1 Compton forms factor with evolution

The factorised expression for Compton form factors (CFF) takes the following form:

$$\mathcal{H}(\xi, t, Q^2) = \int_{-1}^{1} \frac{\mathrm{d}x}{\xi} \sum_{a=q,g} T^a\left(\frac{x}{\xi}, \frac{Q^2}{\mu^2}, \alpha_s(\mu^2)\right) H^a(x, \xi, t, \mu^2) \equiv \sum_{a=q,g} T^a(Q,\mu) \otimes H^a(\mu)\,. \tag{1.1}$$

For the sake of the argument, we assume that there is one single quark generation and that H^q corresponds to the singlet combination([1]). This allows us write the CFF is a matricial form:

$$\mathcal{H}(Q) = \begin{pmatrix} T^q(Q,\mu) & T^g(Q,\mu) \end{pmatrix} \otimes \begin{pmatrix} H^q(\mu) \\ H^g(\mu) \end{pmatrix}. \tag{1.2}$$

The hard cross sections T^q and T^g admit a perturbative expansion whose truncation at $\mathcal{O}(\alpha_s)$ reads:

$$T^q(Q,\mu) = T_0^q + \alpha_s(\mu)\left(T_1^q + T_{\rm coll}^q \log\frac{\mu^2}{Q^2}\right) \quad \text{and} \quad T^g(Q,\mu) = \alpha_s(\mu)\left(T_1^g + T_{\rm coll}^g \log\frac{\mu^2}{Q^2}\right). \tag{1.3}$$

We know that the evolution of GPDs H^q and H^g is governed by the following RG equation:

$$\frac{d}{d\ln\mu^2}\begin{pmatrix} H^q(\mu) \\ H^g(\mu) \end{pmatrix} = \begin{pmatrix} K_{qq}(\mu) & K_{qg}(\mu) \\ K_{gq}(\mu) & K_{gg}(\mu) \end{pmatrix} \otimes \begin{pmatrix} H^q(\mu) \\ H^g(\mu) \end{pmatrix}, \tag{1.4}$$

where the evolution kernels K_{ab} obey the perturbative expansion:

$$K_{ab}(\mu) = \alpha_s(\mu) \sum_{n=0} \alpha_s(\mu) K_{ab}^{(n)}\,. \tag{1.5}$$

Assuming to know GPDs at some initial scale μ_0, the solution to Eq. (1.4) can be written as:

$$\begin{pmatrix} H^q(\mu) \\ H^g(\mu) \end{pmatrix} = \begin{pmatrix} \Gamma_{qq}(\mu,\mu_0) & \Gamma_{qg}(\mu,\mu_0) \\ \Gamma_{gq}(\mu,\mu_0) & \Gamma_{gg}(\mu,\mu_0) \end{pmatrix} \otimes \begin{pmatrix} H^q(\mu_0) \\ H^g(\mu_0) \end{pmatrix}, \tag{1.6}$$

where we have defined the evolution operator as:

$$\begin{pmatrix} \Gamma_{qq}(\mu,\mu_0) & \Gamma_{qg}(\mu,\mu_0) \\ \Gamma_{gq}(\mu,\mu_0) & \Gamma_{gg}(\mu,\mu_0) \end{pmatrix} = \exp\left[\int_{\mu_0}^{\mu} d\ln\mu'^2 \begin{pmatrix} K_{qq}(\mu') & K_{qg}(\mu') \\ K_{gq}(\mu') & K_{gg}(\mu') \end{pmatrix}\right] \tag{1.7}$$

where the exponential function has to be interpreted as a path-ordered exponential. Given the exponential form of the evolution operator, it should clear that the following equality holds:

$$\begin{pmatrix} \Gamma_{qq}(\mu,\mu_0) & \Gamma_{qg}(\mu,\mu_0) \\ \Gamma_{gq}(\mu,\mu_0) & \Gamma_{gg}(\mu,\mu_0) \end{pmatrix} = \begin{pmatrix} \Gamma_{qq}(\mu,Q) & \Gamma_{qg}(\mu,Q) \\ \Gamma_{gq}(\mu,Q) & \Gamma_{gg}(\mu,Q) \end{pmatrix} \otimes \begin{pmatrix} \Gamma_{qq}(Q,\mu_0) & \Gamma_{qg}(Q,\mu_0) \\ \Gamma_{gq}(Q,\mu_0) & \Gamma_{gg}(Q,\mu_0) \end{pmatrix}. \tag{1.8}$$

Now, if the scales μ and Q are not too far apart, the first evolution operator in the r.h.s. of the equation above can be systematically expended in powers of α_s. It is easy to see that to $\mathcal{O}(\alpha_s)$ the expansion is:

$$\begin{pmatrix} \Gamma_{qq}(\mu,Q) & \Gamma_{qg}(\mu,Q) \\ \Gamma_{gq}(\mu,Q) & \Gamma_{gg}(\mu,Q) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \alpha_s(\mu) \begin{pmatrix} K_{qq}^{(0)} & K_{qg}^{(0)} \\ K_{gq}^{(0)} & K_{gg}^{(0)} \end{pmatrix} \ln\frac{\mu^2}{Q^2} + \mathcal{O}(\alpha_s^2)\,. \tag{1.9}$$

We now replace the GPD vector at the final scale μ in Eq. (1.2) with that at the initial scale μ_0 using Eqs. (1.6), (1.8), and (1.9):

$$\mathcal{H}(Q) = \begin{pmatrix} T^q(Q,\mu) & T^g(Q,\mu) \end{pmatrix} \otimes \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \alpha_s(\mu) \begin{pmatrix} K_{qq}^{(0)} & K_{qg}^{(0)} \\ K_{gq}^{(0)} & K_{gg}^{(0)} \end{pmatrix} \ln\frac{\mu^2}{Q^2}\right] \otimes \begin{pmatrix} \Gamma_{qq}(Q,\mu_0) & \Gamma_{qg}(Q,\mu_0) \\ \Gamma_{gq}(Q,\mu_0) & \Gamma_{gg}(Q,\mu_0) \end{pmatrix} \otimes \begin{pmatrix} H^q(\mu_0) \\ H^g(\mu_0) \end{pmatrix}. \tag{1.10}$$

[1] In the presence of more quark generations, also a non-singlet component has to be considered that, to $O(\alpha_s)$, multiplies the same hard cross section T^q as the singlet but evolves multiplicatively through the evolution kernel K_{qq}.

The first two terms in the r.h.s. of the equation above can be combined and only terms up to $\mathcal{O}(\alpha_s)$ retained:

$$\begin{aligned}&\left\{(T^q(Q,\mu) \quad T^g(Q,\mu))\otimes\left[\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}+\alpha_s(\mu)\begin{pmatrix}K_{qq}^{(0)} & K_{qg}^{(0)}\\ K_{gq}^{(0)} & K_{gg}^{(0)}\end{pmatrix}\ln\frac{\mu^2}{Q^2}\right]\right\}^T\\ &=\begin{pmatrix}T_0^q\\ T_0^g\end{pmatrix}+\alpha_s\left[\begin{pmatrix}T_1^q\\ T_1^g\end{pmatrix}+\begin{pmatrix}T_{\rm coll}^q+T_0^q\otimes K_{qq}^{(0)}\\ T_{\rm coll}^g+T_0^q\otimes K_{qg}^{(0)}\end{pmatrix}\ln\frac{\mu^2}{Q^2}\right]+\mathcal{O}(\alpha_s^2)\,.\end{aligned}\tag{1.11}$$

In order for the CFF $\mathcal{H}$ to be independent from the normalisation scale μ up to $\mathcal{O}(\alpha_s)$, we need to require:

$$\begin{aligned}T_{\rm coll}^q &= -T_0^q\otimes K_{qq}^{(0)}\\ T_{\rm coll}^g &= -T_0^q\otimes K_{qg}^{(0)}\end{aligned}\,.\tag{1.12}$$

Finally one has:

$$T^q(Q,\mu)=T_0^q+\alpha_s(\mu)\left(T_1^q-T_0^q\otimes K_{qq}^{(0)}\log\frac{\mu^2}{Q^2}\right)\quad\text{and}\quad T^g(Q,\mu)=\alpha_s(\mu)\left(T_1^g-T_0^q\otimes K_{qg}^{(0)}\log\frac{\mu^2}{Q^2}\right)\,,\tag{1.13}$$

so that:

$$\mathcal{H}(Q)=[(T_0^q \quad 0)+\alpha_s(Q)(T_1^q \quad T_0^g)]\otimes\begin{pmatrix}\Gamma_{qq}(Q,\mu_0) & \Gamma_{qg}(Q,\mu_0)\\ \Gamma_{gq}(Q,\mu_0) & \Gamma_{gg}(Q,\mu_0)\end{pmatrix}\otimes\begin{pmatrix}H^q(\mu_0)\\ H^g(\mu_0)\end{pmatrix}.\tag{1.14}$$

Now, let us also assume that μ_0 and Q are not too far apart. This allows us to expand the evolution operator between μ_0 and Q as above:

$$\begin{pmatrix}\Gamma_{qq}(\mu,Q) & \Gamma_{qg}(\mu,Q)\\ \Gamma_{gq}(\mu,Q) & \Gamma_{gg}(\mu,Q)\end{pmatrix}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}+\alpha_s(Q)\begin{pmatrix}K_{qq}^{(0)} & K_{qg}^{(0)}\\ K_{gq}^{(0)} & K_{gg}^{(0)}\end{pmatrix}\ln\frac{Q^2}{\mu_0^2}+\mathcal{O}(\alpha_s^2)\,.\tag{1.15}$$

Let us also assume that $H^g(\mu_0)=0$. This finally leads to:

$$\begin{aligned}\mathcal{H}(Q) &= T_0^q\otimes H^q(\mu_0)+\alpha_s(Q)\left[T_1^q\otimes H^q(\mu_0)+T_0^q\otimes K_{qq}^{(0)}\otimes H^q(\mu_0)\ln\frac{Q^2}{\mu_0^2}\right]+\mathcal{O}(\alpha_s^2)\\ &= T_0^q\otimes H^q(\mu_0)+\alpha_s(Q)\left[T_1^q\otimes H^q(\mu_0)-T_{\rm coll}^q\otimes H^q(\mu_0)\ln\frac{Q^2}{\mu_0^2}\right]+\mathcal{O}(\alpha_s^2)\,.\end{aligned}\tag{1.16}$$

In conclusion, up to $\mathcal{O}(\alpha_s)$, assuming that the gluon GPD is identically zero at the initial scale μ_0, and for scales Q not too far from μ_0, a shadow GDP needs to simultaneously fulfil the following equalities:

$$\begin{aligned}T_0^q\otimes H^q(\mu_0) &= 0\\ T_1^q\otimes H^q(\mu_0) &= 0\\ T_{\rm coll}^q\otimes H^q(\mu_0) &= -T_0^q\otimes K_{qq}^{(0)}\otimes H^q(\mu_0)=0\,.\end{aligned}\tag{1.17}$$

Conjecture: in order to fulfil the last equalities, it would be enough to require:

$$K_{qq}^{(0)}\otimes H^q(\mu_0)=0\,.\tag{1.18}$$

If this equality holds, I believe that $H^q(\mu_0)$ is unaffected by the evolution, not only at $\mathcal{O}(\alpha_s)$, but to all orders. This would allow us to relax the restriction $Q\simeq\mu_0$. This can be checked numerically.

1.1 Scale variations

The computation of a CFF in terms of coefficient functions, evolution operator, and initial-scale GPDs can be schematically written as:

$$\mathcal{H}(Q,\mu,\mu_0)=\mathbf{T}^T(Q,\mu)\otimes\mathbf{\Gamma}(\mu,\mu_0)\otimes\mathbf{H}(\mu_0)\tag{1.19}$$

where all convolution products also imply a matrix product, with the $\mathbf{T}$ and $\mathbf{H}$ being the column vectors of coefficient functions and GPDs, and $\boldsymbol{\Gamma}$ the evolution matrix. All the irrelevant arguments are dropped. $\mathbf{H}(\mu_0)$ are the input GPDs at some *fixed* initial scale μ_0. The evolution operator $\boldsymbol{\Gamma}(\mu,\mu_0)$, given in Eq. (1.7), has the role of resumming terms of the kind $\alpha_s^m(\mu)\ln^n(\mu/\mu_0)$ to all orders, with $m = n$ being the leading-logarithm (LL) accuracy, $m = n-1$ next-to-leading logarithm (NLL), and so on. Finally, the coefficient functions $\mathbf{T}(Q,\mu)$ can be regarded as a perturbative expansion in powers of $\alpha_s(\mu)$ whose truncation determines the fixed-order perturbative accuracy. For the specific case of a CFF, truncation at $\mathcal{O}(1)$ is said leading-order (LO) approximation, truncation at $\mathcal{O}(\alpha_s)$ next-to-leading order (NLO), and so on. Generalising the discussion above, the coefficient of the contribution to $\mathbf{T}(Q,\mu)$ proportional to $\alpha_s^n(\mu)$, $\mathbf{T}_n$, has the following general structure:

$$\mathbf{T}_n(\mu,Q) = \sum_{k=0}^{n} \mathbf{T}_n^{(k)} \ln^k\left(\frac{\mu}{Q}\right) . \tag{1.20}$$

While the term $\mathbf{T}_n^{(0)}$ requires the calculation on the n-loop diagrams, all other coefficients $\mathbf{T}_n^{(k)}$, with $k \geq 1$, are combinations of $\mathbf{T}_m^{(0)}$, with $m < n$, and the perturbative coefficient of the evolution kernels $\mathbf{K}^{(l)}$, with $l < n$.([2]) This is a direct consequence of the fact that the coefficients $\mathbf{T}_n^{(k)}$, with $k \geq 1$, are specifically engineered to compensate the effect of the evolution beween μ and Q up to order n inclusive. The compensation is such that, if the coefficient functions $\mathbf{T}_n$ are computed to N^{n}LO and the evolution to N^{m}LL with $m \geq n-1$, variations of μ in Eq. (1.19) will generate terms whose size is at worst of order $\mathcal{O}(\alpha_s^{n+1}(\mu)\ln^{n+1}(\mu/Q))$. If we require $\mu \simeq Q$, such that $\ln(\mu/Q) \simeq 1$, the logarithms will no longer inflate the terms thus generated leaving us with $\mathcal{O}(\alpha_s^{n+1})$. An operational way of writing this statement is the following: given two different factorisation scales μ and ν, both in the vicinity of Q, the following relation holds:

$$\mathcal{H}(Q,\mu,\mu_0) - \mathcal{H}(Q,\nu,\mu_0) = C \times \alpha_s^{n+1}(\mu) . \tag{1.21}$$

where $\mathcal{H}$ is computed at N^{n}LO with N^{n-1}LL (or more accurate) evolution, and C is a constant of order one approximately proportional to $\ln^{n+1}(\mu/\nu)$. Notice that the argument in the r.h.s. of the equation above is set to μ. In fact, this is arbitrary. One could have chosen any scale of order Q in that this would give rise to subleading differences in α_s. A possible choice that eliminates one of the scales is of course $\nu = Q$.

Now the question is: how does one probe numerically that Eq. (1.21) is effectively true? My suggestion is the following: let us fix the scales μ_0, Q and μ. For example $\mu_0 = 1$ GeV, $Q = 30$ GeV, $\mu = 60$ GeV.([3]). Since $\mu \simeq M_Z$, *i.e.* $\ln(\mu/M_Z) \simeq 1$, one can write:

$$\mathcal{H}(Q,\mu,\mu_0) - \mathcal{H}(Q,Q,\mu_0) = C' \times \alpha_s^{n+1}(M_Z) . \tag{1.22}$$

The reason for choosing M_Z as a reference scale is that often the evolution of α_s is computed using the value of $\alpha_s(M_Z)$ as a boundary condition (the specific value is typically around $\alpha_s(M_Z) = 0.118$). A way of testing Eq. (1.22) is to numerically compute the r.h.s. changing the value of $\alpha_s(M_Z)$ used as a reference for the evolution of the coupling in a reasonable range (*e.g.* $\alpha_s(M_Z) \in [0.05, 0.2]$) and to check that it scales like $\alpha_s^{n+1}(M_Z)$. The same exercise can be repeated picking different values of Q and μ. By doing so, one should observe that the value of the coefficient C' extracted with different pairs (Q,μ) scales like $\ln^{n+1}(\mu/Q)$.

Of course, it is not necessary to use $\alpha_s(M_Z)$ as a scaling parameter. One can use $\alpha_s(\mu)$ as prescribed by Eq. (1.21) and this value changed by moving the position of the Landau pole.

[2] In principle, also the coefficients of the β-function are present, but since in the following we are interested to NLO accuracy where these coefficients are not present yet, we will not need to discuss them.

[3] Despite μ_0, Q and μ_0 are linearly equally spaced, their ratios, $\mu/\mu_0 = 60$ and $\mu/Q = 2$, are such that the interval $[\mu_0,\mu]$ requires resummation while $[Q,\mu]$ does not.