Artificial starting solutions ⇒

As demonstrated earlier, LPs in which all the constraints are ≤ with non-negative right hand sides offer a convenient all slack starting basic feasible solution. Models involving (=) and/or (≥) constraints donot.

The procedure for starting the "ill-behaved" LPs with (=) and (≥) constraints is to use artificial variables that play the role of slacks at the first iteration. To ~~do~~ implement this procedure, two closely related methods are used: the M-method & the two-phase method.

M-Method ⇒

The M-method starts with the LP in an equation form. If equation i does not have a slack variable (or a variable that can play the role of slack), an artificial variable, R_i, is added to form a starting solution similar to the all-slack basic solution. However the artificial variables are not part of the original problem, and modeling "trick" is needed to force them to zero by the time the optimum iteration is reached. The desired goal is achieved by penalizing these variables in the objective function using the following rule:

Penalty Rule for artificial variables ⇒

Given M, a sufficiently large positive values (mathematically, $M \to \infty$), the objective coefficient of an artificial variable represents an appropriate penalty if

$$\text{Artificial variable objective coefficient} = \begin{cases} -M, & \text{in maximization problem} \\ M, & \text{in minimization problem.} \end{cases}$$

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Example: Minimize $z = 4x_1 + x_2$

Subject to

$$3x_1 + x_2 = 3$$
$$4x_1 + 3x_2 \geq 6$$
$$x_1 + 2x_2 \leq 4$$
$$x_1, x_2 \geq 0.$$

Using x_3 as a surplus in the second constraint and x_4 as a slack in the third constraint, the equation form of the problem is given as

Minimize $z = 4x_1 + x_2$

Subject to

$$3x_1 + x_2 = 3$$
$$4x_1 + 3x_2 - x_3 = 6$$
$$x_1 + 2x_2 + x_4 = 4$$

$$x_1, x_2, x_3, x_4 \geq 0.$$

The ~~first~~ third equation has a slack but first and second have not any slack variable. Thus, we add the artificial variables R_1 & R_2 in the first two equations and penalize them in the objective function with MR_1+MR_2. The resulting problem then takes the form (known as canonical form)

Minimize $z = 4x_1 + x_2 + MR_1 + MR_2$

Subject to

$$3x_1 + x_2 + R_1 = 3$$
$$4x_1 + 3x_2 - x_3 + R_2 = 6$$
$$x_1 + 2x_2 + x_4 = 4$$
$$x_1, x_2, x_3, x_4, R_1, R_2 \geq 0.$$

The starting basic solution is $(R_1, R_2, x_4) = (3, 6, 4)$.

It is convenient to use a numeric value for M to avoid the manipulation of M algebraically. ~~Since~~ The penalty M must be ~~chosen~~ sufficiently large relative to the original objective coefficients such that the artificial variables can be force to zero in the optimal solution. Since, the coefficients of x_1 & x_2 in objective function are 4 & 1, respectively, so we are choosing $M=100$. Hence, the initial simplex tableau is given as follows:

Basic	x_1	x_2	x_3	R_1	R_2	x_4	Solution
z	-4	-1	0	-100	-100	0	0
R_1	3	1	0	1	0	0	3
R_2	4	3	-1	0	1	0	6
x_4	1	2	0	0	0	1	4

Before employing simplex algorithm, the z-row must be made consistent with the rest of the tableau. The right side of z-row in above table ~~currently~~ shows $z=0$. However, since the starting basic solution is $(R_1, R_2, x_4) = (3, 6, 4)$ and along with nonbasic solution $(x_1, x_2, x_3) = (0, 0, 0)$ yields the value of z as

$$z = 4(0) + (0) + 100(3) + 100(6) + 0 = 900.$$

This inconsistency stems from the fact that R_1 & R_2 have nonzero coefficients $(-100, -100)$ in z-row. Hence to eliminate this inconsistency, we need to substitute R_1 & R_2 in the z-row using the following row operations

New z-row = Old z-row $+ 100(R_1\text{-row}) + 100(R_2\text{-row})$. $\Rightarrow$ (*)

(Note that this operation is the same as substituting out $R_1 = 3 - 3x_1 - x_2$ & $R_2 = 6 - 4x_1 - 3x_2 + x_3$ in $z - 4x_1 - x_2 - 100R_1 - 100R_2$).

Therefore (*) $\Rightarrow$

$$\text{New } z\text{-row} = (-4 \quad -1 \quad 0 \quad -100 \quad -100 \quad 0 \quad 0) + \cancel{100}$$
$$100(3 \quad 1 \quad 0 \quad 1 \quad 0 \quad 0 \quad 3)$$
$$+\,100(4 \quad 3 \quad -1 \quad 0 \quad 1 \quad 0 \quad 6).$$

$$\Rightarrow \text{New } z\text{-row} = (-4+300+400 \quad -1+100+300 \quad -100 \quad 0 \quad 0 \quad 0 \quad 900)$$
$$= (696 \quad 399 \quad -100 \quad 0 \quad 0 \quad 0 \quad 900).$$

So the modified tableau thus becomes
(b/c the problem is related to minimization so highest positive coefficient is 69

Basic	x_1 ↓ entering	x_2	x_3	R_1	R_2	x_4	Solution	ratio
z	696	399	-100	0	0	0	900	
leaving ← R_1	3	1	0	1	0	0	3	1
Pivot Row R_2	4	3	-1	0	1	0	6	$6/4$
x_4	1	2	0	0	0	1	4	4

Pivot column

$696-696(1)$ $\quad$ $-100-696(0)$ $\quad$ $0-696(0)$

$399-696(1/3)=167$ $\quad$ $4-4(1)$ $\quad$ $3-4(1/3)$ $\quad$ $0-696(1/3)$ $\quad$ $900-696(1)$

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2-

Basic	x_1	x_2 (entering)	x_3	R_1	R_2	x_4	Solution	Ratio
z	0	167	-100	-232	0	0	204	
x_1	1	$\frac{1}{3}$	0	$\frac{1}{3}$	0	0	1	3
leaving ← R_2	0	$\frac{5}{3}$	-1	$-\frac{4}{3}$	1	0	2	$6/5$
x_4	0	$\frac{5}{3}$	0	$-\frac{1}{3}$	0	1	3	$9/5$

Pivot row $\quad$ Pivot column

$0-\frac{5}{3}(0)=0$ $\quad$ $-\frac{1}{3}-\frac{5}{3}(-4/5)$ $\quad$ $1-\frac{5}{3}(0)$ $\quad$ $\frac{9}{5}-2$

$\frac{5}{3}-\frac{5}{3}(1)=0$ $\quad$ $0-\frac{5}{3}(-3/5)$ $\quad$ $0-\frac{5}{3}\left(\frac{3}{5}\right)$ $\quad$ $\frac{9}{5}-\frac{5}{8}\left(\frac{6}{5}\right)$

Basic	x_1	x_2	x_3 (entering)	R_1	R_2	x_4	Solution	Ratio
z	0	0	$1/5$	$-\frac{492}{5}$	$-\frac{501}{5}$	0	$18/5$	
x_1	1	0	$1/5$	$3/5$	$-1/5$	0	$3/5$	3
~~R_2~~ x_2	0	1	$-3/5$	$-4/5$	$3/5$	0	$6/5$	
leaving x_4	0	0	1	1	-1	1	1	1 *

P.C $\quad$ Pivot row

$1-\frac{1}{3}\left(\frac{6}{5}\right)$ $1-\frac{2}{5}$

$0-\frac{1}{3}\left(\frac{3}{5}\right)$ $\quad$ $1-\frac{1}{3}(0)$, $\quad$ $\frac{1}{3}-\frac{1}{3}(1)$, $\quad$ $0-\frac{1}{3}(-3/5)$ $\quad$ $\frac{1}{3}-\frac{1}{3}(-4/5)=\frac{5+4}{15}=9/15=\frac{3}{5}$ $\quad$ $=18/5$

$0-167(3/5)$ $\quad$ $204-168(6/5)=\frac{1020-1008}{5}$

$-100-167(-3/5)$

$-232+167(4/5)=-232+\frac{668}{5}=\frac{-492}{5}=-\frac{492}{5}$

$\frac{-500+501}{5}=1/5$

$3-\frac{8}{5}\left(\frac{6}{5}\right)=1$

$-\frac{495}{5}-\frac{1}{5}(1)$ $\quad -\frac{501}{5}+\frac{1}{5}$ $\quad -\frac{500}{5}$

3.8

Basic	x_1	x_2	x_3	R_1	R_2	x_4	Solution
z	0	0	0	$-496/5$	-100	$-1/5$	$17/5$
x_1	1	0	0	$2/5$	0	$-1/5$	$2/5$
x_2	0	1	0	$-1/5$	0	$3/5$	$9/5$
x_3	0	0	1	1	-1	1	1

Since none of the coeffient in z-row is positive so the last tableau is optimal. Thus, the optimum solution is

$\frac{1}{5}-\cancel{(\frac{1}{5})(1)}$

$$\boxed{\begin{aligned} z &= 17/5 \\ x_1 &= 2/5 \\ x_2 &= 9/5 \end{aligned}}$$

Ans.