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1 $s \in S_3^?$ on $[0,2]$

$$s(x) = \begin{cases} p(x), & x \in [0,1] \\ (2-x)^3, & x \in [1,2] \end{cases}$$

let $p(x) = ax^3 + bx^2 + cx + d$

$$s(x) = \begin{cases} ax^3+bx^2+cx+d, & x \in [0,1] \\ (2-x)^3, & x \in [1,2] \end{cases}$$

$$s'(x) = \begin{cases} 3ax^2+2bx+c, & x \in [0,1] \\ (2-x)^3, & x \in [1,2] \end{cases}$$

$$s''(x) = \begin{cases} 6ax+2b, & x \in [0,1] \\ (2-x)^3, & x \in [1,2] \end{cases}$$

$$\begin{cases} s(0) = p(0) = 0 \rightarrow d = 0 \\ s(1) = p(1) = 1 \rightarrow a+b+c = 1 \\ s'(1) = p'(1) = -3 \Rightarrow 3a+2b+c = -3 \end{cases}$$

$s''(1) = p''(1) = 6 \rightarrow 6a+2b = 6$

$$\begin{cases} a = 7 \\ b = -18 \\ c = 12 \\ d = 0 \end{cases}$$

$p(x) = 7x^3 - 18x^2 + 12x$

$\because s''(0) = -36 \neq 0$, $s''(2) = 48 \neq 0$

$s(x)$ is not natural cubic spline.

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2 a) $\because$ There are n of knots and $s \in S_2^1$, $(n+1)$ dimension of linear space

$\therefore$ From $f_i = f(x_i)$ provides n of equation, an additional needed to determine s uniquely.

b) $$p_i(x) = f(x_i) + f'(x_i)(x - x_i) + f[x_i, x_i, x_{i+1}](x - x_i)^2$$
$$= f(x_i) + m_i(x - x_i) + \frac{\frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i} - m_i}{x_{i+1} - x_i}(x - x_i)^2$$
$$p_i = f_i + m_i(x - x_i) + \frac{f_{i+1} - f_i - m_i(x_{i+1} - x_i)}{(x_{i+1} - x_i)^2}(x - x_i)^2$$

c) $$p_i'(x) = m_i + 2\,\frac{f_{i+1} - f_i - m_i(x_{i+1} - x_i)}{(x_{i+1} - x_i)^2}(x - x_i)$$
$$p_i'(x_{i+1}) = m_i + 2\,\frac{f_{i+1} - f_i - m_i(x_{i+1} - x_i)}{x_{i+1} - x_i}$$
$$m_{i+1} = m_i + 2\,\frac{f_{i+1} - f_i - m_i(x_{i+1} - x_i)}{x_{i+1} - x_i}$$

3 $$s(x) = \begin{cases} s_1(x) = 1 + c(x+1)^3, & x \in [-1, 0] \\ s_2(x) = ax^3 + bx^2 + ex + d, & x \in [0, 1] \end{cases}$$

$$s'(x) = \begin{cases} s_1(x) = 3c(x+1)^2, & x \in [-1, 0] \\ s_2(x) = 3ax^2 + 2bx + e, & x \in [0, 1] \end{cases}$$

$$s''(x) = \begin{cases} s_1(x) = 6c(x+1), & x \in [-1, 0] \\ s_2(x) = 6ax + 2b, & x \in [0, 1] \end{cases}$$

$$\begin{cases} s(0) = 1 + c = d \\ s'(0) = 3c = e \\ s''(0) = 6c = 2b \\ s''(1) = 0 \to b = -3a \end{cases}$$

$s(1) = -1 \to 6c + 1 = -1$

$$\begin{cases} a = \frac{1}{3} \\ b = -1 \\ e = -1 \\ d = \frac{2}{3} \\ c = -\frac{1}{3} \end{cases}$$

$$s_2(x) = \frac{1}{3}x^3 - x^2 - x + \frac{2}{3}$$

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4 a) $s(x) = \begin{cases} s_1(x) = a_1x^3 + b_1x^2 + c_1x + d_1 & , x \in [-1, 0] \\ s_2(x) = a_2x^3 + b_2x^2 + c_2x + d_2 & , x \in [0, 1] \end{cases}$

$s'(x) = \begin{cases} s_1(x) = 3a_1x^2 + 2b_1x + c_1 & , x \in [-1, 0] \\ s_2(x) = 3a_2x^2 + 2b_2x + c_2 & , x \in [0, 1] \end{cases}$

$s''(x) = \begin{cases} s_1(x) = 6a_1x + 2b_1 & , x \in [-1, 0] \\ s_2(x) = 6a_2x + 2b_2 & , x \in [0, 1] \end{cases}$

"natural cubic spline", $s''(-1) = s''(1) = 0 \rightarrow M_1 = 0, M_3 = 0$

$M_i = s''(f(x_i))$

$\mu_i = \frac{x_i - x_{i-1}}{x_{i+1} - x_{i-1}}$, $\lambda_i = \frac{x_{i+1} - x_i}{x_{i+1} - x_{i-1}}$

$\mu_i M_{i-1} + 2M_i + \lambda_i M_{i+1} = 6f[x_{i-1}, x_i, x_{i+1}]$

$s'(x_i) = f[x_i, x_{i+1}] - \frac{1}{6}(M_{i+1} + 2M_i)(x_{i+1} - x_i)$

$= f[x_{i-1}, x_i] - \frac{1}{6}(M_{i-1} + 2M_i)(x_{i-1} - x_i)$

$x_1 = -1$	0		
$x_2 = 0$	1	1	
$x_3 = 1$	0	-1	-1

$i = 2$, $\mu_2 = \frac{1}{2}$, $\lambda_2 = \frac{1}{2}$, $M_2 = -3$

$\begin{cases} s'(-1) = 3a_1 - 2b_1 - c_1 = \frac{3}{2} \\ s'(0) = c_1 = c_2 = 0 \\ s'(1) = 3a_2 + 2b_2 + c_2 = -\frac{3}{2} \\ s''(-1) = -6a_1 + 2b_1 = 0 \\ s''(1) = 6a_2 + 2b_2 = 0 \\ s(0) = d_1 = d_2 \end{cases}$

$s(1) = s(-1) = \frac{1}{2} - \frac{3}{2} + d_1 = 0$

$\begin{cases} a_1 = -\frac{1}{2} \\ b_1 = -\frac{3}{2} \\ c_1 = 0 \\ d_1 = 1 \\ a_2 = \frac{1}{2} \\ b_2 = -\frac{3}{2} \\ c_2 = 0 \\ d_2 = 1 \end{cases}$

$s(x) = \begin{cases} s_1(x) = -\frac{1}{2}x^3 - \frac{3}{2}x^2 + 1 & , x \in [-1, 0] \\ s_2(x) = \frac{1}{2}x^3 - \frac{3}{2}x^2 + 1 & , x \in [0, 1] \end{cases}$

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b) i) $g(x) = (x+1) - x(x+1) = 1 - x^2$

$g''(x) = -2$

$\int_{-1}^{1} (-2)^2 \, dx = 8$

$$s''(x) = \begin{cases} s_1(x) = -3x - 3, & x \in [-1, 0] \\ s_2(x) = 3x - 3, & x \in [0, 1] \end{cases}$$

$6 = \int_{-1}^{1} [s''(x)]^2 \, dx \le \int_{-1}^{1} (-2)^2 \, dx = 8$

ii) $g(x) = f(x)$

$g''(x) = -\frac{\pi^2}{4} \cos\left(\frac{\pi}{2} x\right)$

$6 = \int_{-1}^{1} [s''(x)]^2 \, dx \le \int_{-1}^{1} \frac{\pi^4}{16} \cos^2\left(\frac{\pi}{2} x\right) dx = \frac{\pi^4}{16}$

5 a) $B_i^{n+1}(x) = \frac{x - t_{i-1}}{t_{i+n} - t_{i-1}} B_i^n(x) + \frac{t_{i+n+1} - x}{t_{i+n+1} - t_i} B_{i+1}^n(x)$

$$B_i^1 = \hat{B}_i \begin{cases} \frac{x - t_{i-1}}{t_i - t_{i-1}}, & x \in (t_{i-1}, t_i] \\ \frac{t_{i+1} - x}{t_{i+1} - t_i}, & x \in (t_i, t_{i+1}] \\ 0, & \text{otherwise} \end{cases}$$

$B_i^2 = \frac{x - t_{i-1}}{t_{i+1} - t_{i-1}} \hat{B}_i + \frac{t_{i+2} - x}{t_{i+2} - t_i} \hat{B}_{i+1}$

$$B_i^2 \begin{cases} \frac{x - t_{i-1}}{t_{i+1} - t_{i-1}} \frac{x - t_{i-1}}{t_i - t_{i-1}}, & x \in (t_{i-1}, t_i] \\ \frac{x - t_{i-1}}{t_{i+1} - t_{i-1}} B_i^1(x) + \frac{t_{i+2} - x}{t_{i+2} - t_i} B_{i+1}^1(x), & x \in (t_i, t_{i+1}] \\ 0, & \text{otherwise} \end{cases}$$

$$B_i^2(x) = \begin{cases} \frac{(x - t_{i-1})^2}{(t_{i+1} - t_{i-1})(t_i - t_{i-1})}, & x \in (t_{i-1}, t_i] \\ \frac{(x - t_{i-1})(t_{i+1} - x)}{(t_{i+1} - t_{i-1})(t_{i+1} - t_i)} + \frac{(t_{i+2} - x)(x - t_i)}{(t_{i+2} - t_i)(t_{i+1} - t_i)}, & x \in (t_i, t_{i+1}] \\ \frac{(t_{i+2} - x)^2}{(t_{i+2} - t_i)(t_{i+2} - t_{i+1})}, & x \in (t_{i+1}, t_{i+2}] \\ 0, & \text{otherwise.} \end{cases}$$

b)

$$\frac{d}{dx} B_i^2(x) = \begin{cases} \frac{2(x - t_{i-1})}{(t_{i+1} - t_{i-1})(t_i - t_{i-1})}, & x \in (t_{i-1}, t_i] \\ \frac{t_{i+1} + t_{i-1} - 2x}{(t_{i+1} - t_{i-1})(t_{i+1} - t_i)} + \frac{t_{i+2} + t_i - 2x}{(t_{i+2} - t_i)(t_{i+1} - t_i)}, & x \in (t_i, t_{i+1}] \\ \frac{2(t_{i+2} - x)}{(t_{i+2} - t_i)(t_{i+2} - t_{i+1})}, & x \in (t_{i+1}, t_{i+2}] \\ 0, & \text{otherwise} \end{cases}$$

$\frac{d}{dx} B_i^2(t_i) = \frac{2}{t_{i+1} - t_{i-1}}$

$\lim_{x \to t_i^+} B_i^2(x) = \frac{2}{t_{i+1} - t_{i-1}} = \lim_{x \to t_i^-} B_i^2 = \frac{d}{dx} B_i^2(t_i)$

$\frac{d}{dx} B_i^2(t_{i+1}) = \frac{t_{i+2} - t_{i+1}}{t_{i+2} - t_i}$

$\lim_{x \to t_{i+1}^+} B_i^2(x) = \frac{t_{i+2} - t_{i+1}}{t_{i+1} - t_i} = \lim_{x \to t_{i+1}^-} B_i^2(x) = \frac{d}{dx} B_i^2(t_{i+1})$

c) $\forall x \in [t_{i+1}, t_i], \frac{d}{dx} B_i^2 > 0$, $\forall x \in (t_i, t_{i+1}], \frac{d}{dx} B_i^2(x) < 0$

$\exists!\, x^* \in (t_{i-1}, t_{i+1})$, s.t. $\frac{d}{dx} B_i^2(x^*) = 0$

$$x^* = \frac{t_{i+2}t_{i+1} - t_i t_{i-1}}{t_{i+2}+t_{i+1}-t_i-t_{i-1}}$$

d) $B_i^2(t_{i+2}) = 0$

$B_i^2(x^*) < 1$

$\therefore B_i^2(x) \in [0,1)$

e)

$$B_{i,\mathbb{Z}}^2(x) = \begin{cases} \frac{(x-i+1)^2}{2}, & x \in (i-1, i] \\ \frac{3}{4} - [x-(i+\frac{1}{2})]^2, & x \in (i, i+1] \\ \frac{(i+2-x)^2}{2}, & x \in (i+1, i+2] \\ 0, & \text{otherwise} \end{cases}$$

6 $B_i^2 = (t_{i+2}-t_{i-1})[t_{i-1}, t_i, t_{i+1}, t_{i+2}](t-x)_+^2$

$= [t_i, t_{i+1}, t_{i+2}](t-x)_+^2 - [t_{i-1}, t_i, t_{i+1}](t-x)_+^2$

$= \frac{[t_{i+1}, t_{i+2}](t-x)_+^2 - [t_i, t_{i+1}](t-x)_+^2}{t_{i+2}-t_i} - \frac{[t_i, t_{i+1}](t-x)_+^2 - [t_{i-1}, t_i](t-x)_+^2}{t_{i+1}-t_{i-1}}$

$= \frac{\frac{(t_{i+2}-x)_+^2 - (t_{i+1}-x)_+^2}{t_{i+2}-t_{i+1}} - \frac{(t_{i+1}-x)_+^2 - (t_i-x)_+^2}{t_{i+1}-t_i}}{t_{i+2}-t_i} - \frac{\frac{(t_{i+1}-x)_+^2 - (t_i-x)^2}{t_{i+1}-t_i} - \frac{(t_i-x)_+^2 - (t_{i-1}-x)_+^2}{t_i-t_{i-1}}}{t_{i+1}-t_{i-1}}$

$$= \begin{cases} \frac{(x-t_{i-1})^2}{(t_{i+1}-t_{i-1})(t_i-t_{i-1})}, & x \in (t_{i-1}, t_i] \\ \frac{(x-t_{i-1})(t_{i+1}-x)}{(t_{i+1}-t_{i-1})(t_{i+1}-t_i)} + \frac{(t_{i+2}-x)(x-t_i)}{(t_{i+2}-t_i)(t_{i+1}-t_i)}, & x \in (t_i, t_{i+1}] \\ \frac{(t_{i+2}-x)^2}{(t_{i+2}-t_i)(t_{i+2}-t_{i+1})}, & x \in (t_{i+1}, t_{i+2}] \\ 0, & \text{otherwise.} \end{cases}$$

7 when $n=0$, $\frac{1}{t_i-t_{i-1}}\int_{t_{i-1}}^{t_i} B_i^0(t)\,dt = \frac{1}{t_i-t_{i-1}} \cdot 1\, dt = 1$

suppose $(n-1)$th case is true, $\frac{1}{t_{i+n-1}-t_{i-1}}\int_{t_{i-1}}^{t_{i+n-1}} B_i^{n-1}(x)\,dx = \frac{1}{n}$

consider nth case, $\int_{t_{i-1}}^{t_{i+n}} B_i^n(t)\,dt = t \cdot B_i^n(t)\Big|_{t_{i-1}}^{t_{i+n}} - \int_{t_{i-1}}^{t_{i+n}} t \cdot \frac{d}{dt} B_i^n(t)\,dt$

$= -\int_{t_{i-1}}^{t_{i+n}} t \cdot \frac{d}{dt} B_i^n(t)\,dt$

$= n\Big[\frac{1}{t_{i+n}-t_i}\int_{t_i}^{t_{i+n}} t\, B_{i+1}^{n-1}(t)\,dt$

$- \frac{1}{t_{i+n-1}-t_{i-1}}\int_{t_{i-1}}^{t_{i+n-1}} t\, B_i^{n-1}(t)\,dt\Big]$ — ①

$B_i^n(t) = \frac{t-t_{i-1}}{t_{i+n-1}-t_{i-1}} B_i^{n-1}(t) + \frac{t_{i+n}-t}{t_{i+n}-t_i} B_{i+1}^{n-1}(t)$

$\int_{t_{i-1}}^{t_{i+n}} B_i^n(t)\,dt = \int_{t_i}^{t_{i+n}} \frac{t_{i+n}-t}{t_{i+n}-t_i} B_{i+1}^{n-1}(t)\,dt + \int_{t_{i-1}}^{t_{i+n-1}} \frac{t-t_{i-1}}{t_{i+n-1}-t_{i-1}} B_i^{n-1}(t)\,dt$ — ②

From ① and ②, $\int_{i-1}^{i+n} B_i^n(t)\,dt = \frac{t_{i+n}-t_{i-1}}{n+1}$

$\frac{1}{t_{i+n}-t_{i-1}}\int_{i-1}^{i+n} B_i^n(t)\,dt = \frac{1}{n+1}$

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8 a)

$$
\begin{array}{c|c|c|c}
x_1 & x_1^4 & & \\
x_2 & x_2^4 & x_1^3+x_2^3+x_1^2x_2+x_2^2x_1 & \\
x_3 & x_3^4 & x_2^3+x_3^3+x_2^2x_3+x_3^2x_2 & x_1^2+x_2^2+x_3^2+x_1x_2+x_2x_3+x_3x_1
\end{array}
$$

$$\tau_2(x_1, x_2, x_3) = x_1x_2+x_2x_3+x_3x_1+x_1^2+x_2^2+x_3^2$$

$$= [x_1, x_2, x_3]x^4$$

b) $\tau_{k+1}(x_1, \dots, x_n, x_{n+1}) = \tau_{k+1}(x_1, \dots, x_n) + x_{n+1}\tau_k(x_1, \dots, x_n, x_{n+1})$

$\tau_{k+1}(x_1, \dots, x_n, x_{n+1}) - x_1\tau_k(x_1, \dots, x_n, x_{n+1}) = \tau_{k+1}(x_1, \dots, x_n) +$

$x_{n+1}\tau_k(x_1, \dots, x_n, x_{n+1})$

$- x_1\tau_k(x_1, \dots, x_n, x_{n+1})$

$(x_{n+1}-x_1)\tau_k(x_1, \dots, x_n, x_{n+1}) = \tau_{k+1}(x_2, \dots, x_n, x_{n+1}) - \tau_{k+1}(x_1, \dots, x_n)$

$$\tau_k(x_1, \dots, x_n, x_{n+1}) = \frac{\tau_{k+1}(x_2, \dots, x_n, x_{n+1}) - \tau_{k+1}(x_1, \dots, x_n)}{x_{n+1}-x_1}$$

for $n=0$, $\tau_m(x_i) = [x_i]x^m = x_i^m$

Suppose $(n-1)$th is true, $\tau_{m-n+1}(x_i, \dots, x_{i+n-1}) = [x_i, \dots, x_{i+n-1}]x^m$

Consider nth case, $\tau_{m-n}(x_i, \dots x_{i+n}) = \dfrac{\tau_{m-n+1}(x_{i+1}, \dots, x_{i+n}) - \tau_{m-n+1}(x_i, \dots, x_{i+n-1})}{x_{i+n}-x_i}$

$$= \frac{[x_{i+1}, \dots, x_{i+n}]x^m - [x_i, \dots, x_{i+n-1}]x^m}{x_{i+n}-x_i}$$

$$= [x_i, \dots, x_{i+n}, x_{i+n}]x^m$$