MOMENT OF MEAN (NORMAL DISTRIBUTION)

$$f_x(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(l-\mu)^2}{2\sigma^2}\right)$$

$$= X \sim N(\mu, \sigma^2)$$

$$E(x) = \mu$$

$$Var(x) = \sigma^2$$

$$\boxed{\hat{\mu} = \bar{X}}$$

$$\because \bar{X} = \frac{1}{N} \sum X_i$$

$$S_x = \frac{1}{N} \sum (X_i - \bar{X})^2$$

$$\boxed{S_x = \sigma^2}$$

MLE OF NORMAL DISTRIBUTION:

For Likelihood function:

$$L(\mu, \sigma^2; x_1, \dots, x_n) = (2\pi\sigma^2)^{-n/2} \exp\left(-\frac{1}{2\sigma^2} \sum (x_i - \mu)^2\right)$$

Proof:

$$L(\mu, \sigma^2; x_1, \dots, x_n) = \prod_{j=1}^{n} f_X(x_j; \mu, \sigma^2)$$

$$= \prod_{j=1}^{n} (2\pi\sigma^2)^{-1/2} \exp\left(-\frac{1}{2} \frac{(x_i - \mu)^2}{\sigma^2}\right)$$

$$\boxed{L(\mu, \sigma^2; x_1, \dots, x_n) = (2\pi\sigma^2)^{-n/2} \exp\left(-\frac{1}{2\sigma^2} \sum_{i=1}^{n} (x_i - \mu)^2\right)}$$

For log likelihood function:

Proof:

$$l(\mu, \sigma^2; x_1, \dots, x_n) = \ln\left(L(\mu, \sigma^2; x_1, \dots, x_n)\right)$$

$$= \ln\left((2\pi\sigma^2)^{-n/2} \exp\left(-\frac{1}{2\sigma^2} \sum_{i=1}^{n} (x_i - \mu)^2\right)\right)$$

$$= \ln\left((2\pi\sigma^2)^{-n/2}\right) + \ln\left(\exp\left(-\frac{1}{2\sigma^2} \sum_{i=1}^{n} (x_i - \mu)^2\right)\right)$$

$$\boxed{l(\mu, \sigma^2; x_1, \dots, x_n) = -\frac{n}{2} \ln(2\pi\sigma^2) - \frac{1}{2\sigma^2} \sum_{i=1}^{n} (x_i - \mu)^2}$$

Derivative w.r.t mean:

$$= \frac{\partial}{\partial \mu}\left(-\frac{n}{2} \ln(2\pi) - \frac{n}{2} \ln(\sigma^2) - \frac{1}{2\sigma^2} \sum_{i=1}^{n} (x_i - \mu)^2\right)$$

$$= \frac{1}{\sigma^2} \sum_{i=1}^{n} (x_i - \mu)$$

$$= \frac{1}{\sigma^2}\left(\sum_{i=1}^{n} x_i - n\mu\right) \Rightarrow \boxed{\mu = \frac{1}{n} \sum_{i=1}^{n} x_i}$$

Derivative w.r.t variance:

$$= \frac{\partial}{\partial \sigma^2}\left(-\frac{n}{2} \ln(2\pi) - \frac{n}{2} \ln(\sigma^2) - \frac{1}{2\sigma^2} \sum_{i=1}^{n} (x_i - \mu)^2\right)$$

$$= -\frac{n}{2\sigma^2} - \left[\frac{1}{2} \sum_{i=1}^{n} (x_i - \mu)^2\right] \frac{\partial}{\partial \sigma^2}\left(\frac{1}{\sigma^2}\right)$$

$$0 = \frac{1}{2\sigma^2}\left[\frac{1}{\sigma^2} \sum_{i=1}^{n} (x_i - \mu)^2 - n\right]$$

$$\boxed{\sigma^2 = \frac{1}{n} \sum_{i=1}^{n} (x_i - \mu)^2}$$

Moment Of Mean (Exponential Distribution)

$$\bar{X} = \frac{1}{N}\sum_{i=1}^{N} x_i$$

$$\because E[x] = \bar{X}$$

$$\frac{1}{\lambda} = \frac{1}{N}\sum_{i=1}^{N} x_i$$

$$\lambda = \frac{1}{\frac{1}{N}\sum_{i=1}^{N} x_i}$$

$$\Rightarrow \boxed{\hat{\lambda} = \frac{1}{\bar{X}}}$$

Maximum Likelihood Estimation (Exponential Distribution)

Let $X_1, X_2, \ldots, X_n$ be random sample from exponential distribution

$$f(x) = \left(\frac{1}{\theta}\right) \times \exp(-x/\theta)$$

$\because L(\theta)$ is function of $x_1, x_2, x_3, \ldots, x_n$ given by:

$$L(\theta) = (1/\theta) \times \exp(-x_1/\theta) \times \exp(-x_2/\theta) \times \ldots \times (1/\theta) \times \exp(-x_n/\theta)$$

$$L(\theta) = (1/\theta^n)\, e\left(\sum_{i=1}^{n} -x_i/\theta\right) \rightarrow \text{Likelihood function}$$

$$\ln[L(\theta)] = -n \cdot \ln(\theta) - (1/\theta) \sum_{i=1}^{n} x_i \rightarrow \text{log likelihood function}$$

Diff. w.r.t θ

$$\frac{d}{d\theta} \ln[L(\theta)] = \frac{d}{d\theta}[-n \cdot \ln(\theta)] - \frac{d}{d\theta}\left(\frac{1}{\theta}\right) \sum_{i=1}^{n} x_i$$

$$= \frac{-n}{\theta} + \sum_{i=1}^{n} (-x_i/\theta^2)$$

let $\theta = 0$

$$\Rightarrow \boxed{\hat{\theta} = \frac{\sum_{i=1}^{n} x_i}{n} = \overline{X}} \Rightarrow \text{Mean of } X_1, X_2, \ldots, X_n$$

(MOM) For Geometric Distribution

$\mu_1' = \frac{1}{p}$

$\bar{X} = \frac{1}{p}$

$$\boxed{p = \frac{1}{\bar{X}}}$$

$\sigma^2 = S^2 = \frac{1-p}{p^2}$

$S^2 = \frac{q}{\left(\frac{1}{\bar{X}}\right)^2}$

$$\boxed{\hat{q} = \frac{S^2}{\bar{X}^2}}$$

MLE FOR GEOMETRIC DISTRIBUTION

Let $X_1, X_2, \dots, X_n$ be random sample from geometric distribution

$$f(x) = (1-p)^{x-1} \cdot p \quad ; \text{ where } x = 1, 2, 3, \dots \ \& \ 0 \leq p \leq 1$$

$$L(p) = (1-p)^{x_1-1} \cdot p \cdot (1-p)^{x_2-1} \cdot p \cdot (1-p)^{x_3-1} \cdot p \dots (1-p)^{x_n-1} \cdot p$$

$$\boxed{L(p) = p^n \cdot (1-p)^{\sum_{i=1}^{n} x_i - n}} \rightarrow \text{Likelihood function}$$

$$\boxed{\ln L(p) = n \cdot \ln(p) + \sum_{i=1}^{n} x_i - n \cdot \ln(1-p)} \rightarrow \text{log likelihood function}$$

Diff w.r.t p ; Setting $p = 0$

$$(d/dp) \ln \cdot L(p) = (n/p) - \left(\sum_{i=1}^{n} x_i - n / (1-p)\right) = 0$$

$$\Rightarrow \boxed{\hat{p} = \frac{n}{\sum_{i=1}^{n} x_i} = \frac{1}{\bar{X}}} \qquad \Rightarrow p = \text{no. of sucess}$$

MOM for BINOMIAL DISTRIBUTION

$$\mu_1' = np$$

$$\bar{X} = np$$

$$\boxed{\hat{p} = \frac{\bar{X}}{n}}$$

Variance:

$$np(1-p) = S^2$$

$$\cancel{n}\left(\frac{\bar{X}}{\cancel{n}}\right)\left(1-\frac{\bar{X}}{n}\right) = S^2$$

$$\bar{X} - \frac{\bar{X}^2}{n} = S^2$$

$$\boxed{n = \frac{\bar{X}}{\bar{X}-S^2}}$$

$$\hat{p} = \frac{\bar{X}}{\frac{\bar{X}^2}{\bar{X}-S^2}} \Rightarrow \boxed{\frac{\bar{X}-S^2}{\bar{X}}}$$

$$q = 1-p \Rightarrow \hat{q} = 1 - \frac{(\bar{X}-S^2)}{\bar{X}}$$

$$q = \frac{\bar{X} - X + S^2}{X}$$

$$\boxed{q = \frac{S^2}{\bar{X}}}$$

MLE FOR BINOMIAL DISTRIBUTION

Let $X_1, X_2, \ldots, X_n$ be samples from Binomial Distribution

$$f(x) = \frac{n!}{x!\,(n-x)!} p^x (1-p)^{n-x}$$

$$L(p) = \prod_{i=1}^{n} f(x_i) = \prod_{i=1}^{N} \frac{n!}{x_i!\,(n-x_i)!} p^{x_i} (1-p)^{n-x_i} \rightarrow \text{likelihood func}$$

The log-likelihood function is:

$$\ln L(p) = \sum_{i=1}^{N} \ln(n!) - \sum_{i=1}^{N} \ln(x_i!) - \sum_{i=1}^{N} \ln(n-x_i!) + \sum_{i=1}^{N} x_i \cdot \ln(p)$$
$$+ \left(n - \sum_{i=1}^{N} x_i\right) \cdot \ln(1-p)$$

Diff w.r.t p $\quad \therefore p = 0$

$$\frac{d}{dp} \ln L(p) = \frac{1}{p} \cdot \sum_{i=1}^{N} x_i - \frac{1}{1-p} \sum_{i=1}^{N} (n - x_i) = 0$$

$$\Rightarrow \frac{1}{p} \cdot \sum_{i=1}^{N} x_i = \left(\frac{1}{1-p}\right)\left(N \cdot n - \sum_{i=1}^{N} x_i\right)$$

$$\boxed{\hat{p} = \frac{1}{N}\left(\frac{\sum_{i=1}^{N} x_i}{n}\right) = \frac{1}{N}\left(\frac{X_1}{n} + \frac{X_2}{n} + \cdots + \frac{X_N}{n}\right)}$$

MOM of Poisson Distribution

$\mu_1' = \lambda$

or

$\boxed{\lambda = \mu_1'}$ -> Mean

$$np = \frac{1}{N}\sum_{i=1}^{N} X(i) = \overline{X}$$

$$np(1-p) = \frac{1}{N}\sum_{i=1}^{N} \left(X(i) - \overline{X}\right)^2$$

$$1-p = \frac{S_x}{\overline{X}} = \frac{\sum_{i=1}^{N} \left(X(i) - \overline{X}\right)^2}{\sum_{i=1}^{N} X(i)}$$

$$\hat{p} = 1 - \frac{S_x}{\overline{X}} = 1 - \frac{\sum_{i=1}^{N} \left(X(i) - \overline{X}\right)^2}{\sum_{i=1}^{N} (X(i))}$$

$$n\hat{p} = \overline{X}$$

$$\hat{n} = \frac{\overline{X}}{\hat{p}} \Rightarrow \frac{\overline{X}}{1 - \frac{S_x}{\overline{X}}}$$

$$\boxed{\hat{n} = \frac{\overline{X}^2}{\overline{X} - S_x}}$$

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MLE OF POISSON DISTRIBUTION

Let $X_1, X_2, \ldots, X_n$ be random sample from Poisson distribution

p.d.f of Poisson distribution is:

$$f(x) = \frac{\lambda^x e^{-\lambda}}{x!} \quad ; \text{ where } x = 0, 1, 2, \ldots$$

Likelihood function is:

$$L(\lambda) = \prod_{i=1}^{n} \frac{\lambda^{x_i} e^{-\lambda}}{x_i!} = e^{-\lambda n} \frac{\lambda^{\sum_{i=1}^{n} x_i}}{\prod_{i=1}^{n} x_i}$$

Log-Likelihood function:

$$\ln L(\lambda) = -\lambda n + \sum_{i=1}^{n} x_i \ln(\lambda) - \ln\left(\prod_{i=1}^{n} x_i\right)$$

Diff w.r.t λ $\Rightarrow \lambda = 0$

$$\frac{d}{d\lambda} \ln L(\lambda) = -n + \sum_{i=1}^{n} x_i \cdot \frac{1}{\lambda} = 0$$

$$\boxed{\hat{\lambda} = \frac{\sum_{i=1}^{n} x_i}{n} = \bar{X}}$$

MOM of Uniform Distribution

$$\mu_1' = \frac{a+b}{2}$$

$$2\bar{X} = a+b$$

$$\boxed{a = 2\bar{X} - b} \quad \text{—} (A)$$

$$\sqrt{S^2} = \frac{(b-a)}{\sqrt{12}} \quad \text{—} (1)$$

or

$$S = \frac{(b-a)^2}{12}$$

Using eq (1)

$$b - a = \sqrt{12}\, S$$

while a is:

$$b - (2\bar{X} - b) = S\sqrt{12}$$

$$b - 2\bar{X} + b = S\sqrt{12}$$

$$\cancel{2}(b - \bar{X}) = \cancel{2}\sqrt{3}\, S$$

$$\boxed{\hat{b} = \sqrt{3}S + \bar{X}}$$

Eq (A) becomes:

$$a = 2\bar{X} - (\sqrt{3}S + \bar{X})$$

$$\boxed{\hat{a} = \bar{X} - \sqrt{3}S}$$

MLE of Uniform Distribution

let $X_1, X_2, X_3 \dots X_n$ for Uniform distributed random variable

$$f(X_i) = \frac{1}{\theta} \quad ; \text{if } 0 \leq x_i \leq \theta$$

$$f(x) = 0 \quad ; \text{ otherwise}$$

Uniform distributed random variable would arranged in following order:

$$0 \leq X_1 \leq X_2 \leq X_3 \dots \leq X_n \leq \theta,$$

Likelihood function:

$$L(\theta) = \prod_{i=1}^{n} f(x_i) = \prod_{i=1}^{n} \frac{1}{\theta} = \theta^{-n}$$

log-likelihood function:

$$\ln L(\theta) = -n \ln(\theta)$$

Diff w.r.t θ ; $\Rightarrow \theta = 0$

$$\frac{d}{d\theta} \ln L(\theta) = \frac{-n}{\theta} \quad ; \Rightarrow \; < 0 \text{ for } \theta > 0$$

Hence $L(\theta)$ is decreasing function and is max. at $\theta = X_n$

Thus,

$$\boxed{\hat{\theta} = X_n}$$

Moment Of Mean (Bernoulli Distribution)

$\because$ Mean $= \bar{X}$

$$\hat{P} = \bar{X}$$

or

$$\boxed{\bar{X} = \hat{P}}$$

$\because$ Variance $= \sigma^2 = S^2$

$$\boxed{S^2 = p(1-p)}$$

$$S^2 = \bar{X}(q)$$

$$\boxed{q = \frac{S^2}{\bar{X}}}$$

MAXIMUM LIKELIHOOD ESTIMATION (BERNOULLI DIST.)

$$f(x;p) = p^x(1-p)^{1-x} \quad ; \quad x = 0,1$$

$$L(p) = \ln \prod_{i=1}^{n} f(x_i;p)$$

$$= \prod_{i=1}^{n} p^x (1-p)^{1-x}$$

Taking $\ln$ on both sides

$$\ln L(p) = \sum_{i=1}^{n} \ln\left[p^x (1-p)^{1-x_i}\right]$$

$$\ln L(p) = \ln p \sum_{i=1}^{n} x_i + \ln(1-p) \sum_{i=1}^{n} (1-x_i)$$

Taking Derivative

$$\frac{dL(p)}{dp} = \frac{\sum_{i=1}^{n} x_i}{p} - \frac{\sum_{i=1}^{n} (1-x_i)}{1-p}$$

Set its value equal to 0

$$l(p) = \frac{\sum_{i=1}^{n} x_i}{p} - \frac{\sum_{i=1}^{n}(1-x_i)}{1-p} = 0$$

$$\frac{1}{1-p}\left(n - \sum_{i=1}^{n} x_i\right) = \frac{1}{p}\sum_{i=1}^{n} x_i$$

$$\frac{n - \sum_{i=1}^{n} X_i}{\sum_{i=1}^{n} X_i} = \frac{1-p}{p}$$

$$\frac{n}{\sum_{i=1}^{n} X_i} - \cancel{1} = \frac{1}{p} - \cancel{1} \Rightarrow \boxed{\hat{p} = \frac{1}{n}\sum_{i=1}^{n} X_i = \bar{X}_n}$$

Moment Of Mean For Gamma Distribution

$$\mu_1' = \alpha\beta$$

$$\bar{X} = \alpha\beta$$

$$\boxed{\hat{\alpha} = \frac{\bar{X}}{\beta}}$$

Variance:

$$\because S^2 = \alpha\beta^2$$

$$\frac{\bar{X}}{\beta}\beta^2 = S^2$$

$$\boxed{\hat{\beta} = \frac{S^2}{\bar{X}}}$$

$$\alpha = \frac{\bar{X}}{\frac{S^2}{\bar{X}}}$$

$$\boxed{\hat{\alpha} = \frac{\bar{X}^2}{S^2}}$$

Maximum Likelihood Estimation For (Gamma Dist.)

$$P(X=x) = \frac{\lambda^{\alpha} e^{-\lambda x} (x)^{\alpha-1}}{(\alpha-1)! \rightarrow \Gamma(\alpha)} \Rightarrow \text{Gamma function}$$

$$L(\alpha,\lambda) = \prod_{i=1}^{n} P(X_i = x_i) = \prod_{i=1}^{n} \frac{\lambda^{\alpha} e^{-\lambda x_i} (x_i)^{\alpha-1}}{\Gamma(\alpha)}$$

Taking ln on b.s:

$$\ell(\alpha,\lambda) = \ln\left(\prod_{i=1}^{n} \frac{\lambda^{\alpha} e^{-\lambda x_i} (x_i)^{\alpha-1}}{\Gamma(\alpha)}\right)$$

$\ln(a \cdot b) = \ln(a) + \ln(b)$

$\ln(a/b) = \ln(a) - \ln(b)$

$$= \ln\left(\frac{\lambda^{n\alpha} e^{-\lambda \sum x_i} \left(\prod_{i=1}^{n} (x_i)\right)^{\alpha-1}}{[\Gamma(\alpha)]^{n}}\right)$$

$$= n\alpha \ln(\lambda) - \lambda \sum x_i + (\alpha-1) \ln\left(\prod_{i=1}^{n} x_i\right) - \ln\left((\Gamma(\alpha))^{n}\right)$$

$$\ell(\alpha,\lambda) = n\alpha \ln(\lambda) - \lambda \sum x_i + (\alpha-1) \sum_{i=1}^{n} \ln(x_i) - n \ln(\Gamma(\alpha))$$

Taking derivative and set up value equals to 0

$$\frac{\partial \ell(\alpha,\lambda)}{\partial \lambda} = \frac{n\alpha}{\lambda} - \sum x_i$$

$$\frac{n\alpha}{\lambda} - \sum x_i = 0$$

$$\Rightarrow \lambda = \frac{n\alpha}{\sum x_i}$$

$$\hat{\lambda} = \frac{\cancel{n}\hat{\alpha}}{\cancel{n}\bar{x}} \Rightarrow \boxed{\hat{\lambda} = \frac{\hat{\alpha}}{\bar{x}}}$$