~~Dated~~: The cauchy - Euler equation.

The eq. of the form

$$a_0 x^n \frac{d^n y}{dx^n} + a_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + a_2 x^{n-2} \frac{d^{n-2} y}{dx^{n-2}} + \cdots + a_{n-1} x \frac{dy}{dx} + a_n y = F(x)$$

called cauchy - Euler eq. or equidimensional eq. Equation can be reduced to a linear differential eq. with constant coefficient by The Transformation

$$x = e^t$$

$$\text{or } t = \ln x$$

$$x \frac{dy}{dx} = \frac{dy}{dt} \qquad \because \frac{d}{dx} = D \text{ and } \frac{d}{dt} = \Delta$$

$$x \cdot Dy = \Delta y$$

$$\boxed{x \cdot D = \Delta}$$

$$\frac{d^2 y}{dx^2} = \frac{1}{x^2}\left(\frac{d^2 y}{dt^2} - \frac{dy}{dt}\right)$$

$$x^2 \frac{d^2 y}{dx^2} = \frac{d^2 y}{dt^2} - \frac{dy}{dt}$$

$$x^2 D^2 y = \Delta^2 y - \Delta y$$

$$x^2 D^2 y = (\Delta^2 - \Delta) y$$

$$x^2 D^2 = \Delta^2 - \Delta$$

$$\boxed{x^2 D^2 = \Delta(\Delta - 1)}$$

Similarly

$$x^3 D^3 = \Delta(\Delta-1)(\Delta-2)$$

$$x^4 D^4 = \Delta(\Delta-1)(\Delta-2)(\Delta-3).$$

Dated: Q 6 EX 10.4

$$x^4 \frac{d^3y}{dx^3} + 2x^3 \frac{d^2y}{dx^2} - x^2 \frac{dy}{dx} + xy = 1$$

Solution:-

This eq is not the cauchy Euler eq so for cauchy Euler eq. divided by x. both side. we get

$$x^3 \frac{d^3y}{dx^3} + 2x^2 \frac{d^2y}{dx^2} - x \frac{dy}{dx} + y = \frac{1}{x} \quad \text{--- (1)}$$

Reduced to linear differential eq.

let $x = e^t$ or $t = \ln x$

$xD = \Delta$

$x^2D^2 = \Delta(\Delta-1)$

$x^3D^3 = \Delta(\Delta-1)(\Delta-2)$

$(1) \Rightarrow [\Delta(\Delta-1)(\Delta-2) + 2\Delta(\Delta-1) - \Delta + 1]y = e^{-t}$

$[\Delta(\Delta^2-3\Delta+2) + 2(\Delta^2-\Delta) - \Delta+1]y = e^{-t}$

$[\Delta^3 - 3\Delta^2 + \cancel{2\Delta} + 2\Delta^2 - \cancel{2\Delta} - \Delta + 1]y = e^{-t}$

$[\Delta^3 - \Delta^2 - \Delta + 1]y = e^{-t}$

$[\Delta^2(\Delta-1) - 1(\Delta-1)]y = e^{-t}$

$(\Delta-1)(\Delta^2-1)y = e^{-t}$

$(\Delta-1)(\Delta-1)(\Delta+1)y = e^{-t}$

$(\Delta+1)(\Delta-1)^2 y = e^{-t}$

Dated:

For complementary function

$$f(\Delta) = 0$$

$$(\Delta+1)(\Delta-1)^2 = 0$$

$$\Delta = -1, 1, 1$$

Hence the complementary function is

$$y_c = c_1 e^{-t} + e^{t}[c_2 + c_3 t]$$

$$y_c = c_1 x^{-1} + x[c_2 + c_3 \ln x]$$

Now for y_p (particular integral)

$$y_p = \frac{1}{f(\Delta)} \cdot F(t)$$

$$y_p = \frac{1}{(\Delta+1)(\Delta-1)^2} \cdot e^{-t} \qquad \because \frac{1}{\phi(\Delta)(\Delta-a)^k} e^{at} = \frac{t^k e^{at}}{k!\,\phi(a)}$$

$$y_p = \frac{t \cdot e^{-t}}{(-1-1)^2} = \frac{t \cdot e^{-t}}{4} = \frac{x^{-1} \cdot \ln x}{4} = \frac{\ln x}{4x}$$

Hence the general solution is

$$y = y_c + y_p$$

$$y = c_1 x^{-1} + x[c_2 + c_3 \ln x] + \frac{\ln x}{4x} \quad \text{Ans.}$$

Dated: Q 9 Ex 10.4

~~$(x+1)^2 \frac{d^2y}{dx^2} + (x+1)\frac{dy}{dx} + y = 4[\cos \ln (x+1)]^2$~~

$$(2x+1)^2 \frac{d^2y}{dx^2} - 6(2x+1)\frac{dy}{dx} + 16y = 8(1+2x)^2$$

Solution:

Reduced to linear differential eq.

Let $(2x+1) = e^t$ or $t = \ln(2x+1)$

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$$

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{2}{(2x+1)}$$

$$\frac{dy}{dx} = \frac{\Delta y \, 2}{(2x+1)} \qquad \text{——— (1)}$$

$$(2x+1)\frac{dy}{dx} = 2\Delta y$$

$$(2x+1)Dy = 2\Delta y$$

$$\boxed{(2x+1)D = 2\Delta}$$

and $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$

From (1)

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{2}{(2x+1)} \cdot \frac{dy}{dt}\right]$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[2(2x+1)^{-1} \cdot \frac{dy}{dt}\right]$$

By using quotient formula for derivative

Dated:

$$\frac{d^2y}{dx^2} = -2(2x+1)^{-2}(2)\frac{dy}{dt} + \frac{2}{(2x+1)}\cdot\frac{d}{dx}\left(\frac{dy}{dt}\right)$$

$$\frac{d^2y}{dx^2} = \frac{-4}{(2x+1)^2}\frac{dy}{dt} + \frac{2}{(2x+1)}\cdot\frac{d}{dt}\left(\frac{dy}{dx}\right)$$

From (1)

$$\frac{d^2y}{dx^2} = \frac{-4}{(2x+1)^2}\frac{dy}{dt} + \frac{2}{(2x+1)}\cdot\frac{d}{dt}\left[\frac{2}{(2x+1)}\cdot\frac{dy}{dt}\right]$$

$$\frac{d^2y}{dx^2} = \frac{-4}{(2x+1)^2}\frac{dy}{dt} + \frac{2}{(2x+1)}\cdot\frac{2}{(2x+1)}\cdot\frac{d^2y}{dt^2}$$

$$\frac{d^2y}{dx^2} = \frac{-4}{(2x+1)^2}\cdot\frac{dy}{dt} + \frac{4}{(2x+1)^2}\cdot\frac{d^2y}{dt^2}$$

$$\frac{d^2y}{dx^2} = \frac{4}{(2x+1)^2}\left[-\frac{dy}{dt} + \frac{d^2y}{dt^2}\right]$$

$$(2x+1)^2\frac{d^2y}{dx^2} = 4\left[\Delta^2 y - \Delta y\right]$$

$$(2x+1)^2 D^2 y = 4\left[\Delta^2 - \Delta\right] y$$

$$\boxed{(2x+1)^2 D^2 = 4\Delta(\Delta-1)}$$

~~therefore question will be~~

Dated:

$$\{4\Delta(\Delta-1) - 6[2\Delta] + 16\}y = 8e^{2t}$$

$$\{4\Delta^2 - 4\Delta - 12\Delta + 16\}y = 8e^{2t}$$

$$\{4\Delta^2 - 16\Delta + 16\}y = 8e^{2t}$$

$$4\{\Delta^2 - 4\Delta + 4\}y = 8e^{2t}$$

$$\{\Delta^2 - 4\Delta + 4\}y = 2e^{2t}$$

$$(\Delta-2)^2 y = 2e^{2t}$$

For complementary function

$$f(\Delta) = 0$$

$$(\Delta-2)^2 = 0$$

$\Delta = 2, 2$ (characteristic roots are repeated)

The complementary function is

$y_c = e^{2t}[c_1 + c_2 t]$ or $\cancel{y_c = x^2\{c_1 + c_2 \ln(x)\}}$

For particular integral

$$y_p = \frac{1}{f(\Delta)} \cdot F(t)$$

$$y_p = \frac{1}{(\Delta-2)^2} \cdot 2e^{2t}$$

$$y_p = 2 \cdot \frac{1}{(\Delta-2)^2} \cdot e^{2t}$$

Dated:

$$y_p = \frac{2 \cdot t^2 e^{2t}}{4}$$

$$\frac{1}{\phi(D)(D-a)^k} \cdot e^{at} = \frac{t^k \cdot e^{at}}{k! \, \phi(a)}$$

$$y_p = \frac{t^2 \cdot e^{2t}}{2}$$

Hence The general solution is

$$y = y_c + y_p$$

$$y = e^{2t}[c_1 + c_2 t] + \frac{t^2 \cdot e^{2t}}{2}$$

~~Again put $t = \ln(2x+1)$ and $(2x+1) = e^t$~~

$$y = (2x+1)^2 [c_1 + c_2 \ln(2x+1)] + \frac{[\ln(2x+1)]^2 \cdot (2x+1)^2}{2}$$

$$y = (2x+1)^2 \left[c_1 + c_2 \ln(2x+1) + \frac{[\ln(2x+1)]^2}{2}\right]$$

Ans