Short note

March 16, 2021

1 DGLAP evolution equations

Let's now check that the classical DGLAP equations for the QCD evolution can be recovered by applying the appropriate limit ($\xi \to 0$) for the GPD's evolution kernels given by Eqs. (4.42-4.44) in Ref. [1]. We will focus here only on the non-singlet-case evolution equation (see Eq.(4.33) of [1])[1],

$$\zeta^2 \frac{d}{d\zeta^2} H^{\rm NS}(x,\xi,t;\zeta) = -\frac{\alpha_s(\zeta)}{4\pi} \int_{-1}^{1} dy K^{qq}_{(0)}\left(x_1,x_2|y_1,y_2\right) H^{\rm NS}(y,\xi,t;\zeta) \tag{1}$$

with

$$x_1 = \frac{\xi+x}{2} \; , \quad x_2 = \frac{\xi-x}{2} \; , \quad y_1 = \frac{\xi+y}{2} \; , \quad y_2 = \frac{\xi-y}{2} \; ; \tag{2}$$

for which the kernel can be read in Eq. (4.42),

$$\begin{aligned} K^{qq}_{(0)}\left(x_1,x_2|y_1,y_2\right) = C_F \Bigg[& \frac{x_1}{x_1-y_1} \theta^0_{11}\left(x_1,x_1-y_1\right) + \frac{x_2}{x_2-y_2}\theta^0_{11}\left(x_2,x_2-y_2\right) \\ & + \theta^0_{111}\left(x_1,-x_2,x_1-y_1\right) \Bigg]_+ \; ; \end{aligned} \tag{3}$$

where the $[\,\cdot\,]_+$ indicates the usual plus-prescription:

$$\left[F(x,a)\right]_+ = F(x,a) - \delta(x-a) \int_\Omega dy F(y,a) \tag{4}$$

Ω standing for the support of the function $F(x,a)$ which includes $x=a$ where the function $F(x,a)$ possesses a singularity. Additionally, Eqs. (G.105) and (G.111) of Ref. [1] bring us the expression for the involved θ-functions,

$$\theta^0_{11}(z_1,z_2) = \frac{\theta(z_1)-\theta(z_2)}{z_1-z_2} \tag{5}$$

$$\theta^0_{111}(z_1,z_2,z_3) = \frac{z_2}{z_1-z_2}\theta^0_{11}(z_2,z_3) - \frac{z_1}{z_1-z_2}\theta^0_{11}(z_1,z_3) \; . \tag{6}$$

[1]Note that Eq. (4.33) involves a derivative $d/d\ln\zeta$ instead of $d/d\ln\zeta^2$, as we used here, but Eq. ,(4.35) expands the kernel in terms of $\alpha_s/[2\pi]$ and we included a factor $\alpha_s/[4\pi]$ in front of the integral, thus absorbing the extra factor 2.

Then, the rescaling properties of the kernel given in Eq. (3) can be readily made explicit by $K^{qq}_{(0)}(x_1, x_2|y_1, y_2) = aK^{qq}_{(0)}(ax_1, ax_2|ay_1, ay_2)$, where a is a positive c-number. Thus, keeping the notation given in Eqs. (2), one can write

$$\begin{aligned} \lim_{\xi\to 0} K^{qq}_{(0)}(x_1, x_2|y_1, y_2) &= 2C_F \left[\frac{x}{x-y} \left(\theta^0_{11}(x, x-y) + \theta^0_{11}(-x, y-x) \right) \right. \\ &\quad \left. + \lim_{\xi\to 0} \theta^0_{111}(x+\xi, x-\xi, x-y) \right]_+ \qquad (7) \\ &= -2C_F \left[\frac{1\!\!1_{[0,1]}(z)}{|y|} \frac{1+z^2}{1-z} \right]_+ \; ; \qquad (8) \end{aligned}$$

where $z = x/y$ and, for the last result, one has applied:

$$\theta^0_{11}(x, x-y) = \frac{\theta(x) - \theta(x-y)}{y} = \frac{1}{|y|} 1\!\!1_{[0,1]} \left(\frac{x}{y} \right) \qquad (9)$$

$$\theta^0_{11}(-x, y-x) = \frac{\theta(-x) - \theta(y-x)}{-y} = \frac{1}{|y|} 1\!\!1_{[0,1]} \left(\frac{x}{y} \right) \qquad (10)$$

$$\lim_{\xi\to 0} \theta^0_{111}(x+\xi, x-\xi, x-y) = \frac{\frac{x}{y} - 1}{|y|} 1\!\!1_{[0,1]} \left(\frac{x}{y} \right) \; . \qquad (11)$$

Then, plugging Eq. (8) into Eq. (1), one is left with

$$\zeta^2 \frac{d}{d\zeta^2} H^{\rm NS}(x, 0, t; \zeta) = \frac{\alpha_s(\zeta)}{4\pi} \int_{-1}^{1} \frac{dy}{|y|} 1\!\!1_{[0,1]} \left(\frac{x}{y} \right) H^{\rm NS}(y, 0, t; \zeta) P \left(\frac{x}{y} \right) \; , \qquad (12)$$

with

$$P(z) = 2C_F \left[\frac{1+z^2}{1-z} \right]_+ \; . \qquad (13)$$

Owing to the step-function $1\!\!1_{[0,1]}(z)$, the positive and negative domains of the non-singlet GPDs decouple and the solutions will be clearly even for $x, y > 0$ and $x, y < 0$. Eventually, when one specializes for $x, y > 0$, the standard DGLAP evolution equation is recovered

$$\zeta^2 \frac{d}{d\zeta^2} H^{\rm NS}(x, 0, t; \zeta) = \frac{\alpha_s(\zeta)}{4\pi} \int_{x}^{1} \frac{dy}{y} H^{\rm NS}(y, 0, t; \zeta) P \left(\frac{x}{y} \right) \; . \qquad (14)$$

References

[1] A. V. Radyushkin. Symmetries and structure of skewed and double distributions. *Phys. Lett.*, B449:81–88, 1999.