The L' HOPITALs Rule

INDETERMINATE FORMS OF TYPE 0/0

Recall that a limit of the form

$$\lim_{x \to a} \frac{f(x)}{g(x)} \tag{1}$$

in which $f(x) \to 0$ and $g(x) \to 0$ as $x \to a$ is called an ***indeterminate form of type* 0/0**. Some examples encountered earlier in the text are

$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1} = 2, \quad \lim_{x \to 0} \frac{\sin x}{x} = 1, \quad \lim_{x \to 0} \frac{1 - \cos x}{x} = 0$$

3.6.1 THEOREM (***L'Hôpital's Rule for Form* 0/0**) *Suppose that f and g are differentiable functions on an open interval containing $x = a$, except possibly at $x = a$, and that*

$$\lim_{x \to a} f(x) = 0 \quad \textit{and} \quad \lim_{x \to a} g(x) = 0$$

If $\lim_{x \to a} [f'(x)/g'(x)]$ exists, or if this limit is $+\infty$ or $-\infty$, then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

Moreover, this statement is also true in the case of a limit as $x \to a^-$, $x \to a^+$, $x \to -\infty$, or as $x \to +\infty$.

Applying L'Hôpital's Rule

Step 1. Check that the limit of $f(x)/g(x)$ is an indeterminate form of type $0/0$.

Step 2. Differentiate f and g separately.

Step 3. Find the limit of $f'(x)/g'(x)$. If this limit is finite, $+\infty$, or $-\infty$, then it is equal to the limit of $f(x)/g(x)$.

Example 1 Find the limit

$$\lim_{x \to 2} \frac{x^2 - 4}{x - 2}$$

using L'Hôpital's rule, and check the result by factoring.

Solution. The numerator and denominator have a limit of 0, so the limit is an indeterminate form of type $0/0$. Applying L'Hôpital's rule yields

$$\lim_{x \to 2} \frac{x^2 - 4}{x - 2} = \lim_{x \to 2} \frac{\dfrac{d}{dx}[x^2 - 4]}{\dfrac{d}{dx}[x - 2]} = \lim_{x \to 2} \frac{2x}{1} = 4$$

This agrees with the computation

$$\lim_{x \to 2} \frac{x^2 - 4}{x - 2} = \lim_{x \to 2} \frac{(x - 2)(x + 2)}{x - 2} = \lim_{x \to 2} (x + 2) = 4$$ ◀

Example 2 In each part confirm that the limit is an indeterminate form of type $0/0$, and evaluate it using L'Hôpital's rule.

(a) $\displaystyle \lim_{x \to 0} \frac{\sin 2x}{x}$ (b) $\displaystyle \lim_{x \to \pi/2} \frac{1 - \sin x}{\cos x}$ (c) $\displaystyle \lim_{x \to 0} \frac{e^x - 1}{x^3}$

(d) $\displaystyle \lim_{x \to 0^-} \frac{\tan x}{x^2}$ (e) $\displaystyle \lim_{x \to 0} \frac{1 - \cos x}{x^2}$ (f) $\displaystyle \lim_{x \to +\infty} \frac{x^{-4/3}}{\sin(1/x)}$

Solution (a). The numerator and denominator have a limit of 0, so the limit is an indeterminate form of type $0/0$. Applying L'Hôpital's rule yields

$$\lim_{x \to 0} \frac{\sin 2x}{x} = \lim_{x \to 0} \frac{\dfrac{d}{dx}[\sin 2x]}{\dfrac{d}{dx}[x]} = \lim_{x \to 0} \frac{2\cos 2x}{1} = 2$$

Observe that this result agrees with that obtained by substitution in Example 4(b) of Section 1.6.

Solution (b). The numerator and denominator have a limit of 0, so the limit is an indeterminate form of type $0/0$. Applying L'Hôpital's rule yields

$$\lim_{x \to \pi/2} \frac{1 - \sin x}{\cos x} = \lim_{x \to \pi/2} \frac{\dfrac{d}{dx}[1 - \sin x]}{\dfrac{d}{dx}[\cos x]} = \lim_{x \to \pi/2} \frac{-\cos x}{-\sin x} = \frac{0}{-1} = 0$$

Solution (c). The numerator and denominator have a limit of 0, so the limit is an indeterminate form of type 0/0. Applying L'Hôpital's rule yields

$$\lim_{x\to 0}\frac{e^x-1}{x^3}=\lim_{x\to 0}\frac{\dfrac{d}{dx}[e^x-1]}{\dfrac{d}{dx}[x^3]}=\lim_{x\to 0}\frac{e^x}{3x^2}=+\infty$$

Solution (d). The numerator and denominator have a limit of 0, so the limit is an indeterminate form of type 0/0. Applying L'Hôpital's rule yields

$$\lim_{x\to 0^-}\frac{\tan x}{x^2}=\lim_{x\to 0^-}\frac{\sec^2 x}{2x}=-\infty$$

Solution (e). The numerator and denominator have a limit of 0, so the limit is an indeterminate form of type 0/0. Applying L'Hôpital's rule yields

$$\lim_{x\to 0}\frac{1-\cos x}{x^2}=\lim_{x\to 0}\frac{\sin x}{2x}$$

Since the new limit is another indeterminate form of type 0/0, we apply L'Hôpital's rule again:

$$\lim_{x\to 0}\frac{1-\cos x}{x^2}=\lim_{x\to 0}\frac{\sin x}{2x}=\lim_{x\to 0}\frac{\cos x}{2}=\frac{1}{2}$$

Solution (f). The numerator and denominator have a limit of 0, so the limit is an indeterminate form of type 0/0. Applying L'Hôpital's rule yields

$$\lim_{x\to +\infty}\frac{x^{-4/3}}{\sin(1/x)}=\lim_{x\to +\infty}\frac{-\frac{4}{3}x^{-7/3}}{(-1/x^2)\cos(1/x)}=\lim_{x\to +\infty}\frac{\frac{4}{3}x^{-1/3}}{\cos(1/x)}=\frac{0}{1}=0 \blacktriangleleft$$

INDETERMINATE FORMS OF TYPE ∞/∞

When we want to indicate that the limit (or a one-sided limit) of a function is $+\infty$ or $-\infty$ without being specific about the sign, we will say that the limit is ∞. For example,

$$\lim_{x\to a^+} f(x)=\infty \quad \text{means} \quad \lim_{x\to a^+} f(x)=+\infty \quad \text{or} \quad \lim_{x\to a^+} f(x)=-\infty$$

$$\lim_{x\to +\infty} f(x)=\infty \quad \text{means} \quad \lim_{x\to +\infty} f(x)=+\infty \quad \text{or} \quad \lim_{x\to +\infty} f(x)=-\infty$$

$$\lim_{x\to a} f(x)=\infty \quad \text{means} \quad \lim_{x\to a^+} f(x)=\pm\infty \quad \text{and} \quad \lim_{x\to a^-} f(x)=\pm\infty$$

The limit of a ratio, $f(x)/g(x)$, in which the numerator has limit ∞ and the denominator has limit ∞ is called an ***indeterminate form of type*** ∞/∞. The following version of L'Hôpital's rule, which we state without proof, can often be used to evaluate limits of this type.

3.6.2 THEOREM (*L'Hôpital's Rule for Form ∞/∞*) *Suppose that f and g are differentiable functions on an open interval containing $x=a$, except possibly at $x=a$, and that*

$$\lim_{x\to a} f(x)=\infty \quad \textit{and} \quad \lim_{x\to a} g(x)=\infty$$

If $\lim_{x\to a}[f'(x)/g'(x)]$ exists, or if this limit is $+\infty$ or $-\infty$, then

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}$$

Moreover, this statement is also true in the case of a limit as $x\to a^-$, $x\to a^+$, $x\to -\infty$, or as $x\to +\infty$.

► Example 3 In each part confirm that the limit is an indeterminate form of type ∞/∞ and apply L'Hôpital's rule.

$$\text{(a)} \quad \lim_{x \to +\infty} \frac{x}{e^x} \qquad \text{(b)} \quad \lim_{x \to 0^+} \frac{\ln x}{\csc x}$$

Solution (a). The numerator and denominator both have a limit of $+\infty$, so we have an indeterminate form of type ∞/∞. Applying L'Hôpital's rule yields

$$\lim_{x \to +\infty} \frac{x}{e^x} = \lim_{x \to +\infty} \frac{1}{e^x} = 0$$

Solution (b). The numerator has a limit of $-\infty$ and the denominator has a limit of $+\infty$, so we have an indeterminate form of type ∞/∞. Applying L'Hôpital's rule yields

$$\lim_{x \to 0^+} \frac{\ln x}{\csc x} = \lim_{x \to 0^+} \frac{1/x}{-\csc x \cot x} \tag{4}$$

This last limit is again an indeterminate form of type ∞/∞. Moreover, any additional applications of L'Hôpital's rule will yield powers of $1/x$ in the numerator and expressions involving $\csc x$ and $\cot x$ in the denominator; thus, repeated application of L'Hôpital's rule simply produces new indeterminate forms. We must try something else. The last limit in (4) can be rewritten as

$$\lim_{x \to 0^+} \left(-\frac{\sin x}{x} \tan x \right) = -\lim_{x \to 0^+} \frac{\sin x}{x} \cdot \lim_{x \to 0^+} \tan x = -(1)(0) = 0$$

Thus,

$$\lim_{x \to 0^+} \frac{\ln x}{\csc x} = 0 \blacktriangleleft$$

INDETERMINATE FORMS OF TYPE $0 \cdot \infty$

Thus far we have discussed indeterminate forms of type $0/0$ and ∞/∞. However, these are not the only possibilities; in general, the limit of an expression that has one of the forms

$$\frac{f(x)}{g(x)}, \quad f(x) \cdot g(x), \quad f(x)^{g(x)}, \quad f(x) - g(x), \quad f(x) + g(x)$$

is called an *indeterminate form* if the limits of $f(x)$ and $g(x)$ individually exert conflicting influences on the limit of the entire expression. For example, the limit

$$\lim_{x \to 0^+} x \ln x$$

is an ***indeterminate form of type*** $\boldsymbol{0 \cdot \infty}$ because the limit of the first factor is 0, the limit of the second factor is $-\infty$, and these two limits exert conflicting influences on the product. On the other hand, the limit

$$\lim_{x \to +\infty} [\sqrt{x}(1 - x^2)]$$

is not an indeterminate form because the first factor has a limit of $+\infty$, the second factor has a limit of $-\infty$, and these influences work together to produce a limit of $-\infty$ for the product.

Indeterminate forms of type $0 \cdot \infty$ can sometimes be evaluated by rewriting the product as a ratio, and then applying L'Hôpital's rule for indeterminate forms of type $0/0$ or ∞/∞.

▶ Example 4 Evaluate

$$\text{(a)} \lim_{x\to 0^+} x \ln x \qquad \text{(b)} \lim_{x\to \pi/4} (1-\tan x)\sec 2x$$

Solution (a). The factor x has a limit of 0 and the factor $\ln x$ has a limit of $-\infty$, so the stated problem is an indeterminate form of type $0\cdot\infty$. There are two possible approaches: we can rewrite the limit as

$$\lim_{x\to 0^+} \frac{\ln x}{1/x} \quad \text{or} \quad \lim_{x\to 0^+} \frac{x}{1/\ln x}$$

the first being an indeterminate form of type ∞/∞ and the second an indeterminate form of type $0/0$. However, the first form is the preferred initial choice because the derivative of $1/x$ is less complicated than the derivative of $1/\ln x$. That choice yields

$$\lim_{x\to 0^+} x\ln x = \lim_{x\to 0^+} \frac{\ln x}{1/x} = \lim_{x\to 0^+} \frac{1/x}{-1/x^2} = \lim_{x\to 0^+} (-x) = 0$$

Solution (b). The stated problem is an indeterminate form of type $0\cdot\infty$. We will convert it to an indeterminate form of type $0/0$:

$$\lim_{x\to\pi/4} (1-\tan x)\sec 2x = \lim_{x\to\pi/4} \frac{1-\tan x}{1/\sec 2x} = \lim_{x\to\pi/4} \frac{1-\tan x}{\cos 2x}$$

$$= \lim_{x\to\pi/4} \frac{-\sec^2 x}{-2\sin 2x} = \frac{-2}{-2} = 1 \blacktriangleleft$$

INDETERMINATE FORMS OF TYPE $\infty-\infty$

A limit problem that leads to one of the expressions

$$(+\infty)-(+\infty), \quad (-\infty)-(-\infty),$$
$$(+\infty)+(-\infty), \quad (-\infty)+(+\infty)$$

is called an ***indeterminate form of type $\infty-\infty$***. Such limits are indeterminate because the two terms exert conflicting influences on the expression: one pushes it in the positive direction and the other pushes it in the negative direction. However, limit problems that lead to one of the expressions

$$(+\infty)+(+\infty), \quad (+\infty)-(-\infty),$$
$$(-\infty)+(-\infty), \quad (-\infty)-(+\infty)$$

are not indeterminate, since the two terms work together (those on the top produce a limit of $+\infty$ and those on the bottom produce a limit of $-\infty$).

Indeterminate forms of type $\infty-\infty$ can sometimes be evaluated by combining the terms and manipulating the result to produce an indeterminate form of type $0/0$ or ∞/∞.

▶ Example 5 Evaluate $\lim_{x\to 0^+} \left(\frac{1}{x} - \frac{1}{\sin x}\right)$.

Solution. Both terms have a limit of $+\infty$, so the stated problem is an indeterminate form of type $\infty-\infty$. Combining the two terms yields

$$\lim_{x\to 0^+} \left(\frac{1}{x} - \frac{1}{\sin x}\right) = \lim_{x\to 0^+} \frac{\sin x - x}{x\sin x}$$

which is an indeterminate form of type $0/0$. Applying L'Hôpital's rule twice yields

$$\lim_{x\to 0^+} \frac{\sin x - x}{x\sin x} = \lim_{x\to 0^+} \frac{\cos x - 1}{\sin x + x\cos x}$$

$$= \lim_{x\to 0^+} \frac{-\sin x}{\cos x + \cos x - x\sin x} = \frac{0}{2} = 0 \blacktriangleleft$$

INDETERMINATE FORMS OF TYPE $0^0, \infty^0, 1^\infty$

Limits of the form

$$\lim f(x)^{g(x)}$$

can give rise to ***indeterminate forms of the types*** $\mathbf{0^0}$, $\boldsymbol{\infty^0}$, ***and*** $\mathbf{1^\infty}$. (The interpretations of these symbols should be clear.) For example, the limit

$$\lim_{x \to 0^+} (1+x)^{1/x}$$

whose value we know to be e [see Formula (1) of Section 3.2] is an indeterminate form of type 1^∞. It is indeterminate because the expressions $1+x$ and $1/x$ exert two conflicting influences: the first approaches 1, which drives the expression toward 1, and the second approaches $+\infty$, which drives the expression toward $+\infty$.

Indeterminate forms of types 0^0, ∞^0, and 1^∞ can sometimes be evaluated by first introducing a dependent variable

$$y = f(x)^{g(x)}$$

and then computing the limit of $\ln y$. Since

$$\ln y = \ln[f(x)^{g(x)}] = g(x) \cdot \ln[f(x)]$$

the limit of $\ln y$ will be an indeterminate form of type $0 \cdot \infty$ (verify), which can be evaluated by methods we have already studied. Once the limit of $\ln y$ is known, it is a straightforward matter to determine the limit of $y = f(x)^{g(x)}$, as we will illustrate in the next example.

▶ Example 6 Find $\displaystyle\lim_{x \to 0} (1+\sin x)^{1/x}$.

Solution. As discussed above, we begin by introducing a dependent variable

$$y = (1+\sin x)^{1/x}$$

and taking the natural logarithm of both sides:

$$\ln y = \ln(1+\sin x)^{1/x} = \frac{1}{x}\ln(1+\sin x) = \frac{\ln(1+\sin x)}{x}$$

Thus,

$$\lim_{x \to 0} \ln y = \lim_{x \to 0} \frac{\ln(1+\sin x)}{x}$$

which is an indeterminate form of type $0/0$, so by L'Hôpital's rule

$$\lim_{x \to 0} \ln y = \lim_{x \to 0} \frac{\ln(1+\sin x)}{x} = \lim_{x \to 0} \frac{(\cos x)/(1+\sin x)}{1} = 1$$

Since we have shown that $\ln y \to 1$ as $x \to 0$, the continuity of the exponential function implies that $e^{\ln y} \to e^1$ as $x \to 0$, and this implies that $y \to e$ as $x \to 0$. Thus,

$$\lim_{x \to 0} (1+\sin x)^{1/x} = e \ ◀$$