Saha ionization equation in early universe

(A brief review)

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شیخِ مکتب ہے اک عمارت گر جس کی صنعت ہے رُوحِ انسانی

Meaning: A teacher is like a builder whose industry is to build and develop the soul of a human

Hydrogen recombination – equilibrium theory

- Immediately after the Big Bang , Quark epoch . At 10-6 seconds , the hadron epoch.

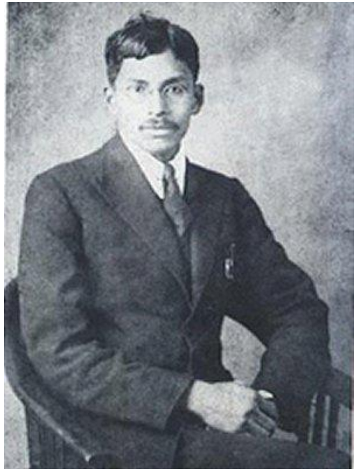
- Plasma was effectively opaque to electromagnetic radiation due to Thomson scattering by free electrons, as the mean free path each photon could travel before encountering an electron was very short

- The universe cooled to the point that the formation of neutral hydrogen was energetically favored.

- Recombination process – how the ionized plasma of protons and electrons turns into (neutral) hydrogen atoms.

- Once photons decoupled from matter, they traveled freely through the universe without interacting with matter and constitute what is observed today as cosmic microwave background radiation

Saha Ionization Equation:



Meghnad N. Saha

- Given by Indian Astrophysicist Meghnad N. Saha in 1920 (at the age of 26)

- Problem: Determine the temperature of stars from spectro-scopic data.

- Solution: Formulated thermal ionisation theory in which the ionization energy U and the thermal energy kBT ought to enter on an equal footing. Thus giving thermal ionization equation.

$$\frac{[(x^{+}) . (x^{-})]}{x} = \left[\frac{2\pi m_e (k_B T)}{h^2}\right]^{3/2} . \left[\frac{(Z_{int})}{(Z^{+}_{int})}\right]$$

India's golden decade: Saha's ionization equation, Bose statistics, Raman Effect, and the Chandrasekhar's limit

Nobel Prize? Nominated 1930, 1937, 1939, 1940, 1951 and 1955 (6 times!)

Thermal Field Theory

- Describe a large ensemble of multi interacting particles (including possible non-Abelian gauge interactions) in a thermodynamical environment

- Different from other theories (kinetic theory or many-body theory) on the basis of following facts: i) it uses a path integral approach, ii) it can treat non-Abelian gauge interactions as QCD and iii) it is Lorentz-covariant.

- Lately, most of the developments of the TFT have been motivated by the study of QCD at finite temperature.

- Main applications of the TFT in high energy physics can be sorted into three classes.

i) Cosmology ii)Astrophysics iii)Heavy-ion collisions

Saha Ionization in the early universe

- Discuss the problem of equilibrium in the early universe
- Published in 2019
- Authored by Aritra Das, Ritesh Ghosh, and Sameer Mallik
- All three working in the **Saha Institute of Nuclear Physics, India** named after **Saha Maghnad**.

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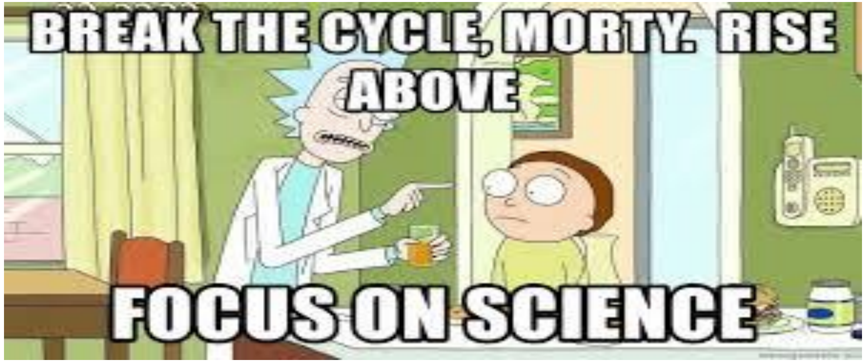

Saha Ionization in the early universe (Summary)

- In the early universe it is essential to compare the interaction rates, which tend to equilibrate particle distributions, with the expansion rate of the universe,given by the Hubble parameter.

- Considered the (recombination) epoch when the universe consists mostly of electrons, protons and hydrogen atoms

- investigated the rate at which the chemical equilibrium of H atoms takes place

$$\gamma + H \rightleftharpoons p^+ + e^-$$

- They used this process as a problem in the thermal field theory to get the ionization and recombination rates.

Ionization and Recombination Rates

- Recombination (or recapture) is the capture of an electron in the continuum by the atomic nucleus with the emission of photon
- Ionization (or photoeffect) is the inverse process, where a photon is absorbed by an atom accompanied by ejection of an electron
- Used thermal quantum field theory to get the recombination and ionization probabilities with appropriate factors involving distribution functions for particles in the medium.
- Construct effective interaction Lagrangian involving all photon, electron and proton fields.

$$\mathcal{L}_{\text{int}} = g(\Phi \overline{\psi}_p \gamma^\mu \psi_e + h.c.) A_\mu$$

$$g = \frac{e}{m}$$

Ionization and Recombination Rates

- The calculation of the relevant reaction rate in thermal field theory can be best approached by considering the self-energy for the H-atom. (the portion of the elementary particle's mass due to the interaction with its environment)

- The self-energy operator in the momentum energy representation is complex. The real part of self-energy is identified with the physical self-energy (referred as particle's "self-energy"); the inverse of the imaginary part is a measure for the lifetime of the particle under investigation.

- Interested in the imaginary part, corresponding to processes with the photon (and H-atom) incoming and electron and proton outgoing for ionization.

Ionization and Recombination Rates

- Usage of three momenta of three H-atom M_H, photon,electron m and proton M by $q, k_i (i = 1, 2, 3)$
- Energies: $\omega = \sqrt{q^2 + M_H^2},\ \omega_1 = k_1,\ \omega_2 = \sqrt{k_2^2 + m^2}$ and $\omega_3 = \sqrt{k_3^2 + M^2}$ respectively.
- Imaginary part of the diagonalised self-energy

$$\mathrm{Im}\overline{\Sigma}_{(1)} = 16\pi g^2 mM \int \frac{d^3k_1}{(2\pi)^3 2\omega_1} \frac{d^3k_2}{(2\pi)^3 2\omega_2} \frac{1}{2\omega_3} [n_1(1 - \widetilde{n}_2)(1 - \widetilde{n}_3) - (1 + n_1)\widetilde{n}_2\widetilde{n}_3] \delta(\omega + \omega_1 - \omega_2 - \omega_3)$$

- It resembles the unitarity relation for the S-matrix in vacuum for the two-particle states of photon and H-atom. (QFT !)
- Dividing it by ω, we convert it to rates

$$\frac{\mathrm{Im}\overline{\Sigma}_{(1)}}{\omega} = \Gamma_d - \Gamma_i$$

Boltzmann Equation

- Equilibrium distribution functions can be written as

$$n_1 = \frac{\exp(-\beta\omega_1/2)}{\exp(\beta\omega_1/2) - \exp(-\beta\omega_1/2)}, \qquad 1 + n_1 = \frac{\exp(\beta\omega_1/2)}{\exp(\beta\omega_1/2) - \exp(-\beta\omega_1/2)}$$

- Similarly for $\tilde{n_2}$ and $\tilde{n_3}$.

$$S_{nm} \propto \delta_{E(n)E(m)}$$

- Considering energy conserving delta function into diagonalised self-energy equation. The decay and inverse decay rates of H-atom can be written as;

where this is the Kronecker delta $\delta_{xy} = \begin{cases} 1 & x = y \\ 0 & x \neq y \end{cases}$.

$$\Gamma_d\ ;\ \Gamma_i = 16\pi g^2 m \frac{M}{\omega}\{\exp[\beta(\omega - \mu_2 - \mu_3)/2]\ ;\ \exp[-\beta(\omega - \mu_2 - \mu_3)/2]\} \times L$$

$$L = \int \frac{d^3k_1}{(2\pi)^3 2\omega_1} \frac{d^3k_2}{(2\pi)^3 2\omega_2} \frac{1}{2\omega_3} \frac{\delta(\omega + \omega_1 - \omega_2 - \omega_3)}{\prod\{\exp[\beta(\omega_i - \mu_i)/2] \mp \exp[-\beta(\omega_i - \mu_i)/2]\}}$$

Boltzmann Equation

- The ratio of decay and inverse decay rates of H-atom is

$$\frac{\Gamma_d}{\Gamma_i} = \exp[\beta(\omega - \mu_2 - \mu_3)]$$

- Uptill now we consider H-atom as a single particle without any distribution in the medium.

- Consider an arbitrary (non-equilibrium) distribution $n(w,t)$ of these particles.

- Boltzmann equation for $n(w,t)$ (with consideration that it decreases at the rate $n\Gamma_d$ increases at the rate , $(1+n)\Gamma_i$

$$\frac{dn(\omega,t)}{dt} = (1+n)\Gamma_i - n\Gamma_d$$

Boltzmann Equation

- The solution is (based on the paper H. Arthur Weldon: Simple rules for discontinuities in finite-temperature field theory)

$$n(\omega,t) = \frac{\Gamma_i}{\Gamma_d - \Gamma_i} + c(\omega)e^{-(\Gamma_d-\Gamma_i)t}$$
$$= \frac{1}{\exp\beta(\omega-\mu_2-\mu_3)-1} + c(\omega)e^{-\Gamma t}, \quad \Gamma = \Gamma_d - \Gamma_i$$
$$\mu = \mu_2 + \mu_3$$

- first term in $n(w,t)$ satisfies the condition of chemical equilibrium for H-atoms

- The distribution function approaches the equilibrium value exponentially in time, irrespective of its initial distribution.

Boltzmann Equation

- Now non-relativistic approximation, the energy can be written as

$$\omega = M_H + \frac{q^2}{2M_H}, \qquad \omega_1 = k_1, \qquad \omega_2 = m + \frac{k_2^2}{2m}, \qquad \omega_3 = M + \frac{k_3^2}{2M} \qquad M_H = M + m - \epsilon_0,$$

- Also assume concentrations of H-atoms, electrons and protons are sufficiently dilute (meaning Fermi-Dirac and Bose-Einstein distributions reduce to Maxwell-Boltzmann distribution)

$$\widetilde{n}_2 = \exp\left[-\beta\left(\frac{k_2^2}{2m} - \mu_2'\right)\right], \qquad \widetilde{n}_3 = \exp\left[-\beta\left(\frac{k_3^2}{2M} - \mu_3'\right)\right], \qquad n = \exp\left[-\beta\left(\frac{q^2}{2M} - \epsilon_0 - \mu_2' - \mu_3'\right)\right]$$

$$\mu_2' = \mu_2 - m \text{ and } \mu_3' = \mu_3 - M.$$

- The the total number of electrons in volume V is

$$N_e = 2V \int \frac{d^3k}{(2\pi)^3} \widetilde{n}(k_2) = 2V \left(\frac{m}{2\pi\beta}\right)^{3/2} \exp(\beta\mu_2')$$

Boltzmann Equation

- Similarly for proton and H-atoms
- incorporates the equilibrium condition

$$N_p = 2V\left(\frac{M}{2\pi\beta}\right)^{3/2}\exp(\beta\mu_3'),$$

$$N_H = 4V\left(\frac{M_H}{2\pi\beta}\right)^{3/2}\exp[\beta(\epsilon_0+\mu_2'+\mu_3')],$$

- The Saha Equation:

$$\frac{N_e N_p}{N_H} = V\left(\frac{m}{2\pi\beta}\right)^{3/2}\exp(-\beta\epsilon_0)$$

- Application of Saha Equation.

$$x_e = \frac{N_e}{N_{H,tot}}$$

$$N_{H,tot} = N_p + N_H$$

$$N_e = N_p = x_e N_{H,tot};\, N_H = (1-x_e)N_{H,tot}$$

Application of Saha Equation

- BBN produced a fraction of 0.76 of the initial mass in hydrogen

$$N_{H,tot} = 0.76 n_b = 8.6 \times 10^{-6} \Omega_b h^2 a^{-3}\,\mathrm{cm}^{-3} = 4.2 \times 10^5 \Omega_b h^2 T_4^3\,\mathrm{cm}^{-3},$$

- Putting all this together.

$$\frac{x_e^2}{1 - x_e} = \frac{5.8 \times 10^{15}}{\Omega_b h^2 T_4^{3/2}} e^{-15.8/T_4}.$$

Half of the hydrogen recombines ($x_e = 0.5$) by $T_4 = 0.374$ or 3740 K; $z = 1370$.

90% of the hydrogen recombines ($x_e = 0.1$) by $T_4 = 0.342$ or 3420 K; $z = 1250$.

99% of the hydrogen recombines ($x_e = 10^{-2}$) by $T_4 = 0.310$ or 3100 K; $z = 1140$.

Early Universe

- The solution for the distribution function of H-atoms shows that if we wait long enough, it approaches equilibrium .
- The cosmic expansion disturbs this equilibrium. So the interaction rate must be much Faster that the expansion rate in order that the equilibrium is maintained
- Estimate roughly the interaction rate Γ in the non-relativistic approximation

$$n(\omega,t) = \frac{\Gamma_i}{\Gamma_d - \Gamma_i} + c(\omega)e^{-(\Gamma_d-\Gamma_i)t}$$
$$= \frac{1}{\exp\beta(\omega-\mu_2-\mu_3)-1} + c(\omega)e^{-\Gamma t}, \quad \Gamma = \Gamma_d - \Gamma_i$$

$$L = \left(\frac{4\pi}{(2\pi)^3}\right)^2 \frac{1}{8Mm}\int_0^\infty dk_1\, k_1\, dk_2\, k_2^2 \exp\left[-\frac{\beta}{2}\left(k_1 + \frac{k_2^2}{2m} - \mu_2' - \mu_3'\right)\right]\delta\left(-\epsilon_0 + k_1 - \frac{k_2^2}{2m}\right)$$

remove the k_1 integral with the delta function, when the k_2 integral reduces to a Gamma function, giving

$$L = \left(\frac{4\pi}{(2\pi)^3}\right)^2 \frac{1}{8Mm}\exp[-\beta(\epsilon_0 - \mu_2' - \mu_3')/2]\frac{\sqrt{\pi}}{4}\left(\frac{m}{\beta}\right)^{3/2}\left(\epsilon_0 + \frac{3}{4\beta}\right).$$

Early Universe

- H-atom concentration is dilute → 1+n ~1 $\frac{dn(\omega,t)}{dt} = (1+n)\Gamma_i - n\Gamma_d$
- The solution shows the equilibrium distribution to be of Maxwell-Boltzmann type and the interaction rate becomes $\Gamma \simeq \Gamma_d$.

$$\Gamma_d = 16\pi g^2 m \frac{M}{\omega}\{\exp[\beta(\omega - \mu_2 - \mu_3)/2] \times L$$

$$L = \left(\frac{4\pi}{(2\pi)^3}\right)^2 \frac{1}{8Mm} \exp[-\beta(\epsilon_0 - \mu_2' - \mu_3')/2] \frac{\sqrt{\pi}}{4}\left(\frac{m}{\beta}\right)^{3/2}\left(\epsilon_0 + \frac{3}{4\beta}\right).$$

$$\Gamma \simeq \frac{e^2}{4\pi}\frac{1}{2\pi^{3/2}}\frac{m}{M}\left(\frac{k_B T}{m}\right)^{3/2} \epsilon_0 \exp(-\epsilon_0/k_B T)$$

- Expansion rate of the Universe is given by the Hubble parameter $H = \dot{a}/a$.
- In the era of interest (T > 1000K), the constant vacuum energy is utterly negligible and we consider the energy density of matter,

$$\rho(t) = \rho_0\left[\Omega_M\left(\frac{T}{T_0}\right)^3 + \Omega_R\left(\frac{T}{T_0}\right)^4\right]$$

Early Universe

- Apply Einstein Equation with the spatially flat metric

$$H^2 = H_0^2 \left[\Omega_M \left(\frac{T}{T_0} \right)^3 + \Omega_R \left(\frac{T}{T_0} \right)^4 \right]$$

$$H = 7.20 \times 10^{-19} T^{3/2} \left(\Omega_M h^2 + 1.52 \times 10^{-5} T \right)^{1/2} s^{-1} \qquad \Omega_M h^2 = 0.15$$

T(in K)	Γ in s^{-1}	H in s^{-1}
6000	2.6×10^{-11}	1.6×10^{-13}
5000	1.0×10^{-13}	1.2×10^{-13}
4000	2.7×10^{-17}	8.4×10^{-14}
3000	3.4×10^{-23}	5.2×10^{-14}

Results and Conclusion

- Equilibrium prevails up to about 5000 K, if we include only the ground state of the H atom in calculating the reaction rate.
- Estimated from the complete calculation of fractional ionization including the excited states and other physical effects presented in the Figure.(Signals From the Epoch of Cosmological Recombination (Karl Schwarzschild Lecture))
- We see that the two calculations start to disagree at redshift z ~ 1500 corresponding to T ~ 4000 K.
- Comparing this result with our calculation, we see that the excited states bring down the equilibrium temperature from 5000 to 4000 K
- Thus, although the equilibrium number density of excited H atoms is negligible compared to that in the ground state, they provide pathways to facilitate attaining the equilibrium

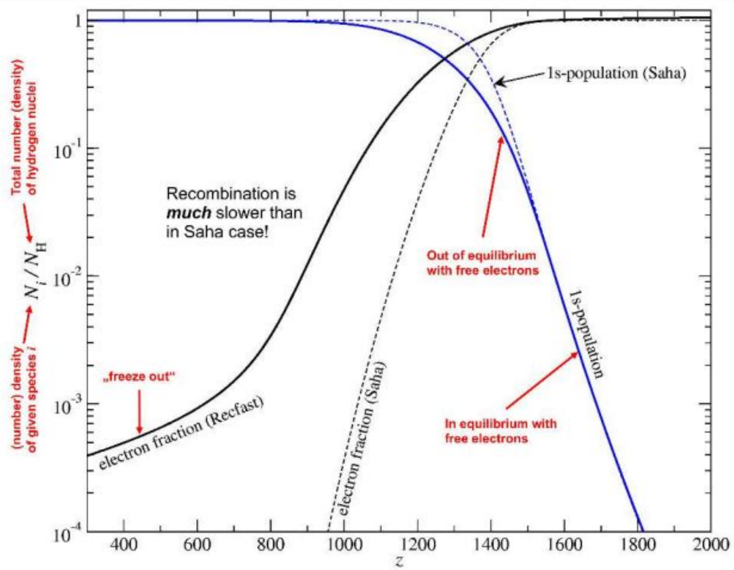

Figure 3: Illustration of the difference in the hydrogen recombination history in comparison with the Saha case. The recombination of hydrogen in the Universe is strongly delayed due to the 'bottleneck' created in the Lyman α resonance and the slow 2s-1s two-photon transition.


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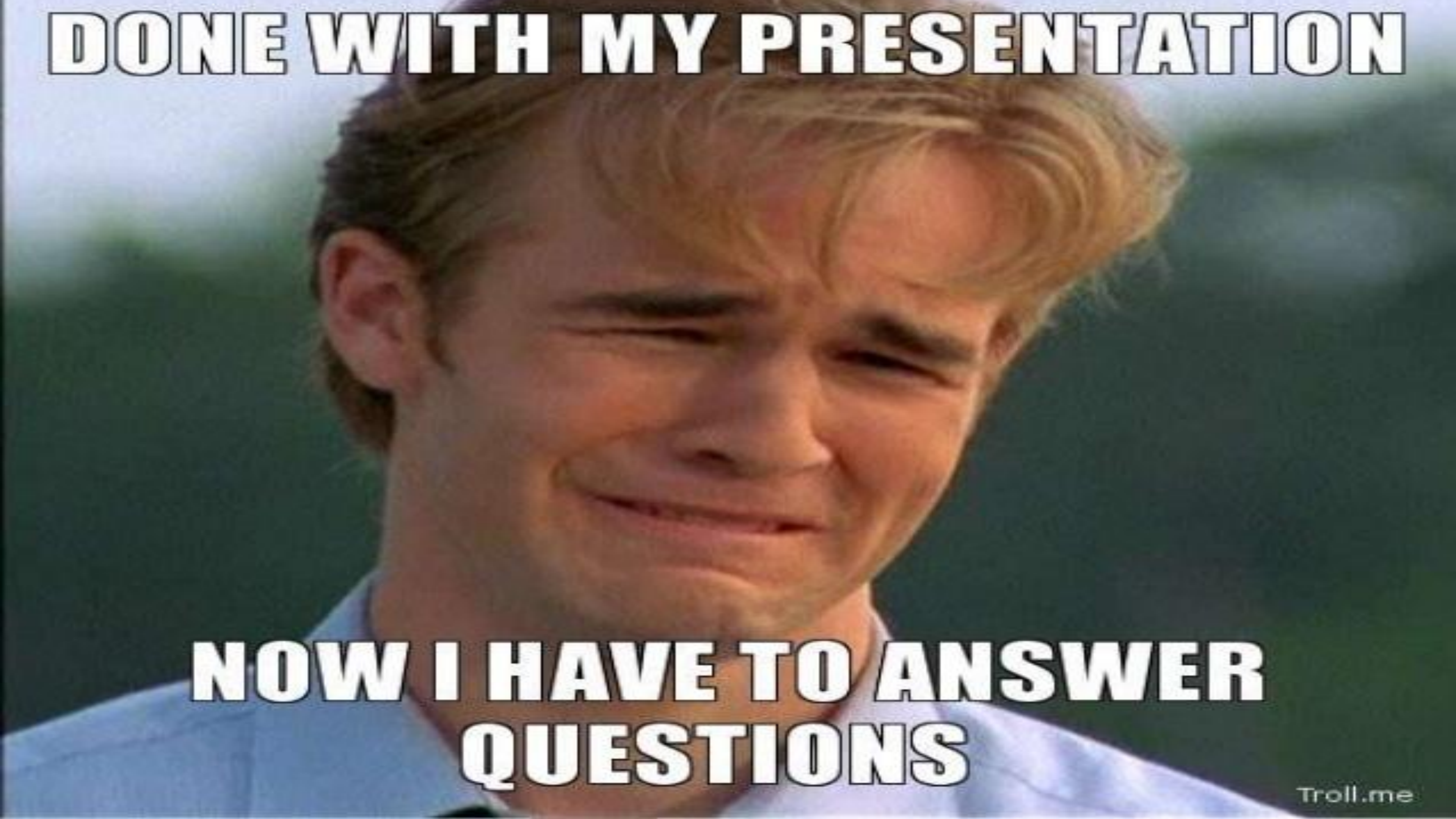
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