Two-phase method ⇒

In the method, the use of the penalty, M, can result in computer round off error. The two-phase method eliminates the use of the constant M altogether. As the name suggests the method solves the LP in two phases: Phase I attempts to find a starting basic feasible solution, and, if one is found, Phase II is invoked to start the original problem.

* Summary

Phase I ⇒ Put the problem in equation form, and add the necessary artificial variables to the constraints (exactly as in M-method) to secure a starting basic solution. Next, compute a basic solution of the resulting equations that always minimizes the sum of the artificial variables, regardless of whether the LP is maximization or minimization. If the minimum value of the sum is positive, the LP has no feasible solution. Otherwise, proceed to phase II.

Phase II → Use the feasible solution from phase I as a starting basic feasible solution for the original problem.

Q. Minimize $z = 4x_1 + x_2$

S/t
$$3x_1 + x_2 = 3$$
$$4x_1 + 3x_2 \geq 6$$
$$x_1 + 2x_2 \leq 4$$
$$x_1, x_2 \geq 0.$$

Sol. Phase I:

Minimize $r = R_1 + R_2$

S/t
$$3x_1 + x_2 + R_1 = 3$$
$$4x_1 + 3x_2 - S_1 + R_2 = 6$$
$$x_1 + 2x_2 + S_2 = 4,$$

$$x_1, x_2, S_1, R_1, R_2, S_2 \geq 0.$$

The corresponding tableau is given as

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Basis	x_1	x_2	S_1	R_1	R_2	$\tilde{S}_2$	Solution
r	0	0	0	-1	-1	0	0
R_1	3	1	0	1	0	0	3
R_2	4	3	-1	0	1	0	6
$\tilde{S}_2$	1	2	0	0	0	1	4

Now New r-row = old r-row + (1 × R_1-row + 1 × R_2-row)

New r-row = (0 0 0 -1 -1 0 0) + (3 1 0 1 0

+ (4 3 -1 0 1 0 6)

$$\boxed{\text{New r-row} = (7\ 4\ -1\ 0\ 0\ 0\ 9)}$$

Basis	entering x_1	x_2	S_1	R_1	R_2	$\tilde{S}_2$	Solution	ratios
r	7	4	-1	0	0	0	9	
R_1	$\boxed{3}$	1	0	1	0	0	3	1
R_2	4	3	-1	0	1	0	6	3/2
$\tilde{S}_2$	1	2	0	0	0	1	4	4

Because minimum $r=0$, Phase I produces the basic feasible solution $x_1 = 3/5$, $x_2 = 6/5$ and $\tilde{S}_2 = 1$. At this point, the artificial variables have completed their mission, and we eliminate their columns altogether from the tableau and move on to phase II.

Phase II ⇒

After deleting the artificial columns, we can write the original problem as

Minimize $z = 4x_1 + x_2$

S/t
$$x_1 + \frac{1}{5}S_1 = 3/5$$
$$x_2 - \frac{3}{5}S_1 = 6/5$$
$$S_1 + \tilde{S}_2 = 1, \quad x_1, x_2, S_1, \tilde{S}_2 \geq 0.$$

Hence the tableau associated with this form is given as

$4-7(\frac{1}{3})$

$-1-7(0)$ $9-7(1)$ $3-4(\frac{1}{3})$

$2-1(\frac{1}{3})$ $-1-4(0)$

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$0-4(\frac{1}{3})$ $\frac{5}{3}$ $1-4(0)$ $6-4(1)$

$-1-\frac{5}{3}(\frac{3}{5})$ $4-1(1)$

II

Basic	x_1	x_2 ↓ entering	S_1	R_1	R_2	$\tilde{S}_2$	Solution	Ratio
r	0	$\frac{5}{3}$	-1	$-\frac{7}{3}$	0	0	2	
x_1	1	$\frac{1}{3}$	0	$\frac{1}{3}$	0	0	1	3
leaving ← R_2 P.R	0	$\frac{5}{3}$	-1	$-\frac{4}{3}$	1	0	2	$\frac{6}{5}$
$\tilde{S}_2$	0	$\frac{5}{3}$	0	$-\frac{1}{3}$	0	1	3	$\frac{9}{5}$

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Basic	x_1	x_2	S_1	R_1	R_2	$\tilde{S}_2$	Solution
r	0	0	0	-1	-1	0	0
x_1	1	0	$\frac{1}{5}$	$\frac{3}{5}$	$-\frac{1}{5}$	0	$\frac{3}{5}$
x_2	0	1	$-\frac{3}{5}$	$-\frac{4}{5}$	$\frac{3}{5}$	0	$\frac{6}{5}$
$\tilde{S}_2$	0	0	1	1	-1	1	1

Basic	x_1	x_2	S_1	$\tilde{S}_2$	Solution
z	-4	-1	0	0	0
x_1	1	0	$1/5$	0	$3/5$
x_2	0	1	$-3/5$	0	$6/5$
$\tilde{S}_2$	0	0	1	1	1.

Again, because the basic variables x_1 and x_2 have nonzero coefficients in the z-row so they must be substituted out from the constraint using the row operation

~~New z-row = current z-row + (4 x x1 row + (1) x2-row)~~

New z-row = current z-row + $4\,x_1$-row + $(1)\,x_2$-row

$$\Rightarrow z\text{-row} = (-4 \quad -1 \quad 0 \quad 0 \quad 0) + (4 \quad 0 \quad 4/5 \quad 0 \quad 12/5) + (0 \quad 1 \quad -3/5 \quad 0 \quad 6/5)$$

$$\Rightarrow z\text{-row} = (0 \quad 0 \quad 1/5 \quad 0 \quad 18/5)$$

Thus,

entering

Basic	x_1	x_2	S_1	$\tilde{S}_2$	Solution	Ratio
z	0	0	$1/5$	0	$18/5$	
x_1	1	0	$1/5$	0	$3/5$	3
x_2	0	1	$-3/5$	0	$6/5$	—
leaving $\leftarrow \tilde{S}_2$	0	0	1	1	1	1

Pivot column

Basic	x_1	x_2	S_1	$\tilde{S}_2$	Solution
z	0	0	0	-1	$17/5$
x_1	1	0	0	$-1/5$	$2/5$
x_2	0	1	0	$3/5$	$9/5$
S_1	0	0	1	1	1

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One can see that the above tableau is optimal as none of the values in z-row is negative. Thus, the optimal solution is

$$\boxed{x_1 = 2/5, \quad x_2 = 9/5, \quad z = 17/5.}$$