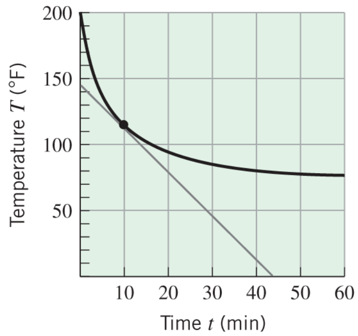


Figure Ex-45

46. Show that $f(x)$ is continuous but not differentiable at the indicated point. Sketch the graph of f.
(a) $f(x) = \sqrt[3]{x}$, $x = 0$
(b) $f(x) = \sqrt[3]{(x-2)^2}$, $x = 2$

47. Show that
$$f(x) = \begin{cases} x^2 + 1, & x \le 1 \\ 2x, & x > 1 \end{cases}$$
is continuous and differentiable at $x = 1$. Sketch the graph of f.

48. Show that
$$f(x) = \begin{cases} x^2 + 2, & x \le 1 \\ x + 2, & x > 1 \end{cases}$$
is continuous but not differentiable at $x = 1$. Sketch the graph of f.

49. Show that
$$f(x) = \begin{cases} x \sin(1/x), & x \ne 0 \\ 0, & x = 0 \end{cases}$$
is continuous but not differentiable at $x = 0$. Sketch the graph of f near $x = 0$. (See Figure 1.6.6 and the remark following Example 3 in Section 1.6.)

50. Show that
$$f(x) = \begin{cases} x^2 \sin(1/x), & x \ne 0 \\ 0, & x = 0 \end{cases}$$
is continuous and differentiable at $x = 0$. Sketch the graph of f near $x = 0$.

FOCUS ON CONCEPTS

51. Suppose that a function f is differentiable at x_0 and that $f'(x_0) > 0$. Prove that there exists an open interval containing x_0 such that if x_1 and x_2 are any two points in this interval with $x_1 < x_0 < x_2$, then $f(x_1) < f(x_0) < f(x_2)$.

52. Suppose that a function f is differentiable at x_0 and define $g(x) = f(mx + b)$, where m and b are constants. Prove that if x_1 is a point at which $mx_1 + b = x_0$, then $g(x)$ is differentiable at x_1 and $g'(x_1) = mf'(x_0)$.

53. Suppose that a function f is differentiable at $x = 0$ with $f(0) = f'(0) = 0$, and let $y = mx$, $m \ne 0$, denote any line of nonzero slope through the origin.
(a) Prove that there exists an open interval containing 0 such that for all nonzero x in this interval $|f(x)| < \left|\frac{1}{2}mx\right|$. [*Hint:* Let $\epsilon = \frac{1}{2}|m|$ and apply Definition 1.4.1 to (5) with $x_0 = 0$.]
(b) Conclude from part (a) and the triangle inequality that there exists an open interval containing 0 such that $|f(x)| < |f(x) - mx|$ for all x in this interval.
(c) Explain why the result obtained in part (b) may be interpreted to mean that the tangent line to the graph of f at the origin is the best *linear* approximation to f at that point.

54. Suppose that f is differentiable at x_0. Modify the argument of Exercise 53 to prove that the tangent line to the graph of f at the point $P(x_0, f(x_0))$ provides the best linear approximation to f at P. [*Hint:* Suppose that $y = f(x_0) + m(x - x_0)$ is any line through $P(x_0, f(x_0))$ with slope $m \ne f'(x_0)$. Apply Definition 1.4.1 to (5) with $x = x_0 + h$ and $\epsilon = \frac{1}{2}|f'(x_0) - m|$.]

55. Writing Write a paragraph that explains what it means for a function to be differentiable. Include examples of functions that are not differentiable as well as examples of functions that are differentiable.

56. Writing Explain the relationship between continuity and differentiability.

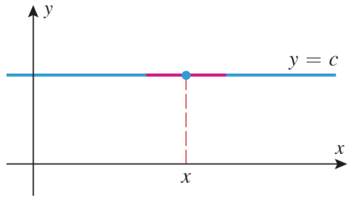
✓ QUICK CHECK ANSWERS 2.2 **1.** $\dfrac{f(x+h) - f(x)}{h}$ **2.** (a) $2x$ (b) $\dfrac{1}{2\sqrt{x}}$ **3.** 1; $-\frac{2}{3}$
4. Theorem 2.2.3: If f is differentiable at x_0, then f is continuous at x_0.

2.3 INTRODUCTION TO TECHNIQUES OF DIFFERENTIATION

In the last section we defined the derivative of a function f as a limit, and we used that limit to calculate a few simple derivatives. In this section we will develop some important theorems that will enable us to calculate derivatives more efficiently.

DERIVATIVE OF A CONSTANT

The simplest kind of function is a constant function $f(x) = c$. Since the graph of f is a horizontal line of slope 0, the tangent line to the graph of f has slope 0 for every x;



The tangent line to the graph of $f(x) = c$ has slope 0 for all x.

Figure 2.3.1

and hence we can see geometrically that $f'(x) = 0$ (Figure 2.3.1). We can also see this algebraically since

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{c - c}{h} = \lim_{h \to 0} 0 = 0$$

Thus, we have established the following result.

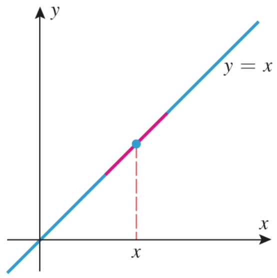
2.3.1 THEOREM *The derivative of a constant function is* 0*; that is, if c is any real number, then*

$$\frac{d}{dx}[c] = 0 \tag{1}$$

▶ Example 1

$$\frac{d}{dx}[1] = 0, \quad \frac{d}{dx}[-3] = 0, \quad \frac{d}{dx}[\pi] = 0, \quad \frac{d}{dx}[-\sqrt{2}] = 0 \quad ◀$$

DERIVATIVES OF POWER FUNCTIONS



The tangent line to the graph of $f(x) = x$ has slope 1 for all x.

Figure 2.3.2

The simplest power function is $f(x) = x$. Since the graph of f is a line of slope 1, it follows from Example 3 of Section 2.2 that $f'(x) = 1$ for all x (Figure 2.3.2). In other words,

$$\frac{d}{dx}[x] = 1 \tag{2}$$

Example 1 of Section 2.2 shows that the power function $f(x) = x^2$ has derivative $f'(x) = 2x$. From Example 2 in that section one can infer that the power function $f(x) = x^3$ has derivative $f'(x) = 3x^2$. That is,

$$\frac{d}{dx}[x^2] = 2x \quad \text{and} \quad \frac{d}{dx}[x^3] = 3x^2 \tag{3–4}$$

These results are special cases of the following more general result.

2.3.2 THEOREM (*The Power Rule*) *If n is a positive integer, then*

$$\frac{d}{dx}[x^n] = nx^{n-1} \tag{5}$$

Verify that Formulas (2), (3), and (4) are the special cases of (5) in which $n = 1$, 2, and 3.

The binomial formula can be found on the front endpaper of the text. Replacing y by h in this formula yields the identity used in the proof of Theorem 2.3.2.

PROOF Let $f(x) = x^n$. Thus, from the definition of a derivative and the binomial formula for expanding the expression $(x + h)^n$, we obtain

$$\begin{aligned}
\frac{d}{dx}[x^n] = f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(x+h)^n - x^n}{h} \\
&= \lim_{h \to 0} \frac{\left[x^n + nx^{n-1}h + \dfrac{n(n-1)}{2!}x^{n-2}h^2 + \cdots + nxh^{n-1} + h^n\right] - x^n}{h} \\
&= \lim_{h \to 0} \frac{nx^{n-1}h + \dfrac{n(n-1)}{2!}x^{n-2}h^2 + \cdots + nxh^{n-1} + h^n}{h} \\
&= \lim_{h \to 0} \left[nx^{n-1} + \frac{n(n-1)}{2!}x^{n-2}h + \cdots + nxh^{n-2} + h^{n-1}\right] \\
&= nx^{n-1} + 0 + \cdots + 0 + 0 \\
&= nx^{n-1} \quad ■
\end{aligned}$$

▶ Example 2

$$\frac{d}{dx}[x^4] = 4x^3, \quad \frac{d}{dx}[x^5] = 5x^4, \quad \frac{d}{dt}[t^{12}] = 12t^{11} \blacktriangleleft$$

Although our proof of the power rule in Formula (5) applies only to *positive* integer powers of x, it is not difficult to show that the same formula holds for all integer powers of x (Exercise 84). Also, we saw in Example 4 of Section 2.2 that

$$\frac{d}{dx}[\sqrt{x}] = \frac{1}{2\sqrt{x}} \tag{6}$$

which can be expressed as

$$\frac{d}{dx}[x^{1/2}] = \frac{1}{2}x^{-1/2} = \frac{1}{2}x^{(1/2)-1}$$

Thus, Formula (5) is valid for $n = \frac{1}{2}$, as well. In fact, it can be shown that this formula holds for any real exponent. We state this more general result for our use now, although we won't be prepared to prove it until Chapter 6.

2.3.3 THEOREM (*Extended Power Rule*) *If r is any real number, then*

$$\frac{d}{dx}[x^r] = rx^{r-1} \tag{7}$$

In words, *to differentiate a power function, decrease the constant exponent by one and multiply the resulting power function by the original exponent.*

▶ Example 3

$$\frac{d}{dx}[x^\pi] = \pi x^{\pi-1}$$

$$\frac{d}{dx}\left[\frac{1}{x}\right] = \frac{d}{dx}[x^{-1}] = (-1)x^{-1-1} = -x^{-2} = -\frac{1}{x^2}$$

$$\frac{d}{dw}\left[\frac{1}{w^{100}}\right] = \frac{d}{dw}[w^{-100}] = -100w^{-101} = -\frac{100}{w^{101}}$$

$$\frac{d}{dx}[x^{4/5}] = \frac{4}{5}x^{(4/5)-1} = \frac{4}{5}x^{-1/5}$$

$$\frac{d}{dx}[\sqrt[3]{x}] = \frac{d}{dx}[x^{1/3}] = \frac{1}{3}x^{-2/3} = \frac{1}{3\sqrt[3]{x^2}} \blacktriangleleft$$

DERIVATIVE OF A CONSTANT TIMES A FUNCTION

Formula (8) can also be expressed in function notation as

$$(cf)' = cf'$$

2.3.4 THEOREM (*Constant Multiple Rule*) *If f is differentiable at x and c is any real number, then cf is also differentiable at x and*

$$\frac{d}{dx}[cf(x)] = c\frac{d}{dx}[f(x)] \tag{8}$$

PROOF

$$\begin{aligned}\frac{d}{dx}[cf(x)] &= \lim_{h\to 0}\frac{cf(x+h)-cf(x)}{h}\\ &= \lim_{h\to 0} c\left[\frac{f(x+h)-f(x)}{h}\right]\\ &= c\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\\ &= c\frac{d}{dx}[f(x)] \quad ■\end{aligned}$$

A constant factor can be moved through a limit sign.

In words, *a constant factor can be moved through a derivative sign.*

▶ Example 4

$$\frac{d}{dx}[4x^8] = 4\frac{d}{dx}[x^8] = 4[8x^7] = 32x^7$$

$$\frac{d}{dx}[-x^{12}] = (-1)\frac{d}{dx}[x^{12}] = -12x^{11}$$

$$\frac{d}{dx}\left[\frac{\pi}{x}\right] = \pi\frac{d}{dx}[x^{-1}] = \pi(-x^{-2}) = -\frac{\pi}{x^2} \quad ◀$$

DERIVATIVES OF SUMS AND DIFFERENCES

2.3.5 THEOREM (*Sum and Difference Rules*) *If f and g are differentiable at x, then so are $f+g$ and $f-g$ and*

$$\frac{d}{dx}[f(x)+g(x)] = \frac{d}{dx}[f(x)] + \frac{d}{dx}[g(x)] \tag{9}$$

$$\frac{d}{dx}[f(x)-g(x)] = \frac{d}{dx}[f(x)] - \frac{d}{dx}[g(x)] \tag{10}$$

Formulas (9) and (10) can also be expressed as

$$(f+g)' = f' + g'$$
$$(f-g)' = f' - g'$$

PROOF Formula (9) can be proved as follows:

$$\begin{aligned}\frac{d}{dx}[f(x)+g(x)] &= \lim_{h\to 0}\frac{[f(x+h)+g(x+h)]-[f(x)+g(x)]}{h}\\ &= \lim_{h\to 0}\frac{[f(x+h)-f(x)]+[g(x+h)-g(x)]}{h}\\ &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} + \lim_{h\to 0}\frac{g(x+h)-g(x)}{h}\\ &= \frac{d}{dx}[f(x)] + \frac{d}{dx}[g(x)]\end{aligned}$$

The limit of a sum is the sum of the limits.

Formula (10) can be proved in a similar manner or, alternatively, by writing $f(x)-g(x)$ as $f(x)+(-1)g(x)$ and then applying Formulas (8) and (9). ■

In words, *the derivative of a sum equals the sum of the derivatives*, and *the derivative of a difference equals the difference of the derivatives.*

▶ Example 5

$$\frac{d}{dx}[2x^6 + x^{-9}] = \frac{d}{dx}[2x^6] + \frac{d}{dx}[x^{-9}] = 12x^5 + (-9)x^{-10} = 12x^5 - 9x^{-10}$$

$$\frac{d}{dx}\left[\frac{\sqrt{x} - 2x}{\sqrt{x}}\right] = \frac{d}{dx}[1 - 2\sqrt{x}\,]$$

$$= \frac{d}{dx}[1] - \frac{d}{dx}[2\sqrt{x}\,] = 0 - 2\left(\frac{1}{2\sqrt{x}}\right) = -\frac{1}{\sqrt{x}}$$

See Formula (6). ◀

Although Formulas (9) and (10) are stated for sums and differences of two functions, they can be extended to any finite number of functions. For example, by grouping and applying Formula (9) twice we obtain

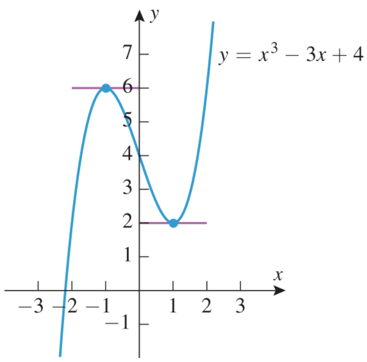
$$(f + g + h)' = [(f + g) + h]' = (f + g)' + h' = f' + g' + h'$$

As illustrated in the following example, the constant multiple rule together with the extended versions of the sum and difference rules can be used to differentiate any polynomial.

▶ Example 6 Find dy/dx if $y = 3x^8 - 2x^5 + 6x + 1$.

Solution.

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}[3x^8 - 2x^5 + 6x + 1] \\ &= \frac{d}{dx}[3x^8] - \frac{d}{dx}[2x^5] + \frac{d}{dx}[6x] + \frac{d}{dx}[1] \\ &= 24x^7 - 10x^4 + 6 \ ◀\end{aligned}$$



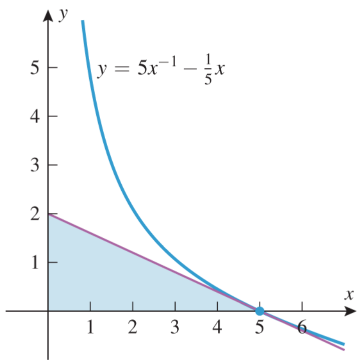
▲ **Figure 2.3.3**

▶ Example 7 At what points, if any, does the graph of $y = x^3 - 3x + 4$ have a horizontal tangent line?

Solution. Horizontal tangent lines have slope zero, so we must find those values of x for which $y'(x) = 0$. Differentiating yields

$$y'(x) = \frac{d}{dx}[x^3 - 3x + 4] = 3x^2 - 3$$

Thus, horizontal tangent lines occur at those values of x for which $3x^2 - 3 = 0$, that is, if $x = -1$ or $x = 1$. The corresponding points on the curve $y = x^3 - 3x + 4$ are $(-1, 6)$ and $(1, 2)$ (see Figure 2.3.3). ◀



▲ **Figure 2.3.4**

▶ Example 8 Find the area of the triangle formed from the coordinate axes and the tangent line to the curve $y = 5x^{-1} - \frac{1}{5}x$ at the point $(5, 0)$.

Solution. Since the derivative of y with respect to x is

$$y'(x) = \frac{d}{dx}\left[5x^{-1} - \frac{1}{5}x\right] = \frac{d}{dx}[5x^{-1}] - \frac{d}{dx}\left[\frac{1}{5}x\right] = -5x^{-2} - \frac{1}{5}$$

the slope of the tangent line at the point $(5, 0)$ is $y'(5) = -\frac{2}{5}$. Thus, the equation of the tangent line at this point is

$$y - 0 = -\frac{2}{5}(x - 5) \quad \text{or equivalently} \quad y = -\frac{2}{5}x + 2$$

Since the y-intercept of this line is 2, the right triangle formed from the coordinate axes and the tangent line has legs of length 5 and 2, so its area is $\frac{1}{2}(5)(2) = 5$ (Figure 2.3.4). ◀

HIGHER DERIVATIVES

The derivative f' of a function f is itself a function and hence may have a derivative of its own. If f' is differentiable, then its derivative is denoted by f'' and is called the ***second derivative*** of f. As long as we have differentiability, we can continue the process of differentiating to obtain third, fourth, fifth, and even higher derivatives of f. These successive derivatives are denoted by

$$f', \quad f''=(f')', \quad f'''=(f'')', \quad f^{(4)}=(f''')', \quad f^{(5)}=(f^{(4)})', \ldots$$

If $y=f(x)$, then successive derivatives can also be denoted by

$$y', \quad y'', \quad y''', \quad y^{(4)}, \quad y^{(5)}, \ldots$$

Other common notations are

$$\begin{aligned}
y' &= \frac{dy}{dx} = \frac{d}{dx}[f(x)] \\
y'' &= \frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{d}{dx}[f(x)]\right] = \frac{d^2}{dx^2}[f(x)] \\
y''' &= \frac{d^3y}{dx^3} = \frac{d}{dx}\left[\frac{d^2}{dx^2}[f(x)]\right] = \frac{d^3}{dx^3}[f(x)] \\
&\vdots \qquad\qquad\qquad\qquad\qquad \vdots
\end{aligned}$$

These are called, in succession, the *first derivative*, the *second derivative*, the *third derivative*, and so forth. The number of times that f is differentiated is called the ***order*** of the derivative. A general nth order derivative can be denoted by

$$\boxed{\frac{d^ny}{dx^n} = f^{(n)}(x) = \frac{d^n}{dx^n}[f(x)]} \tag{11}$$

and the value of a general nth order derivative at a specific point $x=x_0$ can be denoted by

$$\boxed{\left.\frac{d^ny}{dx^n}\right|_{x=x_0} = f^{(n)}(x_0) = \left.\frac{d^n}{dx^n}[f(x)]\right|_{x=x_0}} \tag{12}$$

▶ **Example 9** If $f(x)=3x^4-2x^3+x^2-4x+2$, then

$$\begin{aligned}
f'(x) &= 12x^3-6x^2+2x-4 \\
f''(x) &= 36x^2-12x+2 \\
f'''(x) &= 72x-12 \\
f^{(4)}(x) &= 72 \\
f^{(5)}(x) &= 0 \\
&\vdots \\
f^{(n)}(x) &= 0 \quad (n \geq 5)
\end{aligned}$$ ◀

We will discuss the significance of second derivatives and those of higher order in later sections.

✓QUICK CHECK EXERCISES 2.3

(See page 88 for answers.)

1. In each part, determine $f'(x)$.
(a) $f(x)=\sqrt{6}$ (b) $f(x)=\sqrt{6x}$
(c) $f(x)=6\sqrt{x}$ (d) $f(x)=\sqrt{6x}$

2. In parts (a)–(d), determine $f'(x)$.
(a) $f(x)=x^3+5$ (b) $f(x)=x^2(x^3+5)$
(c) $f(x)=\dfrac{x^3+5}{2}$ (d) $f(x)=\dfrac{x^3+5}{x^2}$

3. The slope of the tangent line to the curve $y=x^2+4x+7$ at $x=1$ is ________.

4. If $f(x)=3x^3-3x^2+x+1$, then $f''(x)=$ ________.

EXERCISE SET 2.3 Graphing Utility

1–8 Find dy/dx.

1. $y = 4x^7$ **2.** $y = -3x^{12}$

3. $y = 3x^8 + 2x + 1$ **4.** $y = \frac{1}{2}(x^4 + 7)$

5. $y = \pi^3$ **6.** $y = \sqrt{2}x + (1/\sqrt{2})$

7. $y = -\frac{1}{3}(x^7 + 2x - 9)$ **8.** $y = \dfrac{x^2 + 1}{5}$

9–16 Find $f'(x)$.

9. $f(x) = x^{-3} + \dfrac{1}{x^7}$ **10.** $f(x) = \sqrt{x} + \dfrac{1}{x}$

11. $f(x) = -3x^{-8} + 2\sqrt{x}$ **12.** $f(x) = 7x^{-6} - 5\sqrt{x}$

13. $f(x) = x^{\pi} + \dfrac{1}{x^{\sqrt{10}}}$ **14.** $f(x) = \sqrt[3]{\dfrac{8}{x}}$

15. $f(x) = (3x^2 + 1)^2$

16. $f(x) = ax^3 + bx^2 + cx + d$ $(a, b, c, d$ constant)

17–18 Find $y'(1)$.

17. $y = 5x^2 - 3x + 1$ **18.** $y = \dfrac{x^{3/2} + 2}{x}$

19–20 Find dx/dt.

19. $x = t^2 - t$ **20.** $x = \dfrac{t^2 + 1}{3t}$

21–24 Find $dy/dx|_{x=1}$.

21. $y = 1 + x + x^2 + x^3 + x^4 + x^5$

22. $y = \dfrac{1 + x + x^2 + x^3 + x^4 + x^5 + x^6}{x^3}$

23. $y = (1 - x)(1 + x)(1 + x^2)(1 + x^4)$

24. $y = x^{24} + 2x^{12} + 3x^8 + 4x^6$

25–26 Approximate $f'(1)$ by considering the difference quotient

$$\frac{f(1 + h) - f(1)}{h}$$

for values of h near 0, and then find the exact value of $f'(1)$ by differentiating.

25. $f(x) = x^3 - 3x + 1$ **26.** $f(x) = \dfrac{1}{x^2}$

27–28 Use a graphing utility to estimate the value of $f'(1)$ by zooming in on the graph of f, and then compare your estimate to the exact value obtained by differentiating.

27. $f(x) = \dfrac{x^2 + 1}{x}$ **28.** $f(x) = \dfrac{x + 2x^{3/2}}{\sqrt{x}}$

29–32 Find the indicated derivative.

29. $\dfrac{d}{dt}[16t^2]$ **30.** $\dfrac{dC}{dr}$, where $C = 2\pi r$

31. $V'(r)$, where $V = \pi r^3$ **32.** $\dfrac{d}{d\alpha}[2\alpha^{-1} + \alpha]$

33–36 True–False Determine whether the statement is true or false. Explain your answer.

33. If f and g are differentiable at $x = 2$, then

$$\left.\frac{d}{dx}[f(x) - 8g(x)]\right|_{x=2} = f'(2) - 8g'(2)$$

34. If $f(x)$ is a cubic polynomial, then $f'(x)$ is a quadratic polynomial.

35. If $f'(2) = 5$, then

$$\left.\frac{d}{dx}[4f(x) + x^3]\right|_{x=2} = \left.\frac{d}{dx}[4f(x) + 8]\right|_{x=2} = 4f'(2) = 20$$

36. If $f(x) = x^2(x^4 - x)$, then

$$f''(x) = \frac{d}{dx}[x^2] \cdot \frac{d}{dx}[x^4 - x] = 2x(4x^3 - 1)$$

37. A spherical balloon is being inflated.

(a) Find a general formula for the instantaneous rate of change of the volume V with respect to the radius r, given that $V = \frac{4}{3}\pi r^3$.

(b) Find the rate of change of V with respect to r at the instant when the radius is $r = 5$.

38. Find $\dfrac{d}{d\lambda}\left[\dfrac{\lambda\lambda_0 + \lambda^6}{2 - \lambda_0}\right]$ (λ_0 is constant).

39. Find an equation of the tangent line to the graph of $y = f(x)$ at $x = -3$ if $f(-3) = 2$ and $f'(-3) = 5$.

40. Find an equation of the tangent line to the graph of $y = f(x)$ at $x = 2$ if $f(2) = -2$ and $f'(2) = -1$.

41–42 Find d^2y/dx^2.

41. (a) $y = 7x^3 - 5x^2 + x$ (b) $y = 12x^2 - 2x + 3$

(c) $y = \dfrac{x + 1}{x}$ (d) $y = (5x^2 - 3)(7x^3 + x)$

42. (a) $y = 4x^7 - 5x^3 + 2x$ (b) $y = 3x + 2$

(c) $y = \dfrac{3x - 2}{5x}$ (d) $y = (x^3 - 5)(2x + 3)$

43–44 Find y'''.

43. (a) $y = x^{-5} + x^5$ (b) $y = 1/x$

(c) $y = ax^3 + bx + c$ $(a, b, c$ constant)

44. (a) $y = 5x^2 - 4x + 7$ (b) $y = 3x^{-2} + 4x^{-1} + x$

(c) $y = ax^4 + bx^2 + c$ $(a, b, c$ constant)

45. Find

(a) $f'''(2)$, where $f(x) = 3x^2 - 2$

(b) $\left.\dfrac{d^2y}{dx^2}\right|_{x=1}$, where $y = 6x^5 - 4x^2$

(c) $\left.\dfrac{d^4}{dx^4}[x^{-3}]\right|_{x=1}$.

46. Find

(a) $y'''(0)$, where $y = 4x^4 + 2x^3 + 3$

(b) $\left.\dfrac{d^4y}{dx^4}\right|_{x=1}$, where $y = \dfrac{6}{x^4}$.

47. Show that $y = x^3 + 3x + 1$ satisfies $y''' + xy'' - 2y' = 0$.

48. Show that if $x \neq 0$, then $y = 1/x$ satisfies the equation $x^3y'' + x^2y' - xy = 0$.

73. (a) Prove:

$$\frac{d^2}{dx^2}[cf(x)] = c\frac{d^2}{dx^2}[f(x)]$$

$$\frac{d^2}{dx^2}[f(x)+g(x)] = \frac{d^2}{dx^2}[f(x)] + \frac{d^2}{dx^2}[g(x)]$$

(b) Do the results in part (a) generalize to nth derivatives? Justify your answer.

74. Let $f(x) = x^8 - 2x + 3$; find

$$\lim_{w \to 2} \frac{f'(w) - f'(2)}{w - 2}$$

75. (a) Find $f^{(n)}(x)$ if $f(x) = x^n$, $n = 1, 2, 3, \ldots$.
(b) Find $f^{(n)}(x)$ if $f(x) = x^k$ and $n > k$, where k is a positive integer.
(c) Find $f^{(n)}(x)$ if

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

76. (a) Prove: If $f''(x)$ exists for each x in (a, b), then both f and f' are continuous on (a, b).
(b) What can be said about the continuity of f and its derivatives if $f^{(n)}(x)$ exists for each x in (a, b)?

77. Let $f(x) = (mx + b)^n$, where m and b are constants and n is an integer. Use the result of Exercise 52 in Section 2.2 to prove that $f'(x) = nm(mx + b)^{n-1}$.

78–79 Verify the result of Exercise 77 for $f(x)$.

78. $f(x) = (2x + 3)^2$ **79.** $f(x) = (3x - 1)^3$

80–83 Use the result of Exercise 77 to compute the derivative of the given function $f(x)$.

80. $f(x) = \dfrac{1}{x - 1}$

81. $f(x) = \dfrac{3}{(2x + 1)^2}$

82. $f(x) = \dfrac{x}{x + 1}$

83. $f(x) = \dfrac{2x^2 + 4x + 3}{x^2 + 2x + 1}$

84. The purpose of this exercise is to extend the power rule (Theorem 2.3.2) to any integer exponent. Let $f(x) = x^n$, where n is any integer. If $n > 0$, then $f'(x) = nx^{n-1}$ by Theorem 2.3.2.
(a) Show that the conclusion of Theorem 2.3.2 holds in the case $n = 0$.
(b) Suppose that $n < 0$ and set $m = -n$ so that

$$f(x) = x^n = x^{-m} = \frac{1}{x^m}$$

Use Definition 2.2.1 and Theorem 2.3.2 to show that

$$\frac{d}{dx}\left[\frac{1}{x^m}\right] = -mx^{m-1} \cdot \frac{1}{x^{2m}}$$

and conclude that $f'(x) = nx^{n-1}$.

✓ **QUICK CHECK ANSWERS 2.3** **1.** (a) 0 (b) $\sqrt{6}$ (c) $3/\sqrt{x}$ (d) $\sqrt{6}/(2\sqrt{x})$ **2.** (a) $3x^2$ (b) $5x^4 + 10x$ (c) $\frac{3}{2}x^2$ (d) $1 - 10x^{-3}$ **3.** 6 **4.** $18x - 6$

2.4 THE PRODUCT AND QUOTIENT RULES

In this section we will develop techniques for differentiating products and quotients of functions whose derivatives are known.

DERIVATIVE OF A PRODUCT

You might be tempted to conjecture that the derivative of a product of two functions is the product of their derivatives. However, a simple example will show this to be false. Consider the functions

$$f(x) = x \quad \text{and} \quad g(x) = x^2$$

The product of their derivatives is

$$f'(x)g'(x) = (1)(2x) = 2x$$

but their product is $h(x) = f(x)g(x) = x^3$, so the derivative of the product is

$$h'(x) = 3x^2$$

Thus, the derivative of the product is not equal to the product of the derivatives. The correct relationship, which is credited to Leibniz, is given by the following theorem.

2.4.1 THEOREM (*The Product Rule*) *If f and g are differentiable at x, then so is the product $f \cdot g$, and*

$$\frac{d}{dx}[f(x)g(x)] = f(x)\frac{d}{dx}[g(x)] + g(x)\frac{d}{dx}[f(x)] \tag{1}$$

Formula (1) can also be expressed as

$$(f \cdot g)' = f \cdot g' + g \cdot f'$$

PROOF Whereas the proofs of the derivative rules in the last section were straightforward applications of the derivative definition, a key step in this proof involves adding and subtracting the quantity $f(x+h)g(x)$ to the numerator in the derivative definition. This yields

$$\begin{aligned}
\frac{d}{dx}[f(x)g(x)] &= \lim_{h\to 0}\frac{f(x+h)\cdot g(x+h) - f(x)\cdot g(x)}{h} \\
&= \lim_{h\to 0}\frac{f(x+h)g(x+h) - f(x+h)g(x) + f(x+h)g(x) - f(x)g(x)}{h} \\
&= \lim_{h\to 0}\left[f(x+h)\cdot\frac{g(x+h)-g(x)}{h} + g(x)\cdot\frac{f(x+h)-f(x)}{h}\right] \\
&= \lim_{h\to 0} f(x+h)\cdot\lim_{h\to 0}\frac{g(x+h)-g(x)}{h} + \lim_{h\to 0} g(x)\cdot\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\
&= \left[\lim_{h\to 0} f(x+h)\right]\frac{d}{dx}[g(x)] + \left[\lim_{h\to 0} g(x)\right]\frac{d}{dx}[f(x)] \\
&= f(x)\frac{d}{dx}[g(x)] + g(x)\frac{d}{dx}[f(x)]
\end{aligned}$$

[*Note:* In the last step $f(x+h)\to f(x)$ as $h\to 0$ because f is continuous at x by Theorem 2.2.3. Also, $g(x)\to g(x)$ as $h\to 0$ because $g(x)$ does not involve h and hence is treated as constant for the limit.] ■

In words, *the derivative of a product of two functions is the first function times the derivative of the second plus the second function times the derivative of the first.*

▶ **Example 1** Find dy/dx if $y = (4x^2 - 1)(7x^3 + x)$.

Solution. There are two methods that can be used to find dy/dx. We can either use the product rule or we can multiply out the factors in y and then differentiate. We will give both methods.

Method 1. *(Using the Product Rule)*

$$\begin{aligned}
\frac{dy}{dx} &= \frac{d}{dx}[(4x^2-1)(7x^3+x)] \\
&= (4x^2-1)\frac{d}{dx}[7x^3+x] + (7x^3+x)\frac{d}{dx}[4x^2-1] \\
&= (4x^2-1)(21x^2+1) + (7x^3+x)(8x) = 140x^4 - 9x^2 - 1
\end{aligned}$$

Method 2. *(Multiplying First)*

$$y = (4x^2-1)(7x^3+x) = 28x^5 - 3x^3 - x$$

Thus,

$$\frac{dy}{dx} = \frac{d}{dx}[28x^5 - 3x^3 - x] = 140x^4 - 9x^2 - 1$$

which agrees with the result obtained using the product rule. ◀

▶ Example 2 Find ds/dt if $s = (1+t)\sqrt{t}$.

Solution. Applying the product rule yields

$$\begin{aligned}\frac{ds}{dt} &= \frac{d}{dt}[(1+t)\sqrt{t}]\\ &= (1+t)\frac{d}{dt}[\sqrt{t}] + \sqrt{t}\frac{d}{dt}[1+t]\\ &= \frac{1+t}{2\sqrt{t}} + \sqrt{t} = \frac{1+3t}{2\sqrt{t}}\end{aligned}$$ ◀

DERIVATIVE OF A QUOTIENT

Just as the derivative of a product is not generally the product of the derivatives, so the derivative of a quotient is not generally the quotient of the derivatives. The correct relationship is given by the following theorem.

2.4.2 THEOREM (*The Quotient Rule*) *If f and g are both differentiable at x and if $g(x) \neq 0$, then f/g is differentiable at x and*

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\dfrac{d}{dx}[f(x)] - f(x)\dfrac{d}{dx}[g(x)]}{[g(x)]^2} \tag{2}$$

Formula (2) can also be expressed as

$$\left(\frac{f}{g}\right)' = \frac{g\cdot f' - f\cdot g'}{g^2}$$

PROOF

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \lim_{h\to 0}\frac{\dfrac{f(x+h)}{g(x+h)} - \dfrac{f(x)}{g(x)}}{h} = \lim_{h\to 0}\frac{f(x+h)\cdot g(x) - f(x)\cdot g(x+h)}{h\cdot g(x)\cdot g(x+h)}$$

Adding and subtracting $f(x)\cdot g(x)$ in the numerator yields

$$\begin{aligned}\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] &= \lim_{h\to 0}\frac{f(x+h)\cdot g(x) - f(x)\cdot g(x) - f(x)\cdot g(x+h) + f(x)\cdot g(x)}{h\cdot g(x)\cdot g(x+h)}\\ &= \lim_{h\to 0}\frac{\left[g(x)\cdot\dfrac{f(x+h)-f(x)}{h}\right] - \left[f(x)\cdot\dfrac{g(x+h)-g(x)}{h}\right]}{g(x)\cdot g(x+h)}\\ &= \frac{\lim\limits_{h\to 0} g(x)\cdot\lim\limits_{h\to 0}\dfrac{f(x+h)-f(x)}{h} - \lim\limits_{h\to 0} f(x)\cdot\lim\limits_{h\to 0}\dfrac{g(x+h)-g(x)}{h}}{\lim\limits_{h\to 0} g(x)\cdot\lim\limits_{h\to 0} g(x+h)}\\ &= \frac{\left[\lim\limits_{h\to 0} g(x)\right]\cdot\dfrac{d}{dx}[f(x)] - \left[\lim\limits_{h\to 0} f(x)\right]\cdot\dfrac{d}{dx}[g(x)]}{\lim\limits_{h\to 0} g(x)\cdot\lim\limits_{h\to 0} g(x+h)}\\ &= \frac{g(x)\dfrac{d}{dx}[f(x)] - f(x)\dfrac{d}{dx}[g(x)]}{[g(x)]^2}\end{aligned}$$

[See the note at the end of the proof of Theorem 2.4.1 for an explanation of the last step.] ■

In words, *the derivative of a quotient of two functions is the denominator times the derivative of the numerator minus the numerator times the derivative of the denominator, all divided by the denominator squared.*

Sometimes it is better to simplify a function first than to apply the quotient rule immediately. For example, it is easier to differentiate

$$f(x) = \frac{x^{3/2} + x}{\sqrt{x}}$$

by rewriting it as

$$f(x) = x + \sqrt{x}$$

as opposed to using the quotient rule.

Example 3 Find $y'(x)$ for $y = \dfrac{x^3 + 2x^2 - 1}{x + 5}$.

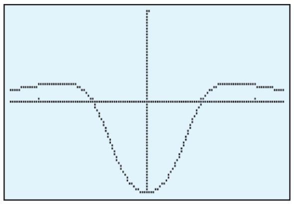
Solution. Applying the quotient rule yields

$$\begin{aligned}
\frac{dy}{dx} = \frac{d}{dx}\left[\frac{x^3 + 2x^2 - 1}{x + 5}\right] &= \frac{(x+5)\dfrac{d}{dx}[x^3 + 2x^2 - 1] - (x^3 + 2x^2 - 1)\dfrac{d}{dx}[x+5]}{(x+5)^2} \\
&= \frac{(x+5)(3x^2 + 4x) - (x^3 + 2x^2 - 1)(1)}{(x+5)^2} \\
&= \frac{(3x^3 + 19x^2 + 20x) - (x^3 + 2x^2 - 1)}{(x+5)^2} \\
&= \frac{2x^3 + 17x^2 + 20x + 1}{(x+5)^2} \blacktriangleleft
\end{aligned}$$

Example 4 Let $f(x) = \dfrac{x^2 - 1}{x^4 + 1}$.

(a) Graph $y = f(x)$, and use your graph to make rough estimates of the locations of all horizontal tangent lines.

(b) By differentiating, find the exact locations of the horizontal tangent lines.



$[-2.5, 2.5] \times [-1, 1]$
$x\text{Scl} = 1, y\text{Scl} = 1$

$$y = \frac{x^2 - 1}{x^4 + 1}$$

▲ **Figure 2.4.1**

Solution (a). In Figure 2.4.1 we have shown the graph of the equation $y = f(x)$ in the window $[-2.5, 2.5] \times [-1, 1]$. This graph suggests that horizontal tangent lines occur at $x = 0$, $x \approx 1.5$, and $x \approx -1.5$.

Solution (b). To find the exact locations of the horizontal tangent lines, we must find the points where $dy/dx = 0$. We start by finding dy/dx:

$$\begin{aligned}
\frac{dy}{dx} = \frac{d}{dx}\left[\frac{x^2 - 1}{x^4 + 1}\right] &= \frac{(x^4+1)\dfrac{d}{dx}[x^2 - 1] - (x^2 - 1)\dfrac{d}{dx}[x^4 + 1]}{(x^4+1)^2} \\
&= \frac{(x^4+1)(2x) - (x^2-1)(4x^3)}{(x^4+1)^2} \\
&= \frac{-2x^5 + 4x^3 + 2x}{(x^4+1)^2} = -\frac{2x(x^4 - 2x^2 - 1)}{(x^4+1)^2}
\end{aligned}$$

The differentiation is complete. The rest is simplification.

Now we will set $dy/dx = 0$ and solve for x. We obtain

$$-\frac{2x(x^4 - 2x^2 - 1)}{(x^4+1)^2} = 0$$

The solutions of this equation are the values of x for which the numerator is 0, that is,

$$2x(x^4 - 2x^2 - 1) = 0$$

The first factor yields the solution $x = 0$. Other solutions can be found by solving the equation

$$x^4 - 2x^2 - 1 = 0$$

This can be treated as a quadratic equation in x^2 and solved by the quadratic formula. This yields

$$x^2 = \frac{2 \pm \sqrt{8}}{2} = 1 \pm \sqrt{2}$$

Derive the following rule for differentiating a reciprocal:

$$\left(\frac{1}{g}\right)' = -\frac{g'}{g^2}$$

Use it to find the derivative of

$$f(x) = \frac{1}{x^2+1}$$

The minus sign yields imaginary values of x, which we ignore since they are not relevant to the problem. The plus sign yields the solutions

$$x = \pm\sqrt{1+\sqrt{2}}$$

In summary, horizontal tangent lines occur at

$$x = 0, \quad x = \sqrt{1+\sqrt{2}} \approx 1.55, \quad \text{and} \quad x = -\sqrt{1+\sqrt{2}} \approx -1.55$$

which is consistent with the rough estimates that we obtained graphically in part (a). ◀

SUMMARY OF DIFFERENTIATION RULES

The following table summarizes the differentiation rules that we have encountered thus far.

Table 2.4.1

RULES FOR DIFFERENTIATION

$\frac{d}{dx}[c] = 0$	$(f+g)' = f' + g'$	$(f \cdot g)' = f \cdot g' + g \cdot f'$	$\left(\frac{1}{g}\right)' = -\frac{g'}{g^2}$
$(cf)' = cf'$	$(f-g)' = f' - g'$	$\left(\frac{f}{g}\right)' = \frac{g \cdot f' - f \cdot g'}{g^2}$	$\frac{d}{dx}[x^r] = rx^{r-1}$

✓ QUICK CHECK EXERCISES 2.4

(See page 93 for answers.)

1. (a) $\frac{d}{dx}[x^2 f(x)] =$ _____ (b) $\frac{d}{dx}\left[\frac{f(x)}{x^2+1}\right] =$ _____
(c) $\frac{d}{dx}\left[\frac{x^2+1}{f(x)}\right] =$ _____

2. Find $F'(1)$ given that $f(1) = -1$, $f'(1) = 2$, $g(1) = 3$, and $g'(1) = -1$.
(a) $F(x) = 2f(x) - 3g(x)$ (b) $F(x) = [f(x)]^2$
(c) $F(x) = f(x)g(x)$ (d) $F(x) = f(x)/g(x)$

EXERCISE SET 2.4 Graphing Utility

1–4 Compute the derivative of the given function $f(x)$ by (a) multiplying and then differentiating and (b) using the product rule. Verify that (a) and (b) yield the same result. ■

1. $f(x) = (x+1)(2x-1)$ **2.** $f(x) = (3x^2-1)(x^2+2)$

3. $f(x) = (x^2+1)(x^2-1)$

4. $f(x) = (x+1)(x^2-x+1)$

5–20 Find $f'(x)$. ■

5. $f(x) = (3x^2+6)\left(2x - \frac{1}{4}\right)$

6. $f(x) = (2 - x - 3x^3)(7 + x^5)$

7. $f(x) = (x^3 + 7x^2 - 8)(2x^{-3} + x^{-4})$

8. $f(x) = \left(\frac{1}{x} + \frac{1}{x^2}\right)(3x^3 + 27)$

9. $f(x) = (x-2)(x^2+2x+4)$

10. $f(x) = (x^2+x)(x^2-x)$

11. $f(x) = \frac{3x+4}{x^2+1}$ **12.** $f(x) = \frac{x-2}{x^4+x+1}$

13. $f(x) = \frac{x^2}{3x-4}$ **14.** $f(x) = \frac{2x^2+5}{3x-4}$

15. $f(x) = \frac{(2\sqrt{x}+1)(x-1)}{x+3}$

16. $f(x) = (2\sqrt{x}+1)\left(\frac{2-x}{x^2+3x}\right)$

17. $f(x) = (2x+1)\left(1+\frac{1}{x}\right)(x^{-3}+7)$

18. $f(x) = x^{-5}(x^2+2x)(4-3x)(2x^9+1)$

19. $f(x) = (x^7+2x-3)^3$ **20.** $f(x) = (x^2+1)^4$

21–24 Find $dy/dx|_{x=1}$. ■

21. $y = \frac{2x-1}{x+3}$ **22.** $y = \frac{4x+1}{x^2-5}$

23. $y = \left(\frac{3x+2}{x}\right)(x^{-5}+1)$ **24.** $y = (2x^7 - x^2)\left(\frac{x-1}{x+1}\right)$

25–26 Use a graphing utility to estimate the value of $f'(1)$ by zooming in on the graph of f, and then compare your estimate to the exact value obtained by differentiating. ■

25. $f(x) = \frac{x}{x^2+1}$ **26.** $f(x) = \frac{x^2-1}{x^2+1}$

27. Find $g'(4)$ given that $f(4) = 3$ and $f'(4) = -5$.
(a) $g(x) = \sqrt{x}f(x)$ (b) $g(x) = \dfrac{f(x)}{x}$

28. Find $g'(3)$ given that $f(3) = -2$ and $f'(3) = 4$.
(a) $g(x) = 3x^2 - 5f(x)$ (b) $g(x) = \dfrac{2x+1}{f(x)}$

29. In parts (a)–(d), $F(x)$ is expressed in terms of $f(x)$ and $g(x)$. Find $F'(2)$ given that $f(2) = -1$, $f'(2) = 4$, $g(2) = 1$, and $g'(2) = -5$.
(a) $F(x) = 5f(x) + 2g(x)$ (b) $F(x) = f(x) - 3g(x)$
(c) $F(x) = f(x)g(x)$ (d) $F(x) = f(x)/g(x)$

30. Find $F'(\pi)$ given that $f(\pi) = 10$, $f'(\pi) = -1$, $g(\pi) = -3$, and $g'(\pi) = 2$.
(a) $F(x) = 6f(x) - 5g(x)$ (b) $F(x) = x(f(x) + g(x))$
(c) $F(x) = 2f(x)g(x)$ (d) $F(x) = \dfrac{f(x)}{4 + g(x)}$

31–36 Find all values of x at which the tangent line to the given curve satisfies the stated property. ■

31. $y = \dfrac{x^2-1}{x+2}$; horizontal **32.** $y = \dfrac{x^2+1}{x-1}$; horizontal

33. $y = \dfrac{x^2+1}{x+1}$; parallel to the line $y = x$

34. $y = \dfrac{x+3}{x+2}$; perpendicular to the line $y = x$

35. $y = \dfrac{1}{x+4}$; passes through the origin

36. $y = \dfrac{2x+5}{x+2}$; y-intercept 2

FOCUS ON CONCEPTS

37. (a) What should it mean to say that two curves intersect at right angles?
(b) Show that the curves $y = 1/x$ and $y = 1/(2-x)$ intersect at right angles.

38. Find all values of a such that the curves $y = a/(x-1)$ and $y = x^2 - 2x + 1$ intersect at right angles.

39. Find a general formula for $F''(x)$ if $F(x) = xf(x)$ and f and f' are differentiable at x.

40. Suppose that the function f is differentiable everywhere and $F(x) = xf(x)$.
(a) Express $F'''(x)$ in terms of x and derivatives of f.
(b) For $n \geq 2$, conjecture a formula for $F^{(n)}(x)$.

41. A manufacturer of athletic footwear finds that the sales of their ZipStride brand running shoes is a function $f(p)$ of the selling price p (in dollars) for a pair of shoes. Suppose that $f(120) = 9000$ pairs of shoes and $f'(120) = -60$ pairs of shoes per dollar. The revenue that the manufacturer will receive for selling $f(p)$ pairs of shoes at p dollars per pair is $R(p) = p \cdot f(p)$. Find $R'(120)$. What impact would a small increase in price have on the manufacturer's revenue?

42. Solve the problem in Exercise 41 under the assumption that $f(120) = 9000$ and $f'(120) = -80$.

43. Use the quotient rule (Theorem 2.4.2) to derive the formula for the derivative of $f(x) = x^{-n}$, where n is a positive integer.

✓ **QUICK CHECK ANSWERS 2.4** **1.** (a) $x^2f'(x) + 2xf(x)$ (b) $\dfrac{(x^2+1)f'(x) - 2xf(x)}{(x^2+1)^2}$ (c) $\dfrac{2xf(x) - (x^2+1)f'(x)}{[f(x)^2]}$
2. (a) 7 (b) -4 (c) 7 (d) $\frac{5}{9}$

2.5 DERIVATIVES OF TRIGONOMETRIC FUNCTIONS

The main objective of this section is to obtain formulas for the derivatives of the six basic trigonometric functions. If needed, you will find a review of trigonometric functions in Appendix A.

We will assume in this section that the variable x in the trigonometric functions $\sin x$, $\cos x$, $\tan x$, $\cot x$, $\sec x$, and $\csc x$ is measured in radians. Also, we will need the limits in Theorem 1.6.3, but restated as follows using h rather than x as the variable:

$$\lim_{h \to 0} \frac{\sin h}{h} = 1 \quad \text{and} \quad \lim_{h \to 0} \frac{1 - \cos h}{h} = 0 \tag{1–2}$$

Let us start with the problem of differentiating $f(x) = \sin x$. Using the definition of the derivative we obtain

$$\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h} \\
&= \lim_{h \to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h} && \boxed{\text{By the addition formula for sine}} \\
&= \lim_{h \to 0} \left[\sin x \left(\frac{\cos h - 1}{h}\right) + \cos x \left(\frac{\sin h}{h}\right)\right] \\
&= \lim_{h \to 0} \left[\cos x \left(\frac{\sin h}{h}\right) - \sin x \left(\frac{1 - \cos h}{h}\right)\right] && \boxed{\text{Algebraic reorganization}} \\
&= \lim_{h \to 0} \cos x \cdot \lim_{h \to 0} \frac{\sin h}{h} - \lim_{h \to 0} \sin x \cdot \lim_{h \to 0} \frac{1 - \cos h}{h} \\
&= \left(\lim_{h \to 0} \cos x\right)(1) - \left(\lim_{h \to 0} \sin x\right)(0) && \boxed{\text{Formulas (1) and (2)}} \\
&= \lim_{h \to 0} \cos x = \cos x && \boxed{\begin{array}{l}\cos x \text{ does not involve the variable } h \text{ and hence} \\ \text{is treated as a constant in the limit computation.}\end{array}}
\end{aligned}$$

Formulas (1) and (2) and the derivation of Formulas (3) and (4) are only valid if h and x are in radians. See Exercise 49 for how Formulas (3) and (4) change when x is measured in degrees.

Thus, we have shown that

$$\frac{d}{dx}[\sin x] = \cos x \tag{3}$$

In the exercises we will ask you to use the same method to derive the following formula for the derivative of $\cos x$:

$$\frac{d}{dx}[\cos x] = -\sin x \tag{4}$$

▶ **Example 1** Find dy/dx if $y = x \sin x$.

Solution. Using Formula (3) and the product rule we obtain

$$\begin{aligned}
\frac{dy}{dx} &= \frac{d}{dx}[x \sin x] \\
&= x\frac{d}{dx}[\sin x] + \sin x \frac{d}{dx}[x] \\
&= x \cos x + \sin x \quad ◀
\end{aligned}$$

▶ **Example 2** Find dy/dx if $y = \dfrac{\sin x}{1 + \cos x}$.

Solution. Using the quotient rule together with Formulas (3) and (4) we obtain

$$\begin{aligned}
\frac{dy}{dx} &= \frac{(1 + \cos x) \cdot \frac{d}{dx}[\sin x] - \sin x \cdot \frac{d}{dx}[1 + \cos x]}{(1 + \cos x)^2} \\
&= \frac{(1 + \cos x)(\cos x) - (\sin x)(-\sin x)}{(1 + \cos x)^2} \\
&= \frac{\cos x + \cos^2 x + \sin^2 x}{(1 + \cos x)^2} = \frac{\cos x + 1}{(1 + \cos x)^2} = \frac{1}{1 + \cos x} \quad ◀
\end{aligned}$$

Since Formulas (3) and (4) are valid only if x is in radians, the same is true for Formulas (5)–(8).

The derivatives of the remaining trigonometric functions are

$$\frac{d}{dx}[\tan x] = \sec^2 x \qquad \frac{d}{dx}[\sec x] = \sec x \tan x \tag{5–6}$$

$$\frac{d}{dx}[\cot x] = -\csc^2 x \qquad \frac{d}{dx}[\csc x] = -\csc x \cot x \tag{7–8}$$

These can all be obtained using the definition of the derivative, but it is easier to use Formulas (3) and (4) and apply the quotient rule to the relationships

$$\tan x = \frac{\sin x}{\cos x}, \quad \cot x = \frac{\cos x}{\sin x}, \quad \sec x = \frac{1}{\cos x}, \quad \csc x = \frac{1}{\sin x}$$

For example,

$$\frac{d}{dx}[\tan x] = \frac{d}{dx}\left[\frac{\sin x}{\cos x}\right] = \frac{\cos x \cdot \frac{d}{dx}[\sin x] - \sin x \cdot \frac{d}{dx}[\cos x]}{\cos^2 x}$$

$$= \frac{\cos x \cdot \cos x - \sin x \cdot (-\sin x)}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x$$

When finding the value of a derivative at a specific point $x = x_0$, it is important to substitute x_0 *after* the derivative is obtained. Thus, in Example 3 we made the substitution $x = \pi/4$ after f'' was calculated. What would have happened had we *incorrectly* substituted $x = \pi/4$ into $f'(x)$ before calculating f''?

Example 3 Find $f''(\pi/4)$ if $f(x) = \sec x$.

$$f'(x) = \sec x \tan x$$

$$f''(x) = \sec x \cdot \frac{d}{dx}[\tan x] + \tan x \cdot \frac{d}{dx}[\sec x]$$

$$= \sec x \cdot \sec^2 x + \tan x \cdot \sec x \tan x$$

$$= \sec^3 x + \sec x \tan^2 x$$

Thus,

$$f''(\pi/4) = \sec^3(\pi/4) + \sec(\pi/4)\tan^2(\pi/4)$$

$$= (\sqrt{2})^3 + (\sqrt{2})(1)^2 = 3\sqrt{2}$$

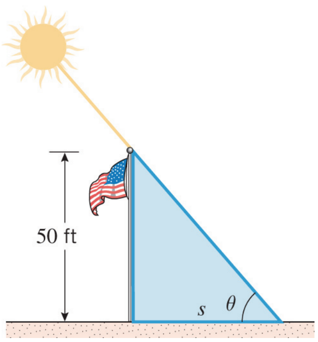


Figure 2.5.1

Example 4 On a sunny day, a 50 ft flagpole casts a shadow that changes with the angle of elevation of the Sun. Let s be the length of the shadow and θ the angle of elevation of the Sun (Figure 2.5.1). Find the rate at which the length of the shadow is changing with respect to θ when $\theta = 45°$. Express your answer in units of feet/degree.

Solution. The variables s and θ are related by $\tan\theta = 50/s$ or, equivalently,

$$s = 50\cot\theta \tag{9}$$

If θ is measured in radians, then Formula (7) is applicable, which yields

$$\frac{ds}{d\theta} = -50\csc^2\theta$$

which is the rate of change of shadow length with respect to the elevation angle θ in units of feet/radian. When $\theta = 45°$ (or equivalently $\theta = \pi/4$ radians), we obtain

$$\left.\frac{ds}{d\theta}\right|_{\theta=\pi/4} = -50\csc^2(\pi/4) = -100 \text{ feet/radian}$$

Converting radians (rad) to degrees (deg) yields

$$-100\,\frac{\text{ft}}{\text{rad}} \cdot \frac{\pi}{180}\frac{\text{rad}}{\text{deg}} = -\frac{5}{9}\pi\,\frac{\text{ft}}{\text{deg}} \approx -1.75 \text{ ft/deg}$$

Thus, when $\theta = 45°$, the shadow length is decreasing (because of the minus sign) at an approximate rate of 1.75 ft/deg increase in the angle of elevation.

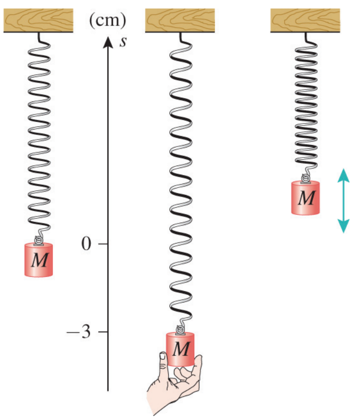


Figure 2.5.2

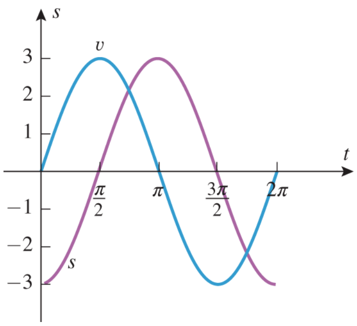
Example 5 As illustrated in Figure 2.5.2, suppose that a spring with an attached mass is stretched 3 cm beyond its rest position and released at time $t = 0$. Assuming that the position function of the top of the attached mass is

$$s = -3\cos t \tag{10}$$

where s is in centimeters and t is in seconds, find the velocity function and discuss the motion of the attached mass.

Solution. The velocity function is

$$v = \frac{ds}{dt} = \frac{d}{dt}[-3\cos t] = 3\sin t$$



▲ **Figure 2.5.3**

Figure 2.5.3 shows the graphs of the position and velocity functions. The position function tells us that the top of the mass oscillates between a low point of $s = -3$ and a high point of $s = 3$ with one complete oscillation occuring every 2π seconds [the period of (10)]. The top of the mass is moving up (the positive s-direction) when v is positive, is moving down when v is negative, and is at a high or low point when $v = 0$. Thus, for example, the top of the mass moves up from time $t = 0$ to time $t = \pi$, at which time it reaches the high point $s = 3$ and then moves down until time $t = 2\pi$, at which time it reaches the low point of $s = -3$. The motion then repeats periodically. ◀

In Example 5, the top of the mass has its maximum speed when it passes through its rest position. Why? What is that maximum speed?

✓ QUICK CHECK EXERCISES 2.5 (See page 98 for answers.)

1. Find dy/dx.
   (a) $y = \sin x$ (b) $y = \cos x$
   (c) $y = \tan x$ (d) $y = \sec x$
2. Find $f'(x)$ and $f'(\pi/3)$ if $f(x) = \sin x \cos x$.
3. Use a derivative to evaluate each limit.
   (a) $\lim_{h\to 0} \frac{\sin\left(\frac{\pi}{2}+h\right)-1}{h}$ (b) $\lim_{h\to 0} \frac{\csc(x+h)-\csc x}{h}$

EXERCISE SET 2.5 Graphing Utility

1–18 Find $f'(x)$. ■

1. $f(x) = 4\cos x + 2\sin x$
2. $f(x) = \dfrac{5}{x^2} + \sin x$
3. $f(x) = -4x^2\cos x$
4. $f(x) = 2\sin^2 x$
5. $f(x) = \dfrac{5-\cos x}{5+\sin x}$
6. $f(x) = \dfrac{\sin x}{x^2+\sin x}$
7. $f(x) = \sec x - \sqrt{2}\tan x$
8. $f(x) = (x^2+1)\sec x$
9. $f(x) = 4\csc x - \cot x$
10. $f(x) = \cos x - x\csc x$
11. $f(x) = \sec x \tan x$
12. $f(x) = \csc x \cot x$
13. $f(x) = \dfrac{\cot x}{1+\csc x}$
14. $f(x) = \dfrac{\sec x}{1+\tan x}$
15. $f(x) = \sin^2 x + \cos^2 x$
16. $f(x) = \sec^2 x - \tan^2 x$
17. $f(x) = \dfrac{\sin x \sec x}{1 + x\tan x}$
18. $f(x) = \dfrac{(x^2+1)\cot x}{3 - \cos x \csc x}$

19–24 Find d^2y/dx^2. ■

19. $y = x\cos x$
20. $y = \csc x$
21. $y = x\sin x - 3\cos x$
22. $y = x^2\cos x + 4\sin x$
23. $y = \sin x \cos x$
24. $y = \tan x$
25. Find the equation of the line tangent to the graph of $\tan x$ at
    (a) $x = 0$ (b) $x = \pi/4$ (c) $x = -\pi/4$.
26. Find the equation of the line tangent to the graph of $\sin x$ at
    (a) $x = 0$ (b) $x = \pi$ (c) $x = \pi/4$.
27. (a) Show that $y = x\sin x$ is a solution to $y'' + y = 2\cos x$.
    (b) Show that $y = x\sin x$ is a solution of the equation $y^{(4)} + y'' = -2\cos x$.
28. (a) Show that $y = \cos x$ and $y = \sin x$ are solutions of the equation $y'' + y = 0$.
    (b) Show that $y = A\sin x + B\cos x$ is a solution of the equation $y'' + y = 0$ for all constants A and B.
29. Find all values in the interval $[-2\pi, 2\pi]$ at which the graph of f has a horizontal tangent line.
    (a) $f(x) = \sin x$ (b) $f(x) = x + \cos x$
    (c) $f(x) = \tan x$ (d) $f(x) = \sec x$
30. (a) Use a graphing utility to make rough estimates of the values in the interval $[0, 2\pi]$ at which the graph of $y = \sin x \cos x$ has a horizontal tangent line.
    (b) Find the exact locations of the points where the graph has a horizontal tangent line.
31. A 10 ft ladder leans against a wall at an angle θ with the horizontal, as shown in the accompanying figure on the next page. The top of the ladder is x feet above the ground. If the bottom of the ladder is pushed toward the wall, find the rate at which x changes with respect to θ when $\theta = 60°$. Express the answer in units of feet/degree.