INTEGRATING RATIONAL FUNCTIONS BY PARTIAL FRACTIONS

PARTIAL FRACTIONS:

In algebra, one learns to combine two or more fractions into a single fraction by finding a common denominator. For example,

$$\frac{2}{x-4}+\frac{3}{x+1}=\frac{2(x+1)+3(x-4)}{(x-4)(x+1)}=\frac{5x-10}{x^2-3x-4} \qquad (1)$$

FINDING THE FORM OF A PARTIAL FRACTION DECOMPOSITION

LINEAR FACTORS:

LINEAR FACTOR RULE For each factor of the form $(ax+b)^m$, the partial fraction decomposition contains the following sum of m partial fractions:

$$\frac{A_1}{ax+b}+\frac{A_2}{(ax+b)^2}+\cdots+\frac{A_m}{(ax+b)^m}$$

where $A_1, A_2, \ldots, A_m$ are constants to be determined. In the case where $m=1$, only the first term in the sum appears.

▸ Example 1 Evaluate $\displaystyle\int \frac{dx}{x^2+x-2}$.

Solution. The integrand is a proper rational function that can be written as

$$\frac{1}{x^2+x-2} = \frac{1}{(x-1)(x+2)}$$

The factors $x-1$ and $x+2$ are both linear and appear to the first power, so each contributes one term to the partial fraction decomposition by the linear factor rule. Thus, the decomposition has the form

$$\frac{1}{(x-1)(x+2)} = \frac{A}{x-1} + \frac{B}{x+2} \tag{5}$$

where A and B are constants to be determined. Multiplying this expression through by $(x-1)(x+2)$ yields

$$1 = A(x+2) + B(x-1) \tag{6}$$

As discussed earlier, there are two methods for finding A and B: we can substitute values of x that are chosen to make terms on the right drop out, or we can multiply out on the right and equate corresponding coefficients on the two sides to obtain a system of equations that can be solved for A and B. We will use the first approach.

Setting $x = 1$ makes the second term in (6) drop out and yields $1 = 3A$ or $A = \frac{1}{3}$; and setting $x = -2$ makes the first term in (6) drop out and yields $1 = -3B$ or $B = -\frac{1}{3}$. Substituting these values in (5) yields the partial fraction decomposition

$$\frac{1}{(x-1)(x+2)} = \frac{\frac{1}{3}}{x-1} + \frac{-\frac{1}{3}}{x+2}$$

The integration can now be completed as follows:

$$\begin{aligned}\int \frac{dx}{(x-1)(x+2)} &= \frac{1}{3}\int \frac{dx}{x-1} - \frac{1}{3}\int \frac{dx}{x+2} \\ &= \frac{1}{3}\ln|x-1| - \frac{1}{3}\ln|x+2| + C = \frac{1}{3}\ln\left|\frac{x-1}{x+2}\right| + C \quad ◂\end{aligned}$$

Example 2 Evaluate $\int \frac{2x+4}{x^3-2x^2}\,dx$.

Solution. The integrand can be rewritten as

$$\frac{2x+4}{x^3-2x^2} = \frac{2x+4}{x^2(x-2)}$$

Although x^2 is a quadratic factor, it is *not* irreducible since $x^2 = xx$. Thus, by the linear factor rule, x^2 introduces two terms (since $m = 2$) of the form

$$\frac{A}{x} + \frac{B}{x^2}$$

and the factor $x - 2$ introduces one term (since $m = 1$) of the form

$$\frac{C}{x-2}$$

so the partial fraction decomposition is

$$\frac{2x+4}{x^2(x-2)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-2} \tag{7}$$

Multiplying by $x^2(x-2)$ yields

$$2x + 4 = Ax(x-2) + B(x-2) + Cx^2 \tag{8}$$

which, after multiplying out and collecting like powers of x, becomes

$$2x + 4 = (A+C)x^2 + (-2A+B)x - 2B \tag{9}$$

Setting $x = 0$ in (8) makes the first and third terms drop out and yields $B = -2$, and setting $x = 2$ in (8) makes the first and second terms drop out and yields $C = 2$ (verify). However, there is no substitution in (8) that produces A directly, so we look to Equation (9) to find this value. This can be done by equating the coefficients of x^2 on the two sides to obtain

$$A + C = 0 \quad \text{or} \quad A = -C = -2$$

Substituting the values $A = -2$, $B = -2$, and $C = 2$ in (7) yields the partial fraction decomposition

$$\frac{2x+4}{x^2(x-2)} = \frac{-2}{x} + \frac{-2}{x^2} + \frac{2}{x-2}$$

Thus,

$$\begin{aligned}\int \frac{2x+4}{x^2(x-2)}\,dx &= -2\int \frac{dx}{x} - 2\int \frac{dx}{x^2} + 2\int \frac{dx}{x-2} \\ &= -2\ln|x| + \frac{2}{x} + 2\ln|x-2| + C = 2\ln\left|\frac{x-2}{x}\right| + \frac{2}{x} + C\end{aligned}$$

QUADRATIC FACTORS:

QUADRATIC FACTOR RULE For each factor of the form $(ax^2 + bx + c)^m$, the partial fraction decomposition contains the following sum of m partial fractions:

$$\frac{A_1 x + B_1}{ax^2 + bx + c} + \frac{A_2 x + B_2}{(ax^2 + bx + c)^2} + \cdots + \frac{A_m x + B_m}{(ax^2 + bx + c)^m}$$

where $A_1, A_2, \ldots, A_m, B_1, B_2, \ldots, B_m$ are constants to be determined. In the case where $m = 1$, only the first term in the sum appears.

▶ **Example 3** Evaluate $\displaystyle\int \frac{x^2 + x - 2}{3x^3 - x^2 + 3x - 1}\,dx$.

Solution. The denominator in the integrand can be factored by grouping:

$$3x^3 - x^2 + 3x - 1 = x^2(3x - 1) + (3x - 1) = (3x - 1)(x^2 + 1)$$

By the linear factor rule, the factor $3x - 1$ introduces one term, namely,

$$\frac{A}{3x - 1}$$

and by the quadratic factor rule, the factor $x^2 + 1$ introduces one term, namely,

$$\frac{Bx + C}{x^2 + 1}$$

Thus, the partial fraction decomposition is

$$\frac{x^2 + x - 2}{(3x - 1)(x^2 + 1)} = \frac{A}{3x - 1} + \frac{Bx + C}{x^2 + 1} \tag{10}$$

Multiplying by $(3x - 1)(x^2 + 1)$ yields

$$x^2 + x - 2 = A(x^2 + 1) + (Bx + C)(3x - 1) \tag{11}$$

We could find A by substituting $x = \frac{1}{3}$ to make the last term drop out, and then find the rest of the constants by equating corresponding coefficients. However, in this case it is just as easy to find *all* of the constants by equating coefficients and solving the resulting system. For this purpose we multiply out the right side of (11) and collect like terms:

$$x^2 + x - 2 = (A + 3B)x^2 + (-B + 3C)x + (A - C)$$

Equating corresponding coefficients gives

$$\begin{aligned} A + 3B \qquad &= 1 \\ - B + 3C &= 1 \\ A \qquad - C &= -2 \end{aligned}$$

To solve this system, subtract the third equation from the first to eliminate A. Then use the resulting equation together with the second equation to solve for B and C. Finally, determine A from the first or third equation. This yields (verify)

$$A = -\frac{7}{5}, \quad B = \frac{4}{5}, \quad C = \frac{3}{5}$$

Thus, (10) becomes

$$\frac{x^2 + x - 2}{(3x-1)(x^2+1)} = \frac{-\frac{7}{5}}{3x-1} + \frac{\frac{4}{5}x + \frac{3}{5}}{x^2+1}$$

and

$$\begin{aligned} \int \frac{x^2 + x - 2}{(3x-1)(x^2+1)}\,dx &= -\frac{7}{5}\int \frac{dx}{3x-1} + \frac{4}{5}\int \frac{x}{x^2+1}\,dx + \frac{3}{5}\int \frac{dx}{x^2+1} \\ &= -\frac{7}{15}\ln|3x-1| + \frac{2}{5}\ln(x^2+1) + \frac{3}{5}\tan^{-1} x + C \end{aligned}$$

▸ Example 4 Evaluate $\displaystyle\int \frac{3x^4 + 4x^3 + 16x^2 + 20x + 9}{(x+2)(x^2+3)^2}\,dx.$

Solution. Observe that the integrand is a proper rational function since the numerator has degree 4 and the denominator has degree 5. Thus, the method of partial fractions is applicable. By the linear factor rule, the factor $x + 2$ introduces the single term

$$\frac{A}{x+2}$$

and by the quadratic factor rule, the factor $(x^2 + 3)^2$ introduces two terms (since $m = 2$):

$$\frac{Bx + C}{x^2 + 3} + \frac{Dx + E}{(x^2+3)^2}$$

Thus, the partial fraction decomposition of the integrand is

$$\frac{3x^4 + 4x^3 + 16x^2 + 20x + 9}{(x+2)(x^2+3)^2} = \frac{A}{x+2} + \frac{Bx + C}{x^2 + 3} + \frac{Dx + E}{(x^2+3)^2} \tag{12}$$

Multiplying by $(x + 2)(x^2 + 3)^2$ yields

$$\begin{aligned} &3x^4 + 4x^3 + 16x^2 + 20x + 9 \\ &\qquad = A(x^2+3)^2 + (Bx + C)(x^2+3)(x+2) + (Dx + E)(x+2) \end{aligned} \tag{13}$$

which, after multiplying out and collecting like powers of x, becomes

$$\begin{aligned} &3x^4 + 4x^3 + 16x^2 + 20x + 9 \\ &\qquad = (A + B)x^4 + (2B + C)x^3 + (6A + 3B + 2C + D)x^2 \\ &\qquad\qquad + (6B + 3C + 2D + E)x + (9A + 6C + 2E) \end{aligned} \tag{14}$$

Equating corresponding coefficients in (14) yields the following system of five linear equations in five unknowns:

$$\begin{aligned} A + B &= 3 \\ 2B + C &= 4 \\ 6A + 3B + 2C + D &= 16 \\ 6B + 3C + 2D + E &= 20 \\ 9A + 6C + 2E &= 9 \end{aligned} \tag{15}$$

in (13), which yields $A = 1$. Substituting this known value of A in (15) yields the simpler system

$$\begin{aligned} B &= 2 \\ 2B + C &= 4 \\ 3B + 2C + D &= 10 \\ 6B + 3C + 2D + E &= 20 \\ 6C + 2E &= 0 \end{aligned} \tag{16}$$

This system can be solved by starting at the top and working down, first substituting $B = 2$ in the second equation to get $C = 0$, then substituting the known values of B and C in the third equation to get $D = 4$, and so forth. This yields

$$A = 1, \quad B = 2, \quad C = 0, \quad D = 4, \quad E = 0$$

Thus, (12) becomes

$$\frac{3x^4 + 4x^3 + 16x^2 + 20x + 9}{(x+2)(x^2+3)^2} = \frac{1}{x+2} + \frac{2x}{x^2+3} + \frac{4x}{(x^2+3)^2}$$

and so

$$\begin{aligned} &\int \frac{3x^4 + 4x^3 + 16x^2 + 20x + 9}{(x+2)(x^2+3)^2}\, dx \\ &\qquad = \int \frac{dx}{x+2} + \int \frac{2x}{x^2+3}\, dx + 4\int \frac{x}{(x^2+3)^2}\, dx \\ &\qquad = \ln|x+2| + \ln(x^2+3) - \frac{2}{x^2+3} + C \end{aligned}$$

INTEGRATING IMPROPER RATIONAL FUNCTIONS

Although the method of partial fractions only applies to proper rational functions, an improper rational function can be integrated by performing a long division and expressing the function as the quotient plus the remainder over the divisor. The remainder over the divisor will be a proper rational function, which can then be decomposed into partial fractions. This idea is illustrated in the following example.

▸ Example 5 Evaluate $\displaystyle\int \frac{3x^4+3x^3-5x^2+x-1}{x^2+x-2}\,dx$.

Solution. The integrand is an improper rational function since the numerator has degree 4 and the denominator has degree 2. Thus, we first perform the long division

$$
\begin{array}{r|l}
 & \phantom{3x^4+3x^3-{}}3x^2 \phantom{{}+x}\;+1 \\
\cline{2-2}
x^2+x-2 & 3x^4+3x^3-5x^2+x-1 \\
 & \underline{3x^4+3x^3-6x^2} \\
 & x^2+x-1 \\
 & \underline{x^2+x-2} \\
 & \phantom{3x^4+3x^3-5x^2+x-{}}1
\end{array}
$$

It follows that the integrand can be expressed as

$$\frac{3x^4+3x^3-5x^2+x-1}{x^2+x-2} = (3x^2+1) + \frac{1}{x^2+x-2}$$

and hence

$$\int \frac{3x^4+3x^3-5x^2+x-1}{x^2+x-2}\,dx = \int (3x^2+1)\,dx + \int \frac{dx}{x^2+x-2}$$

The second integral on the right now involves a proper rational function and can thus be evaluated by a partial fraction decomposition. Using the result of Example 1 we obtain

$$\int \frac{3x^4+3x^3-5x^2+x-1}{x^2+x-2}\,dx = x^3+x+\frac{1}{3}\ln\left|\frac{x-1}{x+2}\right| + C \quad ◂$$

EXERCISE SET 7.5

9–34 Evaluate the integral.

9. $\displaystyle\int \frac{dx}{x^2 - 3x - 4}$

10. $\displaystyle\int \frac{dx}{x^2 - 6x - 7}$

11. $\displaystyle\int \frac{11x + 17}{2x^2 + 7x - 4}\,dx$

12. $\displaystyle\int \frac{5x - 5}{3x^2 - 8x - 3}\,dx$

13. $\displaystyle\int \frac{2x^2 - 9x - 9}{x^3 - 9x}\,dx$

14. $\displaystyle\int \frac{dx}{x(x^2 - 1)}$

15. $\displaystyle\int \frac{x^2 - 8}{x + 3}\,dx$

16. $\displaystyle\int \frac{x^2 + 1}{x - 1}\,dx$

17. $\displaystyle\int \frac{3x^2 - 10}{x^2 - 4x + 4}\,dx$

18. $\displaystyle\int \frac{x^2}{x^2 - 3x + 2}\,dx$

19. $\displaystyle\int \frac{2x - 3}{x^2 - 3x - 10}\,dx$

20. $\displaystyle\int \frac{3x + 1}{3x^2 + 2x - 1}\,dx$

21. $\displaystyle\int \frac{x^5 + x^2 + 2}{x^3 - x}\,dx$

22. $\displaystyle\int \frac{x^5 - 4x^3 + 1}{x^3 - 4x}\,dx$

23. $\displaystyle\int \frac{2x^2 + 3}{x(x - 1)^2}\,dx$

24. $\displaystyle\int \frac{3x^2 - x + 1}{x^3 - x^2}\,dx$

25. $\displaystyle\int \frac{2x^2 - 10x + 4}{(x + 1)(x - 3)^2}\,dx$

26. $\displaystyle\int \frac{2x^2 - 2x - 1}{x^3 - x^2}\,dx$

27. $\displaystyle\int \frac{x^2}{(x + 1)^3}\,dx$

28. $\displaystyle\int \frac{2x^2 + 3x + 3}{(x + 1)^3}\,dx$

29. $\displaystyle\int \frac{2x^2 - 1}{(4x - 1)(x^2 + 1)}\,dx$

30. $\displaystyle\int \frac{dx}{x^3 + 2x}$

SOLUTION SET

9. $\dfrac{1}{(x-4)(x+1)}=\dfrac{A}{x-4}+\dfrac{B}{x+1}$; $A=\dfrac{1}{5}$, $B=-\dfrac{1}{5}$, so $\dfrac{1}{5}\displaystyle\int\dfrac{1}{x-4}dx-\dfrac{1}{5}\int\dfrac{1}{x+1}dx=\dfrac{1}{5}\ln|x-4|-\dfrac{1}{5}\ln|x+1|+C=$ $\dfrac{1}{5}\ln\left|\dfrac{x-4}{x+1}\right|+C$.

11. $\dfrac{11x+17}{(2x-1)(x+4)}=\dfrac{A}{2x-1}+\dfrac{B}{x+4}$; $A=5$, $B=3$, so $5\displaystyle\int\dfrac{1}{2x-1}dx+3\int\dfrac{1}{x+4}dx=\dfrac{5}{2}\ln|2x-1|+3\ln|x+4|+C$.

13. $\dfrac{2x^2-9x-9}{x(x+3)(x-3)}=\dfrac{A}{x}+\dfrac{B}{x+3}+\dfrac{C}{x-3}$; $A=1$, $B=2$, $C=-1$, so $\displaystyle\int\dfrac{1}{x}\,dx+2\int\dfrac{1}{x+3}\,dx-\int\dfrac{1}{x-3}\,dx=$ $\ln|x|+2\ln|x+3|-\ln|x-3|+C=\ln\left|\dfrac{x(x+3)^2}{x-3}\right|+C$. Note that the symbol C has been recycled; to save space this recycling is usually not mentioned.

15. $\dfrac{x^2-8}{x+3}=x-3+\dfrac{1}{x+3}$, $\displaystyle\int\left(x-3+\dfrac{1}{x+3}\right)dx=\dfrac{1}{2}x^2-3x+\ln|x+3|+C$.

17. $\dfrac{3x^2-10}{x^2-4x+4}=3+\dfrac{12x-22}{x^2-4x+4}$, $\dfrac{12x-22}{(x-2)^2}=\dfrac{A}{x-2}+\dfrac{B}{(x-2)^2}$; $A=12$, $B=2$, so $\displaystyle\int 3dx+12\int\dfrac{1}{x-2}dx+$ $2\displaystyle\int\dfrac{1}{(x-2)^2}dx=3x+12\ln|x-2|-2/(x-2)+C$.

19. $u=x^2-3x-10, du=(2x-3)\,dx$, $\displaystyle\int\dfrac{du}{u}=\ln|u|+C=\ln|x^2-3x-10|+C$.

21. $\dfrac{x^5+x^2+2}{x^3-x}=x^2+1+\dfrac{x^2+x+2}{x^3-x}$, $\dfrac{x^2+x+2}{x(x+1)(x-1)}=\dfrac{A}{x}+\dfrac{B}{x+1}+\dfrac{C}{x-1}$; $A=-2$, $B=1$, $C=2$, so $\displaystyle\int(x^2+1)dx-$ $\displaystyle\int\dfrac{2}{x}dx+\int\dfrac{1}{x+1}dx+\int\dfrac{2}{x-1}dx=\dfrac{1}{3}x^3+x-2\ln|x|+\ln|x+1|+2\ln|x-1|+C=\dfrac{1}{3}x^3+x+\ln\left|\dfrac{(x+1)(x-1)^2}{x^2}\right|+C.$

23. $\dfrac{2x^2+3}{x(x-1)^2}=\dfrac{A}{x}+\dfrac{B}{x-1}+\dfrac{C}{(x-1)^2}$; $A=3$, $B=-1$, $C=5$, so $3\displaystyle\int\dfrac{1}{x}\,dx-\int\dfrac{1}{x-1}\,dx+5\int\dfrac{1}{(x-1)^2}dx=$ $3\ln|x|-\ln|x-1|-5/(x-1)+C$.

25. $\dfrac{2x^2-10x+4}{(x+1)(x-3)^2}=\dfrac{A}{x+1}+\dfrac{B}{x-3}+\dfrac{C}{(x-3)^2}$; $A=1$, $B=1$, $C=-2$, so $\displaystyle\int\dfrac{1}{x+1}\,dx+\int\dfrac{1}{x-3}\,dx-\int\dfrac{2}{(x-3)^2}\,dx=$ $\ln|x+1|+\ln|x-3|+\dfrac{2}{x-3}+C_1$.

27. $\dfrac{x^2}{(x+1)^3}=\dfrac{A}{x+1}+\dfrac{B}{(x+1)^2}+\dfrac{C}{(x+1)^3}$; $A=1$, $B=-2$, $C=1$, so $\displaystyle\int\dfrac{1}{x+1}\,dx-\int\dfrac{2}{(x+1)^2}\,dx+\int\dfrac{1}{(x+1)^3}\,dx=$ $\ln|x+1|+\dfrac{2}{x+1}-\dfrac{1}{2(x+1)^2}+C$.

29. $\dfrac{2x^2-1}{(4x-1)(x^2+1)}=\dfrac{A}{4x-1}+\dfrac{Bx+C}{x^2+1}$; $A=-14/17$, $B=12/17$, $C=3/17$, so $\displaystyle\int\dfrac{2x^2-1}{(4x-1)(x^2+1)}dx=$ $-\dfrac{7}{34}\ln|4x-1|+\dfrac{6}{17}\ln(x^2+1)+\dfrac{3}{17}\tan^{-1}x+C$.