Lecture 2.5
Phasors

(Really, using j to our advantage)

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Image: http://en.memory-alpha.org/wiki/Phaser

Phys 365/369

Superposition of oscillations

- Any oscillation can be broken into a sum (superposition) of simple harmonic oscillations.
- Sum of sinusoids at the same frequency: ω:

$$y = A_1 \cos(\underline{\omega} t + \phi_1) + A_2 \sin(\underline{\omega} t + \phi_2)$$

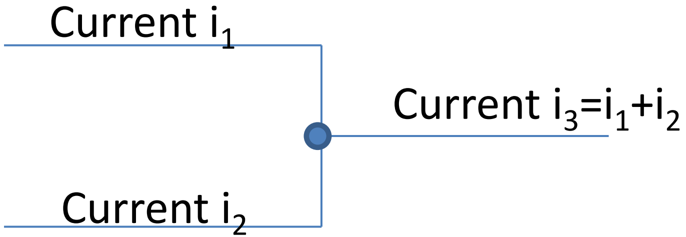
is just another sinusoid:

$$y = A_3 \cos(\omega t + \phi_3)$$

But, A_3 & ϕ_3 are hard to find.

Example

Two alternating (AC) currents enter a 3-wire junction. The first is a current of 1.0 Amps at a phase of 30 degrees. The second is 2.0 Amps at a phase of 60 degrees. Find the resulting current out of the junction.



$$i_1(t) = (1.0A)\cos\left(\omega t + \frac{\pi}{6}\right)$$

$$i_2(t) = (2.0A)\cos\left(\omega t + \frac{\pi}{3}\right)$$

$$i_3(t) = i_1(t) + i_2(t)$$

The hard way

$i_3 = A_1 \cos(\omega t + \phi_1) + A_2 \cos(\omega t + \phi_2)$

1. Break apart using the trig identity $\cos(\alpha + \beta) = \cos\alpha \cos\beta - \sin\alpha \sin\beta$

$i_3 = A_1 \cos(\omega t) \cos(\phi_1)$ - $A_1 \sin(\omega t) \sin(\phi_1) + A_2 \cos(\omega t) \cos(\phi_2)$ - $A_2 \sin(\omega t) \sin(\phi_2)$

2. Collect terms

$i_3 = [A_1 \cos(\phi_1) + A_2 \cos(\phi_2)] \cos(\omega t)$-$[A_1 \sin(\phi_1) + A_2 \sin(\phi_2)] \sin(\omega t)$

3. Define A, δ such that:

$$\begin{cases} A\cos\delta = A_1 \cos(\phi_1) + A_2 \cos(\phi_2) \\ A\sin\delta = A_1 \sin(\phi_1) + A_2 \sin(\phi_2) \end{cases}$$

4. So that:

$$i_3 = A\cos\delta \cos(\omega t) - A\sin\delta \sin(\omega t) = A\cos(\omega t + \delta)$$

5. Solve the simultaneous equations from step 3 for A, δ

5a. Square and add them, use a few more trig identities:

$$A^2 = A_1^2 + A_2^2 + 2A_1A_2\cos(\phi_2 - \phi_1)$$

5b. Divide them:

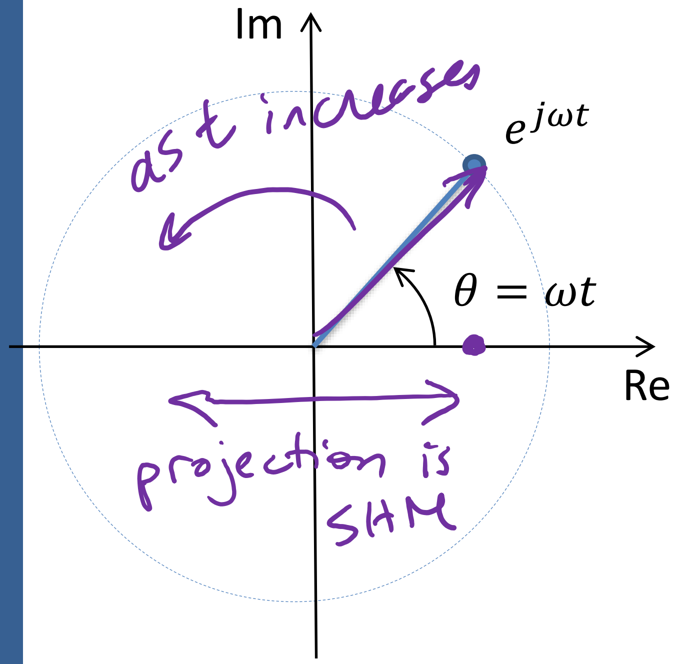
$$\tan\delta = \frac{A_1 \sin\phi_1 + A_2 \sin\phi_2}{A_1 \cos\phi_1 + A_2 \cos\phi_2}$$

6. Collect:

$$i_3 = \underbrace{\sqrt{A_1^2 + A_2^2 + 2A_1A_2\cos(\phi_2 - \phi_1)}}_{A_3} \cos\left[\omega t + \underbrace{\tan^{-1}\left(\frac{A_1 \sin\phi_1 + A_2 \sin\phi_2}{A_1 \cos\phi_1 + A_2 \cos\phi_2}\right)}_{\phi_3}\right]$$

7. Plug in values.

The easier way: Phasors



Using Euler's identity

$$e^{j\theta} = \cos\theta + j\sin\theta$$

we see that

$$e^{j\omega t} = \underline{\cos\omega t} + j\sin\omega t$$

describes a **<u>rotating</u>** unit vector in the complex plane.

$$\mathrm{Re}\{e^{j\omega t}\} = \cos(\omega t)$$

(See animation at http://ophysics.com/w0.html)

The easier way: Phasors

$e^{j\theta} = \cos\theta + j\sin\theta$

Write $x(t) = A\cos(\omega t + \phi_0)$ in **phasor** representation:

$$\tilde{x}(t) = A e^{j(\omega t + \phi_0)}$$
$$= A e^{j\phi_0} e^{j\omega t}$$
$$= \tilde{A}\, e^{j\omega t}$$

"complex amplitude"
- encodes amplitude A and phase ϕ_0
- No time dependence

rotating unit vector
- has time dependence

$$x(t) = \mathrm{Re}\{\tilde{x}(t)\}$$
$$= \mathrm{Re}\{A e^{j(\omega t + \phi_0)}\}$$
$$= \mathrm{Re}\{A\cos(\omega t + \phi_0) + jA\sin(\omega t + \phi_0)\}$$
$$= A\cos(\omega t + \phi_0)$$

Adding SHM with phasors

Problem:

$$\begin{cases} x_1(t) = A_1 \cos(\omega t + \phi_1) \\ x_2(t) = A_2 \cos(\omega t + \phi_2) \end{cases}$$

Looking for: $x_3(t) = x_1(t) + x_2(t)$

$\tilde{x}_1 = A_1 e^{j\phi_1}$

Solution: Express in complex form:

$$x_1(t) = \mathrm{Re}\left[\underline{A_1 e^{j\phi_1}} e^{j\omega t}\right] = \mathrm{Re}\left[\tilde{x}_1 e^{j\omega t}\right]$$

and add the complex amplitudes:

$$x_3(t) = \mathrm{Re}\left[\tilde{x}_1 e^{j\omega t}\right] + \mathrm{Re}\left[\tilde{x}_2 e^{j\omega t}\right]$$
$$= \mathrm{Re}\left[\underline{(\tilde{x}_1 + \tilde{x}_2)}\, e^{j\omega t}\right]$$

Example

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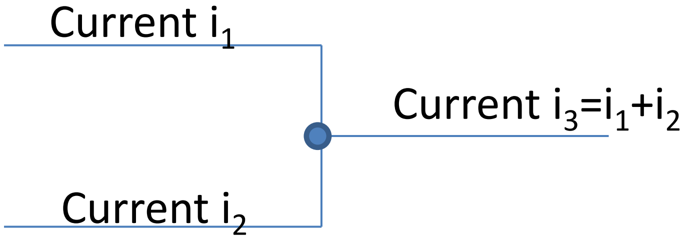


$$i_1(t) = (1\,A)\cos\left(\omega t + \frac{\pi}{6}\right)$$

$$i_2(t) = (2\,A)\cos\left(\omega t + \pi/3\right)$$

$$i_3(t) = i_1(t) + i_2(t)$$

With phasors

$$i_1 = 1.0\cos(\omega t + \pi/6)$$
$$i_2 = 2.0\cos(\omega t + \pi/3)$$

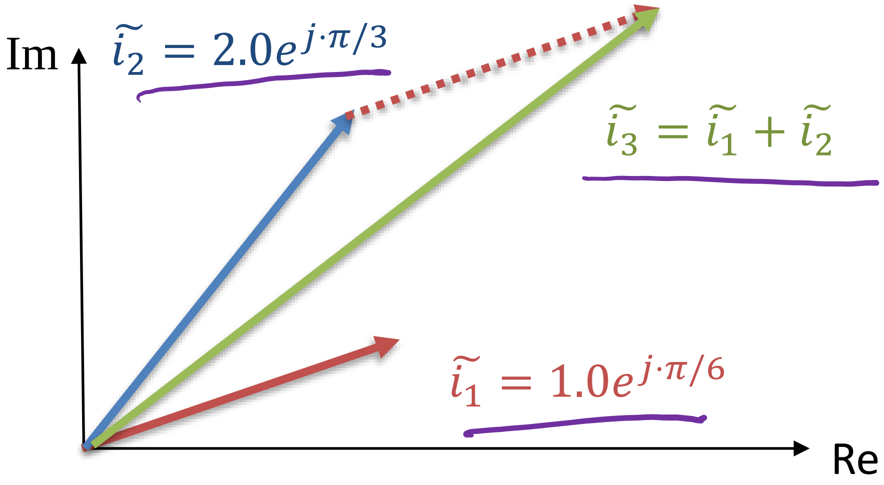
Write as phasors:

$$\tilde{i}_1(t) = (1.0)\, e^{j\pi/6}\, e^{j\omega t} \equiv \hat{i}_1\, e^{j\omega t}$$

$$\hat{i}_2(t) = (2.0)\, e^{j\pi/3}\, e^{j\omega t} \equiv \tilde{i}_2\, e^{j\omega t}$$

$$\tilde{i}_3 = \tilde{i}_1 + \tilde{i}_2$$

The addition of the complex amplitudes looks like this in the complex plane:



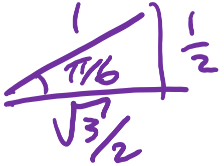
Recall: Euler's formula

$$e^{j\theta} = \cos\theta + j\sin\theta$$

$$\tilde{z}_1 = (1.0)\, e^{j\pi/6}$$

$$\tilde{z}_2 = (2.0)\, e^{j\pi/3}$$

$$\tilde{z}_3 = \hat{z}_1 + \hat{z}_2$$



$$\hat{z}_1 = (1.0)\left(\cos(\pi/6) + j\sin(\pi/6)\right) = \frac{\sqrt{3}}{2} + j\,\frac{1}{2}$$

$$\tilde{z}_2 = (2.0)\left(\cos(\pi/3) + j\sin(\pi/3)\right) = 1 + j\sqrt{3}$$

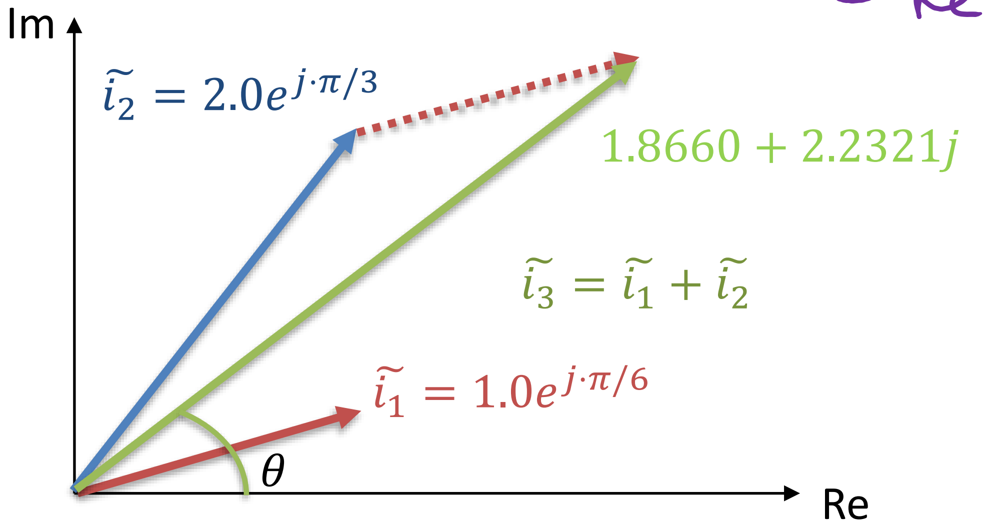
$$\Rightarrow \hat{z}_3 = \left(\frac{\sqrt{3}}{2} + j\frac{1}{2}\right) + \left(1 + j\sqrt{3}\right)$$

$$= 1.8660 + 2.2321\, j$$

To express the results in standard form, switch back to polar form:

- $\tilde{i_3} = 1.8660 + 2.2321j$
- $|\tilde{i_3}| = \sqrt{1.8660^2 + 2.2321^2} = 2.91$
- $\theta = \tan^{-1}(2.2321/1.8660) = 50.1^o = 0.87\,\text{rad}$

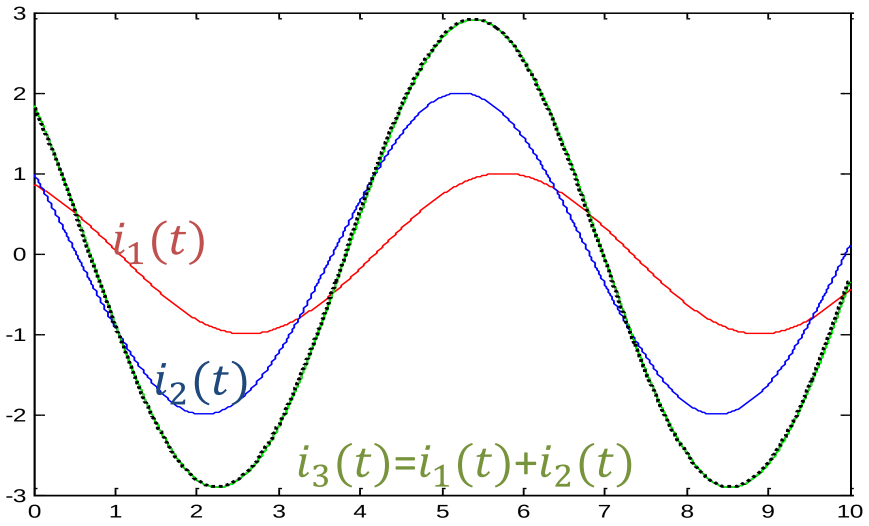
$$\Rightarrow i_3(t) = \mathrm{Re}\left\{ |\tilde{i}_3|\, e^{j\theta} e^{j\omega t} \right\}$$
$$= \mathrm{Re}\left\{ |\tilde{i}_3|\, e^{j(\omega t + \theta)} \right\}$$
$$= |\tilde{i}_3| \cos(\omega t + \theta)$$
$$= (2.91\,A) \cos(\omega t + 0.87)$$



Verify

Matlab:

```
>> wt=0:0.001:10;
>> i1=1.0*cos(wt+30*(pi/180));
>> i2=2.0*cos(wt+60*(pi/180));
>> i3=2.91*cos(wt+50.1*(pi/180));
>> plot(wt,i1,'r')
>> hold on
>> plot(wt,i2,'b')
>> plot(wt,i1+i2,'g')
>> plot(wt,i3,'k:','LineWidth',2)
```



Recipe for phasor addition

Recipe:

1. Write all sinusoidal motion in the form $A\cos(\omega t + \phi_0)$. (Don't use sines!)
2. Convert to phasor form $Ae^{j\phi_0}e^{j\omega t} = \tilde{A}e^{j\omega t}$
3. Add the complex amplitudes $\tilde{A}_1 + \tilde{A}_2$ in rectangular form
4. Convert the result to polar form
   - Careful of the multi-valued arctan()
5. Done: take the real part.