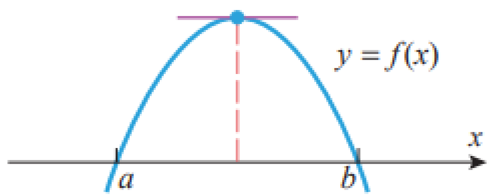
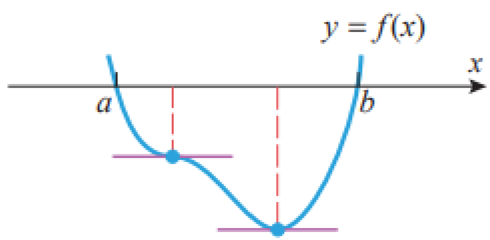
The Rolle's Theorem

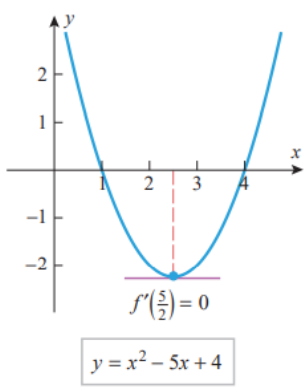
4.8.1 THEOREM (***Rolle's Theorem***) *Let f be continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b). If*

$$f(a) = 0 \quad \textit{and} \quad f(b) = 0$$

then there is at least one point c in the interval (a, b) such that $f'(c) = 0$.







▲ **Figure 4.8.2**

▶ **Example 1** Find the two x-intercepts of the function $f(x) = x^2 - 5x + 4$ and confirm that $f'(c) = 0$ at some point c between those intercepts.

Solution. The function f can be factored as

$$x^2 - 5x + 4 = (x - 1)(x - 4)$$

so the x-intercepts are $x = 1$ and $x = 4$. Since the polynomial f is continuous and differentiable everywhere, the hypotheses of Rolle's Theorem are satisfied on the interval $[1, 4]$. Thus, we are guaranteed the existence of at least one point c in the interval $(1, 4)$ such that $f'(c) = 0$. Differentiating f yields

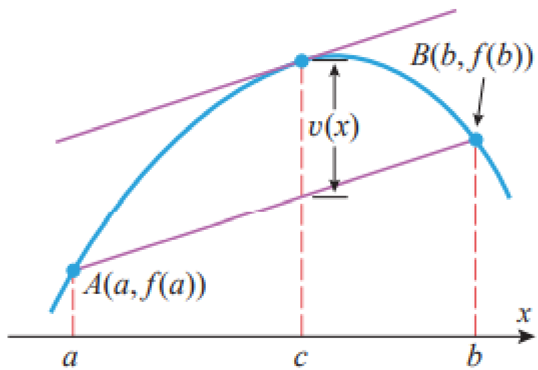
$$f'(x) = 2x - 5$$

Solving the equation $f'(x) = 0$ yields $x = \frac{5}{2}$, so $c = \frac{5}{2}$ is a point in the interval $(1, 4)$ at which $f'(c) = 0$ (Figure 4.8.2). ◀

The Mean Value Theorem

4.8.2 THEOREM (*Mean-Value Theorem*) *Let f be continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b). Then there is at least one point c in (a, b) such that*

$$f'(c) = \frac{f(b) - f(a)}{b - a} \tag{1}$$



The tangent line is parallel to the secant line where the vertical distance $v(x)$ between the secant line and the graph of f is maximum.

Example 4 Show that the function $f(x) = \frac{1}{4}x^3 + 1$ satisfies the hypotheses of the Mean-Value Theorem over the interval $[0, 2]$, and find all values of c in the interval $(0, 2)$ at which the tangent line to the graph of f is parallel to the secant line joining the points $(0, f(0))$ and $(2, f(2))$.

Solution. The function f is continuous and differentiable everywhere because it is a polynomial. In particular, f is continuous on $[0, 2]$ and differentiable on $(0, 2)$, so the hypotheses of the Mean-Value Theorem are satisfied with $a = 0$ and $b = 2$. But

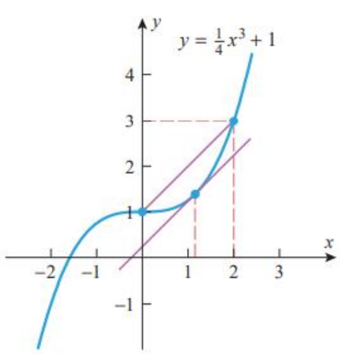
$$f(a) = f(0) = 1, \quad f(b) = f(2) = 3$$

$$f'(x) = \frac{3x^2}{4}, \qquad f'(c) = \frac{3c^2}{4}$$

so in this case Equation (1) becomes

$$\frac{3c^2}{4} = \frac{3-1}{2-0} \quad \text{or} \quad 3c^2 = 4$$

which has the two solutions $c = \pm 2/\sqrt{3} \approx \pm 1.15$. However, only the positive solution lies in the interval $(0, 2)$; this value of c is consistent with Figure 4.8.7. ◄



▲ **Figure 4.8.7**

Question set

1–4 Verify that the hypotheses of Rolle's Theorem are satisfied on the given interval, and find all values of c in that interval that satisfy the conclusion of the theorem.

1. $f(x) = x^2 - 8x + 15;\ [3, 5]$

2. $f(x) = \frac{1}{2}x - \sqrt{x};\ [0, 4]$

3. $f(x) = \cos x;\ [\pi/2, 3\pi/2]$

4. $f(x) = \ln(4 + 2x - x^2);\ [-1, 3]$

Solution

1. f is continuous on $[3,5]$ and differentiable on $(3,5)$, $f(3) = f(5) = 0$; $f'(x) = 2x - 8$, $2c - 8 = 0$, $c = 4$, $f'(4) = 0$.

2. f is continuous on $[0,4]$ and differentiable on $(0,4)$, $f(0) = f(4) = 0$; $f'(x) = \frac{1}{2} - \frac{1}{2\sqrt{x}}$, $\frac{1}{2} - \frac{1}{2\sqrt{c}} = 0$, $c = 1$, $f'(1) = 0$.

3. f is continuous on $[\pi/2, 3\pi/2]$ and differentiable on $(\pi/2, 3\pi/2)$, $f(\pi/2) = f(3\pi/2) = 0$, $f'(x) = -\sin x$, $-\sin c = 0$, $c = \pi$.

4. f is continuous on $[-1,3]$ and differentiable on $(-1,3)$, $f(-1) = f(3) = 0$; $f'(1) = 0$; $f'(x) = 2(1-x)/(4+2x-x^2)$; $2(1-c) = 0$, $c = 1$.

Question set

5–8 Verify that the hypotheses of the Mean-Value Theorem are satisfied on the given interval, and find all values of c in that interval that satisfy the conclusion of the theorem.

5. $f(x) = x^2 - x;\ [-3, 5]$

6. $f(x) = x^3 + x - 4;\ [-1, 2]$

7. $f(x) = \sqrt{25 - x^2};\ [-5, 3]$

8. $f(x) = x - \dfrac{1}{x};\ [3, 4]$

Solution

5. f is continuous on $[-3,5]$ and differentiable on $(-3,5)$, $(f(5)-f(-3))/(5-(-3)) = 1$; $f'(x) = 2x-1$; $2c-1 = 1$, $c = 1$.

6. f is continuous on $[-1,2]$ and differentiable on $(-1,2)$, $f(-1) = -6$, $f(2) = 6$, $f'(x) = 3x^2+1$, $3c^2+1 = \dfrac{6-(-6)}{2-(-1)} = 4$, $c^2 = 1$, $c = \pm 1$ of which only $c = 1$ is in $(-1,2)$.

7. f is continuous on $[-5,3]$ and differentiable on $(-5,3)$, $(f(3)-f(-5))/(3-(-5)) = 1/2$; $f'(x) = -\dfrac{x}{\sqrt{25-x^2}}$; $-\dfrac{c}{\sqrt{25-c^2}} = 1/2$, $c = -\sqrt{5}$.

8. f is continuous on $[3,4]$ and differentiable on $(3,4)$, $f(4) = 15/4$, $f(3) = 8/3$, solve $f'(c) = (15/4-8/3)/1 = 13/12$; $f'(x) = 1+1/x^2$, $f'(c) = 1+1/c^2 = 13/12$, $c^2 = 12$, $c = \pm 2\sqrt{3}$, but $-2\sqrt{3}$ is not in the interval, so $c = 2\sqrt{3}$.