PHYS 421 Midterm Exam 2 (Version B)

2022-11-09

Last Name: Key

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(9.125): $\tilde{k} = k + i\kappa$

(9.126): $k = \omega\sqrt{\frac{\varepsilon\mu}{2}}\left[\sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2}+1\right]^{1/2}$, $\kappa = \omega\sqrt{\frac{\varepsilon\mu}{2}}\left[\sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2}-1\right]^{1/2}$

1 (10 points) A plane wave is travelling downward in the $+z$ direction in seawater, with the xy plane denoting the sea surface and $z = 0$ denoting a point just below the surface. The constitutive parameters of seawater are $\varepsilon_r = 81, \mu_r = 1, \sigma = 4\ s/m$. The electric field at $z = 0$ is given by

$$\vec{E} = \hat{x}100\cos(10^6 t) \quad \text{(V/m)}.$$

1 (a) Can sea water be treated as good conductor? Why or why not?

3 (b) Calculate the phase velocity and wavelength of the wave in seawater. What would be the wavelength if the wave were in vacuum?

4 (c) Determine the complex wave functions of the electric and magnetic fields, $\tilde{\vec{E}}(z,t)$ and $\tilde{\vec{B}}(z,t)$, in the seawater (for $z > 0$).

2 (d) Evaluate the intensity at a depth of 2.5m.

(a). Yes, because $\frac{\sigma}{\omega\varepsilon} = \frac{4}{10^6 \times 8.85\times10^{-12}\times 81} = 5600 \gg 1$.

(b). Good conductor: $k = \kappa \simeq \omega\sqrt{\frac{\varepsilon\mu}{2}}\sqrt{\frac{\sigma}{\varepsilon\omega}} = \sqrt{\frac{\omega\mu\sigma}{2}} = 1.585\ m^{-1}$
$\approx 1.6\ (m^{-1})$

$$v_{ph} = \frac{\omega}{k} = \frac{10^6}{1.585} = 6.3\times10^5\ m/s$$

$$\lambda = \frac{2\pi}{k} = \frac{2\pi}{1.585} = 4.0\ m.$$

In vacuum: $\lambda_0 = \frac{2\pi c}{\omega} = \frac{2\pi\times3\times10^8}{10^6} = 1900\ m$.

(c). $\tilde{\vec{E}}(z,t) = \hat{x}\,100\, e^{i(\tilde{k}z-\omega t)} = \hat{x}\,100\, e^{-\kappa z}\cdot e^{i(kz-\omega t)}$

$\tilde{\vec{B}}(z,t) = \hat{y}\,\frac{\tilde{k}}{\omega}\,100\, e^{-\kappa z}\cdot e^{i(kz-\omega t)}$ $\quad \left(\because \nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}\right)$

where $\tilde{k} = k + i\kappa = \sqrt{k^2+\kappa^2}\, e^{i\phi} = \sqrt{2}\, k\, e^{i\frac{\pi}{4}} = 2.2\, e^{i\frac{\pi}{4}}$

(d).

$$I = \langle S\rangle = \left|\frac{1}{2\mu_0}\mathrm{Re}\left(\tilde{\vec{E}}\times\tilde{\vec{B}}^*\right)\right|$$

$$= 100^2\times\frac{2.2}{10^6\times2\mu_0}\, e^{-2\kappa z}\,\mathrm{Re}\left(e^{-i\frac{\pi}{4}}\right) = \frac{1.1}{100\,\mu_0}\, e^{-2\kappa z}\cdot\cos\frac{\pi}{4}.$$

at $z=2.5m$

$$I = \frac{0.778}{4\pi\times10^{-5}}\, e^{-2\times1.6\times2.5} = 2.08\ \ W/m^2$$

$a = 3\,cm = 0.03\,m, \quad b = 1\,cm = 0.01\,m.$

2 (10 Points). Consider a hollow rectangular wave guide with dimensions 3cm × 1cm.

5. (a) What TE modes will propagate in this wave guide, if the driving frequency is 19 GHz?

3 (b) Suppose you wanted to excite only one TE mode; what range of frequencies could you use?

2 (c) What is the ratio of the lowest TM cutoff frequency to the lowest TE cutoff frequency?

(a) cutoff frequencies: $f_{mn} = \frac{\omega_{mn}}{2\pi} = \frac{c}{2}\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}$. (2)

for $f_{mn} < 19\,GHz$: $f_{10} = \frac{3\times10^8}{2}\sqrt{\left(\frac{1}{0.03}\right)^2} = 5\,GHz$. ($TE_{10}$) 0.5

$f_{20} = 1.5\times10^8\sqrt{\left(\frac{2}{0.03}\right)^2} = 10\,GHz$. ($TE_{20}$) 0.5

$f_{30} = 1.5\times10^8\sqrt{\left(\frac{3}{0.03}\right)^2} = 15\,GHz$. ($TE_{30}$) 0.5

$f_{11} = 1.5\times10^8\sqrt{\left(\frac{1}{0.03}\right)^2+\left(\frac{1}{0.01}\right)^2} = 15.8\,GHz$. ($TE_{11}$, $\cancel{TM_{11}}$) ✓0.5

$f_{01} = 1.5\times10^8\sqrt{\left(\frac{1}{0.01}\right)^2} = 15\,GHz$. ($TE_{01}$) 0.5

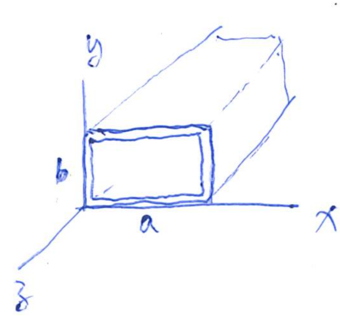
$f_{21} = 1.5\times10^8\sqrt{\left(\frac{2}{0.03}\right)^2+\left(\frac{1}{0.01}\right)^2} = 18\,GHz$. ($TE_{21}$) 0.5

(b). To excite only one TE mode, it will be TE_{10}.

range of frequencies can be used: $f_{20} > f > f_{10}$

i.e: $10\,GHz > f > 5\,GH$

(c) TM_{11}/TE_{10} : $\frac{f_{11}}{f_{10}} = \frac{15.8\,GHz}{5\,GHz} = 3.16$

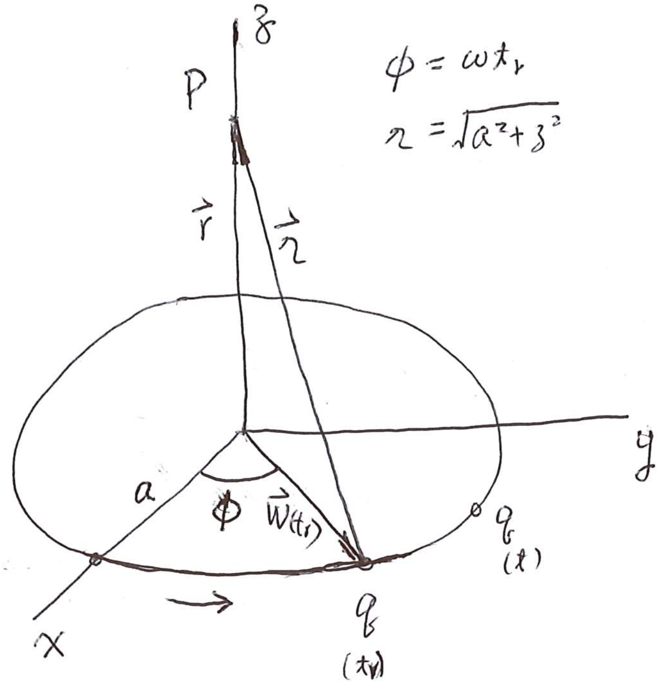


(Extra space for question #2)

3 (10 Points). A particle of charge q moves in a circle of radius a at constant angular velocity $\vec{\omega} = \omega\hat{z}$ (Assume that the circle lies in the xy plane, centered at the origin, and at time $t = 0$ the charge is at $(a, 0, 0)$, on the positive x axis.).

6 (a) Find the Liénard-Wiechert potentials $V(\vec{r}, t)$ and $\vec{A}(\vec{r}, t)$ for points on the z axis (0, 0, z).

4 (b) Find the z-component of the electric field for points on the z axis.



$\phi = \omega t_r$

$\eta = \sqrt{a^2+z^2}$

Location of q at time t:

$$\vec{W}(t) = a\left[\cos(\omega t)\,\hat{x} + \sin(\omega t)\,\hat{y}\right]$$

velocity of q at time t:

$$\vec{v}(t) = \omega a\left[-\sin(\omega t)\,\hat{x} + \cos(\omega t)\,\hat{y}\right]$$

(a). $$V(\vec{r}, t) = \frac{1}{4\pi\varepsilon_0}\frac{qc}{(\eta c - \vec{\eta}\cdot\vec{v})}$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{q}{\eta} \quad (\because \vec{\eta}\cdot\vec{v} = 0)$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{q}{\sqrt{z^2+a^2}}$$

$$\vec{A}(\vec{r}, t) = \frac{\vec{v}}{c^2}V(\vec{r}, t) = \frac{1}{4\pi\varepsilon_0}\frac{q\omega a}{c^2\sqrt{z^2+a^2}}\left[-\sin(\omega t_r)\,\hat{x} + \cos(\omega t_r)\,\hat{y}\right]$$

where $$t_r = t - \frac{\eta}{c} = t - \frac{\sqrt{z^2+a^2}}{c}$$

(b). $$\vec{E} = -\nabla V - \frac{\partial \vec{A}}{\partial t}$$

$$E_z = -\frac{\partial V}{\partial z} - \frac{\partial A_z}{\partial t}$$

$$= -\left(-\frac{1}{2}\right)\frac{q\,2z}{4\pi\varepsilon_0}(z^2+a^2)^{-3/2} = \frac{1}{4\pi\varepsilon_0}\frac{qz}{(z^2+a^2)^{3/2}}.$$