The Exponential Distribution :

A continuous random variable X has an exponential distribution, and it is referred to as an exponential random variable, if and only if its p.density is given by

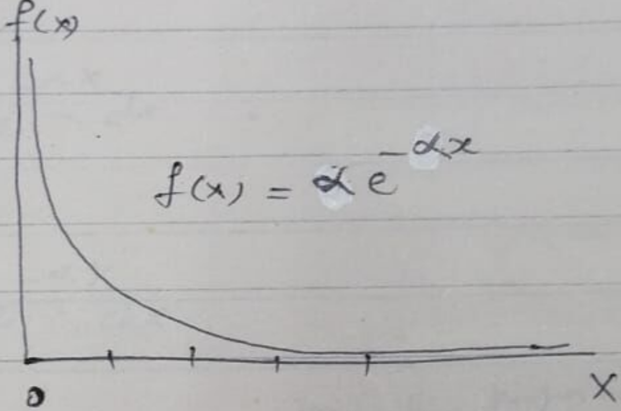
$$f(x) = \begin{cases} \frac{1}{\theta} e^{-\frac{x}{\theta}}, & \text{for } x > 0 \\ 0, & \text{elsewhere} \end{cases}$$

where $\theta > 0$.

$$f(x) = \begin{cases} \alpha e^{-\alpha x}, & \text{for } x > 0 \\ 0, & \text{elsewhere} \end{cases}$$

where $\alpha > 0$.

The p.d.f is



The Distribution function is

$$F(X) = \int_0^{\cancel{x}} f(x)\, dx$$

$$= \alpha \int_0^{X} e^{-\alpha x}\, dx$$

$$= \cancel{\alpha} \left[\frac{e^{-\alpha x}}{-\cancel{\alpha}} \right]_0^X$$

$$= \left[-e^{-\alpha x} \right]_0^X$$

$$\boxed{F(X) = \left[1 - e^{-\alpha X} \right]}$$

p.d.f

$$\int_0^\infty f(x)\,dx = 1$$

$$= \alpha \int_0^\infty e^{-\alpha x}\,dx$$

$$= \alpha \left[\frac{e^{-\alpha x}}{-\alpha} \right]_0^\infty$$

$$= \left| -e^{-\alpha x} \right|_0^\infty$$

$$= 1 - e^{-\infty}$$

$$= 1 - 0$$

$$= 1$$

iii) $E(X) = \int_0^\infty x \,\alpha e^{-\alpha x}\,dx$

$$= \alpha \int_0^\infty x^{2-1} e^{-\alpha x}\,dx$$

$$= \frac{\alpha \Gamma 2}{\alpha^2} = \frac{1}{\alpha}$$

using the properties of definite integral we have gamma function:

$$\Gamma n = \int_0^\infty x^{n-1} e^{-x}\,dx \quad ; n > 0$$

$$\Gamma \tfrac{1}{2} = \sqrt{\pi}$$

$$\frac{\alpha \Gamma n}{\alpha^n} = \alpha \int_0^\infty x^{n-1} e^{-\alpha x}\,dx ;$$

$$\boxed{E(X) = \frac{1}{\alpha}}$$

$$E(X^r) = \int_0^\infty x^r \alpha e^{-\alpha x}\,dx$$

$$= \alpha \int_0^\infty x^{(r+1)-1} e^{-\alpha x}\,dx$$

$$= \frac{\alpha \Gamma (r+1)}{\alpha^{r+1}} = \boxed{\frac{\Gamma (r+1)}{\alpha^r} = \mu_r'} \boxed{= \frac{r!}{\alpha^r}}$$

Put $r=2$

$$\mu_2' = E(x^2) = \frac{\sqrt{2+1}}{\alpha^2} = \frac{\sqrt{3}}{\alpha^2} = \frac{2}{\alpha^2}$$

$$\boxed{\mu_2' = \frac{2}{\alpha^2}}$$

$$\sigma^2 = Var(x) = \mu_2' - \mu_1'^2 = \frac{2}{\alpha^2} - \frac{1}{\alpha^2} = \frac{1}{\alpha^2}$$

$$\boxed{\sigma^2 = \frac{1}{\alpha^2}}$$

<u>Median</u>

$$F(M) = \frac{1}{2} \Rightarrow 1 - e^{-\alpha M} = \frac{1}{2}$$

$$= \alpha \int_0^M e^{-\alpha x} dx = \frac{1}{2}$$

$$= \alpha \left[\frac{e^{-\alpha x}}{-\alpha} \right]_0^M = \frac{1}{2}$$

$$= 1 - e^{-\alpha M} = \frac{1}{2}$$

$$= e^{-\alpha M} = \frac{1}{2}$$

$$= e^{-\alpha M} = 2^{-1}$$

$$= -\alpha M = -\log_e 2$$

$$= \alpha M = \log_e 2$$

$$\boxed{M = \frac{1}{\alpha} \log_e 2}$$

Quartile

$$F(Q_1) = \frac{1}{4}.$$

$$1 - e^{-\alpha Q_1} = \frac{1}{2}$$

$$\int_0^{Q_1} \alpha e^{-\alpha x}\, dx = \frac{1}{4}$$

$$= \alpha \left[\frac{e^{-\alpha x}}{-\alpha} \right|_0^M = \frac{1}{4}$$

$$1 - e^{-\alpha Q_1} = \frac{1}{4}.$$

$$e^{-\alpha Q_1} = \frac{3}{4}.$$

$$-\alpha Q_1 = -\log_e \frac{3}{4}.$$

$$\alpha Q_1 = \log_e \frac{3}{4}.$$

$$\boxed{Q_1 = \frac{1}{\alpha} \log_e \frac{3}{4}.}$$

Similarly.

$$\boxed{Q_3 = \frac{1}{\alpha} \log_e \left(\frac{1}{4}\right)}$$

Mode:

$$f(x) = \alpha e^{-\alpha x}$$

$$f'(x) = -\alpha^2 e^{-\alpha x} = 0.$$

$$= e^{-\alpha x} = 0$$

$$= e^{-\infty} = 0. \Rightarrow x \to \boxed{\infty. = \text{mode}}$$

but on the basis of sketch of pdf, it can be shown that mode exists at '0'

M.G.F

$$M_x(t) = E[e^{tx}] = \int_0^\infty e^{tx} f(x;\alpha)\,dx$$

$$= \alpha \int_0^\infty e^{tx} e^{-\alpha x}\,dx$$

$$= \alpha \int_0^\infty e^{-(\alpha - t)x}\,dx$$

$$= \alpha \left[\frac{e^{-(\alpha-t)x}}{-(\alpha-t)} \right]_0^\infty \qquad \because e^{-0} = 1 \quad e^{-\infty} = 0$$

$$= \frac{\alpha}{(\alpha-t)} \left| - e^{-(\alpha-t)x} \right|_0^\infty$$

$$= \frac{\alpha}{(\alpha-t)} \left| 1 - 0 \right|$$

$$\boxed{M_x(t) = \frac{\alpha}{(\alpha-t)}} = \quad \text{for } t < \alpha$$

$$\boxed{M_x(t) = \frac{1}{\left(1 - \frac{t}{\alpha}\right)}} \; ; \alpha > 0$$

$$M_x(t) = \frac{1}{(1 - t/\alpha)} \quad ; \alpha > 0.$$

$$\mu_1' = \left[\frac{d}{dt} M_x(t)\right]_{t=0}$$

$$= \left\{\frac{d}{dt}\left[1 \cdot \left(1-\frac{t}{\alpha}\right)^{-1}\right]\right\}_{t=0}$$

$$= \left|(-1)\left(1-\frac{t}{\alpha}\right)^{-2} \cdot \left(-\frac{1}{\alpha}\right)\right|_{t=0}$$

$$= \left|\frac{1}{\alpha\left(1-\frac{t}{\alpha}\right)^2}\right|_{t=0}$$

$$\boxed{\mu_1' = \frac{1}{\alpha}}$$

$$\mu_2' = \left\{\frac{d^2}{dt^2}\left[M_x(t)\right]\right\}_{t=0}$$

$$= \left[(-2) \cdot \frac{1}{\alpha(1-t/\alpha)^3} \left(-1/\alpha\right)\right]_{t=0}$$

$$= \left|\frac{2}{\alpha^2} \cdot \frac{1}{(1-t/\alpha)^3}\right|_{t=0}$$

$$\boxed{\mu_2' = \frac{2}{\alpha^2}}$$

Problem: Customers arrive in a certain shop according to an approximate Poisson process at a mean rate of 20 per hour. What is the probability that the shopkeeper will have to wait more than 5 minutes for the arrival of the customers?

Sol: Let X denote the waiting time in minutes until the first customer arrives and note that $\lambda = \frac{20}{60} = \frac{1}{3}$ is the expected number of arrivals per minutes. Thus.

$$\alpha = \frac{1}{3}$$

$$f(x) = \frac{1}{3} e^{-\frac{x}{3}} \quad x \geq 0$$

$$P(X>5) = \int_5^\infty f(x)\,dx = \frac{1}{3}\int_5^\infty e^{-\frac{1}{3}x}\,dx$$

$$= \frac{1}{3}\left[\frac{e^{-\frac{1}{3}x}}{-\frac{1}{3}}\right]_5^\infty$$

$$= \left[e^{-5/3} - 0\right]$$

$$= e^{-5/3}$$

$$\boxed{P(X>5) = 0.189}$$

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Problem: Let the p.d.f of X be

$f(x) = \frac{1}{2} e^{-\frac{1}{2}x}$; $0 \le x < \infty$

a) What are the mean, variance and m.g.f of X?

b) Calculate $P(X > 3)$

$f(x) = \frac{1}{2} e^{-\frac{1}{2}x}$

$E(x) = \frac{1}{2}\int_0^\infty x e^{-\frac{1}{2}x} dx.$

$y = \frac{1}{2}x.$

$x = 2y.$

$dx = 2dy.$

$= \frac{1}{\cancel{2}} \int_0^\infty (2y) e^{-y} \cdot \cancel{2} dy.$

$= 2 \int_0^\infty y^{2-1} e^{-y} dy.$

$= 2 \cdot \Gamma 2$

$= 2 \cdot 1 = 2$

$\boxed{E(x) = 2}$

$E(x^2) = \frac{1}{2} \int_0^\infty x^2 e^{-1/2 x} dx$

$= \frac{1}{\cancel{2}} \int_0^\infty (2y)^2 e^{-y} \cdot \cancel{2} dy$

$= 4 \int_0^\infty y^{3-1} e^{-y} dy.$

$= 4 \Gamma 3$

$\boxed{E(x^2) = 8}$

$Var(x) = E[x^2] - \{E[x]\}^2$

$= 8 - 4$

$\boxed{\sigma_{\cancel{x}}^2 = 4}$

$\boxed{M_x(t) = \frac{1}{1-2t}}$

Problem: The life time in years of a television tube of a certain make is a random variable T and its probability density function f(t) is given by

$$f(t) = A e^{-kt} \quad \text{for } 0 \le t \le \infty \quad (k > 0)$$
$$= 0 \quad , \quad \text{e.w}$$

obtain A interms of k.

a) If the manufacturer, after some research, finds that out of 1000 such tubes 371 failed within first two years of use, estimate the value of k.

b) Using this value of k correct to 3 significant figures, calculate the mean and variance of T, giving answers correct to 2 significant figures.

c) If two such tubes are bought, what is the probability that one fails within its first year and the other lasts longer than six years?

Sol. T is a random variable

$$\int_0^\infty f(t)\,dt = 1$$

$$A\int_0^\infty e^{-kt}\,dt = 1$$

$$A\left[\frac{e^{-kt}}{-k}\right]_0^\infty = 1$$

$$\frac{A}{k}\left[e^{-0} - e^{-\infty}\right] = 1 \qquad e^{-x} \to 0 \text{ as } x \to \infty$$

$$\frac{A}{k} = 1$$

$$\boxed{A = k}$$

$$P(T<1) = \int_0^1 k e^{-kt} dt$$

$$= [-e^{-kt}]_0^1$$

$$= 1 - e^{-k}$$

$$= 1 - 0.793$$

$$\boxed{P(T<1) = 0.207}$$

and

$$P(T>6) = 1 - P(T \leq 6)$$

$$= 1 - k\int_0^6 e^{-kt} dt$$

$$= 1 - [-e^{-kt}]_0^6$$

$$= 1 + e^{-6k} - 1$$

$$= e^{-6k}$$

$$\boxed{P(T>6) = 0.249}$$

$\therefore$ if Two tubes are bought

$$P[(T_1<1) \cap (T_2>6)] + P[(T_2<1) \cap (T_1>6)]$$

$$= 2(0.207)(0.249)$$

$$= 0.103$$

Therefore the probability that one fails within its first years and the other lasts longer than 6 yea[rs] is 0.103

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$\therefore$ if Two tubes are bought

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$$= 2(0.207)(0.249)$$

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Therefore the probability that one fails within its first years and the other lasts longer than 6 years is 0.103

MGF of Exponential Distribution.

$$M_X(t) = E[e^{tx}] = \int_0^\infty f(x)\, e^{tx}\, dx$$

$$= \lambda \int e^{-\lambda x} e^{tx}\, dx = \lambda \int_0^\infty e^{-x(\lambda - t)}\, dx$$

Let $u = x(\lambda - t)$

$du = (\lambda - t)\, dx$ if $x = 0,\ u = 0$

$\Rightarrow dx = \dfrac{du}{(\lambda - t)}$ $x = \infty,\ u = \infty$

$$M_X(t) = \lambda \int_0^\infty e^{-u} \frac{du}{\lambda - t} = \frac{\lambda}{\lambda - t} \int_0^\infty e^{-u}\, du = \frac{\lambda}{\lambda - t} \int_0^\infty u^{1-1} e^{-u}\, du$$

$$= \frac{\lambda}{\lambda - t} \Gamma 1 = \frac{\lambda}{\lambda - t} (0!) = \frac{\lambda}{\lambda - t} (1)$$

$$\boxed{M_X(t) = \frac{\lambda}{\lambda - t}} = \left(\frac{\lambda - t}{\lambda}\right)^{-1} = \left(1 - \frac{t}{\lambda}\right)^{-1}$$

Expanding we have

$$M_X(t) = \sum_{r=0}^{\infty} \left(\frac{t}{\lambda}\right)^r = \sum_{r=0}^{\infty} \left(\frac{r!}{\lambda^r}\right) \cdot \frac{t^r}{r!}$$

Comparing the coefficients of $\frac{t^r}{r!}$ we have

$$\boxed{\mu_r' = \frac{r!}{\lambda^r}}$$

Cumulant Generating function

$$K_X(t) = \log M_X(t) = \log\left(1 - \frac{t}{\lambda}\right)^{-1} = -\log\left(1 - \frac{t}{\lambda}\right)$$

$\ln(1-x) = -x - \frac{x^2}{2} \ldots$

$$= \sum_{r=1}^{\infty} \frac{(t/\lambda)^r}{r} = \sum_{r=1}^{\infty} \frac{(r-1)!}{\lambda^r} \frac{t^r}{r!}$$

Comparing the coefficients of $\frac{t^r}{r!}$

$$k_1 = \frac{1}{\lambda},\ k_2 = \frac{1}{\lambda^2},\ k_3 = \frac{2}{\lambda^3},\ k_4 = \frac{6}{\lambda^4}, \ldots$$

Comparing with moments

$$\mu_1' = k_1 = \frac{1}{\lambda},\ \mu_2 = k_2 = \frac{1}{\lambda^2},\ \mu_3 = k_3 = 2/\lambda^3$$

$$\mu_4 = k_4 + 3k_2^2 = 9/\lambda^4$$

$$\beta_1 = \frac{\mu_3^2}{\mu_2^3} = 4 \qquad \gamma_1 = \sqrt{\beta_1} = 2$$

$$\beta_2 = \frac{\mu_4}{\mu_2^2} = 9 \qquad \gamma_2 = \beta_2 - 3 = 9 - 3 = 6$$

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Problem: Telephone calls enter a college switchboard according to a poisson process on the average of two every 3 minutes. Let X denote the waiting time until the first call that arrives after 10 A.M.

a) What is the p.d.f of X ?

b) Find $P(X > 2)$

a)

$$f(x) = \frac{1}{\theta} e^{-\frac{x}{\theta}}, \quad \text{for } x > 0$$
$$= 0, \quad \text{e.w}$$

$$\theta = \frac{2}{3}.$$

$$f(x) = \frac{3}{2} e^{-\frac{3}{2}x}, \quad \text{for } x > 0$$

b)

$$P(X > 2) = \frac{3}{2}\int_2^\infty e^{-\frac{3}{2}x}\,dx.$$

$$= \frac{3}{2}\left[\frac{e^{-\frac{3}{2}x}}{-\frac{3}{2}}\right]_2^\infty$$

$$= \left[e^{-\frac{3}{2}\cdot 2} - e^{-\infty}\right]$$

$$= e^{-3}$$