

Hypergeometric Distribution:

(7)

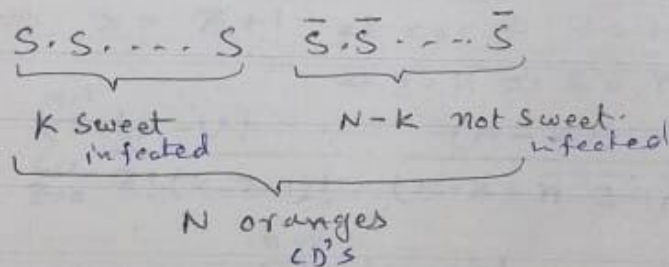
A random variable  $X$  has a hypergeometric distribution iff its probability distribution is given by

$$h(x; n, N, K) = \frac{\binom{K}{x} \binom{N-K}{n-x}}{\binom{N}{n}} ; \quad \begin{matrix} x=0, 1, 2, \dots, K \\ x \leq K ; n-x \leq N-K \end{matrix}$$

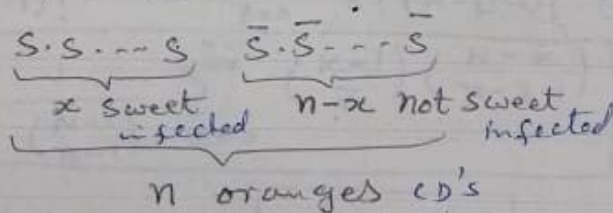
Proof:

Suppose a box of  $N$  oranges contains  $K$  Sweet oranges. A sample of  $n$  oranges ( $n < N$ ) is selected. The probability that the sample contains  $x$  Sweet oranges ( $x \leq K$ ).

Let sample space  $S$  contains  $\binom{N}{n}$  sample points. The population contains



and the sample contains,



The number of possible ways of selecting  $x$  oranges from  $K$  is  $\binom{K}{x}$  and the number of ways of selecting  $n-x$  not Sweet oranges out of  $N-K$  is  $\binom{N-K}{n-x}$  ways.

Thus the probability distribution of selecting  $x$  Sweet out of  $K$  and  $n-x$  not Sweet out of  $N-K$  not Sweet oranges by independent law of Probability is

$$h(x; n, N, K) = \frac{\binom{K}{x} \binom{N-K}{n-x}}{\binom{N}{n}} ; \quad \begin{matrix} x=0, 1, 2, \dots, K \\ x \leq K, n-x \leq N-K \end{matrix}$$

which is hypergeometric distribution with parameters  $n, N$  and  $K$ .

⑧

Moments of Hypergeometric Distribution:

By definition.

$$\mu'_1 = E[X^1] = \sum_{x=0}^n x^1 h(x; n, N, k) = \sum_{x=0}^n x \cdot \frac{\binom{k}{x} \binom{N-k}{n-x}}{\binom{N}{n}}$$

at  $r=1$ 

$$\mu'_1 = \sum_{x=0}^n x \cdot \frac{\binom{k}{x} \binom{N-k}{n-x}}{\binom{N}{n}} = \frac{1}{\binom{N}{n}} \cdot \sum_{x=1}^n x \cdot \frac{\binom{k}{x} \binom{N-k}{n-x}}$$

$$= \frac{1}{\binom{N}{n}} \sum_{x=1}^n \frac{k(k-1)!}{x(x-1)!(k-x)!} \cdot \frac{(N-k)!}{(N-k-n+x)!(n-x)!}$$

$$= \frac{k}{\binom{N}{n}} \sum_{x=1}^n \frac{(k-1)!}{(x-1)!(k-x)!} \cdot \frac{(N-k)!}{[N-k-(n-x)]!(n-x)!}$$

Let  $z = x-1 \Rightarrow x = z+1$  at  $x=1 \Rightarrow z = 1-1 = 0$ .

at  $x=n \Rightarrow z = n-1$ .

$$\therefore \mu'_1 = \frac{k}{\binom{N}{n}} \sum_{z=0}^{n-1} \frac{(k-1)!}{z!(k-z-1)!} \cdot \frac{(N-k)!}{(N-k-n+z-1)!(n-z-1)!}$$

$$= \frac{k \cdot n(n-1)!(N-n)!}{N(N-1)!} \cdot \sum_{z=0}^{n-1} \frac{(k-1)!}{z!(k-z-1)!} \cdot \frac{(N-k)!}{(N-k-n+z-1)!(n-z-1)!}$$

$$= k \cdot \frac{n}{N} \cdot \left[ \frac{1}{\binom{N-1}{n-1}} \sum_{z=0}^{n-1} \binom{k-1}{z} \binom{N-k}{n-z-1} \right]$$

$$= k \cdot \frac{n}{N} \cdot (1) \Rightarrow \boxed{\mu'_1 = \frac{k n}{N}}$$

at  $r=2$ 

$$\mu'_2 = E(X^2) = E[X(X-1) + X] = E[X(X-1)] + E(X) = E[X(X-1)] + \frac{k n}{N}$$

$$\therefore E[X(X-1)] = \sum_{x=0}^n x(x-1) \cdot \frac{\binom{k}{x} \binom{N-k}{n-x}}{\binom{N}{n}}$$

$$= \frac{1}{\binom{N}{n}} \cdot \sum_{x=0}^n x(x-1) \cdot \frac{k!}{x!(k-x)!} \cdot \frac{(N-k)!}{(n-x)!(N-k-n+x)!}$$

$$= \frac{n(n-1)(n-2)!(N-n)!}{N(N-1)(N-2)!} \sum_{x=2}^n \frac{x(x-1) \cdot \frac{k(k-1)(k-2)!}{(k-x)! x(x-1)(x-2)!} \cdot \frac{(N-k)!}{(n-x)!(N-k-n+x)!}}{(N-k)!}$$

$$= \frac{n(n-1)}{N(N-1)} \cdot k(k-1) \sum_{x=2}^n \frac{(k-2)!}{(x-2)!(k-x)!} \cdot \frac{(N-k)!}{(n-x)!(N-k-n+x)!}$$

Let  $z = x-2 \Rightarrow x = z+2$  at  $x=2 \Rightarrow z = 2-2 = 0$ .

at  $x=n \Rightarrow z = n-2$

$$E[X(X-1)] = E[(z+2)(z+1)] = \frac{n(n-1)}{N(N-1)} \cdot k(k-1) \sum_{z=0}^{n-2} \frac{(k-2)!}{z!(k-z-2)!} \cdot \frac{(N-k)!}{(n-z-2)!(N-k-n)!}$$

$$= \frac{n(n-1)}{N(N-1)} \cdot k(k-1) \cdot \frac{(N-k)!}{(n-2)!(N-n)!}$$



⑨

Since the expression  $\binom{k-2}{z} \binom{N-k}{n-z-2} / \binom{N-2}{n-2}$  is p.d.f for hypergeometric distribution for  $z=0, 1, 2, \dots, n-2$  using the property of p.d.f  $\sum_{z=0}^{n-2} \binom{k-2}{z} \binom{N-k}{n-z-2} / \binom{N-2}{n-2} = 1$ .

$$\Rightarrow E[X(X-1)] = E[(z+1)(z+2)] = k \cdot (k-1) \cdot \frac{n(n-1)}{N(N-1)} \cdot (1)$$

$$\therefore \mu_2' = k(k-1) \frac{n(n-1)}{N(N-1)} + k \frac{n}{N}$$

$$= \frac{k n}{N} \left[ (k-1) \frac{(n-1)}{(N-1)} + 1 \right]$$

$$\begin{aligned} \text{Variance} = \mu_2' - \{E[X]\}^2 \\ = \frac{n k}{N} \left[ \frac{(k-1)(n-1)}{(N-1)} + 1 \right] - \frac{k^2 n^2}{N^2} \end{aligned}$$

$$\boxed{\text{Var}(X) = \mu_2' = \frac{n k}{N} \left[ \frac{(k-1)(n-1)}{(N-1)} + 1 - \frac{k n}{N} \right]}$$

Higher order moments can be obtained by proceeding the same manner.

### Hypergeometric Experiment:

An experiment is called as hypergeometric experiment iff

- The outcomes of the experiment classified into two categories, successes and failures
- The probability of successes changes from trial to trial.
- Successive trials are dependent
- The experiment is repeated upto a fixed number of times.

⑨

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$$\Rightarrow E[X(X-1)] = E[(z+1)(z+2)] = k \cdot (k-1) \cdot \frac{n(n-1)}{N(N-1)} \cdot (1)$$

$$\therefore \mu'_2 = k(k-1) \frac{n(n-1)}{N(N-1)} + k \frac{n}{N}$$

$$= \frac{k n}{N} \left[ \frac{(k-1)(n-1)}{(N-1)} + 1 \right]$$

$$\begin{aligned} \text{Variance} = \mu'_2 &= E[X^2] - \{E[X]\}^2 \\ &= \frac{n k}{N} \left[ \frac{(k-1)(n-1)}{(N-1)} + 1 \right] - \frac{k^2 n^2}{N^2} \end{aligned}$$

$$\boxed{\text{Var}(X) = \mu'_2 = \frac{n k}{N} \left[ \frac{(k-1)(n-1)}{(N-1)} + 1 \right] - \frac{k n}{N}}$$

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recurrence relation of probabilities for hypergeometric dist

the p.d. of exactly  $x$  successes is

$$P(X) = \frac{\binom{K}{x} \binom{N-K}{n-x}}{\binom{N}{n}}$$

and the p.d. of  $x+1$  successes is

$$P(X+1) = \frac{\binom{K}{x+1} \binom{N-K}{n-x-1}}{\binom{N}{n}}$$

The ratio

$$\frac{P(X+1)}{P(X)} = \frac{\binom{K}{x+1} \binom{N-K}{n-x-1}}{\binom{N}{n}} \cdot \frac{\binom{N}{n}}{\binom{K}{x} \binom{N-K}{n-x}}$$

simplifying we get

$$P(X+1) = \frac{(n-x)(K-x)}{(x+1)(N-K-n+x+1)} P(X)$$

Problem:

Among the 120 applicants for a job only 80 are actually qualified. If 5 of these applicants are randomly selected for an "indepth" interview, find the probability that only 2 of the 5 will be qualified for the job.

Sol:

Let  $x=2$ ,  $n=5$ ,  $N=120$ ,  $K=80$

$$h(2; 5, 120, 80) = \frac{\binom{80}{2} \binom{40}{3}}{\binom{120}{5}} = 0.164 //$$

using Binomial distribution:

$$b(2; 5, \frac{2}{3}) = \binom{5}{2} \left(\frac{2}{3}\right)^2 \left(1 - \frac{2}{3}\right)^3 = 0.165 //$$

$$p = \frac{80}{120} = \frac{2}{3}$$

Walpole 3rd edition.

Solve	Exercise	page 165
	Exercise	page 173
	Exercise	page 180



What is the probability that a waiter will refuse to serve alcoholic beverages to only 2 minors if she checks the I.D.s of 6 students from among 9 students of which 4 are not of legal age? (11)

Sol.

There are 9 students out of which 6 doesn't reach the particular age. So sample space is  $\binom{9}{6}$ .

$$\frac{\binom{4}{2} \binom{5}{4}}{\binom{9}{6}} = \frac{5}{4} //$$