# Pricing Barrier Options under Local Volatility

**Artur Sepp**
Mail: artursepp@hotmail.com, Web: www.hot.ee/seppar
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## Abstract

We study pricing under the local volatility. Our research is mainly intended for pedagogical purposes. In the first part of our work we study the local volatility modeling. We derive the local volatility formula in terms of the European call prices and in terms of the market implied volatilities. We propose and calibrate to the DAX option data a functional form for the implied volatility which simplifies pricing under the local volatility. In the second part of our work we analyze pricing of vanilla and double barrier options under the local volatility. To carry out our analysis of the pricing problem, we code three finite-difference solvers to compute vanilla and double barrier option prices using the local volatility function. At first, we verify that the local volatility solver produces vanilla prices which are exactly compatible with the Black-Scholes prices. Then we compare prices of double barrier options which are computed using market implied volatility and the corresponding local volatility. We find that using the local volatility yields prices and deltas of double barriers which are considerably higher than those computed using the constant volatility.

## Contents

# 1 Pricing Double Barrier Options

Barrier options are so-called path-dependent options. Their payoff is dependent on the realized asset path. A typical example is the double barrier option with an upper and a lower barrier, the first above and the second below the current underlying price. In a double knock-out option the contract becomes worthless if either of the barriers is reached. In double knock-in option one of the barriers must be reached before expiry, otherwise the option becomes worthless. Sometimes a rebate is paid if the barrier level is reached. Barrier options are quite popular because they are cheaper than the vanilla calls and puts and can be used to hedge specific cashflows. In figure 1 are shown values of vanilla call versus the double barrier knock-out call.

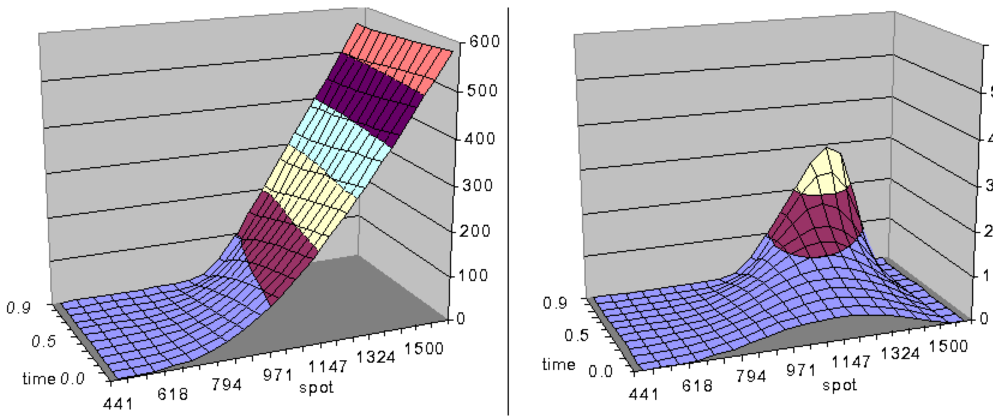


Figure 1: Values of Vanilla Call vs Value of Barrier Out-Call as functions of $S$ and $t$. Respective parameters are: $K = 1000$, $T = 1.0$, $\sigma = 0.3$, $r = 0.05$, $d = 0.02$, $S_u = 1500$, $S_d = 500$. When spot price $S$ is high, the probability of hitting upper barrier $S_u$ increases leading to lower prices of barrier option.

There is also a variety of single barrier options whose payoffs are dependent on whether a single up or down barrier has been reached or not.

Although we have some analytic pricing formulas for single barriers, there are not available simple analytic formulas for pricing double barriers and double-no-touches even in the standard Black-Scholes (1973) model. The pricing problem can be solved via Greens function or Fourier series. We note that the pricing problem can effectively be solved via Laplace transform and we use the latter in our analysis. When volatility is time and space-dependent, the pricing problem can be solved (only) via finite differences.
Let $S_u$ and $S_d$ be upper and lower barrier level, respectively. The pricing PDE corresponding to the value of knock-out barrier option under constant volatility is given by

$$\begin{aligned} &F_t + \frac{1}{2}\sigma^2 S^2 F_{SS} + (r-d)SF_S - rF = 0, \\ &F(S,T) = \max\{\phi[S-K], 0\}, \\ &F(S_u,t) = \varphi_u(t),\ F(S_d,t) = \varphi_d(t). \end{aligned} \tag{1.1}$$

where $S$ is the underlier price, $K$ is strike, $r$ is risk-free (domestic) interest rate, $d$ is dividend (foreign interest) rate, $\sigma$ is volatility, the binary variable $\phi = +1$ for a call option and $\phi = -1$ for a put option, $\varphi_u(t)$ and $\varphi_d(t)$ are contract functions which determine the payoff if the barrier is reached.
For a standard double barrier option we have $\varphi_u(t) = 0$ and $\varphi_d(t) = 0$.

# 2 Local Volatility Modeling

Now we consider local volatility modeling. We assume that the risk-neutral dynamics for $S$ is given by

$$dS(t) = (r(t) - d(t))S(t)dt + \sigma_{loc}(S(t),t)S(t)dW(t),\ S(t) = S_0. \tag{2.1}$$

Standard no-arbitrage conditions give that the value of the European option $V(S,t;K,T)$ at any time before $T$ satisfies the following backward equation

$$\begin{aligned} &\frac{\partial V}{\partial t} + \frac{1}{2}\sigma_{loc}^2(S,t)S^2\frac{\partial^2 V}{\partial S^2} + (r(t)-d(t))S\frac{\partial V}{\partial S} - r(t)V = 0, \\ &V(S,T;K,T) = \max\{\phi[S-K], 0\}, \end{aligned} \tag{2.2}$$

## 2.1 Local Volatility Formula

**Theorem 2.1.** *The conditional probability density function $p(x,t;y,T)$ of a general stochastic process $X(t)$ where $0 \le t \le T$ given by*

$$dX(t) = \mu(x,t)dt + \sigma(x,t)dW(t) \tag{2.3}$$

*satisfies the Fokker-Plank or forward Kolmogorov equation*

$$\begin{aligned}&\frac{\partial}{\partial T}p(y,T)+\frac{\partial}{\partial y}\left(\mu(y,T)p(y,T)\right)=\frac{1}{2}\frac{\partial^2}{\partial y^2}\left(\sigma^2(y,T)p(y,T)\right)\\&p(x,t;y,t)=\delta(y-x).\end{aligned}\tag{2.4}$$

**Corollary 2.1.** *The risk-neutral transition density, $p=p(S,t;y,T)$ associated with the asset price dynamics (2.1) satisfies the Fokker-Plank equation given by*

$$\begin{aligned}&\frac{\partial}{\partial T}p=-\frac{\partial}{\partial y}\left((r(T)-d(T))yp\right)+\frac{\partial^2}{\partial y^2}\left(\frac{1}{2}\sigma^2(y,T)y^2p\right)\\&p(S,t;y,t)=\delta(y-S)\end{aligned}\tag{2.5}$$

**Theorem 2.2 (Dupire local volatility formula).** *Given that the underlier price follows SDE (2.1), the local volatility, $\sigma_{loc}(K,T)$, is given by*

$$\sigma_{loc}^2(K,T)=\frac{\frac{\partial C}{\partial T}+(r(T)-d(T))K\frac{\partial C}{\partial K}+d(T)C}{\frac{1}{2}K^2\frac{\partial^2 C}{\partial K^2}}\tag{2.6}$$

*where $C:=C(S,t;K,T)$ is the value of a European call option.*

*Proof.* **1)** We can write the $t$-value of a call as

$$C(S,t;K,T)=\mathbb{E}[e^{-r(T)(T-t)}\max\{S(T)-K,0\}]=e^{-\int_t^T r(\xi)d\xi}\int_K^\infty(y-K)p(S,t;y,T)dy.\tag{2.7}$$

We use the formula

$$\frac{\partial}{\partial x}\int_0^x f(t,x)dt=f(x,x)+\int_0^x\frac{\partial f(t,x)}{\partial x}dt$$

to get

$$\begin{aligned}\frac{\partial}{\partial K}C(S,t;K,T)&=e^{-\int_t^T r(\xi)d\xi}\frac{\partial}{\partial K}\int_K^\infty(y-K)p(S,t;y,T)dy\\&=-e^{-\int_t^T r(\xi)d\xi}\frac{\partial}{\partial K}\int_\infty^K(y-K)p(S,t;y,T)dy\\&=-e^{-\int_t^T r(\xi)d\xi}\left((K-K)-\int_\infty^K\frac{\partial}{\partial K}(y-K)p(S,t;y,T)dy\right)\\&=e^{-\int_t^T r(\xi)d\xi}\int_\infty^K p(S,t;y,T)dy,\end{aligned}\tag{2.8}$$

and using the fundamental theorem of calculus we get

$$\frac{\partial^2}{\partial K^2}C(S,t;K,T) = e^{-\int_t^T r(\xi)d\xi}p(S,t;K,T). \tag{2.9}$$

Modifying (2.9) we get

$$p(S,t;K,T) = e^{\int_t^T r(\xi)d\xi}\frac{\partial^2}{\partial K^2}C(S,t;K,T). \tag{2.10}$$

Thus, given a continuum of market prices of European calls with different strikes and maturities, we can recover the risk-neutral probability density using equation (2.10). This formula was first derived by Breeden (1978).

**2)** Using equation (2.7) and Fokker-Plank PDE (2.5), we get

$$\frac{\partial}{\partial T}C(S,t;K,T) = -r(T)C(S,t;K,T) + e^{-\int_t^T r(\xi)d\xi}\int_K^\infty (y-K)\frac{\partial}{\partial T}p(S,t;y,T)dy$$
$$= -r(T)C(S,t;K,T) + e^{-\int_t^T r(\xi)d\xi}\times$$
$$\times\int_K^\infty (y-K)\left[-\frac{\partial}{\partial y}\left((r(T)-d(T))yp(S,t;y,T)\right) + \frac{\partial^2}{\partial y^2}\left(\frac{1}{2}\sigma^2(y,T)y^2p(S,t;y,T)\right)\right]dy. \tag{2.11}$$

We assume that $p$ and $p_y$ approach zero fast enough that the boundary terms vanish. We calculate

$$\int_K^\infty (y-K)\frac{\partial^2}{\partial y^2}\left(\frac{1}{2}\sigma^2(y,T)y^2p(S,t;y,T)\right)dy = (y-K)\frac{\partial}{\partial y}\frac{1}{2}\sigma^2(y,T)y^2p(S,t;y,T)\Big|_K^\infty$$
$$-\int_K^\infty \frac{\partial}{\partial y}\left(\frac{1}{2}\sigma^2(y,T)y^2p(S,t;y,T)\right)dy = \frac{1}{2}\sigma^2(K,T)K^2p(S,t;K,T)$$
$$= \frac{1}{2}\sigma^2(K,T)K^2e^{\int_t^T r(\xi)d\xi}\frac{\partial^2}{\partial K^2}C(S,t;K,T)$$

where we use expression (2.10). Finally

$$\int_K^\infty (y-K)\frac{\partial}{\partial y}\left((r(T)-d(T))yp(S,t;y,T)\right)dy = -(r(T)-d(T))\int_K^\infty yp(S,t;y,T)dy$$
$$= -(r(T)-d(T))\left[\int_K^\infty (y-K)p(S,t;y,T)dy + \int_K^\infty Kp(S,t;y,T)dy\right]$$
$$= -(r(T)-d(T))e^{\int_t^T r(\xi)d\xi}\left[C(S,t;K,T) - K\frac{\partial}{\partial K}(S,t;K,T)\right]$$

where we use expressions (2.7) and (2.8).
Substituting these into (2.11) gives the Dupire equation (2.6). $\square$

## 2.2 Local Volatility in Terms of Implied Volatility

It is more convenient to work not with call prices but with market implied volatility.

**Theorem 2.3.** *We assume that the implied volatility function, $\sigma_{imp}(K,T)$, is differentiable with respect to $T$ and twice differentiable with respect to $K$. The local volatility formula, $\sigma_{loc}(K,T)$, in PDE (2.2) in terms of implied volatility is given by*

$$\sigma_{loc}^2(K,T) = \frac{\frac{\sigma_{imp}}{T-t} + 2\frac{\partial \sigma_{imp}}{\partial T} + 2(r(T)-d(T))K\frac{\partial \sigma_{imp}}{\partial K}}{K^2\left(\frac{\partial^2 \sigma_{imp}}{\partial K^2} - d_1\sqrt{T-t}\left(\frac{\partial \sigma_{imp}}{\partial K}\right)^2 + \frac{1}{\sigma_{imp}}\left(\frac{1}{K\sqrt{T-t}} + d_1\frac{\partial \sigma_{imp}}{\partial K}\right)^2\right)} \tag{2.12}$$

*where*

$$d_{1,2} = d_{+,-} = \frac{\ln(S/K) + (r(T) - d(T) \pm \frac{1}{2}\sigma_{imp}^2)(T-t)}{\sigma_{imp}\sqrt{T-t}}. \tag{2.13}$$

*Proof.* We recall that the value of a call option under constant (deterministic) volatility is given by Black-Scholes (1973) formula

$$C(K,T,\sigma) = Se^{-d(T-t)}N(d_1) - Ke^{-r(T-t)}N(d_2) \tag{2.14}$$

where

$$n(t) = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}t^2},\ N(x) = \int_{-\infty}^{x} n(t)dt. \tag{2.15}$$

We note that

$$C^{loc} := C^{loc}(K,T,\sigma_{loc}) \equiv C^{BS}(K,T,\sigma_{imp}(K,T)) := C(K,T,\sigma). \tag{2.16}$$

where $C^{loc}$ is the call price implied by the local volatility model and $C^{BS}$ is the call price implied by formula (2.14). Let $C$ denote $C(K,T,\sigma)$.
The chain rule gives

$$\begin{aligned} C_T^{loc} &= C_T + C_\sigma \sigma_T, \\ C_K^{loc} &= C_K + C_\sigma \sigma_K, \\ C_{KK}^{loc} &= C_{KK} + 2C_{K\sigma}\sigma_K + C_{\sigma\sigma}(\sigma_K)^2 + C_\sigma \sigma_{KK}. \end{aligned} \tag{2.17}$$

Necessary partial derivatives (greeks) of Black-Scholes formula (2.14) are given by

$$\begin{aligned}
C_T &= \frac{\sigma S e^{-d(T-t)} N(d_1)}{2\sqrt{T-t}} - dSN(d_1)e^{-d(T-t)} + rKe^{-r(T-t)}N(d_2), \\
C_K &= -e^{-r(T-t)}N(d_2), \\
C_{KK} &= e^{-r(T-t)} \frac{n(d_2)}{K\sigma\sqrt{T-t}}, \\
C_\sigma &= Se^{-d(T-t)} n(d_1)\sqrt{T-t}, \\
C_{K\sigma} &= \frac{Se^{-d(T-t)} n(d_1) d_1}{K\sigma}, \\
C_{\sigma\sigma} &= Se^{-d(T-t)} n(d_1)\sqrt{T-t}\frac{d_1 d_2}{\sigma}.
\end{aligned} \tag{2.18}$$

Substituting these into the numerator of (2.6) and simplifying, we obtain

$$C_T + (r-d)KC_K + dC = ... = C_\sigma \left( \frac{\sigma}{2(T-t)} + \sigma_T + (r-d)K\sigma_K \right). \tag{2.19}$$

We note that

$$\frac{C_{KK}}{C_\sigma} = \frac{1}{K^2\sigma(T-t)}. \tag{2.20}$$

Substituting greeks into the expression for $C_{KK}^{loc}$ and rearaging yields

$$C_{KK}^{loc} = ... = C_\sigma \left( \sigma_{KK} - d_1\sqrt{T-t}(\sigma_K)^2 + \frac{1}{\sigma}\left( \frac{1}{K\sqrt{T-t}} + d_1\sigma_K \right)^2 \right). \tag{2.21}$$

Further substitution of these results into equation (2.6) and simplification yield formula (2.12) □

## 2.3 Practical Considerations

Lipton (2001) as well as other practitioners state that formulas (2.6) and (2.12) are very difficult to use in practice. The requirement to calculate partial derivatives from the discrete market data makes these formulas very sensitive to the interpolation method. As a result, the local volatility surface is very unstable.

To illustrate pricing under the local volatility surface, we use a functional form for implied volatilities. We calibrate it to the DAX option data and use in our FD solvers.

## 2.4 Functional Form for Marked Implied Volatility

We adapt the following formula for implied volatilities which is often seen in the literature:

$$\sigma_{imp}(K,T) = a + bT + cT^2 + dXT + eX + gX^2 + hX^3 \tag{2.22}$$

where $X$ is a measure of moneyness: $X = \ln \frac{Se^{(r-d)(T-t)}}{K}$.
We find that another specification is more justified and fits the data well.

$$\sigma_{imp}(K,T) = a + ce^{bT} + dX + eX^2 + gX^3 + hX^4. \tag{2.23}$$

The first three terms controll the term structure of the at-the-money (ATM) volatility (we note that if $K \approx Se^{(r-d)(T-t)}$ then $X \approx 0$). Let us consider the ODE:

$$d\sigma(t) = \kappa(\theta - \sigma(t))dt \tag{2.24}$$

where $\theta$ is the long term mean of $\sigma$ and $\kappa$ is the mean-reversion speed. Solution to ODE (2.24) is given by

$$\sigma_t = \theta + (\sigma_0 - \theta)e^{-\kappa t} \tag{2.25}$$

and it is downward or upward sloping.
Thus, interpretation of the parameters in 2.23 is the following: $a$ is the long-term ATM volatility, $c$ is the difference between the current ATM volatility and the long-term one, $b$ is the mean reversion speed. Other parameters control the smile.
We calibrate the model to the DAX (The Deutscher Aktienindex) option data. We obtained the following estimates

$$\sigma_{imp}(K,T) = 0.23 + 0.17e^{-2.65T} + 0.25X + 0.19X^2 - 0.27X^3 + 0.05X^4. \tag{2.26}$$

Except for short maturities the model fits the data well. In Appendix 4 we report the market implied volatility, model implied volatility and the corresponding local volatility surfaces.
The required partial derivatives are given by

$$\begin{aligned}
\frac{\partial \sigma_{imp}(K,T)}{\partial T} &= cbe^{bT} + (r-d)(d + e + 2eX + 3gX^2 + 4hX^3), \\
\frac{\partial \sigma_{imp}(K,T)}{\partial K} &= -\frac{1}{K}\left(d + 2eX + 3gX^2 + 4hX^3\right), \\
\frac{\partial^2 \sigma_{imp}(K,T)}{\partial K^2} &= \frac{1}{K^2}\left(d + 2e(X+1) + 3gX(X+2) + 4hX^2(X+3)\right).
\end{aligned} \tag{2.27}$$

# 3 Numerical Results

For illustrative purposes, we compute a number of vanilla and double barrier call option prices using the market implied volatility given by formula (2.26) and the corresponding local volatility model (2.6). To solve the pricing PDE (2.2) and (1.1), we use Crank-Nicolson finite-difference discretization. The spot price, $S$, is 4468.17. We use constant interest rate $r_0 = 0.0375$ and no dividend yield. We use notation of price differences or price errors which are computed by

$$\frac{\text{FD solver price} - \text{reference model price}}{\text{FD solver price}} * 100\%.$$

## 3.1 Pricing Vanilla Calls under Local Volatility

At first, we compute vanilla call prices with the local volatility solver. We use $S_{min} = 2000$, $S_{max} = 9000$, and number of time and space steps set to 900. We apply the first type of boundary condition, i.e. call values at time step $t$ and nodes $S_{min}$ and $S_{max}$ are given by $\max\left\{S_* e^{-d \cdot t \cdot dt} - K e^{-r \cdot t \cdot dt}, 0\right\}$. We compute a number of call prices for different strikes and maturities.
Figure 3.1 reports price differences between call prices from the local volatility solver and the BS formula with market implied volatility given by formula (2.26).

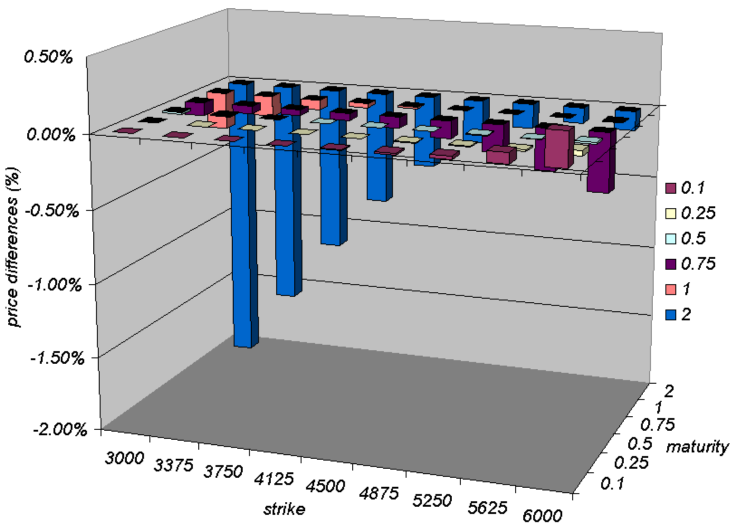


Figure 2: Price differences (%) between the local volatility solver and BS formula with the market implied volatility.

We see that the local volatility solver produces call prices that are compatible with the market prices across all strikes and maturities.

## 3.2 Pricing Double Barriers under Market Implied Volatility

Now we compute double barrier call prices using the double barriers solver with constant volatility and semi-analytical formula obtained by numerical inversion of Laplace transform of PDE (1.1). We use $S_{down} = 3000$, $S_{up} = 6000$, and number of time and space steps set to 900. We compute a number of double barrier call prices for different strikes and maturities using market implied volatility given by formula (2.26).

Figure 3.2 reports price differences between the double barrier solver and our semi-analytical formula.

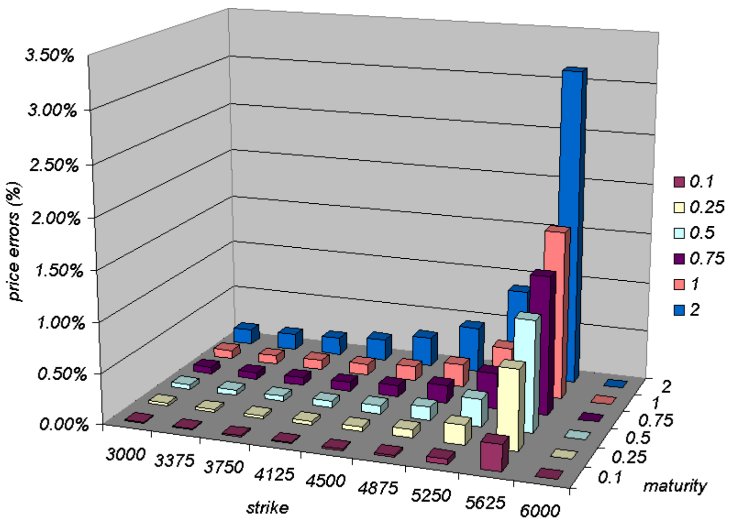


Figure 3: Price differences between the double barrier solver and semi-analytical formula using market implied volatility.

We see that the double barrier solver produces prices that are compatible with the semi-analytical formula across all strikes and maturities.

## 3.3 Pricing Double Barriers under Local Volatility

At last, we compute double barrier call prices using the local volatility solver for pricing double barriers and semi-analytical formula obtained by numerical inversion of Laplace transform of PDE (1.1) with the market implied

volatility given by formula (2.26). We use $S_{down} = 3000$, $S_{up} = 6000$, and number of time and space steps set to 900. We compute a number of double barrier call prices for different strikes and maturities.
Figure 3.3 reports price differences between double barrier solver with local volatility and semi-analytical formula with the market implied volatility.

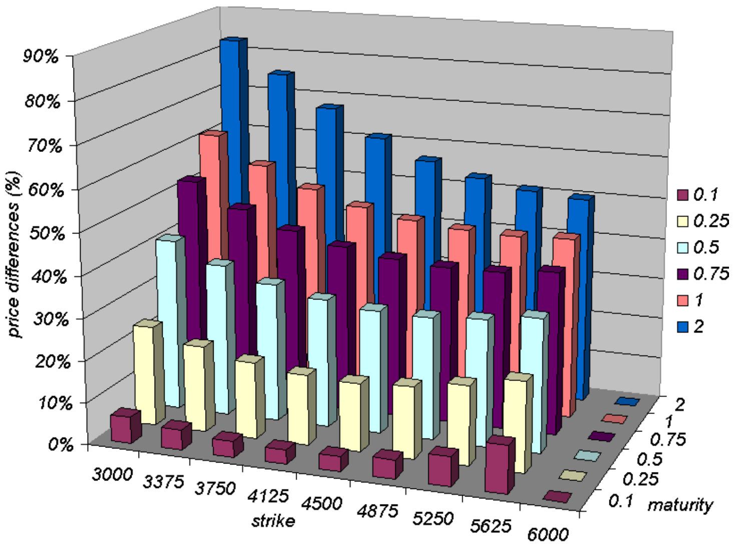


Figure 4: Price differences of double barriers between the local volatility solver and the semi-analytical formula with market implied volatility.

We see that using the local volatility leads to double barrier prices that are considerably higher than prices obtained by using the market implied volatility.

# 4 Conclusions

We studied pricing options under the local volatility. We parametrized the market implied volatility surface and used FD solvers for pricing vanilla and double barrier options under the local volatility. We found that the local volatility solver produces prices of vanillas which are very close to the corresponding Black-Scholes prices. We also found that the double barrier solver with constant volatility prices double barrier calls consistently with the semi-analytical formula obtained via the numerical inversion of Laplace transform of the pricing PDE. We found that the local volatility solver produces double barrier prices and deltas that are considerably higher than the corresponding prices and deltas obtained by using the market implied volatility.

A similar result was obtained by Hirsa *et al* (2002). They calibrated Variance Gamma, CEV, local volatility model, Variance Gamma with stochastic arrival model and used the calibrated models for pricing up-and-out call options. Their summary was: "Regardless of the closeness of the vanilla fits to different models, prices of up-and-out call options (a simple case of exotic options) differ noticeably when using different stochastic processes to calibrate the vanilla options surface."
Accordingly, the value of barrier options is very sensitive to the chosen volatility model.

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# A Volatility Surfaces

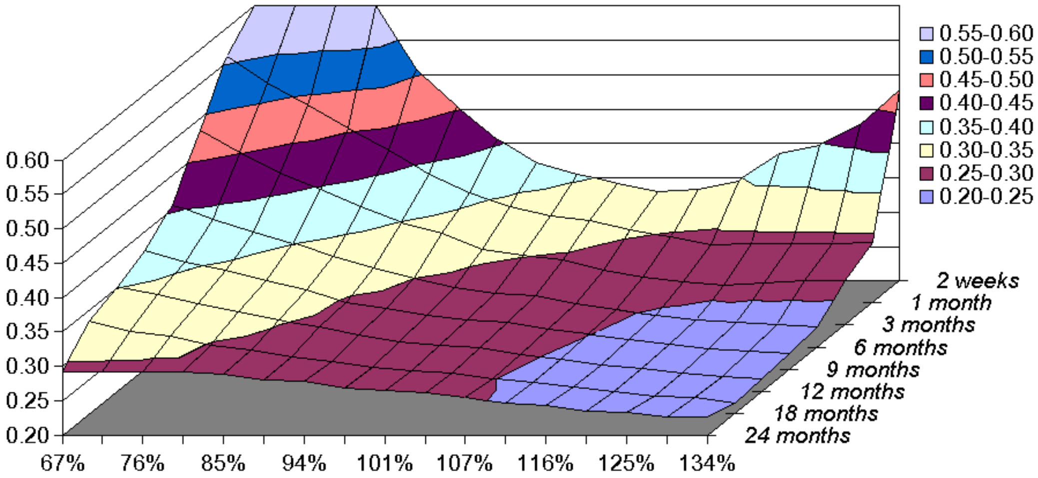


Figure 5: DAX implied volatility surface (July 05, 2002).

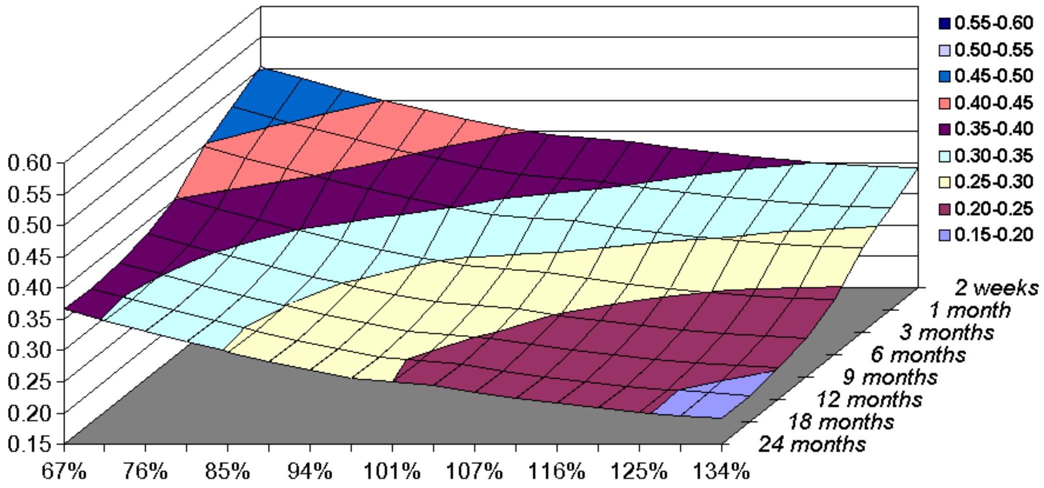


Figure 6: Model (2.26) implied volatility surface.

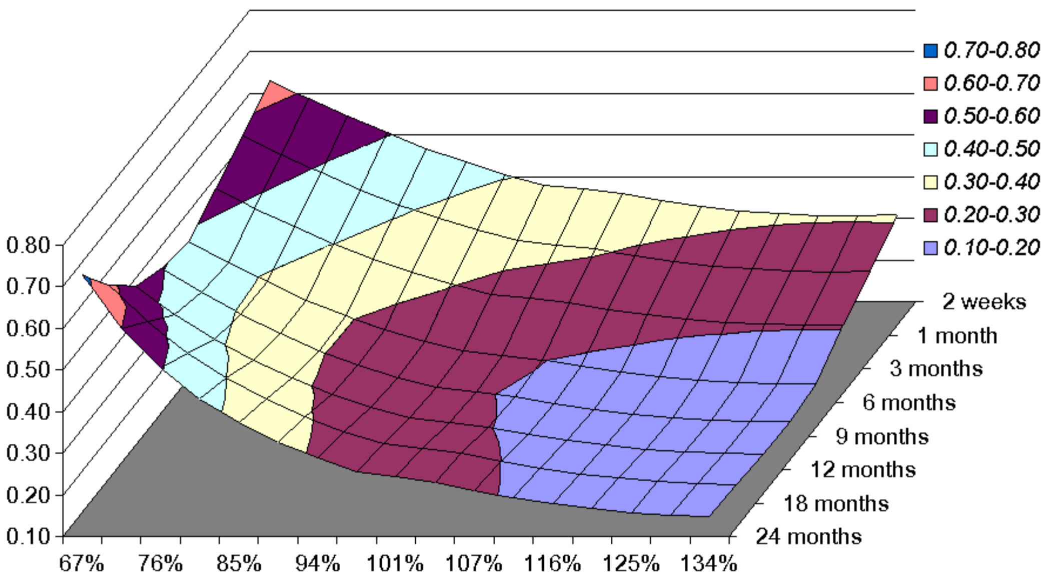


Figure 7: The corresponding local volatility surface.