Saha ionization equation in early universe

(A brief review)

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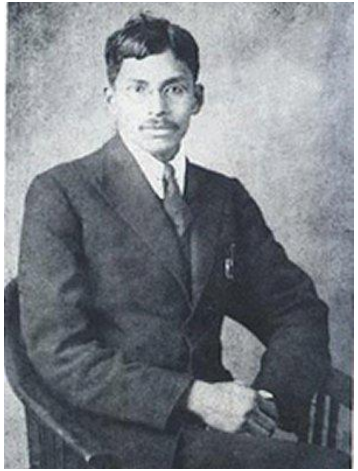
شیخِ مکتب ہے اک عمارت گر جس کی صنعت ہے روحِ انسانی

Meaning: A teacher is like a builder whose industry is to build and develop the soul of a human

Hydrogen recombination – equilibrium theory

- Immediately after the Big Bang , Quark epoch . At 10-6 seconds , the hadron epoch.
- Plasma was effectively opaque to electromagnetic radiation due to Thomson scattering by free electrons, as the mean free path each photon could travel before encountering an electron was very short
- The universe cooled to the point that the formation of neutral hydrogen was energetically favored.
- Recombination process – how the ionized plasma of protons and electrons turns into (neutral) hydrogen atoms.
- Once photons decoupled from matter, they traveled freely through the universe without interacting with matter and constitute what is observed today as cosmic microwave background radiation

Saha Ionization Equation:



Meghnad N. Saha

- Given by Indian Astrophysicist Meghnad N. Saha in 1920 (at the age of 26)
- Problem: Determine the temperature of stars from spectro-scopic data.
- Solution: Formulated thermal ionisation theory in which the ionization energy U and the thermal energy kBT ought to enter on an equal footing. Thus giving thermal ionization equation.

$$\frac{[(x^+) \,.\, (x^-)]}{x} = \left[\frac{2\pi m_e \,(k_B T)}{h^2}\right]^{3/2} \cdot \left[\frac{(Z_{int})}{(Z^+_{int})}\right]$$

India's golden decade: Saha's ionization equation, Bose statistics, Raman Effect, and the Chandrasekhar's limit

Nobel Prize? Nominated 1930, 1937, 1939, 1940, 1951 and 1955 (6 times!)

Thermal Field Theory

- Describe a large ensemble of multi interacting particles (including possible non-Abelian gauge interactions) in a thermodynamical environment

- Different from other theories (kinetic theory or many-body theory) on the basis of following facts: i) it uses a path integral approach, ii) it can treat non-Abelian gauge interactions as QCD and iii) it is Lorentz-covariant.

- Lately, most of the developments of the TFT have been motivated by the study of QCD at finite temperature.

- Main applications of the TFT in high energy physics can be sorted into three classes.

i) Cosmology ii)Astrophysics iii)Heavy-ion collisions

Saha Ionization in the early universe

- Discuss the problem of equilibrium in the early universe
- Published in 2019
- Authored by Aritra Das, Ritesh Ghosh, and Sameer Mallik
- All three working in the **Saha Institute of Nuclear Physics, India** named after **Saha Maghnad**.

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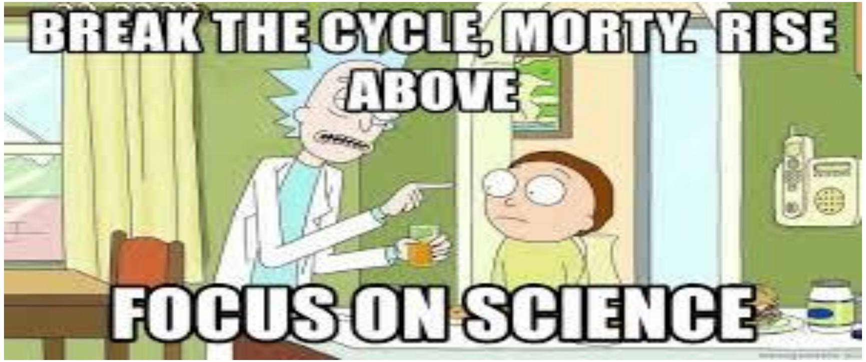


Saha Ionization in the early universe (Summary)

- In the early universe it is essential to compare the interaction rates, which tend to equilibrate particle distributions, with the expansion rate of the universe,given by the Hubble parameter.

- Considered the (recombination) epoch when the universe consists mostly of electrons, protons and hydrogen atoms

- investigated the rate at which the chemical equilibrium of H atoms takes place

$$\gamma + H \rightleftharpoons p^+ + e^-$$

Ionization and Recombination Rates

- Recombination (or recapture) is the capture of an electron in the continuum by the atomic nucleus with the emission of photon
- Ionization (or photoeffect) is the inverse process, where a photon is absorbed by an atom accompanied by ejection of an electron
- Used thermal quantum field theory to get the recombination and ionization probabilities with appropriate factors involving distribution functions for particles in the medium.
- Construct effective interaction Lagrangian involving all photon, electron and proton fields.

$$\mathcal{L}_{\text{int}} = g(\Phi\overline{\psi}_p\gamma^\mu\psi_e + h.c.)A_\mu \qquad g = \frac{e}{m}$$

Ionization and Recombination Rates

- The calculation of the relevant reaction rate in thermal field theory can be best approached by considering the self-energy for the H-atom. (the portion of the elementary particle's mass due to the interaction with its environment)

- The self-energy operator in the momentum energy representation is complex. The real part of self-energy is identified with the physical self-energy (referred as particle's "self-energy"); the inverse of the imaginary part is a measure for the lifetime of the particle under investigation.

- Interested in the imaginary part, corresponding to processes with the photon (and H-atom) incoming and electron and proton outgoing for ionization.

Ionization and Recombination Rates

- Usage of three momenta of three H-atom M_H, photon,electron m and proton M by $q, k_i (i = 1, 2, 3)$
- Energies: $\omega = \sqrt{q^2 + M_H^2},\ \omega_1 = k_1,\ \omega_2 = \sqrt{k_2^2 + m^2}$ and $\omega_3 = \sqrt{k_3^2 + M^2}$ respectively.
- Imaginary part of the diagonalised self-energy

$$\mathrm{Im}\overline{\Sigma}_{(1)} = 16\pi g^2 m M \int \frac{d^3k_1}{(2\pi)^3 2\omega_1} \frac{d^3k_2}{(2\pi)^3 2\omega_2} \frac{1}{2\omega_3} [n_1(1-\widetilde{n}_2)(1-\widetilde{n}_3) - (1+n_1)\widetilde{n}_2\widetilde{n}_3]\delta(\omega + \omega_1 - \omega_2 - \omega_3)$$

- It resembles the unitarity relation for the S-matrix in vacuum for the two-particle states of photon and H-atom. (QFT !)
- Dividing it by ω, we convert it to rates

$$\frac{\mathrm{Im}\overline{\Sigma}_{(1)}}{\omega} = \Gamma_d - \Gamma_i$$

Boltzmann Equation

- Equilibrium distribution functions can be written as

$$n_1 = \frac{\exp(-\beta\omega_1/2)}{\exp(\beta\omega_1/2) - \exp(-\beta\omega_1/2)}, \qquad 1 + n_1 = \frac{\exp(\beta\omega_1/2)}{\exp(\beta\omega_1/2) - \exp(-\beta\omega_1/2)}$$

- Similarly for $\tilde{n_2}$ and $\tilde{n_3}$.

$$S_{nm} \propto \delta_{E(n)E(m)}$$

where this is the Kronecker delta $\delta_{xy} = \begin{cases} 1 & x = y \\ 0 & x \neq y \end{cases}$.

- Considering energy conserving delta function into diagonalised self-energy equation. The decay and inverse decay rates of H-atom can be written as;

$$\Gamma_d \; ; \Gamma_i = 16\pi g^2 m \frac{M}{\omega} \{\exp[\beta(\omega - \mu_2 - \mu_3)/2] \; ; \exp[-\beta(\omega - \mu_2 - \mu_3)/2]\} \times L$$

$$L = \int \frac{d^3k_1}{(2\pi)^3 2\omega_1} \frac{d^3k_2}{(2\pi)^3 2\omega_2} \frac{1}{2\omega_3} \frac{\delta(\omega + \omega_1 - \omega_2 - \omega_3)}{\prod \{\exp[\beta(\omega_i - \mu_i)/2] \mp \exp[-\beta(\omega_i - \mu_i)/2]\}}$$

Boltzmann Equation

- Uptill now we consider H-atom as a single particle without any distribution in the medium.
- Consider an arbitrary (non-equilibrium) distribution $n(w,t)$ of these particles.
- Boltzmann equation for $n(w,t)$ (with consideration that it decreases at the rate $n\Gamma_d$ increases at the rate , $(1+n)\Gamma_i$

$$\frac{dn(\omega,t)}{dt} = (1+n)\Gamma_i - n\Gamma_d$$

Boltzmann Equation

- The solution is (based on the paper H. Arthur Weldon: Simple rules for discontinuities in finite-temperature field theory)

$$
\begin{aligned}
n(\omega, t) &= \frac{\Gamma_i}{\Gamma_d - \Gamma_i} + c(\omega) e^{-(\Gamma_d - \Gamma_i)t} \\
&= \frac{1}{\exp \beta(\omega - \mu_2 - \mu_3) - 1} + c(\omega) e^{-\Gamma t}, \quad \Gamma = \Gamma_d - \Gamma_i \\
&\qquad \mu = \mu_2 + \mu_3
\end{aligned}
$$

- first term in $n(w,t)$ satisfies the condition of chemical equilibrium for H-atoms

- The distribution function approaches the equilibrium value exponentially in time, irrespective of its initial distribution.

Boltzmann Equation

- Now non-relativistic approximation, the energy can be written as

$$\omega = M_H + \frac{q^2}{2M_H}, \qquad \omega_1 = k_1, \qquad \omega_2 = m + \frac{k_2^2}{2m}, \qquad \omega_3 = M + \frac{k_3^2}{2M} \qquad M_H = M + m - \epsilon_0,$$

- Also assume concentrations of H-atoms, electrons and protons are sufficiently dilute (meaning Fermi-Dirac and Bose-Einstein distributions reduce to Maxwell-Boltzmann distribution)

$$\widetilde{n}_2 = \exp\left[-\beta\left(\frac{k_2^2}{2m} - \mu_2'\right)\right], \qquad \widetilde{n}_3 = \exp\left[-\beta\left(\frac{k_3^2}{2M} - \mu_3'\right)\right], \qquad n = \exp\left[-\beta\left(\frac{q^2}{2M} - \epsilon_0 - \mu_2' - \mu_3'\right)\right]$$

$$\mu_2' = \mu_2 - m \text{ and } \mu_3' = \mu_3 - M.$$

- The the total number of electrons in volume V is

$$N_e = 2V \int \frac{d^3k}{(2\pi)^3} \widetilde{n}(k_2) = 2V \left(\frac{m}{2\pi\beta}\right)^{3/2} \exp(\beta\mu_2')$$

Boltzmann Equation

- Similarly for proton and H-atoms

$$N_p = 2V \left(\frac{M}{2\pi\beta}\right)^{3/2} \exp(\beta\mu_3') \,,$$

- incorporates the equilibrium condition

$$N_H = 4V \left(\frac{M_H}{2\pi\beta}\right)^{3/2} \exp[\beta(\epsilon_0 + \mu_2' + \mu_3')] \,,$$

- The Saha Equation:

$$\frac{N_e N_p}{N_H} = V \left(\frac{m}{2\pi\beta}\right)^{3/2} \exp(-\beta\epsilon_0)$$

- Application of Saha Equation.

$$x_e = \frac{N_e}{N_{H,tot}}$$

$$N_{H,tot} = N_p + N_H$$

$$N_e = N_p = x_e N_{H,tot}\,; N_H = (1 - x_e) N_{H,tot}$$

Application of Saha Equation

- BBN produced a fraction of 0.76 of the initial mass in hydrogen

$$N_{H,tot} = 0.76 n_b = 8.6 \times 10^{-6} \Omega_b h^2 a^{-3}\,\mathrm{cm}^{-3} = 4.2 \times 10^5 \Omega_b h^2 T_4^3\,\mathrm{cm}^{-3},$$

- Putting all this together.

$$\frac{x_e^2}{1 - x_e} = \frac{5.8 \times 10^{15}}{\Omega_b h^2 T_4^{3/2}} e^{-15.8/T_4}.$$

Half of the hydrogen recombines ($x_e = 0.5$) by $T_4 = 0.374$ or 3740 K; $z = 1370$.

90% of the hydrogen recombines ($x_e = 0.1$) by $T_4 = 0.342$ or 3420 K; $z = 1250$.

99% of the hydrogen recombines ($x_e = 10^{-2}$) by $T_4 = 0.310$ or 3100 K; $z = 1140$.

Early Universe

- The solution for the distribution function of H-atoms shows that if we wait long enough, it approaches equilibrium .
- The cosmic expansion disturbs this equilibrium. So the interaction rate must be much Faster that the expansion rate in order that the equilibrium is maintained
- Estimate roughly the interaction rate Γ in the non-relativistic approximation

$$n(\omega,t) = \frac{\Gamma_i}{\Gamma_d - \Gamma_i} + c(\omega)e^{-(\Gamma_d-\Gamma_i)t}$$
$$= \frac{1}{\exp\beta(\omega-\mu_2-\mu_3)-1} + c(\omega)e^{-\Gamma t}, \quad \Gamma = \Gamma_d - \Gamma_i$$

$$L = \left(\frac{4\pi}{(2\pi)^3}\right)^2 \frac{1}{8Mm}\int_0^\infty dk_1\, k_1\, dk_2\, k_2^2 \exp\left[-\frac{\beta}{2}\left(k_1 + \frac{k_2^2}{2m} - \mu_2' - \mu_3'\right)\right]\delta\left(-\epsilon_0 + k_1 - \frac{k_2^2}{2m}\right)$$

remove the k_1 integral with the delta function, when the k_2 integral reduces to a Gamma function, giving

$$L = \left(\frac{4\pi}{(2\pi)^3}\right)^2 \frac{1}{8Mm}\exp[-\beta(\epsilon_0 - \mu_2' - \mu_3')/2]\frac{\sqrt{\pi}}{4}\left(\frac{m}{\beta}\right)^{3/2}\left(\epsilon_0 + \frac{3}{4\beta}\right).$$

Early Universe

- H-atom concentration is dilute → 1+n ~1 $$\frac{dn(\omega,t)}{dt} = (1+n)\Gamma_i - n\Gamma_d$$
- The solution shows the equilibrium distribution to be of Maxwell-Boltzmann type and the interaction rate becomes $\Gamma \simeq \Gamma_d$.

$$\Gamma_d = 16\pi g^2 m \frac{M}{\omega}\{\exp[\beta(\omega - \mu_2 - \mu_3)/2] \times L$$

$$L = \left(\frac{4\pi}{(2\pi)^3}\right)^2 \frac{1}{8Mm} \exp[-\beta(\epsilon_0 - \mu_2' - \mu_3')/2] \frac{\sqrt{\pi}}{4}\left(\frac{m}{\beta}\right)^{3/2}\left(\epsilon_0 + \frac{3}{4\beta}\right).$$

$$\Gamma \simeq \frac{e^2}{4\pi}\frac{1}{2\pi^{3/2}}\frac{m}{M}\left(\frac{k_B T}{m}\right)^{3/2} \epsilon_0 \exp(-\epsilon_0/k_B T)$$

- Expansion rate of the Universe is given by the Hubble parameter $H = \dot{a}/a.$
- In the era of interest (T > 1000K), the constant vacuum energy is utterly negligible and we consider the energy density of matter,

$$\rho(t) = \rho_0\left[\Omega_M\left(\frac{T}{T_0}\right)^3 + \Omega_R\left(\frac{T}{T_0}\right)^4\right]$$

Early Universe

- Apply Einstein Equation with the spatially flat metric

$$H^2 = H_0^2 \left[\Omega_M \left(\frac{T}{T_0} \right)^3 + \Omega_R \left(\frac{T}{T_0} \right)^4 \right]$$

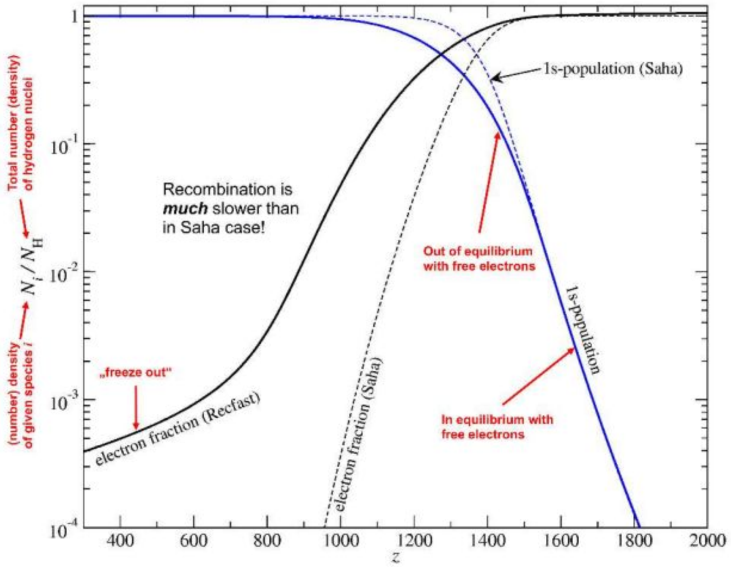
$$H = 7.20 \times 10^{-19} T^{3/2} \left(\Omega_M h^2 + 1.52 \times 10^{-5} T \right)^{1/2} s^{-1}$$

$$\Omega_M h^2 = 0.15$$

T(in K)	Γ in s^{-1}	H in s^{-1}
6000	2.6×10^{-11}	1.6×10^{-13}
5000	1.0×10^{-13}	1.2×10^{-13}
4000	2.7×10^{-17}	8.4×10^{-14}
3000	3.4×10^{-23}	5.2×10^{-14}

Results and Conclusion

- Equilibrium prevails up to about 5000 K, if we include only the ground state of the H atom in calculating the reaction rate.
- Estimated from the complete calculation of fractional ionization including the excited states and other physical effects presented in the Figure.[Signals From the Epoch of Cosmological Recombination (Karl Schwarzschild Lecture)]
- We see that the two calculations start to disagree at redshift z ~ 1500 corresponding to T ~ 4000 K.
- Comparing this result with our calculation, we see that the excited states bring down the equilibrium temperature from 5000 to 4000 K
- Thus, although the equilibrium number density of excited H atoms is negligible compared to that in the ground state, they provide pathways to facilitate attaining the equilibrium

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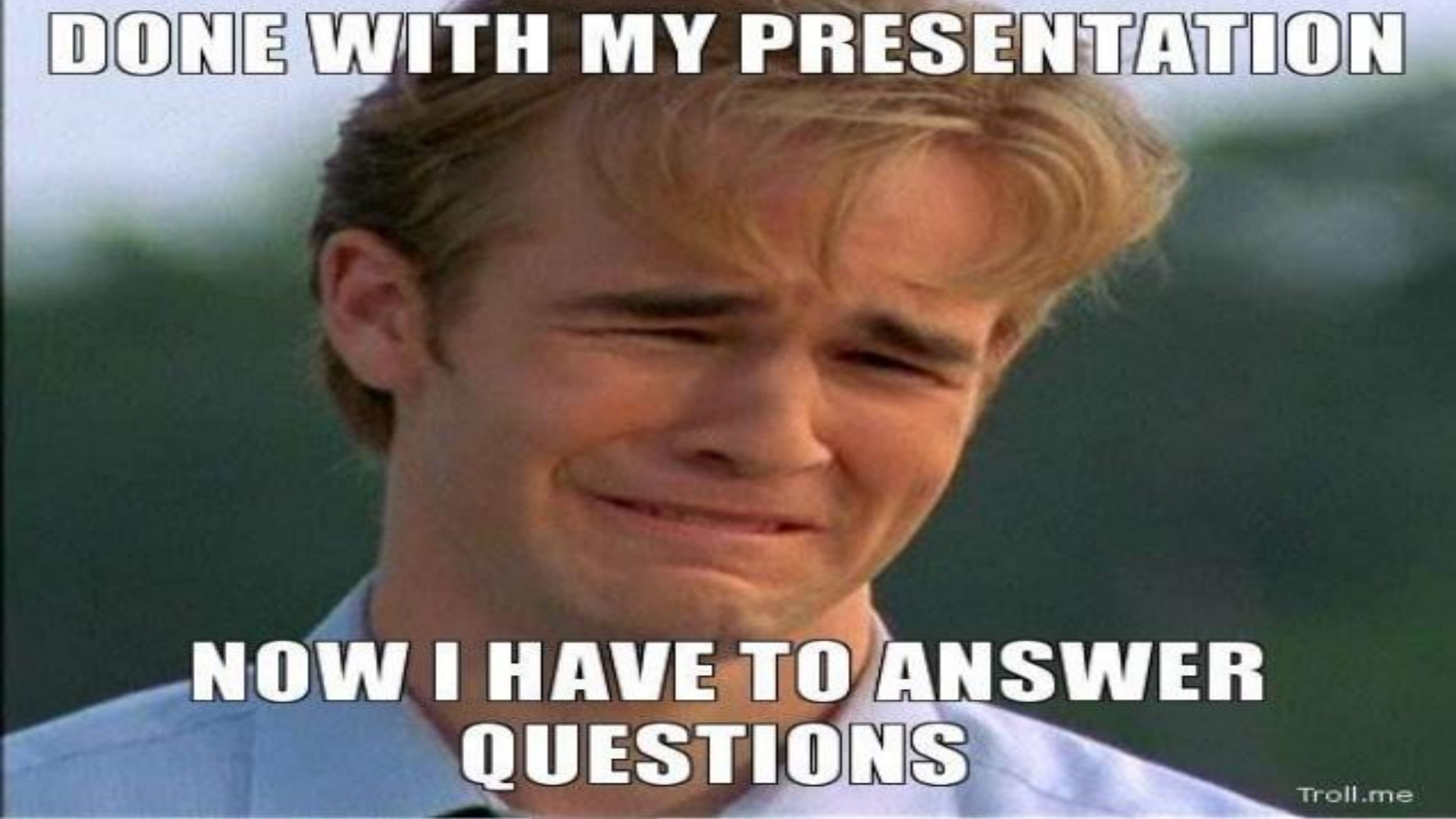
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