National University of Computer & Emerging Sciences
MT-1003 Calculus and Analytical Geometry
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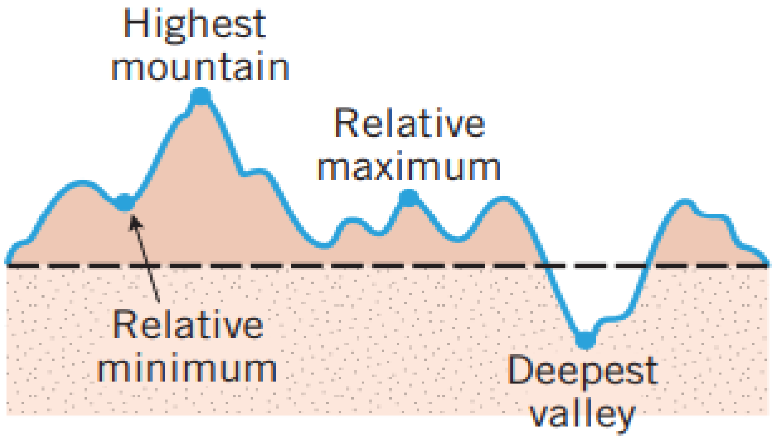
ANALYSIS OF FUNCTIONS-II

RELATIVE EXTREMA

RELATIVE MAXIMA AND MINIMA

If we imagine the graph of a function f to be a two-dimensional mountain range with hills and valleys, then the tops of the hills are called "relative maxima," and the bottoms of the valleys are called "relative minima" (Figure 4.2.1). The relative maxima are the high points in their *immediate vicinity*, and the relative minima are the low points. A relative maximum need not be the highest point in the entire mountain range, and a relative minimum need not be the lowest point—they are just high and low points *relative* to the nearby terrain. These ideas are captured in the following definition.

4.2.1 DEFINITION A function f is said to have a ***relative maximum*** at x_0 if there is an open interval containing x_0 on which $f(x_0)$ is the largest value, that is, $f(x_0) \geq f(x)$ for all x in the interval. Similarly, f is said to have a ***relative minimum*** at x_0 if there is an open interval containing x_0 on which $f(x_0)$ is the smallest value, that is, $f(x_0) \leq f(x)$ for all x in the interval. If f has either a relative maximum or a relative minimum at x_0, then f is said to have a ***relative extremum*** at x_0.



▲ **Figure 4.2.1**

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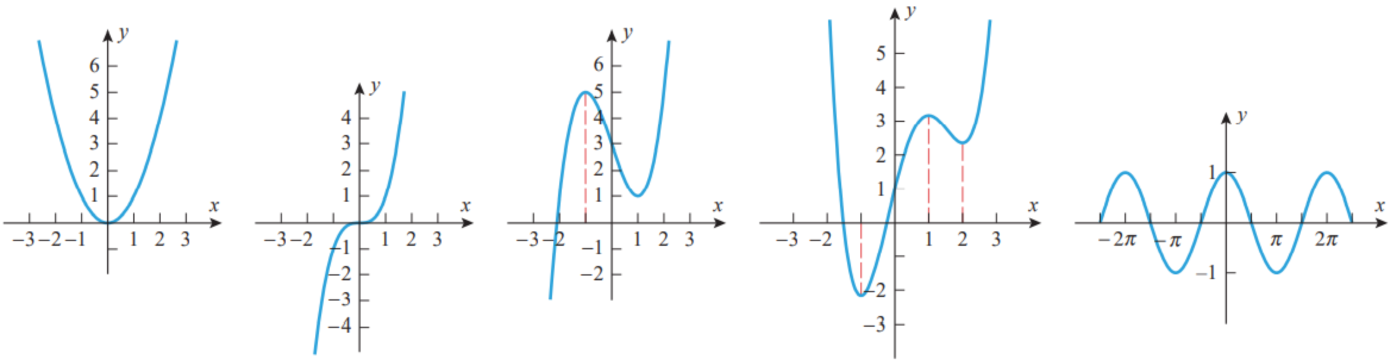
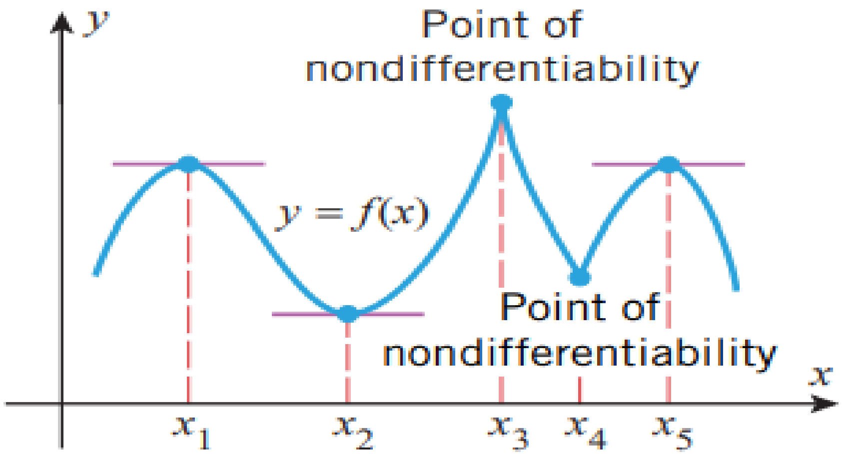
▶ **Example 1** We can see from Figure 4.2.2 that:

- $f(x) = x^2$ has a relative minimum at $x = 0$ but no relative maxima.
- $f(x) = x^3$ has no relative extrema.
- $f(x) = x^3 - 3x + 3$ has a relative maximum at $x = -1$ and a relative minimum at $x = 1$.
- $f(x) = \frac{1}{2}x^4 - \frac{4}{3}x^3 - x^2 + 4x + 1$ has relative minima at $x = -1$ and $x = 2$ and a relative maximum at $x = 1$.
- $f(x) = \cos x$ has relative maxima at all even multiples of π and relative minima at all odd multiples of π. ◀



$y = x^2$ $y = x^3$ $y = x^3 - 3x + 3$ $y = \frac{1}{2}x^4 - \frac{4}{3}x^3 - x^2 + 4x + 1$ $y = \cos x$

▲ **Figure 4.2.2**



▲ **Figure 4.2.3** The points x_1, x_2, x_3, x_4, and x_5 are critical points. Of these, x_1, x_2, and x_5 are stationary points.

The relative extrema for the five functions in Example 1 occur at points where the graphs of the functions have horizontal tangent lines. Figure 4.2.3 illustrates that a relative extremum can also occur at a point where a function is not differentiable. In general, we define a ***critical point*** for a function f to be a point in the domain of f at which either the graph of f has a horizontal tangent line or f is not differentiable. To distinguish between the two types of critical points we call x a ***stationary point*** of f if $f'(x) = 0$. The following theorem, which is proved in Appendix J, states that the critical points for a function form a complete set of candidates for relative extrema on the interior of the domain of the function.

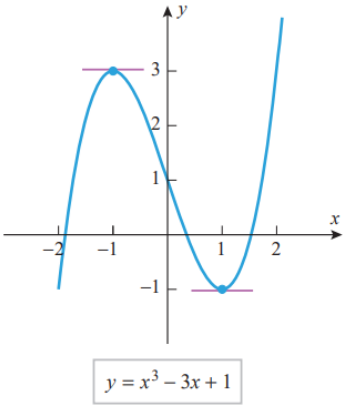
4.2.2 THEOREM *Suppose that f is a function defined on an open interval containing the point x_0. If f has a relative extremum at $x = x_0$, then $x = x_0$ is a critical point of f; that is, either $f'(x_0) = 0$ or f is not differentiable at x_0.*

► **Example 2** Find all critical points of $f(x) = x^3 - 3x + 1$.

Solution. The function f, being a polynomial, is differentiable everywhere, so its critical points are all stationary points. To find these points we must solve the equation $f'(x) = 0$. Since

$$f'(x) = 3x^2 - 3 = 3(x + 1)(x - 1)$$

we conclude that the critical points occur at $x = -1$ and $x = 1$. This is consistent with the graph of f in Figure 4.2.4. ◄

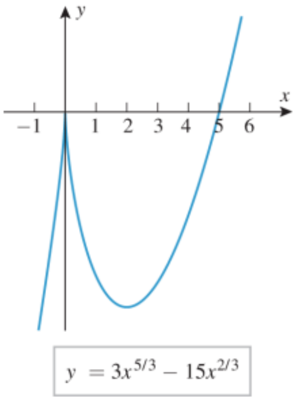


► **Example 3** Find all critical points of $f(x) = 3x^{5/3} - 15x^{2/3}$.

Solution. The function f is continuous everywhere and its derivative is

$$f'(x) = 5x^{2/3} - 10x^{-1/3} = 5x^{-1/3}(x - 2) = \frac{5(x - 2)}{x^{1/3}}$$

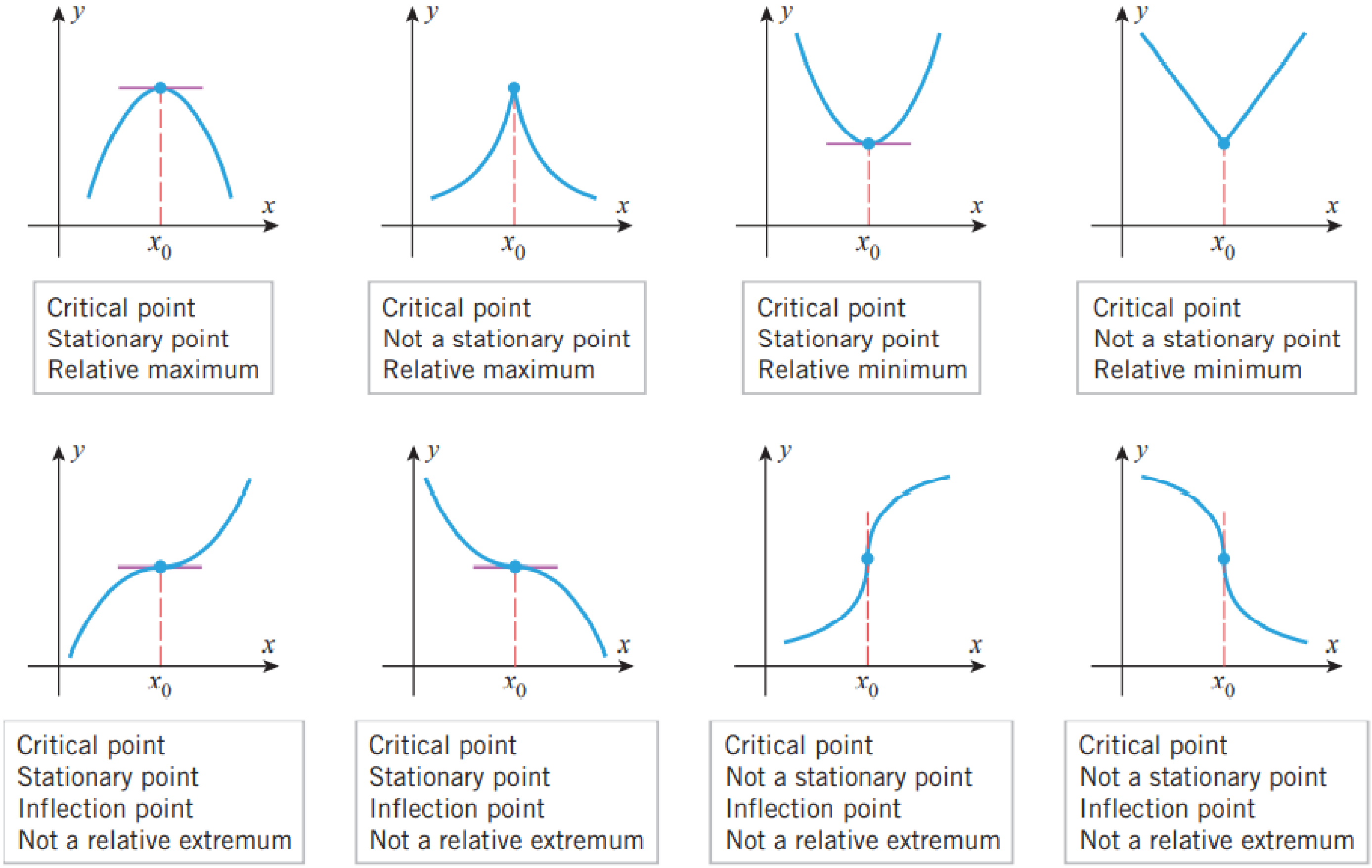
We see from this that $f'(x) = 0$ if $x = 2$ and $f'(x)$ is undefined if $x = 0$. Thus $x = 0$ and $x = 2$ are critical points and $x = 2$ is a stationary point. This is consistent with the graph of f shown in Figure 3.2.5. ◄



▲ **Figure 3.2.5**

FIRST DERIVATIVE TEST

A function f has a relative extremum at those critical points where f' changes sign.



Critical point
Stationary point
Relative maximum

Critical point
Not a stationary point
Relative maximum

Critical point
Stationary point
Relative minimum

Critical point
Not a stationary point
Relative minimum

Critical point
Stationary point
Inflection point
Not a relative extremum

Critical point
Stationary point
Inflection point
Not a relative extremum

Critical point
Not a stationary point
Inflection point
Not a relative extremum

Critical point
Not a stationary point
Inflection point
Not a relative extremum

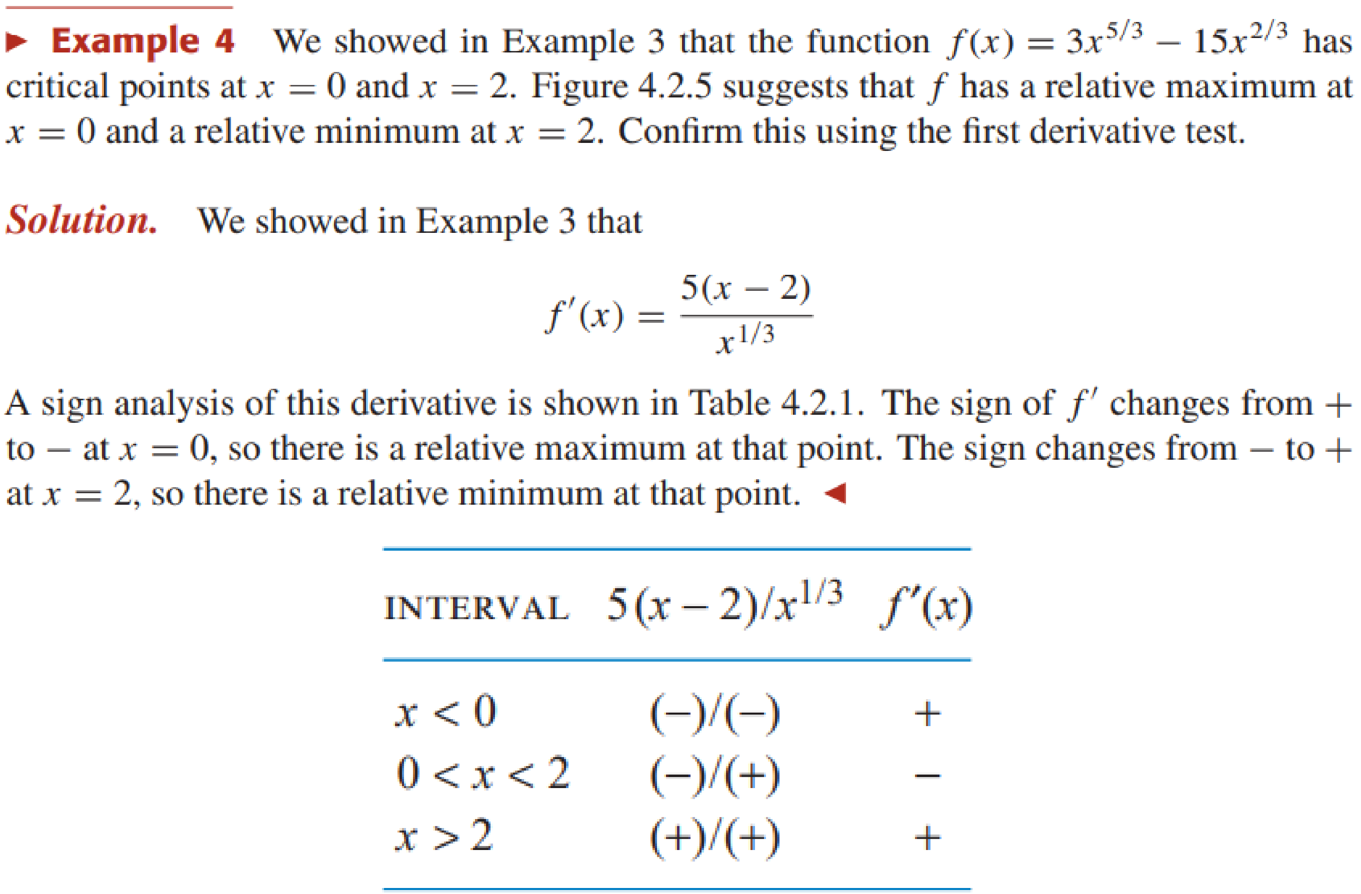
▲ **Figure 4.2.6**

4.2.3 THEOREM ***(First Derivative Test)*** *Suppose that f is continuous at a critical point x_0.*

(a) *If $f'(x) > 0$ on an open interval extending left from x_0 and $f'(x) < 0$ on an open interval extending right from x_0, then f has a relative maximum at x_0.*

(b) *If $f'(x) < 0$ on an open interval extending left from x_0 and $f'(x) > 0$ on an open interval extending right from x_0, then f has a relative minimum at x_0.*

(c) *If $f'(x)$ has the same sign on an open interval extending left from x_0 as it does on an open interval extending right from x_0, then f does not have a relative extremum at x_0.*



▶ Example 4 We showed in Example 3 that the function $f(x) = 3x^{5/3} - 15x^{2/3}$ has critical points at $x = 0$ and $x = 2$. Figure 4.2.5 suggests that f has a relative maximum at $x = 0$ and a relative minimum at $x = 2$. Confirm this using the first derivative test.

Solution. We showed in Example 3 that

$$f'(x) = \frac{5(x-2)}{x^{1/3}}$$

A sign analysis of this derivative is shown in Table 4.2.1. The sign of f' changes from $+$ to $-$ at $x = 0$, so there is a relative maximum at that point. The sign changes from $-$ to $+$ at $x = 2$, so there is a relative minimum at that point. ◀

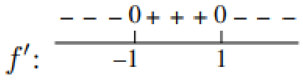
INTERVAL	$5(x-2)/x^{1/3}$	$f'(x)$
$x < 0$	$(-)/(-)$	$+$
$0 < x < 2$	$(-)/(+)$	$-$
$x > 2$	$(+)/(+)$	$+$

Question:

Use the given derivative to find all critical points of *f* , and at each critical point determine whether a relative maximum, relative minimum, or neither occurs. Assume in each case that f is continuous everywhere.

$$f'(x) = \ln\left(\frac{2}{1+x^2}\right)$$

Solution



Critical points: $x = -1, 1$; $x = -1$: relative minimum, $x = 1$: relative maximum.

EXERCISE SET 3.2

7–14 Locate the critical points and identify which critical points are stationary points.

7. $f(x) = 4x^4 - 16x^2 + 17$

8. $f(x) = 3x^4 + 12x$

9. $f(x) = \dfrac{x+1}{x^2+3}$

10. $f(x) = \dfrac{x^2}{x^3+8}$

11. $f(x) = \sqrt[3]{x^2 - 25}$

12. $f(x) = x^2(x-1)^{2/3}$

25–28 Use the given derivative to find all critical points of f, and at each critical point determine whether a relative maximum, relative minimum, or neither occurs. Assume in each case that f is continuous everywhere.

25. $f'(x) = x^2(x^3 - 5)$

26. $f'(x) = 4x^3 - 9x$

27. $f'(x) = \dfrac{2-3x}{\sqrt[3]{x+2}}$

28. $f'(x) = \dfrac{x^2-7}{\sqrt[3]{x^2+4}}$

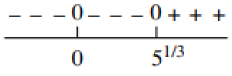
SOLUTION SET:

7. $f'(x) = 16x^3 - 32x = 16x(x^2 - 2)$, so $x = 0, \pm\sqrt{2}$ are stationary points.

9. $f'(x) = \dfrac{-x^2 - 2x + 3}{(x^2+3)^2}$, so $x = -3, 1$ are the stationary points.

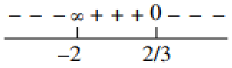
11. $f'(x) = \dfrac{2x}{3(x^2-25)^{2/3}}$; so $x = 0$ is the stationary point; $x = \pm 5$ are critical points which are not stationary points.

25. f':



Critical points: $x = 0, 5^{1/3}$; $x = 0$: neither, $x = 5^{1/3}$: relative minimum.

27. f':



Critical points: $x = -2, 2/3$; $x = -2$: relative minimum, $x = 2/3$: relative maximum.