# Chapter 6

28/4/2020

## Exercise 6.1:- (Q1 – Q20)

## Euclidean Inner Product

or

## Standard Inner Product

let $\underline{u}, \underline{v} \in R^n$ be two vectors then there Euclidean Inner Product is defined as

$$\langle \underline{u}\ \underline{v} \rangle = \underline{u} \cdot \underline{v}$$
$$= (u_1, u_2, u_3, \cdots, u_n) \cdot (v_1, v_2, v_3, \cdots, v_n)$$
$$= u_1v_1 + u_2v_2 + u_3v_3 + \cdots + u_nv_n$$

and

$$\|\underline{u}\| = \sqrt{\langle u, u \rangle} = \sqrt{(u_1, u_2, \cdots u_n) \cdot (u_1, u_2, \cdots u_n)}$$
$$= \sqrt{u_1^2 + u_2^2 + \cdots + u_n^2}$$

$$d(u,v) = \|u - v\| = \sqrt{\langle u-v, u-v \rangle} = \sqrt{(\underline{u}-\underline{v}) \cdot (\underline{u}-\underline{v})}$$

### Four properties

1. $\langle \underline{u}, \underline{v} \rangle = \langle \underline{v}, \underline{u} \rangle$ (Symmetry axioms)
2. $\langle \underline{u} + \underline{v}, \underline{w} \rangle = \langle \underline{u}, \underline{w} \rangle + \langle \underline{v}, \underline{w} \rangle$ (Additivity axiom)
3. $\langle k\underline{u}, v \rangle = k \langle u, v \rangle$ (Homogeneity axiom)
4. $\langle v, v \rangle \geq 0$ & $\langle v, v \rangle = 0 \Leftrightarrow v = 0$.

(02)

Q1 :- If $\underline{u} = (1,1)$
$\underline{v} = (3,2)$
$\underline{w} = (0,-1)$
$k = 3$

Compute

(a) $\langle \underline{u}, \underline{v} \rangle = \underline{u} \cdot \underline{v}$
$= (1,1)\cdot(3,2)$
$= (1)(3) + (1)(2)$
$= 3+2$
$= 5$

(b)
$\langle kv, w \rangle = k\langle v, w \rangle$
$= k(\underline{v}\cdot\underline{w})$
$= k[(3,2)\cdot(0,-1)]$
$= 3[(3)(0) + 2(-1)]$
$= 3(0-2)$
$= -6$

(c)

$\langle u+v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$
$= u\cdot w + v\cdot w$
$= (1,1)\cdot(0,-1) + (3,2)\cdot(0,-1)$
$= (1)(0)+(1)(-1) + 3(0)+2(-1)$
$= 0-1+0-2$
$= -3$

(d)

$\|v\| = \sqrt{\langle v, v \rangle}$
$= \sqrt{\underline{v}\cdot\underline{v}}$
$= \sqrt{(3,2)\cdot(3,2)}$
$= \sqrt{(3)(3)+(2)(2)}$
$= \sqrt{9+4}$
$= \sqrt{13}$

(e)

$d(u,v) = \|\underline{u}-\underline{v}\|$
$= \sqrt{\langle u-\underline{v}, u-v \rangle}$
$= \sqrt{(\underline{u}-\underline{v})\cdot(\underline{u}-\underline{v})}$

Consider
$\underline{u}-\underline{v} = (1,1)-(3,2)$
$= ((1-3),(1-2))$
$= (-2,-1)$

$d(u,v) = \sqrt{(-2,-1)\cdot(-2,-1)}$
$= \sqrt{(-2)(-2)+(-1)(-1)}$
$= \sqrt{4+1} = \sqrt{5}$

(f) $\|u-kv\| = \sqrt{\langle u-kv, u-kv \rangle}$

$= \sqrt{(u-kv)\cdot(u-kv)}$

(03)

Consider

$u - kv = (1,1) - 3(3,2)$
$= (1,1) - (9,6)$
$= (1-9, 1-6)$
$= (-8,-5)$

$\|u - kv\| = \sqrt{(-8,-5)\cdot(-8,-5)}$
$= \sqrt{-8\times-8 + -5\times-5}$
$= \sqrt{64+25}$
$= \sqrt{89}$

$Q_1 \Rightarrow u=(1,1),\ v=(3,2),\ w=(0,-1)$
$k=3$

$Q_2$ :-

Weighted Euclidean Inner product

$\langle u,v\rangle = 2u_1v_1 + 3u_2v_2$

(a) $\langle u,v\rangle = u\cdot v$
$= 23u_1v_1 + 3u_2v_2$
$= 2(1)(3)+3(1)(2)$
$= 6..+6$ =12
$= 15$ ............

(b)

$\langle kv,w\rangle = k\langle v,w\rangle$
$= k(2v_1w_1 + 3v_2w_2)$
$= 3[2(3)(0)+3(2)(-1)]$
$= 3[0-6]$
$= -18$

(c)

$\langle u+v,w\rangle = \langle u,w.\rangle + \langle v,w\rangle$
$= (2u_1w_1 + 3u_2w_2) + (2v_1w_1 + 3v_2w_2)$
$= [2(1)(0)+3(1)(-1)] + [2(3)(0)+3(2)(-1)]$
$= -3 + (-6)$
$= 6....$ -9

(d) $\|v\| = \sqrt{\langle v,v\rangle}$
$= \sqrt{v\cdot v}$
$= \sqrt{2v_1v_1 + 3v_2v_2}$
$= \sqrt{2(3)(3)+3(2)(2)}$
$= \sqrt{18+12}$
$= \sqrt{30}$

(e)

$d(u,v) = \|u-v\|$
$= \sqrt{\langle u-v, u-v\rangle}$
$= \sqrt{(u-v)\cdot(u-v)}$

$u - v = (-2,-1)$ as computed in $Q_1$

$= \sqrt{(-2,-1)\cdot(-2,-1)}$
$= \sqrt{2(-2)(-2) + 3(-1)(-1)}$
$= \sqrt{8+3} = \sqrt{11}$

(f) $\|u-kv\| = \sqrt{\langle u-kv,\ u-kv\rangle}$

$= \sqrt{(u-kv)\cdot(u-kv)}$

As $u-kv = (-8,-5)$ computed in Q1

$= \sqrt{(\overset{u_1}{-8},\overset{u_2}{-5})\cdot(\overset{v_1}{-8},\overset{v_2}{-5})}$

$= \sqrt{2u_1v_1 + 3u_2v_2} = \sqrt{2(-8)(-8)+3(-5)(-5)}$

$= \sqrt{2(64)+3(25)}$

$= \sqrt{128+75}$

$= \sqrt{203}$

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# Euclidean Inner product for matrices

$\underline{U} = \begin{bmatrix} u_1 & u_2 \\ u_3 & u_4 \end{bmatrix}, \quad \underline{V} = \begin{bmatrix} v_1 & v_2 \\ v_3 & v_4 \end{bmatrix}$

$\langle U, \underline{V}\rangle = U_1V_1 + U_2V_2 + U_3V_3 + U_4V_4$

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# Euclidean Inner product for vectors

$\underline{U} = (U_1, U_2)$ & $\underline{V} = (V_1, V_2)$

$\langle U,V\rangle = U_1V_1 + U_2V_2$ as used in Q1 & Q2

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# Euclidean Inner product for Vectors generated by matrix A

$\langle U, V\rangle = AU \cdot AV$

Q5 :- $A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ 05

(a) $\underline{u} = (2,1)$, $\underline{v} = (-1,1)$ & $\underline{w} = (0,-1)$

$$\langle \underline{u}, \underline{v} \rangle = A\underline{u} \cdot A\underline{v}$$
$$= \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$
$$= (5, 3) \cdot (-1, 0)$$
$$= (5)(-1) + (3)(0)$$
$$= -5$$

(b)

$$\langle v, w \rangle = A\underline{v} \cdot A\underline{w}$$
$$= \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ -1 \end{bmatrix}$$
$$= (-1, 0) \cdot (-1, -1)$$
$$= (-1)(-1) + (0)(-1)$$
$$= 1$$

(c)

$$\langle u+v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$$
$$= Au \cdot Aw + Av \cdot Aw$$
$$= \left( \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ -1 \end{bmatrix} \right) + \left( \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ -1 \end{bmatrix} \right)$$
$$= \big( (5,3) \cdot (-1,-1) \big) + \big( (-1,0) \cdot (-1,-1) \big)$$
$$= \big( (5)(-1) + 3(-1) \big) + \big( (-1)(-1) + (0)(-1) \big)$$
$$= (-5-3) + (1+0)$$
$$= -8 + 1$$
$$= -7$$

(d) $\|v\| = \sqrt{\langle v, v\rangle}$

$= \sqrt{Av \cdot Av}$

$= \sqrt{\begin{bmatrix}2&1\\1&1\end{bmatrix}\begin{bmatrix}-1\\1\end{bmatrix} \cdot \begin{bmatrix}2&1\\1&1\end{bmatrix}\begin{bmatrix}-1\\1\end{bmatrix}}$

$= \sqrt{(-1,0)\cdot(-1,0)}$

$= \sqrt{(-1)(-1)+(0)(0)}$

$= \sqrt{1}$

$= 1$

(e) (d6)

$d(v, w) = \|v-w\|$

$= \sqrt{\langle v-w, v-w\rangle}$

$= \sqrt{A(v-w)\cdot A(v-w)}$

we compute $A \underset{=}{v-w}$

$A(v-w) = \begin{bmatrix}2&1\\1&1\end{bmatrix}(-1-0, 1+1)$

$= \begin{bmatrix}2&1\\1&1\end{bmatrix}\begin{bmatrix}-1\\2\end{bmatrix}$

$= (0,1)$

$d(v,w) = \sqrt{(0,1)\cdot(0,1)} = \sqrt{(0)(0)+(1)(1)}$

$= 1$

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(f) $\|v-w\|^2 = \langle v-w, v-w\rangle$

$= A(v-w)\cdot A(v-w)$

$= \begin{bmatrix}2&1\\1&1\end{bmatrix}\begin{bmatrix}-1\\2\end{bmatrix} \cdot \begin{bmatrix}2&1\\1&1\end{bmatrix}\begin{bmatrix}-1\\2\end{bmatrix}$

$= (0,1)\cdot(0,1)$

$= (0)(0)+(1)(1)$

$= 1$

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(7) $u = \begin{bmatrix}3&-2\\4&8\end{bmatrix}, \quad v = \begin{bmatrix}-1&3\\1&1\end{bmatrix}$

(a) $\langle u, v\rangle = (3)(-1) + (-2)(3) + 4(1) + 8(1)$

$= -3 - 6 + 4 + 8$

$= -9 + 12 = \boxed{3}$

07

**Q8** Euclidean Inner product for Polynomials

$P = a_0 + a_1x + a_2x^2$

$q = b_0 + b_1x + b_2x^2$

$\langle P, q\rangle = a_0b_0 + a_1b_1 + a_2b_2$

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$P = -2 + x + 3x^2$

$q = 4 + 0x - 7x^2$

$\langle P, q\rangle = (-2)(4) + (1)(0) + (3)(-7)$

$= -8 + 0 - 21$

$= -29$

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**Q9** $A = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$, $\underline{u} = (u_1, u_2)$

$\underline{v} = (v_1, v_2)$

Consider,

$\langle u, v\rangle = A\underline{u} \cdot A\underline{v}$

$= \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \cdot \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$

$= (3u_1, 2u_2) \cdot (3v_1, 2v_2)$

$= (3u_1)(3v_1) + (2u_2)(2v_2)$

$= 9u_1v_1 + 4u_2v_2$

Proved

Compute $\langle u, v\rangle$

(b) Take

$u = (-3, 2)$

$v = (1, 7)$

$\langle u, \underline{v}\rangle = 9(-3)(1) + 4(2)(7)$

$= -27 + 56$

$= 29$

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**Q11** Find matrix

let $A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$

$\langle u, v\rangle = Au \cdot Av$

$= \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \cdot \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$

$= (au_1, bu_2) \cdot (av_1, bv_2)$

$= a^2\underline{u_1v_1} + b^2\underline{u_2v_2}$

But given

$\langle u, v\rangle = 3\underline{u_1v_1} + 5\underline{u_2v_2}$

comparing

$a^2 = 3, \quad b^2 = 5$

$a = \sqrt{3}$ & , $b = \sqrt{5}$

$\Rightarrow A = \begin{bmatrix} \sqrt{3} & 0 \\ 0 & \sqrt{5} \end{bmatrix}$.

Q12) $P = -2 + 3x + 2x^2$

$\|P\| = \sqrt{\langle P, P \rangle}$

$= \sqrt{(-2)(-2) + 3(3) + 2(2)}$

$= \sqrt{4 + 9 + 4}$

$= \sqrt{17}$

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Q14 $d(P, q) = ?$

$P = 3 - x + x^2$

$q = 2 + 5x^2$

$d(P, q) = \|P - q\|$

$= \sqrt{\langle P-q, P-q \rangle}$

$$\begin{array}{rl} P - q = & 3 - x + x^2 \\ & 2 \quad\;\; + 5x^2 \\ & - \qquad\;\; - \\ \hline = & 1 - x - 4x^2 \end{array}$$

$d(P, q) = \sqrt{(1)(1) + (-1)(-1) + (-4)(-4)}$

$= \sqrt{1 + 1 + 16}$

$= \sqrt{18}$

Q15:- 08

$d(A, B) = \|A - B\|$

$= \sqrt{\langle A-B, A-B \rangle}$

$= x_1 y_1 + x_2 y_2 + x_3 y_3 + \ldots$

$A - B = \begin{bmatrix} 2 & 6 \\ 9 & 4 \end{bmatrix} - \begin{bmatrix} -4 & 7 \\ 1 & 6 \end{bmatrix}$

$= \begin{bmatrix} 2+4 & 6-7 \\ 9-1 & 4-6 \end{bmatrix}$

$= \begin{array}{c} \;\; x_1 \quad x_2 \\ \begin{bmatrix} 6 & -1 \\ 8 & -2 \end{bmatrix} \\ \;\; x_3 \quad x_4 \end{array}$

&

$A - B = \begin{array}{c} \;\; y_1 \quad y_2 \\ \begin{bmatrix} 6 & -1 \\ 8 & -2 \end{bmatrix} \\ \;\; y_3 \quad y_4 \end{array}$

$d(A, B) = \sqrt{(6)(6) + (-1)(-1) + (8)(8) + (-2)(-2)}$

$= \sqrt{36 + 1 + 64 + 4}$

$= \sqrt{41 + 64} = \sqrt{105}$

Q20:-



$\langle u,v\rangle = 2$ , $\|u\|=1$
$\langle v,w\rangle = -3$ , $\|v\|=2$
$\langle u,w\rangle = 5$ , $\|w\|=7$

(a) $\langle u+v, v+w\rangle$

$= \langle u+v, v\rangle + \langle u+v, w\rangle$

$= \langle u,v\rangle + \langle v,v\rangle + \langle u,w\rangle + \langle v,w\rangle$

$= 2 + \|v\|^2 + 5 + (-3)$

$= 4 + (2)^2$

$= 4+4$

$= 8$

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(b) $\langle 2v-w, 3u+2w\rangle$

$= \langle 2v-w, 3u\rangle + \langle 2v-w, 2w\rangle$

$= \langle 2v, 3u\rangle + \langle -w, 3u\rangle + \langle 2v, 2w\rangle + \langle -w, 2w\rangle$

$= 6\langle v,u\rangle + (-1)(3)\langle w,u\rangle + 4\langle v,w\rangle + (-1)(2)\langle w,w\rangle$

$= (6)(2) - 3(5) + 4(-3) - 3\|w\|^2$

$= \cancel{12} - 15 - \cancel{12} - 3(7)^2$

$= -15 - 3(49)$

$= -15 - 147$

$= -162$

(f)

$\|u-2v+4w\|$

$= \sqrt{\langle u-2v+4w, u-2v+4w\rangle}$

Consider

$\langle u-2v+4w, u-2v+4w\rangle$

$= \langle u, u-2v+4w\rangle + \langle -2v, u-2v+4w\rangle + \langle 4w, u-2v+4w\rangle$

$= \langle u,u\rangle - 2\langle u,v\rangle + 4\langle u,w\rangle - 2\langle v,$ $+ (-2)(-2)\langle v,v\rangle$
$+ (-2)(4)\langle v,w\rangle$
$+ 4\langle w,u\rangle + (4)(-2)\langle w,v\rangle$
$+ (4)(4)\langle w,w\rangle$

$= \|u\|^2 - 2\langle u,v\rangle + 4\langle u,w\rangle - 2\langle v,u\rangle$
$+ 4\|v\|^2 - 8\langle v,w\rangle + 4\langle w,u\rangle$
$- 8\langle w,v\rangle + 16\|w\|^2$

putting values ?