Linear Transformation:-

Let V and W be two vector spaces. A function $T: V \to W$ is called linear transformation from V to W if

(i) $T(ku) = kT(u)$, $k \in R$, $u \in V$

(ii) $T(u+v) = T(u) + T(v)$, $u, v \in V$

Note: (i) If 'T' is a LT then $T(0) = 0$

(1) Is linear $T: V \to R$ s.t $T(u) = \|u\|$?

(i) To prove $T(ku) = kT(u)$

(Check $T(-u) = -1T(u)$?

Take $k = -1$

$$T(-u) = \|-u\|$$
$$= \|u\|$$
$$= T(u)$$

we noted that $T(-u) \neq (-1)T(u)$

$\Rightarrow$ T is not Linear Transformation.

Q 2 $T: R^3 \longrightarrow R^3$

$T(U) = V \times V_0$, $V_0 \in R^3$ fixed vector

① $T(\alpha \underline{U}) = \alpha \underline{U} \times V_0$

$= \alpha (\underline{U} \times V_0)$

$= \alpha T(\underline{U})$

② $T(\underline{U} + \underline{V}) = (\underline{U} + \underline{V}) \times V_0$

$= \underline{U} \times V_0 + \underline{V} \times V_0$

$= T(\underline{U}) + T(\underline{V})$

Yes Liner Transformation.

Q6 $T: M_{22} \longrightarrow R$

(a) $T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = 3a - 4b + c - d$

$T(\alpha A) = \alpha T(A)$

① $T\left(\alpha \overset{A}{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}\right)$

$= T\begin{bmatrix} \alpha a & \alpha b \\ \alpha c & \alpha d \end{bmatrix}$

$= 3(\alpha a) - 4(\alpha b) + (\alpha c) - (\alpha d)$

$= \alpha(3a) - \alpha(4b) + \alpha(c) - \alpha(d)$

$= \alpha(3a - 4b + c - d)$

$= \alpha T\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$\alpha T(A)$

② $T(A+B) = T(A) + T(B)$

$T\left(\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} + \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}\right)$

$= T\begin{bmatrix} a_1 + a_2 & b_1 + b_2 \\ c_1 + c_2 & d_1 + d_2 \end{bmatrix}$

$= 3(a_1 + a_2) - 4(b_1 + b_2) + (c_1 + c_2) - (d_1 + d_2)$

$= 3a_1 + 3a_2 - 4b_1 - 4b_2 + c_1 + c_2 - d_1 - d_2$

$= (3a_1 - 4b_1 + c_1 - d_1) + (3a_2 - 4b_2 + c_2 - d_2)$

$= T\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} + T\begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}$

Yes

(b)

$$T\begin{bmatrix} a & b \\ c & d \end{bmatrix} = a^2 + b^2$$

(1) $T\left[\alpha\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right] = T\begin{bmatrix} \alpha a & \alpha b \\ \alpha c & \alpha d \end{bmatrix}$

$= (\alpha a)^2 + (\alpha b)^2$

$= \alpha^2 a^2 + \alpha^2 b^2$

$= \alpha^2 (a^2 + b^2)$

$= \alpha^2 T\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

Non Linear Transformation.

$T(\alpha A) = \alpha T(A)$

but here

$T(\alpha A) = \alpha^2 T(A)$

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(7) $T(a_0 + a_1 x + a_2 x^2) = a_0 + a_1(x+1) + a_2(x+1)^2$

(i) $T[\alpha(a_0 + a_1 x + a_2 x^2)] = T[\alpha a_0 + \alpha a_1 x + \alpha a_2 x^2]$

$= (\alpha a_0) + (\alpha a_1)(x+1) + (\alpha a_2)(x+1)^2$

$T(\alpha p) = \alpha T(p)$

$= \alpha a_0 + \alpha(a_1(x+1)) + \alpha(a_2(x+1)^2)$

$= \alpha[a_0 + a_1(x+1) + a_2(x+1)^2]$

$= \alpha T(a_0 + a_1 x + a_2 x^2)$ OK

(2) $T[(a_0 + a_1 x + a_2 x^2) + (b_0 + b_1 x + b_2 x^2)] = T[(a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2]$

$= (a_0 + b_0) + (a_1 + b_1)(x+1) + (a_2 + b_2)(x+1)^2$

$T(P_1 + P_2)$
$= T(P_1) + T(P_2)$

$= (a_0 + a_1(x+1) + a_2(x+1)^2) + (b_0 + b_1(x+1) + b_2(x+1)^2)$

$= T(a_0 + a_1 x + a_2 x^2) + T(b_0 + b_1 x + b_2 x^2)$

Q8

(a) $T : F(-\infty,\infty) \longrightarrow F(-\infty,\infty)$

$T(f(x)) = 1 + f(x)$

Consider

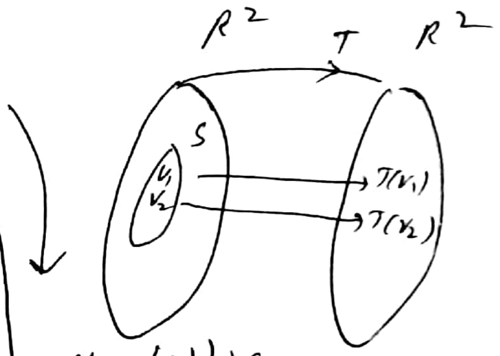
$$T(\alpha f) = T((\alpha f)(x))$$
$$= 1 + (\alpha f)(x)$$
$$\neq \alpha[1 + f(x)]$$
$$\neq \alpha T(f)$$

$\alpha[1+f(x)]$
$\alpha T(f)$

Non-linear.

Q9:-

Since $S = \{v_1, v_2\}$ is <u>basis</u> for R^2



$v_1 = (1,1)$ ✓
$v_2 = (1,0)$ ✓
$T(v_1) = (1,-2)$ ✓
$T(v_2) = (-4,1)$ ✓

Basis
① Lin. indp
② Lin. combinat

Therefore, for $\underline{u}(x,y) \in R^2$

<u>linear combination</u>

$$\underline{u} = K_1 V_1 + K_2 V_2$$
$$\Rightarrow K_1(1,1) + K_2(1,0) = (x,y)$$
$$(K_1, K_1) + (K_2, 0K_2) = (x,y)$$
$$(K_1 + K_2, K_1 + 0K_2) = (x,y)$$

$K_1 + K_2 = x$
$K_1 + 0K_2 = y \Rightarrow \boxed{K_1 = y}$

$\boxed{K_2 = x - y}$

So, $(x,y) = y(1,1) + (x-y)(1,0)$
$= y v_1 + (x-y) v_2$
$= \cancel{(y,y) + (x-y, 0)}$
$= \cancel{(y + x - y, y + 0)}$

operating T

$$T(x,y) = yT(v_1) + (x-y)T(v_2)$$
$$= y(1,-2) + (x-y)(-4,1)$$
$$= (y, -2y) + (-4(x-y), (x-y))$$
$$= (y - 4x + 4y, -2y + x - y)$$
$$= (-4x + 5y, x - 3y)$$

$$T(5,-3) = (-4(5) + 5(-3), 5 - 3(-3))$$
$$= (-20 - 15, 5 + 9)$$
$$= (-35, 14)$$

(10) $S = \{v_1, v_2\}$ is basis for R^2 (5)

$v_1 = (-2,1)$ $T(v_1) = (-1,2,0)$

$v_2 = (1,3)$ $T(v_2) = (0,-3,5)$ $T: R^2 \to R^3$ is Linear Transformation

$\underline{T(x_1,x_2) = ?}$

Since S is basis for R^2

let $u = (x_1, x_2) \in R^2$ by by property of linear combination

$$k_1 v_1 + k_2 v_2 = u$$

$$k_1(-2,1) + k_2(1,3) = (x_1,x_2)$$

$$(-2k_1, k_1) + (k_2, 3k_2) = (x_1,x_2)$$

$$\begin{aligned} -2k_1 + k_2 &= x_1 \\ k_1 + 3k_2 &= x_2 \end{aligned} \Rightarrow \begin{aligned} k_1 + 3k_2 &= x_2 \\ -2k_1 + k_2 &= x_1 \end{aligned} \quad \left[\begin{array}{cc|c} 1 & 3 & x_2 \\ -2 & 1 & x_1 \end{array}\right]$$

$$= \left[\begin{array}{cc|c} 1 & 3 & x_2 \\ 0 & 7 & x_1+2x_2 \end{array}\right] \Rightarrow \left[\begin{array}{cc|c} 1 & 3 & x_2 \\ 0 & 1 & \frac{x_1+2x_2}{7} \end{array}\right]$$

$\Rightarrow$ $k_1 + 3k_2 = x_2$ & $\boxed{k_2 = \frac{x_1+2x_2}{7}}$ $1 - \frac{6}{7} = \frac{7-6}{7} = \frac{1}{7}$

$$k_1 + 3\left(\frac{x_1+2x_2}{7}\right) = x_2$$

$$k_1 = x_2 - \frac{3}{7}x_1 - \frac{6}{7}x_2 = -\frac{3}{7}x_1 + \frac{1}{7}x_2 = \boxed{\frac{-3x_1+x_2}{7} = k_1}$$

$$\Rightarrow \quad u = \frac{-3x_1+x_2}{7}(v_1) + \frac{x_1+2x_2}{7}(v_2)$$

$$\Rightarrow \quad T(u) = \frac{-3x_1+x_2}{7}T(v_1) + \left(\frac{x_1+2x_2}{7}\right)T(v_2)$$

$$= \frac{-3x_1+x_2}{7}(-1,2,0) + \left(\frac{x_1+2x_2}{7}\right)(0,-3,5)$$

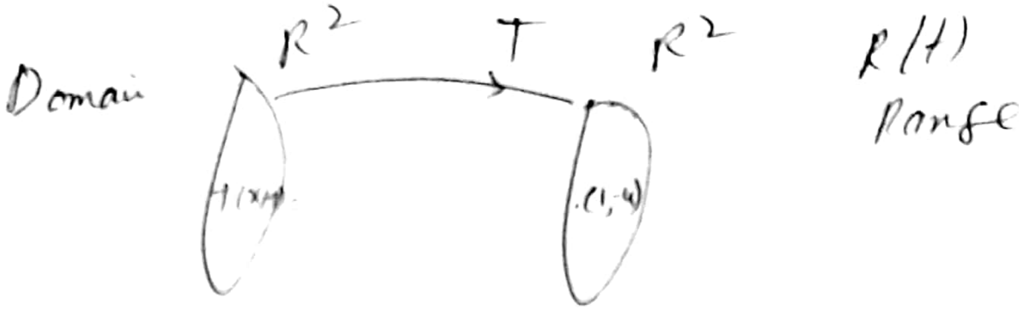
$$= \left(-1\left(\frac{-3x_1+x_2}{7}\right)+0,\ 2\left(\frac{-3x_1+x_2}{7}\right) - 3\left(\frac{x_1+2x_2}{7}\right),\ 0 + 5\left(\frac{x_1+2x_2}{7}\right)\right)$$

$$T(x_1,x_2) = \left(\frac{3x_1-x_2}{7}, \frac{-9x_1-4x_2}{7}, \frac{5x_1+10x_2}{7}\right)$$

$$T(2,-3) = \left(\frac{9}{7}, -\frac{6}{7}, \frac{-20}{7}\right)$$

14

6



Given $T(x,y) = (2x-y, -8x+4y) \longrightarrow (1)$

Assume $T(x,y) = (1,-4)$

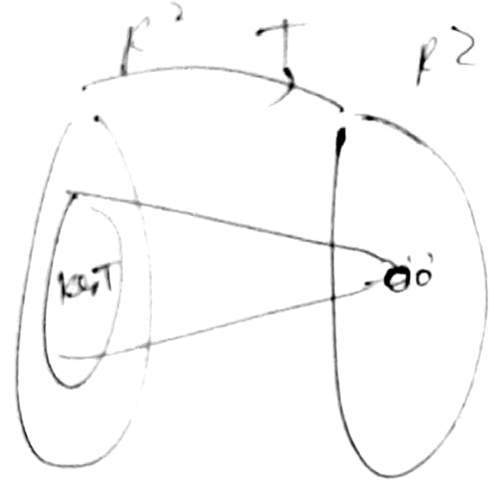
$\Rightarrow$

$2x - y = 1 \Rightarrow 2x - y = 1$

$-8x + 4y = -4 \Rightarrow 2x - y = 1$

$$\left[\begin{array}{cc|c} 2 & -1 & 1 \\ 2 & -1 & 1 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 2 & -1 & 1 \\ 0 & 0 & 0 \end{array}\right]$$



$2x - y = 0$ Take $y = t$

$2x - t = 0$

$2x = t \Rightarrow x = t/2$ Yes

$(x,y) = (t/2, t)$ Exists

Q15 Is $(5,10) \in \text{KerT}$

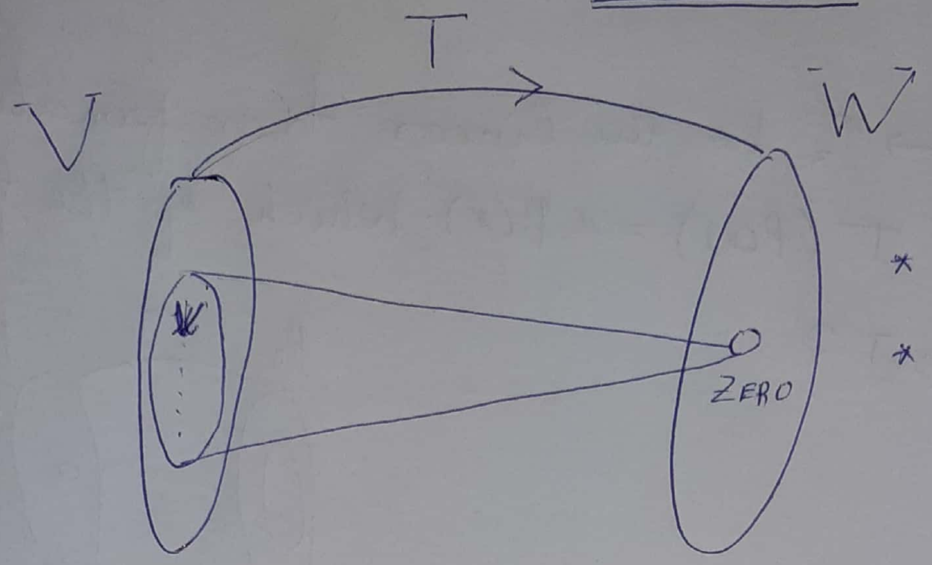
$T(5,10) = (2(5)-10, -8(5)+4(10))$

$= (10-10, -40+40)$

$= (0,0) = \underline{0}$

$\Rightarrow (5,10) \in \text{KerT}$ Yes

Exercise 8.1



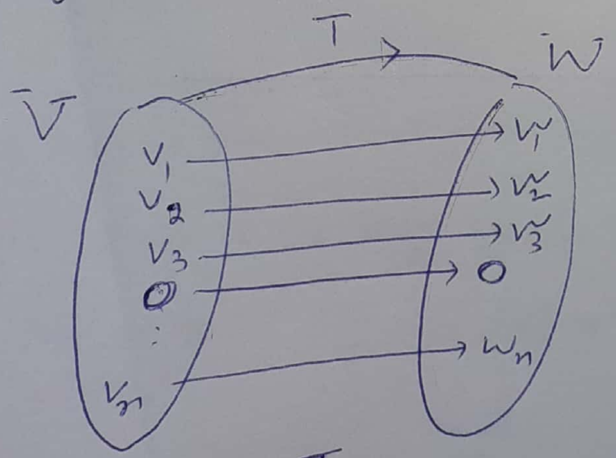
* Domain $T = V$
* Range $T \subseteq W$

Kernel of T:-

$$\text{Ker } T = \{ v \in V : T(v) = 0 \} \quad ; \quad \text{Ker } T \subseteq V$$

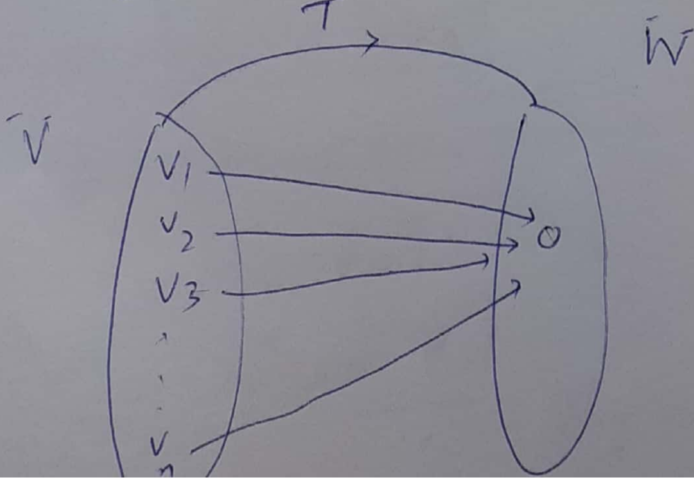
① If $T(v) = v \quad \forall \ v \in V \Rightarrow \text{Ker } T = \{0\}$

② If $T(v) = 0 \quad \forall \ v \in V \Rightarrow \text{Ker } T = V$



Range $T = W$

$\text{Ker } T = \{0\}$

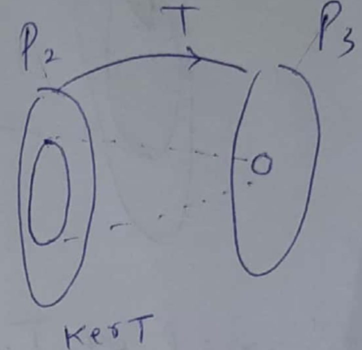


Range $T = \{0\}$

$\text{Ker } T = V$

Q_{10}:-

Let $T: P_2 \longrightarrow P_3$ be the linear transformation defined by $T(P(x)) = x\,P(x)$. Which of the following are in Ker T?



(a) x^2

Since

$$T(P(x)) = x\,P(x)$$

Put $P(x) = x^2$

$$T(P(x)) = T(x^2)$$
$$= x(x^2)$$
$$= x^3 \implies T(x^2) = x^3$$
$$\implies x^2 \notin \text{Ker } T$$

(b) $P(x) = 0$

$$T(P(x)) = x\,P(x)$$
$$T(0) = x(0)$$
$$T(0) = 0$$

$\implies 0 \in \text{Ker } T$

(c) $p(x) = 1 + x$

$T(p(x)) = x\,p(x)$

$T(1+x) = x(1+x)$

$= x + x^2 \neq 0$

$\Rightarrow 1 + x \notin \ker T$

(d) $p(x) = -x$

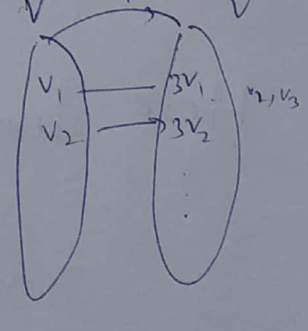
$T(p(x)) = x\,p(x)$

$T(-x) = x(-x)$

$= -x^2 \neq 0$

$\Rightarrow -x \notin \ker T$

Q_{12}:- Let $\bar{V}$ be any vector space and let $T: V \to V$ be defined by $T(v) = 3v$



(a) What is $\ker T$

$\because$ for $0 \in \bar{V}$, $T(0) = 3(0) = 0$

$\Rightarrow \ker T = \{0\}$

(b) What is Range of T: Since $T(v) = 3v$

Range $T = 3\bar{V}$

Dimension theorem for linear Transformation (04)

$\operatorname{rank}(T) + \operatorname{nullity}(T) = \dim(V)$

Q_{13}:- In each part, find the nullity of the linear transformation

(a) $T: R^5 \longrightarrow P_5$ has rank 3

Here $\dim V = \dim(R^5)$

$= 5$

and rank $= 3$

So, $\operatorname{Nullity}(T) = \dim(V) - \operatorname{rank}(T)$

$= 5 - 3$

$= 2$

R^n

$\underline{\underline{n}}$

(b) $T: P_4 \longrightarrow P_3$ has rank 1

Here $\dim(V) = \dim(P_4)$

$= 5$

$\operatorname{rank}(T) = 1$

$\operatorname{nullity}(T) = 5 - 1$

$= 4$

P_n

$\{a, x, x^2, x^3, x^4\}$

$\underline{\underline{n+1}}$

(b

=

$\Rightarrow$

(c) The range of $T: M_{mn} \longrightarrow R^3$ is R^3

$dim(V) = dim(M_{mn})$

$= mn$

M_{mn}

M_{22}

$= \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ (4)

$Range(T) = R^3$

$Dim(Range) = Dim(R^3)$

$= 3$

$= Rank(T)$

Formula

Nullity of T = $dim(V) - Rank\ T$

$= mn - 3$

(d) $T: M_{22} \longrightarrow M_{22}$ has rank 3

$dim(V) = dim(M_{22}) = 2 \times 2$

$= 4$

Nullity of $T = dim(V) - Rank\ T$

$= 4 - 3$

$= 1$

Q_{23} (10th Edi)

For $A = \begin{bmatrix} 1 & -1 & 3 \\ 5 & 6 & -4 \\ 7 & 4 & 2 \end{bmatrix}$, let T be

multiplication by the matrix A. Find,

(a) Basis for the range of T
(b) Basis for the Kernel of T
(c) The Rank and Nullity of T
(d) The Rank and Nullity of A.

Reducing into Echelon form

$$\begin{bmatrix} 1 & -1 & 3 \\ 5 & 6 & -4 \\ 7 & 4 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 3 \\ 0 & 11 & -19 \\ 0 & 11 & -19 \end{bmatrix} \quad \begin{matrix} R_2 - 5R_1 \\ R_3 - 7R_1 \end{matrix}$$

$$\begin{bmatrix} 1 & -1 & 3 \\ 0 & 1 & -\frac{19}{11} \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{matrix} R_3 - R_2 \\ \frac{R_2}{11} \end{matrix}$$

(a)
Bases for Column space $= \left\{ \begin{bmatrix} 1 \\ 5 \\ 7 \end{bmatrix}, \begin{bmatrix} -1 \\ 6 \\ 4 \end{bmatrix} \right\}$

Basis for Rage of $T = \left\{ \begin{bmatrix} 1 \\ 5 \\ 7 \end{bmatrix}, \begin{bmatrix} -1 \\ 6 \\ 4 \end{bmatrix} \right\}$

(b)

$$\begin{bmatrix} 1 & -1 & 3 \\ 0 & 1 & -\frac{19}{11} \\ 0 & 0 & 0 \end{bmatrix} \qquad AX = 0$$

leading variables; x_1 & x_2

Free variable; x_3

$$x_1 - x_2 + 3x_3 = 0$$
$$x_2 - \frac{19}{11}x_3 = 0$$

Put $x_3 = t$

$$x_2 = \frac{19}{11}t$$
$$x_1 = \frac{19}{11}t - \frac{3t}{1} = \frac{-14}{11}t$$

$$S.S = \{(x_1, x_2, x_3) ; x_1, x_2, x_3 \in \mathbb{R}\}$$

$$S.\text{space} = \left\{t\left(-\frac{14}{11}, \frac{19}{11}, 1\right) ; t \in \mathbb{R}\right\} = \text{Null space}$$

$$\text{Bases}_N = \left\{\left(-\frac{14}{11}, \frac{19}{11}, 1\right)\right\} = \text{Bases of Ker } T$$

(c)

Dimension of Null space $= 1 = \text{Dim}(\text{Ker } T)$

Nullity of $T = \underline{1}$

&

$\text{Dim}(R(T)) = 2$

$\text{Rank}(T) = 2$

$\left(\because \text{(a)} \quad \text{Basis for } R(T) = \left\{\begin{bmatrix} 1 \\ 5 \\ 7 \end{bmatrix}, \begin{bmatrix} -1 \\ 6 \\ 4 \end{bmatrix}\right\} \text{ Colum space Basis}\right)$

(d) Dim of Column space of $A = 2$

Rank $A = 2$

Dim of Null space of $A = 1$

Nullity of $A = 1$