Image Processing and Computer Vision (IPCV)

Prof. Dr. Joachim Weickert
Mathematical Image Analysis Group

Summer term 2017
Saarland University

Example Solutions for Homework Assignment 2 (H2)

Problem 1 (Properties of the continuous Fourier transform)

Verify that the following properties of the continuous Fourier transform are true.

Linearity: $\mathcal{F}[a \cdot f(x) + b \cdot g(x)](u) = a \cdot \mathcal{F}[f](u) + b \cdot \mathcal{F}[g](u)$

$$\begin{aligned}
\mathcal{F}[a \cdot f(x) + b \cdot g(x)](u) &= \int_{\mathbb{R}} (a \cdot f(x) + b \cdot g(x))\, e^{-i2\pi ux} \mathrm{d}x \\
&= a \int_{\mathbb{R}} f(x) e^{-i2\pi ux} \mathrm{d}x + b \int_{\mathbb{R}} g(x) e^{-i2\pi ux} \mathrm{d}x \\
&= a \cdot \mathcal{F}[f](u) + b \cdot \mathcal{F}[g](u)
\end{aligned}$$

Spatial Shift: $\mathcal{F}[f(x-a)](u) = e^{-i2\pi ua} \cdot \mathcal{F}[f](u)$

$$\begin{aligned}
\mathcal{F}[f(x-a)](u) &= \int_{\mathbb{R}} f(x-a) e^{-i2\pi ux} \mathrm{d}x \quad t \leftrightarrow x-a \\
&= \int_{\mathbb{R}} f(t) e^{-i2\pi u(t+a)} \mathrm{d}t \\
&= e^{-i2\pi ua} \int_{\mathbb{R}} f(t) e^{-i2\pi ut} \mathrm{d}t \\
&= e^{-i2\pi ua} \mathcal{F}[f](u)
\end{aligned}$$

Frequency Shift: $\mathcal{F}[f(x) \cdot e^{-i2\pi u_0 x}](u) = \mathcal{F}[f](u+u_0)$

$$\begin{aligned}
\mathcal{F}[f(x) \cdot e^{-i2\pi u_0 x}](u) &= \int_{\mathbb{R}} f(x) e^{-i2\pi u_0 x} e^{-i2\pi ux} \mathrm{d}x \\
&= \int_{\mathbb{R}} f(x) e^{-i2\pi (u_0+u)x} \mathrm{d}x \\
&= \mathcal{F}[f](u_0+u)
\end{aligned}$$

Scaling: $\mathcal{F}[f(ax)](u) = \frac{1}{|a|} \cdot \mathcal{F}[f](\frac{u}{a})$

$$\begin{aligned}
\mathcal{F}[f(ax)](u) &= \int_{\mathbb{R}} f(ax) e^{-i2\pi ux} \mathrm{d}x \quad t \leftrightarrow ax \\
&= \int_{\mathbb{R}} f(t) e^{-i2\pi u \frac{t}{a}} \frac{1}{|a|} \mathrm{d}t \\
&= \frac{1}{|a|} \mathcal{F}[f] \left(\frac{u}{a} \right)
\end{aligned}$$

Please note that here, we assume that $a \neq 0$. Furthermore, in the substitution step, we implicitly do a case distinction that results in the absolute value $\frac{1}{|a|}$ as an additional factor. For $a > 0$, the substitution is straightforward since $\frac{dt}{dx} = a \Leftrightarrow dx = \frac{1}{a}dt = \frac{1}{|a|}dt$ and the integral limits stay the same. For $a < 0$, one naturally also gets $\frac{1}{a}$ as an additional factor from the substitution, however the integral limits are flipped, which can be reversed by multiplying the integrand with -1, thus leading to $-\frac{1}{a} = \frac{1}{|a|}$ for $a < 0$.

Convolution: $\mathcal{F}[(f * g)(x)](u) = \mathcal{F}[f](u) \cdot \mathcal{F}[g](u)$

$$\begin{aligned}
\mathcal{F}[(f * g)(x)](u) &= \int_{\mathbb{R}} (f * g)(x) e^{-i2\pi ux} \mathrm{d}x \\
&= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} f(\xi) g(x - \xi) \mathrm{d}\xi \right) e^{-i2\pi ux} \mathrm{d}x \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} f(\xi) g(x - \xi) e^{-i2\pi ux} \mathrm{d}x \mathrm{d}\xi \\
&= \int_{\mathbb{R}} f(\xi) e^{-i2\pi u\xi} \left(\int_{\mathbb{R}} g(x - \xi) e^{-i2\pi u(x-\xi)} \mathrm{d}x \right) \mathrm{d}\xi \quad t \leftrightarrow x - \xi \\
&= \int_{\mathbb{R}} f(\xi) e^{-i2\pi u\xi} \left(\int_{\mathbb{R}} g(t) e^{-i2\pi ut} \mathrm{d}t \right) \mathrm{d}\xi \\
&= \int_{\mathbb{R}} f(\xi) e^{-i2\pi u\xi} \mathrm{d}\xi \mathcal{F}[g](u) \\
&= \mathcal{F}[f](u) \cdot \mathcal{F}[g](u)
\end{aligned}$$

Derivative: $\mathcal{F}[f'](u) = i2\pi u \cdot \mathcal{F}[f](u)$

We know that $f(x) = \mathcal{F}^{-1}[\mathcal{F}[f](u)](x)$, therefore

$$f(x) = \int_{\mathbb{R}} \mathcal{F}[f](u) \cdot e^{i2\pi ux} \, du \, .$$

Taking the derivative w.r.t. x on both sides yields

$$f'(x) = \int_{\mathbb{R}} i2\pi u \cdot \mathcal{F}[f](u) \cdot e^{i2\pi ux} \, du = \mathcal{F}^{-1}[i2\pi u \cdot \mathcal{F}[f](u)](x) \, .$$

Finally, we take the Fourier transform on both sides and obtain

$$\mathcal{F}[f'](u) = \mathcal{F}[\mathcal{F}^{-1}[i2\pi u \cdot \mathcal{F}[f](u)](x)](u) = i2\pi u \cdot \mathcal{F}[f](u) \, .$$

Problem 2 (Continuous Fourier Transform of a Hat Function)

The function f from this exercise corresponds to a convolution $f(x) = g(x) * g(x) * g(x)$ where g is the box function from H1, Problem 1:

$$g(x) := \begin{cases} \frac{1}{2} & -1 \leq x \leq 1 \\ 0 & \text{else} \end{cases}$$

From the lecture we know what form the Fourier transform of a box function h with height A on the interval $[-R, R]$ is:

$$\mathcal{F}[h](u) = \frac{A}{\pi u} \sin(2\pi u R)$$

In the case of our box function g we have $A = \frac{1}{2}$ and $R = 1$, thus yielding

$$\mathcal{F}[g](u) = \frac{1}{2\pi u} \sin(2\pi u) = \operatorname{sinc}(2\pi u)$$

Finally, we apply the convolution theorem and get:

$$\mathcal{F}[f](u) = \mathcal{F}[g * g * g](u) = (\mathcal{F}[g](u))^3 = \operatorname{sinc}^3(2\pi u)$$

Problem 3 (Continuous Fourier Transform of an Even Function)

(a) We first show, that the product of an odd and an even function is an odd function. Let g_e be an even function and let g_o be an odd function. Then we obtain for the product $p = g_e \cdot g_o$ of those functions:

$$p(-x) = g_e(-x) \cdot g_o(-x) = -g_e(x) \cdot g_o(x) = -p(x)$$

which means that p is an odd function.
Now let us consider the integral of an odd and an even function. Let p be defined as above, then

$$\int_{-\infty}^{\infty} g_e(x) g_o(x)\, dx = \int_{-\infty}^{\infty} p(x)\, dx = \int_{-\infty}^{0} p(x)\, dx + \int_{0}^{\infty} p(x)\, dx\,.$$

Substituting $t = -x$ and correspondingly $dt = -dx$ in the first term and $t = x$ in the second term yields

$$\int_{\infty}^{0} -p(-t)\, dt + \int_{0}^{\infty} p(t)\, dt = \int_{0}^{\infty} -p(t)\, dt + \int_{0}^{\infty} p(t)\, dt = \int_{0}^{\infty} p(t) - p(t)\, dt = 0$$

(b) Now we compute the Fourier transform of f_e:

$$\begin{aligned}
\mathcal{F}[f_e](u) &= \int_{-\infty}^{\infty} f_e(x) e^{-i2\pi ux}\, dx \\
&= \int_{-\infty}^{\infty} f_e(x)\cos(2\pi ux)\, dx - i \underbrace{\int_{-\infty}^{\infty} f_e(x)\sin(2\pi ux)\, dx}_{=0} \\
&= \int_{-\infty}^{\infty} f_e(x)\cos(2\pi ux)\, dx\,.
\end{aligned}$$

Thus $\mathcal{F}[f_e](u)$ is a real function.

Problem 4 (Colour Spaces)

(a) The conversion from RGB to YCbCr images requires to supplement the following code:

```
/* computes the YCbCr values */
for (i=1;i<=nx;i++)
  for (j=1;j<=ny;j++)
      {
        u_YCbCr[0][i][j]  =    0.299 * u_RGB[0][i][j]
                             + 0.587 * u_RGB[1][i][j]
                             + 0.114 * u_RGB[2][i][j];
        u_YCbCr[1][i][j] = 127.5 + (- 0.169 * u_RGB[0][i][j]
                                     - 0.331 * u_RGB[1][i][j]
                                     + 0.500 * u_RGB[2][i][j]);
        u_YCbCr[2][i][j] = 127.5 + (  0.500 * u_RGB[0][i][j]
                                     - 0.419 * u_RGB[1][i][j]
                                     - 0.081 * u_RGB[2][i][j]);
      }
```

(b) The compressed variants of the image `baboon.ppm` for subsampling factors of $S = 1, 2, 4$ and 8 are depicted in Tab 1. As one can see, the variants for $S = 2$ and $S = 4$ provide still a quite good quality. This is due to the fact that the details (hairs, patterns) are preserved, since the Y-channel that contains these details is not compressed (downsampled). In the case of $S = 8$, however, slight block artifacts become visible. This is verified by the images in Tab. 2 that depict a zoom of the nose region. Here also for $S = 2$ and $S = 4$ the loss of quality becomes obvious.

(c) While the original RGB image requires $3 \cdot 8 = 24$ bpp to store the information, the requirements of the compressed (subsampled) images is given by

$$v = 8\left(1 + \frac{1}{S^2} + \frac{1}{S^2}\right)\,.$$

Table 1: Compressed variants of the image `baboon.ppm`. *(a) Top left:* Image for $S = 1$ (original image). *(b) Top right:* $S = 2$. *(c) Bottom left:* $S = 4$. *(d) Bottom right:* $S = 8$.

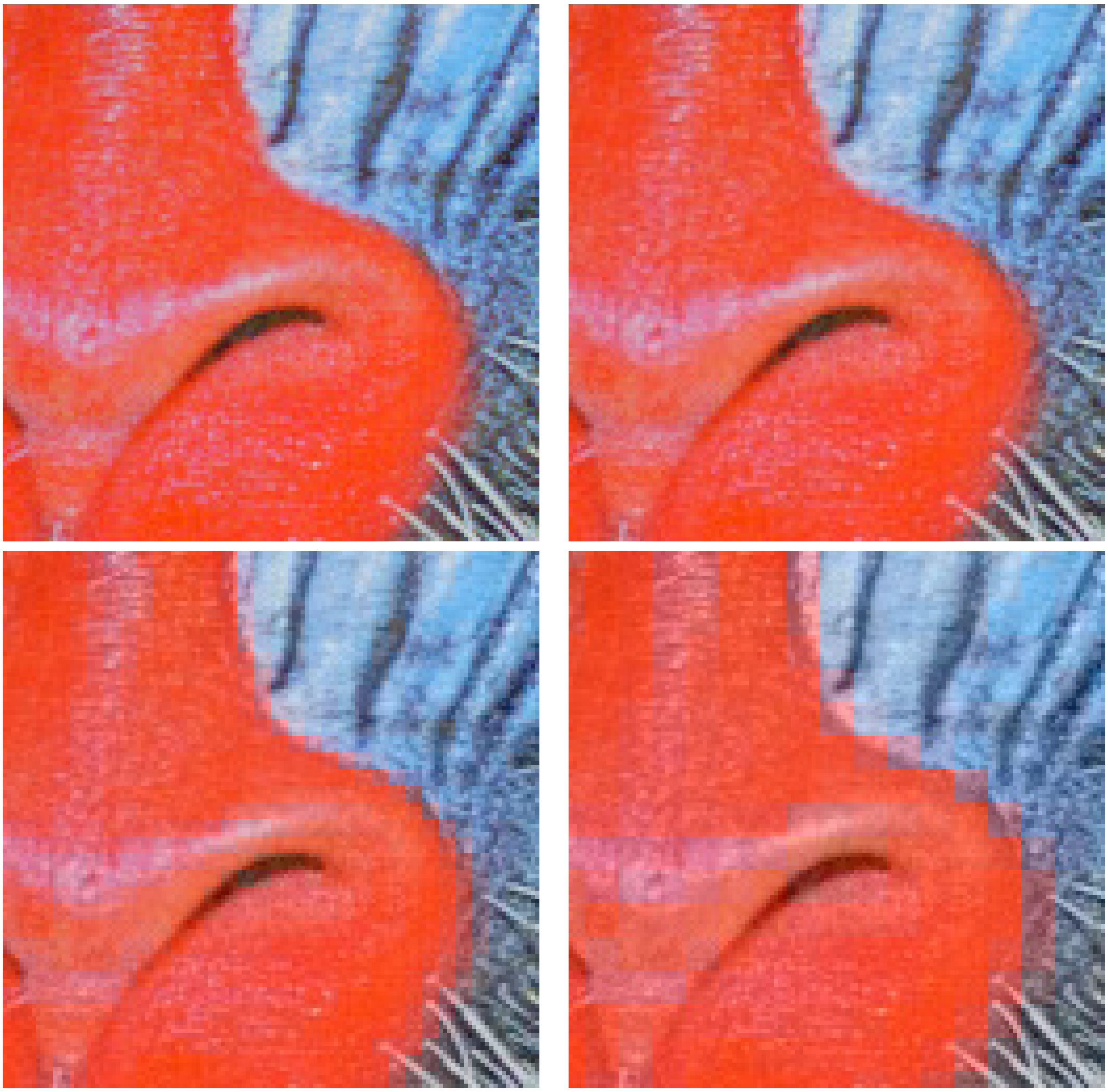

Table 2: Zoom into the compressed variants of the image `baboon.ppm`. *(a) Top left:* Image for $S = 1$ (original image). *(b) Top right:* $S = 2$. *(c) Bottom left:* $S = 4$. *(d) Bottom right:* $S = 8$.

While the Y channels remain uncompressed, the Cb- and Cr-channel are reduced in both dimensions by a factor of S. Thus, 12 bpp for $S = 2$, 9 bpp for $S = 4$, and 8.25 bpp for $S = 8$ are needed. This is in the order of the memory consumption of grey value images.