## Geometric Distribution:

A random variable X has a geometric distribution iff its probability distribution is given by

$$g(x;p) = q^{x-1} \cdot p \quad ; \quad x = 1, 2, \ldots$$

which is called as geometric distribution with parameter p.

**Proof:**

Consider an experiment with two possible outcomes, success (s) and failure (f), with $P(s) = p$ and $P(f) = 1-p = q$. The experiment is repeated until first success appears. Let X be the number of independent trials required to obtain one success. It means last trial must end in success.

$$\underbrace{f\,f \cdots f}_{x-1} \cdot \underset{1}{s}$$

The probability distribution of $(x-1)$ failures and a success in last trial appear is

$$P(X=x) = (1-p)^{x-1} \cdot p \quad ; \quad x = 1, 2, \ldots$$

or

$$g(x,p) = (1-p)^{x-1} \cdot p \qquad \because q = 1-p$$

$$= q^{x-1}\, p \quad ; \quad x = 1, 2, \ldots$$

Which is called as geometric distribution with parameter p.

**Moments**

Let the random variable X have a geometric distribution then the rth moment about origin is obtained as

$$\mu_r' = E[X^r] = \sum_{x=1}^{\infty} x^r g(x;p) = \sum_{x=1}^{\infty} x^r q^{x-1} p.$$

at $r=1$

$$\mu_1' = p + 2qp + 3q^2 p + 4q^3 p + \cdots$$

$$= p[1 + 2q + 3q^2 + 4q^3 + \cdots]$$

$$= p[1-q]^{-2} = p \cdot p^{-2} = p^{-1}$$

$$\boxed{\text{Mean} = \mu_1' = \frac{1}{p}}$$

$$\mu_1 = p$$

at $r=2$

$$\mu_2' = E[X^2] = \sum_{x=1}^{\infty} x^2 q^{x-1} p$$

$$= p + 2^2 qp + 3^2 q^2 p + 4^2 q^3 p + 5^2 q^4 p + \cdots$$

$$= p[1 + 4q + 9q^2 + 16q^3 + \cdots\cdots]$$

$$= p[(1+3q+6q^2+10q^3+\cdots) + (q+3q^2+6q^3+\cdots)]$$

$$= p[(1-q)^{-3} + q(1-q)^{-3}]$$

$$= \frac{p}{p^3} + \frac{pq}{p^3}$$

$$\boxed{\mu_2' = \frac{1}{p^2} + \frac{q}{p^2}}$$

$$\text{Variance} = \mu_2 = \mu_2' - \mu_1'^2 = \cancel{\frac{1}{p^2}} + \frac{q}{p^2} - \cancel{\frac{1}{p^2}} = \frac{q}{p^2}$$

$$\mu_2 = \frac{(1-p)}{p^2} = \frac{1}{p^2} - \frac{1}{p}$$

$$\Rightarrow \boxed{Var(X) = \mu_2 = \frac{1-p}{p^2}} = \text{or}. \boxed{S.D(X) = \frac{\sqrt{1-p}}{p}}$$

<u>M.g.f</u>

The m.g.f of the geometric Distribution is derived as.

By definition: $M_X(t) = E[e^{tx}] = \sum_{x=1}^{\infty} e^{tx} q^{x-1} p$

$$= \sum_{x=1}^{\infty} e^t \cdot e^{t(x-1)} q^{x-1} p = pe^t \sum_{x=1}^{\infty} (e^t q)^{x-1}$$

$$= pe^t [1 + qe^t + (qe^t)^2 + (qe^t)^3 + \cdots]$$

$$= pe^t [1 - qe^t]^{-1}$$

$$\Rightarrow \boxed{M_X(t) = \frac{pe^t}{1-qe^t}} \quad ; \text{ where } qe^t < 1$$

<u>Moments</u> $\mu_r' = \left\{\frac{d^r}{dt^r} M_X(t)\right\}_{t=0}$.

$$\mu_r' = \left\{\frac{d}{dt^r}\left(\frac{pe^t}{1-qe^t}\right)\right\}_{t=0}$$

$$\therefore M_X(t) = \frac{pe^t}{1-qe^t} = \frac{p}{e^{-t}-q} = \boxed{p(e^{-t}-q)^{-1} = M_X(t)}$$

at $r=1$

$$\mu_1' = \left[\frac{d}{dt}\left\{p(e^{-t}-q)^{-1}\right\}\right]_{t=0} = (-1)p(e^{-t}-q)^{-2}(-e^{-t})\Big|_{t=0}$$

$$= pe^{-t}(e^{-t}-q)^{-2}\Big|_{t=0} = p\cdot(1-q)^{-2} = p\cdot p^{-2} = p^{-1}$$

$$\boxed{\mu_1' = \frac{1}{p}}$$

$$\mu_2' = \left\{\frac{d^2}{dt^2}\, p(e^{-t}-q)^{-1}\right\}_{t=0} = \left\{\frac{d}{dt}\left[pe^{-t}(e^{-t}-q)^{-2}\right]\right\}_{t=0}$$

$$= \left[2pe^{-2t}(e^{-t}-q)^{-2-1} - pe^{-t}(e^{-t}-q)^{-2}\right]_{t=0}$$

$$= 2p(1-q)^{-3} - p(1-q)^{-2}$$

$$= \frac{2p}{p^3} - \frac{p}{p^2}$$

$$\boxed{\mu_2' = \frac{2}{p^2} - \frac{1}{p}}$$

$$\mu_2 = \mu_2' - \mu_1'^2 = \frac{2}{p^2} - \frac{1}{p} - \frac{1}{p^2} = \frac{1}{p^2} - \frac{1}{p}.$$

$$\boxed{\mu_2 = \frac{1-p}{p^2}} \Rightarrow \boxed{Var(X) = \frac{1-p}{p^2} = \frac{q}{p^2}} \text{ or } \boxed{S.D(X) = \frac{\sqrt{1-p}}{p} = \frac{\sqrt{q}}{p}}$$

Similarly we can find

$$\mu_3' = \frac{6q^2 + 6pq + p^2}{p^3}.$$

and

$$\mu_4' = \frac{24q^3}{p^4} + \frac{36q^2}{p^3} + \frac{14q}{p^2} + \frac{1}{p}$$

and

$$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2\mu_1'^3$$

we get

$$\boxed{\mu_3 = \frac{q(2-p)}{p^3}}$$

and

$$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'\mu_1'^2 - 3\mu_1'^4$$

we get

$$\boxed{\mu_4 = \frac{3q^2 + q(p^2 - 6p + 6)}{p^4}.}$$

Derive the recurrence relation for geometric distribution and find $\beta_1$ and $\beta_2$.

$$\mu_{r+1} = q\left[\frac{r}{p^2}\mu_{r-1} - q\cdot\frac{d\mu_r}{dp}\right]$$

Proof:

By definition $\mu_r = E[(X-\text{mean})^r] = \sum_{x=1}^{\infty}(x-\frac{1}{p})^r\cdot(1-p)^{x-1}\cdot p$

$$\frac{d\mu_r}{dp} = \sum_{x=1}^{\infty}\left[p\cdot(1-p)^{x-1}\cdot r\cdot(x-\tfrac{1}{p})^{r-1}\cdot(\tfrac{1}{p^2}) + (x-\tfrac{1}{p})^r\cdot(x-1)(1-p)^{x-2}\cdot(-1)p + (x-\tfrac{1}{p})^r\cdot(1-p)^{x-1}\cdot(1)\right]$$

$$= \sum_{x=1}^{\infty}\left[\frac{r}{p^2}\cdot(x-\tfrac{1}{p})^{r-1}\cdot(1-p)^{x-1}\cdot p - (x-\tfrac{1}{p})^r\cdot(1-p)^{x-2}p\cdot(x-1) + (x-\tfrac{1}{p})^r(1-p)^{x-1}\right]$$

$$= \frac{r}{p^2}\mu_{r-1} + \sum_{x=1}^{\infty}(x-\tfrac{1}{p})^r(1-p)^{x-1}\cdot p\left[-\frac{(x-1)}{q} + \frac{1}{p}\right]$$

$$= \frac{r}{p^2}\mu_{r-1} + \sum_{x=1}^{\infty}(x-\tfrac{1}{p})^r(1-p)^{x-1}\cdot p\left[\frac{-px+p+q}{pq}\right]$$

$$= \frac{r}{p^2}\mu_{r-1} + \sum_{x=1}^{\infty}(x-\tfrac{1}{p})^r(1-p)^{x-1}\cdot p\left[-\frac{1}{q}(x-\tfrac{1}{p})\right]$$

$$= \frac{r}{p^2}\mu_{r-1} + \sum_{x=1}^{\infty}(x-\tfrac{1}{p})^r(1-p)^{x-1}\cdot p$$

$$\frac{d\mu_r}{dp} = \frac{r}{p^2}\mu_{r-1} - \frac{1}{q}\mu_{r+1} \Rightarrow \boxed{\mu_{r+1} = q\left[\frac{r}{p^2}\mu_{r-1} - \frac{d\mu_r}{dp}\right]}$$

At $r=1$

$$\mu_2 = q\left[\frac{1}{p^2}\mu_0 - \frac{d\mu_1}{dp}\right] = q\left[\frac{1}{p^2} - 0\right] = \boxed{\frac{q}{p^2} = \mu_2}$$

$\because \mu_2 = \frac{q}{p^2} = (1-p)p^{-2}$

$\frac{d\mu_2}{dp} = -2p^{-3} + p^{-2}$

At $r=2$

$$\mu_3 = q\left[\frac{2}{p^2}\mu_1 - \frac{d\mu_2}{dp}\right] = q\left[0 - (-2p^{-3}+p^{-2})\right]$$

$$\mu_3 = q\left[\frac{2}{p^3} - \frac{1}{p^2}\right] = \boxed{q(2-p)/p^3 = \mu_3}$$

At $r=3$.

$$\mu_4 = q\left[\frac{3}{p^2}\mu_2 - \frac{d\mu_3}{dp}\right] \Rightarrow \because \frac{d\mu_3}{dp} = -\frac{1}{p^2} + \frac{6}{p^3} - \frac{6}{p^4}$$

$$\therefore \mu_4 = q\left[\frac{3q}{p^4} + \frac{1}{p^2} - \frac{6}{p^3} + \frac{6}{p^4}\right] = q\left[\frac{3q+p^2-6p+6}{p^4}\right]$$

$$\boxed{\mu_4 = \frac{3q^2}{p^4} + \frac{q}{p^4}(p^2-6p+6)}$$

$$\beta_1 = \frac{\mu_3^2}{\mu_2^3} = \frac{q^2(2-p)^2}{p^6} \Big/ \frac{q^3}{p^6} = \frac{q^2(2-p)^2}{q^3}$$

$$\boxed{\beta_1 = (1+q)^2/q}$$

$$\beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{3q^2+q(p^2-6p+6)}{q^2} = \boxed{3 + \frac{p^2-6p+6}{q} = \beta_2}$$

$(n-2)!\ (N-n)!$

oblem: If the probability ~~do~~ is 0.75 that an applicant will pass the road test on any try, what is the probability that an applicant will finally pass the test on the fourth try.

Sol: $p = 0.75$, $q = 1 - 0.75 = 0.25$

$x = 4$; Assuming trials are independent.

$g(x;p) = g(4;0.75) = q^{x-1}p = (0.25)^{4-1}(0.75) = 0.0117$

is the probability that an applicant pass the test at 4th trials.

Problem 2: Three people toss a coin and the odd man pays for the coffee. If the coins all turn up the same, they are tossed again. Find the probability that fewer than 4 tosses are needed.

Sol:

The results of tossing of the three coins are listed below

$$S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$$

$$p = \frac{6}{8} = \frac{3}{4}, \quad 1-p = q = \frac{2}{8} = \frac{1}{4}$$

$$P(X<4) = \sum_{x=1}^{3} g(x;p) = \sum_{x=1}^{3} p \cdot q^{x-1} = \sum_{x=1}^{3} \left(\frac{3}{4}\right)\cdot\left(\frac{1}{4}\right)^{x-1}$$

$$= \frac{3}{4}\left[1 + \frac{1}{4} + \frac{1}{16}\right]$$

$$= \frac{3}{4}\left[\frac{16+4+1}{16}\right]$$

$$= \frac{21 \times 3}{64}$$

$$\boxed{P(x<4) = \frac{63}{64}}$$