Virtual Element method for Timoshenko beam

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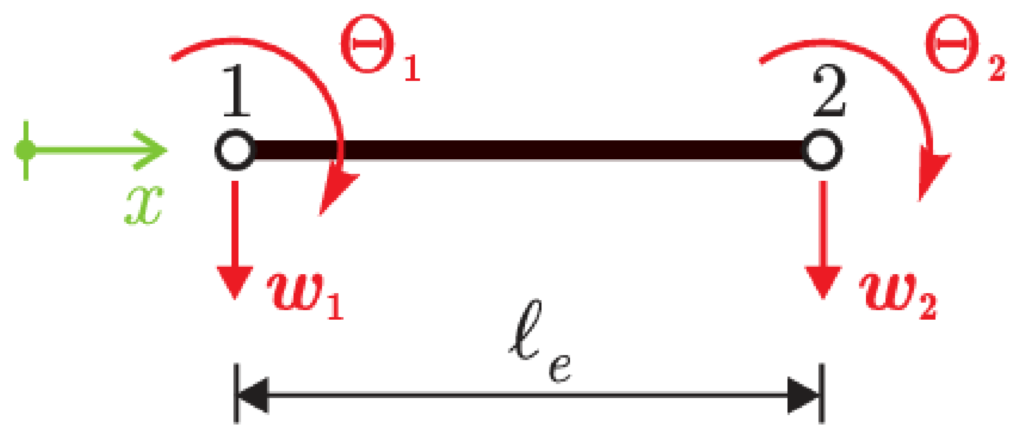
Introduction

- **Virtual Element Method (VEM)**
- The Virtual Element Method is based on an **ansatz space** where:
  - Displacements are known **only at nodal points**
  - Displacements on **element edges** are assumed (edges do not exist in 1D problems)
- Unlike the **Finite Element Method (FEM)**:
  - FEM defines the displacement field over the **entire domain**
  - VEM does **not** require explicit knowledge of the displacement field inside the element

Derivation of Virtual element Method



How can we derive an ansatz for the displacement field uh ?

- The idea is to use a projection Π[uh] of the displacement onto a polynomial space.
  So, approximations displacement uh are written as

$$\mathbf{u_h} = \mathbf{\Pi}[\mathbf{u_h}] + (\mathbf{u_h} - \mathbf{\Pi}[\mathbf{u_h}])$$

Where **projected** part or **consistency terms** $\mathbf{\Pi}[\mathbf{u_h}]$
And **remainder** $(\mathbf{u_h} - \mathbf{\Pi}[\mathbf{u_h}])$

- Also, for short notation

$$\boldsymbol{u_\pi} = \mathbf{\Pi}[\mathbf{u_h}]$$

  Also

$$\boldsymbol{u_h} \approx \boldsymbol{u_\pi}$$

- Note
  Contrary to virtual elements for two- and three-dimensional solids, the consistency term, as we will see in the following, provides element matrices that have full rank and thus the **remainder can be neglected**. This means that the virtual Timoshenko beam elements do not have to be stabilized.

How can we derive projected Π[uh] of ansatz space are derived ?

- Since only **first order derivatives appear in the weak form and potential** it is sufficient to compute e.g. the projection of the displacement by using **two conditions** for the

- **first derivative (gradient)**

$$\int_0^{l_e} p'(u'_\pi - u'_h)\,\mathrm{d}x = 0$$

and the **average of the displacements** related to uh and Π[uh]

$$\int_0^{l_e} (u_\pi - u_h)\,\mathrm{d}x = 0$$

- Note for **Euler–Bernoulli beams** we have to use this condition:

$$\int_0^{l_e} p''\,(w''_h - w''_\pi)\,\mathrm{d}x = 0 \qquad \int_0^{l_e} w_\pi(x)\mathrm{d}x = \int_0^{l_e} w_h(x)\mathrm{d}x \qquad \int_0^{l_e} w'_\pi(x)\mathrm{d}x = \int_0^{l_e} w'_h(x)\mathrm{d}x\,.$$

VEM formulation for second order ansatz

We take gradient equation

$$\int_0^{l_e} p'\,(u_h' - u_\pi')\,\mathrm{d}x = 0 \rightarrow \int_0^{l_e} p'\,u_\pi'\,\mathrm{d}x = \int_0^{l_e} p'\,u_h'\,\mathrm{d}x$$

We apply integration by part on projected part

$$\int_0^{l_e} p'\,u_\pi'\,\mathrm{d}x = (p'\,u_h)\big|_0^{l_e} - \int_0^{l_e} p''\,u_h\,\mathrm{d}x\,.$$

Projected ansatz and its derivative is given by is given by

$$u_\pi = a_1 + a_2\,x + a_3\,x^2$$

$$u_\pi' = a_2 + 2a_3\,x = \langle 1\ \ 2x \rangle \begin{Bmatrix} a_2 \\ a_3 \end{Bmatrix}$$

$$p' = \langle 1\ \ 2x \rangle^T \qquad p'' = \langle 0\ \ 2 \rangle^T$$

- Left hand of equation:

$$\int_0^{l_e} p'\, u'_\pi \,\mathrm{d}x = \int_0^{l_e} \begin{Bmatrix} 1 \\ 2x \end{Bmatrix} \langle 1 \;\; 2x \rangle \,\mathrm{d}x \begin{Bmatrix} a_2 \\ a_3 \end{Bmatrix} = \int_0^{l_e} \begin{bmatrix} 1 & 2x \\ 2x & 4x^2 \end{bmatrix} \mathrm{d}x \begin{Bmatrix} a_2 \\ a_3 \end{Bmatrix}$$

$$\int_0^{l_e} p'\, u'_\pi \,\mathrm{d}x = \begin{bmatrix} l_e & l_e^2 \\ l_e^2 & \frac{4}{3} l_e^3 \end{bmatrix} \begin{Bmatrix} a_2 \\ a_3 \end{Bmatrix} = \mathbf{G}\,\mathbf{a}$$

- Right hand side of equation:

$$(p'\, u_h)\big|_0^{l_e} - \int_0^{l_e} p''\, u_h \,\mathrm{d}x = \left[\begin{Bmatrix} 1 \\ 2x \end{Bmatrix} u_h \right]_0^{l_e} - \begin{Bmatrix} 0 \\ 2 \end{Bmatrix} \int_0^{l_e} u_h \,\mathrm{d}x$$

- Since we don’t uh inside element, so we introduced new internal variable that is not associated with any node, is called **moment** in the virtual element literature

$$m_0 = \frac{1}{l_e} \int_0^{l_e} u_h \,\mathrm{d}x$$

- Note:
  it only appear higher order ansatz **n>= 2** . It has **same unit** as displacement

$$(p'\,u_h)\big|_0^{l_e} - \int_0^{l_e} p''\,u_h\,\mathrm{d}x = \left[\begin{Bmatrix}1\\2x\end{Bmatrix} u_h\right]_0^{l_e} - \begin{Bmatrix}0\\2\end{Bmatrix}\int_0^{l_e} u_h\,\mathrm{d}x \qquad m_0 = \frac{1}{l_e}\int_0^{l_e} u_h\,\mathrm{d}x$$

- Since **uh (0) = u1** and **u(le) = u2 :**

$$(p'\,u_h)\big|_0^{l_e} - \int_0^{l_e} p''\,u_h\,\mathrm{d}x = \begin{Bmatrix}u_2 - u_1\\ 2\,l_e u_2\end{Bmatrix} - \begin{Bmatrix}0\\ 2\,l_e m_0\end{Bmatrix} = \mathbf{r}(u_i\,, m_0)$$

$$\mathbf{G\,a} = \mathbf{r}(u_i\,, m_0) \to \begin{Bmatrix}a_2\\a_3\end{Bmatrix} = \begin{Bmatrix}-\frac{2}{l_e}(2u_1 + u_2 - 3m_0)\\ \frac{3}{l_e^2}(u_1 + u_2 - 2m_0)\end{Bmatrix}$$

- So **derivative of projected ansatz** is given by :

$$u'_\pi = \langle 1\ 2x\rangle \begin{Bmatrix}-\frac{2}{l_e}(2u_1 + u_2 - 3m_0)\\ \frac{3}{l_e^2}(u_1 + u_2 - 2m_0)\end{Bmatrix}$$

- **Now how we computed a1 ?**

- Now **average of displacement** is given by :

$$\int_0^{l_e} u_\pi \, \mathrm{d}x = \int_0^{l_e} u_h \mathrm{d}x \, .$$

$$u_\pi = a_1 + a_2\, x + a_3\, x^2$$

$$m_0 = \frac{1}{l_e} \int_0^{l_e} u_h \, \mathrm{d}x$$

$$\begin{Bmatrix} a_2 \\ a_3 \end{Bmatrix} = \begin{Bmatrix} -\frac{2}{l_e}(2u_1 + u_2 - 3m_0) \\ \frac{3}{l_e^2}(u_1 + u_2 - 2m_0) \end{Bmatrix}$$

$$a_1 l_e + \frac{1}{2} a_2\, l_e^2 + \frac{1}{3} a_3\, l_e^3 = l_e\, m_0 \quad \rightarrow a_1 = u_1$$

- Also **projection** ansatz is given by

$$u_\pi = \langle 1 \;\; x \;\; x^2 \rangle \begin{Bmatrix} u_1 \\ -\frac{2}{l_e}(2u_1 + u_2 - 3m_0) \\ \frac{3}{l_e^2}(u_1 + u_2 - 2m_0) \end{Bmatrix}$$

Projected ansatz and derivative in term of nodal displacements and internal variable

$$u_\pi = \langle 1 \; x \; x^2 \rangle \begin{Bmatrix} u_1 \\ -\frac{2}{l_e}(2u_1 + u_2 - 3m_0) \\ \frac{3}{l_e^2}(u_1 + u_2 - 2m_0) \end{Bmatrix}$$

$$u'_\pi = \langle 1 \; 2x \rangle \begin{Bmatrix} -\frac{2}{l_e}(2u_1 + u_2 - 3m_0) \\ \frac{3}{l_e^2}(u_1 + u_2 - 2m_0) \end{Bmatrix}$$

$$\mathbf{N}_\pi^{(2)}(x) = \langle 1 \; x \; x^2 \rangle$$

$$\nabla \mathbf{N}_\pi^{(2)}(x) = \langle 1 \; 2x \rangle$$

$$\mathbb{P}^{(2)} = \frac{1}{l_e^2} \begin{bmatrix} l_e^2 & 0 & 0 \\ -4l_e & -2l_e & 6l_e \\ 3 & 3 & -6 \end{bmatrix}$$

$$\mathbb{B}^{(2)} = \frac{1}{l_e^2} \begin{bmatrix} -4l_e & -2l_e & 6l_e \\ 3 & 3 & -6 \end{bmatrix}$$

$$u_\pi = \mathbf{N}_\pi^{(2)}(x)\, \mathbb{P}^{(2)}\, \widehat{\mathbf{u}}_e$$

$$u'_\pi = \nabla \mathbf{N}_\pi^{(2)}(x)\, \mathbb{B}^{(2)}\, \widehat{\mathbf{u}}_e$$

Generalization for higher-order ansatz functions

$$w_\pi(x) = \sum_{k=0}^{n} a_{k+1}\, x^k = \mathbf{N}_\pi^{(n)}(x)\, \mathbf{a} \qquad \mathbf{N}_\pi^{(n)}(x) = \langle 1 \quad x \quad x^2 \quad \cdots \quad x^n \rangle \qquad \mathbf{a}^T = \langle a_1 \quad a_2 \quad a_3 \quad \ldots \quad a_{n+1} \rangle^T$$

$$w'_\pi = \mathbf{B}_\pi^{(n)}(x)\, \hat{\mathbf{a}} \qquad \mathbf{B}_\pi^{(n)}(x) = \langle 1 \quad 2x \quad \cdots \quad n\, x^{n-1} \rangle \qquad \hat{\mathbf{a}}^T = \langle a_2 \quad a_3 \quad \ldots \quad a_{n+1} \rangle^T$$

- **Gradient equation**

$$\int_0^{l_e} p' \left\{ (\Pi[w_h])' - w'_h \right\} \mathrm{d}x = 0$$

$$\int_0^{l_e} p'\, w'_h \, \mathrm{d}x = (p'\, w_h)\Big|_0^{l_e} - \int_0^{l_e} p''\, w_h \, \mathrm{d}x$$

$$p' = [\mathbf{B}_\pi^{(n)}(x)]^T = \langle 1 \quad 2x \quad \cdots \quad n\, x^{n-1} \rangle^T$$

$$p'' = \langle 0 \quad 2 \quad \ldots \quad n(n-1)x^{n-2} \rangle^T$$

- **Here n is order of ansatz**

$$\int_0^{l_e} p'\, w'_\pi \,\mathrm{d}x = \int_0^{l_e} [\mathbf{B}_\pi^{(n)}(x)]^T [\mathbf{B}_\pi^{(n)}(x)] \,\mathrm{d}x\, \hat{\mathbf{a}}$$

$$= \int_0^{l_e} \begin{bmatrix} 1 & 2x & \cdots & nx^{n-1} \\ 2x & 4x^2 & \cdots & 2nx^n \\ \vdots & \vdots & \vdots & \vdots \\ nx^{n-1} & 2nx^n & \cdots & n^2x^{2n-2} \end{bmatrix} \mathrm{d}x \begin{Bmatrix} a_2 \\ a_3 \\ \vdots \\ a_{n+1} \end{Bmatrix}$$

$$\int_0^{l_e} p'\, w'_\pi \,\mathrm{d}x = \begin{bmatrix} l_e & l_e^2 & \cdots & l_e^n \\ l_e^2 & \frac{4}{3} l_e^3 & \cdots & \frac{2n}{n+1} l_e^{n+1} \\ \vdots & \vdots & \vdots & \vdots \\ l_e^n & \frac{2n}{n+1} l_e^{n+1} & \cdots & \frac{n^2}{2n-1} l_e^{2n-1} \end{bmatrix} \begin{Bmatrix} a_2 \\ a_3 \\ \vdots \\ a_{k+1} \end{Bmatrix} = \mathbf{G}\,\hat{\mathbf{a}}$$

$$\mathbf{r}(w_h) = (p'\, w_h)\Big|_0^{l_e} - \int_0^{l_e} p''\, w_h \,\mathrm{d}x = \left[\begin{Bmatrix} 1 \\ 2x \\ \vdots \\ n\,x^{n-1} \end{Bmatrix} w'_h \right]_0^{l_e} - \begin{Bmatrix} 0 \\ 2l_e\, m_0 \\ \vdots \\ n(n-1)\, l_e^{n-1}\, m_{n-2} \end{Bmatrix}$$

$$p' = [\mathbf{B}_\pi^{(n)}(x)]^T = \langle 1 \quad 2x \quad \cdots \quad n\,x^{n-1} \rangle^T \quad p'' = \langle 0 \quad 2 \quad \ldots \quad n(n-1)x^{n-2} \rangle^T \qquad m_k = \frac{1}{l_e^{k+1}} \int_0^{l_e} x^k\, w_h \,\mathrm{d}x \quad \text{with } k = 0, 1, \ldots$$

Here mk is internal variable called **moment** same unit as displacement. Also, **k = n-2**.

$$\mathbf{r}(w_h) = (p'\, w_h)\Big|_0^{l_e} - \int_0^{l_e} p''\, w_h\, \mathrm{d}x = \left[\begin{Bmatrix} 1 \\ 2x \\ \vdots \\ n\, x^{n-1} \end{Bmatrix} w_h' \right]_0^{l_e} - \begin{Bmatrix} 0 \\ 2l_e\, m_0 \\ \vdots \\ n(n-1)\, l_e^{n-1}\, m_{n-2} \end{Bmatrix}$$

$$\mathbf{r}(\mathbf{w}_e, \mathbf{m}_e) = \begin{Bmatrix} (w_2 - w_1) \\ 2l_e\, w_2 \\ \vdots \\ n l_e^{n-1}\, w_2 \end{Bmatrix} - \begin{Bmatrix} 0 \\ 2l_e\, m_0 \\ \vdots \\ n(n-1)\, l_e^{n-1}\, m_{n-2} \end{Bmatrix}$$

$$w_h(0) = w_1$$

$$w_h(l_e) = w_2$$

$$\mathbf{G}\, \hat{\mathbf{a}} = \mathbf{r}(\mathbf{w}_e, \mathbf{m})$$

$$\hat{\mathbf{a}} = \mathbf{G}^{-1}\, \mathbf{r}(\mathbf{w}_e, \mathbf{m}) = \mathbb{B}^{(n)} \widehat{\mathbf{w}}_e^{(n)}$$

$$\hat{\mathbf{a}}^T = \langle a_2 \quad a_3 \quad \dots \quad a_{n+1} \rangle^T \qquad \widehat{\mathbf{w}}_e^{(n)} = \langle w_1 \quad w_2 \quad m_0 \quad \cdots \quad m_{n-2} \rangle^T$$

- **So gradient of projected ansatz in term of nodal and internal variable is given by**

$$w_\pi' = \mathbf{B}_\pi^{(n)}(x)\, \mathbb{B}^{(n)} \widehat{\mathbf{w}}_e^{(n)} .$$

- **Average displacement equation is given by**

$$\int_0^{l_e} (\Pi[w_h] - w_h)\,\mathrm{d}x = 0\,.$$

$$w_\pi(0) = w_1$$

$$a_1 = w_1$$

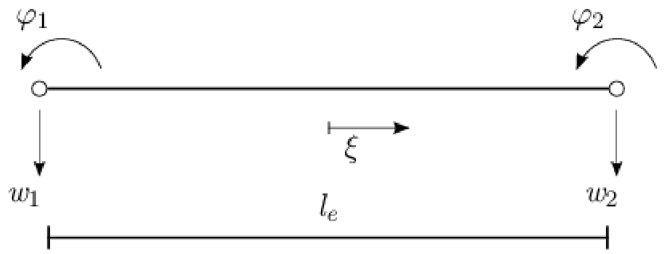
- **So projected ansatz is given by**

$$w_\pi(x) = \mathbf{N}_\pi^{(n)}(x)\,\mathbf{a} = \mathbf{N}_\pi^{(n)}(x)\,\mathbb{P}^{(n)}\widehat{\mathbf{w}}_e^{(n)}$$

$$\mathbf{a}^T = \langle a_1 \quad a_2 \quad a_3 \quad \ldots \quad a_{n+1}\rangle^T \qquad \mathbb{P}^{(n)} = \begin{bmatrix} 1 & \mathbf{o}^T \\ \mathbb{B}^{(n)} & \end{bmatrix} \qquad \mathbf{o}^T = \langle 0 \quad 0 \quad \cdots \quad 0\rangle$$

- Here o vector has **n-1 zeros**

Implementation of a Timoshenko beam element based on VEM



Timoshenko beam element

- **Strong form**

$$V(x) = GA_s\left[w'(x) - \psi(x)\right] \qquad\qquad M(x) = -EI\psi'(x)$$

- **Weak form**

$$\int_0^l \left\{\phi'(x)EI\,\psi'(x) + [v'(x) - \phi(x)]GA_s\left[w'(x) - \psi(x)\right]\right\}\mathrm{d}x - \int_0^l f(x)\,v(x)\mathrm{d}x = 0$$

- **Implementation of VEM is same as fem . We need to change the FEM ansatz to VEM(projected) ansatz in weak form**

Python Implementation & Live Demo

- Demonstrate the **implementation of FEM and VEM in Python**
- Present the **stiffness matrix formulations** for FEM and VEM:
  - **Linear formulation**
  - **Quadratic formulation**
  - **Reduced integration formulation**
- Analyze **cantilever beams and simply supported beams** under **uniformly distributed loading**
- Investigate locking phenomena
- Study **locking-free schemes**, including:
  - **Reduced integration methods**
  - **Mixed formulations**
- Evaluate relative L2 error for **h-convergence**
- Evaluate relative L2 error with respect to **slenderness ratio** for a fixed number of elements

Conclusion

- For the Timoshenko beam, **FEM and VEM give identical stiffness matrix for linear ansatz**; differences only appear for higher-order formulations.
- In **higher-order VEM**, stiffness matrix differ from FEM unless **internal moment variables are condensed out**. After condensation, VEM becomes equivalent to **FEM static condensation**.
- **FEM and VEM produce nearly identical results and L2 errors** for both **linear and quadratic approximations**, indicating no significant advantage of one over the other in these cases.
- In VEM, a **numerical quadratic scheme can be constructed using linear mapping** , due to the presence of **two nodal degrees of freedom and internal moment variables**.
- In contrast, FEM requires **higher-order (isoparametric) mapping** to achieve the same accuracy for higher-order approximations.
- However, for **1D problems**, linear mapping are sufficient for higher order FEM ansatz, so **VEM provides no clear benefit over FEM** in mapping.
- Both FEM and VEM can be implemented in **commercial solvers**, but for linear and quadratic Timoshenko beam problems, **VEM offers no practical advantage**.
- Differences between results of FEM and VEM may appear for **higher-order formulations**, but most commercial solvers **do not use polynomial orders higher than two**.

- The **VEM formulation does not inherently eliminate shear locking**; locking must be addressed using **special techniques** such as mixed formulations or reduced integration, similar to FEM.
- The source of locking lies in the **Timoshenko beam equations themselves**, particularly the shear term with **one variable and one derivative**, not in the numerical formulation.
- FEM and VEM show **similar h-convergence behavior and L2 errors**, even for varying beam slenderness ratios.
- VEM can be viewed as an **alternative formulation to FEM**, based on exact geometry and projection operators.
- In VEM, the displacement field u_h is defined through a **projection operator** $u_h = \Pi_h u$, and is **not explicitly known inside the element**, whereas in FEM the variation of u_h within the element is explicitly defined.

Literature review

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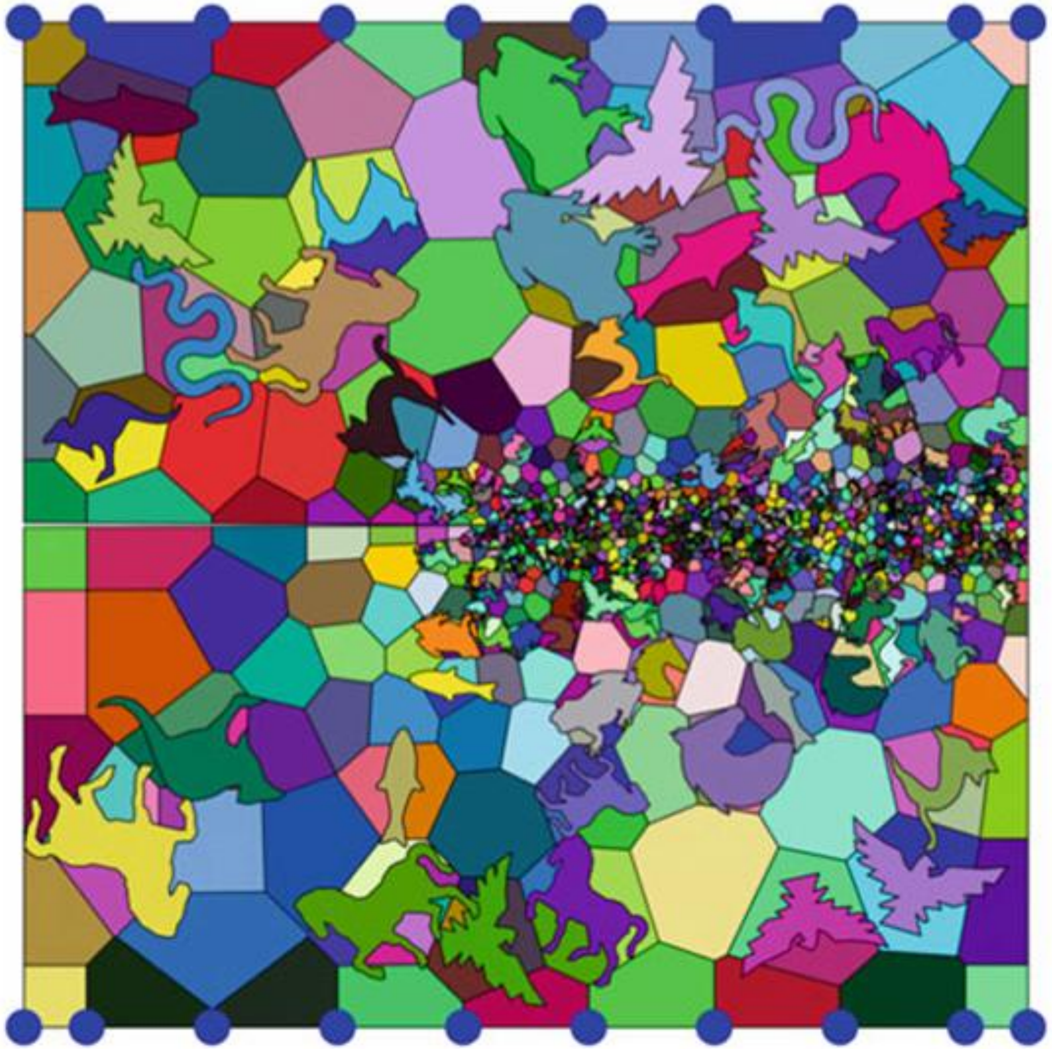

Any Question?