RIEMANN SUM

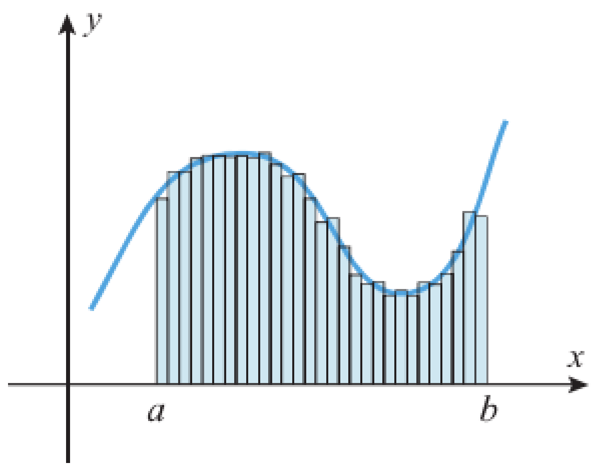
5.4.3 DEFINITION ***(Area Under a Curve)*** If the function f is continuous on $[a, b]$ and if $f(x) \geq 0$ for all x in $[a, b]$, then the ***area*** A under the curve $y = f(x)$ over the interval $[a, b]$ is defined by

$$A = \lim_{n \to +\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x \qquad (2)$$

The limit in (2) is interpreted to mean that given any number $\epsilon > 0$ the inequality

$$\left| A - \sum_{k=1}^{n} f(x_k^*)\Delta x \right| < \epsilon$$

holds when n is sufficiently large, no matter how the points x_k^* are selected.

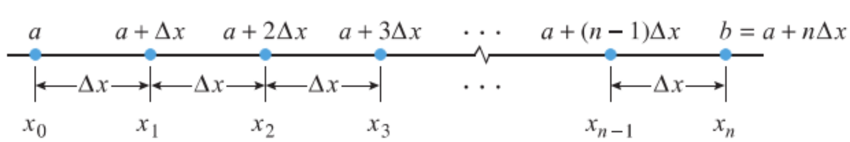


Thus, the left endpoint, right endpoint, and midpoint choices for $x_1^*, x_2^*, \ldots, x_n^*$ are given by

$$x_k^* = x_{k-1} = a + (k-1)\Delta x \qquad \boxed{\text{Left endpoint}} \qquad (3)$$

$$x_k^* = x_k = a + k\Delta x \qquad \boxed{\text{Right endpoint}} \qquad (4)$$

$$x_k^* = \tfrac{1}{2}(x_{k-1} + x_k) = a + \left(k - \tfrac{1}{2}\right)\Delta x \qquad \boxed{\text{Midpoint}} \qquad (5)$$



▶ **Figure 5.4.6**

5.4.2 THEOREM

(a) $\displaystyle\sum_{k=1}^{n} k = 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$

(b) $\displaystyle\sum_{k=1}^{n} k^2 = 1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$

(c) $\displaystyle\sum_{k=1}^{n} k^3 = 1^3 + 2^3 + \cdots + n^3 = \left[\frac{n(n+1)}{2}\right]^2$

5.4.4 THEOREM

(a) $\displaystyle\lim_{n\to+\infty} \frac{1}{n}\sum_{k=1}^{n} 1 = 1$ (b) $\displaystyle\lim_{n\to+\infty} \frac{1}{n^2}\sum_{k=1}^{n} k = \frac{1}{2}$

(c) $\displaystyle\lim_{n\to+\infty} \frac{1}{n^3}\sum_{k=1}^{n} k^2 = \frac{1}{3}$ (d) $\displaystyle\lim_{n\to+\infty} \frac{1}{n^4}\sum_{k=1}^{n} k^3 = \frac{1}{4}$

Question:

41–44 Use Definition 5.4.3 with x_k^* as the *left* endpoint of each subinterval to find the area under the curve $y = f(x)$ over the specified interval. ■

44. $f(x) = 4 - \frac{1}{4}x^2;\ [0, 3]$

Solution:

$\Delta x = \dfrac{3}{n},\ x_k^* = (k-1)\dfrac{3}{n};\ f(x_k^*)\Delta x = \left[4 - \dfrac{1}{4}(x_k^*)^2\right]\Delta x = \left[4 - \dfrac{1}{4}\dfrac{9(k-1)^2}{n^2}\right]\dfrac{3}{n} = \dfrac{12}{n} - \dfrac{27k^2}{4n^3} + \dfrac{27k}{2n^3} - \dfrac{27}{4n^3},$

$\displaystyle\sum_{k=1}^{n} f(x_k^*)\Delta x = \sum_{k=1}^{n}\frac{12}{n} - \frac{27}{4n^3}\sum_{k=1}^{n}k^2 + \frac{27}{2n^3}\sum_{k=1}^{n}k - \frac{27}{4n^3}\sum_{k=1}^{n}1 = 12 - \frac{27}{4n^3}\cdot\frac{1}{6}n(n+1)(2n+1) + \frac{27}{2n^3}\frac{n(n+1)}{2} - \frac{27}{4n^2} =$

$= 12 - \dfrac{9}{8}\dfrac{(n+1)(2n+1)}{n^2} + \dfrac{27}{4n} + \dfrac{27}{4n^2} - \dfrac{27}{4n^2},$

$A = \displaystyle\lim_{n\to+\infty}\left[12 - \frac{9}{8}\left(1 + \frac{1}{n}\right)\left(2 + \frac{1}{n}\right)\right] + 0 + 0 - 0 = 12 - \frac{9}{8}(1)(2) = 39/4.$

Question:

45–48 Use Definition 5.4.3 with x_k^* as the *midpoint* of each subinterval to find the area under the curve $y = f(x)$ over the specified interval. ■

46. $f(x) = 6 - x;\ [1, 5]$

Solution:

Endpoints $1, 1+\frac{4}{n}, 1+\frac{8}{n}, \ldots, 1+\frac{4(n-1)}{n}, 1+4 = 5$, and midpoints $1+\frac{2}{n}, 1+\frac{6}{n}, 1+\frac{10}{n}, \ldots, 1+\frac{4(n-1)-2}{n}, \frac{4n-2}{n}$. Approximate the area with the sum $\sum_{k=1}^{n}\left(6-\left(1+\frac{4k-2}{n}\right)\right)\frac{4}{n} = \sum_{k=1}^{n}\left(5\frac{4}{n} - \frac{16}{n^2}k + \frac{8}{n^2}\right) = 20 - \frac{16}{n^2}\frac{n(n+1)}{2} + \frac{8}{n} = 20 - 8 = 12$, which is exact, because f is linear.