Homework1:L'Hôpital's rule's describe and proof

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L'Hôpital's rule is a theorem which provides a technique to evalute limits of indeterminate forms.Application of the rule often converts an indeterminate form to an expression that can be easily evaluted by substitution.

1 Question Describe

Question describe as below:

1.1 If function $f(x)$ and $g(x)$ in the area of $U(a, \delta)$ are derivable and satisfy :

$$\lim_{x \to a} f(x) = 0 \tag{1}$$

$$\lim_{x \to a} g(x) = 0 \tag{2}$$

$$\lim_{x \to a} \frac{f'(x)}{g'(x)} = l \tag{3}$$

$$\text{then exist : } \lim_{x \to a} \frac{f(x)}{g(x)} = l \tag{4}$$

1.2 If function $f(x)$ and $g(x)$ in the area of $U(\infty, \delta)$ are derivable and satisfy :

$$\lim_{x\to\infty} f(x) = 0 \tag{5}$$

$$\lim_{x\to\infty} g(x) = 0 \tag{6}$$

$$\lim_{x\to\infty} \frac{f'(x)}{g'(x)} = l \tag{7}$$

$$\text{then exist :} \lim_{x\to\infty} \frac{f(x)}{g(x)} = l \tag{8}$$

1.3 If function $f(x)$ and $g(x)$ in the area of $U(a, \delta)$ are derivable and satisfy :

$$\lim_{x\to a} g(x) = \infty \tag{9}$$

$$\lim_{x\to a} \frac{f'(x)}{g'(x)} = l \tag{10}$$

$$\text{then exist :} \lim_{x\to a} \frac{f(x)}{g(x)} = l \tag{11}$$

1.4 If function $f(x)$ and $g(x)$ in the area of $U(\infty, \delta)$ are derivable and satisfy :

$$\lim_{x\to\infty} g(x) = \infty \tag{12}$$

$$\lim_{x\to\infty} \frac{f'(x)}{g'(x)} = l \tag{13}$$

$$\text{then exist :} \lim_{x\to\infty} \frac{f(x)}{g(x)} = l \tag{14}$$

2 Proof the Rule

Just proof the situation when $\lim_{x\to a} \frac{f'(x)}{g'(x)} = l < \infty$,when $l = \infty$,both are same.

2.1 First proof (5) and (6)

$$\text{Let } \lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = 0$$
$$\forall \epsilon > 0, \exists \delta > 0, \text{when } x \in (a, a+\delta)$$
$$l - \epsilon < \frac{f'(x)}{g'(x)} < l + \epsilon, \tag{15}$$
$$\text{for } (x, x_0) \subset (a, a+\delta), \text{from Cauchy's mean value theorem ,got}$$
$$\xi \in (x, x_0), l - \epsilon < \frac{f(x) - f(x_0)}{g(x) - g(x_0)} = \frac{f'(\xi)}{g'(\xi)} < l + \epsilon \tag{16}$$
$$\text{let } x_0 \to a, 1 - \frac{g(x_0)}{g(x)} \to 1 \tag{17}$$
$$\text{so that ,} \limsup_{x\to a^+} \frac{f(x)}{g(x)} \le l + \epsilon \tag{18}$$
$$\text{from arbitrariness of } \epsilon, \limsup_{x\to a^+} \frac{f(x)}{g(x)} \le l \tag{19}$$
$$\text{similarly ,} \liminf_{x\to a^+} \frac{f(x)}{g(x)} \ge l \tag{20}$$
$$\text{from (19) and (20) ,} \lim_{x\to a^+} \frac{f(x)}{g(x)} = l \tag{21}$$

2.2 Now proof (12)

$$\begin{aligned}
\frac{f(x)}{g(x)} &= \frac{f(x) - f(x_0)}{g(x)} + \frac{f(x_0)}{g(x)} \\
&= \frac{g(x) - g(x_0)}{g(x)} * \frac{f(x) - f(x_0)}{g(x) - g(x_0)} + \frac{f(x_0)}{g(x)} \\
&= [1 - \frac{g(x_0)}{g(x)}]\frac{f(x) - f(x_0)}{g(x) - g(x_0)} + \frac{f(x_0)}{g(x)}
\end{aligned} \tag{22}$$
$$\begin{aligned}
|\frac{f(x)}{g(x)} - l| &= |[1 - \frac{g(x_0)}{g(x)}]\frac{f(x) - f(x_0)}{g(x) - g(x_0)} + \frac{f(x_0)}{g(x)} - l| \\
&\le |1 - \frac{g(x_0)}{g(x)}| * |\frac{f(x) - f(x_0)}{g(x) - g(x_0)} - l| + |\frac{f(x_0) - lg(x_0)}{g(x)}|
\end{aligned} \tag{23}$$

then $,|\frac{f(x)-f(x_0)}{g(x)-g(x_0)}-l|=|\frac{f'(\xi)}{g'(\xi)}-l|<\epsilon$

another , $\lim_{x\to a^+} g(x)=\infty$, when x near a

$$|1-\frac{g(x_0)}{g(x)}|<C, \forall C\in R \tag{24}$$

$$|\frac{f(x)-f(x_0)}{g(x)-g(x_0)}-l|\to 0 \tag{25}$$

$$|\frac{f(x_0)-lg(x_0)}{g(x)}|\to 0 \tag{26}$$

from (23)(24)(25)(26) , $\lim_{x\to a^+}\frac{f(x)}{g(x)}=l=\lim_{x\to a^+}\frac{f'(x)}{g'(x)}$ (27)