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Probability (Advanced):

(x1. Binomial x)

1. Bernoulli Distribution:

Consider an experiment with two possible outcomes; call them succes (s) and failure (f) [Alive or dead, sweet or sour, Defective or not-defective, solved or unsolved] with probabilities :-

$$P(s) = p \quad \text{and} \quad P(f) = 1 - p = q$$

Let x be a discrete random variable takes value 0 if failure occur and 1 if succes occurs with probabilities,

$$P(X=0) = 1-p = q$$
$$P(X=1) = P$$

The probability distribution of the values which the random variable ~~takes~~ (X) takes is given by;

$$P(X=x) = p^x(1-p)^{1-x} \quad ; \quad x = 0, 1$$
$$\Rightarrow p^x q^{1-x} \quad ; \quad p+q=1$$

This is called a Bernoulli distribution or point binomial distribution and the random variable is called Bernoulli variate.

1.1. Moments of Bernoulli distribution:

By definitions;

$$\mu'_r = E[x^r] = \sum_x x^r P(X=x)$$

$\because P(X=x) = p^x(1-p)^{1-x}$ or $p^x \cancel{(1-p)} q^{1-x}$

so;

$$\mu'_1 = E[x] = \sum_{x=0,1} x \cdot p^x q^{1-x}$$
$$\mu'_1 = 0 \cdot p^0 \cdot q^{1-0} + 1 \cdot p^{(1)} q^{(1-1)}$$
$$\mu'_1 = P$$

now;

$$\mu'_2 = E[x^2] = \sum_{x=0,1} x^2 p^x q^{1-x}$$
$$\mu'_2 = (0)^2 p^{(0)} q^{1-0} + (1)^2 p^2 q^{(1-2)}$$
$$\cancel{\mu'_2 = \frac{p^2}{q}}$$

$$\mu'_2 = (0)^2 p^{(0)} q^{1-0} + (1)^2 p^1 q^{(1-1)}$$
$$\mu'_2 = P$$

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$$\mu'_3 = E[x^3] = \sum_{x=0,1} x^3 (P(x=x))$$

$$\mu'_3 = E[x^3] = (0)^3 \cdot p^{(0)} q^{(1-0)} + (1)^3 p^{(1)} q^{(1-1)}$$

$$\mu'_3 = E[x^3] = p$$

$$\mu'_3 = p$$

$$\mu'_4 = E[x^4] = \sum_{x=0,1} x^4 P(x=x)$$

$$\mu'_4 = E[x^4] = (0)^4 \cdot p^{(0)} q^{(1-1-0)} + (1)^4 p^{(1)} q^{(4-1)}$$

$$\mu'_4 = E[x^4] = p$$

$$\because \mu_2 = \mu'_2 - (\mu'_1)^2$$

$$\mu_2 = p - p^2$$

$$\mu_2 = p(1-p) \Rightarrow pq$$

$$\because \mu_3 = \mu'_3 - 3\mu'_2\mu'_1 + 2(\mu'_1)^3$$

$$\mu_3 = p - 3p^2 + 2p^3$$

$$\mu_3 = p(1 - 3p + 2p^2)$$

$$\mu_3 = p(p-1)(p-1/2)$$

$$\mu_3 = pq\left(p - \frac{1}{2}\right)$$

$$\because \mu_4 = \mu'_4 - 4\mu'_3\mu'_1 + 6\mu'_2(\mu'_1)^2 - 3(\mu'_1)^4$$

$$\mu_4 = p - 4p^2 + 6p^3 - 3p^4$$

$$\mu_4 = p(1 - 4p + 6p^2 - 3p^3)$$

$$\cancel{\mu_4 = p(p-1)}$$

$$\mu_4 = p(1 - p(4 + 6p - 3p^2))$$

$$\cancel{\mu_4 = p(p-1}$$

$$\cancel{\mu_4 = p(1-p)(}$$

$$\mu_4 = p(1 - p(4 + 3p(2-p)))$$

or

$$\mu_4 = p(1-p)(3p^2 - 3p + 1)$$

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~~2. Binomial Distribution:~~

1.3. Skewness (β_1) and kurtosis (β_2)

$$\because \beta_1 = \frac{\mu_3^2}{\mu_2^3} = \frac{\cancel{p^2}}{p^3} = \frac{1}{p}$$

$$\because \beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{\cancel{p}}{p^2} = \frac{1}{p}$$

2. Binomial Distribution:

Consider an experiment with two possible outcomes call them succes (s) and failure (f) with ~~propab~~ probability $P(S) = P$ and $P(f) = f$, such that $p+q=1$. Let 'x' be a random variable denotes the number of successes in n independent repeated trials i.e.g:

consider:

$$\overbrace{\underbrace{s, s \ldots s}_{\text{success}(x)} \quad \underbrace{f, f \ldots f}_{\text{failure}(n-x)}}$$

The probability distribution of the particular sequence (by multiplicative law of independent events) is

$$p^x q^{(n-x)} \quad \text{or} \quad p^x (1-p)^{(n-x)}$$

The number of sequence in which 'x' succes and $n-x$ failures are observed in some order is $\binom{n}{x}$ ways, which is the binomial co-efficient.

Thus the probability distribution of exactly n successes and $n-x$ failures are observed/occur in n independent trials is

$$b(x; n, p) = \binom{n}{x} p^x q^{n-x} \quad ; \quad x = 0, 1, 2 \ldots n$$

which is known as binomial distribution with index n and parameter p.

$$\because b = \binom{n}{x}$$

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Prove that;

$$\Rightarrow \sum_{x=0}^{n} b(x; n, p) = 1$$

$$\Rightarrow \sum_{x=0}^{n} \binom{n}{x} p^x q^{n-x}$$

$$\Rightarrow q^n + \binom{n}{1} p q^{n-1} + \binom{n}{2} p^2 q^{n-2} + \dots + \binom{n}{n} p^n$$

$$\because (a+b)^n = a^n + \binom{n}{1} a^{n-1} b + \binom{n}{2} a^{n-2} b^2 + \dots + \binom{n}{n} b^n$$

$$\Rightarrow (q+p)^n \qquad \because p+q=1$$

$$= 1^n = 1 \qquad \because 1^n = 1$$

$$\Rightarrow 1 = 1 \qquad \text{proved!}$$

1.2 Moments of Binomial Distribution:

$$\mu'_1 = E[x] = \sum_{x=0}^{n} x\, b(x; n, p)$$

$$= \sum_{x=0}^{n} x \binom{n}{x} p^x q^{n-x}$$

$$= 0 \cdot p^{(0)} q^{n-0} + 1 \binom{n}{1} p^1 q^{n-1} + \cancel{2 \binom{n}{2}} \, 2 \binom{n}{2} p^2 q^{n-2} \dots n \binom{n}{n} p^n$$

$$= npq^{n-1} + 2\left[\frac{n!}{(n-2)!\,2!}\right] p^2 q^{n-2} \dots n \binom{n}{n} p^n$$

$$= npq^{n-1} + \cancel{2}\left[\frac{n(n-1)\cancel{(n-2)!}}{\cancel{(n-2)!} \cdot \cancel{2}}\right] p^2 q^{n-2} \dots np^n$$

$$= npq^{n-1} + n \binom{n-1}{1} p^2 q^{n-2} \dots np^n$$

$$\Rightarrow np\left[q^{n-1} + \binom{n-1}{1} p q^{n-2} \dots p^{n-1}\right]$$

$$\Rightarrow np\,[q+p]^{n-1} \qquad \because p+q=1$$

$$\Rightarrow np$$

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at $r=2$;

$\mu_2' = E[x^2] = \sum_{x=0}^{n} x^2 \binom{n}{x} p^x q^{n-x}$

$= \sum_{x=0}^{n} [x + x(x-1)] \binom{n}{x} p^x q^{n-x}$

$= \sum_{x=0}^{n} x \binom{n}{x} p^x q^{n-x} + \sum_{x=0}^{n} x(x-1) \binom{n}{x} p^x q^{n-x}$

$= np + 0(0-1)\binom{n}{0}p^0 q^{n-0} + 1(1-1)\binom{n}{1}p^1 q^{n-1} + 2(2-1)\binom{n}{2}p^2 q^{n-2}$
$+ 3(3-1)\binom{n}{3} p^3 q^{n-3} \dots n(n-1)\binom{n}{n} p^n q^{n-n}$

$= np + 2\binom{n}{2}p^2 q^{n-2} + 6\binom{3}{3}p^3 q^{n-3} \dots n(n-1)p^n$

~~$= np + 2\left[\frac{n!}{(n-2)!2!}\right] + 6\left[\frac{n!}{(n-3)!3!}\right]$~~

$= np + 2\left[\frac{n!}{(n-2)!2!}\right]p^2 q^{n-2} + 6\left[\frac{n!}{(n-3)!3!}\right]p^3 q^{n-3} \dots n(n-1)p^n$

$= np + \cancel{2}\left[\frac{n(n-1)\cancel{(n-2)!}}{\cancel{(n-2)!}\cancel{2!}}\right]p^2 q^{n-2} + \cancel{6}\left[\frac{n(n-1)(n-2)\cancel{(n-3)!}}{\cancel{(n-3)!}\cancel{3!}}\right]p^3 q^{n-3}$
$\dots n(n-1)p^n$

$= np + n(n-1)p^2 q^{n-2} + n(n-1)(n-2)p^3 q^{n-3} \dots n(n-1)p^n$

$= np + n(n-1)p^2 [q^{n-2} + \binom{n-2}{1} p q^{n-3} \dots p^{n-2}]$

$= np + n(n-1)p^2 [p+q]^{n-2}$

$\because p+q = 1$

$\Rightarrow np[1 + (n-1)p]$

$= np[1 + np - p]$

$= np[np + 1 - p]$ $\qquad \because 1-p = q$

$= np[np + q]$

$\mu_2' = n^2p^2 + npq$

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at $r=3$

$$E[x^3] = \mu_3' = \sum_{x=0}^{n} x^3 \binom{n}{x} p^x q^{n-x}$$

$$\because x^3 = [x(x-1)(x-2) + 3x(x-1) + x]$$

$$\mu_3' = \sum_{x=0}^{n} x(x-1)(x-2)x$$

$$\mu_3' = \sum_{x=0}^{n} x \binom{n}{x} p^x q^{n-x} + \sum_{x=0}^{n} x(x-1)(x-2)\binom{n}{x} p^x q^{n-x} + 3\sum_{x=0}^{n} x(x-1)\binom{n}{x} p^x q^{n-x} \cancel{+ \sum_{x=0}^{n} x \binom{n}{x} p^x q^{n-x}}$$

$$\because \sum_{x=0}^{n} x \binom{n}{x} p^x q^{n-x} = np$$

$$\because \sum_{x=0}^{n} x(x-1)\binom{n}{x} p^x q^{n-x} = n(n-1)p^2$$

$$\mu_3' = np + \sum_{x=0}^{n} x(x-1)(x-2)\binom{n}{x} p^x q^{n-x} + 3n(n-1)p^2 \cancel{+np}$$

$$\mu_3' = \cancel{2}np + 3n(n-1)p^2 + \sum_{x=0}^{n} x(x-1)(x-2)\binom{n}{x} p^x q^{n-x}$$

considering:

$$\sum_{x=0}^{n} x(x-1)(x-2)\binom{n}{x} p^x q^{n-x}$$

$$\Rightarrow 3(3-1)(3-2)\binom{n}{3} p^3 q^{n-3} + 4(\cancel{3}\,4-1)(4-2)\binom{n}{4} p^4 q^{n-4} \ldots n(n-1)(n-2)\binom{n}{n}$$

$$\Rightarrow \cancel{6}\left[\frac{n!}{(n-3)!\,\cancel{3!}}\right] p^3 q^{n-3} + 24\left[\frac{n!}{(n-4)!\,\cancel{4!}}\right] p^4 q^{n-4} \ldots n(n-1)(n-2)p^n$$

$$\Rightarrow \frac{n(n-1)(n-2)\cancel{(n-3)!}}{\cancel{(n-3)!}} \cdot p^3 q^{n-3} + \frac{n(n-1)(n-2)(n-3)\cancel{(n-4)!}}{\cancel{(n-4)!}} \cdot p^4 q^{n-4} \ldots n(n-1)(n-2)p^n$$

$$\Rightarrow n(n-1)(n-2)p^3 q^{n-3} + n(n-1)(n-2)(n-3)p^4 q^{n-4} \ldots n(n-1)(n-2)p^n$$

$$\Rightarrow n(n-1)(n-2)p^3\left[q^{n-3} + \binom{n-3}{1} p q^{n-4} \ldots p^{n-3}\right]$$

$$\Rightarrow n(n-1)(n-2)p^3 [q+p]^{n-3} \qquad \because p+q=1$$

$$\Rightarrow n(n-1)(n-2)p^3$$

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$\mu'_3 = \cancel{2}np + 3(\cancel{n})(n-1)p^2 + n(n-1)(n-2)p^3$

$\mu'_3 = np[2 + 3(n-1)p + (n-1)(n-2)p^2]$

$\mu'_3 = np\ [3(n-1)p$

$\mu'_3 = n(n-1)(n-2)p^3 + 3n(n-1)p^2 + np$

$r = 4$

$\mu'_4 = E[x^4] = \sum_{x=0}^{n} x^4 \binom{n}{x} p^x q^{n-x}$

$\because x^4 = [x(x-1)(x-2)(x-3) + 6x(x-1)(x-2) + 7x(x-1) + x]$

$\mu'_4 = \sum_{x=0}^{n} x(x-1)(x-2)(x-3)\binom{n}{x}p^x q^{n-x} + 6\sum_{x=0}^{n} x(x-1)(x-2)\binom{n}{x}p^x q^{n-x} + 7\sum_{x=0}^{n} x(x-1)\binom{n}{x}p^x q^{n-x}$
$+ \sum_{x=0}^{n} x\binom{n}{x}p^x q^{n-x}$

~~$\mu_4 = \sum^n$~~ $\because \sum_{x=0}^{n} x(x-1)(x-2)\binom{n}{x}p^x q^{n-x} = n(n-1)(n-2)p^3$

$\because \sum_{x=0}^{n} x(x-1)\binom{n}{x}p^x q^{n-x} = n(n-1)p^2$

$\because \sum_{x=0}^{n} x\binom{n}{x}p^x q^{n-x} = np$

Considering:

$\sum_{x=0}^{n} x(x-1)(x-2)(x-3)\binom{n}{x}p^x q^{n-x}$

$\Rightarrow 4(4-1)(4-2)(4-3)\binom{n}{4}p^4 q^{n-4} + 5(5-1)(5-2)(5-3)\binom{n}{5}p^5 q^{n-5} \ldots n(n-1)(n-2)(n-3)\binom{n}{n}p^n$

$\Rightarrow 24\left[\frac{n(n-1)(n-2)(n-3)\cancel{(n-4)!}}{\cancel{(n-4)!}\,4!}\right]p^4 q^{n-4} + 120\left[\frac{n(n-1)(n-2)(n-3)(n-4)\cancel{(n-5)!}}{\cancel{(n-5)!}\,5!}\right]p^5 q^{n-5}$

$\ldots n(n-1)(n-2)(n-3)p^n$

$\Rightarrow n(n-1)(n-2)(n-3)p^4 q^{n-4} + n(n-1)(n-2)(n-3)(n-4)p^5 q^{n-5} \ldots n(n-1)(n-2)(n-3)p^n$

$\Rightarrow n(n-1)(n-2)(n-3)p^4\ [q^{n-4} + \binom{n-4}{1}p^5 q^{n-5} \ldots p^{n-4}]$

$\Rightarrow n(n-1)(n-2)(n-3)p^4\ [p+q]^{n-4}$ $\because p+q=1$

$\Rightarrow n(n-1)(n-2)(n-3)p^4$

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nows;

$\mu_4' = n(n-1)(n-2)(n-3)p^4 + 6n(n-1)(n-2)p^3 + 7n(n-1)p^2 + np$

$\mu_4' = np[(n-1)(n-2)(n-3)p^3 + 6(n-1)(n-2)p^3 + 7(n-1)p + 1]$

1.3. Mean Moments:

$\rightarrow \mu_2 = \mu_2' - (\mu_1')^2$

$\mu_2 = n^2p^2 + npq - n^2p^2$

$\mu_2 = npq$

$\rightarrow \mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2(\mu_1')^3$

$\mu_3 = n(n-1)(n-2)p^3 + 3n(n-1)p^2 + np - 3(n^2p^2 + npq)(np) + 2n^3p^3$

$\mu_3 = n(n-1)(n-2)p^3 + 3n(n-1)p^2 + np - 3n^3p^3 - 3n^2p^2q + 2n^3p^3$

$\mu_3 = (n^2-n)(n-2)p^3 + (3n^2-3n)p^3 + np - 3n^2p^2q - n^3p^3$

$\mu_3 = (n^3 - 2n^2 - n^2 + 2n)p^3 + (3n^2p^3 - 3p^3n) + np - 3n^2p^2q - n^3p^3$

$\mu_3 = \cancel{n^3p^3} - 2n^2p^3 - n^2p^3 + 2np^3 + 3n^2p^3 - 3np^3 + np - 3n^2p^2q - \cancel{n^3p^3}$

$\mu_3 = \cancel{-3n^2p^3} + \cancel{3n^2p^3} + 2np^3 - 3np^3 + np - 3n^2p^2q$

$\mu_3 = np - 3n^2p^2q - np^3$

$\mu_3 = np - 3n^2p^2(1-p) - np^3$

$\mu_3 = np - 3n^2p^2 + 3n^2p^3 - np^3 \Rightarrow np[1 - 3np + 3np^2 - p^2]$

$\mu_3 = 2n^2p^3 - 3n^2p^2 + np$

$\mu_3 = np[2np^2 - 3np + 1]$

$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'(\mu_1')^2 - 3(\mu_1')^4$

$\mu_4 = n(n-1)(n-2)(n-3)p^4 + 6n(n-1)(n-2)p^3 + 7n(n-1)p^2 + np$
$\quad - 4(n(n-1)(n-2)p^3)(np) + 6(n^2p^2 + npq)(np)^2 - 3(np)^4$

~~$\mu_4 = n^2 - n(n-2)(n-3)$~~

$\mu_4 = (n^2-n)(n^2-3n-2n+6)p^4 + 6(n^2-n)(n-2)p^3 + 7(n^2-n)p^2 + np$
$\quad - 4(n^2-n)(n-2)p^3(np) + 6n^2p^2(n^2p^2 + npq) - 3n^4p^4$

$\mu_4 = p^4(n^2-n)(n^2-5n+6) + 6p^3(n^3-2n^2-n^2+2n) + 7(n^2p^2 - np^2) + n$
$\quad - 4p^3(n^3-2n^2-n^2+2n)(np) + 6(n^4p^4 + n^3p^3q) - 3n^4p^4$

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$\mu_3 = \mu'_3 - 3\mu'_2\mu'_1 + 2(\mu'_1)^3$

$\mu_3 = [n(n-1)(n-2)p^3 + 3(n-1)p^2 + np] - 3[n(n-1)p^2 + np]np$
$+ 2n^3p^3$

$\mu_3 = [n^3p^3 - 3n^2p^3 + 2np^3] + 3n^2p^2 - 3np^2 + np - [3n^3p^3 - 3n^2p^3 + 3n^2p^2]$
$+ 2n^3p^3$

$\mu_3 = n^3p^3 - 3n^2p^3 + 2np^3 + 3n^2p^2 - 3np^2 + np - 3n^3p^3 + 3n^2p^3 - 3n^2p^2 + 2n^3p^3$

$\mu_3 = np[2p^2 - 3p + 1]$

$\mu_3 = np[(1-p)(1-2p)] \rightarrow (1-p-p)$

$\mu_3 = np(1-p)(q-p)$

$\mu_3 = npq(q-p)$

$\mu_4 = \mu'_4 - 4\mu'_3\mu'_1 + 6\mu'_2(\mu'_1)^2 - 3(\mu'_1)^4$

$\mu_4 = [n(n-1)(n-2)(n-3)p^4 + 6n(n-1)(n-2)p^3 + 7n(n-1)p^2 + np]$
$- 4np(n(n-1)(n-2)p^3 + 3n(n-1)p^2 + np] + 6n^2p^2[np + n(n-1)p^2]$
$- 3n^4p^4$

$\because n(n-1)(n-2)(n-3) = n^4 - 6n^3 + 11n^2 - 6n$

$\because n(n-1)(n-2) = n^3 - 3n^2 + 2n$

$\because n(n-1) = n^2 - n$

$\mu_4 = p^4(n^4 - 6n^3 + 11n^2 - 6n) + 6p^3(n^3 - 3n^2 + 2n) + 7p^2(n^2 - n) + np$
$- 4np[p^3(n^3 - 3n^2 + 2n) + 3p^2(n^2 - n) + np] + 6n^2p^2[np + p^2(n^2 - n)]$
$- 3n^4p^4$

$\mu_4 = n^4p^4 - 6n^3p^4 + 11n^2p^4 - 6np^4 + 6n^3p^3 - 18n^2p^3 + 12np^3 + 7n^2p^2 - 7np^2$
$+ np - 4np[n^3p^3 - 3n^2p^3 + 2np^3 + 3n^2p^2 - 3np^2 + np]$
$+ 6n^2p^2[np + n^2p^2 - np^2] - 3n^4p^4$

$\mu_4 = n^4p^4 - 6n^3p^4 + 11n^2p^4 - 6np^4 + 6n^3p^3 - 18n^2p^3 + 12np^3 + 7n^2p^2 - 7np^2$
$+ np - 4n^4p^4 + 12n^3p^4 - 8n^2p^4 - 12n^3p^3 + 12n^2p^3 - 4n^2p^2$
$+ 6n^3p^3 + 6n^4p^4 - 6n^3p^4 - 3n^4p^4$

$\mu_4 = n^4(p^4 - 4p^4 + 6p^4 - 3p^4) + n^3(-6p^4 + 6p^3 + 12p^4 - 12p^3 + 6p^3 - 6p^4)$
$+ n^2(11p^4 - 18p^3 + 7p^2 - 8p^4 + 12p^3 - 4p^2) + n(-6p^4 + 12p^3 - 7p^2 + p)$

$\mu_4 = n^2(3p^4 - 6p^3 + 3p^2) + np(-6p^3 + 12p^2 - 7p + 1)$

$\mu_4 = 3n^2p^2(p^2 - 2p + 1) + np(-6p^3 + 12p^2 - 7p + 1)$

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$\mu_4 = 3n^2p^2(1-p)^2 + np(1-p)(6p^2-6p+1)$

$\mu_4 = 3n^2p^2q^2 + npq\,(1+6p(p-1))$

$\mu_4 = 3n^2p^2q^2 + npq\,(1-6p(1-p))$

$\mu_4 = 3n^2p^2q^2 + npq\,(1-6pq)$

$\mu_4 = 3n^2p^2q^2 + npq - 6np^2q^2$

$\mu_4 = npq\,(3npq - 6pq + 1)$

$\mu_4 = npq\,(3pq(n-2) + 1)$

~~* Mgf (Moment generating function): x)~~

$\beta_1 = \dfrac{\mu_3^2}{\mu_2^3} = \dfrac{(npq(q-p))^2}{[npq]^3} = \dfrac{(q-p)^2}{npq} = \dfrac{(1-2p)^2}{npq}$

$\sqrt{\beta_1} = \sqrt{\gamma} = \dfrac{1-2p}{\sqrt{npq}}$

$\beta_2 = \dfrac{\mu_4}{\mu_2^2} = \dfrac{\cancel{npq}(3npq-6pq+1)}{n^{\cancel{2}}\cancel{p^2}\cancel{q^2}} = \dfrac{3npq-6pq+1}{npq}$

$= \dfrac{\cancel{pq}(3n-6)}{n\cancel{pq}} + \dfrac{1}{npq} = \dfrac{3(n-2)}{n} + \dfrac{1}{npq}$

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1.4 M.G.F.

The mgf of the binomial distribution $b(x; n, p)$ is derived as.

$$M_x(t) = E[e^{tx}]$$
$$= \sum_{x=0}^{n} e^{tx} \binom{n}{x} p^x q^{n-x}$$
$$= \sum_{x=0}^{n} \binom{n}{x} [pe^t]^x q^{n-x}$$
$$= \binom{n}{0}[pe^t]^0 q^{n-0} + \binom{n}{1}(pe^t)^1 q^{n-1} + \binom{n}{2}(pe^t)^2 q^{n-2}$$
$$\dots [pe^t]^n$$
$$M_x(t) = [q + pe^t]^n$$

which is simplified from MGF of binomial distribution.

1.5 Moments of Binomial Distribution.

The moments of binomial distribution is obtained by differentiating $M_x(t)$ (mgf) rth time with respect to t, and putting $t=0$ thus:

$$\mu'_r = E[x^r] = \left[\frac{d^r}{dt^r}(q + pe^t)^n\right]_{t=0}$$

at $r=1$

$$\mu'_1 = n(q + pe^t)^{n-1} \frac{d}{dt}(q + pe^t)$$
$$= n(q + pe^t)^{n-1}\left(0 + p\frac{de^t}{dt} + e^t \frac{dp}{dt}\right)$$
$$= n(q + pe^t)^{n-1}(pe^t) \quad \text{at } t=0$$
$$= n(q + p)^{n-1}(p) \quad \because q + p = 1$$
$$= np$$

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r=2

$$\mu_2' = E[x^2] = \left[\frac{d^2}{dt^2}\left[(q+pe^t)^n\right]\right]_{t=0}$$

$$= \frac{d}{dt}\left[\frac{d}{dt}(q+pe^t)^n\right]_{t=0}$$

$$\cancel{= \frac{d}{dt}\left[\frac{dq}{dt} + p\frac{de^t}{dt} + e^t\frac{dp}{dt}\right]}$$

$$= \frac{d}{dt}\left[n(q+pe^t)^{n-1}\cdot\left[\frac{dq}{dt} + p\frac{de^t}{dt} + e^t\frac{dp}{dt}\right]\right]_{t=0}$$

$$= \frac{d}{dt}\left[n(q+pe^t)^{n-1}\cdot pe^t\right]_{t=0}$$

$$\Rightarrow n(q+pe^t)^{n-1}\cdot p\frac{de^t}{dt} + pe^t\left[n\frac{d}{dt}(q+pe^t)^{n-1} + \frac{dn}{dt}\right]$$

$$\Rightarrow npe^t(q+pe^t)^{n-1} + npe^t(n-1)(q+pe^t)^{n-2}\cdot\left(\frac{dq}{dt} + p\frac{de^t}{dt}\right)$$

$$\Rightarrow npe^t(q+pe^t)^{n-1} + np^2e^{2t}(n-1)(q+pe^t)^{n-2} \quad \because t=0$$

$$\Rightarrow npe^0(q+pe^0)^{n-1} + np^2e^{2(0)}(n-1)(q+p)e^0)^{n-2} \quad \because q+p=1$$

$$\Rightarrow np + np^2(n-1)$$

$$\Rightarrow np + n^2p^2 - np^2$$

$$\Rightarrow n^2p^2 + np(1-p)$$

$$\Rightarrow n^2p^2 + npq$$

r=3

$$\mu_3' = E[x^3] = \frac{d^3}{dt^3}\left[(q+pet)^n\right]$$

$$\mu_3' = E[x^3] = \frac{d^2}{dt^2}\left[n(q+pet)^{n-1}\cdot pe^t\right]$$

$$\cancel{\mu_3' = E[x^3] = \frac{d}{dt}\left[npe^t(q+pe^t)^{n-1}\cdot pe^t\left[n(n-1)(q+pe^t)^{n-2}\right]\right]}$$

$$\cancel{\mu_3' = E[x^3] = \frac{d}{dt}\left[n^2p^2e^{2t}(n-1)(q+pe^t)^{2n-3}\right]}$$

$$\cancel{\mu_3' = E[x^3] = \frac{d}{dt}\left[e^{2t}\cdot n^2p^2(n-1)(q+pe^t)^{2n-3}\right]}$$

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$\mu_3' = E[x^3] = \frac{d^3}{dt^3}\left[(q+pe^t)^n\right]$

$\mu_3' = E[x^3] = \frac{d^2}{dt^2}\left[npe^t(q+pe^t)^{n-1}\right]$

$\mu_3' = E[x^3] = \frac{\partial}{\partial t}\left[np\left[e^t(n-1)(q+pe^t)^{n-2}(pe^t) + (q+pe^t)^{n-1}.e^t\right]\right]$

$\mu_3' = \frac{\partial}{\partial t}\left[np^2e^{2t}(n-1)(q+pe^t)^{n-2} + npe^t(q+pe^t)^{n-1}\right]$

$\mu_3' = np^2(n-1)\left[e^{2t}(n-2)(q+pe^t)^{n-3}(pe^t) + (q+pe^t)^{n-2}.2e^{2t}\right]$

$+ np\left[e^t(n-1)(q+pe^t)^{n-2}(pe^t) + (q+pe^t)^{n-1}(e^t)\right]$

$\mu_3' = np^3e^{3t}(n-1)(n-2)(q+pe^t)^{n-3} + 2np^2e^{2t}(n-1)(q+pe^t)^{n-2}$
$+ np^2e^{2t}(n-1)(q+pe^t)^{n-2} + npe^t(q+pe^t)^{n-1}$

at $t=0$; $\because q+p=1$

$\mu_3' = E[x^3] = np^3(n-1)(n-2) + 3np^2(n-1) + np$

$r=4$

$\mu_4' = E[x^4] = \frac{d^4}{dt^4}\left[(q+pe^t)^n\right]$

$= \frac{\partial}{\partial t}\left[np^3e^{3t}(n-1)(n-2)(q+pe^t)^{n-3} + 3np^2e^{2t}(n-1)(q+pe^t)^{n-2} + npe^t(q+pe^t)^{n-1}\right]$

$= np^3(n-1)(n-2)\left[e^{3t}(n-3)(q+pe^t)^{n-4}(pe^t) + (q+pe^t)^{n-3}\,3e^{3t}\right]$

$+ 3np^2(n-1)\left[e^{2t}(n-2)(q+pe^t)^{n-3}(pe^t) + (q+pe^t)^{n-2}2e^{2t}\right]$
$+ np\left[e^t(n-1)(q+pe^t)^{n-2}(pe^t) + (q+pe^t)^{n-1}(e^t)\right]$

$\Rightarrow np^4e^{4t}(n-1)(n-2)(n-3)(q+pe^t)^{n-4} + 3np^3e^{3t}(q+pe^t)^{n-3}(n-1)(n-2)$
$+ 3np^3e^{3t}(n-1)(n-2)(q+pe^t)^{n-3} + 6np^2e^{2t}(n-1)(q+pe^t)^{n-2}$
$+ np^2e^{2t}(n-1)(q+pe^t)^{n-1} + npe^t(q+pe^t)^{n-1}$

$\because t=0$, $\because p+q=1$

$\mu_4' = E[x^4] = np^4(n-1)(n-2)(n-3) + 3np^3(n-1)(n-2) + 3np^3(n-1)(n-2)$
$+ 6np^2(n-1) + np^2(n-1) + np$

$\mu_4' = E[x^4] = np^4(n-1)(n-2)(n-3) + 6np^3(n-1)(n-2) + 7np^2(n-1) + np$

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Poisson Distribution:-

Suppose that an experiment with two possible outcomes s and f with probabilities $P(s) = p$ and $P(f) = q = 1-p$ is repeated independently and indefinitely. Let p be small $(p \to 0)$, such that $np \to \lambda$ as $n \to \infty$. Then, the probability distribution of the number of successes is;

$$P(X=x) = \frac{e^{-\lambda}\lambda^x}{x!} \quad ; \quad x = 0, 1, 2, 3 \ldots \infty$$

which is called poisson distribution with parameter λ.

Derivation:

The Probability of x successes in n trials is given by the binomial distribution $b(x; n, p)$;

$$P(X=x) = \binom{n}{x}(p)^x q^{n-x}$$

Let; $p = \frac{\lambda}{n}$, $q = \left(1 - \frac{\lambda}{n}\right)$

considering that $\lim_{n\to\infty}$

$$P(X=x) = \frac{n!}{(n-x)!\,x!}\left(\frac{\lambda^x}{n^x}\right)\left(1-\frac{\lambda}{n}\right)^n\left(1-\frac{\lambda}{n}\right)^{-x}$$

$$P(X=x) = \lim_{n\to\infty} \frac{n(n-1)(n-2)\ldots(n-x+1)\cancel{(n-x)!}}{\cancel{(n-x)!}\,x!} \cdot \frac{\lambda^x}{n^x}\left(1-\frac{\lambda}{n}\right)^n\left(1-\frac{\lambda}{n}\right)^{-x}$$

$$P(X=x) = \frac{\lambda^x}{x!} \cdot \frac{\cancel{n^x}}{\cancel{n^x}} \lim_{n\to\infty}\left[\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\ldots\left(1-\frac{x}{n}+\frac{1}{n}\right)\left(1-\frac{\lambda}{n}\right)^n\left(1-\frac{\lambda}{n}\right)^{-x}\right]$$

$$\because \lim_{n\to\infty}\left[\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\ldots\left(1-\frac{x}{n}+\frac{1}{n}\right)\right] = 1$$

$$\because \lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^n = e^{-\lambda} \qquad \because \text{using identity}$$

$$\because \lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^{-x} = 1^{-x} = 1$$

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$P(X=x) = \frac{e^{-\lambda}\lambda^{x}}{x!}$ hence proved!

→ Events occur randomly and independently
↳ Probability that an event occurs in a given length of time /area / volume doesn't change through lengths of time / area / volume.

1) Moments about Origin

Let X be a random variable with poisson distribution $P(X;\lambda)$ then;

$$\mu'_r = E[x^r] = \sum_{x=0}^{\infty} x^r P(x;\lambda)$$

at; $r=1$ $$\mu'_1 = \sum_{x=0}^{\infty} x \cdot \frac{e^{-\lambda}\lambda^{x}}{x!}$$

$$\Rightarrow e^{-\lambda}\sum_{x=0}^{\infty} \frac{\cancel{e}x\lambda^{x}}{x!}$$

$$\Rightarrow e^{-\lambda}\left[0+\lambda+\frac{2\lambda^2}{2!}+\frac{3\lambda^3}{3!}+\frac{4\lambda^4}{4!}\cdots\right]$$

$$\Rightarrow \lambda e^{-\lambda}\left[0+1+\frac{2\lambda}{2!}+\frac{3\lambda^2}{3!}+\frac{4\lambda^3}{4!}\cdots\right]$$

$$\cancel{\because e^{\theta} = \left[1+\frac{2\theta}{2!}\lambda+\frac{\lambda^2}{2!}+\frac{\lambda^3}{3!}\right]}$$

$$e^{\theta} = \left[1+\theta+\frac{\theta^2}{2!}+\frac{\theta^3}{3!}+\frac{\theta^4}{4!}\cdots\right]$$

so;

$$\Rightarrow \lambda e^{-\lambda} e^{\lambda}$$

$$\Rightarrow \lambda = E[x] = \mu.$$

at $r=2$;

$$\mu'_2 = \sum_{x=0}^{\infty} x^2 \frac{e^{-\lambda}\lambda^{x}}{x!}$$

$$\because x^2 = x(x-1)+x$$

$$\mu'_2 = \sum_{x=0}^{\infty}\left[\frac{x(x-1)e^{-\lambda}\lambda^{x}}{x!}+\frac{xe^{-\lambda}\lambda^{x}}{x!}\right]$$

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$$\Rightarrow \sum_{x=0}^{\infty} x(x-1)\frac{e^{-\lambda}\lambda^x}{x!} + \sum_{x=0}^{\infty} x\frac{e^{-\lambda}\lambda^x}{x!}$$

we know; $E[x] = \sum_{x=0}^{\infty} x\frac{e^{-\lambda}\lambda^x}{x!} = \lambda$

considering $\sum_{x=0}^{\infty} x(x-1)\frac{e^{-\lambda}\lambda^x}{x!}$

$$\Rightarrow e^{-\lambda}\sum_{x=0}^{\infty} \frac{x(x-1)\lambda^x}{x!}$$

$$\Rightarrow e^{-\lambda}\left[0+0+\frac{2\lambda^2}{2!}+\cancel{3}\frac{6\lambda^3}{3!}+\frac{12\lambda^4}{4!}\dots\right]$$

$$\Rightarrow e^{-\lambda}.\lambda^2\left[1+\lambda+\frac{\lambda^2}{2!}+\frac{\lambda^3}{3!}\dots\right]$$

$$\Rightarrow \because e^{\theta} = \left[1+\theta+\frac{\theta^2}{2!}+\frac{\theta^3}{3!}\dots\right]$$

$$\Rightarrow \lambda^2 \cancel{e^{-\lambda}}\,\cancel{e^{\lambda}}$$

so; $E[x(x-1)+x] = E[x^2] = \lambda^2+\lambda$

$\mu_2' = \lambda^2+\lambda$

at $r=3$;

$E[x^3] = E[x(x-1)(x-2)+3x(x-1)+x]$

considering $E[x(x-1)(x-2)]$

so;

$$E[x(x-1)(x-2)] = \sum_{x=0}^{\infty} x(x-1)(x-2)\frac{e^{-\lambda}\lambda^x}{x!}$$

$$\Rightarrow e^{-\lambda}\sum_{x=0}^{\infty} x(x-1)(x-2)\frac{\lambda^x}{x!}$$

$$\Rightarrow e^{-\lambda}\left[0+0+0+\frac{6\lambda^3}{3!}+\frac{24\lambda^4}{4!}+\frac{60\lambda^5}{5!}+\frac{120\lambda^6}{6!}\dots\right]$$

$$\Rightarrow \lambda^3 e^{-\lambda}\left[1+\lambda+\frac{\lambda^2}{2!}+\frac{\lambda^3}{3!}\dots\right]$$

$$\Rightarrow \lambda^3 e^{-\lambda}.e^{\lambda}$$

$$\Rightarrow \lambda^3$$

so

$\cancel{E[x^3] = \lambda^3 + 3\lambda^2 + 3\lambda + \lambda}$

$\cancel{E[x^3] = \lambda^3 + 3\lambda^2 + 4\lambda}$

$E[x^3] = \lambda^3 + 3\lambda^2 + \lambda$

at $r = 4$

$E[x^4] = E[x(x-1)(x-2)(x-3) + 6x(x-1)(x-2) + 7x(x-1) + x]$

$E[x^4] = E[x(x-1)(x-2)(x-3)] + 6E[x(x-1)(x-2)] \cancel{+ 7E[x(x-1)] +}$
$+ 7E[x(x-1)] + E[x]$

considering $E[x(x-1)(x-2)(x-3)]$

$E[x(x-1)(x-2)(x-3)] = \sum_{x=0}^{\infty} x(x-1)(x-2)(x-3) \frac{e^{-\lambda}\lambda^x}{x!}$

$\Rightarrow e^{-\lambda} \sum_{x=0}^{\infty} x(x-1)(x-2)(x-3) \frac{\lambda^x}{x!}$

$\Rightarrow e^{-\lambda} \left[0 + 0 + 0 + 0 + \frac{24\lambda^4}{4!} + \frac{\overset{120}{\cancel{60}}\lambda^5}{5!} + \frac{360\lambda^6}{6!} + \frac{840\lambda^7}{7!} \dots \right]$

$\Rightarrow \lambda^4 e^{-\lambda} \left[1 + \lambda + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} \dots \right]$

$\Rightarrow \lambda^4 \cancel{e^{-\lambda}} \cdot \cancel{e^{\lambda}}$

$\Rightarrow E[x(x-1)(x-2)(x-3)] = \lambda^4$

$\Rightarrow E[x^4] = \lambda^4 + 6\lambda^3 + 7\lambda^2 + \lambda$

2) Moments about Mean:

$\because \mu_2 = \mu_2' - (\mu_1')^2$

$\mu_2 = \lambda^2 + \lambda - \lambda^2$

$\boxed{\mu_2 = \lambda}$

$\because \mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2(\mu_1')^3$

$\mu_3 = \lambda^3 + 3\lambda^2 + \lambda - 3(\lambda^2 + \lambda)(\lambda) + 2\lambda^3$

$\mu_3 = \cancel{\lambda^3} + \cancel{3\lambda^2} + \lambda - \cancel{3\lambda^3} - \cancel{3\lambda^2} + \cancel{2\lambda^3}$

$\boxed{\mu_3 = \lambda}$

$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'(\mu_1')^2 - 3(\mu_1')^4$

$\mu_4 = \lambda^4 + 6\lambda^3 + 7\lambda^2 + \lambda - 4(\lambda^3 + 3\lambda^2 + \lambda) \cancel{+ 6(} (\lambda)$
$+ 6(\lambda^2 + \lambda)(\lambda^2) - 3\lambda^4$

$\mu_4 = \cancel{\lambda^4} + \cancel{6\lambda^3} + 7\lambda^2 + \lambda - \cancel{4\lambda^4} - 12\lambda^3 - 4\lambda^2 + \cancel{6\lambda^4} + \cancel{6\lambda^3} - \cancel{3\lambda^4}$

$\mu_4 = 3\lambda^2 + \lambda$

3) Skewness and kurtosis

$\because \beta_1 = \frac{\mu_3^2}{\mu_2^3} = \frac{\lambda^2}{\lambda^3} = \beta_1 = \frac{1}{\lambda}$

$\because \beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{3\lambda^2 + \lambda}{\lambda^2} = 3 + \frac{1}{\lambda}$

Geometric Distribution:

A random variable X has a geometric distribution if and only if it's probability distribution is given by,

$$g(x; p) = q^{x-1} . p \quad ; x = 1, 2, 3 \ldots$$

Proof:

Consider an experiment with two possible outcomes, succes (s) and failure (f), with $p(s) = p$ and $p(f) = 1-p = q$. The experiment is repeated until first succes appears. Let X be the number of independent trials required to obtain one success. It means last trial must end in succes.

$$\underbrace{ff \ldots f \ldots}_{x-1} \underbrace{s}_{1}$$

The probability distribution of (x-1) failures and a succes in last trial appears is

$$P(X=x) = (1-p)^{x-1} . p \quad ; \quad x = 1, 2, 3 \ldots$$

or

$$g(x, p) = (1-p)^{x-1} . p$$
$$= q^{x-1} . p$$

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which is called geometric distribution with parameter p.

1.) Moment about Origin

Let the random variable X have a geometric distribution then the r^{th} moment about origin is obtained as;

$$\mu'_r = E[X^r] = \sum_{x=1}^{\infty} x^r g(x;p) = \sum_{x=1}^{\infty} x^r q^{x-1} p.$$

at $r=1$

$$\mu'_1 = p + 2qp + 3q^2p + 4q^3p + \dots$$
$$= p[1 + 2q + 3q^2 + 4q^3 + \dots]$$
$$= p[1-q]^{-2}$$
$$= p \cdot p^{-2}$$
$$\boxed{\mu'_1 = 1/p}$$

at $r=2$

$$\mu'_2 = E[X^2] = \sum_{x=1}^{\infty} x^2 q^{x-1} p$$
$$= p + 2^2qp + 3^2q^2p + 4^2q^3p + 5^2q^4p + \dots$$
$$= p[\cancel{4}\,1 + 4q + 9q^2 + 16q^3 + \dots]$$
$$= p[(1 + 3q + 6q^2 + 10q^3 \dots) + (q + 3q^2 + 6q^3 + \dots)]$$
$$= p[(1-q)^{-3} + q(1-q)^{-3}]$$
$$= p[p^{-3} + qp^{-3}]$$
$$= p^{-2} + qp^{-2}$$
$$\Rightarrow \frac{1}{p^2} + \frac{q}{p^2}$$
$$\mu'_2 \Rightarrow \frac{1+q}{p^2}$$

so; $Var(X) = \mu_2 = \mu'_2 - (\mu'_1)^2$

$$= \frac{1+q}{p^2} - \frac{1}{p^2}$$

$$\boxed{\Rightarrow \frac{q}{p^2} \Rightarrow \frac{1-p}{p^2}}$$

$$\boxed{S.D(X) = \frac{\sqrt{1-p}}{p}}$$

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Moment Generating function:

The m.g.f of the geomtric distribution is derived as.

By defination;
$$M_X(t) = E[e^{tx}] = \sum_{x=1}^{\infty} e^{tx} q^{x-1} p$$

$$\Rightarrow \sum_{x=1}^{\infty} e^{t} . e^{t(x-1)} q^{x-1} p$$

$$\Rightarrow pe^t \sum_{x=1}^{\infty} (e^t q)^{x-1}$$

$$\Rightarrow pe^t [1 + qe^t + (qe^t)^2 + (qe^t)^3 + \ldots]$$

$$\Rightarrow pe^t [1-qe^t]^{-1}$$

$$\boxed{M_X(t) = \frac{pe^t}{1-qe^t}}$$

<u>Moments:</u>

$$\mu'_r = \left\{\frac{d^r}{dt^r}\left(\frac{pe^t}{1-qe^t}\right)\right\}_{t=0}$$

which is $\Rightarrow$ $$\mu'_r = \left\{\frac{d^r}{dt^r} p(e^{-t}-q)^{-1}\right\}_{t=0}$$

μ'_1 ; $r=1$

$$\mu'_1 = p\left[\frac{d}{dt}(e^{-t}-q)^{-1}\right]_{t=0}$$

$$\mu'_1 = \left[p . -(e^{-t}-q)^{-2}(-e^{-t})\right]_{t=0}$$

$$\mu'_1 = [pe^{-t}(e^{-t}-q)^{-2}]_{t=0}$$

$$\boxed{\mu'_1 = \frac{p}{(1-q)^2} = \frac{p}{p^2} = \frac{1}{p}}$$

μ'_2 ; $r=2$

$$\Rightarrow p\frac{d}{dt}\left[e^{-t}(e^{-t}-q)^{-2}\right]_{t=0}$$

$$\Rightarrow p\left[\left(e^{-t} . -2(e^{-t}-q)^{-3} . -e^{-t}\right) + (e^{-t}-q)^{-2} . -e^{-t}\right]_{t=0}$$

$$\Rightarrow p\left[2e^{-2t}(e^{-t}-q)^{-3} - e^{-t}(e^{-t}-q)^{-2}\right]_{t=0}$$

$$\Rightarrow p[2(1-q)^{-3} - (1-q)^{-2}]$$

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$$= \mu_2' = p\left[\frac{2}{p^3} - \frac{1}{p^2}\right]$$

$$\mu_2' = \frac{2}{p^2} - \frac{1}{p} \Rightarrow \frac{2-p}{p^2} = \frac{1+1-p}{p^2} = \frac{1+q}{p^2}$$

$$\boxed{\mu_2' = \frac{1+q}{p^2}}$$

μ_3' ; $r=3$

$$\mu_3' = p\frac{d}{dt}\left[2e^{-2t}(e^{-t}-q)^{-3} - e^{-t}(e^{-t}-q)^{-2}\right]_{t=0}$$

$$\mu_3' = p\Big[(2e^{-2t}\cdot -3(e^{-t}-q)^{-4}\cdot -e^{-t} + (e^{-t}-q)^{-3}(-4e^{-2t}) - (e^{-t}\cdot -2(e^{-t}-q)^{-3}\cdot -e^{-t} + (e^{-t}-q)^{-2}(-e^{-t}))\Big]_{t=0}$$

$$\mu_3' = p\left[6e^{-3t}(e^{-t}-q)^{-4} - 6e^{-2t}(e^{-t}-q)^{-3} + e^{-t}(e^{-t}-q)^{-2}\right]_{t=0}$$

$$\mu_3' = p\left[\frac{6e^{-3t}}{(e^{-t}-q)^4} - \frac{6e^{-2t}}{(e^{-t}-q)^3} + \frac{e^{-t}}{(e^{-t}-q)^2}\right]_{t=0}$$

$$\mu_3' = p\left[\frac{6}{p^4} - \frac{6}{p^3} + \frac{1}{p^2}\right]$$

$$\mu_3' = \frac{6}{p^3} - \frac{6}{p^2} + \frac{1}{p}$$

$$\boxed{\mu_3' = \frac{6-6p+p^2}{p^3}}$$

μ_4' ; $r=4$

$$\mu_4' = p\frac{d}{dt}\left[6e^{-3t}(e^{-t}-q)^{-4} - 6e^{-2t}(e^{-t}-q)^{-3} + e^{-t}(e^{-t}-q)^{-2}\right]_{t=0}$$

$$\mu_4' = p\Big[6e^{-3t}\cdot -4(e^{-t}-q)^{-5}\cdot -e^{-t} + (e^{-t}-q)^{-4}(-18e^{-3t})$$
$$-(6e^{-2t}\cdot -3(e^{-t}-q)^{-4}\cdot -e^{-t} + (e^{-t}-q)^{-3}(-12e^{-2t}))$$
$$+(e^{-t}\cdot -2(e^{-t}-q)^{-3}\cdot -e^{-t} + (e^{-t}-q)^{-2}(-e^{-t}))\Big]_{t=0}$$

$$\mu_4' = p\Big[24e^{-4t}(e^{-t}-q)^{-5} - 18e^{-3t}(e^{-t}-q)^{-4}$$
$$-(18e^{-3t}(e^{-t}-q)^{-4} - 12e^{-2t}(e^{-t}-q)^{-3})$$
$$+(2e^{-2t}(e^{-t}-q)^{-3} - e^{-t}(e^{-t}-q)^{-2})\Big]_{t=0}$$

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$$\mu_4' = p\left[24e^{-4t}(e^{-t}-q)^{-5} - 36e^{-3t}(e^{-t}-q)^{-4} + 14e^{-2t}(e^{-t}-q)^{-3} - e^{t}(e^{-t}-q)^{-2}\right]_{t=0}$$

$$\mu_4' = p\left[\frac{24}{p^5} - \frac{36}{p^4} + \frac{14}{p^3} - \frac{1}{p^2}\right]$$

$$\mu_4' = \frac{24}{p^4} - \frac{36}{p^3} + \frac{14}{p^2} - \frac{1}{p}$$

$$\boxed{\mu_4' = \frac{24 - 36p + 14p^2 - p^4}{p^4}}$$

Moments about mean:

$$\mu_2 = \mu_2' - (\mu_1')^2$$

$$\mu_2 = \frac{1}{p^2} + \frac{q}{p^2} - \frac{1}{p^2}$$

$$\boxed{\mu_2 = \frac{q}{p^2} = \frac{1-p}{p^2} = Var(x)}$$

$$S.D = \frac{\sqrt{1-p}}{p}$$

$$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2(\mu_1')^3$$

$$\mu_3 = \frac{6}{p^3} - \frac{6}{p^2} + \frac{1}{p} - 3\left(\frac{1}{p^2} + \frac{q}{p^2}\right)\left(\frac{1}{p}\right) + \frac{2}{p^3}$$

$$\mu_3 = \frac{6}{p^3} - \frac{6}{p^2} + \frac{1}{p} - \frac{3}{p^3} - \frac{3q}{p^3} + \frac{2}{p^3}$$

$$\mu_3 = \frac{6-3-3q+2}{p^3} - \frac{6}{p^2} + \frac{1}{p}$$

$$\mu_3 = \frac{3p+2}{p^3} - \frac{6}{p^2} + \frac{1}{p} \Rightarrow \frac{3p+2-6p+p^2}{p^3} \Rightarrow \frac{p^2-3p+2}{p^3}$$

$$\mu_3 = \frac{(p-2)(p-1)}{p^3} \Rightarrow \frac{(2-p)(1-p)}{p^3}$$

$$\boxed{\mu_3 = \frac{q(2-p)}{p^3}}$$

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$$\mu_4 = \mu'_4 - 4\mu'_3\mu'_1 + 6\mu'_2(\mu'_1)^2 - 3(\mu'_1)^4$$

$$\mu_4 = \frac{24}{p^4} - \frac{36}{p^3} + \frac{14}{p^2} - \frac{1}{p} - 4\left(\frac{6}{p^3} - \frac{6}{p^2} + \frac{1}{p}\right)\left(\frac{1}{p}\right)$$
$$+ 6\left(\frac{1}{p^2} + \frac{q}{p^2}\right)\left(\frac{1}{p^2}\right) - 3\left(\frac{1}{p^4}\right)$$

$$\mu_4 = \frac{24}{p^4} - \frac{36}{p^3} + \frac{14}{p^2} - \frac{1}{p} - \frac{24}{p^4} + \frac{24}{p^3} - \frac{4}{p^2} + \frac{6}{p^4} + \frac{6q}{p^4} - \frac{3}{p^4}$$

$$\mu_4 = \frac{24-24+6+6q-3}{p^4} + \frac{24-36}{p^3} + \frac{14-4}{p^2} - \frac{1}{p}$$

$$\mu_4 = \frac{9-6p}{p^4} - \frac{12}{p^3} + \frac{10}{p^2} - \frac{1}{p}$$

$$\mu_4 = \frac{9-6p-12p+10p^2-p^3}{p^4}$$

$$\mu_4 = \frac{9-18p+10p^2-p^3}{p^4}$$

$$\begin{array}{r|rrrr} 1 & -1 & 10 & -18 & 9 \\ & & -1 & 9 & -9 \\ \hline & -1 & 9 & -9 & 0 \end{array}$$

$$(p-1)(-p^2+9p-9)$$

$$\mu_4 = \frac{(p-1)(-p^2+9p-9)}{p^4}$$

$$\mu_4 = \frac{(1-p)(p^2-9p+9)}{p^4}$$

$$\mu_4 = \frac{q(p^2+9(1-p))}{p^4}$$

$$\boxed{\mu_4 = \frac{q(p^2+9q)}{p^4}}$$

Hypergeometric Distribution:

A random variable X has a hypergeometric distribution iff it's probability distribution is given by;

$$h(x; n, N, K) = \frac{\binom{K}{x}\binom{N-K}{n-x}}{\binom{N}{n}} \quad ; \quad x = 0, 1, 2 \dots K \quad x \le K \; ; \; n-x \le N-K$$

Proof:

Suppose a box of N oranges contains K sweet oranges. A sample of n oranges $(n < N)$ is selected. The probability that the sample contains x sweet oranges $(x \le K)$.

Let sample space S contain $\binom{N}{n}$ sample points. The population contains;

$$\underbrace{\underbrace{S.S \dots S}_{\substack{K \text{ sweet} \\ \text{(infected)}}} \quad \underbrace{\bar{S}.\bar{S} \dots \bar{S}}_{\substack{N-K \text{ not sweet} \\ \text{(infected)}}}}_{N \text{ oranges}}$$

and the sample contains.

$$\underbrace{\underbrace{S.S \dots S}_{x \text{ sweet}} \quad \underbrace{\bar{S}.\bar{S} \dots \bar{S}}_{n-x \text{ not sweet}}}_{n \text{ ~~oranges~~ oranges}}$$

The number of possible ways of selecting x oranges from K is $\binom{K}{x}$ and the number of ways of selecting $n-x$ not sweet oranges from N-K is $\binom{N-K}{n-x}$ ways. Thus the probability distribution of selecting x sweet oranges out of K and $n-x$ not sweet oranges out of N-K not sweet oranges by independent law of probability is given by;

$$h(x; n, N, K) = \frac{\binom{K}{x}\binom{N-K}{n-x}}{\binom{N}{n}} \quad ; \quad x = 0, 1, 2 \dots K \quad x \le K \; ; \; n-x \le N-K$$

which is hypergeometric distribution with parameters n, N and K.

$E[x^2] = \mu_2'$

→ Moments about mean

general formula: $E[x^r] = \sum_{x=0}^{n} x^r \frac{\binom{K}{x}\binom{N-K}{n-x}}{\binom{N}{n}}$

∴ $r=1$

$$E[X] = \frac{1}{\binom{N}{n}} \sum_{x=0}^{n} \frac{\cancel{x}\, K(K-1)!}{\cancel{x}\,(x-1)!\,(K-x)!} \cdot \frac{(N-K)!}{(N-K-n-x)!\,(n-x)!}$$

$$\Rightarrow \frac{K}{\binom{N}{n}} \sum_{x=0}^{n} \frac{(K-1)!}{(x-1)!\,(K-x)!} \cdot \frac{(N-K)!}{(N-K-n-x)!\,(n-x)!}$$

let; $z = x-1$; if $x=1$; if $x=n$

$x = z+1$; $z = 1-1=0$; $z = n-1$

$$\Rightarrow K\frac{(N-n)!\,n!}{N!} \sum_{z=0}^{n-1} \frac{(K-1)!}{z!\,(K-1-z)!} \cdot \frac{(N-K)!}{(N-K-n-z-1)!\,(n-z-1)!}$$

$$\Rightarrow K \cdot \frac{1}{\frac{N(N-1)!}{(N-n)!\,n(n-1)!}} \sum_{z=0}^{n-1} \binom{K-1}{z}\binom{N-K}{n-z-1}$$

$$\Rightarrow \frac{Kn}{N} \sum_{z=0}^{n-1} \frac{\binom{K-1}{z}\binom{N-K}{n-z-1}}{\binom{N-1}{n-1}}$$

∵ $\sum_{z=0}^{n-1} \binom{K-1}{z}\binom{N-K}{n-z-1} \Big/ \binom{N-1}{n-1}$ gives hypergeometric distributi for the entire area under the graph w.r.t z. Hence from the axiom of probability it equals 1.

$$\boxed{E[X] = \mu_1' = \frac{Kn}{N}}$$

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$\because r = 2$

$E[x^2] = \mu_2'$

$\because x^2 = x(x-1) + x$

$E[x(x-1)+x] = E[x(x-1)] + E[x]$

considering;

$$E[x(x-1)] = \sum_{x=0}^{n} x(x-1) \frac{\binom{K}{x}\binom{N-K}{n-x}}{\binom{N}{n}}$$

$$\Rightarrow \frac{1}{\binom{N}{n}} \sum_{x=0}^{n} x(x-1) \frac{K(K-1)(K-2)!}{(K-x)!\,x(x-1)(x-2)!} \cdot \frac{(N-K)!}{(N-K-n-x)!\,(n-x)!}$$

$$\Rightarrow \frac{K(K-1)}{\frac{N(N-1)(N-2)!}{(N-n)!\,n(n-1)(n-2)!}} \sum_{x=0}^{n} \frac{(K-2)!}{(K-x)!\,(x-2)!} \cdot \frac{(N-K)!}{(N-K-n-x)!\,(n-x)!}$$

$$\Rightarrow \frac{K(K-1).n(n-1)}{N(N-1)} \cdot \frac{1}{\frac{(N-2)!}{(N-n)!\,(n-2)!}} \sum_{x=0}^{n} \frac{(K-2)!}{(K-x)!\,(x-2)!} \cdot \frac{(N-K)!}{(N-K-n-x)!\,(n-x)!}$$

Let; $z = x-2$; if $x=2$; if $x=n$

$x = z+2$; $z = 2-2=0$; $z = n-2$

$$\Rightarrow \frac{Kn}{N}\frac{(K-1)(n-1)}{(N-1)} \cdot \frac{1}{\binom{N-2}{n-2}} \sum_{z=0}^{n-2} \frac{(K-2)!}{(K-2-z)!\,z!} \cdot \frac{(N-K)!}{(N-K-n-z-2)!\,(n-z-2)!}$$

$$\Rightarrow \frac{Kn}{N}\frac{(K-1)(n-1)}{(N-1)} \sum_{z=0}^{n-2} \frac{\binom{K-2}{z}\binom{N-K}{n-z-2}}{\binom{N-2}{n-2}}$$

$$\because \sum_{z=0}^{n-2} \frac{\binom{K-2}{z}\binom{N-K}{n-z-2}}{\binom{N-2}{n-2}} = 1$$

$$\Rightarrow E[x(x-1)] = \frac{Kn}{N}\frac{(K-1)(n-1)}{(N-1)}$$

so;

$$\boxed{\Rightarrow E[x(x-1)+x] = E[x^2] = \mu_2' = \frac{Kn}{N}\left[\frac{(K-1)(n-1)}{(N-1)} + 1\right]}$$

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$\because r=3$

$$E[x^3] = E[x(x-1)(x-2) + 3x(x-1) + x]$$

$$E[x^3] = E[x(x-1)(x-2)] + 3E[x(x-1)] + E[x]$$

Considering $E[x(x-1)(x-2)]$

$$\Rightarrow E[x(x-1)(x-2)] = \sum_{x=0}^{n} \cancel{x}\cancel{(x-1)}\cancel{(x-2)} \frac{K(K-1)(K-2)(K-3)!}{(K-x)!\,\cancel{x}\cancel{(x-1)}\cancel{(x-2)}(x-3)!} \cdot \frac{(N-K)!}{(N-K-n-x)!(n-x)!}$$

$$\Rightarrow \frac{K(K-1)(K-2)n(n-1)(n-2)}{N(N-1)(N-2)} \cdot \frac{1}{\frac{(N-3)!}{(N-n)!(n-3)!}} \sum_{x=0}^{n} \frac{(K-3)!}{(K-x)!(x-3)!} \cdot \frac{(N-K)!}{(N-K-n-x)!(n-x)!}$$

Let; $z = x-3$; if $x=3$; if $x=n$

$x = z+3$; $z = 3-3=0$; $z = n-3$

$$\Rightarrow \frac{K(K-1)(K-2)n(n-1)(n-2)}{N(N-1)(N-2)} \cdot \frac{1}{\binom{N-3}{n-3}} \sum_{z=0}^{n-3} \frac{(K-3)!}{(K-3-z)!\,z!} \cdot \frac{(N-K)!}{(N-K-n-z-3)!(n-z-3)!}$$

$$\Rightarrow \frac{K(K-1)(K-2)n(n-1)(n-2)}{N(N-1)(N-2)} \sum_{z=0}^{n-3} \frac{\binom{K-3}{z}\binom{N-K}{n-z-3}}{\binom{N-3}{n-3}} :$$

$$\because \sum_{z=0}^{n-3} \frac{\binom{K-3}{z}\binom{N-K}{n-3-z}}{\binom{N-3}{n-3}} = 1$$

so;

$$\Rightarrow \cancel{K}\ E[x(x-1)(x-2)] = \frac{Kn}{N} \frac{(K-1)(K-2)(n-1)(n-2)}{(N-1)(N-2)}$$

so;

$$E[x(x-1)(x-2)] + 3E[x(x-1)] + E[x] = E[x^3]$$

$$E[x^3] = \frac{Kn}{N} \frac{(K-1)(K-2)(n-1)(n-2)}{(N-1)(N-2)} + 3\frac{Kn}{N}\frac{(K-1)(n-1)}{(N-1)} + \frac{Kn}{N}$$

$$\boxed{E[x^3] = \frac{Kn}{N}\left[\frac{(K-1)(K-2)(n-1)(n-2)}{(N-1)(N-2)} + 3\frac{(K-1)(n-1)}{(N-1)} + 1\right]} = \mu_3'$$

Date

now; $r = 4$

$$E[x^4] = E[x(x-1)(x-2)(x-3)] + 6E[x(x-1)(x-2)] + 7E[x(x-1)] + E[x]$$

considering; $E[x(x-1)(x-2)(x-3)]$.

$$\Rightarrow \sum_{x=0}^{n} \frac{\cancel{x(x-1)(x-2)(x-3)}\, K(K-1)(K-2)(K-3)(K-4)!}{(K-x)!\,\cancel{x(x-1)(x-2)(x-3)}\,(x-4)!} \cdot \frac{\frac{(N-K)!}{(N-K-n-x)!\,(n-x)!}}{\frac{N(N-1)(N-2)(N-3)(N-4)!}{(N-n)!\,n(n-1)(n-2)(n-3)(n-4)!}}$$

$$\Rightarrow \frac{K(K-1)(K-2)(K-3)n(n-1)(n-2)(n-3)}{N(N-1)(N-2)(N-3)} \sum_{x=0}^{n} \frac{\frac{(K-4)!}{(K-x)!(x-4)!} \cdot \frac{(N-K)!}{(N-K-n-x)!\,(n-x)!}}{\frac{(N-4)!}{(N-n)!\,(n-4)!}}$$

Let; $z = x-4$; if $x = 4$; if $x = n$

$x = z+4$; $z = 4-4 = 0$; $z = n-4$

$$\Rightarrow \frac{Kn}{N} \frac{(K-1)(K-2)(K-3)(n-1)(n-2)(n-3)}{(N-1)(N-2)(N-3)} \cdot \sum_{z=0}^{n-4} \frac{\frac{(K-4)!}{(K-4-z)!\,z!} \cdot \frac{(N-K)!}{(N-K-n-z-4)!\,(n-z-4)!}}{\binom{N-4}{n-4}}$$

$$\Rightarrow \frac{Kn}{N} \frac{(K-1)(K-2)(K-3)(n-1)(n-2)(n-3)}{(N-1)(N-2)(N-3)} \frac{\sum_{z=0}^{n-4} \binom{K-4}{z}\binom{N-K}{n-4-z}}{\binom{N-4}{n-4}}$$

$$\sum_{z=0}^{n-4} \binom{K-4}{z}\binom{N-K}{n-4-\cancel{K}z} \Big/ \binom{N-4}{n-4} = 1$$

$$E[x(x-1)(x-2)(x-3)] = \frac{Kn}{N} \frac{(K-1)(K-2)(K-3)(n-1)(n-2)(n-3)}{(N-1)(N-2)(N-3)} \quad (1)$$

so; $E[x(x-1)(x-2)(x-3)] + 6E[x(x-1)(x-2)] + 7E[x(x-1)] + E[x]$

$$E[x^4] = \frac{Kn}{N} \frac{(K-1)(K-2)(K-3)(n-1)(n-2)(n-3)}{(N-1)(N-2)(N-3)} + \frac{6Kn}{N} \frac{(K-1)(K-2)(n-1)(n-2)}{(N-1)(N-2)}$$

$$+ \frac{7Kn}{N} \frac{(K-1)(n-1)}{(N-1)} + \frac{Kn}{N}$$

$$\boxed{E[x^4] = \mu'_4 = \frac{Kn}{N}\left[\frac{(K-1)(K-2)(K-3)(n-1)(n-2)(n-3)}{(N-1)(N-2)(N-3)} + \frac{6(K-1)(K-2)(n-1)(n-2)}{(N-1)(N-2)} + \frac{7(K-1)(n-1)}{(N-1)} + 1\right]}$$

Date

Moments about Mean:

$\because \mu_2 = \mu_2' - (\mu_1')^2$

$$\mu_2 = \frac{kn}{N}\left[\frac{(K-1)(n-1)}{(N-1)} + 1\right] - \frac{k^2n^2}{N^2}$$

$$\boxed{M_2 = \frac{Kn}{N}\left[\frac{(K-1)(n-1)}{(N-1)} + 1 - \frac{kn}{N}\right] = var(x) \qquad S.D = \sqrt{\frac{Kn}{N}\left[\frac{(k-1)(n-1)}{(N-1)} + - \frac{kn}{N}\right]}}$$

$\because \mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2(\mu_1')^3$

$$\mu_3 = \frac{kn}{N}\left[\frac{(K-1)(K-2)(n-1)(n-2)}{(N-1)(N-2)} + \frac{3(K-1)(n-1)}{(N-1)} + 1\right] - \frac{3Kn}{N}\left[\frac{(K-1)(n-1)}{(N-1)} + 1\right]\left[\frac{Kn}{N}\right] + \frac{2K^3n^3}{N^3}$$

$$\boxed{\mu_3 = \frac{Kn}{N}\left\{\frac{(K-1)(K-2)(n-1)(n-2)}{(N-1)(N-2)} + \frac{3(K-1)(n-1)}{(N-1)} + 1 - \frac{3Kn}{N}\left[\frac{(K-1)(n-1)}{(N-1)} + 1\right] + \frac{2K^2n^2}{N^2}\right\}}$$

$\because \mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'(\mu_1')^2 - 3(\mu_1')^4$

$$\mu_4 = \frac{Kn}{N}\left[\frac{(K-1)(K-2)(K-3)(n-1)(n-2)(n-3)}{(N-1)(N-2)(N-3)} + \frac{6(K-1)(K-2)(n-1)(n-2)}{(N-1)(N-2)} + \frac{7(K-1)(n-1)}{(N-1)} + 1\right]$$
$$- \frac{4K^2n^2}{N^2}\left[\frac{(K-1)(K-2)(n-1)(n-2)}{(N-1)(N-2)} + \frac{3(K-1)(n-1)}{(N-1)} + 1\right] + \frac{6K^3n^3}{N^3}\left[\frac{(K-1)(n-1)}{(N-1)} + 1\right]$$
$$- \frac{3K^4n^4}{N^4}$$

$$\Rightarrow \mu_4 = \frac{Kn}{N}\left\{\frac{(K-1)(K-2)(K-3)(n-1)(n-2)(n-3)}{(N-1)(N-2)(N-3)} + \frac{6(K-1)(K-2)(n-1)(n-2)}{(N-1)(N-2)} + \frac{7(K-1)(n-1)}{(N-1)} + 1\right.$$
$$- \frac{4Kn}{N}\left[\frac{(K-1)(K-2)(n-1)(n-2)}{(N-1)(N-2)} + \frac{3(K-1)(n-1)}{(N-1)} + 1\right] + \frac{6K^2n^2}{N^2}\left[\frac{(K-1)(n-1)}{(N-1)} + 1\right]$$
$$\left. - \frac{3K^3n^3}{N^3}\right\}$$

Date

Skewness and kurtosis:

$$\Rightarrow \beta_1 = \frac{(\mu_3)^2}{(\mu_2)^3}$$

sq

$$\beta_1 = \frac{\left[\frac{(k-1)(k-2)(n-1)(n-2)}{(N-1)(N-2)} + \frac{3(k-1)(n-1)}{(N-1)} + 1 - \frac{3kn}{N}\left[\frac{(k-1)(n-1)}{(N-1)} + 1\right] + \frac{2k^2n^2}{N^2}\right]^2}{\frac{kn}{N}\left[\frac{(k-1)(n-1)}{(N-1)} + 1 - \frac{kn}{N}\right]^3}$$

$$\Rightarrow \beta_2 = \frac{\mu_4}{(\mu_2)^2}$$

$$\beta_2 = \frac{\left[\frac{(k-1)(k-2)(k-3)(n-1)(n-2)(n-3)}{(N-1)(N-2)(N-3)} + \frac{6(k-1)(k-2)(n-1)(n-2)}{(N-1)(N-2)} + \frac{7(k-1)(n-1)}{(N-1)} + 1 - \frac{4kn}{N}\left[\frac{(k-1)(k-2)(n-1)(n-2)}{(N-1)(N-2)} + \frac{3(k-1)(n-1)}{(N-1)} + 1\right] + \frac{6k^2n^2}{N^2}\left[\frac{(k-1)(n-1)}{(N-1)} + 1\right] - \frac{3k^3n^3}{N^3}\right]}{\frac{kn}{N}\left[\frac{(k-1)(n-1)}{(N-1)} - \frac{kn}{N} + 1\right]}$$

Hypergeometric test(Experiment):

An experiment is called hypergeometric experiment iff

a) The outcome of the experiment is classified into two categories, successes and failures.

b) The probability of successes changes from trial to trial

c) Successive trials are dependent

d) The experiment is repeated upto a fixed number of time.

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Uniform Distribution:

A random variable 'x' has the discrete uniform distribution if it has a finite number of possible values i.e; $x_1, x_2, x_3 \dots x_n$ and

$$f(x_i) = f(X = x_i) = \frac{1}{n} \quad \text{where; } i = 1, 2, 3 \dots n$$

If 'x' has discrete uniform distribution on the consecutive integers $a, a+1, a+2, \dots, b$ then;

$$\mu = E[X] = \frac{b+a}{2}$$

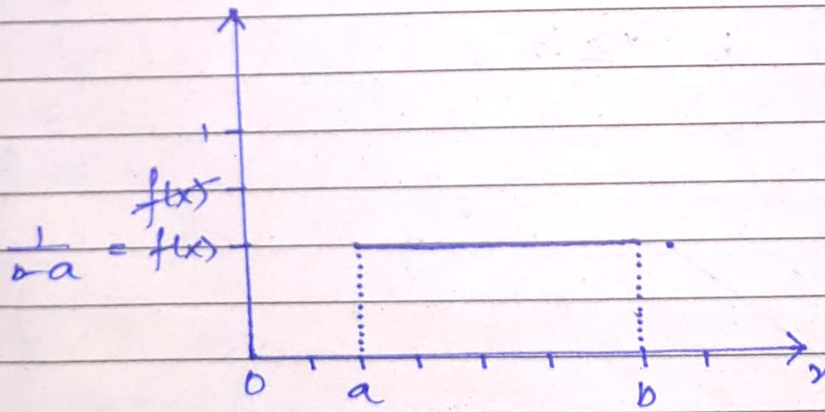
and

$$var(x) = \sigma^2 = \frac{(b-a+1)^2 - 1}{12}$$

Continous Uniform Distribution:

A random variable 'x' has a Uniform distribution if it's pdf is given by;

$$f(x;, a, b) = f(x) = \frac{1}{b-a} \quad ; \quad a \leq x \leq b$$



Since in Uniform distribution, the Area is of a rectangles; $\therefore$ $A = L \times b$ and equals 1 over the interval (a, b) so;

$$(b-a) \times f(x) = 1 \quad \text{so;}$$

$$\boxed{f(x) = \frac{1}{b-a}}$$

Forms of Uniform Distribution :

1. $f(x) = \frac{1}{b-a}$; $a < x < b$

2. $f(x) = \begin{cases} 1/\theta & ; 0 < x < \theta \\ 0 & ; \text{elsewhere} \end{cases}$ $\because b = \theta,\ a = 0$

3. $f(x) = \begin{cases} \frac{1}{2\theta} & ; -\theta < x < \theta \\ 0 & ; \text{else where} \end{cases}$ $\because b = \theta,\ a = -\theta$

4. $f(x) = \begin{cases} 1 & ; 0 < x < 1 \\ 0 & ; \text{elsewhere} \end{cases}$ $\because b = 1,\ a = 0$

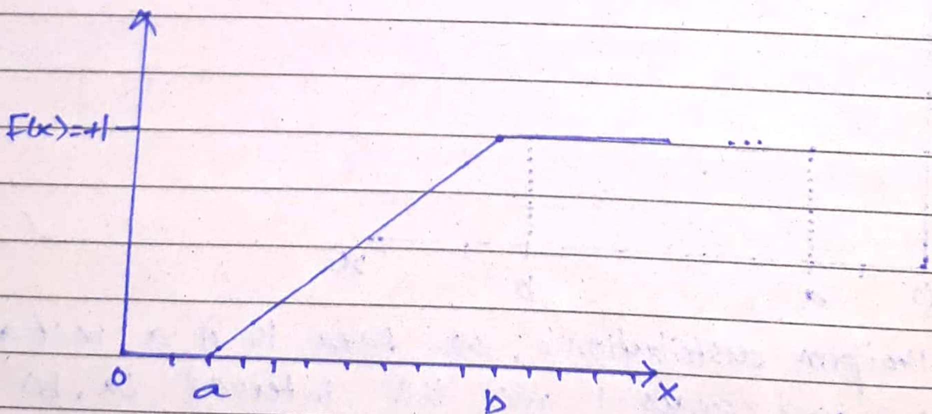
Cumulative distribution function : (cdf)

$$F(x) = \int_a^x f(x)\,dx = \frac{1}{b-a}\int_a^x dx = \frac{1}{b-a}[x]_a^x$$

$$F(x) = \frac{x-a}{b-a}$$

Thus;

$$F(x) = \begin{cases} 0 & ; x < a \\ \frac{x-a}{b-a} & ; a \le x \le b \\ 1 & ; x \ge b \end{cases}$$



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Moments about origin:

r^{th} moment about origin is given by:

$$\mu'_r = E[x^r] = \int_a^b f(x)\, x^r\, dx = \frac{1}{b-a}\int_a^b x^r dx = \frac{1}{b-a}\left[\frac{x^{r+1}}{r+1}\right]_a^b$$

$$\mu'_r = E[x^r] = \frac{1}{b-a}\left[\frac{b^{r+1}}{r+1} - \frac{a^{r+1}}{r+1}\right]$$

$$\mu'_r = E[x^r] = \frac{b^{r+1} - a^{r+1}}{(b-a)(r+1)}$$

at $r=1$

$$E[X] = \frac{b^2-a^2}{(b-a).2} = \frac{\cancel{(b-a)}(b+a)}{\cancel{(b-a)}.2} \boxed{= \frac{b+a}{2}}$$

at $r=2$

$$E[x^2] = \frac{b^3-a^3}{(b-a)(3)} \Rightarrow \frac{\cancel{(b-a)}(a^2+ab+b^2)}{3.\cancel{(b-a)}} = \boxed{\frac{b^2+ab+a^2}{3}}$$

at $r=3$

$$E[x^3] = \frac{b^4-a^4}{(b-a).(4)} \Rightarrow \frac{(b^2)^2-(a^2)^2}{(b-a).(4)} = \frac{(b^2-a^2)(b^2+a^2)}{4.(b-a)}$$

$$E[x^3] = \frac{\cancel{(b-a)}(b+a)(b^2+a^2)}{4.\cancel{(b-a)}} = \boxed{\frac{b^3+a^2b+ab^2+a^3}{4}}$$

at $r=4$

$$E[x^4] = \frac{b^5-a^5}{5(b-a)} = \frac{\cancel{(b-a)}(b^4+ab^3+a^2b^2+a^3b+a^4)}{5\cancel{(b-a)}}$$

$$E[x^4] = \boxed{\frac{a^4+a^3b+a^2b^2+ab^3+b^4}{5}}$$

Moments about Mean:

$$\because \mu_2 = \mu'_2 - (\mu'_1)^2$$

$$\mu_2 = \frac{b^2+ab+a^2}{3} - \left(\frac{b+a}{2}\right)^2$$

$$\mu_2 = \frac{b^2+ab+a^2}{3} - \frac{a^2+2ab+b^2}{4}$$

$$\mu_2 = \frac{4b^2+4ab+4a^2-3a^2-6ab-3b^2}{12}$$

$$\mu_2 = \frac{b^2-2ab+a^2}{12}$$

Date

$$\boxed{\mu_2 = \frac{(b-a)^2}{12}}$$

$$\because \mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2(\mu_1')^3$$

$$\mu_3 = \frac{b^3+a^2b+ab^2+a^3}{4} - \cancel{3}\left[\frac{b^2+ab+a^2}{\cancel{3}}\left(\frac{b+a}{2}\right)\right] + 2\left(\frac{b+a}{2}\right)^3$$

$$\mu_3 = \frac{b^3+a^2b+ab^2+a^3}{4} - \frac{b^3+ab^2+ab^2+a^2b+a^2b+a^3}{2} + \frac{(b+a)^3}{4}$$

$$\mu_3 = \frac{b^3+a^2b+ab^2+a^3-2b^3-4ab^2-4a^2b-2a^3}{4} + \frac{(b+a)^3}{4}$$

$$\mu_3 = \frac{-b^3-3a^2b-3ab^2-a^3}{4} + \frac{(b+a)^3}{4}$$

$$\mu_3 = -\frac{(b+a)^3}{4} + \frac{(b+a)^3}{4}$$

$$\boxed{\mu_3 = 0}$$

$$\because \mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'(\mu_1')^2 - 3(\mu_1')^4$$

$$\mu_4 = \frac{a^4+a^3b+a^2b^2+ab^3+b^4}{5} - \cancel{4}\left[\frac{b+a}{2}\right]\left[\frac{b^3+ab^2+a^2b+a^3}{\cancel{4}}\right]$$

$$+6\left[\frac{b^2+ab+a^2}{3}\right]\left[\frac{b+a}{2}\right]^2 - 3\left[\frac{b+a}{2}\right]^4$$

$$\mu_4 = \frac{a^4+a^3b+a^2b^2+ab^3+b^4}{5} - \left[\frac{b^4+ab^3+a^2b^2+a^3b+ab^3+a^2b^2+a^3b+a^4}{2}\right]$$

$$+6\left[\frac{(b^2+ab+a^2)(b^2+2ab+a^2)}{12}\right] - 3\left[\frac{(a^2+2ab+b^2)(a^2+2ab+b^2)}{16}\right]$$

$$\mu_4 = \frac{2a^4+2a^3b+2a^2b^2+2ab^3+2b^4-5b^4-10ab^3-10a^2b^2-10a^3b-5a^4}{10}$$

$$+\frac{b^4+2ab^3+a^2b^2+ab^3+2a^2b^2+a^3b+a^2b^2+2a^3b+a^4}{2}$$

$$-3\left[\frac{a^4+2a^3b+a^2b^2+2a^3b+4a^2b^2+2ab^3+a^2b^2+2ab^3+b^4}{16}\right]$$

$$\mu_4 = \frac{-3b^4-3a^4-8a^3b-8a^2b^2-8ab^3}{10} + \frac{b^4+a^4+3ab^3+4a^2b^2+3a^3b}{2}$$

$$-3\left[\frac{b^4+a^4+4a^3b+6a^2b^2+4ab^3}{16}\right]$$

Date

$$\mu_4 = \frac{-3b^4-3a^4-8a^3b-8a^2b^2-8ab^3+5b^4+5a^4+15ab^3+20a^2b^2+15a^3b}{10}$$

$$+\left[\frac{-3b^4-3a^4-12a^3b-18a^2b^2-12ab^3}{16}\right]$$

$$\mu_4 = \frac{2b^4+2a^4+7a^3b+7ab^3+12a^2b^2}{10}+\left[\frac{-3b^4-3a^4-12a^3b-18a^2b^2-12ab^3}{16}\right]$$

$$\mu_4 = \frac{16b^4+16a^4+56a^3b+56ab^3+96a^2b^2-15b^4-15a^4-60a^3b-\cancel{90}-90a^2b^2-60ab^3}{80}$$

$$\mu_4 = \frac{b^4+a^4-4a^3b-4ab^3+6a^2b^2}{80} = \frac{(b-a)^4}{80}$$

$$\boxed{\mu_4 = \frac{(b-a)^4}{80}}$$

Skewness and kurtosis:

$$\beta_1 = \frac{(\mu_3)^2}{(\mu_2)^3} = 0$$

$$\beta_2 = \frac{\mu_4}{(\mu_2)^4} = \frac{\frac{\cancel{(b-a)^4}}{80}}{\frac{\cancel{(b-a)^4}}{144}} \Rightarrow \frac{144}{80} = \frac{18}{10} = \frac{9}{5}$$

$$\boxed{\beta_2 = 9/5}$$

Date

Mgf (Moment generating function):

By definition;

$$M_x(t) = E[e^{tx}] = \int_a^b f(x)\, e^{tx}\, dx$$

$$M_x(t) = \frac{1}{b-a}\int_a^b e^{tx}\cdot dx$$

$$M_x(t) = \frac{1}{t(b-a)}\left[e^{tx}\right]_a^b$$

$$M_x(t) = \frac{e^{tb} - e^{ta}}{t\,(b-a)}$$

$$\because e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} \dots$$

so;

$$M_x(t) = \frac{1}{t(b-a)}\left[\left(1 + \frac{tb}{1!} + \frac{t^2b^2}{2!} + \frac{t^3b^3}{3!} + \frac{t^4b^4}{4!} + \frac{t^5b^5}{5!} \dots\right) - \left(1 + \frac{ta}{1!} + \frac{t^2a^2}{2!} + \frac{t^3a^3}{3!} + \frac{t^4a^4}{4!} + \frac{t^5a^5}{5!} \dots\right)\right]$$

$$M_x(t) = \frac{1}{t(b-a)}\left[\frac{t(b-a)}{1!} + \frac{t^2(b^2-a^2)}{2!} + \frac{t^3(b^3-a^3)}{3!} + \frac{t^4(b^4-a^4)}{4!} + \frac{t^5(b^5-a^5)}{5!} \dots\right]$$

$$M_x(t) = \frac{1}{\cancel{t(b-a)}}\cdot t(b-a)\left[\cancel{1 + \frac{t(b+a)}{2!}} + \frac{t^2(b+a)(b^2+a)}{3!} + t^3\right.$$

$$M_x(t) = \frac{1}{t(b-a)}\cdot t(b-a)\left[1 + \frac{t(b+a)}{2!} + \frac{t^2(a^2+ab+b^2)}{3!} + \frac{t^3(b+a)(b^2+a^2)}{4!} + \frac{t^4(a^4+a^3b+a^2b^2+ab^3+b^4)}{5!} \dots\right]$$

finding and comparing of coefficients of $\frac{t^x}{x!}$

$$\mu_1' = \frac{t}{1!} = \frac{b+a}{2} \qquad \mu_4' = \frac{t^4}{4!} = \frac{a^4+a^3b+a^2b^2+ab^3+b^4}{5}$$

$$\mu_2' = \frac{t^2}{2!} = \frac{a^2+ab+b^2}{3}$$

$$\mu_3' = \frac{t^3}{3!} = \frac{(b+a)(b^2+a^2)}{4}$$

Date

Median:

$$F(M) = \int_a^M f(x)\,dx = \frac{1}{2}$$

$$= \frac{1}{b-a}\int_a^M dx = \frac{1}{2}$$

$$\Rightarrow \frac{1}{b-a}[x]_a^M = \frac{M-a}{b-a} = \frac{1}{2}$$

$$\Rightarrow 2M-2a = b-a$$

$$2M = b+a$$

$$\boxed{M = \frac{b+a}{2}}$$

∵ according to the graph it is obsewed that it is symetric Mean = Median = Mode.

Date:

Exponential Distribution:

A continous Random vaiiable x has an exponential distribution, and it is reffered to as an exponential random vaiiable, if and only if it's pdf is given by;

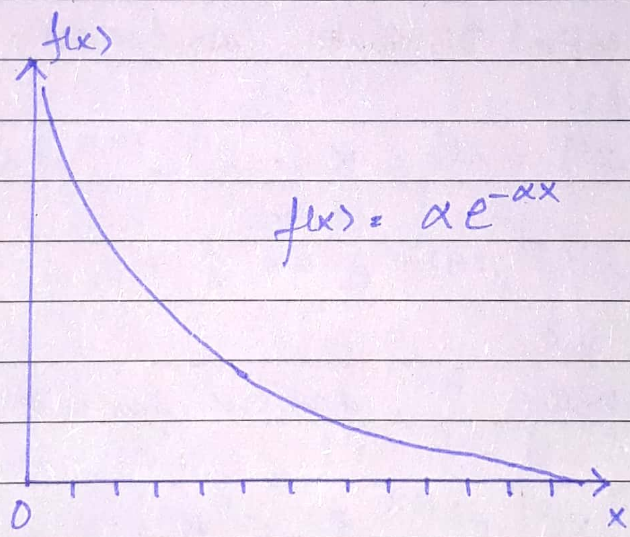
$$f(x) = \begin{cases} \frac{1}{\theta} e^{-x/\theta} & ; \text{ for } x > 0 \\ 0 & ; \text{ elsewhere} \end{cases}$$

or;

$$f(x) = \begin{cases} \alpha e^{-\alpha x} & ; \text{ for } x > 0 \\ 0 & ; \text{ elsewhere} \end{cases}$$

where $x > 0$

The p.d.f is



The distribution function:

$$F(x) = \int_0^x f(x)\,dx$$

$$= \alpha \int_0^x e^{-\alpha x}\,dx$$

$$= \cancel{\alpha} \left[\frac{e^{-\alpha x}}{-\cancel{\alpha}} \right]_0^x$$

$$= \left[-e^{-\alpha x} \right]_0^x$$

$$\boxed{F(x) = 1 - e^{-\alpha x}}$$

Page

Date:

P.d.f:

$$\Rightarrow \int_0^\infty f(x)\,dx = 1 \qquad \because f(x) = \alpha e^{-\alpha x}$$

$$\Rightarrow \alpha \int_0^\infty e^{-\alpha x}\,dx = 1$$

$$\Rightarrow \cancel{\alpha}\left[-\frac{e^{-\alpha x}}{\cancel{\alpha}}\right]_0^\infty$$

$$\Rightarrow \left[-e^{-\alpha x}\right]_0^\infty$$

$$\Rightarrow -e^{-\alpha(\infty)} - (-e^{-\alpha(0)})$$

$$\Rightarrow 0 - (-1)$$

$$\Rightarrow 1$$

Moments of Exponential Distribution (about origin):

general formula:

$$E[x^r] = \mu'_r = \alpha \int_0^\infty x^r e^{-\alpha x}\,dx$$

$$\Rightarrow \alpha \int_0^\infty x^{(r+1)-1} e^{-\alpha x}\,dx$$

using property of gamma function;

$$\because \alpha \int_0^\infty x^{n-1} e^{-\alpha x}\,dx = \frac{\alpha\,\Gamma n}{\alpha^n}$$

so;

$$\Rightarrow \frac{\alpha\,\Gamma(r+1)}{\alpha^{r+1}} = \frac{\Gamma(r+1)}{\alpha^r}$$

$$\boxed{E[x^r] = \mu'_r = \frac{\Gamma(r+1)}{\alpha^r}}$$

Date: ____

$\because$ property of gamma function;

$$\Gamma(n+1) = n!$$

or $$\Gamma(n) = (n-1)!$$

so;

at $r=1$

$$\boxed{E[X] = \mu_1' = \frac{\Gamma(2)}{\alpha} = \frac{1}{\alpha}}$$

at $r=2$

$$\boxed{E[X^2] = \mu_2' = \frac{\Gamma(3)}{\alpha^2} = \frac{2}{\alpha^2}}$$

at $r=3$

$$\boxed{E[X^3] = \mu_3' = \frac{\Gamma(4)}{\alpha^3} = \frac{6}{\alpha^3}}$$

at $r=4$

$$\boxed{E[X^4] = \mu_4' = \frac{\Gamma(5)}{\alpha^4} = \frac{24}{\alpha^4}}$$

Moments about mean:

$\because \mu_2 = \mu_2' - (\mu_1')^2$

$$\mu_2 = \frac{2}{\alpha^2} - \frac{1}{\alpha^2}$$

$$\boxed{\mu_2 = Var(X) = \frac{1}{\alpha^2}}$$

$$\boxed{S.D = 1/\alpha}$$

$\because \mu_3 = \mu_3' - 3\mu_1'\mu_2' + 2(\mu_1')^3$

$$\mu_3 = \frac{6}{\alpha^3} - 3\left(\frac{2}{\alpha^2}\right)\left(\frac{1}{\alpha}\right) + \frac{2}{\alpha^3}$$

$$\mu_3 = \frac{6}{\alpha^3} - \frac{6}{\alpha^3} + \frac{2}{\alpha^3}$$

$$\boxed{\mu_3 = 2/\alpha^3}$$

Date:

$\because \mu_4 = \mu'_4 - 4\mu'_3\mu'_1 + 6\mu'_2(\mu'_1)^2 - 3(\mu'_1)^4$

$$\mu_4 = \frac{24}{\alpha^4} - 4\left(\frac{6}{\alpha^3}\right)\left(\frac{1}{\alpha}\right) + 6\left(\frac{2}{\alpha^2}\right)\left(\frac{1}{\alpha^2}\right) - \frac{3}{\alpha^4}$$

$$\boxed{\mu_4 = 9/\alpha^4}$$

Skewness and kurtosis:

$$\because \beta_1 = \frac{(\mu_3)^2}{(\mu_2)^3} = \frac{4/\cancel{\alpha^6}}{1/\cancel{\alpha^6}} = 4$$

$$\boxed{\beta_1 = 4}$$

$$\because \beta_2 = \frac{\mu_4}{(\mu_2)^2} = \frac{9/\cancel{\alpha^4}}{1/\cancel{\alpha^4}} = 9$$

$$\boxed{\beta_2 = 9}$$

Median:

By definition;

$$\int_0^M f(x)\,dx = 1/2$$

$$\Rightarrow \alpha \int_0^M e^{-\alpha x}\,dx = \frac{1}{2}$$

$$\Rightarrow \cancel{\alpha}\left[-\frac{e^{-\alpha x}}{\cancel{\alpha}}\right]_0^M = \frac{1}{2}$$

$$\Rightarrow 1 - e^{-M\alpha} = \frac{1}{2}$$

$$\Rightarrow e^{-M\alpha} = \frac{1}{2}$$

$$-M\alpha = \ln(1/2)$$

$$M\alpha = \ln(2)$$

$$\boxed{M = \frac{\ln(2)}{\alpha}}$$

Date:

Quartiles:

By definition:

$\because Q_1 = 1/4$

so;

$$\int_0^{Q_1} f(x)\,dx = \frac{1}{4}$$

$$= \alpha \int_0^{Q_1} e^{-\alpha x}\,dx = \frac{1}{4}$$

$$= \cancel{\alpha}\left[-\frac{e^{-\alpha x}}{\cancel{\alpha}}\right]_0^{Q_1} = \frac{1}{4}$$

$$= 1 - e^{-Q_1 \alpha} = \frac{1}{4}$$

$$e^{-Q_1\alpha} = \frac{3}{4}$$

$$Q_1\alpha = \ln(4/3)$$

$$\boxed{Q_1 = \frac{\ln(4/3)}{\alpha}}$$

for Q_3;

$\because Q_3 = 3/4$

$$\because \int_0^{Q_3} f(x)\,dx = \frac{3}{4}$$

$$\Rightarrow \alpha \int_0^{Q_3} e^{-\alpha x}\,dx = 3/4$$

$$\Rightarrow \cancel{\alpha}\left[-\frac{e^{-\alpha x}}{\cancel{\alpha}}\right]_0^{Q_3} = 3/4$$

$$\Rightarrow 1 - e^{-Q_3\alpha} = 3/4$$

$$e^{-Q_3\alpha} = 1/4$$

$$Q_3\alpha = \ln(4)$$

$$\boxed{Q_3 = \frac{\ln(4)}{\alpha}}$$

Mode:

$\because f(x) = \alpha e^{-\alpha x}$

$f'(x) = \alpha \frac{d}{dx}(e^{-\alpha x})$

$f'(x) = -\alpha^2 e^{-\alpha x}$

$\because f'(x) = 0$

$\because \alpha = \infty$

$\Rightarrow e^{-\infty}$

$\Rightarrow 0$

Moment Generating function (M.G.F):

$\because$ By definition:

$$M_X(t) = E[e^{tx}] = \alpha \int_0^\infty e^{tx} e^{-\alpha x} dx$$

$$= \alpha \int_0^\infty e^{-x(\alpha - t)} dx$$

$$= \alpha \left[-\frac{e^{-x(\alpha-t)}}{\alpha - t} \right]_0^\infty$$

$$= \alpha \left[0 + \frac{1}{\alpha - t} \right]$$

$$\Rightarrow \frac{\alpha}{\alpha - t} = \frac{1}{1 - t/\alpha}$$

$$M_X(t) = \frac{1}{1 - t/\alpha}$$

So;

$$\because \mu'_r = \frac{d^r}{dt^r}\left[\left(\frac{1}{1 - t/\alpha}\right)\right]_{t=0}$$

Date:

at r=1;

$$\mu'_1 = \frac{d}{dt}\left[(1-t/\alpha)^{-1}\right]_{t=0}$$

$$\mu'_1 = \left[(-1)(1-t/\alpha)^{-2}\ (-1/\alpha)\right]_{t=0}$$

$$\mu'_1 = \left[\frac{1}{\alpha(1-t/\alpha)^2}\right]_{t=0}$$

$$\boxed{\mu'_1 = 1/\alpha}$$

at r=2;

$$\mu'_2 = \frac{d}{dt}\left[(-1)(1-t/\alpha)^{-2}(-1/\alpha)\right]_{t=0}$$

$$\mu'_2 = \left[2(1-t/\alpha)^{-3}\ (1/\alpha^2)\right]_{t=0}$$

$$\mu'_2 = \left[\frac{2}{\alpha^2(1-t/\alpha)^3}\right]_{t=0}$$

$$\boxed{\mu'_2 = \frac{2}{\alpha^2}}$$

at r=3;

$$\cancel{\mu'_3 = \frac{d}{dt}\left[-6(1-t/\alpha)^{-4}\ (-1/\alpha^3)\right]_{t=0}}$$

$$\mu'_3 = \frac{d}{dt}\left[2(1-t/\alpha)^{-3}\ (1/\alpha^2)\right]_{t=0}$$

$$\mu'_3 = \left[-6(1-t/\alpha)^{-4}(-1\alpha^3)\right]_{t=0}$$

$$\mu'_3 = \left[\frac{6}{\alpha^3(1-t/\alpha)^4}\right]_{t=0}$$

$$\boxed{\mu'_3 = 6/\alpha^3}$$

Date:

at $X_r = 4$

$$\mu_4' = \frac{d}{dt}\left[6\left(1-t/\alpha\right)^{-4}\left(1/\alpha^3\right)\right]_{t=0}$$

$$\mu_4' = \left[-24\left(1-t/\alpha\right)^{-5}\left(-1/\alpha^4\right)\right]_{t=0}$$

$$\mu_4' = \left[\frac{24}{\alpha^4\left(1-t/\alpha\right)^5}\right]_{t=0}$$

$$\boxed{\mu_4' = \frac{24}{\alpha^4}}$$