Volume of Surface of Revolution

6.2.2 VOLUME FORMULA Let S be a solid bounded by two parallel planes perpendicular to the x-axis at $x = a$ and $x = b$. If, for each x in $[a, b]$, the cross-sectional area of S perpendicular to the x-axis is $A(x)$, then the volume of the solid is

$$V = \int_a^b A(x)\,dx \tag{3}$$

provided $A(x)$ is integrable.

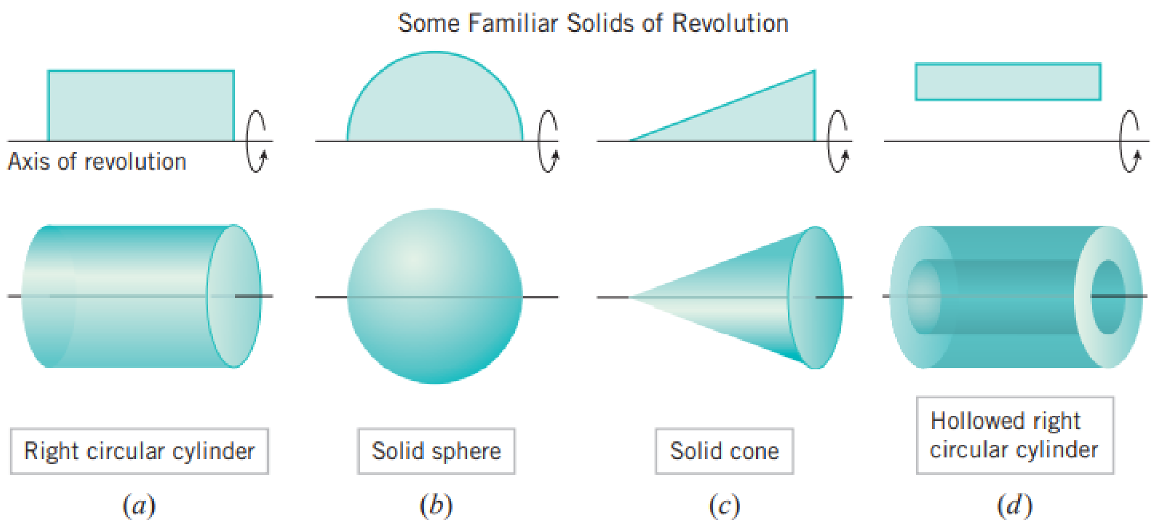
6.2.3 VOLUME FORMULA Let S be a solid bounded by two parallel planes perpendicular to the y-axis at $y = c$ and $y = d$. If, for each y in $[c, d]$, the cross-sectional area of S perpendicular to the y-axis is $A(y)$, then the volume of the solid is

$$V = \int_c^d A(y)\,dy \tag{4}$$

provided $A(y)$ is integrable.

SOLIDS OF REVOLUTION

A ***solid of revolution*** is a solid that is generated by revolving a plane region about a line that lies in the same plane as the region; the line is called the ***axis of revolution***. Many familiar solids are of this type (Figure 6.2.8).



Disk Method:

VOLUMES BY DISKS PERPENDICULAR TO THE x-AXIS

We will be interested in the following general problem.

> **6.2.4 PROBLEM** Let f be continuous and nonnegative on $[a, b]$, and let R be the region that is bounded above by $y = f(x)$, below by the x-axis, and on the sides by the lines $x = a$ and $x = b$ (Figure 6.2.9a). Find the volume of the solid of revolution that is generated by revolving the region R about the x-axis.

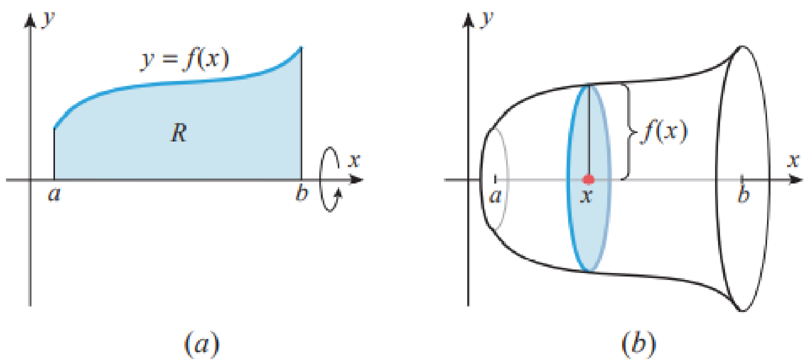


Figure 6.2.9

We can solve this problem by slicing. For this purpose, observe that the cross section of the solid taken perpendicular to the x-axis at the point x is a circular disk of radius $f(x)$ (Figure 5.2.9b). The area of this region is

$$A(x) = \pi[f(x)]^2$$

Thus, from (3) the volume of the solid is

$$V = \int_a^b \pi[f(x)]^2\,dx \qquad (5)$$

Because the cross sections are disk shaped, the application of this formula is called the ***method of disks***.

Example 2 Find the volume of the solid that is obtained when the region under the curve $y = \sqrt{x}$ over the interval $[1, 4]$ is revolved about the x-axis (Figure 6.2.10).

Solution. From (5), the volume is

$$V = \int_a^b \pi[f(x)]^2\,dx = \int_1^4 \pi x\,dx = \left.\frac{\pi x^2}{2}\right]_1^4 = 8\pi - \frac{\pi}{2} = \frac{15\pi}{2}$$

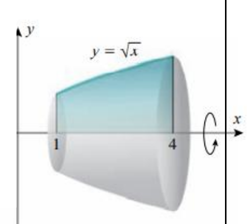


Figure 6.2.10

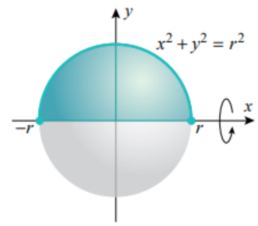
▶ **Example 3** Derive the formula for the volume of a sphere of radius r.

Solution. As indicated in Figure 6.2.11, a sphere of radius r can be generated by revolving the upper semicircular disk enclosed between the x-axis and

$$x^2 + y^2 = r^2$$

about the x-axis. Since the upper half of this circle is the graph of $y = f(x) = \sqrt{r^2 - x^2}$, it follows from (5) that the volume of the sphere is

$$V = \int_a^b \pi[f(x)]^2\,dx = \int_{-r}^{r} \pi(r^2 - x^2)\,dx = \pi\left[r^2x - \frac{x^3}{3}\right]_{-r}^{r} = \frac{4}{3}\pi r^3 \blacktriangleleft$$



Washers Method:

6.2.5 PROBLEM Let f and g be continuous and nonnegative on $[a, b]$, and suppose that $f(x) \geq g(x)$ for all x in the interval $[a, b]$. Let R be the region that is bounded above by $y = f(x)$, below by $y = g(x)$, and on the sides by the lines $x = a$ and $x = b$ (Figure 6.2.12a). Find the volume of the solid of revolution that is generated by revolving the region R about the x-axis (Figure 6.2.12b).

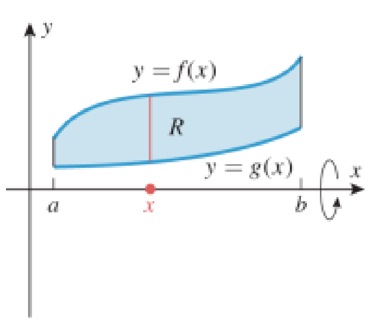
We can solve this problem by slicing. For this purpose, observe that the cross section of the solid taken perpendicular to the x-axis at the point x is the annular or "washer-shaped" region with inner radius $g(x)$ and outer radius $f(x)$ (Figure 5.2.12b); its area is

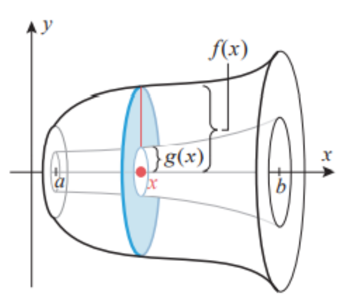
$$A(x) = \pi[f(x)]^2 - \pi[g(x)]^2 = \pi([f(x)]^2 - [g(x)]^2)$$

Thus, from (3) the volume of the solid is

$$V = \int_a^b \pi([f(x)]^2 - [g(x)]^2)\,dx \tag{6}$$

Because the cross sections are washer shaped, the application of this formula is called the ***method of washers***.



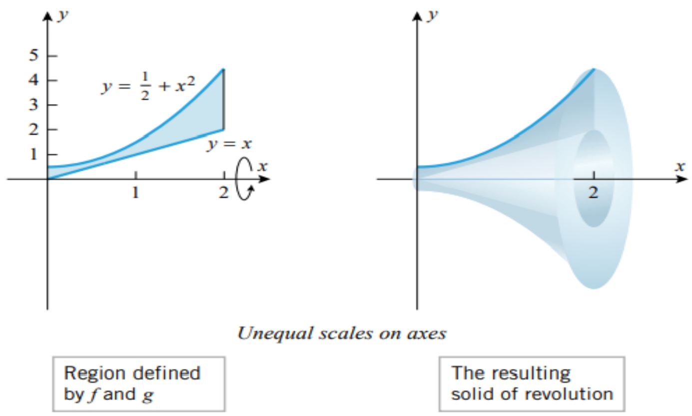


▶ **Example 4** Find the volume of the solid generated when the region between the graphs of the equations $f(x) = \frac{1}{2} + x^2$ and $g(x) = x$ over the interval $[0, 2]$ is revolved about the x-axis.

Solution. First sketch the region (Figure 6.2.13*a*); then imagine revolving it about the x-axis (Figure 6.2.13*b*). From (6) the volume is

$$V = \int_a^b \pi([f(x)]^2 - [g(x)]^2)\,dx = \int_0^2 \pi\left(\left[\tfrac{1}{2} + x^2\right]^2 - x^2\right) dx$$

$$= \int_0^2 \pi\left(\frac{1}{4} + x^4\right) dx = \pi\left[\frac{x}{4} + \frac{x^5}{5}\right]_0^2 = \frac{69\pi}{10}$$ ◀

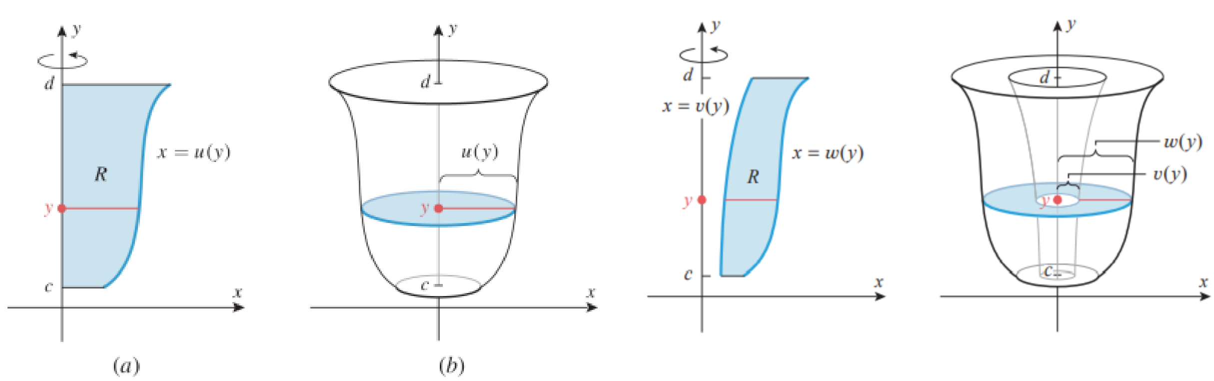


VOLUMES BY DISKS AND WASHERS PERPENDICULAR TO THE *y*-AXIS

The methods of disks and washers have analogs for regions that are revolved about the y-axis (Figures 5.2.14 and 5.2.15). Using the method of slicing and Formula (4), you should be able to deduce the following formulas for the volumes of the solids in the figures.

$$V = \int_c^d \pi[u(y)]^2\,dy \qquad \text{Disks}$$

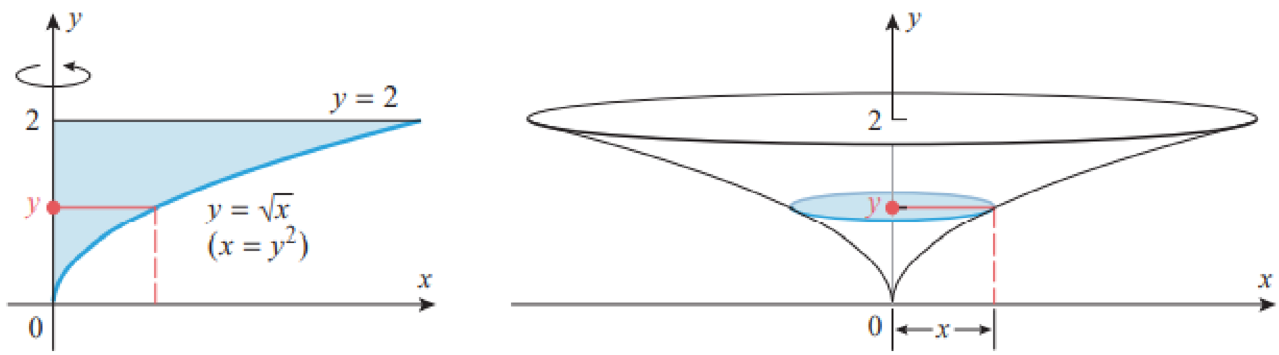
$$V = \int_c^d \pi([w(y)]^2 - [v(y)]^2)\,dy \qquad \text{Washers} \qquad (7\text{–}8)$$



▶ Example 5 Find the volume of the solid generated when the region enclosed by $y = \sqrt{x}$, $y = 2$, and $x = 0$ is revolved about the y-axis.

Solution. First sketch the region and the solid (Figure 6.2.16). The cross sections taken perpendicular to the y-axis are disks, so we will apply (7). But first we must rewrite $y = \sqrt{x}$ as $x = y^2$. Thus, from (7) with $u(y) = y^2$, the volume is

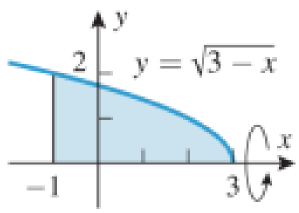
$$V = \int_c^d \pi[u(y)]^2\,dy = \int_0^2 \pi y^4\,dy = \frac{\pi y^5}{5}\bigg]_0^2 = \frac{32\pi}{5} \blacktriangleleft$$



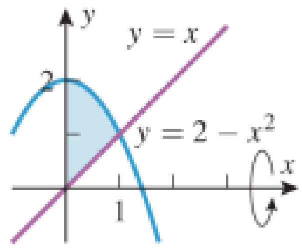
EXERCISE SET 5.2

1–8 Find the volume of the solid that results when the shaded region is revolved about the indicated axis.

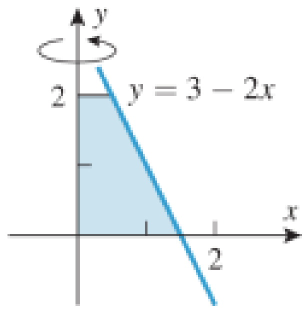
1.



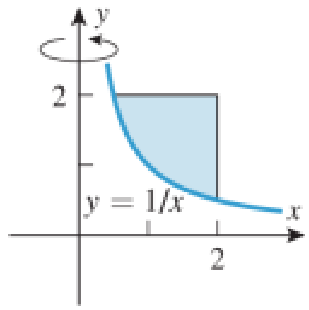
2.



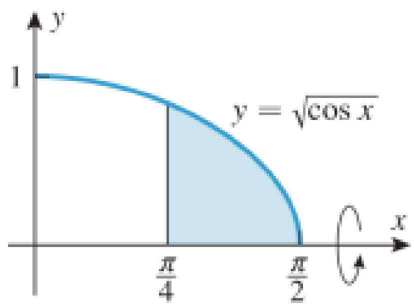
3.



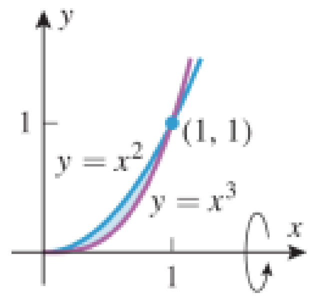
4.



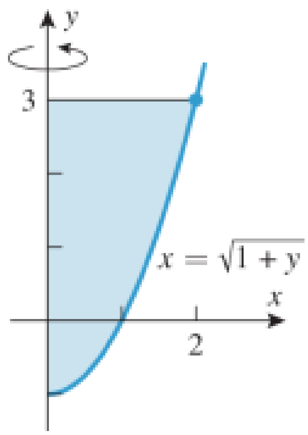
5.



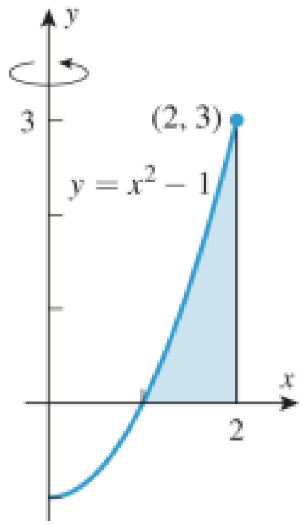
6.



7.



8.



9. Find the volume of the solid whose base is the region bounded between the curve $y = x^2$ and the x-axis from $x = 0$ to $x = 2$ and whose cross sections taken perpendicular to the x-axis are squares.

10. Find the volume of the solid whose base is the region bounded between the curve $y = \sec x$ and the x-axis from $x = \pi/4$ to $x = \pi/3$ and whose cross sections taken perpendicular to the x-axis are squares.

11–14 Find the volume of the solid that results when the region enclosed by the given curves is revolved about the x-axis.

11. $y = \sqrt{25 - x^2},\ y = 3$

12. $y = 9 - x^2,\ y = 0$

13. $x = \sqrt{y},\ x = y/4$

14. $y = \sin x,\ y = \cos x,\ x = 0,\ x = \pi/4$
[*Hint:* Use the identity $\cos 2x = \cos^2 x - \sin^2 x$.]

15. Find the volume of the solid whose base is the region bounded between the curve $y = x^3$ and the y-axis from $y = 0$ to $y = 1$ and whose cross sections taken perpendicular to the y-axis are squares.

16. Find the volume of the solid whose base is the region enclosed between the curve $x = 1 - y^2$ and the y-axis and whose cross sections taken perpendicular to the y-axis are squares.

17–20 Find the volume of the solid that results when the region enclosed by the given curves is revolved about the y-axis.

17. $x = \csc y,\ y = \pi/4,\ y = 3\pi/4,\ x = 0$

18. $y = x^2,\ x = y^2$

19. $x = y^2,\ x = y + 2$

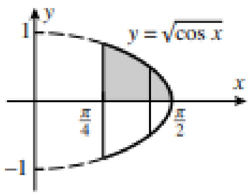
20. $x = 1 - y^2,\ x = 2 + y^2,\ y = -1,\ y = 1$

SOLUTION SET

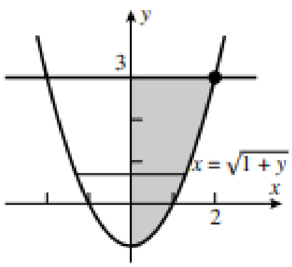
1. $V = \pi \int_{-1}^{3} (3 - x)\, dx = 8\pi.$

3. $V = \pi \int_{0}^{2} \frac{1}{4}(3 - y)^2\, dy = 13\pi/6.$

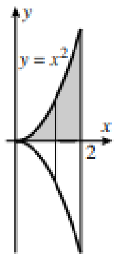
5. $V = \pi \int_{\pi/4}^{\pi/2} \cos x\, dx = (1 - \sqrt{2}/2)\pi.$



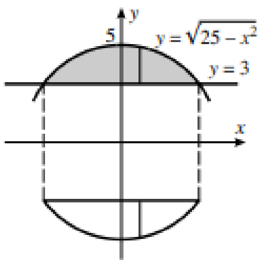
7. $V = \pi \int_{-1}^{3} (1 + y)\, dy = 8\pi.$



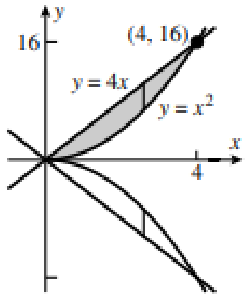
9. $V = \int_{0}^{2} x^4\, dx = 32/5.$



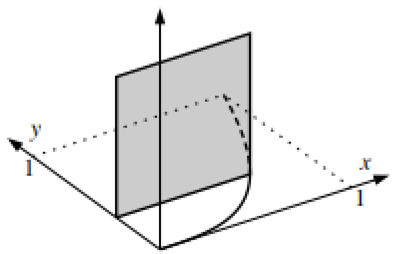
11. $V = \pi \int_{-4}^{4} [(25 - x^2) - 9]\, dx = 2\pi \int_{0}^{4} (16 - x^2)\, dx = 256\pi/3.$



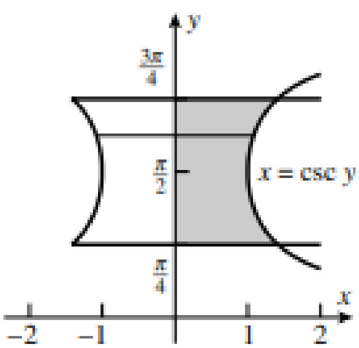
13. $V = \pi \int_0^4 [(4x)^2 - (x^2)^2]\, dx = \pi \int_0^4 (16x^2 - x^4)\, dx = 2048\pi/15.$



15. $V = \int_0^1 \left(y^{1/3}\right)^2 dy = \frac{3}{5}.$



17. $V = \pi \int_{\pi/4}^{3\pi/4} \csc^2 y\, dy = 2\pi.$



19. $V = \pi \int_{-1}^{2} [(y+2)^2 - y^4]\, dy = 72\pi/5.$