INTEGRATION BY SPECIAL SUBSTITUTION

For rational functions of sin x and cos x:

Functions that consist of finitely many sums, differences, quotients, and products of $\sin x$ and $\cos x$ are called ***rational functions of sin x and cos x***. Some examples are

$$\frac{\sin x + 3\cos^2 x}{\cos x + 4\sin x}, \quad \frac{\sin x}{1 + \cos x - \cos^2 x}, \quad \frac{3\sin^5 x}{1 + 4\sin x}$$

The Endpaper Integral Table gives a few formulas for integrating rational functions of $\sin x$ and $\cos x$ under the heading *Reciprocals of Basic Functions*. For example, it follows from Formula (18) that

$$\int \frac{1}{1 + \sin x}\, dx = \tan x - \sec x + C \tag{2}$$

However, since the integrand is a rational function of $\sin x$, it may be desirable in a particular application to express the value of the integral in terms of $\sin x$ and $\cos x$ and rewrite (2) as

$$\int \frac{1}{1 + \sin x}\, dx = \frac{\sin x - 1}{\cos x} + C$$

Many rational functions of $\sin x$ and $\cos x$ can be evaluated by an ingenious method that was discovered by the mathematician Karl Weierstrass (see p. 102 for biography). The idea is to make the substitution

$$u = \tan(x/2), \quad -\pi/2 < x/2 < \pi/2$$

from which it follows that

$$x = 2\tan^{-1} u, \quad dx = \frac{2}{1 + u^2}\, du$$

To implement this substitution we need to express $\sin x$ and $\cos x$ in terms of u. For this purpose we will use the identities

$$\sin x = 2\sin(x/2)\cos(x/2) \tag{3}$$
$$\cos x = \cos^2(x/2) - \sin^2(x/2) \tag{4}$$

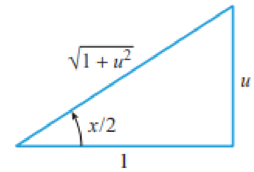
and the following relationships suggested by Figure 7.6.1:

$$\sin(x/2) = \frac{u}{\sqrt{1 + u^2}} \quad \text{and} \quad \cos(x/2) = \frac{1}{\sqrt{1 + u^2}}$$

Substituting these expressions in (3) and (4) yields

$$\sin x = 2\left(\frac{u}{\sqrt{1+u^2}}\right)\left(\frac{1}{\sqrt{1+u^2}}\right) = \frac{2u}{1+u^2}$$

$$\cos x = \left(\frac{1}{\sqrt{1+u^2}}\right)^2 - \left(\frac{u}{\sqrt{1+u^2}}\right)^2 = \frac{1-u^2}{1+u^2}$$



▲ Figure 7.6.1

In summary, we have shown that the substitution $u = \tan(x/2)$ can be implemented in a rational function of $\sin x$ and $\cos x$ by letting

$$\sin x = \frac{2u}{1+u^2}, \quad \cos x = \frac{1-u^2}{1+u^2}, \quad dx = \frac{2}{1+u^2}\,du \tag{5}$$

▶ **Example 5** Evaluate $\displaystyle\int \frac{dx}{1-\sin x+\cos x}$.

Solution. The integrand is a rational function of $\sin x$ and $\cos x$ that does not match any of the formulas in the Endpaper Integral Table, so we make the substitution $u = \tan(x/2)$. Thus, from (5) we obtain

$$\begin{aligned}\int \frac{dx}{1-\sin x+\cos x} &= \int \frac{\dfrac{2\,du}{1+u^2}}{1-\left(\dfrac{2u}{1+u^2}\right)+\left(\dfrac{1-u^2}{1+u^2}\right)} \\ &= \int \frac{2\,du}{(1+u^2)-2u+(1-u^2)} \\ &= \int \frac{du}{1-u} = -\ln|1-u|+C = -\ln|1-\tan(x/2)|+C \quad ◀\end{aligned}$$

EXERCISE SET 7.6

65–70 (a) Make u-substitution (5) to convert the integrand to a rational function of u, and then evaluate the integral. (b) If you have a CAS, use it to evaluate the integral (no substitution), and then confirm that the result is equivalent to that in part (a).

65. $\displaystyle\int \frac{dx}{1+\sin x+\cos x}$

66. $\displaystyle\int \frac{dx}{2+\sin x}$

67. $\displaystyle\int \frac{d\theta}{1-\cos\theta}$

68. $\displaystyle\int \frac{dx}{4\sin x-3\cos x}$

69. $\displaystyle\int \frac{dx}{\sin x+\tan x}$

70. $\displaystyle\int \frac{\sin x}{\sin x+\tan x}\,dx$

SOLUTION SET

65. $u=\tan(x/2)$, $\displaystyle\int \frac{1}{1+\dfrac{2u}{1+u^2}+\dfrac{1-u^2}{1+u^2}}\frac{2}{1+u^2}du=\int\frac{1}{u+1}du=\ln|\tan(x/2)+1|+C.$

67. $u=\tan(\theta/2)$, $\displaystyle\int\frac{d\theta}{1-\cos\theta}=\int\frac{1}{u^2}du=-\frac{1}{u}+C=-\cot(\theta/2)+C.$

69. $u=\tan(x/2)$, $\displaystyle\frac{1}{2}\int\frac{1-u^2}{u}du=\frac{1}{2}\int(1/u-u)du=\frac{1}{2}\ln|\tan(x/2)|-\frac{1}{4}\tan^2(x/2)+C.$