The Chain Rule

2.6.1 THEOREM (*The Chain Rule*) *If g is differentiable at x and f is differentiable at $g(x)$, then the composition $f \circ g$ is differentiable at x. Moreover, if*

$$y = f(g(x)) \quad \text{and} \quad u = g(x)$$

then $y = f(u)$ and

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \tag{1}$$

AN ALTERNATIVE VERSION OF THE CHAIN RULE

Formula (1) for the chain rule can be unwieldy in some problems because it involves so many variables. As you become more comfortable with the chain rule, you may want to dispense with writing out the dependent variables by expressing (1) in the form

$$\frac{d}{dx}[f(g(x))] = (f \circ g)'(x) = f'(g(x))g'(x) \tag{2}$$

A convenient way to remember this formula is to call f the "outside function" and g the "inside function" in the composition $f(g(x))$ and then express (2) in words as:

The derivative of $f(g(x))$ is the derivative of the outside function evaluated at the inside function times the derivative of the inside function.

$$\frac{d}{dx}[f(g(x))] = \underbrace{f'(g(x))}_{\substack{\text{Derivative of the outside}\\ \text{function evaluated at the}\\ \text{inside function}}} \cdot \underbrace{g'(x)}_{\substack{\text{Derivative of the}\\ \text{inside function}}}$$

Table 2.6.1
GENERALIZED DERIVATIVE FORMULAS

$$\frac{d}{dx}[u^r] = ru^{r-1}\frac{du}{dx}$$

$\frac{d}{dx}[\sin u] = \cos u \frac{du}{dx}$	$\frac{d}{dx}[\cos u] = -\sin u \frac{du}{dx}$
$\frac{d}{dx}[\tan u] = \sec^2 u \frac{du}{dx}$	$\frac{d}{dx}[\cot u] = -\csc^2 u \frac{du}{dx}$
$\frac{d}{dx}[\sec u] = \sec u \tan u \frac{du}{dx}$	$\frac{d}{dx}[\csc u] = -\csc u \cot u \frac{du}{dx}$

Question: Find $f'(x)$.

$$f(x) = \tan^4(x^3)$$

Answer:

$$f'(x) = 4\tan^3(x^3)\frac{d}{dx}[\tan(x^3)] = 4\tan^3(x^3)\sec^2(x^3)\frac{d}{dx}(x^3) = 12x^2\tan^3(x^3)\sec^2(x^3).$$

Question: Find $f'(x)$.

$$f(x) = \cos^3\left(\frac{x}{x+1}\right)$$

Answer:

$$f'(x) = 3\cos^2\left(\frac{x}{x+1}\right)\frac{d}{dx}\cos\left(\frac{x}{x+1}\right) = 3\cos^2\left(\frac{x}{x+1}\right)\left[-\sin\left(\frac{x}{x+1}\right)\right]\frac{(x+1)(1)-x(1)}{(x+1)^2} =$$
$$= -\frac{3}{(x+1)^2}\cos^2\left(\frac{x}{x+1}\right)\sin\left(\frac{x}{x+1}\right).$$

Question: Find $f'(x)$.

$$y = \sqrt{x}\tan^3(\sqrt{x})$$

Answer:

$$\frac{dy}{dx} = \sqrt{x}\left[3\tan^2(\sqrt{x})\sec^2(\sqrt{x})\frac{1}{2\sqrt{x}}\right] + \frac{1}{2\sqrt{x}}\tan^3(\sqrt{x}) = \frac{3}{2}\tan^2(\sqrt{x})\sec^2(\sqrt{x}) + \frac{1}{2\sqrt{x}}\tan^3(\sqrt{x}).$$

Question:

$$y = [1+\sin^3(x^5)]^{12}$$

Answer:

$$\frac{dy}{dx} = 12[1+\sin^3(x^5)]^{11}\frac{d}{dx}[1+\sin^3(x^5)] = 12[1+\sin^3(x^5)]^{11}3\sin^2(x^5)\frac{d}{dx}\sin(x^5) =$$
$$= 180x^4[1+\sin^3(x^5)]^{11}\sin^2(x^5)\cos(x^5).$$