Gamma Distribution:

The random variable 'x' has a gamma dist if its p.d.f is

$$f(x;\beta,\alpha) = \frac{1}{\Gamma\alpha\ \beta^{\alpha}} x^{\alpha-1} e^{-\frac{x}{\beta}}, \quad 0 \le x < \infty, \quad \alpha, \beta > 0.$$

1. Pdf

$$\int_{-\infty}^{\infty} f(x)dx = 1 = \frac{1}{\Gamma\alpha\,\beta^{\alpha}} \int_0^{\infty} x^{\alpha-1} e^{-\frac{x}{\beta}} dx$$

Let $y = \frac{x}{\beta} \Rightarrow x = \beta y$

$dy = \frac{dx}{\beta} \Rightarrow dx = \beta dy$

when $x = 0$ $\Rightarrow$ $y = 0$

$x = \infty \Rightarrow y = \infty$

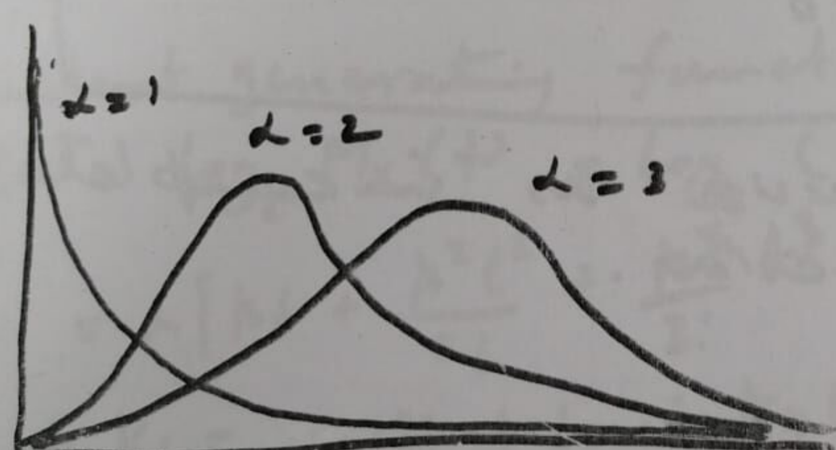
$$= \frac{1}{\Gamma\alpha\,\beta^{\alpha}} \int_0^{\infty} (\beta y)^{\alpha-1} e^{-y} (\beta dy)$$

$$= \frac{1}{\Gamma\alpha\,\cancel{\beta^{\alpha}}} \cancel{\beta^{\alpha}} \int_0^{\infty} y^{\alpha-1} e^{-y} dy$$

$\because \int_0^{\infty} x^{\alpha-1} e^{-x} dx = \Gamma\alpha \ ; \alpha > 0$

$$= \frac{1}{\cancel{\Gamma\alpha}} \cancel{\Gamma\alpha} = 1$$

2. Shape



Incomplete Gamma Function:

we know that

$$F(x) = \frac{1}{\Gamma\alpha}\int_0^x x^{\alpha-1}e^{-x}\,dx$$

For different values of x and α where α is a +ve integer.

$$\because e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \cdots\cdots$$

$$F(x) = \frac{1}{\Gamma\alpha}\int_0^x x^{\alpha-1}\left[1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \cdots\right]dx$$

$$= \frac{1}{\Gamma\alpha}\left[\frac{x^\alpha}{\alpha} - \frac{x^{\alpha+1}}{\alpha+1} + \frac{1}{2}\frac{x^{\alpha+2}}{\alpha+2} - \frac{1}{3!}\frac{x^{\alpha+3}}{\alpha+3} + \frac{1}{4!}\frac{x^{\alpha+4}}{\alpha+4} - \cdots\cdots\right]$$

$$\boxed{F(x) = \frac{1}{\Gamma\alpha}\sum_{i=0}^{\infty}(-1)^i\frac{x^{\alpha+i}}{\alpha+i}\cdot\frac{1}{i!}}$$

which is known as incomplete gamma function.

iii. c d f

It is called incomplete gamma function and is only obtained by tables. Mathematically:

$$F(x) = \begin{cases} \frac{1}{\Gamma\alpha\, \beta^{\alpha}} \int_0^x x^{\alpha-1} e^{-\frac{x}{\beta}}\, dx & ; x > 0 \\ 0 & , e \cdot w. \end{cases}$$

4. Moments :

r^{th} moment about origin:

$$\mu_r' = E(X^r) = \int_{-\infty}^{\infty} x^r f(x)\, dx \quad \text{(By definition)}$$

$$= \frac{1}{\Gamma\alpha\, \beta^{\alpha}} \int_0^{\infty} x^r \left[x^{\alpha-1} e^{-\frac{x}{\beta}}\right] dx$$

Let $y = \frac{x}{\beta} \Rightarrow x = \beta y$

$dx = \beta\, dy$

when $x = 0 \Rightarrow y = 0$

$x = \infty \Rightarrow y = \infty$

$$= \frac{1}{\Gamma\alpha\, \beta^{\alpha}} \int_0^{\infty} (\beta y)^{\alpha+r-1} e^{-y} (\beta\, dy)$$

$$= \frac{\beta^{\alpha+r}}{\Gamma\alpha\, \beta^{\alpha}} \int_0^{\infty} y^{\alpha+r-1} e^{-y}\, dy \quad ; \because \int_0^{\infty} x^{\alpha-1} e^{-x}\, dx = \Gamma\alpha$$

$$\boxed{\mu_r' = \frac{\beta^r}{\Gamma\alpha} \Gamma(\alpha+r) \quad ; r = 1, 2, 3, 4, \ldots} \text{ — (A)}$$

Note : when $\beta = 1 \Rightarrow$ Mean = Variance.

$\bar{x} = \sigma^2$ like Poisson dist.

mgf

$$M_x(t) = E[e^{tx}] = \int_{-\infty}^{\infty} e^{tx} f(x)\,dx$$ (By definition

$$= \int_0^\infty e^{tx}\left[\frac{1}{\Gamma\alpha\,\beta^\alpha} x^{\alpha-1} e^{-\frac{x}{\beta}}\right]dx$$

$$= \frac{1}{\Gamma\alpha\,\beta^\alpha}\int_0^\infty x^{\alpha-1} e^{-(\frac{1}{\beta}-t)x}\,dx$$ (as $\int_0^\infty x^{\alpha-1} e^{-\frac{x}{\beta}}\,dx = \frac{\Gamma\alpha}{\beta^\alpha}$)

$$= \frac{1}{\Gamma\alpha\,\beta^\alpha}\left[\frac{\Gamma\alpha}{(\frac{1}{\beta}-t)^\alpha}\right]$$

$$= \frac{1}{\beta^\alpha\cdot\frac{1}{\beta^\alpha}(1-\beta t)^\alpha} = (1-\beta t)^{-\alpha}$$

$$\boxed{M_x(t) = (1-\beta t)^{-\alpha}}$$

Characteristic Function:

$$\phi_x(t) = (1-i\beta t)^{-\alpha}$$

Cumulant generating function:

$$K_x(t) = \log_e M_x(t) = \log_e (1-\beta t)^{-\alpha} = -\alpha\log_e(1-\beta$$

$$= \alpha\{\beta t + \frac{\beta^2 t^2}{2!} + \frac{\beta^3}{3!}t^3 + \frac{\beta^4 t^4}{4!} + \dots]$$

Mean $= k_1 =$ coeff of t in $K_x(t) = \alpha\beta$

$\mu_2 = k_2 =$ " " $t^2/2!$ " " $= \alpha\beta^2$

$\mu_3 = k_3 =$ " " $t^3/3!$ " " $= 2\alpha\beta^3$

$\mu_4 = k_4 =$ " " $t^4/4!$ " " $= 6\alpha\beta^4$

$M_x(t) = \left(1 - \frac{t}{\alpha}\right)^{-1}$

Expanding we have

$$M_x(t) = \sum_{r=0}^{\infty} \left(\frac{t}{\alpha}\right)^r = \sum_{r=0}^{\infty} \left(\frac{r!}{\alpha^r}\right) \cdot \frac{t^r}{r!}$$

comparing the coefficients

$$\boxed{\mu_r' = \frac{r!}{\alpha^r}}$$

Show that

$\beta_1 = 4$; $\gamma_1 = 2 = \sqrt{\beta_1}$

$\beta_2 = 9$; $\gamma_2 = 6 = \beta_2 - 3$

Cumulant Generating Function

$$K_x(t) = \log_e M_x(t) = \log_e \left(1 - \frac{t}{\alpha}\right)^{-1}$$

$$= -\log_e \left(1 - \frac{t}{\alpha}\right)$$

$\because \ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \dots$

$$= \sum_{r=1}^{\infty} \frac{(t/\alpha)^r}{r}$$

$$\boxed{K_x(t) = \sum_{r=1}^{\infty} \frac{(r-1)!}{\alpha^r} \cdot \frac{t^r}{r!}}$$

Comparing the coefficients of $\frac{t^r}{r!}$ we get

$\mu_1' = k_1 = \frac{1}{\alpha}$, $\mu_2 = k_2 = \frac{1}{\alpha^2}$

$\mu_3 = k_3 = \frac{2}{\alpha^3}$, $\mu_4 = k_4 + 3k_2^2 = \frac{9}{\alpha^4}$

$\beta_1 = \frac{\mu_3^2}{\mu_2^3} = 4$ | $\gamma_1 = \sqrt{\beta_1} = 2$

$\beta_2 = \frac{\mu_4}{\mu_2^2} = 9$ | $\gamma_2 = \beta_2 - 3 = 9 - 3 = 6$

Mean:

Put $r=1$ in Eq (A)

$$\mu_1' = E(X) = \frac{\beta}{\Gamma\alpha}\Gamma\alpha+1 = \frac{\alpha!\,\beta}{(\alpha-1)!} = \alpha\beta.$$

Variance:

Put $r=2$ in eq (A)

$$\mu_2' = E(X^2) = \frac{\beta^2}{\Gamma\alpha}\Gamma\alpha+2 = \frac{(\alpha+1)\alpha(\alpha-1)!\,\beta^2}{(\alpha-1)!}$$

$$= \alpha(\alpha+1)\beta^2$$

$$\sigma^2 = \mu_2 = \mu_2' - \mu_1'^2 = \cancel{\alpha^2\beta^2} + \alpha\beta^2 - \cancel{(\alpha\beta)^2}$$

$$\boxed{\sigma^2 = \mu_2 = \alpha\beta^2}$$

$\mu_3' =$

$\mu_4' =$

$\mu_3 =$

$\mu_4 =$

$\beta_1 =$

$\beta_2 =$

Skewness

$= \sqrt{\beta_1} = \frac{2}{\sqrt{\alpha}}$

Kurtosis

$\beta_2 - 3 = 3\left(1+\frac{2}{\alpha}\right)$

$\beta_2 - 3 =$

Median:

$$\int_{-\infty}^{M} f(x)\,dx = \frac{1}{2} = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{M} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \left[\int_{-\infty}^{\mu} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx + \int_{\mu}^{M} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx \right] = \frac{1}{2}$$

But

$$\frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx = \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-\frac{1}{2}z^2} dz$$

$$= \frac{1}{2} + \frac{1}{\sigma\sqrt{2\pi}} \int_{\mu}^{M} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

$$= \frac{1}{2} + 0 \qquad \because \frac{1}{\sigma\sqrt{2\pi}} \int_{\mu}^{M} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx = 0$$

$$\Rightarrow \boxed{\mu = M} \text{ or } \boxed{M = \mu}$$

Hence $\boxed{\text{Mean} = \text{Median} = \text{Mode}}$

Mean Deviation about mean:

$$M.D_{\bar{x}} = \int_{-\infty}^{\infty} |x-\mu|\, f(x)\,dx = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} |x-\mu|\, e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} |\sigma z|\, e^{-\frac{1}{2}z^2} \sigma\, dz$$

$$= \frac{\sigma}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |z|\, e^{-\frac{1}{2}z^2} dz$$

$$= \frac{\sigma}{\sqrt{2\pi}} \cdot 2 \int_{0}^{\infty} z\, e^{-\frac{1}{2}z^2} dz \qquad \because |z|\, e^{-\frac{1}{2}z^2} \text{ is an even function}$$

$|z| = z$

$0 < z < \infty$

$$= \sqrt{\frac{2}{\pi}}\, \sigma \left(-e^{-y} \Big|_{0}^{\infty} \right)$$

$$= \sqrt{\frac{2}{\pi}}\, \sigma$$

$$\boxed{M.D_{\bar{x}} \simeq \frac{4}{5}\sigma}$$

Problem :

Suppose that on average 30 customer per hour arrive at a shop in accordance with a Poisson process. That is, if minute is our unit, then $\lambda = \frac{1}{2}$.
What is the probability that the shopkeeper will wait more than 5 minutes before both of the first two customers arrive?

Sol. :

Let 'X' denotes the waiting time in minutes until the second customer arrives. Then

$$X \sim \text{gamma}\left(\alpha = 2,\ \beta = \tfrac{1}{\lambda} = 2\right)$$

Hence

$$P(X>5) = \int_5^\infty \frac{x^{\alpha-1} e^{-x/\beta}}{\Gamma\alpha\ \beta^\alpha}\,dx$$

$$= \frac{1}{\Gamma 2\ 2^2} \int_5^\infty x^{2-1} e^{-x/2}\,dx$$

$$= \frac{1}{4}\int_5^\infty x\,e^{-x/2}\,dx$$

$$= \frac{1}{4}\left[\frac{x e^{-x/2}}{-\frac{1}{2}} - \frac{e^{-x/2}}{(-\frac{1}{2})(-\frac{1}{2})}\right]_5^\infty$$

$$= \frac{1}{4}\left[-2x e^{-x/2} - 4e^{-x/2}\right]_5^\infty$$

$$\boxed{P(X>5) = \frac{7}{2} e^{-5/2} = 0.287}$$