Robust MPC via System Level Synthesis

Consider robust MPC of an LTV system with *time-varying* uncertainty and disturbance.

$$x(t+1) = (\hat{A}(t) + D_A(t))x(t) + (\hat{B}(t) + D_B(t))u(t) + w(t)$$

Assumptions:

- Estimates $\hat{A}(t), \hat{B}(t)$ are known.
- $\|D_A(t)\|_{\infty\to\infty} \leq \epsilon_A, \|D_B(t)\|_{\infty\to\infty} \leq \epsilon_B, \|w(t)\|_\infty \leq \sigma_w, \forall t.$
- Polytopic constraints: $x(t) \in \mathcal{X}, u(t) \in \mathcal{U}, x(T) \in \mathcal{X}_T, \forall t$ where T is the horizon.

Robust Optimal Control Problem

Problem formulation (the robust optimal control problem at iteration k):

$$
\begin{aligned}
\min_{\pi} \quad & J_{\text{nom}}(x(k), \pi) \\
\text{s.t.} \quad & \left.\begin{aligned}
& x_{t+1} = (\hat{A}_t + D_{A\,t})x_t + (\hat{B}_t + D_{B\,t})u_t + w_t \\
& u_t = \pi_t(x_{0:t}, u_{0:t-1}) \\
& x_t \in \mathcal{X}, u_t \in \mathcal{U}, x_T \in \mathcal{X}_T
\end{aligned}\right\} \\
& \forall t = 0, 1, \cdots, T-1, \forall w_t \in \mathcal{W} \\
& \forall \|D_{A\,t}\|_{\infty \to \infty} \leq \epsilon_A, \forall \|D_{B\,t}\|_{\infty \to \infty} \leq \epsilon_B \\
& x_0 = x(k)
\end{aligned}
$$

- Restract π_t to time-varying state feedback controllers $u_t = \sum_{i=0}^{t} K^{t,t-i} x_i$.
- Use SLS to optimize over closed-loop system responses to avoid nonconvexity.

System Level Synthesis

Use $\mathbf{\Phi}_x, \mathbf{\Phi}_u$ to denote the closed-loop system responses $\mathbf{w} \mapsto (\mathbf{x}, \mathbf{u})$.

Synthesize the state feedback controller by $\mathbf{K} = \mathbf{\Phi}_u \mathbf{\Phi}_x^{-1}$.

The achievability constraint:

$$\begin{bmatrix} I - Z\mathcal{A} & -Z\mathcal{B} \end{bmatrix} \begin{bmatrix} \mathbf{\Phi}_x \\ \mathbf{\Phi}_u \end{bmatrix} = I \Leftrightarrow \begin{bmatrix} \mathbf{x} \\ \mathbf{u} \end{bmatrix} = \begin{bmatrix} \mathbf{\Phi}_x \\ \mathbf{\Phi}_u \end{bmatrix} \mathbf{w}.$$

The inexact achievability constraint:

$$\begin{bmatrix} I - Z\mathcal{A} & -Z\mathcal{B} \end{bmatrix} \begin{bmatrix} \mathbf{\Phi}_x \\ \mathbf{\Phi}_u \end{bmatrix} = I - \mathbf{\Delta} \Leftrightarrow \begin{bmatrix} \mathbf{x} \\ \mathbf{u} \end{bmatrix} = \begin{bmatrix} \mathbf{\Phi}_x \\ \mathbf{\Phi}_u \end{bmatrix} (I - \mathbf{\Delta})^{-1} \mathbf{w}. \quad (1)$$

- (1) is applied to obtain a convex relaxation of the robust OCP.
- The structure of $\mathbf{\Delta}$ and $\{\mathbf{\Phi}_x, \mathbf{\Phi}_u\}$ are heavily exploited to reduce conservativeness.

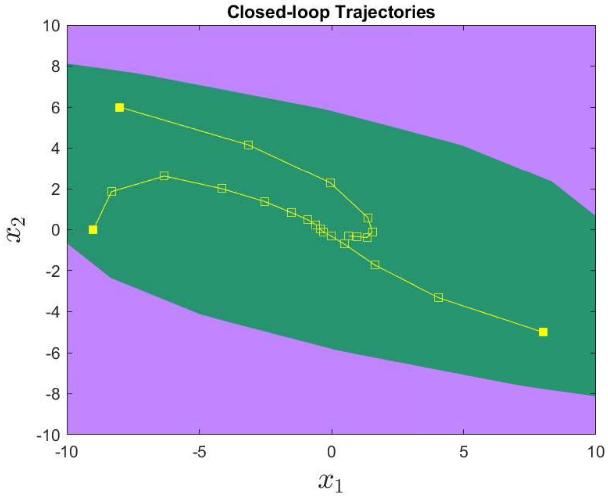
SLS Reformulation of the Robust OCP

Convex relaxation of the robust OCP through SLS:

$$
\begin{aligned}
\min_{\mathbf{\Phi}} \quad & \left\| \begin{bmatrix} \mathcal{Q}^{\frac{1}{2}} & \\ & \mathcal{R}^{\frac{1}{2}} \end{bmatrix} \begin{bmatrix} \mathbf{\Phi}_x(:,0) \\ \mathbf{\Phi}_u(:,0) \end{bmatrix} x_0 \right\|_2^2 \\
\text{s.t.} \quad & \begin{bmatrix} I - Z\hat{\mathcal{A}} & -Z\hat{\mathcal{B}} \end{bmatrix} \begin{bmatrix} \mathbf{\Phi}_x \\ \mathbf{\Phi}_u \end{bmatrix} = I \\
& F_j^\top \mathbf{\Phi}^0 x_0 + \|F_j^\top \mathbf{\Phi}^{\tilde{\mathbf{w}}}\|_1 \frac{1-\tau^T}{1-\tau}\gamma + \beta\sigma_w \le b_j, \forall j \\
& \|F_j^\top \mathbf{\Phi}^{\tilde{\mathbf{w}}}\|_{\infty\to\infty} + \beta\|\epsilon \mathbf{\Phi}^{\tilde{\mathbf{w}}}\|_{\infty\to\infty} \le \beta, \forall j \\
& \left\| \begin{bmatrix} \frac{\epsilon_A}{\alpha} \mathbf{\Phi}_x^{\tilde{\mathbf{w}}} \\ \frac{\epsilon_B}{1-\alpha} \mathbf{\Phi}_u^{\tilde{\mathbf{w}}} \end{bmatrix} \right\|_{\infty\to\infty} \le \tau, \quad \left\| \begin{bmatrix} \frac{\epsilon_A}{\alpha} \mathbf{\Phi}_x^{0} \\ \frac{\epsilon_B}{1-\alpha} \mathbf{\Phi}_u^{0} \end{bmatrix} x_0 \right\|_{\infty} \le \gamma
\end{aligned}
$$

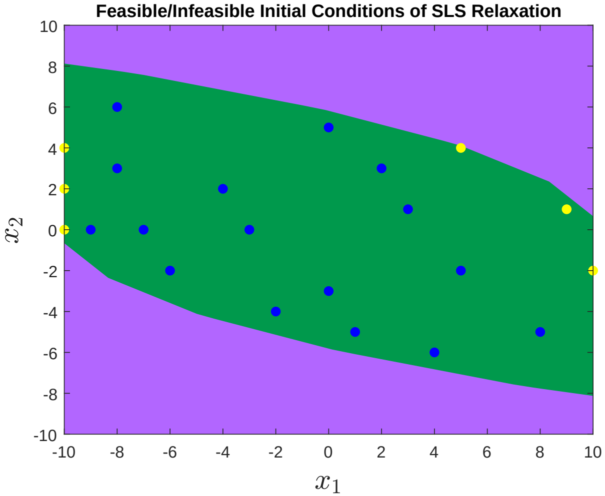
Decision variable $\mathbf{\Phi} = [\mathbf{\Phi}_x; \mathbf{\Phi}_u] \in \mathbb{R}^{(T+1)(n+m)\times(T+1)n}$.
Hyperparameters: $(\alpha, \beta, \gamma, \tau)$. Can find upper and lower bounds.

Example: Double Integrator



Purple: state constraint; Green: maximum robust invariant set.
Simulated for 7 steps by solving the SLS relaxations recursively.

Example: Double Integrator



Randomly chosen x_0 in the maximum RIS (green). Grid-search hyperparameters.
Blue: feasible; Yellow: infeasible.