ASSIGNMENT #3

23K-2023

Ex: 6.1 (Q9, 15, 21)

(Q9)

$$U = \begin{bmatrix} 3 & -2 \\ 4 & 8 \end{bmatrix}, \quad V = \begin{bmatrix} -1 & 3 \\ 1 & 1 \end{bmatrix}$$

$$\langle U, V \rangle = u_1 v_1 + u_2 v_2 + u_3 v_3 + u_4 v_4$$
$$= -3 - 6 + 4 + 8 = 3$$

(Q15) $p = x + x^3$, $q = 1 + x^2$

$x_0 = -2$, $x_1 = -1$, $x_2 = 0$, $x_3 = 1$

$$\langle p, q \rangle = p(x_0)q(x_0) + p(x_1)q(x_1) + p(x_2)q(x_2) + p(x_3)q(x_3)$$
$$= (-10)(5) + (-2)(2) + (0)(1) + (2)(2)$$
$$= -50 - 4 + 4 = -50 \text{ Ans}$$

(Q21) $U = \begin{bmatrix} 3 & -2 \\ 4 & 8 \end{bmatrix}$, $V = \begin{bmatrix} -1 & 3 \\ 1 & 1 \end{bmatrix}$

$$\langle U, V \rangle = -3 - 6 + 4 + 8 = 3$$
$$\|U\| = \sqrt{9 + 4 + 16 + 64} = \sqrt{93} \text{ Ans}$$

Ex: 6.2 (1, 4, 6, 8, 12, 14, 17, 18)

(Q1)

$$\cos\theta = \frac{\langle u \cdot v \rangle}{\|u\| \cdot \|v\|}$$

(a) $u = (1, -3)$, $v = (2, 4)$

$\langle u \cdot v \rangle = 2 - 12 = -10$

$\|u\| = \sqrt{10}$, $\|v\| = \sqrt{20}$

$$\cos\theta = \frac{-10}{\sqrt{200}} = \frac{-\cancel{10}}{\cancel{10}\sqrt{2}} = \frac{-1}{\sqrt{2}} \text{ Ans}$$

(b) $u = (-1, 5, 2)$, $v = (2, 4, -9)$

$\langle u \cdot v \rangle = -2 + 20 - 18 = 0$

$\|u\| = \sqrt{30}$ $\|v\| = \sqrt{101}$

$$\cos\theta = \frac{0}{\sqrt{30} \cdot \sqrt{101}} = 0 \text{ Ans}$$

(c) $u = (1, 0, 1, 0)$, $v = (-3, -3, -3, -3)$

$\langle u \cdot v \rangle = -3 - 3 = -6$

$\|u\| = \sqrt{2}$ $\|v\| = 6$

$$\cos\theta = \frac{-\cancel{6}}{\cancel{6}\sqrt{2}} = \frac{-1}{\sqrt{2}} \text{ Ans}$$

(Q4) $p = x - x^2$, $q = 7 + 3x + 3x^2$

$\langle p.q \rangle = 0 + \cancel{3} - \cancel{3} = 0$

$\|p\| \cdot \|q\| = \sqrt{2} \cdot \sqrt{67}$

$\cos\theta = \frac{0}{\sqrt{134}} = 0$ Ans

(Q6) $A = \begin{bmatrix} 2 & 4 \\ -1 & 3 \end{bmatrix}$, $B = \begin{bmatrix} -3 & 1 \\ 4 & 2 \end{bmatrix}$

$\langle A \cdot B \rangle = -\cancel{6} + \cancel{4} - \cancel{4} + \cancel{6} = 0$

$\|A\| \cdot \|B\| = \sqrt{30} \cdot \sqrt{30} = 30$

$\cos\theta = \frac{0}{30} = 0$

(Q8) a) $u = (u_1, u_2, u_3)$, $v = (0,0,0)$

For orthogonal $\langle u.v \rangle = 0$

$\langle u \cdot v \rangle = 0 + 0 + 0 = 0$ (orthogonal)

(b) $u = (-4, 6, -10, 1)$, $v = (2, 1, -2, 9)$

$\langle u \cdot v \rangle = -8 + 6 + 20 + 9 = 27$ (not orthogonal

(c) $u = (a, b, c)$, $v = (-c, 0, a)$

$\langle u.v \rangle = -ac + 0 + ac = 0$ (orthogonal)

(Q12) $U = \begin{bmatrix} 5 & -1 \\ 2 & -2 \end{bmatrix}$, $V = \begin{bmatrix} 1 & 3 \\ -1 & 0 \end{bmatrix}$

$\langle U.V \rangle = 5 - 3 - 2 = 0$ (orthogonal)

(Q14) $\langle U.V \rangle = 2u_1v_1 + ku_2v_2$

$u = (2, -4)$, $v = (0, 3)$

$\langle U.V \rangle = 2(2)(0) + k(-4)(3)$

$= 0 - 12k \Rightarrow k = 0$

The value of k should be $= 0$ for orthogonal

(Q17) $p_1 = 2 + kx + 6x^2$, $p_2 = l + 5x + 3x^2$

$p_3 = 1 + 2x + 3x^2$

$\langle p_1 . p_2 \rangle = 2l + 5k + 18$ ① $\Rightarrow -38 - 50 + 18 = -70$

$\langle p_1 . p_3 \rangle = 2 + 2k + 18 \Rightarrow k = -10$ put in

$\langle p_2 . p_3 \rangle = l + 10 + 9 \Rightarrow l = -19$ ①

since $-70 \neq 0$ so it is not orthogonal

no scalor k, l exist that make it orthogonal

(Q18) $u = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$, $V = \begin{bmatrix} 5 \\ -8 \end{bmatrix}$; $A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$

$$\langle u \cdot v \rangle = \left(\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 3 \end{bmatrix} \right) \cdot \left(\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ -8 \end{bmatrix} \right)$$

$$= \begin{bmatrix} 9 \\ 6 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -3 \end{bmatrix} = 0 \quad \text{proved}$$

EX: 6.3 (3a, 6, 13, 30, 47, 49)

(Q3) a) $p_1(x) = \frac{2}{3} - \frac{2}{3}x + \frac{1}{3}x^2$, $p_2(x) = \frac{2}{3} + \frac{1}{3}x - \frac{2}{3}x^2$

$p_3(x) = \frac{1}{3} + \frac{2}{3}x + \frac{2}{3}x^2$

$$\langle p_1 \cdot p_2 \rangle = \frac{4}{9} - \frac{2}{9} - \frac{2}{9} = 0$$

$$\langle p_1 \cdot p_3 \rangle = \frac{2}{9} - \frac{4}{9} + \frac{2}{9} = 0$$

$$\langle p_2 \cdot p_3 \rangle = \frac{2}{9} + \frac{2}{9} - \frac{4}{9} = 0$$

It is orthogonal

$$\begin{array}{cccc} & & u_2 & u_3 \\ \text{6b) } A = & \left[\begin{array}{ccc} 1/5 & -1/2 & 1/3 \\ 1/5 & 1/2 & 1/3 \\ 1/5 & 0 & -2/3 \end{array}\right] \end{array}$$

$u_1 = \left(\frac{1}{5}, \frac{1}{5}, \frac{1}{5}\right)$, $u_2 = \left(\frac{-1}{2}, \frac{1}{2}, 0\right)$, $u_3 = \left(1/3, 1/3, \frac{-2}{3}\right)$

checking for orthogonal:

$\langle u_1 \cdot u_2 \rangle = -\frac{1}{10} + \frac{1}{10} + 0 = 0$

$\langle u_1 \cdot u_3 \rangle = \frac{1}{15} + \frac{1}{15} - \frac{2}{15} = 0$

$\langle u_2 \cdot u_3 \rangle = -\frac{1}{6} + \frac{1}{6} + 0 = 0$

column vectors are orthogonal
As values are "0" so they are linear ind.
so they form orthogonal basis for the column space of A.

since vectors are already orthogonal
$(u_1, u_2, u_3) = (v_1, v_2, v_3)$

$q_1 = \frac{u_1}{\|u_1\|} = \frac{\left(\frac{1}{5}, \frac{1}{5}, \frac{1}{5}\right)}{\frac{\sqrt{3}}{5}}$

$$= \frac{8}{\sqrt{3}}\left(\frac{1}{8}, \frac{1}{8}, \frac{1}{8}\right)$$

$$q_1 = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$$

$$q_2 = \frac{u_2}{\|u_2\|} = \frac{\left(\frac{-1}{2}, \frac{1}{2}, 0\right)}{\frac{1}{\sqrt{2}}} \Rightarrow \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0\right)$$

$$q_3 = \frac{u_3}{\|u_3\|} = \frac{\left(\frac{1}{3}, \frac{1}{3}, \frac{-2}{3}\right)}{\sqrt{\frac{1}{9}+\frac{1}{9}+\frac{4}{9}}}$$

$$= \frac{\sqrt{3}}{\sqrt{2}}\left(\frac{1}{3}, \frac{1}{3}, \frac{-2}{3}\right)$$

$$\frac{\sqrt{2} \times \sqrt{3}}{\sqrt{3} \times \sqrt{3}} = \frac{\sqrt{6}}{3}$$

The orthonormal basis are for the column space are,

$$\begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}, \begin{bmatrix} \sqrt{2}/2 \\ \sqrt{2}/2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1/3 \\ 1/3 \\ -2/3 \end{bmatrix}$$

Q13) Find coordinate vector $(u)_S$ for the vector u and the basis S.

$v_1 = (2,-2,1)$, $v_2 = (2,1,-2)$, $v_3 = (1,2,2)$
also express vector u as linear combination of v_1, v_2, v_3.
$u = (-1,0,2)$

To find the coordinate vector we have to show L.C.
Since we are finding for orthogonal basis.

$$u = \frac{\langle u \cdot v_1\rangle}{\|v_1\|^2} v_1 + \frac{\langle u \cdot v_2\rangle}{\|v_2\|} v_2 + \frac{\langle u \cdot v_3\rangle}{\|v_3\|} v_3$$

$$= 0 v_1 + \frac{-\cancel{6}^{\,2}}{\cancel{3}} v_2 + \frac{\cancel{3}^{\,1}}{3\cancel{9}} v_3$$

$$u = 0 v_1 - 2 v_2 + \frac{1}{3} v_3$$

$(u)_S = (0, -2, 1/3)$ Ans (coordinate vector)
$S = (v_1, v_2, v_3)$ (Basis S)

(Q30) $u_1 = (1,0,0)$, $u_2 = (3,7,-2)$
$u_3 = (0,4,1)$

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$$v_1 = u_1 = (1,0,0)$$

$$v_2 = u_2 - \frac{\langle u_2 . v_1 \rangle v_1}{|v_1|^2}$$

$$= (3,7,-2) - \left[\frac{3(1,0,0)}{}\right]$$

$$= (3,7,-2) - (3,0,0)$$

$$v_2 = \cancel{8}(0,7,-2)$$

$$v_3 = u_3 - \left[\frac{(u_3 . v_1) v_1}{|v_1|^2}\right] + \left[\frac{\langle u_3 . v_2 \rangle v_2}{|v_2|^2}\right] \cancel{+ u_3 v_3}$$

$$= (0,4,1) - [0] + \left(\frac{26}{53}(0,7,-2)\right)$$

$$= (0,4,1) - \left(\cancel{\frac{182}{53}0}\,(0, \frac{182}{153}, \frac{-52}{53}\right)$$

$$v_3 = \left(0, \frac{30}{53}, \frac{105}{53}\right)$$

$$q_1 = \frac{V_1}{|V_1|} \Rightarrow \frac{(1,0,0)}{1} \Rightarrow (1,0,0)$$

$$q_2 = \frac{V_2}{|V_2|} = \frac{(0,7,-2)}{\sqrt{53}} \Rightarrow \left(0, \frac{7}{\sqrt{53}}, \frac{-2}{\sqrt{53}}\right)$$

$$q_3 = \frac{V_3}{|V_3|} = \frac{\left(0, \frac{30}{53}, \frac{105}{53}\right)}{\frac{15\sqrt{53}}{53}}$$

$$q_3 = \left(0, \frac{2}{\sqrt{53}}, \frac{7}{\sqrt{53}}\right)$$

Q47) $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 1 & 2 & 0 \end{bmatrix}$, $Q = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{3} & 1/\sqrt{6} \\ 0 & 1/\sqrt{3} & 2/\sqrt{6} \\ 1/\sqrt{2} & 1/\sqrt{3} & -1/\sqrt{6} \end{bmatrix}$

$$R = \begin{bmatrix} \langle u_1 \cdot q_1 \rangle & \langle u_2 \cdot q_1 \rangle & \langle u_3 \cdot q_1 \rangle \\ 0 & \langle u_2 \cdot q_2 \rangle & \langle u_3 \cdot q_2 \rangle \\ 0 & 0 & \langle u_3 \cdot q_3 \rangle \end{bmatrix}$$

$\langle u_1 \cdot q_1 \rangle = \boxed{2/\sqrt{2}}$, $\langle u_2 \cdot q_1 \rangle = \boxed{2/\sqrt{2}}$ ~~$u_3 q_1$~~

$\langle u_3 \cdot q_1 \rangle = \boxed{2/\sqrt{2}}$, $\langle u_2 \cdot q_2 \rangle = \boxed{3/\sqrt{3}}$

$\langle u_3 \cdot q_2 \rangle = \boxed{2/\sqrt{2}}$, $\langle u_3 \cdot q_3 \rangle = \boxed{4/\sqrt{6}}$

$$R = \begin{bmatrix} 2/\sqrt{2} & 2/\sqrt{2} & 2/\sqrt{2} \\ 0 & 3/\sqrt{3} & 2/\sqrt{2} \\ 0 & 0 & 4/\sqrt{6} \end{bmatrix}$$ Ans

(Q49) $A = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ 1 & 0 & 1 \\ -1 & 1 & 1 \end{bmatrix}$ $\begin{matrix} R_1+R_2 = \\ \Rightarrow -R_1+R_3 \\ R_1+R_4 \end{matrix}$ $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 1 & 2 \end{bmatrix}$

$-R_2+R_4 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

since the matrix is linear dependen so A does not Q decompose.