Duality Analysis

A generalized format of the linear programming problem is represented here.

Maximize or $z = C_1X_1 + C_2X_2 + \dots + C_nX_n$.

s/t

$$a_{11}X_1 + a_{12}X_2 + \dots + a_{1n}X_n \leq \text{or} \geq b_1$$

$$a_{21}X_1 + a_{22}X_2 + \dots + a_{2n}X_n \leq \text{or} \geq b_2$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$a_{m1}X_1 + a_{m2}X_2 + \dots a_{mn}X_n \leq \text{or} \geq b_m$$

where $X_1, X_2, \dots, X_n \geq 0$.

Let this problem be called as a primal linear programming problem. If the constraints in the primal problem are too many, then the time taken to solve the problem is expected to be higher. Under such situation, the primal linear programming problem can be converted into its dual linear programming problem which requires relatively less time to solve. Then the solution of the primal problem can be obtained from the optimal table of its dual problem by following certain rules.

Formulation of Dual Problem ⇒

The primal problem is again reproduced below:

Maximize or Minimize $Z = c_1X_1 + c_2X_2 + \dots + c_nX_n$

$$\text{s/t}\quad a_{11}X_1 + a_{12}X_2 + \dots + a_{1n}X_n \lesseqgtr b_1 \longleftarrow Y_1$$

$$a_{12}X_1 + a_{22}X_2 + \dots + a_{1n}X_n \lesseqgtr b_2 \longleftarrow Y_2$$

$$\vdots$$

$$a_{i1}X_1 + a_{i2}X_2 + \dots + a_{in}X_n \lesseqgtr b_i \longleftarrow Y_i$$

$$\vdots$$

$$a_{m1}X_1 + a_{m2}X_2 + \dots + a_{mn}X_n \lesseqgtr b_m \longleftarrow Y_m$$

where $X_1, X_2, \dots, X_n \geq 0$.

Note that Y_i are the dual variables associated with constraints i's.

Objective function ⇒

The number of variables in the dual problem is equal to the number of constraints in the primal problem.

The objective function of the dual problem is constructed by adding the multiples of the right hand side constants of constraints of the primal problem with the respective dual ~~problem~~ variables.

Constraints ⇒ The number of constraints in the dual problem is equal to the number of variables in the primal problem. Each dual constraint corresponds to each primal variable. The left hand side of the dual constraint corresponding

~~constr~~

to jth primal variable is the sum of the multiple of the left hand side constraint coefficients of the jth primal variable with the corresponding dual variables.
The right hand side constant of the dual constraint corresponding to the jth primal variable is the objective function coefficient of the jth-primal variable.

Guidlines for Dual formation ⇒

	Types of problem	Objective function	Constraint type	Nature of variables
(A)	Primal	Maximize	$\leq$	Restricted in sign
	dual	Minimize	$\geq$	Restricted in sign
(B)	Primal	Minimize	$\geq$	//
	dual	Maximize	$\leq$	//
(C)	Primal	Maximize	$=$	Restricted in sign
	dual	Minimize	$\geq$	Unrestricted in sign
(d)	Primal	Maximize	$\leq$	Unrestricted in sign
	dual	minimize	$=$	restricted in sign
(e)	Primal	Minimize	$=$	Restricted in sign
	dual	Maximize	$\leq$	unrestricted in sign
(F)	Primal	Minimize	$\geq$	Unrestricted in sign
	dual	Maximize	$=$	Restricted in sign-

Example: Form the dual of the following problem

Maximize $Z = 4X_1 + 10X_2 + 25X_3$

S/t

$$2X_1 + 4X_2 + 8X_3 \leq 25$$
$$4X_1 + 9X_2 + 8X_3 \leq 30$$
$$6X_1 + 8X_2 + 2X_3 \leq 40$$
$$X_1, X_2, X_3 \geq 0.$$

Sol: Let Y_i be the dual variable associated with ith constraint of the primal problem, i.e.

Maximize $Z = 4\,X_1 + 10\,X_2 + 25\,X_3$

S/t

$$2\,X_1 + 4\,X_2 + 8\,X_3 \leq 25 \leftarrow Y_1$$
$$4\,X_1 + 9\,X_2 + 8\,X_3 \leq 30 \leftarrow Y_2$$
$$6\,X_1 + 8\,X_2 + 2\,X_3 \leq 40 \leftarrow Y_3$$

$$X_1, X_2, X_3 \geq 0.$$

Thus, the corresponding dual problem is

Minimize $Y = 25Y_1 + 30Y_2 + 40Y_3$

S/t

$$2Y_1 + 4Y_2 + 6Y_3 \geq 4$$
$$4Y_1 + 9Y_2 + 8Y_3 \geq 10 \qquad Y_1, Y_2, Y_3 \geq 0.$$
$$8Y_1 + 8Y_2 + 2Y_3 \geq 25.$$

Example: Form the dual of the following primal problem

Minimize $Z = 20X_1 + 40X_2$

S/t
$$2X_1 + 20X_2 \geq 40$$
$$20X_1 + 3X_2 \geq 20$$
$$4X_1 + 15X_2 \geq 30$$
$$X_1 \,\&\, X_2 \geq 0.$$

Solution: Let Y_i be the dual variable associated with the ith constraint of the given primal problem. Thus, the dual problem is presented as

Maximize $Y = 40Y_1 + 20Y_2 + 30Y_3$

S/t
$$2Y_1 + 20Y_2 + 4Y_3 \leq 20$$
$$20Y_1 + 3Y_2 + 15Y_3 \leq 40.$$
$$Y_1, Y_2, Y_3 \geq 0.$$

Q. Form the dual of the following problem

Maximize $Z = 4X_1 + 10X_2 + 25X_3$

S/t
$$2X_1 + 4X_2 + 8X_3 = 25$$
$$4X_1 + 9X_2 + 8X_3 = 30$$
$$6X_1 + 8X_2 + 2X_3 = 40$$
$$X_1, X_2 \ \& \ X_3 \geq 0.$$

Soln The associated dual problem can be constructed as

Minimize $Y = 25Y_1 + 30Y_2 + 40Y_3$

S/t
$$2Y_1 + 4Y_2 + 6Y_3 \geq 4$$
$$4Y_1 + 9Y_2 + 8Y_3 \geq 10$$
$$8Y_1 + 8Y_2 + 2Y_3 \geq 25$$

$Y_1, Y_2 \ \& \ Y_3$ are unrestricted.

Example. **Primal problem**

Minimize $Z = 20X_1 + 40X_2$

S/t

$2X_1 + 20X_2 = 40$

$20X_1 + 3X_2 = 20$

$4X_1 + 15X_2 = 30$

$X_1 \& X_2 \geq 0$.

Dual problem

Maximize $Y = 40Y_1 + 20Y_2 + 30Y_3$

S/t

$2Y_1 + 20Y_2 + 4Y_3 \leq 20$

$20Y_1 + 3Y_2 + 15Y_3 \leq 40$.

Y_1, Y_2 are unrestricted.

Example. Primal problem

Minimize $Z = 5X_1 + 8X_2$, S/t $4X_1 + 9X_2 \geq 100$

$2X_1 + X_2 \leq 20 \rightarrow (2)$

$2X_1 + 5X_2 \geq 120 \rightarrow (3)$

$X_1 \& X_2 \geq 0$

~~Dual problem~~

~~Maxim~~

P.T.O

Sol: One can also write the primal problem as

$$\text{Minimize} \quad Z = 5X_1 + 8X_2$$
$$\text{S/t} \quad 4X_1 + 9X_2 \geq 100$$
$$-2X_1 - X_2 \geq -20$$
$$2X_1 + 5X_2 \geq 120$$
$$X_1, X_2 \geq 0.$$

Since all the constraints are of type $\geq$, so the dual problem is constructed as

$$\text{Maximize} \quad Y = 100Y_1 - 20Y_2 + 120Y_3$$
$$\text{S/t} \quad 4Y_1 - 2Y_2 + 2Y_3 \leq 5$$
$$9Y_1 - Y_2 + 5Y_3 \leq 8$$
$$Y_1, Y_2 \ \& \ Y_3 \geq 0 \ \cancel{\text{are unrestricted in sign}}.$$

Example: Minimize $Z = 2X_1 + 6X_2$

$$\text{S/t} \quad 9X_1 + 3X_2 \geq 20$$
$$2X_1 + 7X_2 = 40$$
$$X_1 \ \& \ X_2 \geq 0.$$

Since the constraint ② can be represented by two constraints i.e

$$2X_1+7X_2=40 \Rightarrow \begin{cases} 2X_1+7X_2 \geq 40 \\ 2X_1+7X_2 \leq 40. \end{cases}$$

Thus, the modified form of the problem is

Minimize $Z = 2X_1+6X_2$

S/t

$$9X_1+3X_2 \geq 20$$
$$2X_1+7X_2 \geq 40$$
$$2X_1+7X_2 \leq 40.$$

In order to obtain minimization model with all constraints of type $\geq$, we can also write

Minimize $Z = 2X_1+6X_2$

S/t

$$9X_1+3X_2 \geq 20$$
$$2X_1+7X_2 \geq 40$$
$$-2X_1-7X_2 \geq 40$$

Thus, the dual problem is given by

Maximize $\cancel{Z = 2X_1 + 6X_2}$ $Y = 20Y_1 + 40Y_2' - 40Y_2''$

S/t $9Y_1 + 2Y_2' - 2Y_2'' \leq 2$

$3Y_1 + 7Y_2' - 7Y_2'' \leq 6$

$Y_1, Y_2' \& Y_2'' \geq 0.$

By substituting $Y_2 = Y_2' - Y_2''$ we have

Maximize $Y = 20Y_1 + 40Y_2$

S/t $9Y_1 + 2Y_2 \leq 2$

$3Y_1 + 7Y_2 \leq 6$

$Y_1 \geq 0$ & Y_2 is unrestricted.