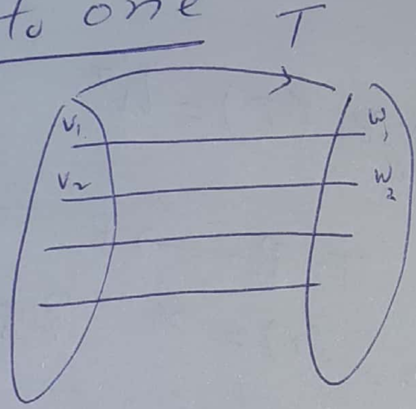
Exercise 8-3

Isomorphism :-

A linear Transformation $T: V \longrightarrow W$ that is both one-to-one and onto is said to be an isomorphism.

one to one

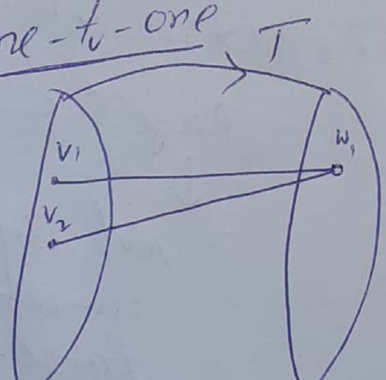


$T(v_1) = w_1$
$T(v_2) = w_2$

Exp
$T: R \longrightarrow R$
$T(x) = x$

Distinct pre-images have distinct images

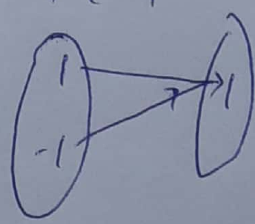
Not one-to-one



$T(v_1) = w_1$
$T(v_2) = w_1$

Exp
$T: R \longrightarrow R$
$T(x) = x^2$

$T(1) = (1)^2 = 1$
$T(-1) = (-1)^2 = 1$

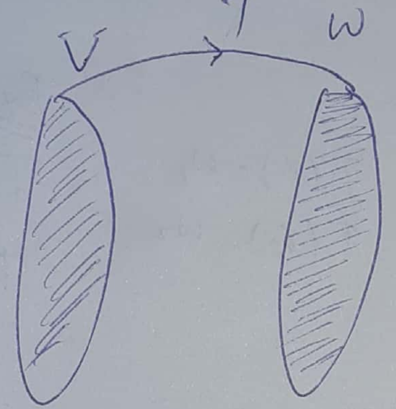


onto A linear transformation

$T : \bar{V} \longrightarrow \bar{W}$ is called onto

if $Range(T) = \bar{W}$

Eeach element of $\bar{W}$ has some pre-image in $\bar{V}$ under T



$Range(T) = \bar{W}$

$T : R \longrightarrow R$
$T(x) = x$ $Rage\, T = R$

$T : R \longrightarrow R$
$T(x) = x^2$

$Rage\, T \subset R$

$\downarrow$

$\because$ Not negative numbers are in Range.

Note:- If $\operatorname{Ker} T = \{0\} \Rightarrow T$ is one-to-one

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Q_1:- State whether the transformation is isomorphism $T: P_1 \to R^2$

$$c_0 + c_1 x \longrightarrow (c_0 - c_1, c_1)$$

i.e

$$T(c_0 + c_1 x) = (c_0 - c_1, c_1)$$

One-to-one :-

Put $c_0 = c_1 = 0$

$T(\underline{0}_P) = (0,0)$, we note that for any other values of c_0 & c_1 we cannot get $(0,0)$.

Consequently, $\operatorname{Ker} T = \{\underline{0}\}$

$\Rightarrow T$ is one to one

onto :- For each $(a,b) \in R^2$, we have $(a+b) + bx$ in P_1 s.t $T((a+b)+bx) = (a+b-b, b)$
$= (a,b)$

$\Rightarrow$ T is onto

Consequently T is isomorphism.

Q2 $T: R^2 \longrightarrow R^3$ defined by

$T(x,y) = (x,y,0)$ is Isomorphism?

one to one:

we note that for only $x=0=y$

$T(0,0) = (0,0,0)$

$\Rightarrow$ $\text{Ker}\,T = \{\underline{0}\} \Rightarrow$ T is one to one.

onto For each (x,y,z) we can not

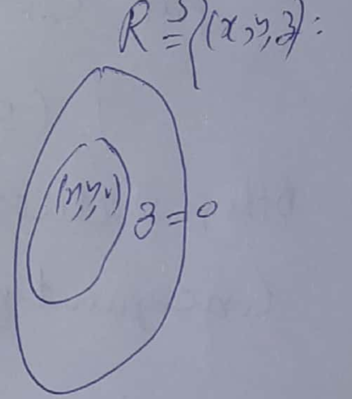
have $(x,y) \in R^2$ s.t $T(x,y) = (x,y,z)$

$\because$ it is only true if $z=0$

$\text{Rang}\,T \subset R^3$

So, Not onto

Not Isomorphism.



Q_3:-

$$T: P_3 \longrightarrow M_{22}$$

$$T(a+bx+cx^2+dx^3) = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$ Isomorph?

<u>on to one:</u>

Since for only $a=0=b=c=d$,

$T(0_p) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow \ker T = \{0_p\} \Rightarrow$ T is one to one.

<u>onto</u>

Since for each $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_{22}$, we have

$a+bx+cx^2+dx^3 \in P_3$ s.t

$T(a+bx+cx^2+dx^3) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Rightarrow$ Range $T = M_{22}$

$\Rightarrow$ T is onto

$\Rightarrow$ T is isomorphism.

Q_4:-

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad-bc$$

$$T: M_{22} \longrightarrow R$$

Is isomorphism?

one-to-one

Since for $a=d=0$ & $b=c=0$

$$T\left(\begin{bmatrix}0&0\\0&0\end{bmatrix}\right)=(0)(0)-(0)(0)$$
$$=0$$

for $a=d=1$ & $b=c=1$

$$T\left(\begin{bmatrix}1&1\\1&1\end{bmatrix}\right)=(1)(1)-(1)(1)$$
$$=0$$

also for
$a=d=2$ & $b=c=2$

$$T\left(\begin{bmatrix}2&2\\2&2\end{bmatrix}\right)=(2)(2)-(2)(2)$$
$$=4-4$$
$$=0$$

So,

$$\ker T=\left\{\begin{bmatrix}0&0\\0&0\end{bmatrix},\begin{bmatrix}1&1\\1&1\end{bmatrix},\begin{bmatrix}2&2\\2&2\end{bmatrix},\cdots\right\}$$
$$=\left\{\begin{bmatrix}a&a\\a&a\end{bmatrix}: a\in R\right\}\neq\left\{\begin{bmatrix}0&0\\0&0\end{bmatrix}\right\}$$

T is not one-to one

$\Rightarrow$ T is not Isomorphism.

(07)

Q.11 Determine whether the matrix transformation $T_A : R^3 \longrightarrow R^3$ is an isomorphism

$$A = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & 2 \\ -1 & 1 & 0 \end{bmatrix}$$

$T_A(\underline{x}) = A\underline{x}$ ✓

$T_A(\underline{u}) = A\underline{u}$ ✓

<u>R-Echelon form</u>

$$\begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & 2 \\ -1 & 0 & 1 \end{bmatrix} R_3 - R_1$$

$$\begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & 2 \\ 0 & 0 & -1 \end{bmatrix} R_3 + R_2$$

$$\begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix} \frac{R_3}{-1}$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{matrix} R_1 + R_3 \\ R_2 - 2R_3 \end{matrix}$$

Rank $(A) = 3$ Nullity $= 0$

$\Rightarrow$ Ker $T = \{0_m\}$ $\Rightarrow$ T is one to one.

Also onto $\Longrightarrow$ Isomorphism.

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Q_{16}:- $T: R_{22} \longrightarrow \mathbb{R}^4$ (08)

$$T\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \left[(a+b), (a+b), (a+b+c), (a+b+c+d)\right]$$

<u>one-to-one</u>

For $a=b=0$ & $c=d=0$

$$T\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = (0,0,0,0)$$

for $a=1$ & $b=-1$ & $c=d=0$

$$T\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} = (0,0,0,0)$$

$$\Rightarrow \text{Ker}\,T = \left\{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}\right\} \neq \text{Ker}\{0_m\}$$

No one to one

Not isomorphic

Q 18

(a) for what value of K,

$M_{mn} \cong R^{K}$ $\qquad$ $M_{22} \cong R^{4}$

$M_{23} \cong R^{6}$

Ans $K = mn$

(b) for what value of K

$M_{mn} \cong P_{K}$ $\qquad$ $T(a+bx+cx^2+dx^3) = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

Ans: $K = mn - 1$ $\qquad$ $T \begin{bmatrix} a & c \\ b & d \end{bmatrix} =$ —
