancel{x = -y}$

$f_z = 0$ $\quad 3z^2 = 0$ (not possible, since $z > 0$)

So no stationary points. And no boundary points since ~~it~~ S is an open set.

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Q5) $f(x,y,z) = x^4 + y^4 + z^4 + x^2 - xy + y^2 + yz + z^2$

$f_x = 4x^3 + 2x - y$, $f_y = 4y^3 \cancel{\;} + 2y - x + z$, $f_z = 4z^3 + 2z + y$

$f_{xx} = 12x^2 + 2$, $f_{yy} = 12y^2 + 2$, $f_{zz} = 12z^2 + 2$

$f_{xy} = -1$, $f_{xz} = 0$, $f_{yz} = 1$

$$H = \begin{bmatrix} 12x^2+2 & -1 & 0 \\ -1 & 12y^2+2 & 1 \\ 0 & 1 & 12z^2+2 \end{bmatrix}$$

$D_1 = 12x^2 + 2 > 0$

$D_2 = (12y^2+2)(12x^2+2) - 1$

$= 144x^2y^2 + 24x^2 + 24y^2 + 4 - 1$

$= 144x^2y^2 + 24x^2 + 24y^2 + 3 > 0$

$\cancel{D_3 = -1(12x^2+2) + (12z^2+2)D_2}$

$\cancel{= 1728x^2y^2z^2 + 288(x^2y^2 + x^2z^2 + y^2z^2)}$

$D_3 = -1(12x^2+2) + (12z^2+2)D_2$

$= 1728x^2y^2z^2 + 288(x^2y^2 + x^2z^2 + y^2z^2) + 36x^2 + 48y^2 + 36z^2 + 4 > 0$

$D_1, D_2, D_3 > 0$

$\therefore$ H is +ve definite so is strictly convex.

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Q6) $\therefore f(x,y,z) = -2x^4 + 2yz - y^2 + 8x$

$f_x = -8x^3 + 8$, $f_y = 2z - 2y$, $f_z = 2y$

$-8x^3 + 8 = 0$ $\quad$ $2z - 2y = 0$ $\quad$ $2y = 0$

$x^3 - 1 = 0$ $\quad$ $z = y$ $\quad$ $\boxed{y = 0}$

$\boxed{x = 1}$ $\quad$ $\boxed{z = 0}$

CP $(1,0,0)$

$f_{xx} = -24x^2$, $f_{yy} = -2$, $f_{zz} = 0$

$f_{xy} = 0$, $f_{xz} = 0$, $f_{yz} = 2$

$$H = \begin{bmatrix} -24x^2 & 0 & 0 \\ 0 & -2 & 2 \\ 0 & 2 & 0 \end{bmatrix} = \begin{bmatrix} -24 & 0 & 0 \\ 0 & -2 & 2 \\ 0 & 2 & 0 \end{bmatrix}$$

$D_1 = -24 < 0$

$D_2 = 48 > 0$

$D_3 = 96 > 0$

H is not Definite so CP$(1,0,0)$ is saddle point.

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Q7) $f(x,y,z) = x^2 + y^2 + 3z^2 - xy + 2xz + yz$

$f_x = 2x - y + 2z$, $f_y = 2y - x + z$, $f_z = 6z + 2x + y$

$f_{xx} = 2$, $f_{yy} = 2$, $f_{zz} = 6$

$f_{xy} = -1$, $f_{xz} = 2$, $f_{yz} = 1$

$$H = \begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & 1 \\ 2 & 1 & 6 \end{bmatrix}$$

$D_1 = 2 > 0$

$D_2 = 3 > 0$

$D_3 = 22 - 8 - 10$

$= 4 > 0$

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H is +ve definite, and since we only need to prove the CP is minima and not find the point.
H being +ve definite means it is minima

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Q.8 $f(x,y) = x^3 - 3xy + y^3$

$f_x = 3x^2 - 3y$, $f_y = 3y^2 - 3x$

$3x^2 - 3y = 0$, $3y^2 - 3x = 0$

$x^2 = y$, $y^2 = x$

$y^4 = y$

$y^4 - y = 0$

$y(y^3 - 1) = 0$

$y = 0, y = 1$ $x = 0, x = 1$

$(0,0), (1,1)$

$f_{xx} = 6x$, $f_{yy} = 6y$

$f_{xy} = -3$

$$H = \begin{bmatrix} 6x & -3 \\ -3 & 6y \end{bmatrix}$$

$D_1 = 6x$

$D_2 = 36xy - 9$

for $(0,0)$:-

$D_1 = 0$

$D_2 = -9 < 0$

$(0,0)$ is saddle point

for $(1,1)$:-

$D_1 = 6 > 0$

$D_2 = 27 > 0$

H is +ve definite , $(1,1)$ is local min.

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Q9) $f(x,y,z) = x^3 + 3xy + 3xz + y^3 + 3yz + z^3$

$f_x = 3x^2 + 3y + 3z$, $f_y = 3y^2 + 3x + 3z$, $f_z = 3z^2 + 3x + 3y$

$f_x = 0$, $f_y = 0$, $f_z = 0$

$x^2 + y + z = 0$, $x + y^2 + z = 0$, $x + y + z^2 = 0$

$z = -x^2 - y$

$x + y^2 - x^2 - y = 0$

$x - y = x^2 - y^2$

$x - y = (x-y)(x+y)$

$x - y = 0$

$x + y = 1$ (Impossible as, if we put it in $x + y + z^2 = 0$, it will become $z^2 + 1 = 0$, which is not possible.)

$x = y$

$z = -x^2 - \cancel{y}x$

$\cancel{x} - x + x + (x^2+x)^2 = 0$

$2x + x^4 + 2x^3 + x^2 = 0$

$x^4 + 2x^3 + x^2 + 2x = 0$

$x(x^3 + 2x^2 + x + 2) = 0$

$x^3 + 2x^2 + x + 2 = 0$

$x^2(x+2) + (x+2) = 0$

$(x^2+1)(x+2) = 0$

$x = 0$, $y = 0$, $z = 0$

$\boxed{x = -2}$, $y = -2$, $z = -2$

+135
+135

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CP $(0,0,0)$, $(-2,-2,-2)$

$f_{xx} = 6x$, $f_{yy} = 6y$, $f_{zz} = 6z$

$f_{xy} = 3$, $f_{xz} = 3$, $f_{yz} = 3$

$$H = \begin{bmatrix} 6x & 3 & 3 \\ 3 & 6y & 3 \\ 3 & 3 & 6z \end{bmatrix}$$

for $(0,0,0)$:-

$$H = \begin{bmatrix} 0 & 3 & 3 \\ 3 & 0 & 3 \\ 3 & 3 & 0 \end{bmatrix}$$

$D_1 = 0$

$D_2 = -9 < 0$

H is ~~in de~~ indefinite, so it is saddle point

for $(-2,-2,-2)$:-

$$H = \begin{bmatrix} -12 & 3 & 3 \\ 3 & -12 & 3 \\ 3 & 3 & -12 \end{bmatrix}$$

$D_1 = -12 < 0$

$D_2 = 135 > 0$

$D_3 = -1350 < 0$

$-ve$ definite, ~~local~~ $(-2,-2,-2)$ is local max.

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