4.11 Geometry of Matrix Operators on R^2

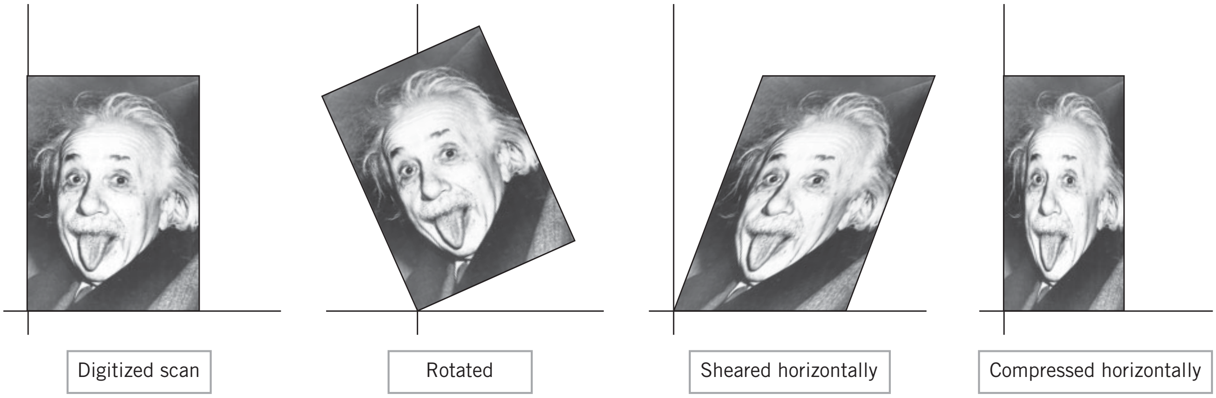
In applications such as computer graphics it is important to understand not only how linear operators on R^2 and R^3 affect individual vectors but also how they affect two-dimensional or three-dimensional *regions*. That is the focus of this section.

Transformations of Regions

Figure 4.11.1 shows a famous picture of Albert Einstein that has been transformed in various ways using matrix operators on R^2. The original image was scanned and then digitized to decompose it into a rectangular array of pixels. Those pixels were then transformed as follows:

- The program MATLAB was used to assign coordinates and a gray level to each pixel.
- The coordinates of the pixels were transformed by matrix multiplication.
- The pixels were then assigned their original gray levels to produce the transformed picture.

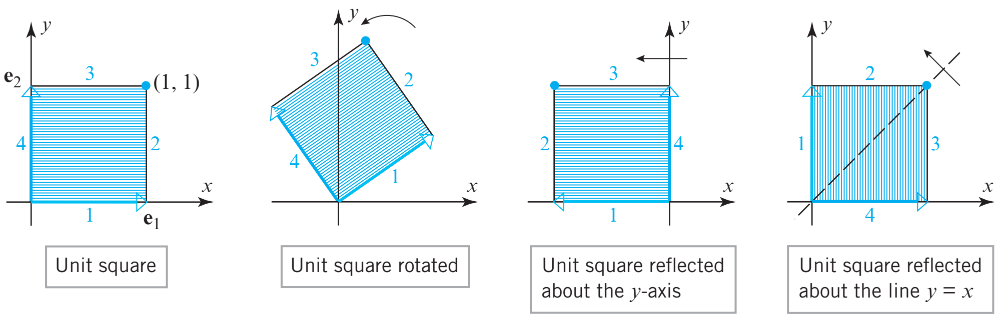
In computer games a perception of motion is created by using matrices to rapidly and repeatedly transform the arrays of pixels that form the visual images.



▲ Figure 4.11.1 [*Image: ARTHUR SASSE/AFP/Getty Images*]

Images of Lines Under Matrix Operators

The effect of a matrix operator on R^2 can often be deduced by studying how it transforms the points that form the unit square. The following theorem, which we state without proof, shows that if the operator is invertible, then it maps each line segment in the unit square into the line segment connecting the images of its endpoints. In particular, the edges of the unit square get mapped into edges of the image (see Figure 4.11.2 in which the edges of a unit square and the corresponding edges of its image have been numbered).



▲ Figure 4.11.2

THEOREM 4.11.1 *If $T: R^2 \to R^2$ is multiplication by an invertible matrix, then:*

(*a*) *The image of a straight line is a straight line.*

(*b*) *The image of a line through the origin is a line through the origin.*

(*c*) *The images of parallel lines are parallel lines.*

(*d*) *The image of the line segment joining points P and Q is the line segment joining the images of P and Q.*

(*e*) *The images of three points lie on a line if and only if the points themselves lie on a line.*

EXAMPLE 1 Image of a Line

According to Theorem 4.11.1, the invertible matrix

$$A = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$$

maps the line $y = 2x + 1$ into another line. Find its equation.

Solution Let (x, y) be a point on the line $y = 2x + 1$, and let (x', y') be its image under multiplication by A. Then

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}^{-1}\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} x' \\ y' \end{bmatrix}$$

so

$$\begin{aligned} x &= x' - y' \\ y &= -2x' + 3y' \end{aligned}$$

Substituting these expressions in $y = 2x + 1$ yields

$$-2x' + 3y' = 2(x' - y') + 1$$

or, equivalently,

$$y' = \tfrac{4}{5}x' + \tfrac{1}{5}$$

EXAMPLE 2 Transformation of the Unit Square

Sketch the image of the unit square under multiplication by the invertible matrix

$$A = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}$$

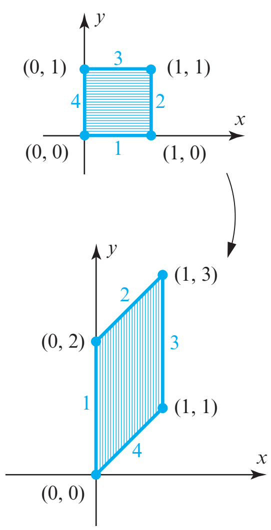
Label the vertices of the image with their coordinates, and number the edges of the unit square and their corresponding images (as in Figure 4.11.2).

Solution Since

$$\begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix},$$

$$\begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

the image of the unit square is a parallelogram with vertices (0, 0), (0, 2), (1, 1), and (1, 3) (Figure 4.11.3). ◀



▲ Figure 4.11.3

The next example illustrates a transformation of the unit square under a composition of matrix operators.

EXAMPLE 3 Transformation of the Unit Square

(a) Find the standard matrix for the operator on R^2 that first shears by a factor of 2 in the x-direction and then reflects the result about the line $y = x$. Sketch the image of the unit square under this operator.

(b) Find the standard matrix for the operator on R^2 that first reflects about $y = x$ and then shears by a factor of 2 in the x-direction. Sketch the image of the unit square under this operator.

(c) Confirm that the shear and the reflection in parts (a) and (b) do not commute.

Solution (a) The standard matrix for the shear is

$$A_1 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

and for the reflection is

$$A_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

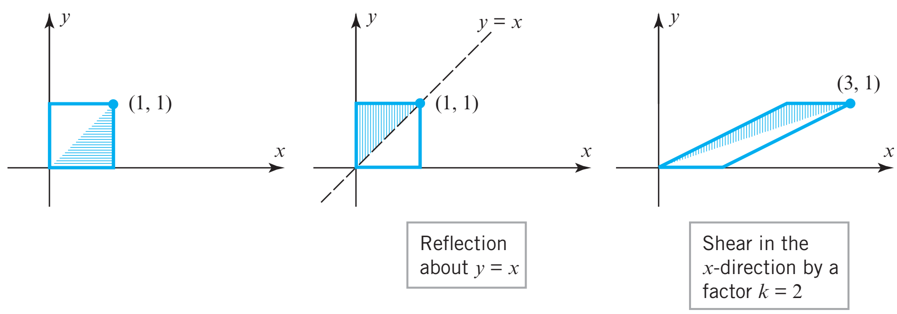
Thus, the standard matrix for the shear followed by the reflection is

$$A_2A_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}$$

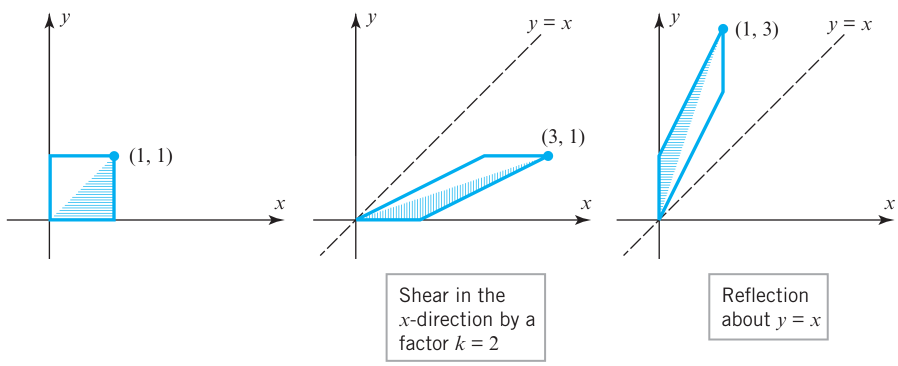
Solution (b) The standard matrix for the reflection followed by the shear is

$$A_1A_2 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$$

Solution (c) The computations in Solutions (*a*) and (*b*) show that $A_1A_2 \neq A_2A_1$, so the standard matrices, and hence the operators, do not commute. The same conclusion follows from Figures 4.11.4 and 4.11.5 since the two operators produce different images of the unit square. ◀



▶ Figure 4.11.4

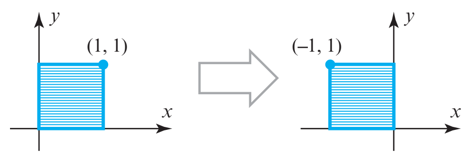
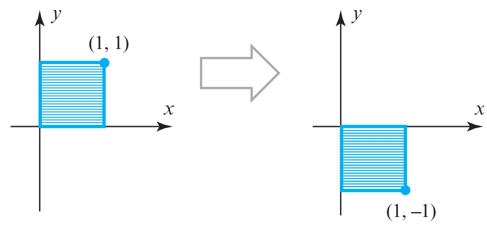
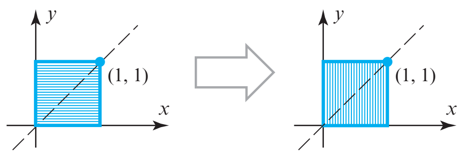
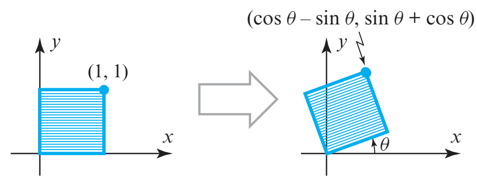
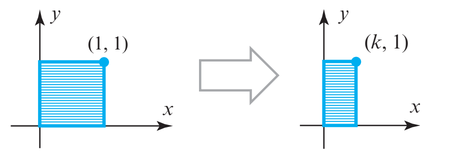
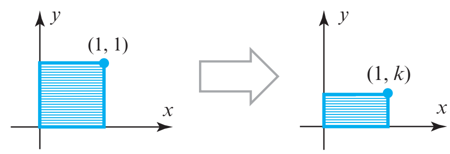
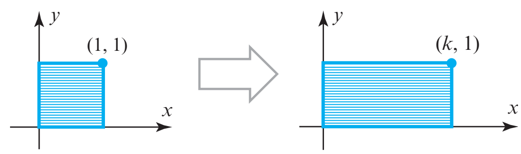


▶ Figure 4.11.5

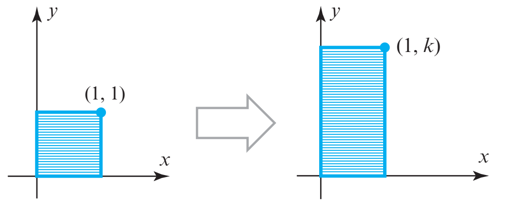
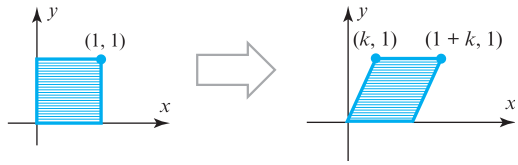
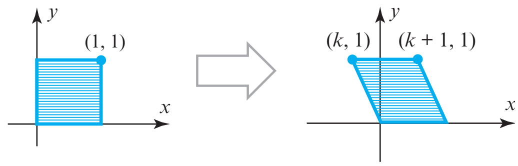
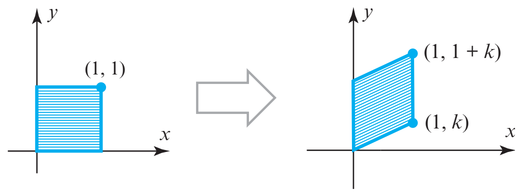
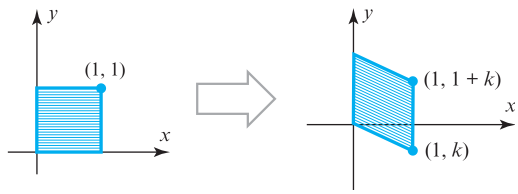
Geometry of Invertible Matrix Operators

In Example 3 we illustrated the effect on the unit square in R^2 of a composition of shears and reflections. Our next objective is to show how to decompose *any* 2×2 invertible matrix into a product of matrices in Table 1, thereby allowing us to analyze the geometric effect of a matrix operator in R^2 as a composition of simpler matrix operators. The next theorem is our first step in this direction.

Table 1

Operator	Standard Matrix	Effect on the Unit Square
Reflection about the y-axis	$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$	
Reflection about the x-axis	$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$	
Reflection about the line $y = x$	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	
Rotation about the origin through a positive angle θ	$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$	
Compression in the x-direction with factor k $(0 < k < 1)$	$\begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$	
Compression in the y-direction with factor k $(0 < k < 1)$	$\begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$	
Expansion in the x-direction with factor k $(k > 1)$	$\begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$	

(Continued on the following page.)

Operator	Standard Matrix	Effect on the Unit Square
Expansion in the y-direction with factor k $(k > 1)$	$\begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$	
Shear in the positive x-direction by a factor k $(k > 0)$	$\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$	
Shear in the negative x-direction by a factor k $(k < 0)$	$\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$	
Shear in the positive y-direction by a factor k $(k > 0)$	$\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$	
Shear in the negative y-direction by a factor k $(k < 0)$	$\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$	

THEOREM 4.11.2 *If E is an elementary matrtix, then $T_E\colon R^2 \to R^2$ is one of the following:*

(*a*) *A shear along a coordinate axis.*

(*b*) *A reflection about $y = x$.*

(*c*) *A compression along a coordinate axis.*

(*d*) *An expansion along a coordinate axis.*

(*e*) *A reflection about a coordinate axis.*

(*f*) *A compression or expansion along a coordinate axis followed by a reflection about a coordinate axis.*

Proof Because a 2×2 elementary matrix results from performing a single elementary row operation on the 2×2 identity matrix, such a matrix must have one of the following forms (verify):

$$\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$$

The first two matrices represent shears along coordinate axes, and the third represents a reflection about $y = x$. If $k > 0$, the last two matrices represent compressions or expansions along coordinate axes, depending on whether $0 \leq k < 1$ or $k > 1$. If $k < 0$, and if we express k in the form $k = -k_1$, where $k_1 > 0$, then the last two matrices can be written as

$$\begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -k_1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} k_1 & 0 \\ 0 & 1 \end{bmatrix} \tag{1}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -k_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & k_1 \end{bmatrix} \tag{2}$$

Since $k_1 > 0$, the product in (1) represents a compression or expansion along the x-axis followed by a reflection about the y-axis, and (2) represents a compression or expansion along the y-axis followed by a reflection about the x-axis. In the case where $k = -1$, transformations (1) and (2) are simply reflections about the y-axis and x-axis, respectively. ◀

We know from Theorem 4.10.2(d) that an invertible matrix can be expressed as a product of elementary matrices, so Theorem 4.11.2 implies the following result.

THEOREM 4.11.3 *If $T_A: R^2 \to R^2$ is multiplication by an invertible matrix A, then the geometric effect of T_A is the same as an appropriate succession of shears, compressions, expansions, and reflections.*

The next example will illustrate how Theorems 4.11.2 and 4.11.3 together with Table 1 can be used to analyze the geometric effect of multiplication by a 2×2 invertible matrix.

▶ EXAMPLE 4 Decomposing a Matrix Operator

In Example 2 we illustrated the effect on the unit square of multiplication by

$$A = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}$$

(see Figure 4.11.3). Express this matrix as a product of elementary matrices, and then describe the effect of multiplication by A in terms of shears, compressions, expansions, and reflections.

Solution The matrix A can be reduced to the identity matrix as follows:

$$\begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Interchange the first and second rows.

Multiply the first row by $\frac{1}{2}$.

Add $-\frac{1}{2}$ times the second row to the first.

These three successive row operations can be performed by multiplying A on the left successively by

$$E_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad E_2 = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & 1 \end{bmatrix}$$

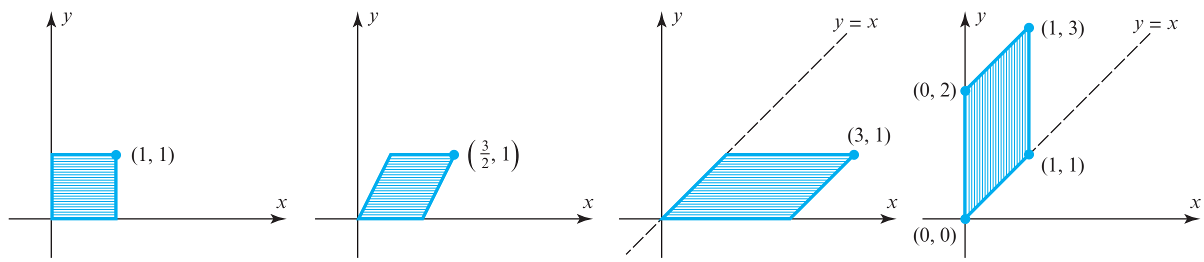
Inverting these matrices and using Formula (4) of Section 1.5 yields

$$A = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} = E_1^{-1}E_2^{-1}E_3^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{bmatrix}$$

Reading from right to left we can now see that the geometric effect of multiplying by A is equivalent to successively

1. shearing by a factor of $\frac{1}{2}$ in the x-direction;
2. expanding by a factor of 2 in the x-direction;
3. reflecting about the line $y = x$.

This is illustrated in Figure 4.11.6, whose end result agrees with that in Example 2. ◀



▲ Figure 4.11.6

▶ EXAMPLE 5 Transformations with Diagonal Matrices

Discuss the geometric effect on the unit square of multiplication by a diagonal matrix

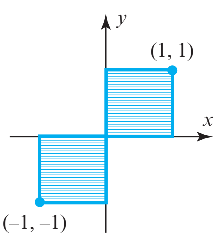
$$A = \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix}$$

in which the entries k_1 and k_2 are positive real numbers ($\neq 1$).

Solution The matrix A is invertible and can be expressed as

$$A = \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & k_2 \end{bmatrix}\begin{bmatrix} k_1 & 0 \\ 0 & 1 \end{bmatrix}$$

which show that multiplication by A causes a compression or expansion of the unit square by a factor of k_1 in the x-direction followed by an expansion or compression of the unit square by a factor of k_2 in the y-direction.



▲ Figure 4.11.7

▶ EXAMPLE 6 Reflection About the Origin

As illustrated in Figure 4.11.7, multiplication by the matrix

$$A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

has the geometric effect of reflecting the unit square about the origin. Note, however, that the matrix equation

$$A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

together with Table 1 shows that the same result can be obtained by first reflecting the unit square about the x-axis and then reflecting that result about the y-axis. You should be able to see this as well from Figure 4.11.7.

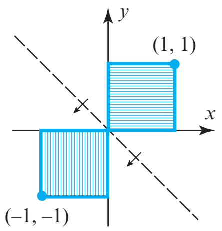


Figure 4.11.8

EXAMPLE 7 Reflection About the Line $y = -x$

We leave it for you to verify that multiplication by the matrix

$$A = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

reflects the unit square about the line $y = -x$ (Figure 4.11.8).

Exercise Set 4.11

1. Use the method of Example 1 to find an equation for the image of the line $y = 4x$ under multiplication by the matrix

$$A = \begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix}$$

2. Use the method of Example 1 to find an equation for the image of the line $y = -4x + 3$ under multiplication by the matrix

$$A = \begin{bmatrix} 4 & -3 \\ 3 & -2 \end{bmatrix}$$

In Exercises 3–4, find an equation for the image of the line $y = 2x$ that results from the stated transformation.

3. A shear by a factor 3 in the x-direction.

4. A compression with factor $\frac{1}{2}$ in the y-direction.

In Exercises 5–6, sketch the image of the unit square under multiplication by the given invertible matrix. As in Example 2, number the edges of the unit square and its image so it is clear how those edges correspond.

5. $\begin{bmatrix} 3 & -1 \\ 1 & -2 \end{bmatrix}$ **6.** $\begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}$

In each part of Exercises 7–8, find the standard matrix for a single operator that performs the stated succession of operations.

7. (a) Compresses by a factor of $\frac{1}{2}$ in the x-direction, then expands by a factor of 5 in the y-direction.

(b) Expands by a factor of 5 in the y-direction, then shears by a factor of 2 in the y-direction.

(c) Reflects about $y = x$, then rotates through an angle of 180° about the origin.

8. (a) Reflects about the y-axis, then expands by a factor of 5 in the x-direction, and then reflects about $y = x$.

(b) Rotates through 30° about the origin, then shears by a factor of -2 in the y-direction, and then expands by a factor of 3 in the y-direction.

In each part of Exercises 9–10, determine whether the stated operators commute.

9. (a) A reflection about the x-axis and a compression in the x-direction with factor $\frac{1}{3}$.

(b) A reflection about the line $y = x$ and an expansion in the x-direction with factor 2.

10. (a) A shear in the y-direction by a factor $\frac{1}{4}$ and a shear in the y-direction by a factor $\frac{3}{5}$.

(b) A shear in the y-direction by a factor $\frac{1}{4}$ and a shear in the x-direction by a factor $\frac{3}{5}$.

In Exercises 11–14, express the matrix as a product of elementary matrices, and then describe the effect of multiplication by A in terms of shears, compressions, expansions, and reflections.

11. $A = \begin{bmatrix} 4 & 4 \\ 0 & -2 \end{bmatrix}$ **12.** $A = \begin{bmatrix} 1 & 4 \\ 2 & 9 \end{bmatrix}$

13. $A = \begin{bmatrix} 0 & -2 \\ 4 & 0 \end{bmatrix}$ **14.** $A = \begin{bmatrix} 1 & -3 \\ 4 & 6 \end{bmatrix}$

In each part of Exercises 15–16, describe, in words, the effect on the unit square of multiplication by the given diagonal matrix.

15. (a) $A = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$ (b) $A = \begin{bmatrix} 1 & 0 \\ 0 & -5 \end{bmatrix}$

16. (a) $A = \begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix}$ (b) $A = \begin{bmatrix} -3 & 0 \\ 0 & -1 \end{bmatrix}$

17. (a) Show that multiplication by
$$A = \begin{bmatrix} 3 & 1 \\ 6 & 2 \end{bmatrix}$$
maps each point in the plane onto the line $y = 2x$.

(b) It follows from part (a) that the noncollinear points $(1, 0)$, $(0, 1)$, $(-1, 0)$ are mapped onto a line. Does this violate part (*e*) of Theorem 4.11.1?

18. Find the matrix for a shear in the x-direction that transforms the triangle with vertices $(0, 0)$, $(2, 1)$, and $(3, 0)$ into a right triangle with the right angle at the origin.

19. In accordance with part (*c*) of Theorem 4.11.1, show that multiplication by the invertible matrix
$$A = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$$
maps the parallel lines $y = 3x + 1$ and $y = 3x - 2$ into parallel lines.

20. Draw a figure that shows the image of the triangle with vertices $(0, 0)$, $(1, 0)$, and $(0.5, 1)$ under a shear by a factor of 2 in the x-direction.

21. (a) Draw a figure that shows the image of the triangle with vertices $(0, 0)$, $(1, 0)$, and $(0.5, 1)$ under multiplication by
$$A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

(b) Find a succession of shears, compressions, expansions, and reflections that produces the same image.

22. Find the endpoints of the line segment that results when the line segment from $P(1, 2)$ to $Q(3, 4)$ is transformed by

(a) a compression with factor $\frac{1}{2}$ in the y-direction.

(b) a rotation of $30°$ about the origin.

23. Draw a figure showing the italicized letter "*T*" that results when the letter in the accompanying figure is sheared by a factor $\frac{1}{4}$ in the x-direction.

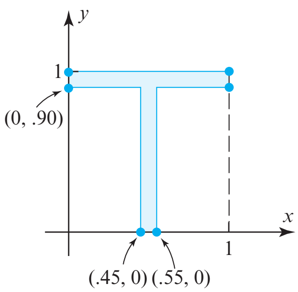


Figure Ex-23

24. Can an invertible matrix operator on R^2 map a square region into a triangular region? Justify your answer.

25. Find the image of the triangle with vertices $(0, 0)$, $(1, 1)$, $(2, 0)$ under multiplication by
$$A = \begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix}$$
Does your answer violate part (*e*) of Theorem 4.11.1? Explain.

26. In R^3 the ***shear in the xy-direction by a factor k*** is the matrix transformation that moves each point (x, y, z) parallel to the xy-plane to the new position $(x + kz, y + kz, z)$. (See the accompanying figure.)

(a) Find the standard matrix for the shear in the xy-direction by a factor k.

(b) How would you define the shear in the xz-direction by a factor k and the shear in the yz-direction by a factor k? What are the standard matrices for these matrix transformations?

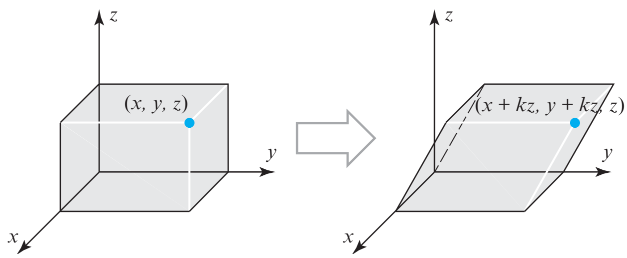


Figure Ex-26

Working with Proofs

27. Prove part (*a*) of Theorem 4.11.1. [*Hint:* A line in the plane has an equation of the form $Ax + By + C = 0$, where A and B are not both zero. Use the method of Example 1 to show that the image of this line under multiplication by the invertible matrix
$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$
has the equation $A'x + B'y + C = 0$, where
$$A' = (dA - cB)/(ad - bc)$$
and
$$B' = (-bA + aB)/(ad - bc)$$
Then show that A' and B' are not both zero to conclude that the image is a line.]

28. Use the hint in Exercise 27 to prove parts (*b*) and (*c*) of Theorem 4.11.1.

True-False Exercises

TF. In parts (a)–(g) determine whether the statement is true or false, and justify your answer.

(a) The image of the unit square under a one-to-one matrix operator is a square.

(b) A 2×2 invertible matrix operator has the geometric effect of a succession of shears, compressions, expansions, and reflections.

(c) The image of a line under an invertible matrix operator is a line.

(d) Every reflection operator on R^2 is its own inverse.

(e) The matrix $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ represents reflection about a line.

(f) The matrix $\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$ represents a shear.

(g) The matrix $\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$ represents an expansion.

Chapter 4 Supplementary Exercises

1. Let V be the set of all ordered triples of real numbers, and consider the following addition and scalar multiplication operations on $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$:

$$\mathbf{u} + \mathbf{v} = (u_1 + v_1, u_2 + v_2, u_3 + v_3), \quad k\mathbf{u} = (ku_1, 0, 0)$$

(a) Compute $\mathbf{u} + \mathbf{v}$ and $k\mathbf{u}$ for $\mathbf{u} = (3, -2, 4)$, $\mathbf{v} = (1, 5, -2)$, and $k = -1$.

(b) In words, explain why V is closed under addition and scalar multiplication.

(c) Since the addition operation on V is the standard addition operation on R^3, certain vector space axioms hold for V because they are known to hold for R^3. Which axioms in Definition 1 of Section 4.1 are they?

(d) Show that Axioms 7, 8, and 9 hold.

(e) Show that Axiom 10 fails for the given operations.

2. In each part, the solution space of the system is a subspace of R^3 and so must be a line through the origin, a plane through the origin, all of R^3, or the origin only. For each system, determine which is the case. If the subspace is a plane, find an equation for it, and if it is a line, find parametric equations.

(a) $0x + 0y + 0z = 0$

(b) $\begin{aligned} 2x - 3y + z &= 0 \\ 6x - 9y + 3z &= 0 \\ -4x + 6y - 2z &= 0 \end{aligned}$

(c) $\begin{aligned} x - 2y + 7z &= 0 \\ -4x + 8y + 5z &= 0 \\ 2x - 4y + 3z &= 0 \end{aligned}$

(d) $\begin{aligned} x + 4y + 8z &= 0 \\ 2x + 5y + 6z &= 0 \\ 3x + y - 4z &= 0 \end{aligned}$

3. For what values of s is the solution space of

$$\begin{aligned} x_1 + x_2 + sx_3 &= 0 \\ x_1 + sx_2 + x_3 &= 0 \\ sx_1 + x_2 + x_3 &= 0 \end{aligned}$$

the origin only, a line through the origin, a plane through the origin, or all of R^3?

4. (a) Express $(4a, a - b, a + 2b)$ as a linear combination of $(4, 1, 1)$ and $(0, -1, 2)$.

(b) Express $(3a + b + 3c, -a + 4b - c, 2a + b + 2c)$ as a linear combination of $(3, -1, 2)$ and $(1, 4, 1)$.

(c) Express $(2a - b + 4c, 3a - c, 4b + c)$ as a linear combination of three nonzero vectors.

5. Let W be the space spanned by $\mathbf{f} = \sin x$ and $\mathbf{g} = \cos x$.

(a) Show that for any value of θ, $\mathbf{f}_1 = \sin(x + \theta)$ and $\mathbf{g}_1 = \cos(x + \theta)$ are vectors in W.

(b) Show that $\mathbf{f}_1$ and $\mathbf{g}_1$ form a basis for W.

6. (a) Express $\mathbf{v} = (1, 1)$ as a linear combination of $\mathbf{v}_1 = (1, -1)$, $\mathbf{v}_2 = (3, 0)$, and $\mathbf{v}_3 = (2, 1)$ in two different ways.

(b) Explain why this does not violate Theorem 4.4.1.

7. Let A be an $n \times n$ matrix, and let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ be linearly independent vectors in R^n expressed as $n \times 1$ matrices. What must be true about A for $A\mathbf{v}_1, A\mathbf{v}_2, \ldots, A\mathbf{v}_n$ to be linearly independent?

8. Must a basis for P_n contain a polynomial of degree k for each $k = 0, 1, 2, \ldots, n$? Justify your answer.

9. For the purpose of this exercise, let us define a "checkerboard matrix" to be a square matrix $A = [a_{ij}]$ such that

$$a_{ij} = \begin{cases} 1 & \text{if } i + j \text{ is even} \\ 0 & \text{if } i + j \text{ is odd} \end{cases}$$

Find the rank and nullity of the following checkerboard matrices.

(a) The 3×3 checkerboard matrix.

(b) The 4×4 checkerboard matrix.

(c) The $n \times n$ checkerboard matrix.

10. For the purpose of this exercise, let us define an "X-matrix" to be a square matrix with an odd number of rows and columns that has 0's everywhere except on the two diagonals where it has 1's. Find the rank and nullity of the following X-matrices.

(a) $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix}$

(c) the X-matrix of size $(2n + 1) \times (2n + 1)$

11. In each part, show that the stated set of polynomials is a subspace of P_n and find a basis for it.

(a) All polynomials in P_n such that $p(-x) = p(x)$.

(b) All polynomials in P_n such that $p(0) = p(1)$.

12. (***Calculus required***) Show that the set of all polynomials in P_n that have a horizontal tangent at $x = 0$ is a subspace of P_n. Find a basis for this subspace.

13. (a) Find a basis for the vector space of all 3×3 symmetric matrices.

(b) Find a basis for the vector space of all 3×3 skew-symmetric matrices.

14. Various advanced texts in linear algebra prove the following determinant criterion for rank: *The rank of a matrix A is r if and only if A has some $r \times r$ submatrix with a nonzero determinant, and all square submatrices of larger size have determinant zero.* [*Note:* A submatrix of A is any matrix obtained by deleting rows or columns of A. The matrix A itself is also considered to be a submatrix of A.] In each part, use this criterion to find the rank of the matrix.

(a) $\begin{bmatrix} 1 & 2 & 0 \\ 2 & 4 & -1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 & 1 \\ 2 & -1 & 3 \\ 3 & -1 & 4 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & -1 & 2 & 0 \\ 3 & 1 & 0 & 0 \\ -1 & 2 & 4 & 0 \end{bmatrix}$

15. Use the result in Exercise 14 above to find the possible ranks for matrices of the form

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & a_{16} \\ 0 & 0 & 0 & 0 & 0 & a_{26} \\ 0 & 0 & 0 & 0 & 0 & a_{36} \\ 0 & 0 & 0 & 0 & 0 & a_{46} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & a_{56} \end{bmatrix}$$

16. Prove: If S is a basis for a vector space V, then for any vectors $\mathbf{u}$ and $\mathbf{v}$ in V and any scalar k, the following relationships hold.

(a) $(\mathbf{u} + \mathbf{v})_S = (\mathbf{u})_S + (\mathbf{v})_S$ (b) $(k\mathbf{u})_S = k(\mathbf{u})_S$

17. Let D_k, R_θ, and S_k be a dilation of R^2 with factor k, a counterclockwise rotation about the origin of R^2 through an angle θ, and a shear of R^2 by a factor k, respectively.

(a) Do D_k and R_θ commute?

(b) Do R_θ and S_k commute?

(c) Do D_k and S_k commute?

18. A vector space V is said to be the ***direct sum*** of its subspaces U and W, written $V = U \oplus W$, if every vector in V can be expressed in exactly one way as $\mathbf{v} = \mathbf{u} + \mathbf{w}$, where $\mathbf{u}$ is a vector in U and $\mathbf{w}$ is a vector in W.

(a) Prove that $V = U \oplus W$ if and only if every vector in V is the sum of some vector in U and some vector in W and $U \cap W = \{\mathbf{0}\}$.

(b) Let U be the xy-plane and W the z-axis in R^3. Is it true that $R^3 = U \oplus W$? Explain.

(c) Let U be the xy-plane and W the yz-plane in R^3. Can every vector in R^3 be expressed as the sum of a vector in U and a vector in W? Is it true that $R^3 = U \oplus W$? Explain.