35–36 Find the amplitude and period, and sketch at least two periods of the graph by hand. If you have a graphing utility, use it to check your work. ■

35. (a) $y = 3\sin 4x$ (b) $y = -2\cos \pi x$
(c) $y = 2 + \cos\left(\dfrac{x}{2}\right)$

36. (a) $y = -1 - 4\sin 2x$ (b) $y = \frac{1}{2}\cos(3x - \pi)$
(c) $y = -4\sin\left(\dfrac{x}{3} + 2\pi\right)$

37. Equations of the form

$$x = A_1 \sin \omega t + A_2 \cos \omega t$$

arise in the study of vibrations and other periodic motion. Express the equation

$$x = 5\sqrt{3}\sin 2\pi t + \tfrac{5}{2}\cos 2\pi t$$

in the form $x = A\sin(\omega t + \theta)$, and use a graphing utility to confirm that both equations have the same graph.

38. Determine the number of solutions of $x = 2\sin x$, and use a graphing or calculating utility to estimate them.

✔ QUICK CHECK ANSWERS 0.3

1. even; odd; negative **2.** $(-\infty, +\infty)$ **3.** (a) $[0, +\infty)$ (b) $(-\infty, +\infty)$ (c) $(0, +\infty)$ (d) $(-\infty, 0) \cup (0, +\infty)$ **4.** (a) algebraic (b) polynomial (c) not algebraic (d) rational (e) rational **5.** $|A|$; $2\pi/|B|$

0.4 INVERSE FUNCTIONS; INVERSE TRIGONOMETRIC FUNCTIONS

In everyday language the term "inversion" conveys the idea of a reversal. For example, in meteorology a temperature inversion is a reversal in the usual temperature properties of air layers, and in music a melodic inversion reverses an ascending interval to the corresponding descending interval. In mathematics the term ***inverse*** *is used to describe functions that reverse one another in the sense that each undoes the effect of the other. In this section we discuss this fundamental mathematical idea. In particular, we introduce inverse trigonometric functions to address the problem of recovering an angle that could produce a given trigonometric function value.*

■ INVERSE FUNCTIONS

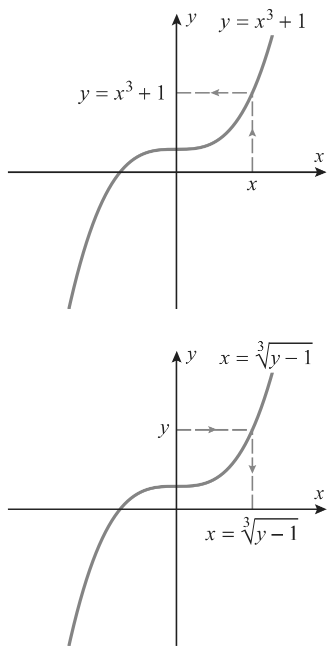
The idea of solving an equation $y = f(x)$ for x as a function of y, say $x = g(y)$, is one of the most important ideas in mathematics. Sometimes, solving an equation is a simple process; for example, using basic algebra the equation

$$y = x^3 + 1 \qquad \boxed{y = f(x)}$$

can be solved for x as a function of y:

$$x = \sqrt[3]{y - 1} \qquad \boxed{x = g(y)}$$

The first equation is better for computing y if x is known, and the second is better for computing x if y is known (Figure 0.4.1).



▲ **Figure 0.4.1**

Our primary interest in this section is to identify relationships that may exist between the functions f and g when an equation $y = f(x)$ is expressed as $x = g(y)$, or conversely. For example, consider the functions $f(x) = x^3 + 1$ and $g(y) = \sqrt[3]{y - 1}$ discussed above. When these functions are composed in either order, they cancel out the effect of one another in the sense that

$$\begin{aligned} g(f(x)) &= \sqrt[3]{f(x) - 1} = \sqrt[3]{(x^3 + 1) - 1} = x \\ f(g(y)) &= [g(y)]^3 + 1 = (\sqrt[3]{y - 1})^3 + 1 = y \end{aligned} \tag{1}$$

Pairs of functions with these two properties are so important that there is special terminology for them.

0.4.1 DEFINITION If the functions f and g satisfy the two conditions

$$g(f(x)) = x \text{ for every } x \text{ in the domain of } f$$
$$f(g(y)) = y \text{ for every } y \text{ in the domain of } g$$

then we say that ***f is an inverse of g*** and ***g is an inverse of f*** or that ***f and g are inverse functions***.

WARNING

If f is a function, then the -1 in the symbol f^{-1} always denotes an inverse and *never* an exponent. That is,

$$f^{-1}(x) \quad \textit{never means} \quad \frac{1}{f(x)}$$

It can be shown (Exercise 62) that if a function f has an inverse, then that inverse is unique. Thus, if a function f has an inverse, then we are entitled to talk about "the" inverse of f, in which case we denote it by the symbol f^{-1}.

▶ **Example 1** The computations in (1) show that $g(y) = \sqrt[3]{y-1}$ is the inverse of $f(x) = x^3 + 1$. Thus, we can express g in inverse notation as

$$f^{-1}(y) = \sqrt[3]{y-1}$$

and we can express the equations in Definition 0.4.1 as

$$\begin{aligned} f^{-1}(f(x)) &= x \quad \text{for every } x \text{ in the domain of } f \\ f(f^{-1}(y)) &= y \quad \text{for every } y \text{ in the domain of } f^{-1} \end{aligned} \tag{2}$$

We will call these the ***cancellation equations*** for f and f^{-1}. ◀

CHANGING THE INDEPENDENT VARIABLE

The formulas in (2) use x as the independent variable for f and y as the independent variable for f^{-1}. Although it is often convenient to use different independent variables for f and f^{-1}, there will be occasions on which it is desirable to use the same independent variable for both. For example, if we want to graph the functions f and f^{-1} together in the same xy-coordinate system, then we would want to use x as the independent variable and y as the dependent variable for both functions. Thus, to graph the functions $f(x) = x^3 + 1$ and $f^{-1}(y) = \sqrt[3]{y-1}$ of Example 1 in the same xy-coordinate system, we would change the independent variable y to x, use y as the dependent variable for both functions, and graph the equations

$$y = x^3 + 1 \quad \text{and} \quad y = \sqrt[3]{x-1}$$

We will talk more about graphs of inverse functions later in this section, but for reference we give the following reformulation of the cancellation equations in (2) using x as the independent variable for both f and f^{-1}:

$$\begin{aligned} f^{-1}(f(x)) &= x \quad \text{for every } x \text{ in the domain of } f \\ f(f^{-1}(x)) &= x \quad \text{for every } x \text{ in the domain of } f^{-1} \end{aligned} \tag{3}$$

▶ **Example 2** Confirm each of the following.

(a) The inverse of $f(x) = 2x$ is $f^{-1}(x) = \frac{1}{2}x$.

(b) The inverse of $f(x) = x^3$ is $f^{-1}(x) = x^{1/3}$.

The results in Example 2 should make sense to you intuitively, since the operations of multiplying by 2 and multiplying by $\frac{1}{2}$ in either order cancel the effect of one another, as do the operations of cubing and taking a cube root.

Solution (a).

$$f^{-1}(f(x)) = f^{-1}(2x) = \tfrac{1}{2}(2x) = x$$
$$f(f^{-1}(x)) = f\left(\tfrac{1}{2}x\right) = 2\left(\tfrac{1}{2}x\right) = x$$

Solution (b).

$$f^{-1}(f(x)) = f^{-1}(x^3) = \left(x^3\right)^{1/3} = x$$
$$f(f^{-1}(x)) = f(x^{1/3}) = \left(x^{1/3}\right)^3 = x \quad \blacktriangleleft$$

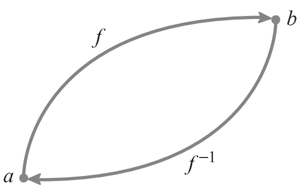
In general, if a function f has an inverse and $f(a) = b$, then the procedure in Example 3 shows that $a = f^{-1}(b)$; that is, f^{-1} maps each output of f back into the corresponding input (Figure 0.4.2).

▶ **Example 3** Given that the function f has an inverse and that $f(3) = 5$, find $f^{-1}(5)$.

Solution. Apply f^{-1} to both sides of the equation $f(3) = 5$ to obtain

$$f^{-1}(f(3)) = f^{-1}(5)$$

and now apply the first equation in (3) to conclude that $f^{-1}(5) = 3$. ◀



▲ **Figure 0.4.2** If f maps a to b, then f^{-1} maps b back to a.

DOMAIN AND RANGE OF INVERSE FUNCTIONS

The equations in (3) imply the following relationships between the domains and ranges of f and f^{-1}:

$$\begin{aligned} &\text{domain of } f^{-1} = \text{range of } f \\ &\text{range of } f^{-1} = \text{domain of } f \end{aligned} \tag{4}$$

One way to show that two sets are the same is to show that each is a subset of the other. Thus we can establish the first equality in (4) by showing that the domain of f^{-1} is a subset of the range of f and that the range of f is a subset of the domain of f^{-1}. We do this as follows: The first equation in (3) implies that f^{-1} is defined at $f(x)$ for all values of x in the domain of f, and this implies that the range of f is a subset of the domain of f^{-1}. Conversely, if x is in the domain of f^{-1}, then the second equation in (3) implies that x is in the range of f because it is the image of $f^{-1}(x)$. Thus, the domain of f^{-1} is a subset of the range of f. We leave the proof of the second equation in (4) as an exercise.

A METHOD FOR FINDING INVERSE FUNCTIONS

At the beginning of this section we observed that solving $y = f(x) = x^3 + 1$ for x as a function of y produces $x = f^{-1}(y) = \sqrt[3]{y-1}$. The following theorem shows that this is not accidental.

0.4.2 THEOREM *If an equation $y = f(x)$ can be solved for x as a function of y, say $x = g(y)$, then f has an inverse and that inverse is $g(y) = f^{-1}(y)$.*

PROOF Substituting $y = f(x)$ into $x = g(y)$ yields $x = g(f(x))$, which confirms the first equation in Definition 0.4.1, and substituting $x = g(y)$ into $y = f(x)$ yields $y = f(g(y))$, which confirms the second equation in Definition 0.4.1. ■

Theorem 0.4.2 provides us with the following procedure for finding the inverse of a function.

A Procedure for Finding the Inverse of a Function f

Step 1. Write down the equation $y = f(x)$.

Step 2. If possible, solve this equation for x as a function of y.

Step 3. The resulting equation will be $x = f^{-1}(y)$, which provides a formula for f^{-1} with y as the independent variable.

Step 4. If y is acceptable as the independent variable for the inverse function, then you are done, but if you want to have x as the independent variable, then you need to interchange x and y in the equation $x = f^{-1}(y)$ to obtain $y = f^{-1}(x)$.

An alternative way to obtain a formula for $f^{-1}(x)$ with x as the independent variable is to reverse the roles of x and y at the outset and solve the equation $x = f(y)$ for y as a function of x.

▶ **Example 4** Find a formula for the inverse of $f(x) = \sqrt{3x-2}$ with x as the independent variable, and state the domain of f^{-1}.

Solution. Following the procedure stated above, we first write

$$y = \sqrt{3x-2}$$

Then we solve this equation for x as a function of y:

$$y^2 = 3x - 2$$
$$x = \tfrac{1}{3}(y^2+2)$$

which tells us that

$$f^{-1}(y) = \tfrac{1}{3}(y^2+2) \tag{5}$$

Since we want x to be the independent variable, we reverse x and y in (5) to produce the formula

$$f^{-1}(x) = \tfrac{1}{3}(x^2+2) \tag{6}$$

We know from (4) that the domain of f^{-1} is the range of f. In general, this need not be the same as the natural domain of the fomula for f^{-1}. Indeed, in this example the natural domain of (6) is $(-\infty, +\infty)$, whereas the range of $f(x) = \sqrt{3x-2}$ is $[0, +\infty)$. Thus, if we want to make the domain of f^{-1} clear, we must express it explicitly by rewriting (6) as

$$f^{-1}(x) = \tfrac{1}{3}(x^2+2), \quad x \geq 0$$ ◀

EXISTENCE OF INVERSE FUNCTIONS

The procedure we gave above for finding the inverse of a function f was based on solving the equation $y = f(x)$ for x as a function of y. This procedure can fail for two reasons—the function f may not have an inverse, or it may have an inverse but the equation $y = f(x)$ cannot be solved explicitly for x as a function of y. Thus, it is important to establish conditions that ensure the existence of an inverse, even if it cannot be found explicitly.

If a function f has an inverse, then it must assign distinct outputs to distinct inputs. For example, the function $f(x) = x^2$ cannot have an inverse because it assigns the same value to $x = 2$ and $x = -2$, namely,

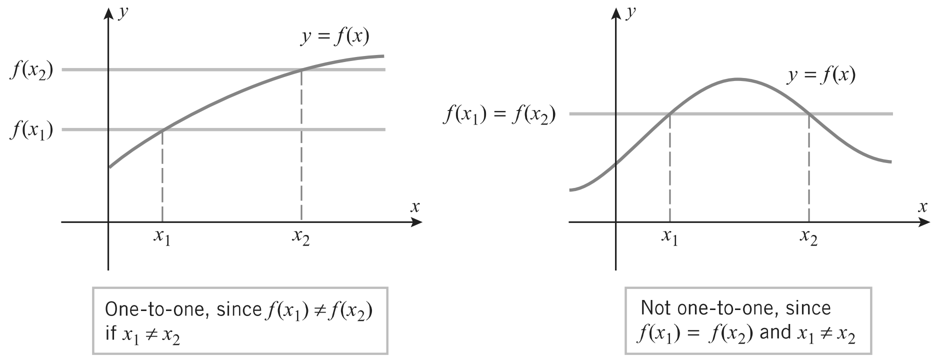
$$f(2) = f(-2) = 4$$

Thus, if $f(x) = x^2$ were to have an inverse, then the equation $f(2) = 4$ would imply that $f^{-1}(4) = 2$, and the equation $f(-2) = 4$ would imply that $f^{-1}(4) = -2$. But this is impossible because $f^{-1}(4)$ cannot have two different values. Another way to see that $f(x) = x^2$ has no inverse is to attempt to find the inverse by solving the equation $y = x^2$ for x as a function of y. We run into trouble immediately because the resulting equation $x = \pm\sqrt{y}$ does not express x as a *single* function of y.

A function that assigns distinct outputs to distinct inputs is said to be ***one-to-one*** or ***invertible***, so we know from the preceding discussion that if a function f has an inverse, then it must be one-to-one. The converse is also true, thereby establishing the following theorem.

0.4.3 THEOREM *A function has an inverse if and only if it is one-to-one.*

Stated algebraically, a function f is one-to-one if and only if $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$; stated geometrically, a function f is one-to-one if and only if the graph of $y = f(x)$ is cut at most once by any horizontal line (Figure 0.4.3). The latter statement together with Theorem 0.4.3 provides the following geometric test for determining whether a function has an inverse.

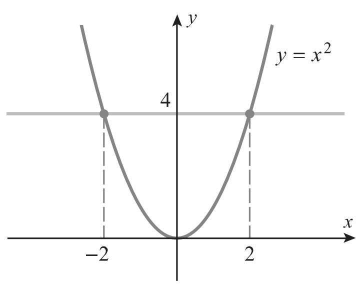


▶ **Figure 0.4.3**

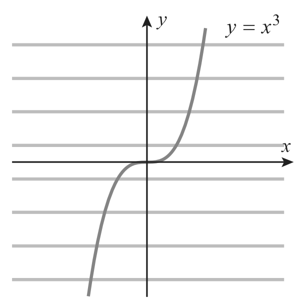
> **0.4.4 THEOREM (*The Horizontal Line Test*)** *A function has an inverse function if and only if its graph is cut at most once by any horizontal line.*

▶ **Example 5** Use the horizontal line test to show that $f(x) = x^2$ has no inverse but that $f(x) = x^3$ does.

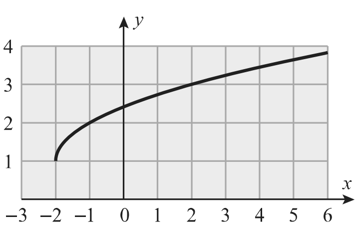
Solution. Figure 0.4.4 shows a horizontal line that cuts the graph of $y = x^2$ more than once, so $f(x) = x^2$ is not invertible. Figure 0.4.5 shows that the graph of $y = x^3$ is cut at most once by any horizontal line, so $f(x) = x^3$ is invertible. [Recall from Example 2 that the inverse of $f(x) = x^3$ is $f^{-1}(x) = x^{1/3}$.] ◀



▲ **Figure 0.4.4**



▲ **Figure 0.4.5**



▲ **Figure 0.4.6**

▶ **Example 6** Explain why the function f that is graphed in Figure 0.4.6 has an inverse, and find $f^{-1}(3)$.

Solution. The function f has an inverse since its graph passes the horizontal line test. To evaluate $f^{-1}(3)$, we view $f^{-1}(3)$ as that number x for which $f(x) = 3$. From the graph we see that $f(2) = 3$, so $f^{-1}(3) = 2$. ◀

> The function $f(x) = x^3$ in Figure 0.4.5 is an example of an increasing function. Give an example of a decreasing function and compute its inverse.

INCREASING OR DECREASING FUNCTIONS ARE INVERTIBLE

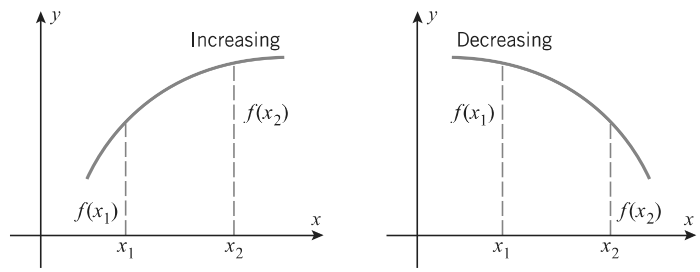
A function whose graph is always rising as it is traversed from left to right is said to be an ***increasing function***, and a function whose graph is always falling as it is traversed from left to right is said to be a ***decreasing function***. If x_1 and x_2 are points in the domain of a function f, then f is increasing if

$$f(x_1) < f(x_2) \quad \text{whenever } x_1 < x_2$$

and f is decreasing if

$$f(x_1) > f(x_2) \quad \text{whenever } x_1 < x_2$$

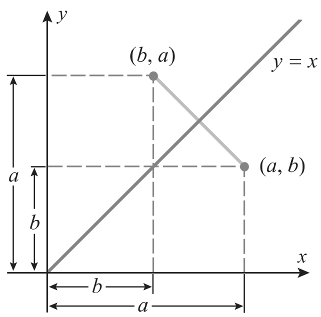
(Figure 0.4.7). It is evident geometrically that increasing and decreasing functions pass the horizontal line test and hence are invertible.



$f(x_1) < f(x_2)$ if $x_1 < x_2$

$f(x_1) > f(x_2)$ if $x_1 < x_2$

▶ **Figure 0.4.7**



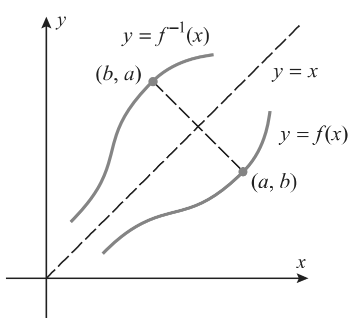
The points (a, b) and (b, a) are reflections about $y = x$.

▲ **Figure 0.4.8**

GRAPHS OF INVERSE FUNCTIONS

Our next objective is to explore the relationship between the graphs of f and f^{-1}. For this purpose, it will be desirable to use x as the independent variable for both functions so we can compare the graphs of $y = f(x)$ and $y = f^{-1}(x)$.

If (a, b) is a point on the graph $y = f(x)$, then $b = f(a)$. This is equivalent to the statement that $a = f^{-1}(b)$, which means that (b, a) is a point on the graph of $y = f^{-1}(x)$. In short, reversing the coordinates of a point on the graph of f produces a point on the graph of f^{-1}. Similarly, reversing the coordinates of a point on the graph of f^{-1} produces a point on the graph of f (verify). However, the geometric effect of reversing the coordinates of a point is to reflect that point about the line $y = x$ (Figure 0.4.8), and hence the graphs of $y = f(x)$ and $y = f^{-1}(x)$ are reflections of one another about this line (Figure 0.4.9). In summary, we have the following result.

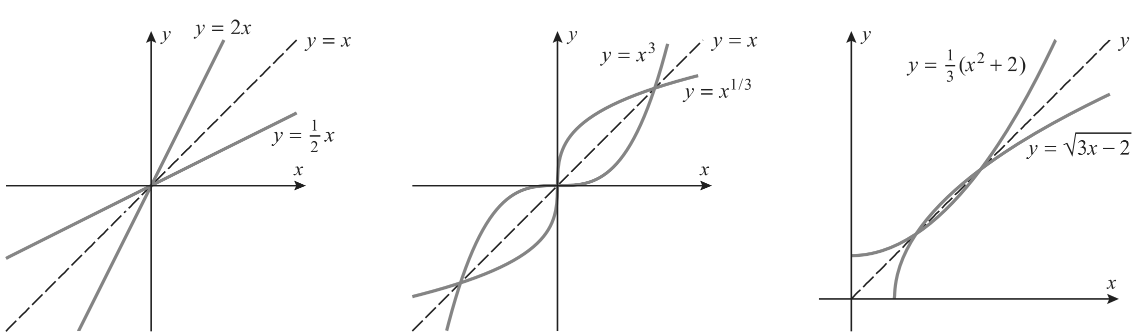


The graphs of f and f^{-1} are reflections about $y = x$.

▲ **Figure 0.4.9**

> **0.4.5 THEOREM** *If f has an inverse, then the graphs of $y = f(x)$ and $y = f^{-1}(x)$ are reflections of one another about the line $y = x$; that is, each graph is the mirror image of the other with respect to that line.*

▶ **Example 7** Figure 0.4.10 shows the graphs of the inverse functions discussed in Examples 2 and 4. ◀



▲ **Figure 0.4.10**

RESTRICTING DOMAINS FOR INVERTIBILITY

If a function g is obtained from a function f by placing restrictions on the domain of f, then g is called a ***restriction*** of f. Thus, for example, the function

$$g(x) = x^3, \quad x \geq 0$$

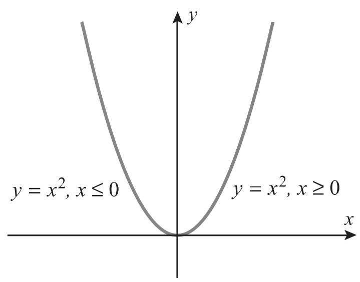
is a restriction of the function $f(x) = x^3$. More precisely, it is called the restriction of x^3 to the interval $[0, +\infty)$.

Sometimes it is possible to create an invertible function from a function that is not invertible by restricting the domain appropriately. For example, we showed earlier that $f(x) = x^2$ is not invertible. However, consider the restricted functions

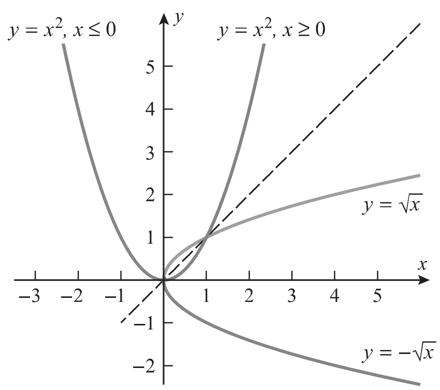
$$f_1(x) = x^2, \quad x \geq 0 \quad \text{and} \quad f_2(x) = x^2, \quad x \leq 0$$

the union of whose graphs is the complete graph of $f(x) = x^2$ (Figure 0.4.11). These restricted functions are each one-to-one (hence invertible), since their graphs pass the horizontal line test. As illustrated in Figure 0.4.12, their inverses are

$$f_1^{-1}(x) = \sqrt{x} \quad \text{and} \quad f_2^{-1}(x) = -\sqrt{x}$$



▲ **Figure 0.4.11**



▲ **Figure 0.4.12**

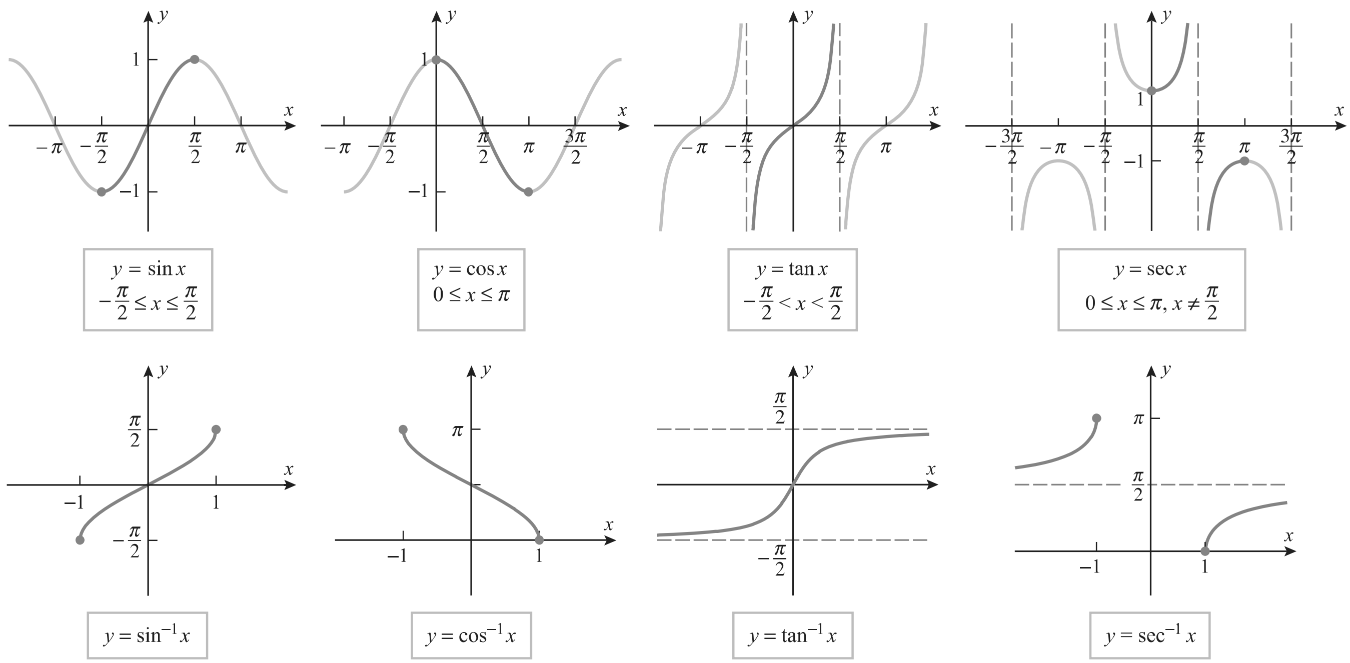
INVERSE TRIGONOMETRIC FUNCTIONS

A common problem in trigonometry is to find an angle x using a known value of $\sin x$, $\cos x$, or some other trigonometric function. Recall that problems of this type involve the computation of "arc functions" such as $\arcsin x$, $\arccos x$, and so forth. We will conclude this section by studying these arc functions from the viewpoint of general inverse functions.

The six basic trigonometric functions do not have inverses because their graphs repeat periodically and hence do not pass the horizontal line test. To circumvent this problem we will restrict the domains of the trigonometric functions to produce one-to-one functions and then define the "inverse trigonometric functions" to be the inverses of these restricted functions. The top part of Figure 0.4.13 shows geometrically how these restrictions are made for $\sin x$, $\cos x$, $\tan x$, and $\sec x$, and the bottom part of the figure shows the graphs of the corresponding inverse functions

$$\sin^{-1} x, \quad \cos^{-1} x, \quad \tan^{-1} x, \quad \sec^{-1} x$$

(also denoted by $\arcsin x$, $\arccos x$, $\arctan x$, and $\text{arcsec}\, x$). Inverses of $\cot x$ and $\csc x$ are of lesser importance and will be considered in the exercises.



▲ **Figure 0.4.13**

If you have trouble visualizing the correspondence between the top and bottom parts of Figure 0.4.13, keep in mind that a reflection about $y = x$ converts vertical lines into horizontal lines, and vice versa; and it converts x-intercepts into y-intercepts, and vice versa.

The following formal definitions summarize the preceding discussion.

0.4.6 DEFINITION The ***inverse sine function***, denoted by $\sin^{-1}$, is defined to be the inverse of the restricted sine function

$$\sin x, \quad -\pi/2 \le x \le \pi/2$$

0.4.7 DEFINITION The ***inverse cosine function***, denoted by $\cos^{-1}$, is defined to be the inverse of the restricted cosine function

$$\cos x, \quad 0 \le x \le \pi$$

0.4.8 DEFINITION The ***inverse tangent function***, denoted by $\tan^{-1}$, is defined to be the inverse of the restricted tangent function

$$\tan x, \quad -\pi/2 < x < \pi/2$$

WARNING

The notations $\sin^{-1} x, \cos^{-1} x, \ldots$ are reserved exclusively for the inverse trigonometric functions and are not used for reciprocals of the trigonometric functions. If we want to express the reciprocal $1/\sin x$ using an exponent, we would write $(\sin x)^{-1}$ and *never* $\sin^{-1} x$.

0.4.9 DEFINITION* The ***inverse secant function***, denoted by $\sec^{-1}$, is defined to be the inverse of the restricted secant function

$$\sec x, \quad 0 \le x \le \pi \text{ with } x \ne \pi/2$$

*There is no universal agreement on the definition of $\sec^{-1} x$, and some mathematicians prefer to restrict the domain of $\sec x$ so that $0 \le x < \pi/2$ or $\pi \le x < 3\pi/2$, which was the definition used in some earlier editions of this text. Each definition has advantages and disadvantages, but we will use the current definition to conform with the conventions used by the CAS programs *Mathematica*, *Maple*, and *Sage*.

Table 0.4.1 summarizes the basic properties of the inverse trigonometric functions we have considered. You should confirm that the domains and ranges listed in this table are consistent with the graphs shown in Figure 0.4.13.

Table 0.4.1
PROPERTIES OF INVERSE TRIGONOMETRIC FUNCTIONS

FUNCTION	DOMAIN	RANGE	BASIC RELATIONSHIPS
$\sin^{-1}$	$[-1, 1]$	$[-\pi/2, \pi/2]$	$\sin^{-1}(\sin x) = x$ if $-\pi/2 \le x \le \pi/2$ $\sin(\sin^{-1} x) = x$ if $-1 \le x \le 1$
$\cos^{-1}$	$[-1, 1]$	$[0, \pi]$	$\cos^{-1}(\cos x) = x$ if $0 \le x \le \pi$ $\cos(\cos^{-1} x) = x$ if $-1 \le x \le 1$
$\tan^{-1}$	$(-\infty, +\infty)$	$(-\pi/2, \pi/2)$	$\tan^{-1}(\tan x) = x$ if $-\pi/2 < x < \pi/2$ $\tan(\tan^{-1} x) = x$ if $-\infty < x < +\infty$
$\sec^{-1}$	$(-\infty, -1] \cup [1, +\infty)$	$[0, \pi/2) \cup (\pi/2, \pi]$	$\sec^{-1}(\sec x) = x$ if $0 \le x \le \pi, x \ne \pi/2$ $\sec(\sec^{-1} x) = x$ if $\lvert x\rvert \ge 1$

EVALUATING INVERSE TRIGONOMETRIC FUNCTIONS

A common problem in trigonometry is to find an angle whose sine is known. For example, you might want to find an angle x in radian measure such that

$$\sin x = \tfrac{1}{2} \tag{7}$$

and, more generally, for a given value of y in the interval $-1 \le y \le 1$ you might want to solve the equation

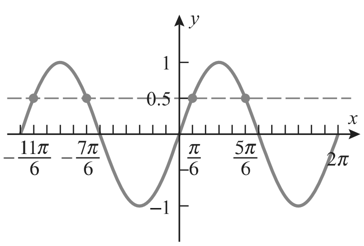
$$\sin x = y \tag{8}$$

Because $\sin x$ repeats periodically, this equation has infinitely many solutions for x; however, if we solve this equation as

$$x = \sin^{-1} y$$

then we isolate the specific solution that lies in the interval $[-\pi/2, \pi/2]$, since this is the range of the inverse sine. For example, Figure 0.4.14 shows four solutions of Equation (7), namely, $-11\pi/6$, $-7\pi/6$, $\pi/6$, and $5\pi/6$. Of these, $\pi/6$ is the solution in the interval $[-\pi/2, \pi/2]$, so

$$\sin^{-1}\left(\tfrac{1}{2}\right) = \pi/6 \tag{9}$$



▲ **Figure 0.4.14**

TECHNOLOGY MASTERY

Refer to the documentation for your calculating utility to determine how to calculate inverse sines, inverse cosines, and inverse tangents; and then confirm Equation (9) numerically by showing that

$$\sin^{-1}(0.5) \approx 0.523598775598\ldots \approx \pi/6$$

In general, if we view $x = \sin^{-1} y$ as an angle in radian measure whose sine is y, then the restriction $-\pi/2 \le x \le \pi/2$ imposes the geometric requirement that the angle x in standard position terminate in either the first or fourth quadrant or on an axis adjacent to those quadrants.

If $x = \cos^{-1} y$ is viewed as an angle in radian measure whose cosine is y, in what possible quadrants can x lie? Answer the same question for

$$x = \tan^{-1} y \quad \text{and} \quad x = \sec^{-1} y$$

▶ **Example 8** Find exact values of

$$\text{(a)}\ \sin^{-1}(1/\sqrt{2}) \qquad \text{(b)}\ \sin^{-1}(-1)$$

by inspection, and confirm your results numerically using a calculating utility.

Solution (a). Because $\sin^{-1}(1/\sqrt{2}) > 0$, we can view $x = \sin^{-1}(1/\sqrt{2})$ as that angle in the first quadrant such that $\sin\theta = 1/\sqrt{2}$. Thus, $\sin^{-1}(1/\sqrt{2}) = \pi/4$. You can confirm this with your calculating utility by showing that $\sin^{-1}(1/\sqrt{2}) \approx 0.785 \approx \pi/4$.

Solution (b). Because $\sin^{-1}(-1) < 0$, we can view $x = \sin^{-1}(-1)$ as an angle in the fourth quadrant (or an adjacent axis) such that $\sin x = -1$. Thus, $\sin^{-1}(-1) = -\pi/2$. You can confirm this with your calculating utility by showing that $\sin^{-1}(-1) \approx -1.57 \approx -\pi/2$. ◀

TECHNOLOGY MASTERY Most calculators do not provide a direct method for calculating inverse secants. In such situations the identity

$$\sec^{-1} x = \cos^{-1}(1/x) \tag{10}$$

is useful (Exercise 50). Use this formula to show that

$$\sec^{-1}(2.25) \approx 1.11 \quad \text{and} \quad \sec^{-1}(-2.25) \approx 2.03$$

If you have a calculating utility (such as a CAS) that can find $\sec^{-1} x$ directly, use it to check these values.

IDENTITIES FOR INVERSE TRIGONOMETRIC FUNCTIONS

If we interpret $\sin^{-1} x$ as an angle in radian measure whose sine is x, and if that angle is *nonnegative*, then we can represent $\sin^{-1} x$ geometrically as an angle in a right triangle in which the hypotenuse has length 1 and the side opposite to the angle $\sin^{-1} x$ has length x (Figure 0.4.15a). Moreover, the unlabeled acute angle in Figure 0.4.15a is $\cos^{-1} x$, since the cosine of that angle is x, and the unlabeled side in that figure has length $\sqrt{1-x^2}$ by the Theorem of Pythagoras (Figure 0.4.15b). This triangle motivates a number of useful identities involving inverse trigonometric functions that are valid for $-1 \le x \le 1$; for example,

$$\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2} \tag{11}$$

$$\cos(\sin^{-1} x) = \sqrt{1-x^2} \tag{12}$$

$$\sin(\cos^{-1} x) = \sqrt{1-x^2} \tag{13}$$

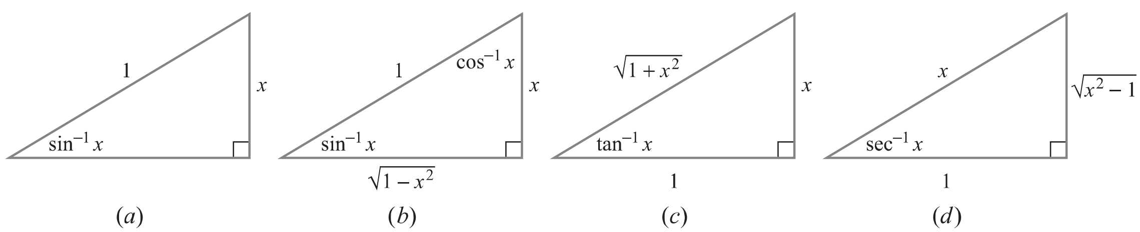
$$\tan(\sin^{-1} x) = \frac{x}{\sqrt{1-x^2}} \tag{14}$$

There is little to be gained by memorizing these identities. What is important is the mastery of the *method* used to obtain them.

In a similar manner, $\tan^{-1} x$ and $\sec^{-1} x$ can be represented as angles in the right triangles shown in Figures 0.4.15c and 0.4.15d (verify). Those triangles reveal additional useful identities; for example,

$$\sec(\tan^{-1} x) = \sqrt{1+x^2} \tag{15}$$

$$\sin(\sec^{-1} x) = \frac{\sqrt{x^2-1}}{x} \quad (x \ge 1) \tag{16}$$



▲ **Figure 0.4.15**

REMARK The triangle technique does not always produce the most general form of an identity. For example, in Exercise 61 we will ask you to derive the following extension of Formula (16) that is valid for $x \le -1$ as well as $x \ge 1$:

$$\sin(\sec^{-1} x) = \frac{\sqrt{x^2-1}}{|x|} \quad (|x| \ge 1) \tag{17}$$

Referring to Figure 0.4.13, observe that the inverse sine and inverse tangent are odd functions; that is,

$$\sin^{-1}(-x) = -\sin^{-1}(x) \quad \text{and} \quad \tan^{-1}(-x) = -\tan^{-1}(x) \tag{18–19}$$

▶ **Example 9** Figure 0.4.16 shows a computer-generated graph of $y = \sin^{-1}(\sin x)$. One might think that this graph should be the line $y = x$, since $\sin^{-1}(\sin x) = x$. Why isn't it?

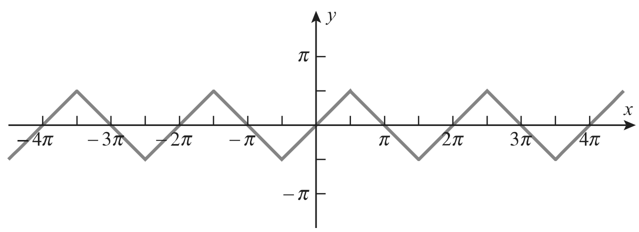
Solution. The relationship $\sin^{-1}(\sin x) = x$ is valid on the interval $-\pi/2 \le x \le \pi/2$, so we can say with certainty that the graphs of $y = \sin^{-1}(\sin x)$ and $y = x$ coincide on this interval (which is confirmed by Figure 0.4.16). However, outside of this interval the relationship $\sin^{-1}(\sin x) = x$ does not hold. For example, if the quantity x lies in the interval $\pi/2 \le x \le 3\pi/2$, then the quantity $x - \pi$ lies in the interval $-\pi/2 \le x \le \pi/2$, so

$$\sin^{-1}[\sin(x - \pi)] = x - \pi$$

Thus, by using the identity $\sin(x - \pi) = -\sin x$ and the fact that $\sin^{-1}$ is an odd function, we can express $\sin^{-1}(\sin x)$ as

$$\sin^{-1}(\sin x) = \sin^{-1}[-\sin(x - \pi)] = -\sin^{-1}[\sin(x - \pi)] = -(x - \pi)$$

This shows that on the interval $\pi/2 \le x \le 3\pi/2$ the graph of $y = \sin^{-1}(\sin x)$ coincides with the line $y = -(x - \pi)$, which has slope -1 and an x-intercept at $x = \pi$. This agrees with Figure 0.4.16. ◀



▶ **Figure 0.4.16**

✔ QUICK CHECK EXERCISES 0.4

(See page 52 for answers.)

1. In each part, determine whether the function f is one-to-one.
(a) $f(t)$ is the number of people in line at a movie theater at time t.
(b) $f(x)$ is the measured high temperature (rounded to the nearest °F) in a city on the xth day of the year.
(c) $f(v)$ is the weight of v cubic inches of lead.

2. A student enters a number on a calculator, doubles it, adds 8 to the result, divides the sum by 2, subtracts 3 from the quotient, and then cubes the difference. If the resulting number is x, then ________ was the student's original number.

3. If $(3, -2)$ is a point on the graph of an odd invertible function f, then ________ and ________ are points on the graph of f^{-1}.

4. In each part, determine the exact value without using a calculating utility.
(a) $\sin^{-1}(-1) =$ ________
(b) $\tan^{-1}(1) =$ ________
(c) $\sin^{-1}\left(\frac{1}{2}\sqrt{3}\right) =$ ________
(d) $\cos^{-1}\left(\frac{1}{2}\right) =$ ________
(e) $\sec^{-1}(-2) =$ ________

5. In each part, determine the exact value without using a calculating utility.
(a) $\sin^{-1}(\sin \pi/7) =$ ________
(b) $\sin^{-1}(\sin 5\pi/7) =$ ________
(c) $\tan^{-1}(\tan 13\pi/6) =$ ________
(d) $\cos^{-1}(\cos 12\pi/7) =$ ________

EXERCISE SET 0.4

Graphing Utility

1. In (a)–(d), determine whether f and g are inverse functions.
(a) $f(x) = 4x,\ g(x) = \frac{1}{4}x$
(b) $f(x) = 3x + 1,\ g(x) = 3x - 1$
(c) $f(x) = \sqrt[3]{x - 2},\ g(x) = x^3 + 2$
(d) $f(x) = x^4,\ g(x) = \sqrt[4]{x}$

2. Check your answers to Exercise 1 with a graphing utility by determining whether the graphs of f and g are reflections of one another about the line $y = x$.

3. In each part, use the horizontal line test to determine whether the function f is one-to-one.
(a) $f(x) = 3x + 2$ (b) $f(x) = \sqrt{x - 1}$
(c) $f(x) = |x|$ (d) $f(x) = x^3$
(e) $f(x) = x^2 - 2x + 2$ (f) $f(x) = \sin x$

4. In each part, generate the graph of the function f with a graphing utility, and determine whether f is one-to-one.
(a) $f(x) = x^3 - 3x + 2$ (b) $f(x) = x^3 - 3x^2 + 3x - 1$

FOCUS ON CONCEPTS

5. In each part, determine whether the function f defined by the table is one-to-one.

(a)

x	1	2	3	4	5	6
$f(x)$	−2	−1	0	1	2	3

(b)

x	1	2	3	4	5	6
$f(x)$	4	−7	6	−3	1	4

6. A face of a broken clock lies in the xy-plane with the center of the clock at the origin and 3:00 in the direction of the positive x-axis. When the clock broke, the tip of the hour hand stopped on the graph of $y = f(x)$, where f is a function that satisfies $f(0) = 0$.

(a) Are there any times of the day that cannot appear in such a configuration? Explain.

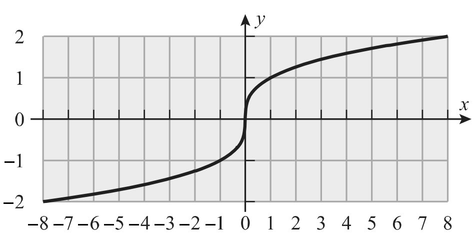
(b) How does your answer to part (a) change if f must be an invertible function?

(c) How do your answers to parts (a) and (b) change if it was the tip of the minute hand that stopped on the graph of f?

7. (a) The accompanying figure shows the graph of a function f over its domain $-8 \le x \le 8$. Explain why f has an inverse, and use the graph to find $f^{-1}(2)$, $f^{-1}(-1)$, and $f^{-1}(0)$.

(b) Find the domain and range of f^{-1}.

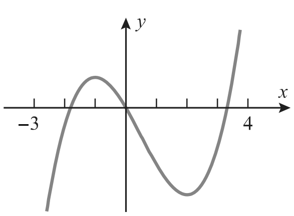
(c) Sketch the graph of f^{-1}.



▲ **Figure Ex-7**

8. (a) Explain why the function f graphed in the accompanying figure has no inverse function on its domain $-3 \le x \le 4$.

(b) Subdivide the domain into three adjacent intervals on each of which the function f has an inverse.



◀ **Figure Ex-8**

9–16 Find a formula for $f^{-1}(x)$. ■

9. $f(x) = 7x - 6$

10. $f(x) = \dfrac{x+1}{x-1}$

11. $f(x) = 3x^3 - 5$

12. $f(x) = \sqrt[5]{4x+2}$

13. $f(x) = 3/x^2, \quad x < 0$

14. $f(x) = 5/(x^2+1), \quad x \ge 0$

15. $f(x) = \begin{cases} 5/2 - x, & x < 2 \\ 1/x, & x \ge 2 \end{cases}$

16. $f(x) = \begin{cases} 2x, & x \le 0 \\ x^2, & x > 0 \end{cases}$

17–20 Find a formula for $f^{-1}(x)$, and state the domain of the function f^{-1}. ■

17. $f(x) = (x+2)^4, \quad x \ge 0$

18. $f(x) = \sqrt{x+3}$

19. $f(x) = -\sqrt{3-2x}$

20. $f(x) = x - 5x^2, \quad x \ge 1$

21. Let $f(x) = ax^2 + bx + c$, $a > 0$. Find f^{-1} if the domain of f is restricted to

(a) $x \ge -b/(2a)$ (b) $x \le -b/(2a)$.

FOCUS ON CONCEPTS

22. The formula $F = \frac{9}{5}C + 32$, where $C \ge -273.15$ expresses the Fahrenheit temperature F as a function of the Celsius temperature C.

(a) Find a formula for the inverse function.

(b) In words, what does the inverse function tell you?

(c) Find the domain and range of the inverse function.

23. (a) One meter is about 6.214×10^{-4} miles. Find a formula $y = f(x)$ that expresses a length y in meters as a function of the same length x in miles.

(b) Find a formula for the inverse of f.

(c) Describe what the formula $x = f^{-1}(y)$ tells you in practical terms.

24. Let $f(x) = x^2$, $x > 1$, and $g(x) = \sqrt{x}$.

(a) Show that $f(g(x)) = x$, $x > 1$, and $g(f(x)) = x$, $x > 1$.

(b) Show that f and g are *not* inverses by showing that the graphs of $y = f(x)$ and $y = g(x)$ are not reflections of one another about $y = x$.

(c) Do parts (a) and (b) contradict one another? Explain.

25. (a) Show that $f(x) = (3-x)/(1-x)$ is its own inverse.

(b) What does the result in part (a) tell you about the graph of f?

26. Sketch the graph of a function that is one-to-one on $(-\infty, +\infty)$, yet not increasing on $(-\infty, +\infty)$ and not decreasing on $(-\infty, +\infty)$.

27. Let $f(x) = 2x^3 + 5x + 3$. Find x if $f^{-1}(x) = 1$.

28. Let $f(x) = \dfrac{x^3}{x^2+1}$. Find x if $f^{-1}(x) = 2$.