SECOND DERIVATIVE TEST

SECOND DERIVATIVE TEST

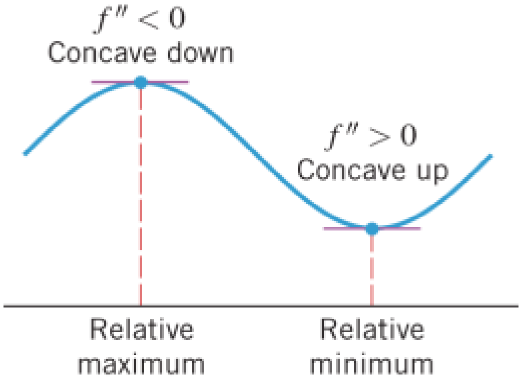
There is another test for relative extrema that is based on the following geometric observation: A function f has a relative maximum at a stationary point if the graph of f is concave down on an open interval containing that point, and it has a relative minimum if it is concave up (Figure 3.2.7).

3.2.4 THEOREM (*Second Derivative Test*) *Suppose that f is twice differentiable at the point x_0.*

(*a*) *If $f'(x_0) = 0$ and $f''(x_0) > 0$, then f has a relative minimum at x_0.*

(*b*) *If $f'(x_0) = 0$ and $f''(x_0) < 0$, then f has a relative maximum at x_0.*

(*c*) *If $f'(x_0) = 0$ and $f''(x_0) = 0$, then the test is inconclusive; that is, f may have a relative maximum, a relative minimum, or neither at x_0.*



▲ **Figure 3.2.7**

Question: Locate the critical points and identify which critical points are stationary points.

10. $f(x) = \dfrac{x^2}{x^3+8}$

Solution:

$$f'(x) = -\frac{x(x^3-16)}{(x^3+8)^2}, \text{ so stationary points at } x = 0, 2^{4/3}.$$

▶ **Example 5** Find the relative extrema of $f(x) = 3x^5 - 5x^3$.

Solution. We have

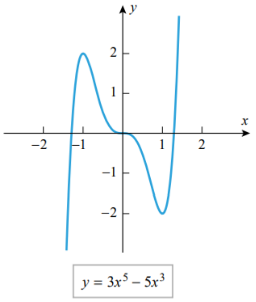
$$f'(x) = 15x^4 - 15x^2 = 15x^2(x^2-1) = 15x^2(x+1)(x-1)$$
$$f''(x) = 60x^3 - 30x = 30x(2x^2-1)$$

Solving $f'(x) = 0$ yields the stationary points $x = 0$, $x = -1$, and $x = 1$. As shown in the following table, we can conclude from the second derivative test that f has a relative maximum at $x = -1$ and a relative minimum at $x = 1$.

STATIONARY POINT	$30x(2x^2-1)$	$f''(x)$	SECOND DERIVATIVE TEST
$x = -1$	-30	$-$	f has a relative maximum
$x = 0$	0	0	Inconclusive
$x = 1$	30	$+$	f has a relative minimum

The test is inconclusive at $x = 0$, so we will try the first derivative test at that point. A sign analysis of f' is given in the following table:

INTERVAL	$15x^2(x+1)(x-1)$	$f'(x)$
$-1 < x < 0$	$(+)(+)(-)$	$-$
$0 < x < 1$	$(+)(+)(-)$	$-$



Since there is no sign change in f' at $x = 0$, there is neither a relative maximum nor a relative minimum at that point. All of this is consistent with the graph of f shown in Figure 4.2.8. ◀

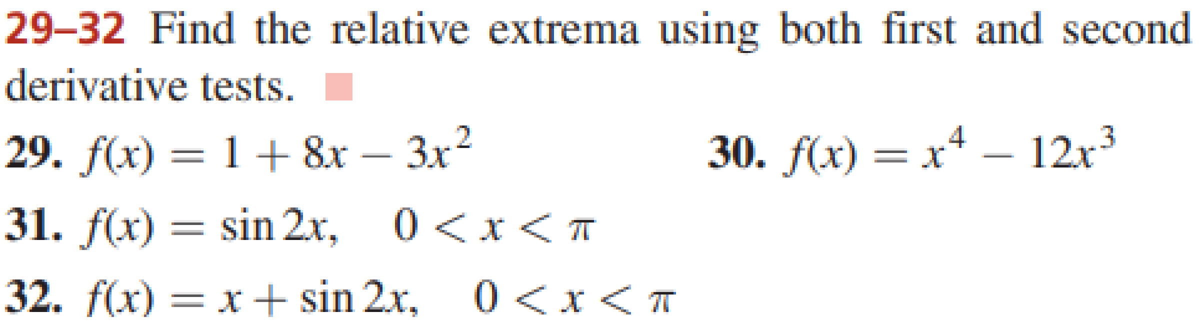
Question: Find the relative extrema using both first and second derivative tests.

$$\textbf{35.}\quad f(x) = \sin 2x, \quad 0 < x < \pi$$

Solution:

$f'(x) = 2\cos 2x$: critical points at $x = \pi/4, 3\pi/4$, $f''(\pi/4) = -4$: f has a relative maximum of 1 at $x = \pi/4$, $f''(3\pi/4) = 4$: f has a relative minimum of -1 at $x = 3\pi/4$.

EXERCISE SET 3.2



29–32 Find the relative extrema using both first and second derivative tests. ■

29. $f(x) = 1 + 8x - 3x^2$

30. $f(x) = x^4 - 12x^3$

31. $f(x) = \sin 2x, \quad 0 < x < \pi$

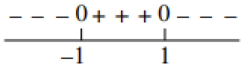
32. $f(x) = x + \sin 2x, \quad 0 < x < \pi$

SOLUTION SET:

29. f':

Critical point: $x = 0$; $x = 0$: relative minimum.

31. f':



Critical points: $x = -1, 1$; $x = -1$: relative minimum, $x = 1$: relative maximum.