# A SEQUENCE

Stated informally, an ***infinite sequence***, or more simply a ***sequence***, is an unending succession of numbers, called ***terms***. It is understood that the terms have a definite order; that is, there is a first term $a_1$, a second term $a_2$, a third term $a_3$, a fourth term $a_4$, and so forth. Such a sequence would typically be written as

$$a_1, a_2, a_3, a_4, \ldots$$

where the dots are used to indicate that the sequence continues indefinitely. Some specific examples are

$$1, 2, 3, 4, \ldots, \qquad 1, \tfrac{1}{2}, \tfrac{1}{3}, \tfrac{1}{4}, \ldots,$$
$$2, 4, 6, 8, \ldots, \qquad 1, -1, 1, -1, \ldots$$

Each of these sequences has a definite pattern that makes it easy to generate additional terms if we assume that those terms follow the same pattern as the displayed terms. However, the most common way to specify a sequence is to give a rule or formula that relates each term in the sequence to its term number. For example, in the sequence

$$2, 4, 6, 8, \ldots$$

each term is twice the term number; that is, the $n$th term in the sequence is given by the formula $2n$. We denote this by writing the sequence as

$$2, 4, 6, 8, \ldots, 2n, \ldots$$

We refer to $2n$ as the *general term* of this sequence. Now, if we want to know a specific term in the sequence, we need only substitute its term number in the formula for the general term. For example, the 37th term in the sequence is $2 \cdot 37 = 74$.

**9.1.1 DEFINITION** A ***sequence*** is a function whose domain is a set of integers.

**▶ Example 1** In each part, find the general term of the sequence.

(a) $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots$ (b) $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots$

(c) $\frac{1}{2}, -\frac{2}{3}, \frac{3}{4}, -\frac{4}{5}, \ldots$ (d) $1, 3, 5, 7, \ldots$

***Solution (a).*** In Table 9.1.1, the four known terms have been placed below their term numbers, from which we see that the numerator is the same as the term number and the denominator is one greater than the term number. This suggests that the general term has numerator $n$ and denominator $n+1$, as indicated in the table.

**Table 9.1.1**

| TERM NUMBER | 1 | 2 | 3 | 4 | ⋯ | $n$ | ⋯ |
|---|---|---|---|---|---|---|---|
| TERM | $\frac{1}{2}$ | $\frac{2}{3}$ | $\frac{3}{4}$ | $\frac{4}{5}$ | ⋯ | $\frac{n}{n+1}$ | ⋯ |

***Solution (b).*** In Table 9.1.2, the denominators of the four known terms have been expressed as powers of 2 and the first four terms have been placed below their term numbers, from which we see that the exponent in the denominator is the same as the term number. This suggests that the general term is $1/2^n$, as indicated in the table.

**Table 9.1.2**

| TERM NUMBER | 1 | 2 | 3 | 4 | ⋯ | $n$ | ⋯ |
|---|---|---|---|---|---|---|---|
| TERM | $\frac{1}{2}$ | $\frac{1}{2^2}$ | $\frac{1}{2^3}$ | $\frac{1}{2^4}$ | ⋯ | $\frac{1}{2^n}$ | ⋯ |

***Solution (c).*** This sequence is identical to that in part (a), except for the alternating signs. Thus, the $n$th term in the sequence can be obtained by multiplying the $n$th term in part (a) by $(-1)^{n+1}$. This factor produces the correct alternating signs, since its successive values, starting with $n = 1$, are $1, -1, 1, -1, \ldots$. Thus, the sequence can be written as

$$\frac{1}{2}, -\frac{2}{3}, \frac{3}{4}, -\frac{4}{5}, \ldots, (-1)^{n+1}\frac{n}{n+1}, \ldots$$

and has general term $(-1)^{n+1}\frac{n}{n+1}$.

***Solution (d).*** In Table 9.1.3, the four known terms have been placed below their term numbers, from which we see that each term is one less than twice its term number. This suggests that the general term in the sequence is $2n - 1$, as indicated in the table.

**Table 9.1.3**

| TERM NUMBER | 1 | 2 | 3 | 4 | $\cdots$ | $n$ | $\cdots$ |
|---|---|---|---|---|---|---|---|
| TERM | 1 | 3 | 5 | 7 | $\cdots$ | $2n-1$ | $\cdots$ |

## THE BRACE NOTATION

When the general term of a sequence

$$a_1, a_2, a_3, \ldots, a_n, \ldots \tag{1}$$

is known, there is no need to write out the initial terms, and it is common to write only the general term enclosed in braces. Thus, (1) might be written as

$$\{a_n\}_{n=1}^{+\infty} \quad \text{or as} \quad \{a_n\}_{n=1}^{\infty}$$

For example, here are the four sequences in Example 1 expressed in brace notation.

| SEQUENCE | BRACE NOTATION |
|---|---|
| $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots, \frac{n}{n+1}, \ldots$ | $\left\{\frac{n}{n+1}\right\}_{n=1}^{+\infty}$ |
| $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots, \frac{1}{2^n}, \ldots$ | $\left\{\frac{1}{2^n}\right\}_{n=1}^{+\infty}$ |
| $\frac{1}{2}, -\frac{2}{3}, \frac{3}{4}, -\frac{4}{5}, \ldots, (-1)^{n+1}\frac{n}{n+1}, \ldots$ | $\left\{(-1)^{n+1}\frac{n}{n+1}\right\}_{n=1}^{+\infty}$ |
| $1, 3, 5, 7, \ldots, 2n-1, \ldots$ | $\{2n-1\}_{n=1}^{+\infty}$ |

The letter $n$ in (1) is called the ***index*** for the sequence. It is not essential to use $n$ for the index; any letter not reserved for another purpose can be used. Moreover, it is not essential to start the index at 1; sometimes it is more convenient to start it at 0 (or some other integer). For example, consider the sequence

$$1, \frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \ldots$$

One way to write this sequence is

$$\left\{\frac{1}{2^{n-1}}\right\}_{n=1}^{+\infty}$$

## LIMIT OF A SEQUENCE

**9.1.2 DEFINITION** A sequence $\{a_n\}$ is said to ***converge*** to the ***limit*** $L$ if given any $\epsilon > 0$, there is a positive integer $N$ such that $|a_n - L| < \epsilon$ for $n \geq N$. In this case we write

$$\lim_{n \to +\infty} a_n = L$$

A sequence that does not converge to some finite limit is said to ***diverge***.

**9.1.3 THEOREM** *Suppose that the sequences $\{a_n\}$ and $\{b_n\}$ converge to limits $L_1$ and $L_2$, respectively, and $c$ is a constant. Then:*

(a) $\lim_{n \to +\infty} c = c$

(b) $\lim_{n \to +\infty} ca_n = c \lim_{n \to +\infty} a_n = cL_1$

(c) $\lim_{n \to +\infty} (a_n + b_n) = \lim_{n \to +\infty} a_n + \lim_{n \to +\infty} b_n = L_1 + L_2$

(d) $\lim_{n \to +\infty} (a_n - b_n) = \lim_{n \to +\infty} a_n - \lim_{n \to +\infty} b_n = L_1 - L_2$

(e) $\lim_{n \to +\infty} (a_n b_n) = \lim_{n \to +\infty} a_n \cdot \lim_{n \to +\infty} b_n = L_1 L_2$

(f) $\lim_{n \to +\infty} \left(\dfrac{a_n}{b_n}\right) = \dfrac{\lim_{n \to +\infty} a_n}{\lim_{n \to +\infty} b_n} = \dfrac{L_1}{L_2} \quad (\text{if } L_2 \neq 0)$

**9.1.4 THEOREM** *A sequence converges to a limit $L$ if and only if the sequences of even-numbered terms and odd-numbered terms both converge to $L$.*

**▶ Example 3** In each part, determine whether the sequence converges or diverges by examining the limit as $n \to +\infty$.

$$\text{(a)}\ \left\{\frac{n}{2n+1}\right\}_{n=1}^{+\infty} \qquad \text{(b)}\ \left\{(-1)^{n+1}\frac{n}{2n+1}\right\}_{n=1}^{+\infty}$$

$$\text{(c)}\ \left\{(-1)^{n+1}\frac{1}{n}\right\}_{n=1}^{+\infty} \qquad \text{(d)}\ \{8-2n\}_{n=1}^{+\infty}$$

***Solution (a).*** Dividing numerator and denominator by $n$ and using Theorem 9.1.3 yields

$$\lim_{n\to+\infty}\frac{n}{2n+1} = \lim_{n\to+\infty}\frac{1}{2+1/n} = \frac{\displaystyle\lim_{n\to+\infty} 1}{\displaystyle\lim_{n\to+\infty}(2+1/n)} = \frac{\displaystyle\lim_{n\to+\infty} 1}{\displaystyle\lim_{n\to+\infty} 2 + \lim_{n\to+\infty} 1/n}$$

$$= \frac{1}{2+0} = \frac{1}{2}$$

Thus, the sequence converges to $\frac{1}{2}$.

***Solution (b).*** This sequence is the same as that in part (a), except for the factor of $(-1)^{n+1}$, which oscillates between $+1$ and $-1$. Thus, the terms in this sequence oscillate between positive and negative values, with the odd-numbered terms being identical to those in part (a) and the even-numbered terms being the negatives of those in part (a). Since the sequence in part (a) has a limit of $\frac{1}{2}$, it follows that the odd-numbered terms in this sequence approach $\frac{1}{2}$, and the even-numbered terms approach $-\frac{1}{2}$. Therefore, this sequence has no limit—it diverges.

***Solution (c).*** Since $1/n \to 0$, the product $(-1)^{n+1}(1/n)$ oscillates between positive and negative values, with the odd-numbered terms approaching 0 through positive values and the even-numbered terms approaching 0 through negative values. Thus,

$$\lim_{n\to+\infty}(-1)^{n+1}\frac{1}{n} = 0$$

so the sequence converges to 0.

***Solution (d).*** $\displaystyle\lim_{n\to+\infty}(8-2n) = -\infty$, so the sequence $\{8-2n\}_{n=1}^{+\infty}$ diverges. ◀

**▶ Example 4** In each part, determine whether the sequence converges, and if so, find its limit.

$$\text{(a)}\ 1, \frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \dots, \frac{1}{2^n}, \dots \qquad \text{(b)}\ 1, 2, 2^2, 2^3, \dots, 2^n, \dots$$

***Solution.*** Replacing $n$ by $x$ in the first sequence produces the power function $(1/2)^x$, and replacing $n$ by $x$ in the second sequence produces the power function $2^x$. Now recall that if $0 < b < 1$, then $b^x \to 0$ as $x \to +\infty$, and if $b > 1$, then $b^x \to +\infty$ as $x \to +\infty$ (Figure 1.8.1). Thus,

$$\lim_{n\to+\infty} \frac{1}{2^n} = 0 \quad \text{and} \quad \lim_{n\to+\infty} 2^n = +\infty$$

So, the sequence $\{1/2^n\}$ converges to 0, but the sequence $\{2^n\}$ diverges. ◀

**▶ Example 5** Find the limit of the sequence $\left\{\dfrac{n}{e^n}\right\}_{n=1}^{+\infty}$.

***Solution.*** The expression

$$\lim_{n\to+\infty} \frac{n}{e^n}$$

is an indeterminate form of type $\infty/\infty$, so L'Hôpital's rule is indicated. However, we cannot apply this rule directly to $n/e^n$ because the functions $n$ and $e^n$ have been defined here only at the positive integers, and hence are not differentiable functions. To circumvent this problem we extend the domains of these functions to all real numbers, here implied by replacing $n$ by $x$, and apply L'Hôpital's rule to the limit of the quotient $x/e^x$. This yields

$$\lim_{x\to+\infty} \frac{x}{e^x} = \lim_{x\to+\infty} \frac{1}{e^x} = 0$$

from which we can conclude that

$$\lim_{n\to+\infty} \frac{n}{e^n} = 0 \blacktriangleleft$$

**▶ Example 6** Show that $\lim\limits_{n\to+\infty} \sqrt[n]{n} = 1$.

***Solution.***

$$\lim_{n\to+\infty} \sqrt[n]{n} = \lim_{n\to+\infty} n^{1/n} = \lim_{n\to+\infty} e^{(1/n)\ln n} = e^0 = 1$$

By L'Hôpital's rule applied to $(1/x)\ln x$ ◀

**▶ Example 7** The sequence

$$\frac{1}{2}, \frac{1}{3}, \frac{1}{2^2}, \frac{1}{3^2}, \frac{1}{2^3}, \frac{1}{3^3}, \ldots$$

converges to 0, since the even-numbered terms and the odd-numbered terms both converge to 0, and the sequence

$$1, \tfrac{1}{2}, 1, \tfrac{1}{3}, 1, \tfrac{1}{4}, \ldots$$

diverges, since the odd-numbered terms converge to 1 and the even-numbered terms converge to 0. ◀

# EXERCISE SET 9.1

**1.** In each part, find a formula for the general term of the sequence, starting with $n = 1$.

(a) $1, \frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \ldots$ (b) $1, -\frac{1}{3}, \frac{1}{9}, -\frac{1}{27}, \ldots$

(c) $\frac{1}{2}, \frac{3}{4}, \frac{5}{6}, \frac{7}{8}, \ldots$ (d) $\frac{1}{\sqrt{\pi}}, \frac{4}{\sqrt[3]{\pi}}, \frac{9}{\sqrt[4]{\pi}}, \frac{16}{\sqrt[5]{\pi}}, \ldots$

**7–22** Write out the first five terms of the sequence, determine whether the sequence converges, and if so find its limit. ■

**7.** $\left\{\frac{n}{n+2}\right\}_{n=1}^{+\infty}$ **8.** $\left\{\frac{n^2}{2n+1}\right\}_{n=1}^{+\infty}$ **9.** $\{2\}_{n=1}^{+\infty}$

**10.** $\left\{\ln\left(\frac{1}{n}\right)\right\}_{n=1}^{+\infty}$ **11.** $\left\{\frac{\ln n}{n}\right\}_{n=1}^{+\infty}$ **12.** $\left\{n \sin \frac{\pi}{n}\right\}_{n=1}^{+\infty}$

**13.** $\{1 + (-1)^n\}_{n=1}^{+\infty}$ **14.** $\left\{\frac{(-1)^{n+1}}{n^2}\right\}_{n=1}^{+\infty}$

**15.** $\left\{(-1)^n \frac{2n^3}{n^3+1}\right\}_{n=1}^{+\infty}$ **16.** $\left\{\frac{n}{2^n}\right\}_{n=1}^{+\infty}$

**17.** $\left\{\frac{(n+1)(n+2)}{2n^2}\right\}_{n=1}^{+\infty}$ **18.** $\left\{\frac{\pi^n}{4^n}\right\}_{n=1}^{+\infty}$

**19.** $\{n^2 e^{-n}\}_{n=1}^{+\infty}$ **20.** $\{\sqrt{n^2+3n} - n\}_{n=1}^{+\infty}$

**21.** $\left\{\left(\frac{n+3}{n+1}\right)^n\right\}_{n=1}^{+\infty}$ **22.** $\left\{\left(1 - \frac{2}{n}\right)^n\right\}_{n=1}^{+\infty}$

**23–30** Find the general term of the sequence, starting with $n = 1$, determine whether the sequence converges, and if so find its limit.

**23.** $\frac{1}{2}, \frac{3}{4}, \frac{5}{6}, \frac{7}{8}, \ldots$

**24.** $0, \frac{1}{2^2}, \frac{2}{3^2}, \frac{3}{4^2}, \ldots$

**25.** $\frac{1}{3}, -\frac{1}{9}, \frac{1}{27}, -\frac{1}{81}, \ldots$

**26.** $-1, 2, -3, 4, -5, \ldots$

**27.** $\left(1 - \frac{1}{2}\right), \left(\frac{1}{3} - \frac{1}{2}\right), \left(\frac{1}{3} - \frac{1}{4}\right), \left(\frac{1}{5} - \frac{1}{4}\right), \ldots$

**28.** $3, \frac{3}{2}, \frac{3}{2^2}, \frac{3}{2^3}, \ldots$

**29.** $(\sqrt{2} - \sqrt{3}), (\sqrt{3} - \sqrt{4}), (\sqrt{4} - \sqrt{5}), \ldots$

**30.** $\frac{1}{3^5}, -\frac{1}{3^6}, \frac{1}{3^7}, -\frac{1}{3^8}, \ldots$

# SOLUTION SET

**1.** (a) $\dfrac{1}{3^{n-1}}$ (b) $\dfrac{(-1)^{n-1}}{3^{n-1}}$ (c) $\dfrac{2n-1}{2n}$ (d) $\dfrac{n^2}{\pi^{1/(n+1)}}$

**7.** $1/3,\ 2/4,\ 3/5,\ 4/6,\ 5/7, \ldots$; $\lim\limits_{n\to+\infty} \dfrac{n}{n+2} = 1$, converges.

**9.** $2, 2, 2, 2, 2, \ldots$; $\lim\limits_{n\to+\infty} 2 = 2$, converges.

**11.** $\dfrac{\ln 1}{1}, \dfrac{\ln 2}{2}, \dfrac{\ln 3}{3}, \dfrac{\ln 4}{4}, \dfrac{\ln 5}{5}, \ldots$; $\lim\limits_{n\to+\infty} \dfrac{\ln n}{n} = \lim\limits_{n\to+\infty} \dfrac{1}{n} = 0$ $\left(\text{apply L'Hôpital's Rule to } \dfrac{\ln x}{x}\right)$, converges.

**13.** $0, 2, 0, 2, 0, \ldots$; diverges.

**15.** $-1,\ 16/9,\ -54/28,\ 128/65,\ -250/126, \ldots$; diverges because odd-numbered terms approach $-2$, even-numbered terms approach 2.

**17.** $6/2,\ 12/8,\ 20/18,\ 30/32,\ 42/50, \ldots$; $\lim\limits_{n\to+\infty} \dfrac{1}{2}(1+1/n)(1+2/n) = 1/2$, converges.

**19.** $e^{-1},\ 4e^{-2},\ 9e^{-3},\ 16e^{-4},\ 25e^{-5}, \ldots$; using L'Hospital's rule, $\lim\limits_{x\to+\infty} x^2e^{-x} = \lim\limits_{x\to+\infty} \dfrac{x^2}{e^x} = \lim\limits_{x\to+\infty} \dfrac{2x}{e^x} = \lim\limits_{x\to+\infty} \dfrac{2}{e^x} = 0$, so $\lim\limits_{n\to+\infty} n^2e^{-n} = 0$, converges.

**21.** $2,\ (5/3)^2,\ (6/4)^3,\ (7/5)^4,\ (8/6)^5, \ldots$; let $y = \left[\dfrac{x+3}{x+1}\right]^x$, converges because $\lim\limits_{x\to+\infty} \ln y = \lim\limits_{x\to+\infty} \dfrac{\ln \dfrac{x+3}{x+1}}{1/x} =$ $\lim\limits_{x\to+\infty} \dfrac{2x^2}{(x+1)(x+3)} = 2$, so $\lim\limits_{n\to+\infty} \left[\dfrac{n+3}{n+1}\right]^n = e^2$.

**23.** $\left\{\dfrac{2n-1}{2n}\right\}_{n=1}^{+\infty}$; $\lim\limits_{n\to+\infty} \dfrac{2n-1}{2n} = 1$, converges.

**25.** $\left\{(-1)^{n-1}\dfrac{1}{3^n}\right\}_{n=1}^{+\infty}$; $\lim\limits_{n\to+\infty} \dfrac{(-1)^{n-1}}{3^n} = 0$, converges.

**27.** $\left\{(-1)^{n+1}\left(\dfrac{1}{n} - \dfrac{1}{n+1}\right)\right\}_{n=1}^{+\infty}$; the sequence converges to 0.

**29.** $\left\{\sqrt{n+1} - \sqrt{n+2}\right\}_{n=1}^{+\infty}$; converges because $\lim\limits_{n\to+\infty} (\sqrt{n+1} - \sqrt{n+2}) = \lim\limits_{n\to+\infty} \dfrac{(n+1)-(n+2)}{\sqrt{n+1}+\sqrt{n+2}} =$

$$= \lim_{n\to+\infty} \frac{-1}{\sqrt{n+1}+\sqrt{n+2}} = 0.$$