Navier-Stokes flow past a rigid body: attainability of steady solutions as limits of unsteady weak solutions, starting and landing cases

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Abstract

Consider the Navier-Stokes flow in 3-dimensional exterior domains, where a rigid body is translating with prescribed translational velocity $-h(t)u_\infty$ with constant vector $u_\infty \in \mathbb{R}^3 \setminus \{0\}$. Finn raised the question whether his steady solutions are attainable as limits for $t \to \infty$ of unsteady solutions starting from motionless state when $h(t) = 1$ after some finite time and $h(0) = 0$ (starting problem). This was affirmatively solved by Galdi, Heywood and Shibata [19] for small u_∞. We study some generalized situation in which unsteady solutions start from large motions being in L^3. We then conclude that the steady solutions for small u_∞ are still attainable as limits of evolution of those fluid motions which are found as a sort of weak solutions. The opposite situation, in which $h(t) = 0$ after some finite time and $h(0) = 1$ (landing problem), is also discussed. In this latter case, the rest state is attainable no matter how large u_∞ is.

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1 Introduction and results

Let us consider a viscous incompressible flow past an obstacle in 3D, which is a translating rigid body with a prescribed velocity $-hu_\infty$, where $u_\infty \in \mathbb{R}^3\setminus\{0\}$ is a constant vector and the function $h = h(t)$ describes the transition of the translational velocity of the body. In the frame attached to the body, the motion of the fluid obeys the exterior problem for the Navier-Stokes system

$$\begin{aligned}
&\partial_t u + u\cdot\nabla u = \Delta u - \nabla p_u - hu_\infty\cdot\nabla u,\\
&\text{div } u = 0,\\
&u|_{\partial\Omega} = -hu_\infty,\\
&u \to 0 \quad \text{as } |x|\to\infty,
\end{aligned} \tag{1.1}$$

where Ω denotes the exterior of the body in $\mathbb{R}^3$ with smooth boundary $\partial\Omega$. The unknown functions are the velocity field $u = (u_1(x,t), u_2(x,t), u_3(x,t))$ and the associated pressure $p_u = p_u(x,t)$.

Suppose both the fluid and the body are initially at rest, that is, $u(\cdot,0) = 0$ and $h(0) = 0$. If the body starts to move from the rest state until the terminal velocity $-u_\infty$ at an instant $T_0 > 0$ and, afterwards, $h(t) = 1$ for $t \ge T_0$, then the large time behavior of the solution $u(x,t)$ to (1.1) subject to the initial condition $u(\cdot,0) = 0$ would be related to the steady problem

$$\begin{aligned}
&u_s\cdot\nabla u_s = \Delta u_s - \nabla p_{u_s} - u_\infty\cdot\nabla u_s,\\
&\text{div } u_s = 0,\\
&u_s|_{\partial\Omega} = -u_\infty,\\
&u_s \to 0 \quad \text{as } |x|\to\infty.
\end{aligned} \tag{1.2}$$

Indeed, in this situation, Finn [15] raised the question whether $u(x,t)$ converges to $u_s(x)$ as $t\to\infty$ in a sense as long as $u_\infty \in \mathbb{R}^3\setminus\{0\}$ is small enough (Finn's starting problem). If that is the case, the steady flow $u_s(x)$ is said to be "attainable" by following the terminology of Heywood [23], who gave a partial answer to the starting problem. Note that the steady problem (1.2) with sufficiently small $u_\infty \in \mathbb{R}^3\setminus\{0\}$ possesses a unique solution u_s, what is called the physically reasonable solution, due to Finn [16] himself. On account of its anisotropic behavior with wake property, the solution $u_s(x)$ enjoys better summability $u_s \in L^q(\Omega)$ for every $q > 2$ (than the case where the body is at rest), see (3.1) below, however, still infinite energy $u_s \notin L^2(\Omega)$ because the net force exerted by the fluid to $\partial\Omega$ cannot vanish when the external force is absent, see Finn [14] and Galdi [18]. It is reasonable to look for a solution $u(x,t)$ of the form $u(x,t) = h(t)u_s(x) + v(x,t)$ and to expect $u(t) \in L^2(\Omega)$ since $u(0) = 0$, however, in this case, $v(t) \notin L^2(\Omega)$ follows from $u_s \notin L^2(\Omega)$ and thus the energy method is not enough to construct the perturbation $v(t)$. Thus the problem had remained open until Kobayashi and Shibata [29] developed the L^q-L^r decay estimate of the Oseen semigroup, see (2.5)–(2.6) below. Finally, by making use of this estimate, the starting problem from the rest state was completely solved by Galdi, Heywood and Shibata [19].

In the present paper we intend to provide further contributions to this issue for its better understanding. It would be worth while studying more possibilities of attainablity of the steady flow u_s. The aim is to find out many solutions to (1.1), which converge to u_s as $t \to \infty$, even if starting from large motions of both the fluid and the body, that is, the initial velocity

$$u(x,0) = u_0(x) \tag{1.3}$$

can be large with infinite energy and $h(0)$ is large, too. We take u_0 from $L^3(\Omega)$, as usual, or even from $L^{3,\infty}_0(\Omega)$, the completion of $C_0^\infty(\Omega)$ in the Lorentz space (weak-L^3 space) $L^{3,\infty}(\Omega)$, together with the compatibility conditions

$$\operatorname{div} u_0 = 0, \qquad \nu \cdot (u_0 + h(0)u_\infty)|_{\partial\Omega} = 0, \tag{1.4}$$

where ν stands for the outer unit normal to $\partial\Omega$ and the latter condition is understood in the sense of normal trace. The function $h = h(t)$ is assumed to satisfy

$$h \in C^{1,\theta}([0,\infty)) \quad \text{for some } \theta \in (0,1), \tag{1.5}$$

$$h(t) = 1 \quad \text{on } [T_0,\infty) \text{ for some } T_0 > 0. \tag{1.6}$$

The main result on the starting problem reads as follows.

Theorem 1.1. *There exists a constant $\delta > 0$ with the following property: If $u_\infty \in \mathbb{R}^3 \setminus \{0\}$ fulfills $|u_\infty| \le \delta$, then, for every $u_0 \in L^{3,\infty}_0(\Omega)$ with* (1.4) *and for every function $h(t)$ satisfying* (1.5)–(1.6)*, problem* (1.1) *subject to* (1.3) *admits at least one solution $u(x,t)$ which enjoys*

$$\|u(t) - u_s\|_{L^\infty(\Omega)} = O(t^{-1/2}) \tag{1.7}$$

as $t \to \infty$, where u_s is a unique solution to (1.2).

We stress that the small constant δ in Theorem 1.1 is independent of u_0 and h. Our global solution is a sort of weak solution, to be precise, it is of the form

$$u(x,t) = h(t)u_s + \widetilde{U}(x,t) + w(x,t), \tag{1.8}$$

where $\widetilde{U}(x,t)$ is an auxiliary function (regular enough for $t > 0$), while $w(x,t)$ is the so-called Leray-Hopf weak solution [31], [25], [36]. The idea to solve the Navier-Stokes initial value problem with large initial data in L^3 (or $L^{3,\infty}_0$) is due to Maremonti [34], in which a solution to (1.1) with $u_\infty = 0$ subject to (1.3) is constructed in the form $u(t) = e^{-tA}u_0 + w(t)$ with a Leray-Hopf weak solution $w(t)$, where e^{-tA} denotes the Stokes semigroup. The similar approach was adopted also by [2], [39]. In the case under consideration of this paper, the pair

$$v(x,t) := u(x,t) - h(t)u_s(x), \qquad p_v(x,t) := p_u(x,t) - h(t)p_{u_s}(x)$$

should obey

$$\begin{aligned}
&\partial_t v + v\cdot\nabla v + h(u_s\cdot\nabla v + v\cdot\nabla u_s) = \Delta v - \nabla p_v - hu_\infty\cdot\nabla v + g,\\
&\mathrm{div}\ v = 0,\\
&v|_{\partial\Omega} = 0,\\
&v \to 0 \quad \text{as } |x|\to\infty,\\
&v(\cdot,0) = v_0 := u_0 - h(0)u_s
\end{aligned} \tag{1.9}$$

with the forcing term

$$g(x,t) := -h'u_s + (h-h^2)(u_s+u_\infty)\cdot\nabla u_s, \tag{1.10}$$

where $h' = \frac{dh}{dt}$. There would be several possibilities of choice of the auxiliary function $\widetilde{U}(x,t)$ in (1.8), which plays the same role as $e^{-tA}u_0$ in [34]. With any choice of $\widetilde{U}(x,t)$ at hand, we subtract this function from $v(x,t)$ to see that the remaining part $w(x,t) := v(x,t) - \widetilde{U}(x,t)$ together with the associated pressure p_w satisfies

$$\begin{aligned}
&\partial_t w + w\cdot\nabla w + \widetilde{U}\cdot\nabla w + w\cdot\nabla\widetilde{U} + h(u_s\cdot\nabla w + w\cdot\nabla u_s)\\
&\qquad = \Delta w - \nabla p_w - hu_\infty\cdot\nabla w + f,\\
&\mathrm{div}\ w = 0,\\
&w|_{\partial\Omega} = 0,\\
&w \to 0 \quad \text{as } |x|\to\infty,\\
&w(\cdot,0) = w_0 := v_0 - \widetilde{U}(\cdot,0),
\end{aligned} \tag{1.11}$$

for some vector field $f = f(x,t)$ as the new forcing term whenever

$$\mathrm{div}\, \widetilde{U} = 0, \qquad \widetilde{U}|_{\partial\Omega} = 0, \qquad \widetilde{U} \to 0 \ (|x| \to \infty).$$

Besides these conditions, the auxiliary function $\widetilde{U}(x,t)$ must be taken so that $f \in L^2_{loc}([0,\infty); H^{-1}(\Omega))$ as well as $w_0 \in L^2(\Omega)$ in order to look for $w(x,t)$ as the Leray-Hopf weak solution with the strong energy inequality

$$\begin{aligned} &\frac{1}{2}\|w(t)\|^2_{L^2(\Omega)} + \int_s^t \|\nabla w\|^2_{L^2(\Omega)} d\tau \\ &\le \frac{1}{2}\|w(s)\|^2_{L^2(\Omega)} + \int_s^t \langle (hu_s + \widetilde{U}) \otimes w, \nabla w \rangle d\tau + \int_s^t \langle f, w \rangle d\tau \end{aligned} \tag{1.12}$$

for $s = 0$, a.e. $s > 0$ and all $t \ge s$. As the auxiliary function, in this paper, we will take the solution of the non-autonomous Oseen initial value problem in the whole space $\mathbb{R}^3$ together with a correction term, see (3.12) and (3.14). Then the forcing term $f(x,t)$ is given by (4.1) together with (3.15).

For the proof of attainability (1.7) of the steady flow, a crucial step is to find out a large instant $\bar{t} > 0$ such that $w(\bar{t})$ is small enough in $L^3(\Omega)$. It is then possible to construct a global strong solution from $\bar{t}$ with some decay properties, particularly L^∞-decay like $O(t^{-1/2})$, which can be identified with the weak solution $w(t)$ by the strong energy inequality (1.12). Indeed this strategy itself is quite classical since the celebrated paper by Leray [31], but there are some details to make $\|w(\bar{t})\|_{L^3(\Omega)}$ small at a suitable $\bar{t}$. This is by no means obvious since the RHS of (1.12) is growing for $t \to \infty$. One would raise the question whether Theorem 1.1 still holds for $u_0 \in L^{3,\infty}(\Omega)$ (that is strictly larger than $L^{3,\infty}_0(\Omega)$). For such data, unfortunately, the behavior of the auxiliary function $\widetilde{U}(t)$ near $t = 0$ is critical and this prevents us from constructing the weak solution $w(t)$.

It is also interesting to consider the opposite situation (landing problem), in which the body is initially translating with velocity $-u_\infty$ and it stops at an instant T_0 and is kept afterwards at rest, that is,

$$h(t) = 0 \quad \text{on } [T_0, \infty) \text{ for some } T_0 > 0; \qquad h(0) = 1. \tag{1.13}$$

The following result on the landing problem tells us that the rest state is attainable no matter how large u_∞ is.

Theorem 1.2. *For every $u_\infty \in \mathbb{R}^3 \setminus \{0\}$, $u_0 \in L^{3,\infty}_0(\Omega)$ with* (1.4) *and $h(t)$ satisfying* (1.13) *as well as* (1.5)*, problem* (1.1) *subject to* (1.3) *admits at least one solution $u(x,t)$ which enjoys*

$$\|u(t)\|_{L^\infty(\Omega)} = O(t^{-1/2}) \tag{1.14}$$

as $t \to \infty$.

The idea of the proof of Theorem 1.2 is the same as the one for the starting problem. For every $u_\infty \in \mathbb{R}^3 \setminus \{0\}$ the steady problem (1.2) admits at least one solution $u_s(x)$ with finite Dirichlet integral $\nabla u_s \in L^2(\Omega)$ (the Leray class), see Leray [30]. It also follows from the result of Babenko [1], Galdi [17], [18], Farwig and Sohr [13] that any solution of the Leray class eventually becomes the physically reasonable solution in the sense of Finn [15], [16]. Since we would have several solutions unless u_∞ is small, we fix a steady flow $u_s(x)$ arbitrarily among them and look for the solution $u(x,t)$ to (1.1) of the form (1.8). It would be interesting to ask sharper L^∞-decay like $o(t^{-1/2})$ in (1.14) as well as (1.7); in fact, this is possible for (1.1) with $u_\infty = 0$ subject to (1.3) when $u_0 \in L^{3,\infty}_0$ is small enough, see [33]. On account of the presence of the forcing term (especially $\widetilde{U} \cdot \nabla \widetilde{U}$, see (4.1)), it does not seem to be clear whether $\|w(t)\|_{L^\infty(\Omega)} = o(t^{-1/2})$, however, one could take another way in which one constructs directly a strong solution $v(t)$ on $[\bar{t}, \infty)$ with a suitable $\bar{t}$ for (1.9), instead of $w(t)$, such that $\|v(t)\|_{L^\infty(\Omega)} = o(t^{-1/2})$ as $t \to \infty$.

This paper concerns the attainability, while the stability of the steady flow was extensively studied, see for instance [41], [10], [28] and the references therein. The paper is organized as follows. After some preliminaries in the next section, we choose the auxiliary function $\widetilde{U}(x,t)$ in (1.8) and derive several properties in section 3. In section 4 we construct a weak solution $w(t)$ to the initial value problem (1.11) and deduce the strong energy inequality (1.12). In section 5 we make use of the L^q-L^r decay estimate of the Oseen semigroup ([29]) to construct a strong solution to (1.11) on $[\bar{t}, \infty)$ whenever $w(\bar{t})$ is small in $L^3(\Omega)$. We further show that this solution is identified with the weak solution on $[\bar{t}, \infty)$. The final section is devoted to finding $\bar{t} > 0$, at which $\|w(\bar{t})\|_{L^3(\Omega)}$ is actullay small enough, to accomplish the proof of Theorems 1.1 and 1.2.

2 Preliminaries

We start with introducing notation. Given a domain $D \subset \mathbb{R}^3$, $1 \le q \le \infty$, and integer $k \ge 0$, we denote by $L^q(D)$ and by $W^{k,q}(D)$ the standard Lebesgue and Sobolev spaces, respectively. We simply write the norm $\|\cdot\|_{q,D} = \|\cdot\|_{L^q(D)}$ and even $\|\cdot\|_q = \|\cdot\|_{q,\Omega}$, where Ω is the exterior domain under consideration. Let $C_0^\infty(D)$ be the class of smooth functions with compact support in D. We denote by $W_0^{k,q}(D)$ the completion of $C_0^\infty(D)$ in $W^{k,q}(D)$, and by $W^{-1,q}(D)$ the dual space of $W_0^{1,q'}(D)$, where $1/q' + 1/q = 1$ and $q \in (1,\infty)$. By $\langle \cdot, \cdot \rangle$ we denote various duality pairings on Ω. When $q = 2$, we write $H^k(D) = W^{k,2}(D)$, $H_0^1(D) = W_0^{1,2}(D)$ and $H^{-1}(D) = W^{-1,2}(D)$, respectively.

Let us introduce the Lorentz spaces (for details, see Bergh and Löfström [3]). Given a measurable function f on a domain D, we set

$$
\begin{aligned}
&m_f(\tau) = |\{x \in D;\ |f(x)| > \tau\}|\,, \qquad \tau > 0,\\
&f^*(t) = \inf\{\tau > 0;\ m_f(\tau) \le t\}, \qquad t > 0,
\end{aligned}
$$

where $|\cdot|$ stands for the Lebesgue measure. Let $1 < q < \infty$ and $1 \le r \le \infty$, then the space $L^{q,r}(D)$ consists of all measurable functions f on D which satisfy

$$
\begin{aligned}
&\left(\int_0^\infty \left\{t^{1/q} f^*(t)\right\}^r \frac{dt}{t}\right)^{1/r} < \infty \qquad (1 \le r < \infty),\\
&\sup_{t>0} t^{1/q} f^*(t) < \infty \qquad (r = \infty).
\end{aligned}
\tag{2.1}
$$

Each of those quantities is a quasi-norm, however, it is possible to introduce an equivalent norm $\|\cdot\|_{q,r,D}$ by use of the average function. Then $L^{q,r}(D)$ endowed with $\|\cdot\|_{q,r,D}$ is a Banach space, called the Lorentz space. We simply write $\|\cdot\|_{q,r} = \|\cdot\|_{q,r,\Omega}$. Note that $L^{q,q}(D) = L^q(D)$ and that $L^{q,r_0}(D) \subset L^{q,r_1}(D)$ if $r_0 \le r_1$. The space $L^{q,\infty}(D)$ is well known as the weak-L^q space, in which $C_0^\infty(D)$ is not dense. Let us define the space $L_0^{q,\infty}(D)$ by the completion of $C_0^\infty(D)$ in $L^{q,\infty}(D)$. The Lorentz space can be also constructed via real interpolation

$$
L^{q,r}(D) = \big(L^1(D), L^\infty(D)\big)_{1-1/q,r}
$$

from which the reiteration theorem in the interpolation theory leads to

$$
L^{q,r}(D) = \big(L^{q_0,r_0}(D), L^{q_1,r_1}(D)\big)_{\theta,r}
$$

together with

$$
\|f\|_{q,r,D} \le C\|f\|_{q_0,r_0,D}^{1-\theta}\|f\|_{q_1,r_1,D}^{\theta} \tag{2.2}
$$

for all $f \in L^{q_0,r_0}(D) \cap L^{q_1,r_1}(D) \subset L^{q,r}(D)$ provided that

$$
1 < q_0 < q < q_1 < \infty, \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}, \quad 1 \le r_0, r_1, r \le \infty.
$$

We have the Lorentz-Hölder and Lorentz-Sobolev inequalities, but the only cases we need in this paper are

$$
\|fg\|_{r,s,D} \le \|f\|_{3,\infty,D}\|g\|_{q,s,D}, \qquad \frac{1}{r} = \frac{1}{3} + \frac{1}{q}, \quad q,\, r \in (1,\infty), \tag{2.3}
$$

$$
\|g\|_{q_*,s} \le C\|\nabla g\|_{q,s}, \qquad \frac{1}{q_*} = \frac{1}{q} - \frac{1}{3}, \quad q \in (1,3), \tag{2.4}
$$

where $1 \leq s \leq \infty$. In what follows the same symbols for vector and scalar function spaces are adopted as long as there is no confusion.

Let us introduce the solenoidal function spaces over the exterior domain Ω. The space $C_{0,\sigma}^\infty(\Omega)$ consists of all divergence free vector fields whose components are in $C_0^\infty(\Omega)$. Let $1 < q < \infty$. We denote by $L_\sigma^q(\Omega)$ the completion of $C_{0,\sigma}^\infty(\Omega)$ in $L^q(\Omega)$. Then it is characterized as

$$L_\sigma^q(\Omega) = \{u \in L^q(\Omega);\ \mathrm{div}\ u = 0,\ \nu \cdot u|_{\partial\Omega} = 0\},$$

where $\nu \cdot u|_{\partial\Omega}$ stands for the normal trace of u. The space $L^q(\Omega)$ of vector fields admits the Helmholtz decomposition

$$L^q(\Omega) = L_\sigma^q(\Omega) \oplus \{\nabla p \in L^q(\Omega);\ p \in L_{loc}^q(\overline{\Omega})\}$$

which was proved by Miyakawa [37] and by Simader and Sohr [42]. When $q = 2$, it is the orthogobal decomposition. We have the same result for the whole space $\mathbb{R}^3$ as well.

By using the projection $\mathbb{P} : L^q(\Omega) \to L_\sigma^q(\Omega)$ associated with the decomposition above, we define the Stokes operator A by

$$D_q(A) = W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega) \cap L_\sigma^q(\Omega), \qquad Af = -\mathbb{P}\Delta f.$$

When $q = 2$, it is a nonnegative self-adjoint operator in $L_\sigma^2(\Omega)$ and

$$\langle A^{1/2}f, A^{1/2}g\rangle = \langle \nabla f, \nabla g\rangle, \qquad \text{for } f,\ g \in D_2(A^{1/2}) = H_{0,\sigma}^1(\Omega),$$

where the space $H_{0,\sigma}^1(\Omega)$ denotes the completion of $C_{0,\sigma}^\infty(\Omega)$ in $H^1(\Omega)$. Due to Solonnikov [43], Giga [21] and Farwig and Sohr [12], we know the generation of an analytic semigroup (the Stokes semigroup) $\{e^{-tA}\}_{t\geq 0}$ on $L_\sigma^q(\Omega)$. Furthermore, it is uniformly bounded $\|e^{-tA}f\|_q \leq C\|f\|_q$ by the result of Borchers and Sohr [5]. Given a constant vector $u_\infty \in \mathbb{R}^3$, let us define the Oseen operator A_{u_∞} by

$$D_q(A_{u_\infty}) = D_q(A), \qquad A_{u_\infty}f = -\mathbb{P}[\Delta f - u_\infty \cdot \nabla f].$$

Then, by a simple perturbation argument, see Miyakawa [37], it is verified that the operator $-A_{u_\infty}$ also generates an analytic semigroup (the Oseen semigroup) $\{e^{-tA_{u_\infty}}\}_{t\geq 0}$ on $L_\sigma^q(\Omega)$. In [29] Kobayashi and Shibata (see also Enomoto and Shibata [9], [10]) developed the L^q-L^r estimates

$$\|e^{-tA_{u_\infty}}f\|_r \leq Ct^{-\alpha}\|f\|_q \qquad (1 < q \leq r \leq \infty,\ q \neq \infty), \tag{2.5}$$

$$\|\nabla e^{-tA_{u_\infty}}f\|_r \leq Ct^{-\alpha-1/2}\|f\|_q \qquad (1 < q \leq r \leq 3), \tag{2.6}$$

for all $t > 0$, where $\alpha = (3/q - 3/r)/2$. They also showed that, for each $K > 0$, the constant $C = C(K; q, r) > 0$ in (2.5)–(2.6) can be taken uniformly with respect to $u_\infty \in \mathbb{R}^3$ satisfying $|u_\infty| \le K$. Therefore, their result includes the L^q-L^r estimates of the Stokes semigroup as a special case, however, even before, both (2.5) and (2.6) (case $u_\infty = 0$) had been established by Iwashita [26], Chen [7] (case $r = \infty$) and Maremonti and Solonnikov [35]. For later use, let us give a supplement about the Oseen operator, which is m-accretive in $L^2_\sigma(\Omega)$. Since both $1 + A_{u_\infty}$ and $1 + A$ are invertible, we have

$$\|Af\|_2 \le C\|(1 + A_{u_\infty})f\|_2, \qquad \|A_{u_\infty} f\|_2 \le C\|(1 + A)f\|_2,$$

for $f \in D_2(A)$. Then the Heinz-Kato inequality for m-accretive operators implies that

$$\|\nabla f\|_2 = \|A^{1/2} f\|_2 \le C\|(1 + A_{u_\infty})^{1/2} f\|_2 \tag{2.7}$$

for all $f \in D_2(A_{u_\infty}^{1/2}) = D_2(A^{1/2}) = H^1_{0,\sigma}(\Omega)$ with some constant $C = C(|u_\infty|) > 0$.

We next consider the boundary value problem for the equation of continuity

$$\text{div } w = f \ \text{ in } D, \qquad w|_{\partial D} = 0,$$

where D is a bounded domain in $\mathbb{R}^3$ with Lipschitz boundary ∂D. Let $1 < q < \infty$. Given $f \in L^q(D)$ with compatibility condition $\int_D f = 0$, there are a lot of solutions, some of which were found by many authors, see Galdi [18, Notes for Chapter III]. Among them a particular solution discovered by Bogovskii [4] is useful to recover the solenoidal condition in a cut-off procedure on account of some fine properties of his solution. The operator $f \mapsto$ his solution w, called the Bogovskii operator, is well defined as follows (for details, see Galdi [18], Borchers and Sohr [6]): there is a linear operator $\mathbb{B} : C_0^\infty(D) \to C_0^\infty(D)^3$ such that, for $1 < q < \infty$ and $k \ge 0$ integers,

$$\|\nabla^{k+1}\mathbb{B}f\|_{q,D} \le C\|\nabla^k f\|_{q,D} \tag{2.8}$$

with some $C = C(D, q, k) > 0$ and that

$$\text{div } \mathbb{B}f = f \qquad \text{if} \quad \int_D f(x)\, dx = 0, \tag{2.9}$$

where the constant C is invariant with respect to dilation of the domain D. By continuity, $\mathbb{B}$ is extended uniquely to a bounded operator from $W_0^{k,q}(D)$ to $W_0^{k+1,q}(D)^3$. It is obvious by real interpolation that several estimates in the Lorentz norm similar to (2.8) are available as well; for instance, we have

$$\|\nabla\mathbb{B}f\|_{q,\infty,D} \le C\|f\|_{q,\infty,D}, \tag{2.10}$$

for every $f \in L^{q,\infty}(D)$ and $q \in (1,\infty)$. By Geissert, Heck and Hieber [20, Theorem 2.5], $\mathbb{B}$ can be also extended to a bounded operator from $W^{1,q'}(D)^*$ to $L^q(D)^3$, that is,

$$\|\mathbb{B}f\|_{q,D} \le C\|f\|_{W^{1,q'}(D)^*}, \tag{2.11}$$

where $1/q'+1/q=1$. Note that this is not true from $W^{-1,q}(D)$ to $L^q(D)^3$, see Galdi [18, Chapter III]. Finally, we mention a sort of commutator estimate between $\mathbb{B}$ and the Laplacian. Let $f \in W^{2,q}(D)$. We fix $\eta \in C_0^\infty(D)$ to find

$$\|\Delta\mathbb{B}[\eta f] - \mathbb{B}[\Delta(\eta f)]\|_{q,D} \le C\|f\|_{q,D}. \tag{2.12}$$

Indeed this is rather restricted form, but it is enough for later use, see Lemma 3.3. By the condition above on the domain D, see Galdi [18, Lemma III.3.4], analysis can be reduced to the case in which D is star-shaped with respect to a ball B, where $\overline{B} \subset D$. In this case, the solution found by Bogovskii [4] is of the form (in 3D case)

$$\mathbb{B}[\eta f](x) = \int_D \Gamma_\kappa(x-y,y)(\eta f)(y)dy$$

with

$$\Gamma_\kappa(z,y) = z\int_1^\infty \kappa(y+\tau z)\tau^2 d\tau,$$

where $\kappa \in C_0^\infty(B)$ is fixed so that $\int_B \kappa = 1$. Set

$$\mathcal{B}_j[\eta f](x) = \int_D \Gamma_{\partial_j\kappa}(x-y,y)(\eta f)(y)dy \qquad (j=1,2,3).$$

Then we have

$$\partial_j\mathbb{B}[\eta f] - \mathbb{B}[\partial_j(\eta f)] = \mathcal{B}_j[\eta f]$$

for each $j=1,2,3$, and, thereby,

$$\Delta\mathbb{B}[\eta f] - \mathbb{B}[\Delta(\eta f)] = \sum_j \mathcal{B}_j[\partial_j(\eta f)] + \sum_j \partial_j\mathcal{B}_j[\eta f].$$

Since the operator $\mathcal{B}_j$ satisfies the same estimates as in (2.8) and (2.11) in spite of $\int_B \partial_j\kappa \ne 1$ (which is related only to whether (2.9) holds), the formula above leads to (2.12).

3 Auxiliary function

In this section we construct an auxiliary function $\widetilde{U}(x,t)$ in (1.8). We begin with knowledge about the steady problem (1.2). Due to Finn [16], Galdi [18], Farwig [11] and Shibata [41], there are constants $\delta_0 > 0$, $C = C(q) > 0$ and $C' = C'(r) > 0$ such that the steady problem (1.2) admits a unique solution

$$\begin{aligned} &u_s \in L^q(\Omega) \cap C^\infty(\Omega), \quad \|u_s\|_q \le C|u_\infty|^{1/2}, \quad \forall q \in (2, \infty], \\ &\nabla u_s \in L^r(\Omega), \quad \|\nabla u_s\|_r \le C'|u_\infty|^{1/2}, \quad \forall r \in (4/3, \infty], \\ &\text{provided } 0 < |u_\infty| \le \delta_0. \end{aligned} \tag{3.1}$$

Specifically, the rate $|u_\infty|^{1/2}$ above was deduced by Shibata as a consequence of his anisotropic pointwise estimates [41, Theorem 1.1]. For the starting problem, we take this solution u_s. For the landing problem, there is at least one solution to (1.2) having finite Dirichlet integral for every $u_\infty \in \mathbb{R}^3 \setminus \{0\}$ (see [30]) and, from now on, we fix a solution u_s; then, it possesses the summability properties in (3.1), no matter which we may choose, see Galdi [18, Section X.6].

Given $u_0 \in L^{3,\infty}_0(\Omega)$ with (1.4), we set $v_0 = u_0 - h(0)u_s \in L^{3,\infty}_0(\Omega)$ which fulfills $\nu \cdot v_0|_{\partial\Omega} = 0$ as well as $\mathrm{div}\, v_0 = 0$, see (1.9). We take the extension $\bar{v}_0$ of v_0 by setting zero outside Ω; then, we have $\bar{v}_0 \in L^{3,\infty}_0(\mathbb{R}^3)$ with $\mathrm{div}\, \bar{v}_0 = 0$. We fix $R > 0$ such that

$$\mathbb{R}^3 \setminus \Omega \subset B_R := \{x \in \mathbb{R}^3; |x| < R\}, \tag{3.2}$$

and take a cut-off function $\phi_0 \in C_0^\infty(B_{2R})$ so that $\phi_0(x) = 1$ in B_R. Set

$$\begin{aligned} &\bar{g}(x,t) = (1 - \phi_0(x))g(x,t), \\ &G(y,t) = \bar{g}\left(y + u_\infty \int_0^t h(\tau)d\tau, t\right), \end{aligned}$$

where g is given by (1.10). Then it follows from (3.1) that $\bar{g}(t)$ belongs to $L^q(\mathbb{R}^3) \cap C^\infty(\mathbb{R}^3)$ for every $q \in (2, \infty]$ and, therefore, so does $G(t)$. We also have

$$\mathrm{div}\, \bar{g} = (1 - \phi_0)(h - h^2) \sum_j (\partial_j u_s) \cdot \nabla u_{sj} - g \cdot \nabla \phi_0,$$

and, thereby, $\mathrm{div}\, G(t) \in L^q(\mathbb{R}^3)$ for every $q \in [1, \infty]$, which together with the Hardy-Littlewood-Sobolev inequality implies that

$$Q(\cdot, t) := \left(\frac{-1}{4\pi|\cdot|} * \mathrm{div}\, G\right)(\cdot, t) \in L^q(\mathbb{R}^3), \quad \forall q \in (3, \infty),$$

where $*$ stands for the convolution on $\mathbb{R}^3$. Set

$$\mathbb{P}_{\mathbb{R}^3} G(t) = G(t) - \nabla Q(t)$$

which satisfies

$$
\begin{aligned}
\|\mathbb{P}_{\mathbb{R}^3} G(t)\|_{q,\mathbb{R}^3} &\le \|G(t)\|_{q,\mathbb{R}^3} + \|\nabla Q(t)\|_{q,\mathbb{R}^3} \\
&\le C\|G(t)\|_{q,\mathbb{R}^3} = C\|\bar{g}(t)\|_{q,\mathbb{R}^3} \le C\|g(t)\|_q \le CM_q
\end{aligned}
\tag{3.3}
$$

for every $q \in (2, \infty)$ with

$$
M_q = |h'|_\infty \|u_s\|_q + (|h|_\infty + |h|_\infty^2)(\|u_s\|_\infty + |u_\infty|)\|\nabla u_s\|_q, \tag{3.4}
$$

where

$$
|h|_\infty = \sup_{t \ge 0} |h(t)|, \qquad |h'|_\infty = \sup_{t \ge 0} |h'(t)|.
$$

By using the heat semigroup

$$
e^{t\Delta} = (4\pi t)^{-3/2} e^{-|\cdot|^2/4t} * (\cdot),
$$

we set

$$
\begin{aligned}
V(t) &= \int_0^t e^{(t-\tau)\Delta} (\mathbb{P}_{\mathbb{R}^3} G)(\tau)\, d\tau, \\
W(t) &= e^{t\Delta} \bar{v}_0 + V(t).
\end{aligned}
\tag{3.5}
$$

Then the pair $W(y,t), Q(y,t)$ solves the Stokes initial value problem

$$
\begin{aligned}
&\partial_t W = \Delta W - \nabla Q + G, \quad \text{div } W = 0 \qquad (y \in \mathbb{R}^3,\, t > 0), \\
&W \to 0 \quad \text{as } |y| \to \infty, \\
&W(y,0) = \bar{v}_0(y).
\end{aligned}
\tag{3.6}
$$

By (1.5) we know

$$
G \in C^\theta([0,\infty); L^q(\mathbb{R}^3)), \quad \forall q \in (2, \infty],
$$

which implies that

$$
\begin{aligned}
&W \in C^1((0,\infty); L^{3,\infty}(\mathbb{R}^3) \cap L^q_\sigma(\mathbb{R}^3)), \quad \forall q \in (3, \infty), \\
&\nabla^2 W \in C((0,\infty); L^{3,\infty}(\mathbb{R}^3) \cap L^q(\mathbb{R}^3)), \quad \forall q \in (3, \infty).
\end{aligned}
\tag{3.7}
$$

We also find

$$
\nabla W \in C^\mu_{loc}((0,\infty); L^{3,\infty}(\mathbb{R}^3) \cap L^q(\mathbb{R}^3)), \quad \forall q \in (3,\infty), \forall \mu \in (0, 1/2). \tag{3.8}
$$

We then make the change of variable as

$$
\begin{aligned}
U(x,t) &= W\left(x - u_\infty \int_0^t h(\tau)\, d\tau, t\right), \\
P(x,t) &= Q\left(x - u_\infty \int_0^t h(\tau)\, d\tau, t\right),
\end{aligned}
\tag{3.9}
$$

to see from (3.7)–(3.8) that

$$\begin{cases} U \in C^1((0,\infty); L^{3,\infty}(\mathbb{R}^3) \cap L^q_\sigma(\mathbb{R}^3)), & \forall q \in (3,\infty), \\ \nabla^2 U \in C((0,\infty); L^{3,\infty}(\mathbb{R}^3) \cap L^q(\mathbb{R}^3)), & \forall q \in (3,\infty), \\ \nabla U \in C^\mu_{loc}((0,\infty); L^{3,\infty}(\mathbb{R}^3) \cap L^q(\mathbb{R}^3)), & \forall q \in (3,\infty),\ \forall \mu \in (0,1/2), \end{cases} \tag{3.10}$$

and that the pair (3.9) satisfies the non-autonomous Oseen initial value problem

$$\begin{aligned} &\partial_t U = \Delta U - \nabla P - hu_\infty \cdot \nabla U + \bar{g}, \quad \text{div } U = 0 \qquad (x \in \mathbb{R}^3,\, t > 0), \\ &U \to 0 \quad \text{as } |x| \to \infty, \\ &U(x,0) = \bar{v}_0(x). \end{aligned} \tag{3.11}$$

Let us take another cut-off function $\phi \in C_0^\infty(B_{3R})$ so that $\phi(x) = 1$ in B_{2R}. Our auxiliary function is then given by

$$\widetilde{U}(x,t) = (1-\phi(x))U(x,t) + \mathbb{B}[U(\cdot,t)\cdot\nabla\phi](x) = U(x,t) + E(x,t), \tag{3.12}$$

see (3.16) below, where $\mathbb{B}$ denotes the Bogovskii operator in the bounded domain $A_R = B_{3R} \setminus \overline{B_R}$. Since div $U = 0$, we observe $\int_{A_R} U \cdot \nabla\phi = 0$, which yields div $\widetilde{U} = 0$. By (3.10) we find that

$$\begin{cases} \widetilde{U} \in C^1((0,\infty); L^{3,\infty}(\Omega) \cap L^q_\sigma(\Omega)), & \forall q \in (3,\infty), \\ \nabla^2 \widetilde{U} \in C((0,\infty); L^{3,\infty}(\Omega) \cap L^q(\Omega)), & \forall q \in (3,\infty), \\ \nabla \widetilde{U} \in C^\mu_{loc}((0,\infty); L^{3,\infty}(\Omega) \cap L^q(\Omega)), & \forall q \in (3,\infty),\ \forall \mu \in (0,1/2), \end{cases} \tag{3.13}$$

and that

$$\begin{aligned} &\partial_t \widetilde{U} = \Delta \widetilde{U} - \nabla P - hu_\infty \cdot \nabla \widetilde{U} + g - F, \quad \text{div } \widetilde{U} = 0 \qquad (x \in \Omega,\, t > 0), \\ &\widetilde{U}|_{\partial\Omega} = 0, \\ &\widetilde{U} \to 0 \quad \text{as } |x| \to \infty, \\ &\widetilde{U}(\cdot,0) = (1-\phi)v_0 + \mathbb{B}[v_0 \cdot \nabla\phi], \end{aligned} \tag{3.14}$$

with

$$F(x,t) := \phi_0 g - \partial_t E + \Delta E - hu_\infty \cdot \nabla E, \tag{3.15}$$

where

$$E = -\phi U + \mathbb{B}[U \cdot \nabla\phi]. \tag{3.16}$$

For later use, we collect some properties of U and $\widetilde{U}$.

Lemma 3.1. *Let $j = 0, 1$. The function U given by* (3.9) *enjoys*

$$\begin{aligned}
&\|U(t)\|_{\infty,\mathbb{R}^3} \le C(\|v_0\|_{3,\infty} + M_3)\, t^{-1/2}, \\
&\|\nabla^j U(t)\|_{r,\mathbb{R}^3} \le C(\|v_0\|_{3,\infty} + M_3)\, t^{-1/2+3/2r-j/2}, \quad \forall r \in (3,\infty), \\
&\|\nabla^j U(t)\|_{3,\infty,\mathbb{R}^3} \le C(\|v_0\|_{3,\infty} + M_3)\, t^{-j/2}
\end{aligned} \tag{3.17}$$

for all $t > 0$, where M_3 is as in (3.4)*, and*

$$\begin{aligned}
&\|U(t)\|_{r,\mathbb{R}^3} = o(t^{-1/2+3/2r}), \quad \forall r \in (3,\infty], \\
&\|U(t)\|_{3,\infty,\mathbb{R}^3} = o(1),
\end{aligned} \tag{3.18}$$

as $t \to \infty$.

Proof. Since

$$\|\nabla^j U(t)\|_{r,\mathbb{R}^3} = \|\nabla^j W(t)\|_{r,\mathbb{R}^3}, \qquad \|\nabla^j U(t)\|_{3,\infty,\mathbb{R}^3} = \|\nabla^j W(t)\|_{3,\infty,\mathbb{R}^3},$$

it suffices to show the desired properties for $W(t)$ given by (3.5). By the Hausdorff-Young inequality and by real interpolation, we easily see that

$$\|\nabla^j e^{t\Delta}\bar{v}_0\|_{r,\mathbb{R}^3} \le Ct^{-1/2+3/2r-j/2}\|v_0\|_{3,\infty}, \quad \|\nabla^j e^{t\Delta}\bar{v}_0\|_{3,\infty,\mathbb{R}^3} \le Ct^{-j/2}\|v_0\|_{3,\infty},$$

for $3 < r \le \infty$. We use the assumption (1.6) and (3.3) with $q = 3$ to observe

$$\|\nabla^j V(t)\|_{r,\mathbb{R}^3} \le CM_3 \int_0^{T_0} (t-\tau)^{-1/2+3/2r-j/2} d\tau \le CM_3T_0\, t^{-1/2+3/2r-j/2} \tag{3.19}$$

for $t \ge 2T_0$, while

$$\|\nabla^j V(t)\|_{r,\mathbb{R}^3} \le CM_3T_0^{1/2+3/2r-j/2} \tag{3.20}$$

for $t \in (0, 2T_0]$ (except for the case $(j,r) = (1,\infty)$). Similarly, we obtain

$$\|\nabla^j V(t)\|_{3,\infty,\mathbb{R}^3} \le CM_3T_0t^{-j/2}$$

for $t \ge 2T_0$ and

$$\|\nabla^j V(t)\|_{3,\infty,\mathbb{R}^3} \le CM_3T_0^{1-j/2}$$

for $t \in (0, 2T_0]$. This shows (3.17).

The sharp behavior (3.18) was observed by [32], but let us give the proof for completeness. For $v_0 \in L^{3,\infty}_0(\Omega)$ and every $\varepsilon > 0$, one can take $v_{0\varepsilon} \in C_0^\infty(\Omega) \subset C_0^\infty(\mathbb{R}^3)$ such that

$$\|v_{0\varepsilon} - \bar{v}_0\|_{3,\infty,\mathbb{R}^3} = \|v_{0\varepsilon} - v_0\|_{3,\infty} \le \varepsilon. \tag{3.21}$$

Then we have

$$\|e^{t\Delta}\bar{v}_0\|_{3,\infty,\mathbb{R}^3} \le C\|v_{0\varepsilon}\|_{1,\mathbb{R}^3}\, t^{-1} + C\varepsilon,$$

yielding $\limsup_{t\to\infty} \|e^{t\Delta}\bar{v}_0\|_{3,\infty,\mathbb{R}^3} \le C\varepsilon$, which also implies

$$\|e^{t\Delta}\bar{v}_0\|_{r,\mathbb{R}^3} \le Ct^{-1/2+3/2r}\|e^{\frac{t}{2}\Delta}\bar{v}_0\|_{3,\infty,\mathbb{R}^3} = o(t^{-1/2+3/2r})$$

as $t\to\infty$. In (3.19) one can use (3.3) with $p\in(2,3)$ to replace M_3 by M_p; then,

$$\begin{aligned}
&\|V(t)\|_{r,\mathbb{R}^3} \le CM_pT_0\, t^{-3/2p+3/2r},\\
&\|V(t)\|_{3,\infty,\mathbb{R}^3} \le CM_pT_0\, t^{-3/2p+1/2},
\end{aligned}$$

for $t\ge 2T_0$, which proves (3.18). □

Corollary 3.1. *Let $j=0,1$. The function $\widetilde{U}$ given by* (3.12) *enjoys*

$$\|\widetilde{U}(t)\|_r \le C(\|v_0\|_{3,\infty} + M_3)\, t^{-1/2+3/2r}, \quad \forall r\in(3,\infty], \tag{3.22}$$

$$\|\nabla\widetilde{U}(t)\|_r \le C(\|v_0\|_{3,\infty} + M_3)\, t^{-1+3/2r}(1+t)^{1/2-3/2r}, \quad \forall r\in(3,\infty), \tag{3.23}$$

$$\|\nabla^j\widetilde{U}(t)\|_{3,\infty} \le C(\|v_0\|_{3,\infty} + M_3)\, t^{-j/2}, \tag{3.24}$$

for all $t>0$, where M_3 is as in (3.4)*, and*

$$\begin{aligned}
&\|\widetilde{U}(t)\|_r = o(t^{-1/2+3/2r}), \quad \forall r\in(3,\infty],\\
&\|\widetilde{U}(t)\|_{3,\infty} = o(1),
\end{aligned} \tag{3.25}$$

as $t\to\infty$.

Let $\bar{t}\in[T_0,\infty)$, where T_0 is as in (1.6) *or* (1.13)*, then*

$$\begin{aligned}
&\|\widetilde{U}(t)\|_r \le C(t-\bar{t})^{-1/2+3/2r}\|U(\bar{t})\|_{3,\infty,\mathbb{R}^3}, \quad \forall r\in(3,\infty],\\
&\|\nabla\widetilde{U}(t)\|_{3,\infty} \le C(t-\bar{t})^{-1/2}\|U(\bar{t})\|_{3,\infty,\mathbb{R}^3},
\end{aligned} \tag{3.26}$$

for all $t>\bar{t}$.

Proof. On account of (2.8) (combined with the Gagliardo-Nirenberg inequality for $r=\infty$) we have

$$\begin{aligned}
&\|\widetilde{U}(t)\|_r \le C\|U(t)\|_{r,\mathbb{R}^3},\\
&\|\nabla\widetilde{U}(t)\|_r \le C\|\nabla U(t)\|_{r,\mathbb{R}^3} + C\|U(t)\|_{\infty,A_R}.
\end{aligned}$$

for $r\in(3,\infty]$ as well as the similar inequalities for $\|\nabla^j\widetilde{U}(t)\|_{3,\infty}$, see (2.10). Then Lemma 3.1 concludes (3.22), (3.24), (3.23) and (3.25).

By (1.6) or (1.13) we have $G(y,t)=0$ for $t\geq T_0$ and, therefore, deduce from (3.6) that $W(t)=e^{(t-\bar{t})\Delta}W(\bar{t})$. In view of (3.9) and (3.12) we find

$$\|\widetilde{U}(t)\|_r\leq C\|W(t)\|_{r,\mathbb{R}^3}\leq C(t-\bar{t})^{-1/2+3/2r}\|W(\bar{t})\|_{3,\infty,\mathbb{R}^3},$$

for $3<r\leq\infty$. Similarly, we have

$$\|\nabla\widetilde{U}(t)\|_{3,\infty}\leq C\|\nabla W(t)\|_{3,\infty,\mathbb{R}^3}+C\|W(t)\|_{\infty,\mathbb{R}^3}\leq C(t-\bar{t})^{-1/2}\|W(\bar{t})\|_{3,\infty,\mathbb{R}^3}.$$

These estimates together with $\|W(\bar{t})\|_{3,\infty,\mathbb{R}^3}=\|U(\bar{t})\|_{3,\infty,\mathbb{R}^3}$ imply (3.26). $\square$

Remark 3.1. *Actually, $\widetilde{U}(t)$ does not possess any singular behavior near $t=\bar{t}$, however, it is convenient to use* (3.26) *in the proof of Proposition 5.1.*

We will be faced with some troubles a few times arising from the behavior of $\widetilde{U}(t)$ such as $\|\widetilde{U}(t)\|_\infty^2\leq Ct^{-1}$ near $t=0$, see (3.22). In order to get around this unpleasant situation, it is convenient to carry out the following simple approximation procedure.

Lemma 3.2. *Let $\varepsilon>0$. Then there is a function*

$$\widetilde{U}_\varepsilon\in L^\infty(0,\infty;L^q(\Omega))$$

with

$$\nabla\widetilde{U}_\varepsilon\in L^\infty(0,\infty;L^q(\Omega))$$

for every $q\in(3,\infty]$ such that

$$\begin{aligned}
&\sup_{t>0}\|\widetilde{U}_\varepsilon(t)-\widetilde{U}(t)\|_{3,\infty}\leq C\varepsilon,\\
&\sup_{t>0}t^{1/2-3/2q}\|\widetilde{U}_\varepsilon(t)-\widetilde{U}(t)\|_q\leq C\varepsilon,\\
&\sup_{0<t\leq 1}t^{1-3/2q}\|\nabla\widetilde{U}_\varepsilon(t)-\nabla\widetilde{U}(t)\|_q\leq C\varepsilon,
\end{aligned}$$

for every $q\in(3,\infty)$.

Proof. We use $v_{0\varepsilon}$ in (3.21). We replace $\bar{v}_0$ by $v_{0\varepsilon}$ in (3.5) to define W_ε, which leads to $\widetilde{U}_\varepsilon$ by (3.12) via (3.9). Then we have

$$\begin{aligned}
&\|\widetilde{U}_\varepsilon(t)-\widetilde{U}(t)\|_q\leq C\|W_\varepsilon(t)-W(t)\|_{q,\mathbb{R}^3}=C\|e^{t\Delta}(v_{0\varepsilon}-\bar{v}_0)\|_{q,\mathbb{R}^3},\\
&\|\widetilde{U}_\varepsilon(t)-\widetilde{U}(t)\|_{3,\infty}\leq C\|W_\varepsilon(t)-W(t)\|_{3,\infty,\mathbb{R}^3}=C\|e^{t\Delta}(v_{0\varepsilon}-\bar{v}_0)\|_{3,\infty,\mathbb{R}^3},
\end{aligned}$$

and

$$\begin{aligned}\|\nabla\widetilde{U}_\varepsilon(t)-\nabla\widetilde{U}(t)\|_q &\le C\|\nabla W_\varepsilon(t)-\nabla W(t)\|_{q,\mathbb{R}^3}+C\|W_\varepsilon(t)-W(t)\|_{q,A_R}\\ &= C\|\nabla e^{t\Delta}(v_{0\varepsilon}-\bar{v}_0)\|_{q,\mathbb{R}^3}+C\|e^{t\Delta}(v_{0\varepsilon}-\bar{v}_0)\|_{q,\mathbb{R}^3},\end{aligned}$$

as well as

$$\begin{aligned}\|\widetilde{U}_\varepsilon(t)\|_q &\le C\|W_\varepsilon(t)\|_{q,\mathbb{R}^3}\le C\|e^{t\Delta}v_{0\varepsilon}\|_{q,\mathbb{R}^3}+C\|V(t)\|_{q,\mathbb{R}^3},\\ \|\nabla\widetilde{U}_\varepsilon(t)\|_q &\le C\|\nabla W_\varepsilon(t)\|_{q,\mathbb{R}^3}+C\|W_\varepsilon(t)\|_{q,A_R}\\ &\le C\|e^{t\Delta}\nabla v_{0\varepsilon}\|_{q,\mathbb{R}^3}+C\|e^{t\Delta}v_{0\varepsilon}\|_{q,\mathbb{R}^3}+C\|V(t)\|_{W^{1,q}(\mathbb{R}^3)},\end{aligned}$$

for every $q\in(3,\infty]$. Concerning $\|\nabla^j V(t)\|_{q,\mathbb{R}^3}$ for $j=0,1$, we have (3.19) and (3.20) except for the case $(j,q)=(1,\infty)$, in which $\|\nabla V(t)\|_{\infty,\mathbb{R}^3}$ can be estimated similarly by use of (3.3) with $q\in(3,\infty)$. The proof is thus complete. $\square$

Remark 3.2. *Both $\widetilde{U}_\varepsilon$ and $\nabla\widetilde{U}_\varepsilon$ belong to $L^\infty(0,\infty;L^q(\Omega))$ for every $q>2$ since we have* (3.3) *for such q, however, for later use, the only cases we need are $q=\infty$ and $q=6$.*

We next deduce some estimates and regularity of the function F.

Lemma 3.3. *The function F given by* (3.15) *satisfies*

$$\|F(t)\|_2\le C(\|v_0\|_{3,\infty}+M_3)\,t^{-1/2}, \tag{3.27}$$

$$\|F(t)\|_{H^{-1}(\Omega)}\le C(\|v_0\|_{3,\infty}+M_3)(1+t)^{-1/2}, \tag{3.28}$$

$$|\langle F(t),\varphi\rangle|\le C(\|v_0\|_{3,\infty}+M_3)(1+t)^{-1/2}\|\nabla\varphi\|_2,\quad \forall\varphi\in H^1_0(\Omega), \tag{3.29}$$

for all $t>0$, where M_3 is as in (3.4), *and thereby*

$$F\in L^2(0,T;H^{-1}(\Omega)) \tag{3.30}$$

for every $T\in(0,\infty)$. Furthermore,

$$F\in C^\mu_{loc}((0,\infty);L^2(\Omega)) \tag{3.31}$$

for every $\mu\in(0,1/2)$ with $\mu\le\theta$, where θ is as in (1.5).

Let $q\in(1,3)$ and $\bar{t}\in[T_0,\infty)$, where T_0 is as in (1.6) *or* (1.13). *Then*

$$\|F(t)\|_q\le C(t-\bar{t})^{-1/2}\|U(\bar{t})\|_{3,\infty,\mathbb{R}^3} \tag{3.32}$$

for all $t>\bar{t}$.

Proof. Using the equation (3.11), we split F into

$$F(x,t) = F_1 + F_2 + F_3 + F_4 + F_5$$

with

$$\begin{aligned}
F_1 &= \phi(g - \nabla P) - \mathbb{B}[(g - \nabla P) \cdot \nabla \phi], \\
F_2 &= -2\nabla \phi \cdot \nabla U, \\
F_3 &= -(\Delta \phi)U + h(u_\infty \cdot \nabla \phi)U - h u_\infty \cdot \nabla \mathbb{B}[U \cdot \nabla \phi], \\
F_4 &= h\mathbb{B}[(u_\infty \cdot \nabla U) \cdot \nabla \phi], \\
F_5 &= -\mathbb{B}[\Delta U \cdot \nabla \phi] + \Delta \mathbb{B}[U \cdot \nabla \phi].
\end{aligned}$$

Here, we have used $\phi_0 g + \phi \bar{g} = \phi g$. It is easily seen from (3.3) that

$$\|F_1\|_2 \le C\|g(t)\|_3 + C\|\nabla Q(t)\|_{3,\mathbb{R}^3} \le CM_3.$$

Note that

$$F_1 = 0 \qquad (t \ge T_0)$$

by (1.6) or (1.13). We also have

$$\begin{aligned}
&\|F_2\|_2 \le C\|\nabla U(t)\|_{2,A_R} \le C\|\nabla U(t)\|_{3,\infty,\mathbb{R}^3}, \\
&\|F_2\|_{H^{-1}(\Omega)} + \|F_3\|_2 \le C\|U(t)\|_{2,A_R}.
\end{aligned}$$

Thanks to (2.11), we obtain

$$\|F_4\|_2 \le C\|(u_\infty \cdot \nabla U) \cdot \nabla \phi\|_{H^1(A_R)^*} \le C\|U(t)\|_{2,A_R}.$$

The last term is further modified as

$$F_5 = F_{51} + F_{52},$$

where

$$\begin{aligned}
F_{51} &= -\mathbb{B}[\Delta(U \cdot \nabla \phi)] + \Delta \mathbb{B}[U \cdot \nabla \phi], \\
F_{52} &= \mathbb{B}[2\nabla U \cdot \nabla(\nabla \phi) + U \cdot \nabla(\Delta \phi)].
\end{aligned}$$

From (2.11) as well as (2.8) we observe

$$\|F_{52}\|_2 \le C\|U(t)\|_{2,A_R}.$$

By virtue of (2.12) we find

$$\|F_{51}\|_2 \le C\|U(t)\|_{2,A_R}.$$

All the computation above tells us that

$$|\langle F(t),\varphi\rangle| \le \|F_1+F_3+F_4+F_5\|_2\|\varphi\|_{2,\Omega_{3R}} + C\|U(t)\|_{2,A_R}\|\varphi\|_{H^1(\Omega_{3R})},$$

the latter of which comes from F_2, where $\Omega_{3R}=\Omega\cap B_{3R}$. Since $\|\varphi\|_{2,\Omega_{3R}} \le C\|\nabla\varphi\|_2$ for $\varphi\in H^1_0(\Omega)$, we get

$$|\langle F(t),\varphi\rangle| \le C\|U(t)\|_{2,A_R}\|\nabla\varphi\|_2.$$

Using

$$\|U(t)\|_{2,A_R} \le \begin{cases} C\|U(t)\|_{3,\infty,\mathbb{R}^3}, \\ C\|U(t)\|_{\infty,\mathbb{R}^3}, \end{cases}$$

we conclude (3.27)–(3.29) from (3.17).

Estimates above in $L^2(\Omega)$ imply that

$$\begin{aligned}\|F(t)-F(s)\|_2 \le C\|g(t)-g(s)\|_3 + C\|\nabla U(t)-\nabla U(s)\|_{2,A_R} \\ + C\|U(t)-U(s)\|_{2,A_R} + C|h(t)-h(s)|,\end{aligned}$$

which leads us to (3.31) on account of (1.5), (1.10) and (3.10).

Finally, let $q\in(1,3)$, $\bar t\in[T_0,\infty)$ and $t>\bar t$. Since estimates above in $L^2(\Omega)$ replaced by $L^q(\Omega)$ hold true, we have

$$\|F(t)\|_q \le C\|\nabla U(t)\|_{q,A_R} + C\|U(t)\|_{q,A_R} \le C\|\nabla U(t)\|_{3,\infty,\mathbb{R}^3} + C\|U(t)\|_{\infty,\mathbb{R}^3}.$$

Then the same reasoning as in the proof of (3.26) yields (3.32). $\square$

4 Weak solution

Let us take the auxiliary function $\widetilde{U}(x,t)$ given by (3.12) and look for a solution to (1.1) of the form (1.8). Then (1.9) and (3.14) imply that $w(x,t)$ should obey (1.11) with

$$\begin{aligned} &f = F - \widetilde{U}\cdot\nabla\widetilde{U} - h(u_s\cdot\nabla\widetilde{U} + \widetilde{U}\cdot\nabla u_s), \\ &w_0 = \phi v_0 - \mathbb{B}[v_0\cdot\nabla\phi] \in L^2_\sigma(\Omega), \end{aligned} \tag{4.1}$$

where $p_w = p_v - P$ is the pressure associated with w, while F is given by (3.15). By (2.2), (2.3), (2.4), (3.22) and (3.24) we have

$$\begin{aligned} &\|\widetilde{U}\cdot\nabla\widetilde{U}\|_2 \le \|\widetilde{U}\|_{6,2}\|\nabla\widetilde{U}\|_{3,\infty} \le C(\|v_0\|_{3,\infty}+M_3)^2\, t^{-3/4}, \\ &\|u_s\cdot\nabla\widetilde{U}\|_2 \le C\|u_s\|_{6,2}(\|v_0\|_{3,\infty}+M_3)\, t^{-1/2}, \\ &\|\widetilde{U}\cdot\nabla u_s\|_2 \le C(\|\nabla u_s\|_{6,2}+\|\nabla u_s\|_2)(\|v_0\|_{3,\infty}+M_3)(1+t)^{-1/2}, \end{aligned}$$

for all $t > 0$. These estimates together with (3.27) imply

$$\kappa_f := \sup_{t>0} t^{3/4}(1+t)^{-1/4}\|f(t)\|_2 < \infty. \tag{4.2}$$

By (3.22) and (3.25) we know

$$\|\widetilde{U} \otimes \widetilde{U} + h(\widetilde{U} \otimes u_s + u_s \otimes \widetilde{U})\|_2 \begin{cases} \leq Ct^{-1/4} & \text{for all } t > 0, \\ = o(t^{-1/4}) & \text{as } t \to \infty, \end{cases} \tag{4.3}$$

which together with (3.30) yields

$$f \in L^2(0, T; H^{-1}(\Omega)) \tag{4.4}$$

for every $T \in (0, \infty)$. Furthermore, by (1.5), (3.13) and (3.31) we find

$$f \in C^{\mu}_{loc}((0, \infty); L^2(\Omega)), \tag{4.5}$$

for every $\mu \in (0, 1/2)$ with $\mu \leq \theta$.

In this section we show the existence of weak solution with the strong energy inequality (1.12). Let us recall the definition of the Leray-Hopf weak solution ([31], [25], [36]).

Definition 4.1. *We say that $w(x, t)$ is a weak solution to* (1.11) *with* (4.1) *if*

$$w \in L^\infty(0, T; L^2_\sigma(\Omega)) \cap L^2(0, T; H^1_{0,\sigma}(\Omega)) \cap C_w([0, \infty); L^2_\sigma(\Omega))$$

for all $T \in (0, \infty)$ together with $\lim_{t\to 0} \|w(t) - w_0\|_2 = 0$ and w satisfies (1.12) *for $s = 0$ as well as*

$$\begin{aligned} &\langle w(t), \varphi(t)\rangle + \int_s^t \Big[\langle \nabla w, \nabla \varphi\rangle + \langle \{h(u_\infty + u_s) + \widetilde{U}\} \cdot \nabla w, \varphi\rangle \\ &\qquad - \langle (hu_s + \widetilde{U}) \otimes w, \nabla \varphi\rangle + \langle w \cdot \nabla w, \varphi\rangle\Big] d\tau \\ &= \langle w(s), \varphi(s)\rangle + \int_s^t \Big[\langle w, \partial_\tau \varphi\rangle + \langle f, \varphi\rangle\Big] d\tau \end{aligned} \tag{4.6}$$

for all $0 \leq s < t < \infty$ and φ, which is of class

$$\begin{aligned} &\varphi \in C([0, \infty); L^2_\sigma(\Omega)) \cap L^\infty_{loc}([0, \infty); L^{3,\infty}(\Omega)), \\ &\nabla \varphi \in L^2_{loc}([0, \infty); L^2(\Omega)), \quad \partial_t \varphi \in L^2_{loc}([0, \infty); L^2_\sigma(\Omega)). \end{aligned} \tag{4.7}$$

We will follow in principle the argument of Miyakawa and Sohr [38], whose idea partially goes back to Leray [31]. Set

$$J_k = e^{-\frac{1}{k}A}, \qquad (k = 1, 2, ...)$$

and consider the approximate problem

$$\begin{aligned} &\partial_t w + Aw + \mathbb{P}[Sw + (J_k w) \cdot \nabla w] = \mathbb{P}f, \\ &w(0) = w_0, \end{aligned} \tag{4.8}$$

where

$$Sw = \{h(u_\infty + u_s) + \widetilde{U}\} \cdot \nabla w + w \cdot \nabla(hu_s + \widetilde{U}).$$

The following lemma provides a solution with the a priori estimate.

Lemma 4.1. *For each $k = 1, 2, ...$, problem (4.8) admits a unique global strong solution $w = w_k$ of class*

$$w_k \in C([0, \infty); L^2_\sigma(\Omega)) \cap C((0, \infty); D_2(A)) \cap C^1((0, \infty); L^2_\sigma(\Omega))$$

subject to $\lim_{t\to 0} \|w_k(t) - w_0\|_2 = 0$, which satisfies

$$\|w_k(t)\|_2^2 + \int_0^t \|\nabla w_k\|_2^2 d\tau \le Y(t) \tag{4.9}$$

for all $t > 0$ with

$$\begin{aligned} &Y(t) := \left(\|w_0\|_2^2 + C\|f\|^2_{L^2(0,t;H^{-1}(\Omega))}\right) e^{CNt}, \\ &N := 1 + |h|_\infty^2 \|u_s\|_\infty^2 + \|\widetilde{U}_{\varepsilon_0}\|^2_{L^\infty(0,\infty;L^\infty(\Omega))}, \end{aligned} \tag{4.10}$$

where $\widetilde{U}_{\varepsilon_0}$ is the function given by Lemma 3.2 for some $\varepsilon_0 > 0$.

Proof. We fix $T \in (1, \infty)$ arbitrarily, and let us construct a solution on $(0, T]$. We first establish the local existence of solutions. Let $T_* \in (0, 1]$ and set

$$\begin{aligned} E_{T_*} =&\{w \in C((0, T_*]; H^1_{0,\sigma}(\Omega)); \\ &\|w\|_{E_{T_*}} := \sup_{0<t\le T_*} \left(\|w(t)\|_2 + t^{1/2}\|\nabla w(t)\|_2\right) < \infty\} \end{aligned}$$

which is a Banach space endowed with norm $\|\cdot\|_{E_{T_*}}$. We set

$$(\Phi w)(t) = H(t) - \int_0^t e^{-(t-\tau)A}\mathbb{P}[Sw + (J_k w) \cdot \nabla w](\tau) d\tau,$$

where

$$H(t) = e^{-tA}w_0 + \int_0^t e^{-(t-\tau)A}\mathbb{P}f(\tau)d\tau,$$

and intend to solve the integral equation $w = \Phi w$ in E_{T_*} by using (2.5)–(2.6) (for the Stokes semigroup). For $w \in E_{T_*}$, we easily find

$$\begin{aligned} &\Phi w \in C^\mu_{loc}((0,T_*]; L^q_\sigma(\Omega)), \quad \forall q \in [2,\infty), \forall \mu \in (0,\mu_0), \\ &\nabla\Phi w \in C^\mu_{loc}((0,T_*]; L^q(\Omega)), \quad \forall q \in [2,6), \forall \mu \in (0,\mu_0 - 1/2), \end{aligned} \tag{4.11}$$

where $\mu_0 = \frac{3}{2q} + \frac{1}{4}$. By (4.2) we have $H \in E_{T_*}$ with

$$\begin{aligned} &\|H(t) - w_0\|_2 \le \|e^{-tA}w_0 - w_0\|_2 + C\kappa_f\, t^{1/4}(1+t)^{1/4}, \\ &\|H\|_{E_{T_*}} \le C_0\left(\|w_0\|_2 + \kappa_f\sqrt{T}\right). \end{aligned}$$

Let $w \in E_{T_*}$, then we have

$$\begin{aligned} \|\nabla^j \int_0^t e^{-(t-\tau)A}\mathbb{P}[(J_k w)\cdot\nabla w](\tau)d\tau\|_2 &\le C\int_0^t (t-\tau)^{-j/2}\|J_k w\|_\infty \|\nabla w\|_2 d\tau \\ &\le C_1 k^{3/4}\sqrt{T_*}\, t^{-j/2}\|w\|^2_{E_{T_*}} \end{aligned}$$

for $t \in (0,T_*]$ and $j = 0,1$. Let $\varepsilon > 0$. We fix $r \in (3,\infty)$ and employ $\widetilde{U}_\varepsilon$ in Lemma 3.2 to find

$$\begin{aligned} &\|\nabla^j \int_0^t e^{-(t-\tau)A}\mathbb{P}Sw(\tau)d\tau\|_2 \\ &\le C\int_0^t (t-\tau)^{-j/2}\left(|h|_\infty\|u_\infty + u_s\|_\infty + \|\widetilde{U}_\varepsilon\|_\infty\right)\|\nabla w\|_2 d\tau \\ &\quad + C\int_0^t (t-\tau)^{-3/2r - j/2}\|\widetilde{U}_\varepsilon - \widetilde{U}\|_r\|\nabla w\|_2 d\tau \\ &\quad + C\int_0^t (t-\tau)^{-j/2}\left(|h|_\infty\|\nabla u_s\|_\infty + \|\nabla\widetilde{U}_\varepsilon\|_\infty\right)\|w\|_2 d\tau \\ &\quad + C\int_0^t (t-\tau)^{-3/2r - j/2}\|\nabla\widetilde{U}_\varepsilon - \nabla\widetilde{U}\|_r\|w\|_2 d\tau \\ &\le \left\{C_2^{(\varepsilon)}(\sqrt{T_*} + T_*) + C_2'\varepsilon\right\} t^{-j/2}\|w\|_{E_{T_*}} \end{aligned}$$

for $t \in (0,T_*]$ and $j = 0,1$. As a consequence, we obtain

$$\begin{aligned} \|\Phi w\|_{E_{T_*}} \le C_0\left(\|w_0\|_2 + \kappa_f\sqrt{T}\right) &+ C_1 k^{3/4}\sqrt{T_*}\|w\|^2_{E_{T_*}} \\ &+ \left(2C_2^{(\varepsilon)}\sqrt{T_*} + C_2'\varepsilon\right)\|w\|_{E_{T_*}} \end{aligned}$$

as well as

$$\limsup_{t\to 0} \|(\Phi w)(t) - w_0\|_2 \le C\varepsilon \|w\|_{E_{T_*}}$$

for $w \in E_{T_*}$. The latter for arbitrary $\varepsilon > 0$ yields

$$\lim_{t\to 0} \|(\Phi w)(t) - w_0\|_2 = 0. \tag{4.12}$$

We next choose $\varepsilon = 1/8C_2'$ in the former, so that $2C_2^{(\varepsilon)}\sqrt{T_*} + C_2'\varepsilon \le 1/4$ when $T_* \le (1/16C_2^{(\varepsilon)})^2$. We set

$$E_{T_*,\rho} = \{w \in E_{T_*};\, \|w\|_{E_{T_*}} \le \rho\}$$

with

$$\rho = 2C_0\left(\|w_0\|_2 + \kappa_f\sqrt{T}\right), \quad T_* = \min\left\{\left(4C_1 k^{3/4}\rho\right)^{-2},\, (16C_2^{(\varepsilon)})^{-2},\, 1\right\}. \tag{4.13}$$

Then $w \in E_{T_*,\rho}$ implies $\Phi w \in E_{T_*,\rho}$. Furthermore, we find

$$\|\Phi w_1 - \Phi w_2\|_{E_{T_*}} \le \frac{3}{4}\|w_1 - w_2\|_{E_{T_*}}$$

for $w_1, w_2 \in E_{T_*,\rho}$. We thus get a unique fixed point $w \in E_{T_*,\rho}$ of the map Φ, which fulfills the initial condition by (4.12). It also follows from (4.11) together with (1.5), (3.13) and (4.5) that the local solution $w(t)$ satisfies

$$\mathbb{P}[f - Sw - (J_k w)\cdot\nabla w] \in C^{\mu}_{loc}((0,T_*]; L^2_\sigma(\Omega)), \quad \forall \mu \in (0,1/2) \text{ with } \mu \le \theta.$$

Therefore, $w(t)$ is a strong solution of class

$$w \in C([0,T_*]; L^2_\sigma(\Omega)) \cap C((0,T_*]; D_2(A)) \cap C^1((0,T_*]; L^2_\sigma(\Omega)).$$

In view of (4.13), it suffices to derive a priori estimate of strong solutions in $L^2(\Omega)$ for continuation of the solution globally in time. Let $\varepsilon > 0$. By (4.8) we have

$$\frac{1}{2}\frac{d}{dt}\|w(t)\|_2^2 + \|\nabla w(t)\|_2^2 = \langle (hu_s + \widetilde{U})\otimes w, \nabla w\rangle + \langle f, w\rangle. \tag{4.14}$$

We use Lemma 3.2 again to find that it is bounded from above by

$$\begin{aligned} &C\|f(t)\|^2_{H^{-1}(\Omega)} + C\left(1 + |h(t)|^2\|u_s\|_\infty^2 + \|\widetilde{U}_\varepsilon(t)\|_\infty^2\right)\|w(t)\|_2^2 \\ &+ \frac{1}{4}\|\nabla w(t)\|_2^2 + C_3\|\widetilde{U}_\varepsilon(t) - \widetilde{U}(t)\|_{3,\infty}\|\nabla w(t)\|_2^2. \end{aligned}$$

We choose $\varepsilon = \varepsilon_0$ such that $\sup_{t>0}\|\widetilde{U}_{\varepsilon_0}(t) - \widetilde{U}(t)\|_{3,\infty} \le 1/4C_3$ to conclude (4.9). □

Let $T \in (0,\infty)$. By (4.9) one can find a subsequence of $\{w_k\}$, which is denoted by itself, as well as a function

$$w \in L^\infty(0,T;L^2_\sigma(\Omega)) \cap L^2(0,T;H^1_{0,\sigma}(\Omega)) \tag{4.15}$$

so that

$$\begin{aligned} &w_k \to w \quad \text{weakly-star in } L^\infty(0,T;L^2_\sigma(\Omega)), \\ &w_k \to w \quad \text{weakly in } L^2(0,T;H^1_{0,\sigma}(\Omega)), \end{aligned} \tag{4.16}$$

as $k \to \infty$. Let us deduce further convergence of $\{w_k\}$

Lemma 4.2. *Let $T \in (0,\infty)$, and let w be the function obtained in* (4.15). *There is a subsequence of $\{w_k\}$, which we denote by itself, such that*

$$\lim_{k\to\infty} \sup_{0\le t\le T} |\langle w_k(t) - w(t), \phi\rangle| = 0, \qquad \forall \phi \in L^2_\sigma(\Omega), \tag{4.17}$$

$$\lim_{k\to\infty} \|w_k - w\|_{L^2(0,T;L^2(\Omega_L))} = 0, \qquad \forall L \in [R,\infty), \tag{4.18}$$

$$\lim_{k\to\infty} \|J_k w_k - w\|_{L^2(0,T;L^2(\Omega_L))} = 0, \qquad \forall L \in [R,\infty), \tag{4.19}$$

where $\Omega_L = \Omega \cap B_L$ and R is as in (3.2). *Furthermore, we have*

$$w \in C_w([0,T];L^2_\sigma(\Omega)), \tag{4.20}$$

$$\lim_{t\to 0} \|w(t) - w_0\|_2 = 0. \tag{4.21}$$

Proof. We first fix $\phi \in C^\infty_{0,\sigma}(\Omega)$. By (4.9) it is obvious that $\langle w_k, \phi\rangle$ is uniformly bounded. Let $0 \le s < t \le T$, then we see from (2.3), (2.4), (3.24), (4.2), (4.8) and (4.9) that

$$\begin{aligned} &|\langle w_k(t), \phi\rangle - \langle w_k(s), \phi\rangle| \\ &\le \int_s^t \Big[\|\nabla w_k\|_2\|\nabla\phi\|_2 + |h|_\infty(|u_\infty| + \|u_s\|_\infty)\|\nabla w_k\|_2\|\phi\|_2 \\ &\quad + \|\widetilde{U}\|_{3,\infty}\|\nabla w_k\|_2\|\phi\|_{6,2} + |h|_\infty\|\nabla u_s\|_\infty\|w_k\|_2\|\phi\|_2 + \|\nabla\widetilde{U}\|_{3,\infty}\|w_k\|_2\|\phi\|_{6,2} \\ &\quad + C\|w_k\|_2^{1/2}\|\nabla w_k\|_2^{3/2}\|\phi\|_6 + \|f\|_2\|\phi\|_2\Big] d\tau \\ &\le CY(T)^{1/2}\Big\{(\|\nabla\phi\|_2 + \|\phi\|_2)(t-s)^{1/2} + \|\phi\|_2(t-s) + \|\nabla\phi\|_2(t^{1/2} - s^{1/2})\Big\} \\ &\quad + CY(T)\|\nabla\phi\|_2(t-s)^{1/4} + C\|\phi\|_2(t^{1/4} - s^{1/4}). \end{aligned}$$

This shows that $\langle w_k, \phi\rangle$ is equi-continuous on $[0,T]$. By the Ascoli-Arzelà theorem, $\{\langle w_k, \phi\rangle\}$ contains a subsequence (dependent of $\phi \in C^\infty_{0,\sigma}(\Omega)$) which is uniformly convergent on $[0,T]$. Since $L^2_\sigma(\Omega)$ is separable, the diagonal method concludes that one can further take a subsequence of $\{w_k\}$ (independent of $\phi \in L^2_\sigma(\Omega)$), which is denoted by itself, such that (4.17) holds true. This immediately implies (4.20), and thereby $\|w_0\|_2^2 \le \liminf_{t\to 0}\|w(t)\|_2^2$. On the other hand, $\|w(t)\|_2^2$ is bounded from above by the RHS of (4.9), which implies that $\limsup_{t\to 0}\|w(t)\|_2^2 \le \|w_0\|_2^2$. We thus obtain (4.21).

Let $L \in [R,\infty)$, and fix a cut-off function $\psi \in C_0^\infty(B_{2L})$ satisfying $\psi = 1$ on B_L. We utilize the Friedrichs inequality ([8, p.489]) to see that, for every $\varepsilon > 0$, there are finite number of elements $\phi_1, \cdots, \phi_m \in L^2(\Omega_{2L})$ such that

$$\begin{aligned}
&\|w_k(t) - w(t)\|_{2,\Omega_L}^2 \\
&\le \|\psi(w_k(t) - w(t))\|_{2,\Omega_{2L}}^2 \\
&\le \varepsilon\|\nabla[\psi(w_k(t) - w(t))]\|_{2,\Omega_{2L}}^2 + \sum_{j=1}^m |\langle \psi(w_k(t) - w(t)), \phi_j\rangle|^2 .
\end{aligned}$$

Using (4.9), we find

$$\begin{aligned}
&\int_0^T \|w_k(t) - w(t)\|_{2,\Omega_L}^2\,dt \\
&\le C(1+T)Y(T)\,\varepsilon + \sum_{j=1}^m \int_0^T |\langle w_k(t) - w(t), \mathbb{P}(\psi\phi_j)\rangle|^2 .
\end{aligned}$$

By virtue of (4.17) with $\mathbb{P}(\psi\phi_j) \in L^2_\sigma(\Omega)$ we obtain

$$\limsup_{k\to\infty} \int_0^T \|w_k(t) - w(t)\|_{2,\Omega_L}^2\,dt \le C_T\varepsilon,$$

which yields (4.18). Finally, by (4.9) we have

$$\begin{aligned}
\int_0^T \|J_k w_k(t) - w_k(t)\|_{2,\Omega_L}^2\,dt &\le \int_0^T \left(\int_0^{1/k} \|\frac{d}{d\tau}e^{-\tau A}w_k(t)\|_2\,d\tau\right)^2 dt \\
&\le \frac{C}{k}\int_0^T \|\nabla w_k(t)\|_2^2\,dt.
\end{aligned}$$

This combined with (4.18) completes the proof of (4.19). □

We are in a position to provide a weak solution.

Proposition 4.1. *Problem* (1.11) *with* (4.1) *admits at least one weak solution.*

Proof. The solution w_k to (4.8) obtained in Lemma 4.1 fulfills

$$\begin{aligned}\langle w_k(t), \varphi(t)\rangle &+ \int_s^t \Big[\langle \nabla w_k, \nabla\varphi\rangle + \langle\{h(u_\infty + u_s) + \widetilde{U}\}\cdot\nabla w_k, \varphi\rangle \\ &- \langle (hu_s + \widetilde{U})\otimes w_k, \nabla\varphi\rangle + \langle (J_k w_k)\cdot\nabla w_k, \varphi\rangle\Big] d\tau \\ &= \langle w_k(s), \varphi(s)\rangle + \int_s^t \Big[\langle w_k, \partial_\tau\varphi\rangle + \langle f, \varphi\rangle\Big] d\tau\end{aligned}$$

for all $0 \le s < t < \infty$ and φ satisfying (4.7). It suffices to show (4.6) under the additional condition $\varphi \in L^\infty_{loc}([0,\infty); L^\infty(\Omega))$; in fact, (4.6) with $J_m\varphi\, (m = 1, 2, ...)$ implies (4.6) for general φ of class (4.7) by passing to the limit as $m \to \infty$. We fix $T \in (0, \infty)$, and let $0 \le s < t \le T$. As in the standard Navier-Stokes theory, it follows from (4.16) together with Lemma 4.2 that

$$\lim_{k\to\infty} \int_s^t \langle (J_k w_k)\cdot\nabla w_k, \varphi\rangle d\tau = \int_s^t \langle w\cdot\nabla w, \varphi\rangle d\tau. \tag{4.22}$$

Indeed, for every $\varepsilon > 0$, one can take $L = L(\varepsilon, T) \in [R, \infty)$ so large, independent of k on account of (4.9), that

$$\begin{aligned}&\left|\int_s^t \langle (J_k w_k - w)\cdot\nabla w_k, (1 - \chi_{B_L})\varphi\rangle d\tau\right| \\ &\le CY(T)\left(\int_0^T \|\varphi(\tau)\|^4_{6,\mathbb{R}^3\setminus B_L}\, d\tau\right)^{1/4} \le \varepsilon,\end{aligned}$$

where χ_{B_L} stands for the characteristic function on B_L. We then find from (4.19) that

$$\begin{aligned}&\limsup_{k\to\infty}\left|\int_s^t \langle (J_k w_k - w)\cdot\nabla w_k, \varphi\rangle d\tau\right| \\ &\le \varepsilon + \lim_{k\to\infty}\left|\int_s^t \langle (J_k w_k - w)\cdot\nabla w_k, \chi_{B_L}\varphi\rangle d\tau\right| = \varepsilon,\end{aligned}$$

which yields (4.22). Given $\varepsilon > 0$, we take $\widetilde{U}_\varepsilon$ in Lemma 3.2. Then we have

$$\begin{aligned}&\left|\int_s^t \langle \widetilde{U}\otimes(w_k - w), \nabla\varphi\rangle d\tau\right| \\ &\le \left|\int_s^t \langle \widetilde{U}_\varepsilon\otimes(w_k - w), \nabla\varphi\rangle d\tau\right| + C\varepsilon\, Y(T)^{1/2}\|\nabla\varphi\|_{L^2(0,T;L^2(\Omega))}.\end{aligned}$$

Since $\sum_j \widetilde{U}_{\varepsilon,j}(\nabla\varphi_j) \in L^1(0, T; L^2(\Omega))$ and since $\varepsilon > 0$ is arbitrary, it follows from (4.16) that

$$\lim_{k\to\infty} \int_s^t \langle \widetilde{U}\otimes(w_k - w), \nabla\varphi\rangle d\tau = 0.$$

The convergence of the other terms is easily verified. Thus the function w obtained in (4.15) satisfies (4.6).

It remains to show (1.12) for $s=0$. By (4.14) we have

$$\begin{aligned}&\frac{1}{2}\|w_k(t)\|_2^2+\int_0^t\|\nabla w_k\|_2^2 d\tau\\&=\frac{1}{2}\|w_0\|_2^2+\int_0^t\big[\langle(hu_s+\widetilde{U})\otimes w_k,\nabla w_k\rangle+\langle f,w_k\rangle\big]d\tau\end{aligned}$$

for all $t\geq 0$ and it suffices to prove

$$\lim_{k\to\infty}\int_0^t\langle(hu_s+\widetilde{U})\otimes w_k,\nabla w_k\rangle d\tau=\int_0^t\langle(hu_s+\widetilde{U})\otimes w,\nabla w\rangle d\tau. \tag{4.23}$$

We fix $T\in(0,\infty)$, and let $t\in(0,T)$. We also fix $\varepsilon>0$ arbitrarily and use the function $\widetilde{U}_\varepsilon$ in Lemma 3.2 again to obtain

$$\left|\int_0^t\langle(\widetilde{U}-\widetilde{U}_\varepsilon)\otimes(w_k-w),\nabla w_k\rangle d\tau\right|\leq CY(T)\varepsilon.$$

One can choose $L=L(\varepsilon,T)\in[R,\infty)$, independent of k, such that

$$\begin{aligned}&\left|\int_0^t\langle(1-\chi_{B_L})(hu_s+\widetilde{U}_\varepsilon)\otimes(w_k-w),\nabla w_k\rangle d\tau\right|\\&\leq CY(T)\left(\int_0^T\left\{\|u_s\|_{6,\mathbb{R}^3\setminus B_L}+\|\widetilde{U}_\varepsilon(\tau)\|_{6,\mathbb{R}^3\setminus B_L}\right\}^4 d\tau\right)^{1/4}\leq\varepsilon.\end{aligned}$$

Hence, we obtain from (4.18) that

$$\begin{aligned}&\limsup_{k\to\infty}\left|\int_0^t\langle(hu_s+\widetilde{U})\otimes(w_k-w),\nabla w_k\rangle d\tau\right|\\&\leq(CY(T)+1)\,\varepsilon+\lim_{k\to\infty}\int_0^T\|hu_s+\widetilde{U}_\varepsilon\|_\infty\|\chi_{B_L}(w_k-w)\|_2\|\nabla w_k\|_2 d\tau\\&=(CY(T)+1)\,\varepsilon.\end{aligned}\tag{4.24}$$

On the other hand, since

$$\|(hu_s+\widetilde{U})\otimes w\|_2\leq C(|h|_\infty\|u_s\|_3+\|\widetilde{U}\|_{3,\infty})\|\nabla w\|_2\in L^2(0,T),$$

we have

$$\lim_{k\to\infty}\int_0^t\langle(hu_s+\widetilde{U})\otimes w,\nabla w_k-\nabla w\rangle d\tau=0.$$

This together with (4.24) concludes (4.23). □

We conclude this section with the proof of the strong energy inequality (1.12).

Proposition 4.2. *The solution obtained in Proposition 4.1 enjoys* (1.12) *for* $s = 0$*, a.e.* $s > 0$ *and all* $t \geq s$*.*

Proof. The case $s = 0$ has been already shown in the proof of Proposition 4.1. Let $T \in (0, \infty)$. To consider the other case $s \in (0, T)$, let us take a subsequence of $\{w_k\}$, which is still denoted by itself, and a set $J \subset (0, T)$ with the Lebesgue measure $|J| = 0$ such that

$$\lim_{k\to\infty} \|w_k(t) - w(t)\|_{2,\Omega_L} = 0, \qquad \forall L \in [R, \infty),\, \forall t \in (0, T) \setminus J, \tag{4.25}$$

where $\Omega_L = \Omega \cap B_L$ and R is as in (3.2). This is in fact verified as follows: For each $i = 1, 2, ...$, it follows from (4.18) that one can take a subsequence of $\{w_k\}$, denoted by itself, and a set $J_i \subset (0, T)$ with $|J_i| = 0$ such that

$$\lim_{k\to\infty} \|w_k(t) - w(t)\|_{2,\Omega_{R+i}} = 0, \qquad \forall t \in (0, T) \setminus J_i.$$

Then, by the diagonal method, we are led to (4.25) for a suitable subsequence of $\{w_k\}$, where $J = \cup_{i=1}^{\infty} J_i$.

Let us go back to the approximate problem (4.8) together with the pressure p_k associated with the strong solution w_k obtained in Lemma 4.1:

$$\begin{aligned}
&\partial_t w_k + (J_k w_k) \cdot \nabla w_k + S w_k = \Delta w_k - \nabla p_k + f, \\
&\mathrm{div}\; w_k = 0, \\
&w_k|_{\partial\Omega} = 0, \\
&w_k \to 0 \quad \text{as } |x| \to \infty, \\
&w_k(\cdot, 0) = w_0.
\end{aligned} \tag{4.26}$$

In order to control the behavior of the pressure p_k at infinity uniformly in k, it is convenient to split the solution w_k into three parts

$$w_k = w_k^1 + w_k^2 + w_k^3, \qquad p_k = p_k^1 + p_k^2 + p_k^3,$$

where

$$\partial_t w_k^1 - \Delta w_k^1 + \nabla p_k^1 = -h u_\infty \cdot \nabla w_k + f, \quad w_k^1(\cdot, 0) = w_0, \tag{4.27}$$

$$\partial_t w_k^2 - \Delta w_k^2 + \nabla p_k^2 = -(J_k w_k) \cdot \nabla w_k, \quad w_k^2(\cdot, 0) = 0, \tag{4.28}$$

$$\partial_t w_k^3 - \Delta w_k^3 + \nabla p_k^3 = -(h u_s + \widetilde{U}) \cdot \nabla w_k - w_k \cdot \nabla (h u_s + \widetilde{U}), \quad w_k^3(\cdot, 0) = 0, \tag{4.29}$$

subject to

$$\operatorname{div} w_k^j = 0, \qquad w_k^j|_{\partial\Omega} = 0, \qquad w_k^j \to 0 \quad \text{as } |x| \to \infty$$

for $j = 1, 2, 3$.

Let us begin with (4.27). By the standard energy method together with (4.9), (4.4) and the Gronwall argument, we have

$$\|w_k^1(t)\|_2^2 + \int_0^t \|\nabla w_k^1\|_2^2 d\tau \le CY(T)e^T, \tag{4.30}$$

and

$$\|\nabla w_k^1(t)\|_2^2 + \int_s^t \|Aw_k^1\|_2^2 d\tau \le \|\nabla w_k^1(s)\|_2^2 + 2\int_s^t \|f\|_2^2 d\tau + CY(T),$$

for $0 < s < t \le T$. Integration of the latter with respect to s over $(0, t)$ together with (4.2) and (4.30) yield

$$t\|\nabla w_k^1(t)\|_2^2 + \int_0^t \tau\|Aw_k^1\|_2^2 d\tau \le C_T,$$

which implies

$$\int_s^t \|Aw_k^1\|_2^2 d\tau \le \frac{C_T}{s} \tag{4.31}$$

for $0 < s < t \le T$. In view of the equation of (4.27) and by use of estimate $\|\nabla^2 g\|_2 \le C(\|Ag\|_2 + \|\nabla g\|_2)$ for $g \in D_2(A)$ (see Heywood [24]), we gather (4.2), (4.9), (4.30) and (4.31) to find

$$\sup_k \int_s^T \|\nabla p_k^1\|_2^2 d\tau < \infty.$$

By the embedding relation, there are constants c_k^1 $(k = 1, 2, ...)$ such that

$$\sup_k \int_s^T \|p_k^1 + c_k^1\|_6^2 d\tau < \infty.$$

Hence, one finds a subsequence of $\{p_k^1\}$ (dependent of each $s \in (0, T)$), which one denotes by itself, as well as $p^1 \in L^2(s, T; L^6(\Omega))$ with $\nabla p^1 \in L^2(s, T; L^2(\Omega))$ so that

$$\begin{aligned} &p_k^1 + c_k^1 \to p^1 \quad \text{weakly in } L^2(s, T; L^6(\Omega)), \\ &\nabla p_k^1 \to \nabla p^1 \quad \text{weakly in } L^2(s, T; L^2(\Omega)), \end{aligned} \tag{4.32}$$

as $k \to \infty$.

We next consider (4.28), but this part is exactly the same as in [38]. From (4.9) we deduce

$$\sup_k \int_0^T \|(J_k w_k) \cdot \nabla w_k\|_{5/4}^{5/4} \, d\tau < \infty.$$

Then the maximal regularity for the Stokes system (see Solonnikov [43], Giga and Sohr [22]) leads to

$$\sup_k \int_0^T \|p_k^2 + c_k^2\|_{15/7}^{5/4} \, d\tau \le C \sup_k \int_0^T \|\nabla p_k^2\|_{5/4}^{5/4} \, d\tau < \infty \tag{4.33}$$

for some constants $c_k^2 \, (k = 1, 2, ...)$.

We turn to (4.29). We fix $q \in (1, 2)$ and take $p \in (3, \infty)$ satisfying $3/2p + 1/q > 1$. By (4.9) and by (3.22)–(3.23) we see that

$$\begin{aligned}
&\|(hu_s + \widetilde{U}) \cdot \nabla w_k + w_k \cdot \nabla(hu_s + \widetilde{U})\|_r \\
&\le \big(|h|_\infty \|u_s\|_p + \|\widetilde{U}\|_p\big) \|\nabla w_k\|_2 + \big(|h|_\infty \|\nabla u_s\|_p + \|\nabla \widetilde{U}\|_p\big) \|w_k\|_2 \\
&\le C\big(1 + \tau^{-1/2+3/2p}\big) \|\nabla w_k\|_2 + C\big\{1 + \tau^{-1+3/2p}(1+T)^{1/2-3/2p}\big\} Y(T)^{1/2},
\end{aligned}$$

for $\tau \in (0, T)$, where $r \in (6/5, 2)$ satisfies $1/r = 1/p + 1/2$, and therefore

$$\sup_k \int_0^T \|(hu_s + \widetilde{U}) \cdot \nabla w_k + w_k \cdot \nabla(hu_s + \widetilde{U})\|_r^q \, d\tau < \infty.$$

By the same reasoning as above, we obtain

$$\sup_k \int_0^T \|p_k^3 + c_k^3\|_{r_*}^q \, d\tau \le C \sup_k \int_0^T \|\nabla p_k^3\|_r^q \, d\tau < \infty, \tag{4.34}$$

for some constants $c_k^3 \, (k = 1, 2, ...)$, where $1/r_* = 1/r - 1/3$.

We now fix $s \in (0, T) \setminus J$, and let $t \in (s, T]$, where J is as in (4.25). We take a cut-off function $\psi \in C_0^\infty(B_2)$ such that $\psi = 1$ on B_1 as well as $\psi \ge 0$, and set $\psi_L(x) = \psi(x/L)$ for $L \ge R$, where R is as in (3.2). We multiply the equation of (4.26) by $\psi_L w_k$ and integrate the resulting formula over $\Omega \times (s, t)$

to find

$$
\begin{aligned}
&\frac{1}{2}\|\sqrt{\psi_L}w_k(t)\|_2^2+\int_s^t\Big(\|\sqrt{\psi_L}\nabla w_k\|_2^2+\langle\nabla p_k^1,\psi_L w_k\rangle\Big)d\tau\\
&=\frac{1}{2}\|\sqrt{\psi_L}w_k(s)\|_2^2+\int_s^t\Big(-\langle\nabla\psi_L\cdot\nabla w_k,w_k\rangle\\
&\quad+\langle p_k^2+c_k^2,w_k\cdot\nabla\psi_L\rangle+\langle p_k^3+c_k^3,w_k\cdot\nabla\psi_L\rangle\\
&\quad+\left\langle\frac{|w_k|^2}{2},\{J_kw_k+h(u_\infty+u_s)+\widetilde{U}\}\cdot\nabla\psi_L\right\rangle\\
&\quad+\langle(hu_s+\widetilde{U})\cdot w_k,w_k\cdot\nabla\psi_L\rangle+\langle(hu_s+\widetilde{U})\otimes w_k,(\nabla w_k)\psi_L\rangle\\
&\quad+\langle f,\psi_Lw_k\rangle\Big)d\tau.
\end{aligned}
\tag{4.35}
$$

On account of (4.33) and (4.34), we observe

$$
\begin{aligned}
&\left|\int_s^t\langle p_k^2+c_k^2,w_k\cdot\nabla\psi_L\rangle+\langle p_k^3+c_k^3,w_k\cdot\nabla\psi_L\rangle d\tau\right|\\
&\le C_T\|w_k\cdot\nabla\psi_L\|_{L^5(0,T;L^{15/8}(\Omega))}+C_T\|w_k\cdot\nabla\psi_L\|_{L^{q'}(0,T;L^{(r_*)'}(\Omega))}\\
&\le C_T\big(\|\nabla\psi_L\|_{30}+\|\nabla\psi_L\|_\sigma\big),
\end{aligned}
\tag{4.36}
$$

where $1/q'+1/q=1$, $1/(r_*)'+1/r_*=1$ and $1/\sigma=5/6-1/r$. Note that $\sigma\in(3,\infty)$. Making use of (2.3), (2.4), (3.24) and (4.9), we find

$$
\begin{aligned}
&\Big|\int_s^t\Big(-\langle\nabla\psi_L\cdot\nabla w_k,w_k\rangle+\left\langle\frac{|w_k|^2}{2},\{J_kw_k+h(u_\infty+u_s)+\widetilde{U}\}\cdot\nabla\psi_L\right\rangle\\
&\qquad+\langle(hu_s+\widetilde{U})\cdot w_k,w_k\cdot\nabla\psi_L\rangle\Big)d\tau\Big|\\
&\le C\int_s^t\Big[\Big(1+|h|_\infty\|u_s\|_3+\|\widetilde{U}\|_{3,\infty}\Big)\|w_k\|_2\|\nabla w_k\|_2+|h|_\infty|u_\infty|\|w_k\|_2^2\\
&\qquad+\|w_k\|_2^{3/2}\|\nabla w_k\|_2^{3/2}\Big]d\tau\,\|\nabla\psi_L\|_\infty\\
&\le C\Big\{Y(T)(T^{1/2}+T)+Y(T)^{3/2}T^{1/4}\Big\}\|\nabla\psi_L\|_\infty,
\end{aligned}
\tag{4.37}
$$

from which together with (4.36), we see that (4.35) yields

$$
\begin{aligned}
&\frac{1}{2}\|\sqrt{\psi_L}w_k(t)\|_2^2+\int_s^t\Big(\|\sqrt{\psi_L}\nabla w_k\|_2^2+\langle\nabla p_k^1,\psi_Lw_k\rangle\Big)d\tau\\
&\le\frac{1}{2}\|\sqrt{\psi_L}w_k(s)\|_2^2+\int_s^t\Big(\langle(hu_s+\widetilde{U})\otimes w_k,(\nabla w_k)\psi_L\rangle+\langle f,\psi_Lw_k\rangle\Big)d\tau\\
&\quad+C_T(\|\nabla\psi_L\|_{30}+\|\nabla\psi_L\|_\sigma+\|\nabla\psi_L\|_\infty).
\end{aligned}
\tag{4.38}
$$

We now let $k \to \infty$ along the subsequence above. Since $s \in (0,T) \setminus J$, we know by (4.25) that $\lim_{k\to\infty} \|\sqrt{\psi_L} w_k(s)\|_2 = \|\sqrt{\psi_L} w(s)\|_2$. We split

$$\int_s^t \Big(\langle (hu_s + \widetilde{U}) \otimes w_k, (\nabla w_k)\psi_L \rangle - \langle (hu_s + \widetilde{U}) \otimes w, (\nabla w)\psi_L \rangle \Big) d\tau$$

into two parts $I + II$, where

$$\begin{aligned} |I| &= \left| \int_s^t \langle (hu_s + \widetilde{U}) \otimes (w_k - w), (\nabla w_k)\psi_L \rangle \right| \\ &\le CY(T)^{1/2} \left(|h|_\infty \|u_s\|_\infty + \sup_{s\le\tau\le t} \|\widetilde{U}(\tau)\|_\infty \right) \|w_k - w\|_{L^2(0,T;L^2(\Omega_{2L}))}, \end{aligned}$$

while

$$|II| = \left| \int_s^t \langle (hu_s + \widetilde{U}) \otimes w, (\nabla w_k - \nabla w)\psi_L \rangle \right| \to 0 \quad (k \to \infty)$$

is easily verified by (4.16). Since $\|\widetilde{U}(\tau)\|_\infty \le Cs^{-1/2}$ for $\tau \ge s > 0$ by (3.22), Lemma 4.2 implies that $\lim_{k\to\infty} I = 0$, too. From (4.16), (4.17), (4.18) and (4.32) as well as the observation above we deduce that (4.38) leads to

$$\begin{aligned} &\frac{1}{2}\|\sqrt{\psi_L} w(t)\|_2^2 + \int_s^t \Big(\|\sqrt{\psi_L}\nabla w\|_2^2 + \langle \nabla p^1, \psi_L w \rangle \Big) d\tau \\ &\le \frac{1}{2}\|\sqrt{\psi_L} w(s)\|_2^2 + \int_s^t \Big(\langle (hu_s + \widetilde{U}) \otimes w, (\nabla w)\psi_L \rangle + \langle f, \psi_L w \rangle \Big) d\tau \\ &\quad + C_T(\|\nabla\psi_L\|_{30} + \|\nabla\psi_L\|_\sigma + \|\nabla\psi_L\|_\infty). \end{aligned} \tag{4.39}$$

Here, we have

$$\begin{aligned} \left| \int_s^t \langle \nabla p^1, \psi_L w \rangle d\tau \right| &= \left| - \int_s^t \langle p^1, w \cdot \nabla\psi_L \rangle d\tau \right| \\ &\le \|\nabla\psi_L\|_3 \int_s^t \|w\|_{L^2(A_L)} \|p^1\|_{L^6(A_L)} d\tau, \end{aligned}$$

where $A_L = B_{2L} \setminus \overline{B_L}$. By the Lebesgue convergence theorem, we see that $\int_s^t \cdots \to 0$ as $L \to \infty$. Therefore, by passing to the limit as $L \to \infty$ in (4.39), we arrive at (1.12) for all $s \in (0,T) \setminus J$ and $t \in (s,T]$. $\square$

5 Strong solution

Let $\bar{t} \in (T_0, \infty)$, where T_0 is as in (1.6) (resp. (1.13)) for the starting (resp. landing) problem. In this section we construct a strong solution to (1.11) with (4.1) on the interval $[\bar{t},\infty)$ under a certain smallness condition on $w(\bar{t})$. And then, it is identified on $[\bar{t},\infty)$ with the weak solution obtained in the previous section.

The first two propositions in this section are independent of the argument in the previous section. By (1.6) problem (1.11) on $[\bar{t},\infty)$ is formally converted into the integral equation

$$w = \Psi w \qquad (t \geq \bar{t}) \tag{5.1}$$

with

$$\begin{aligned}
(\Psi w)(t) = H(t) - \int_{\bar{t}}^{t} T(t-\tau)\mathbb{P}[(u_s + \widetilde{U})\cdot \nabla w \\
+ w\cdot\nabla(u_s+\widetilde{U}) + w\cdot\nabla w](\tau)d\tau, \\
H(t) = T(t-\bar{t})w(\bar{t}) + H_f(t), \\
H_f(t) = \int_{\bar{t}}^{t} T(t-\tau)\mathbb{P}f(\tau)d\tau,
\end{aligned}$$

where the term $u_s\cdot\nabla w + w\cdot\nabla u_s$ is absent for the landing problem and

$$T(t) = \begin{cases} e^{-tA_{u_\infty}} & \text{(starting problem)}, \\ e^{-tA} & \text{(landing problem)}. \end{cases}$$

We take a small $w(\bar{t})$ from $L^3_\sigma(\Omega)$ and look for a solution in a closed ball

$$E_\rho = \{w \in E; \|w\|_E \leq \rho\} \tag{5.2}$$

of the Banach space

$$\begin{aligned}
E =& \{w \in C((\bar{t},\infty); L^6_\sigma(\Omega)\cap L^\infty(\Omega));\ \nabla w \in C((\bar{t},\infty); L^3(\Omega)), \\
& \|w\|_E := \sup_{t\in(\bar{t},\infty)} \phi_w(t) < \infty,\ \lim_{t\to\bar{t}+0}\phi_w(t) = 0\}
\end{aligned} \tag{5.3}$$

endowed with norm $\|\cdot\|_E$, where

$$\phi_w(t) := (t-\bar{t})^{1/2}\big(\|w(t)\|_\infty + \|\nabla w(t)\|_3\big) + (t-\bar{t})^{1/4}\|w(t)\|_6.$$

Since we need the smallness of $|u_\infty|$ to get a unique steady flow u_s for the starting problem, see (3.1), we may assume at the beginning that $|u_\infty| \leq \delta_0$. This is not needed for the landing problem.

Let us start with the following lemma on $H_f(t)$.

Lemma 5.1. *Let*

$$\frac{4}{3} < q < \frac{3}{2} < r < 3.$$

Then we have $H_f \in E$ and

$$\|H_f\|_E \le \gamma(1 + \|\nabla u_s\|_q + \|\nabla u_s\|_r)(\|U(\bar{t})\|_{3,\infty,\mathbb{R}^3} + \|U(\bar{t})\|^2_{3,\infty,\mathbb{R}^3}), \qquad (5.4)$$

with some constant $\gamma = \gamma(q, r) > 0$. For the landing problem, the term $\|\nabla u_s\|_q + \|\nabla u_s\|_r$ is absent.

Proof. We derive only (5.4) since the continuity in t (as in (4.11)) and $\lim_{t\to\bar{t}+0} \phi_{H_f}(t) = 0$ are easily verified so that $H_f \in E$ (the latter follows from the fact that $f(t)$ does not possess any singular behavior near $t = \bar{t}$). We divide the external force, see (4.1), into two parts:

$$f = f_0 - \widetilde{U} \cdot \nabla\widetilde{U}, \quad f_0 = F - (u_s \cdot \nabla\widetilde{U} + \widetilde{U} \cdot \nabla u_s) \quad (t \ge \bar{t}).$$

By (3.26) we obtain

$$\|u_s \cdot \nabla\widetilde{U} + \widetilde{U} \cdot \nabla u_s\|_p \le C(t - \bar{t})^{-1/2}\|\nabla u_s\|_p\|U(\bar{t})\|_{3,\infty,\mathbb{R}^3}$$

for all $t > \bar{t}$ and $p \in (4/3, 3)$, which combined with (3.32) leads to

$$\|f_0(t)\|_p \le Cm_p(t - \bar{t})^{-1/2}\|U(\bar{t})\|_{3,\infty,\mathbb{R}^3} \qquad (5.5)$$

for the same p as above, where we put $m_p = 1 + \|\nabla u_s\|_p$ for notational simplicity ($m_p = 1$ for the landing problem). We fix q and r such that

$$\frac{4}{3} < q < \frac{3}{2} < r < 3$$

and split $H_{f_0}(t)$ into

$$H_{f_0}(t) = \left(\int_{\bar{t}}^{(\bar{t}+t)/2} + \int_{(\bar{t}+t)/2}^{t}\right) T(t - \tau)\mathbb{P}f_0(\tau)d\tau =: H_{f_0,1}(t) + H_{f_0,2}(t).$$

We are going to employ (2.5) and (2.6). From (5.5) we deduce

$$\begin{aligned}\|H_{f_0,1}(t)\|_\infty + \|\nabla H_{f_0,1}(t)\|_3 &\le C\int_{\bar{t}}^{(\bar{t}+t)/2} (t - \tau)^{-1}\|f_0(\tau)\|_{3/2}d\tau \\ &\le Cm_{3/2}(t - \bar{t})^{-1/2}\|U(\bar{t})\|_{3,\infty,\mathbb{R}^3} \qquad (t > \bar{t})\end{aligned}$$

and

$$\begin{aligned}
&\|H_{f_0,2}(t)\|_\infty + \|\nabla H_{f_0,2}(t)\|_3 \\
&\leq C\int_{(\bar{t}+t)/2}^t (t-\tau)^{-3/2r}\|f_0(\tau)\|_r d\tau \\
&\leq Cm_r(t-\bar{t})^{1/2-3/2r}\|U(\bar{t})\|_{3,\infty,\mathbb{R}^3} \qquad (\bar{t} < t \leq \bar{t}+2).
\end{aligned}$$

To estimate $H_{f_0,2}(t)$ for $t > \bar{t}+2$, we further split it into

$$H_{f_0,2}(t) = \int_{(\bar{t}+t)/2}^{t-1} + \int_{t-1}^t =: H_{f_0,21}(t) + H_{f_0,22}(t).$$

Then we find

$$\begin{aligned}
\|H_{f_0,21}(t)\|_\infty + \|\nabla H_{f_0,21}(t)\|_3 &\leq C\int_{(\bar{t}+t)/2}^{t-1} (t-\tau)^{-3/2q}\|f_0(\tau)\|_q d\tau \\
&\leq Cm_q(t-\bar{t})^{-1/2}\|U(\bar{t})\|_{3,\infty,\mathbb{R}^3} \qquad (t > \bar{t}+2),
\end{aligned}$$

and

$$\begin{aligned}
\|H_{f_0,22}(t)\|_\infty + \|\nabla H_{f_0,22}(t)\|_3 &\leq C\int_{t-1}^t (t-\tau)^{-3/2r}\|f_0(\tau)\|_r d\tau \\
&\leq Cm_r(t-\bar{t})^{-1/2}\|U(\bar{t})\|_{3,\infty,\mathbb{R}^3} \qquad (t > \bar{t}+2).
\end{aligned}$$

It is easy to estimate $\|H_{f_0}(t)\|_6$ without any splitting by use of (5.5) for $p = 3/2$. The other term $\widetilde{U}\cdot\nabla\widetilde{U}$ should be treated separately because it does not belong to $L^q(\Omega)$ with $q \leq 3/2$; however, the treatment is easier without any splitting on account of the faster decay

$$\|\widetilde{U}\cdot\nabla\widetilde{U}\|_2 \leq C(t-\bar{t})^{-3/4}\|U(\bar{t})\|_{3,\infty,\mathbb{R}^3}^2 \qquad (t > \bar{t})$$

which follows from (2.2), (2.3) and (3.26). The proof is complete. $\square$

The following proposition provides a solution to (5.1) with some decay properties. Indeed we know by (3.18) that (5.8) below is accomplished for large $\bar{t}$, but this will be taken into consideration together with the other smallness condition (5.7) in the proof of the main theorems.

Proposition 5.1. *Let*

$$\frac{4}{3} < q < \frac{3}{2} < r < 3.$$

There are constants $\delta_j = \delta_j(q,r) > 0$ ($j = 1,3$) and $\delta_2 > 0$ (independent of q, r) such that if

$$|u_\infty| \leq \delta_0, \qquad \|\nabla u_s\|_q + \|\nabla u_s\|_r \leq \delta_1, \tag{5.6}$$

$$w(\bar{t}) \in L^3_\sigma(\Omega), \qquad \|w(\bar{t})\|_3 \le \delta_2, \tag{5.7}$$

$$\|U(\bar{t})\|_{3,\infty,\mathbb{R}^3} \le \delta_3, \tag{5.8}$$

where δ_0 *is as in* (3.1)*, then equation* (5.1) *admits a unique solution*

$$w \in E \cap C([\bar{t}, \infty); L^3_\sigma(\Omega)), \tag{5.9}$$

see (5.3)*, subject to*

$$\lim_{t \to \bar{t}+0} \|w(t) - w(\bar{t})\|_3 = 0, \qquad \|w(t)\|_3 \le C\|w(\bar{t})\|_3 \quad (t \ge \bar{t}).$$

For the landing problem, the condition (5.6) *is redundant.*

Proof. We follow the method of Kato [27] by use of (2.5) and (2.6). Let $w \in E$. Then the continuity of Ψw (as in (4.11)) and $\lim_{t\to\bar{t}+0} \phi_{\Psi w}(t) = 0$ as the properties of elements of E are easily verified. By using

$$\nabla u_s \in L^q(\Omega) \cap L^r(\Omega), \qquad u_s \in L^{q_*}(\Omega) \cap L^{r_*}(\Omega),$$

where $1/q_* = 1/q - 1/3$ and $1/r_* = 1/r - 1/3$, and by splitting the integral over $(\bar{t}, t)$ in the same way as in the proof of Lemma 5.1 (see also Chen [7], Enomoto and Shibata [10]), the term $u_s \cdot \nabla w + w \cdot \nabla u_s$ can be treated. From this together with (5.4) (in which $\|U(\bar{t})\|^2_{3,\infty,\mathbb{R}^3}$ is replaced just by $\|U(\bar{t})\|_{3,\infty,\mathbb{R}^3}$ if assuming that it is less than one, see (5.8) with (5.10) below) and (3.26) we deduce

$$\begin{aligned}\|\Psi w\|_E \le c_1\|w(\bar{t})\|_3 &+ 2\gamma(1 + \|\nabla u_s\|_q + \|\nabla u_s\|_r)\|U(\bar{t})\|_{3,\infty,\mathbb{R}^3} \\ &+ c_2\big(\|\nabla u_s\|_q + \|\nabla u_s\|_r + \|U(\bar{t})\|_{3,\infty,\mathbb{R}^3}\big)\|w\|_E + c_3\|w\|_E^2,\end{aligned}$$

where the only term one uses the Lorentz norm is $w \cdot \nabla\widetilde{U}$, that is,

$$\|w \cdot \nabla\widetilde{U}\|_{2,6} \le \|w\|_6\|\nabla\widetilde{U}\|_{3,\infty},$$

see (2.3), which is combined with $L^{2,6}$-L^r estimate ($r = 3, 6, \infty$) of the semigroup; indeed, such estimate is a simple consequence of (2.5) and (2.6) by real interpolation. Similarly, we have

$$\begin{aligned}\|\Psi w_1 - \Psi w_2\|_E \le c_2\big(&\|\nabla u_s\|_q + \|\nabla u_s\|_r + \|U(\bar{t})\|_{3,\infty,\mathbb{R}^3}\big)\|w_1 - w_2\|_E \\ &+ c_3(\|w_1\|_E + \|w_2\|_E)\|w_1 - w_2\|_E\end{aligned}$$

for $w_1, w_2 \in E$, where c_2 and c_3 are the same constants as above. Let us take

$$\rho = 2\Big\{c_1\|w(\bar{t})\|_3 + 2\gamma(1 + \|\nabla u_s\|_q + \|\nabla u_s\|_r)\|U(\bar{t})\|_{3,\infty,\mathbb{R}^3}\Big\}$$

and $w \in E_\rho$, see (5.2). We set

$$\delta_1 = \frac{1}{8c_2}, \qquad \delta_2 = \frac{1}{16c_1c_3}, \qquad \delta_3 = \min\left\{\delta_1,\ \frac{1}{32\gamma(1+\delta_1)c_3}\right\}. \tag{5.10}$$

Then the conditions (5.6), (5.7) and (5.8) imply $\rho \le 1/4c_3$, so that

$$\begin{aligned} &\|\Psi w\|_E \le \rho \qquad \text{for } w \in E_\rho, \\ &\|\Psi w_1 - \Psi w_2\|_E \le \frac{3}{4}\|w_1 - w_2\|_E \qquad \text{for } w_1, w_2 \in E_\rho. \end{aligned}$$

We thus obtain a unique solution $w \in E_\rho$ to (5.1). The proof of additional properties of $w(t)$ in the statement is standard and may be omitted. $\square$

Indeed the solution obtained in Proposition 5.1 is a strong solution with values in $L^3_\sigma(\Omega)$, but we need the following L^2-strong solution for later use rather than the L^3-strong solution.

Proposition 5.2. *Let $w(t)$ be the solution to* (5.1) *obtained in Proposition 5.1. We further assume that $w(\bar{t}) \in L^2_\sigma(\Omega)$.*

1. *The solution is of class*

$$w \in C([\bar{t},\infty); L^2_\sigma(\Omega)) \cap C((\bar{t},\infty); D_2(A)) \cap C^1((\bar{t},\infty); L^2_\sigma(\Omega)) \tag{5.11}$$

subject to $\lim_{t\to\bar{t}+0} \|w(t) - w(\bar{t})\|_2 = 0$. It also satisfies the equation

$$\partial_t w + Aw + \mathbb{P}[(u_\infty + u_s + \widetilde{U})\cdot\nabla w + w\cdot\nabla(u_s + \widetilde{U}) + w\cdot\nabla w] = \mathbb{P}f \tag{5.12}$$

in $L^2_\sigma(\Omega)$ and the energy equality

$$\begin{aligned} &\frac{1}{2}\|w(t)\|_2^2 + \int_{\bar{t}}^t \|\nabla w\|_2^2 d\tau \\ &= \frac{1}{2}\|w(\bar{t})\|_2^2 + \int_{\bar{t}}^t \Big[\langle (u_s + \widetilde{U})\otimes w, \nabla w\rangle + \langle f, w\rangle\Big] d\tau \end{aligned} \tag{5.13}$$

for all $t \ge \bar{t}$ as well as

$$\nabla w \in L^2_{loc}([\bar{t},\infty); L^2(\Omega)). \tag{5.14}$$

For the landing problem, the steady flow u_s is absent in (5.12) *and* (5.13).

2. *If, in addition, $w(\bar{t}) \in H^1_{0,\sigma}(\Omega)$, then we have*

$$\begin{aligned} &w \in L^2_{loc}([\bar{t},\infty); L^\infty(\Omega)), \qquad \nabla w \in L^\infty_{loc}([\bar{t},\infty); L^2(\Omega)), \\ &\partial_t w,\ Aw \in L^2_{loc}([\bar{t},\infty); L^2_\sigma(\Omega)). \end{aligned} \tag{5.15}$$

Proof. Concerning the first assertion, it suffices to show that

$$\mathbb{P}[f - (u_s + \widetilde{U}) \cdot \nabla w - w \cdot \nabla(u_s + \widetilde{U}) - w \cdot \nabla w](t)$$

is locally Hölder continuous in t on the interval $(\bar{t}, \infty)$ with values in $L^2_\sigma(\Omega)$ as well as summable near $t = \bar{t}$ with values there. The latter is obvious, for $\|f(t)\|_2$, $\|\widetilde{U}(t)\|_6$ and $\|\nabla\widetilde{U}(t)\|_{3,\infty}$ do not possess any singular behavior near $t = \bar{t}$, see (3.22), (3.24) and (4.2). It is easy to verify the Hölder continuity locally on $(\bar{t}, \infty)$ of $w(t)$ with values in $L^q_\sigma(\Omega)$ for $q \geq 3$ and that of $\nabla w(t)$ with values in $L^3(\Omega)$. This together with (3.13) and (4.5) lead to the desired result.

For deduction of the second assertion, we use the standard energy method for (5.12) combined with

$$\sup_{\bar{t} \leq t \leq T} \|\nabla w(t)\|_2 \leq c_T, \qquad \forall T \in (\bar{t}, \infty),$$

which follows from estimates of the integral equation (5.1) together with (2.7) by use of $w(\bar{t}) \in H^1_{0,\sigma}(\Omega)$, to find

$$\begin{aligned} &\frac{d}{dt}\|\nabla w(t)\|_2^2 + \|Aw(t)\|_2^2 \\ &\leq C\Big(\|u_\infty + u_s\|_\infty^2 + \|\widetilde{U}(t)\|_\infty^2 + \|\nabla u_s\|_3^2 + \|\nabla\widetilde{U}(t)\|_{3,\infty}^2 + c_T^2 + c_T^4\Big)\|\nabla w(t)\|_2^2 \\ &\quad + C\|f(t)\|_2^2 \end{aligned}$$

for all $t \in (\bar{t}, T]$, where $T \in (\bar{t}, \infty)$ is fixed. Note that the coefficient of $\|\nabla w\|_2^2$ as well as $\|f\|_2^2$ in the RHS above belongs to $L^\infty(\bar{t}, T)$. We thus employ (5.14) to see that

$$\nabla w \in L^\infty(\bar{t}, T; L^2(\Omega)), \qquad Aw \in L^2(\bar{t}, T; L^2_\sigma(\Omega)).$$

By the equation (5.12) and by

$$\|w\|_\infty^2 \leq C\|Aw\|_2\|\nabla w\|_2 + C\|\nabla w\|_2^2$$

(see Heywood [24]), we conclude the others in (5.15) as well. □

The following proposition plays an important role in the proof of the main theorems. For the weak solution constructed in the previous section, the existence of $\bar{t}$ satisfying the requirement below will be shown in the following section.

Proposition 5.3. *Let $\bar{t} \in (T_0, \infty)$, where T_0 is as in* (1.6) *or* (1.13). *Let $w(t)$ be a weak solution to* (1.11) *on $[\bar{t}, \infty)$ with* (1.12) *for $s = \bar{t}$, and $w(\bar{t})$ satisfy* (5.7) *as well as $w(\bar{t}) \in H^1_{0,\sigma}(\Omega)$. Assume further* (5.6) *and* (5.8). *By $\widetilde{w}(t)$ we denote the strong solution on $[\bar{t}, \infty)$ to* (1.11) *with initial condition $\widetilde{w}(\bar{t}) = w(\bar{t})$, which is obtained in Proposition 5.2. Then we have*

$$w(t) = \widetilde{w}(t) \qquad on\ [\bar{t}, \infty),$$

and thereby

$$\|w(t)\|_\infty = O(t^{-1/2}) \qquad as\ t \to \infty.$$

For the landing problem, the condition (5.6) *is redundant.*

Proof. We follow the argument of Serrin [40]. In view of (5.9), (5.11), (5.14) and (5.15) one can take the strong solution $\widetilde{w}(t)$ as a test function, see (4.7), in the relation (4.6) (with $s = \bar{t}$) for the weak solution $w(t)$. We gather the resulting formula, (5.12), (5.13) for $\widetilde{w}(t)$ and (1.12) (with $s = \bar{t}$) for $w(t)$ to find

$$\begin{aligned}&\frac{1}{2}\|w(t) - \widetilde{w}(t)\|_2^2 + \int_{\bar{t}}^t \|\nabla w - \nabla \widetilde{w}\|_2^2 d\tau \\ &\le \int_{\bar{t}}^t \langle (\widetilde{w} + \widetilde{U} + u_s) \otimes (w - \widetilde{w}), \nabla w - \nabla \widetilde{w} \rangle d\tau\end{aligned}$$

for all $t \ge \bar{t}$. By (5.15) together with (3.22) we know

$$\widetilde{w} + \widetilde{U} + u_s \in L^2_{loc}([\bar{t}, \infty); L^\infty(\Omega)).$$

Hence, we deduce from the inequality

$$\|w(t) - \widetilde{w}(t)\|_2^2 \le \int_{\bar{t}}^t \|\widetilde{w} + \widetilde{U} + u_s\|_\infty^2 \|w - \widetilde{w}\|_2^2 d\tau$$

that both solutions must coincide for $t \ge \bar{t}$. Thus, the large time behavior of the weak solution $w(t)$ follows from (5.9). □

6 Proof of main theorems

We are now in a position to prove the main theorems. Let $w(t)$ be the weak solution to (1.11) with (4.1) obtained in Proposition 4.1. Let us start with the energy inequality (1.12) for $s = 0$. By (3.29) we have

$$|\langle f, w \rangle| \le \left\{ C(1+t)^{-1/2} + \|\widetilde{U} \otimes \widetilde{U} + h(\widetilde{U} \otimes u_s + u_s \otimes \widetilde{U})\|_2 \right\} \|\nabla w\|_2.$$

Given small $\varepsilon > 0$, to be determined later, see (6.8), we deduce from (4.3) that there is $T_\varepsilon > 0$ such that

$$\|\widetilde{U} \otimes \widetilde{U} + h(\widetilde{U} \otimes u_s + u_s \otimes \widetilde{U})\|_2^2 \le \varepsilon t^{-1/2}, \qquad \forall t \ge T_\varepsilon,$$

which implies that

$$\begin{aligned} &\left| \int_0^t \langle f, w \rangle d\tau \right| \\ &\le \frac{1}{4} \int_0^t \|\nabla w\|_2^2 d\tau + C \int_0^t \frac{d\tau}{1+\tau} + C \int_0^{T_\varepsilon} \tau^{-1/2} d\tau + 2\varepsilon \int_{T_\varepsilon}^t \tau^{-1/2} d\tau \end{aligned} \tag{6.1}$$

for all $t > T_\varepsilon$. As for the second term of the RHS of (1.12), we observe

$$\left| \int_0^t \langle (hu_s + \widetilde{U}) \otimes w, \nabla w \rangle d\tau \right| \le c_0 \int_0^t \left(|h(\tau)| \|u_s\|_3 + \|\widetilde{U}(\tau)\|_{3,\infty} \right) \|\nabla w\|_2^2 d\tau.$$

Thanks to (3.25), there is $T_1 \in (T_0, \infty)$ such that

$$\|\widetilde{U}(t)\|_{3,\infty} \le \frac{1}{8c_0} \qquad \forall t \ge T_1,$$

where T_0 is as in (1.6) or (1.13). Suppose that the steady flow u_s is so small that

$$\|u_s\|_3 \le \frac{1}{8c_0}. \tag{6.2}$$

Then we get

$$\begin{aligned} &\left| \int_0^t \langle (hu_s + \widetilde{U}) \otimes w, \nabla w \rangle d\tau \right| \\ &\le \frac{1}{4} \int_{T_1}^t \|\nabla w\|_2^2 d\tau + C\left(|h|_\infty \|u_s\|_3 + \|v_0\|_{3,\infty} + M_3 \right) \int_0^{T_1} \|\nabla w\|_2^2 d\tau \end{aligned} \tag{6.3}$$

for all $t > T_1$. From (1.12) for $s = 0$ together with (6.1) and (6.3) we find

$$\|w(t)\|_2^2 + \int_0^t \|\nabla w\|_2^2 d\tau \le K_\varepsilon + C \log(1+t) + 8\varepsilon\sqrt{t} \tag{6.4}$$

for all $t > \max\{T_\varepsilon, T_1\}$, and, therefore,

$$\int_t^{2t} \|\nabla w\|_2^2 d\tau \le K_\varepsilon + C \log(1+2t) + 8\varepsilon\sqrt{2t} \tag{6.5}$$

for all $t > \max\{T_\varepsilon/2, T_1/2\}$, where

$$K_\varepsilon = \|w_0\|_2^2 + C\left(|h|_\infty \|u_s\|_3 + \|v_0\|_{3,\infty} + M_3\right) \int_0^{T_1} \|\nabla w\|_2^2 d\tau + C\sqrt{T_\varepsilon}.$$

Let us recall the condition (5.8) in the previous section. By (3.18) there is

$$T_2 > \max\{T_\varepsilon, T_1\} \tag{6.6}$$

such that

$$\|U(t)\|_{3,\infty,\mathbb{R}^3} \le \delta_3, \qquad \forall t \ge T_2, \tag{6.7}$$

where δ_3 is the constant in Propositon 5.1. By Proposition 4.2 we know that there is a set $J \subset (0,\infty)$ with the Lebesgue measure $|J| = 0$ such that $w(t)$ satisfies (1.12) for all $s \in (0,\infty) \setminus J$ and $t > s$. On account of (6.5) as well as (6.4), for every $t > T_2$, one can find $\bar{t} \in (t, 2t) \setminus J$ such that

$$\|\nabla w(\bar{t})\|_2^2 \le \frac{2}{t}\left(K_\varepsilon + C\log(1+2t) + 8\varepsilon\sqrt{2t}\right),$$

$$\|w(\bar{t})\|_2^2 \le K_\varepsilon + C\log(1+2t) + 8\varepsilon\sqrt{2t},$$

which yield

$$\|w(\bar{t})\|_3^4 \le C\|\nabla w(\bar{t})\|_2^2 \|w(\bar{t})\|_2^2 \le \frac{c_*}{t}\left[\{K_\varepsilon + \log(1+2t)\}^2 + \varepsilon^2 t\right].$$

Let $\delta_2 > 0$ be the constant in Proposition 5.1. We first choose and fix $\varepsilon > 0$ such that

$$c_*\varepsilon^2 \le \frac{\delta_2^4}{2}. \tag{6.8}$$

For such $\varepsilon > 0$, we take T_2 satisfying (6.6)–(6.7) and then find $T_3 \in (T_2, \infty)$ so that

$$\frac{c_*}{t}\{K_\varepsilon + \log(1+2t)\}^2 \le \frac{\delta_2^4}{2} \qquad \forall t \ge T_3.$$

Let us fix $t \ge T_3\,(> T_2)$, for which we find $\bar{t} \in (t, 2t) \setminus J$ such that $w(\bar{t}) \in H^1_{0,\sigma}(\Omega)$ with

$$\|w(\bar{t})\|_3 \le \delta_2. \tag{6.9}$$

Suppose that the steady flow u_s is so amall that (5.6) as well as (6.2) holds. By (3.1) there is a constant $\delta \in (0, \delta_0]$ such that the condition $|u_\infty| \le \delta$ implies both of them. Then, by virtue of (6.9) together with (6.7), all the assumptions in Proposition 5.3 are fulfilled. We thus obtain the decay property

$$\|w(t)\|_\infty = O(t^{-1/2}) \qquad \text{as } t \to \infty$$

which together with (3.25) leads us to (1.7) in view of (1.8). For the landing problem, it is obvious to obtain (1.14) without any smallness condition on the steady flow u_s. We have thus completed the proof of both Theorems 1.1 and 1.2. $\square$

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References

[1] K.I. Babenko, On staitonary solutions of the problem of flow past a body of viscous incompressible fluid, *Math. Sb.* **91** (1973), 3–27; English Translation: *Math. USSR Sbornik* **20** (1973), 1–25.

[2] T. Baker and G. Seregin, On global solutions to the Navier-Stokes system with large $L^{3,\infty}$ initial data, arXiv:1603.03211.

[3] J. Bergh and J. Löfström, *Interpolation Spaces*, Springer, Berlin, 1976.

[4] M. E. Bogovskiĭ, Solution of the first boundary value problem for the equation of continuity of an incompressible medium, *Soviet Math. Dokl.* **20** (1979), 1094–1098.

[5] W. Borchers and H. Sohr, On the semigroup of the Stokes opeartor for exterior domains in L^q-spaces, *Math. Z.* **196** (1987), 415–425.

[6] W. Borchers and H. Sohr, On the equations rot $v = g$ and div $u = f$ with zero boundary conditions, *Hokkaido Math. J.* **19** (1990), 67–87.

[7] Z.M. Chen, Solutions of the stationary and nonstationary Navier-Stokes equations in exterior domains, *Pacific J. Math.* **159** (1993), 227–240.

[8] R. Courant and D. Hilbert, *Methoden der Mathematischen Physik II*, Berlin-Heidelberg-New York, Springer, 1968.

[9] Y. Enomoto and Y. Shibata, Local energy decay of solutions to the Oseen equation in the exterior domains, *Indiana Univ. Math. J.* **53** (2004), 1291–1330.

[10] Y. Enomoto and Y. Shibata, On the rate of decay of the Oseen semigroup in exterior domains and its applications to the Navier-Stokes equation, *J. Math. Fluid Mech.* **7** (2005), 339–367.

[11] R. Farwig, The stationary exterior 3D-problem of Oseen and Navier-Stokes equations in anisotropically weighted Sobolev spaces, *Math. Z.* **211** (1992), 409–447.

[12] R. Farwig and H. Sohr, Generalized resolvent estimates for the Stokes system in bounded and unbounded domains, *J. Math. Soc. Japan* **46** (1994), 607–643.

[13] R. Farwig and H. Sohr, Weighted estimates for the Oseen equations and the Navier-Stokes equations in exterior domains, 11–30, *Theory of the Navier-Stokes Equations, Ser. Adv. Math. Appl. Sci.* **47**, World. Sci. Publ., River Edge, NJ, 1998.

[14] R. Finn, An energy theorem for viscous fluid motions, *Arch. Rational Mech. Anal.* **6** (1960), 371–381.

[15] R. Finn, Stationary solutions of the Navier-Stokes equations, *Proc. Symp. Appl. Math.* **17** (1965), 121–153.

[16] R. Finn, On the exterior statioonary problem for the Navier-Stokes equations, and associated perturbation problems, *Arch. Rational Mech. Anal.* 1965 **19** (1965), 363–406.

[17] G.P. Galdi, On the asymptotic structure of D-solutions to steady Navier-Stokes equations in exterior domains, 81–104, *Mathematical Problems relating to the Navier-Stokes Equations*, *Ser. Adv. Math. Appl. Sci.* **11**, World. Sci. Publ., River Edge, NJ, 1992.

[18] G. P. Galdi, *An Introduction to the Mathematical Theory of the Navier-Stokes Equations*, *Steady-State Problems*, Second Edition, Springer, 2011.

[19] G. P. Galdi, J. G. Heywood and Y. Shibata, On the global existence and convergence to steady state of Navier-Stokes flow past an obstacle that is started from rest, *Arch. Rational Mech. Anal.* **138** (1997), 307–318.

[20] M, Geissert, H. Heck, M. Hieber, On the equation div $u = g$ and Bogovskiĭ's operator in Sobolev spaces of negative order, *Operator Theory: Advances and Applications* **168** (2006), 113–121.

[21] Y. Giga, Analyticity of the semigroup generated by the Stokes operator in L_r spaces, *Math. Z.* **178** (1981), 297–329.

[22] Y. Giga and H. Sohr, Abstract L^p estimates for the Cauchy problem with applications to the Navier-Stokes equations in exterior domains, *J. Funct. Anal.* **102** (1991), 72–94.

[23] J. G. Heywood, The exterior nonstationary problem for the Navier-Stokes equations, *Acta Math.* **129** (1972), 11–34.

[24] J. G. Heywood, The Navier-Stokes equations: on the existence, regularity and decay, *Indiana Univ. Math. J.* **29** (1980), 639–681.

[25] E. Hopf, Über die Aufangswertaufgabe für die hydrodynamischen Grundgleichungen, *Math. Nachr.* **4** (1950), 213–231.

[26] H. Iwashita, L_q-L_r estimates for solutions of the nonstationary Stokes equations in an exterior domain and the Navier-Stokes initial value problems in L_q spaces, *Math. Ann.* **285** (1989), 265–288.

[27] T. Kato, Strong L^p solutions of the Navier-Stokes equation in $\mathbb{R}^m$, with applications to weak solutions, *Math. Z.* **187** (1984), 471–480.

[28] H. Koba, On $L^{3,\infty}$-stability of the Navier-Stokes system in exterior domains, *J. Differential Equations* **262** (2017), 2618–2683.

[29] T. Kobayashi and Y. Shibata, On the Oseen equation in the three dimensional exterior domains, *Math. Ann.* **310** (1998), 1–45.

[30] J. Leray, Etude de diverses equations integrales non lineaires et de quelques problemes que pose l'Hydrodynamique, *J. Math. Pures Appl.* **12** (1933), 1–82.

[31] J. Leray, Sur le mouvement d'un fluide visqueux emplissant l'espace, *Acta Math.* **63** (1934), 193–248.

[32] P. Maremonti, A remark on the Stokes problem in Lorentz spaces, *Disc. Conti. Dyna. Systems Ser. S.* **6** (2013), 1323–1342.

[33] P. Maremonti, Regular solutions to the Navier-Stokes equations with an initial data in $L(3,\infty)$, *Ricerche Mat.*, DOI 10.1007/s11587-016-0287-7.

[34] P. Maremonti, Weak solutions to the Navier-Stokes equations with data in $\mathbb{L}(3,\infty)$, "*Mathematics for Nonlinear Phenomena: Analysis and Computation*", Proceedings in honor of Professor Giga's 60th birthday, Springer (to appear).

[35] P. Maremonti and V.A. Solonnikov, On nonstationary Stokes problems in exterior domains, *Ann. Sc. Norm. Sup. Pisa* **24** (1997), 395–449.

[36] K. Masuda, Weak solutions of Navier-Stokes equations, *Tohoku Math. J.* **36** (1984), 623–646.

[37] T. Miyakawa, On nonstationary solutions of the Navier-Stokes equations in an exterior domain, *Hiroshima Math. J.* **12** (1982), 115–140.

[38] T. Miyakawa and H. Sohr, On energy inequality, smoothness and large time behavior in L^2 for weak solutions of the Navier-Stokes equations in exterior domains, *Math. Z.* **199** (1988), 455–478.

[39] G. Seregin and V. Sverak, On global weak solutions to the Cauchy problem for the Navier-Stokes equations with Large L_3-initial data, arXiv:1601.03096.

[40] J. Serrin, The initial value problem for the Navier-Stokes equation, 69–98, *Nonlinear Problems*, Univ. Wisconsin Press, 1963.

[41] Y. Shibata, On an exterior initial value boundary problem for Navier-Stokes equations, *Quart. Appl. Math.* **57** (1999), 117–155.

[42] C.G. Simader and H. Sohr, A new approach to the Helmholtz decomposition and the Neumann problem in L^q-spaces for bounded and exterior domains, 1–35, *Mathematical Problems relating to the Navier-Stokes Equations*, *Ser. Adv. Math. Appl. Sci.* **11**, World Sci. Publ., River Edge, NJ, 1992.

[43] V.A. Solonnikov, Estimates for solutions of nonstationary Navier-Stokes equations, *J. Sov. Math.* **8** (1977), 467–529.