1 Numerical computation of the imaginary part of a Compton form factor

The basic integral to be computed when evaluating a Compton form factor has the following structure:

$$\mathcal{H}(\xi) = \int_{-1}^{1} dx\, [C(x,\xi) - C(-x,\xi)]\, H(x)\,. \tag{1.1}$$

Defining $H^{(+)}(x) = H(x) - H(-x)$[1], it is easy to see that:

$$\mathcal{H}(\xi) = \int_{0}^{1} dx\, [C(x,\xi) - C(-x,\xi)]\, H^{(+)}(x)\,. \tag{1.2}$$

This step is of fundamental importance because restricts the integration to positive values of x allowing, as we will see below, for important simplifications. In addition it tuns out that the coefficient function C enjoys the following property:

$$C(x,\xi) = \frac{1}{x}\widetilde{C}\left(\frac{\xi}{x}\right)\,. \tag{1.3}$$

This equality is actually strictly true only if one neglects the $i\varepsilon$ prescription required to regularise the propagator poles. However, depending on the sign of x, one can always reinstate the $i\varepsilon$ prescription taking care of matching the sign of $i\varepsilon$ with the sign of x. This identity allows one to write:

$$\mathcal{H}(\xi) = \int_{0}^{1} \frac{dx}{x} \left[\widetilde{C}\left(\frac{\xi}{x}\right) + \widetilde{C}\left(-\frac{\xi}{x}\right)\right] H^{(+)}(x)\,, \tag{1.4}$$

that with a simple change of variable becomes:

$$\mathcal{H}(\xi) = \int_{\xi}^{\infty} \frac{dy}{y} \left[\widetilde{C}\left(y\right) + \widetilde{C}\left(-y\right)\right] H^{(+)}\left(\frac{\xi}{y}\right)\,. \tag{1.5}$$

Now, if:

$$C(x,\xi) = \frac{1}{x+\xi-i\varepsilon}\left(-3 - 2\ln\left(\frac{x+\xi}{2\xi} - i\varepsilon\right)\right)\,, \tag{1.6}$$

using Eq. (1.3) it is easy to see that:

$$\widetilde{C}(y) = \frac{1}{1+y-i\varepsilon}\left(-3 - 2\ln\left(\frac{1+y}{2y} - i\varepsilon\right)\right)\,, \tag{1.7}$$

and:

$$\widetilde{C}(-y) = \frac{1}{1-y+i\varepsilon}\left(-3 - 2\ln\left(-\frac{1-y}{2y} + i\varepsilon\right)\right)\,. \tag{1.8}$$

Notice that in $C(-y)$ we have swapped the sign of both y and $i\varepsilon$ as appropriate due to the definition of $\widetilde{C}$ in Eq. (1.3). Since the integral in Eq. (1.5) runs over positive values of y, $i\varepsilon$ inside $C(y)$ can be dropped because $C(y)$ has no pole over the integration range. As a direct consequence $C(y)$ becomes a real function. Since we are interested in computing the imaginary part of $\mathcal{H}$, owing to the fact that H is also a real function, one finds that:

$$\mathrm{Im}\left[\mathcal{H}(\xi)\right] = \int_{\xi}^{\infty} \frac{dy}{y} \mathrm{Im}\left[\widetilde{C}\left(-y\right)\right] H^{(+)}\left(\frac{\xi}{y}\right)\,. \tag{1.9}$$

Now we use the well-know relation:

$$\frac{1}{1-y+i\varepsilon} = \mathrm{P.V.}\frac{1}{1-y} - i\pi\delta(1-y)\,, \tag{1.10}$$

and the slightly more subtle equality:

$$\ln\left(-\frac{1-y}{2y} + i\varepsilon\right) = \ln\left(\left|\frac{1-y}{2y}\right|\right) - i\pi\theta(1-y)\,. \tag{1.11}$$

[1] Using Ji's convention, the function H can be either a quark or a gluon distribution.

The presence of the θ-function in front of $i\pi$ is consequence of the fact that that term only arises if the argument of the logarithm is negative that only happens if $y < 1$. This allows one to write:

$$\mathrm{Im}\left[\widetilde{C}(-y)\right] = 2\pi\left[\frac{\theta(1-y)}{1-y} + \delta(1-y)\left(\frac{3}{2} + \ln(1-y) - \ln(2)\right)\right] . \tag{1.12}$$

Notice that we already set $\delta(1-y)\ln y = 0$ and that, due to the presence of the θ-function, we dropped the principal value. Despite the θ-function exposes the singularity of $1/(1-y)$, this is exactly what is needed to cancel the singularity generated by $\delta(1-y)\ln(1-y)$. To see this write:

$$\delta(1-y)\ln(1-y) = -\delta(1-y)\int_0^1 \frac{dz}{1-z} , \tag{1.13}$$

and using the definition of +-prescription:

$$\left(\frac{1}{1-y}\right)_+ = \frac{1}{1-y} - \delta(1-y)\int_0^1 \frac{dz}{1-z} , \tag{1.14}$$

we finally have:

$$\mathrm{Im}\left[\widetilde{C}(-y)\right] = 2\pi\left[\theta(1-y)\left(\frac{1}{1-y}\right)_+ + \delta(1-y)\left(\frac{3}{2} - \ln(2)\right)\right] . \tag{1.15}$$

that plugged into Eq. (1.5) gives:

$$\begin{aligned}\frac{1}{2\pi}\mathrm{Im}\left[\mathcal{H}(\xi)\right] &= \int_\xi^1 \frac{dy}{y}\left[\left(\frac{1}{1-y}\right)_+ + \delta(1-y)\left(\frac{3}{2} - \ln(2)\right)\right] H^{(+)}\left(\frac{\xi}{y}\right) \\ &= \int_\xi^1 \frac{dy}{1-y}\left[\frac{1}{y}H^{(+)}\left(\frac{\xi}{y}\right) - H^{(+)}(\xi)\right] + \left(\frac{3}{2} - \ln(2) + \ln(1-\xi)\right) H^{(+)}(\xi) .\end{aligned} \tag{1.16}$$