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Homework 1

Question 1

I show the proof that the Laplacian operator is not influenced by rotations.

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 f}{\partial x'^2} + \frac{\partial^2 f}{\partial y'^2} \tag{1}$$

I selected rotation equation following.

$$x' = xcos\theta - ysin\theta \tag{2}$$
$$y' = xsin\theta + ycos\theta \tag{3}$$

(For x)Showing that equation $\frac{\partial^2 f}{\partial x^2}$ in evidence:

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial x'}\frac{\partial x'}{\partial x} + \frac{\partial f}{\partial y'}\frac{\partial y'}{\partial x} = \frac{\partial f}{\partial x'}cos\theta + \frac{\partial f}{\partial y'}sin\theta \tag{4}$$

(For x)Calculating a derivative that equation again $\frac{\partial f}{\partial x}$ in evidence:

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial f}{\partial x}(\frac{\partial f}{\partial x'}cos\theta + \frac{\partial f}{\partial y'}sin\theta) = \frac{\partial f}{\partial x}\frac{\partial f}{\partial x'}cos\theta + \frac{\partial f}{\partial x}\frac{\partial f}{\partial y'}sin\theta \tag{5}$$

(For x)These are equal $\frac{\partial}{\partial x}\frac{\partial f}{\partial x'} = \frac{\partial}{\partial x'}\frac{\partial f}{\partial x}$ and $\frac{\partial}{\partial x}\frac{\partial f}{\partial y'} = \frac{\partial}{\partial y'}\frac{\partial f}{\partial x}$

$$\frac{\partial}{\partial x'}\frac{\partial f}{\partial x}cos\theta = cos\theta\frac{\partial}{\partial x'}(\frac{\partial f}{\partial x'}cos\theta + \frac{\partial f}{\partial y'}sin\theta) = \frac{\partial^2 f}{\partial x'^2}cos^2\theta + \frac{\partial^2 f}{\partial x'y'}sin\theta cos\theta \tag{6}$$

$$\frac{\partial}{\partial y'}\frac{\partial f}{\partial x}sin\theta = sin\theta\frac{\partial}{\partial y'}(\frac{\partial f}{\partial x'}cos\theta + \frac{\partial f}{\partial y'}sin\theta) = \frac{\partial^2 f}{\partial y'^2}sin^2\theta + \frac{\partial^2 f}{\partial x'y'}sin\theta cos\theta \tag{7}$$

(For x)Consequently, $\frac{\partial^2 f}{\partial x^2}$ is equal the following equation,

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 f}{\partial x'^2}cos^2\theta + \frac{\partial^2 f}{\partial x'y'}sin\theta cos\theta + \frac{\partial^2 f}{\partial y'^2}sin^2\theta + \frac{\partial^2 f}{\partial x'y'}sin\theta cos\theta \tag{8}$$

(For y)Showing that equation $\frac{\partial^2 f}{\partial y^2}$ in evidence:

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial x'}\frac{\partial x'}{\partial y} + \frac{\partial f}{\partial y'}\frac{\partial y'}{\partial y} = -\frac{\partial f}{\partial x'}sin\theta + \frac{\partial f}{\partial y'}cos\theta \tag{9}$$

(For y)Calculating a derivative that equation again $\frac{\partial f}{\partial y}$ in evidence:

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial f}{\partial y}(-\frac{\partial f}{\partial x'}sin\theta + \frac{\partial f}{\partial y'}cos\theta) = -\frac{\partial f}{\partial y}\frac{\partial f}{\partial x'}sin\theta + \frac{\partial f}{\partial y}\frac{\partial f}{\partial y'}cos\theta \tag{10}$$

(For y)These are equal $\frac{\partial}{\partial y}\frac{\partial f}{\partial x'} = \frac{\partial}{\partial x'}\frac{\partial f}{\partial y}$ and $\frac{\partial}{\partial y}\frac{\partial f}{\partial y'} = \frac{\partial}{\partial y'}\frac{\partial f}{\partial y}$

$$-\frac{\partial}{\partial x'}\frac{\partial f}{\partial y}sin\theta = -sin\theta\frac{\partial}{\partial x'}(-\frac{\partial f}{\partial x'}sin\theta + \frac{\partial f}{\partial y'}cos\theta) = \frac{\partial^2 f}{\partial x'^2}sin^2\theta - \frac{\partial^2 f}{\partial x'y'}sin\theta cos\theta \tag{11}$$

$$\frac{\partial}{\partial y'}\frac{\partial f}{\partial y}cos\theta = cos\theta\frac{\partial}{\partial y'}(-\frac{\partial f}{\partial x'}sin\theta + \frac{\partial f}{\partial y'}cos\theta) = \frac{\partial^2 f}{\partial y'^2}cos^2\theta - \frac{\partial^2 f}{\partial x'y'}sin\theta cos\theta \tag{12}$$

(For y)Consequently, $\frac{\partial^2 f}{\partial y^2}$ is equal the following equation,

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 f}{\partial x'^2}sin^2\theta - \frac{\partial^2 f}{\partial x'y'}sin\theta cos\theta + \frac{\partial^2 f}{\partial y'^2}cos^2\theta - \frac{\partial^2 f}{\partial x'y'}sin\theta cos\theta \tag{13}$$

(Sum up $\frac{\partial^2 f}{\partial x'^2} + \frac{\partial^2 f}{\partial y'^2}$) We sum up 8 and 13.

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 f}{\partial x'^2}sin^2\theta + \frac{\partial^2 f}{\partial y'^2}cos^2\theta + \frac{\partial^2 f}{\partial x'^2}cos^2\theta + \frac{\partial^2 f}{\partial y'^2}sin^2\theta \tag{14}$$

(Note : $cos^2\theta + sin^2\theta = 1$) Finally ,

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 f}{\partial x'^2}(sin^2\theta + cos^2\theta) + \frac{\partial^2 f}{\partial y'^2}(sin^2\theta + cos^2\theta) \tag{15}$$

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = (sin^2\theta + cos^2\theta)(\frac{\partial^2 f}{\partial x'^2} + \frac{\partial^2 f}{\partial y'^2}) \tag{16}$$

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 f}{\partial x'^2} + \frac{\partial^2 f}{\partial y'^2} \tag{17}$$

Question 2

I will prove triangle inequality for ℓ^1(city block) metric on distance function. .I have 2 proofs. The city block distance function is given by:

$$d(a, b) = |a_1 - b_1| + |a_2 - b_2|$$

<u>Proof-1</u>
I will use absolute value rule that is subadditivity : $|x + y| \leq |x| + |y|$.
Triangle Inequality:

$$d(x, z) \leq d(x, y) + d(y, z) \tag{18}$$

$$|x_1 - z_1| + |x_2 - z_2| \leq |y_1 - z_1| + |y_2 - z_2| + |x_1 - y_1| + |x_2 - y_2| \tag{19}$$

Let's use our absolute value rule to the right side of the equation

$$\rightarrow |y_1 - z_1| + |y_2 - z_2| + |x_1 - y_1| + |x_2 - y_2| \tag{20}$$

$$|y_1 - z_1 + x_1 - y_1| + |y_2 - z_2 + x_2 - y_2| \leq |y_1 - z_1| + |y_2 - z_2| + |x_1 - y_1| + |x_2 - y_2| \tag{21}$$

$$|x_1 - z_1| + |x_2 - z_2| \leq |y_1 - z_1| + |y_2 - z_2| + |x_1 - y_1| + |x_2 - y_2| \tag{22}$$

We see that, If we use subadditivity rule ,we get same equation (19,22).
<u>Proof-2</u>
The city block distance function is given by:

$$d(a, b) = |a_1 - b_1| + \cdots + |a_n - b_n|$$

Note : given points could be metric and x,y and z on $\mathbb{R}^n$.
Triangle Inequality For d(x,z):

$$d(x, z) \leq d(x, y) + d(y, z) \tag{23}$$

Triangle Inequality For d(x,y):

$$d(x, y) \leq d(x, z) + d(y, z) \tag{24}$$

Triangle Inequality For d(y,z):

$$d(y,z) \leq d(x,y) + d(x,z) \tag{25}$$

Sum up 23 , 24 and 25

$$\begin{gathered}|x_1 - z_1| + |x_1 - y_1| + |y_1 - z_1| + \cdots + |x_n - z_n| + |x_n - y_n| + |y_n - z_n| \\ \leq \\ 2|x_1 - z_1| + 2|x_1 - y_1| + 2|y_1 - z_1| + \cdots + 2|x_n - z_n| + 2|x_n - y_n| + 2|y_n - z_n|\end{gathered}$$

Finally, we subtract left side to right side. Then we see that left side where is zero which shows that any edge is not bigger than any other 2 edges.

$$0 \leq |x_1 - z_1| + |x_1 - y_1| + |y_1 - z_1| + \cdots + |x_n - z_n| + |x_n - y_n| + |y_n - z_n|$$

$$0 \leq d(x,z) + d(y,z) + d(x,y)$$

Question 3

The transformation matrix C that registers B with A.

$$C = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & 0 & 1 \end{bmatrix}$$

There are 3 points which will be registered by transformation matrix.

$$\begin{aligned} B(x,y) &= [1,2],[2,1],[3,1] \\ A(x,y) &= [2,2],[-1,4],[-4,4] \end{aligned}$$

We multiply matrix C and $B(1,2) = A(2,2)$

$$C.B = A : \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$$

$$a_{11} + 2a_{12} + a_{13} = 2 \tag{26}$$
$$a_{21} + 2a_{22} + a_{23} = 2 \tag{27}$$

We multiply matrix C and $B(2, 1) = A(-1, 4)$

$$C.B = A : \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix}$$

$$2a_{11} + a_{12} + a_{13} = -1 \tag{28}$$
$$2a_{21} + a_{22} + a_{23} = 4 \tag{29}$$

We multiply matrix C and $B(3, 1) = A(-4, 4)$

$$C.B = A : \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -4 \\ 4 \\ 1 \end{bmatrix}$$

$$3a_{11} + a_{12} + a_{13} = -4 \tag{30}$$
$$3a_{21} + a_{22} + a_{23} = 4 \tag{31}$$

Let's apply gauss elimination 26, 28 and 30 to find a_{11}, a_{12} and a_{13}.

$$\left[\ C_a\ \right] = \left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 2 & 1 & 1 & -1 \\ 3 & 1 & 1 & -4 \end{array}\right]$$

Let's apply gauss elimination 27, 29 and 31 to find a_{21}, a_{22} and a_{23}.

$$\left[\ C_b\ \right] = \left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 2 & 1 & 1 & 4 \\ 3 & 1 & 1 & 4 \end{array}\right]$$

We look for proportion that helps to make zero.(C_a)

$$k_{21} = \frac{C_{a21}}{C_{a11}} = \frac{2}{1} = 2 \tag{32}$$
$$k_{31} = \frac{C_{a31}}{C_{a11}} = \frac{3}{1} = 3 \tag{33}$$
$$Ca_2 - k_{21}.Ca_1 \to C_{a2} \tag{34}$$
$$Ca_3 - k_{31}.Ca_1 \to C_{a3} \tag{35}$$

$$\left[\begin{array}{c} C_a \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & -3 & -1 & -5 \\ 0 & -5 & -2 & -10 \end{array}\right]$$

We are adding 3rd line : $C_{a3} - k_{32}.C_{a2} \rightarrow C_{a3}$

$$k_{32} = \frac{C_{a32}}{C_{a22}} = \frac{-5}{-3} = 1.666 \tag{36}$$

$$C_{a3} - k_{32}.C_{a2} \rightarrow C_{a3} \tag{37}$$

$$\tag{38}$$

$$\left[\begin{array}{c} C_a \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & -3 & -1 & -5 \\ 0 & 0 & -0.333 & -1.665 \end{array}\right]$$

Finally, we found a_{11}, a_{12} and a_{13}.

$$a_{11} = -3 \tag{39}$$

$$a_{12} = 0 \tag{40}$$

$$a_{13} = 5 \tag{41}$$

We look for proportion that helps to make zero.(C_b)

$$k_{21} = \frac{C_{b21}}{C_{b11}} = \frac{2}{1} = 2 \tag{42}$$

$$k_{31} = \frac{C_{b31}}{C_{b11}} = \frac{3}{1} = 3 \tag{43}$$

$$C_{b2} - k_{21}.C_{b1} \rightarrow C_{b2} \tag{44}$$

$$C_{b3} - k_{31}.C_{b1} \rightarrow C_{b3} \tag{45}$$

$$\left[\begin{array}{c} C_b \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & -3 & -1 & 0 \\ 0 & -5 & -2 & -2 \end{array}\right]$$

We are adding 3rd line : $C_{b3} - k_{32}.C_{b2}$

$$k_{32} = \frac{C_{b32}}{C_{b22}} = \frac{-5}{-3} = 1.6667 \tag{46}$$

$$C_{b3} - k_{32}.C_{b2} \rightarrow C_{b3} \tag{47}$$

$$\tag{48}$$

$$\left[\begin{array}{c} C_b \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & -3 & -1 & 0 \\ 0 & 0 & -0.333 & -2 \end{array}\right]$$

Finally, we found a_{21}, a_{22} and a_{23}.

$$a_{21} = 0 \tag{49}$$
$$a_{22} = -2 \tag{50}$$
$$a_{23} = 6 \tag{51}$$

Transformation matrix C is:

$$C = \begin{bmatrix} -3 & 0 & 5 \\ 0 & -2 & 6 \\ 0 & 0 & 1 \end{bmatrix}$$

Question 4

I prove that the C and D be two sets and B a structuring element.

$$(C \cup D) \oplus \overset{\vee}{B} = (C \oplus \overset{\vee}{B}) \cup (D \oplus \overset{\vee}{B})$$

I will use following equality that dilation of a binary image :

$$A \oplus B = \bigcup_{a \in A}$$

It consists of all displacements z, such that $\overset{\vee}{B}_{\mathrm{z}}$ and A overlap by at least one pixel.
$(\mathrm{z}|\overset{\vee}{B}_{\mathrm{z}} \cap A \neq 0)$
I will use this equality.

$$(C \cup D) \oplus \overset{\vee}{B} = \bigcup_{b \in \overset{\vee}{B}} (C \cup D)_{\mathrm{b}}$$

$$= \bigcup_{b \in \overset{\vee}{B}} (C_{\mathrm{b}} \cup D_{\mathrm{b}})$$

$$= (\bigcup_{b \in \overset{\vee}{B}} C_{\mathrm{b}}) \cup (\bigcup_{b \in \overset{\vee}{B}} D_{\mathrm{b}})$$

Let's use the other equality.Finally, we get the same right side of equality.

$$= (C \oplus \overset{\vee}{B}) \cup (D \oplus \overset{\vee}{B})$$